Hilbert Space and Propagator in Thermal Field Theory

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By use of general principles (Kubo-Martin-Schwinger condition, breakdown of Lorentz invariance, triviality arguments), the nature of diagrammatic perturbation theory in relativistic field theory at nonzero temperature is investigated. By operator-algebraic techniques it is found that the conventional method is inconsistent, and an essentially unique alternative is given.

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Relativistic quantum field theory at nonzero temperature (T) is a basic theoretical tool for the study of the quark-gluon plasma, which may be created in heavy-ion collisions, and, more generally, for the description of the early Universe. In contrast to the vacuum case ($T=0$), perturbation theory has been of little help in this area. For example, in quantum chromodynamics the naive thermal perturbation series blows up as a consequence of uncontrollable infrared divergences.[1]

Even though problems like this might be remedied by certain partial resummations, it seems worthwhile to formulate a perturbation method that is well defined and consistent without the need of sometimes rather *ad hoc* rescue operations. This amounts to the identification of the correct free Hamiltonian (or Lagrangean), as it is the free propagator that determines the nature of the perturbation series.

Now conventionally[2] one just takes over the free Hamiltonian H_0 from the vacuum theory, and goes on to expand in terms of the renormalized coupling constant λ. At $T=0$, H_0 is uniquely determined for field theories describing scattering of particles of mass $m>0$. Namely, H_0 should be such that the free (scalar) propagator becomes $D(p)=(p^2-m^2+i\epsilon)^{-1}$, as only this choice implies that the S matrix is unitary order by order in perturbation theory as a consequence of the cutting equations.[3] Moreover, the expansion parameter λ has a direct physical meaning as the scattering amplitude at some given energy.

At nonzero temperature, however, this picture dramatically changes. In order to discuss structural issues like this it is convenient to assume that the system is infinitely extended; the same approximation is always made in quantum scattering theory. Infinite systems can be studied by operator-algebraic methods; our recourse to operator and Hilbert-space techniques is further motivated by the nonperturbative solution[4] of the infrared problem in vacuum QED, in which the identification of the nature of the photon Hilbert space turned out to be crucial.

The operator-algebraic approach to quantum statistical mechanics[5] (including thermal field theory) starts with a state ω (a positive linear functional over the abstract operator algebra $\mathfrak{A}$) satisfying the Kubo-Martin-Schwinger (KMS) condition[2,5] at $T=\beta^1$. Subsequently, $\mathfrak{A}$ is represented by operators $\pi_\omega(\mathfrak{A})$ on a concrete Hilbert space $\mathcal{H}_\omega$ with cyclic ground states $|0(\beta)\rangle$ (also called Ω_ω), such that $\omega(A)=\langle 0(\beta)|\pi_\omega(A)|0(\beta)\rangle$ [Gel'fand-Naimark-Segal (GNS) representation induced by ω [2,5]]. The essence of this construction is that the thermal average is equal to a generalized vacuum expectation value. If $\mathfrak{A}$ describes a field theory this construction leads to the so-called thermo field dynamics.[2,6,7] Ordinary ($T=0$) quantum field theory can be set up similary; then, of course, ω is chosen to be a vacuum state. Most importantly, under a mass gap assumption, $\mathcal{H}_\omega$ can be shown to be (more precisely, to include) a Fock space spanned by particle states. This, in turn, leads to the perturbation theory described above.

I would like to have an analog of this structural result at nonzero T as well. So let us ask, given a KMS state ω, what is the concrete realization of $\mathcal{H}_\omega$, and what kind of consistent perturbation theory is implied by it? This question is decidedly nontrivial even if the answer is known at zero temperature, because representations of $\mathfrak{A}$ at different temperatures are disjoint[5] (a strong form of unitary inequivalence). In fact, the identification of the correct perturbation theory crucially depends on the representation (vacuum, thermal) as well as on the dynamics.

The KMS condition implies[2,5] three basic differences between zero- and finite-temperature representations: (i) There is a one-to-one correspondence $\mathfrak{M}\in A \leftrightarrow \tilde{A}\in\mathfrak{M}'$ between $\mathfrak{M}=\pi_\omega(\mathfrak{A})''$ and its commutant $\mathfrak{M}'$. In particular, $|0(\beta)\rangle$ is cyclic for $\mathfrak{M}'$, contrary to (pure) vacuum representations, where $\mathfrak{M}'$ is trivial. (ii) The generator of time translations $\hat{H}$ in $\mathcal{H}_\omega$, which acts as the effective Hamiltonian, is unbounded from below, and its spectrum is[8] $\mathbb{R}$. This follows from the relation[2] $\hat{H}=\lim_{V\to\infty} H_V-\tilde{H}_V$, where H_V is the ordinary, finite-volume Hamiltonian. This means that all (would-be) particles are unstable in the presence of interaction. (iii) Lorentz invariance is spontaneously broken, and boost symmetry cannot be unitarily implemented.[9,10] Although in KMS representations this does not imply the existence of Goldstone bosons,[10,11] it does indicate the absence of a dispersion relation (see below).

An important property of $\mathcal{H}_\omega$ is that it carries a unitary representation of the unbroken symmetry group G; in the vacuum case one has $G=P$, the Poincaré group, and in the thermal KMS case $G=\mathrm{SO}(3)(\times T_4$, which is P with boosts left out. If G is a type-I group[12] (as in these cases), Mautner's theorem[12] states that $\mathcal{H}_\omega$ can be decomposed as a direct integral over carrier spaces $\mathcal{H}(\hat{g})$ of inequivalent irreducible unitary representations of G, labeled by $\hat{g}\in\hat{G}$, the dual of G. So up to unitary equivalence,

$$\mathcal{H}_\omega=\int_{\hat{G}} d\mu(\hat{g})\, n(\hat{g})\mathcal{H}(\hat{g}), \qquad (1)$$

where μ is a measure on $\hat{G}$ and $n(\hat{g})$ is the multiplicity of the representation $\hat{g}$. In the vacuum case $\hat{G}=\hat{P}$ is labeled by the mass m^2 (and the spin for $m^2>0$, etc.), and (if we ignore the spin for simplicity) the carrier space $\mathcal{H}(m^2)$ consists of square-integrable functions of four-momentum p restricted to the orbit $p^2=m^2$, $p_0>0$. In the language of (1), the existence of one-particle states with mass gap implies that $\mu(m^2)$ is the sum of a point measure and a continuous measure. It is important to notice that the dispersion relation $p_0=\omega_p=(\mathbf{p}^2+m^2)^{1/2}$ is implicit in the structure of $\hat{P}$ and $\mathcal{H}(\hat{p})$.

At nonzero T we need to know the unitary irreps of $\mathrm{SO}(3)(\times T_4$; simple application of the theory of induced representations[12] shows that these are labeled by three numbers $E\in\mathbb{R}$, $\sigma^2\in\mathbb{R}^+$, and $n\in\mathbb{Z}$ (hence $\hat{G}=\mathbb{R}\times\mathbb{R}^+\times\mathbb{Z}$). The carrier space $\mathcal{H}(E,\sigma^2,n)$ consists of square-integrable functions $\phi_{E,\sigma}(p)$ restricted to the orbit $p_0=E$, $\mathbf{p}^2=\sigma^2$ (n enters in the explicit action of group elements on these functions). One immediately notes the complete decoupling of energy and momentum and, hence, the absence of a dispersion relation.[9] The crucial question is now: What is $d\mu(E,\sigma^2,n)$? Conventional perturbation theory[2] tacitly and implicitly presumes that $d\mu$ contains a point contribution $dE\, d\sigma^2\delta(E-\epsilon(\sigma^2))$ for some "dispersion relation" $\epsilon(\mathbf{p}^2)$. (By this I mean that standard perturbation theory can only be rigorously justified if this is the case.) Equivalently, this means that the operator $\hat{H}-\epsilon(\hat{\mathbf{P}}^2)$ has proper eigenstates other than $|0(\beta)\rangle$.

Most distressingly, such an assumption cannot be justified in an interacting system. First, contrary to the vacuum case, a dispersion relation is not suggested by the group theory. Second, spectral analysis[7] shows that the full two-point function should then have a pole in $p_0=\epsilon(\mathbf{p}^2)$, which in reality is not seen in sufficiently high order of (standard) perturbation theory.[2,7] Third, it can be shown[13] that, as a consequence of the operator property (i) above, such "quasiparticles" with energy $\epsilon(\mathbf{p}^2)$ would not scatter if they existed. This, in fact, is the major contrast to the corresponding situation at $T=0$: There is no room for interaction between exact quasiparticles at nonzero T. Indeed, point (ii) above already contradicts the notion of stable, asymptotic on-shell particles.

This argument purports to show that the description of interacting field theories at nonzero temperature should be drastically modified. Now that we are denied interacting particle states in $\mathcal{H}_\omega$, the part of $\mathcal{H}_\omega$ containing interacting states must have the structure

$$\mathcal{H}_\omega^{\mathrm{int}}=\mathcal{H}(0,0,0)\oplus\infty\cdot\mathcal{H}^{(1)}, \qquad (2)$$

where I have abbreviated $\mathcal{H}^{(1)}=\int_{-\infty}^{\infty}dE\int_0^\infty d\sigma^2\mathcal{H}(E,\sigma^2,0)$, and for simplicity I have restricted myself to scalar field theories. The first term in (2) accounts for the state $|0(\beta)\rangle$, while the second term, in fact, summarizes what we know about the spectrum of $\hat{H}$ and $\hat{\mathbf{P}}$. Note that any other multiplicity than ∞ in (2) would be most puzzling. It is a highly nontrivial feature of the Clebsch-Gordan series of the group $\mathrm{SO}(3)(\times T_4$ that $\mathcal{H}_\omega^{\mathrm{int}}$ in (2) can be written[14] as a Fock space over $\mathcal{H}^{(1)}$. Since $\mathcal{H}^{(1)}$ is isomorphic to $L^2(\mathbb{R}^4)$, our results boil down to the proposal that the Hilbert space of the GNS representation induced by a KMS state ω is a Fock space $[L^2(\mathbb{R}^4)]$ if the operator algebra describes an interesting scalar field theory without collective modes, Goldstone bosons, etc. If the latter exist, as in more complicated field theories, they contribute further terms to $\mathcal{H}_\omega$ involving point measures in (1); such modes would have to propagate undamped in accordance with the result in Ref. 13. In the following, I shall ignore these and set $\mathcal{H}_\omega=\mathcal{H}_\omega^{\mathrm{int}}$. For free fields $\mathcal{H}_\omega$ is known[2,7] to be the direct product of two identical copies of the $T=0$ Hilbert-Fock space $[L^2(\mathbb{R}^3)]$, and so interactions distinctly alter the Hilbert space; this occasionally happens at $T=0$ as well.[7]

The next step is to determine the perturbation theory consistent with this result. To do so, let us first identify a complete set of operators on $\mathcal{H}_\omega$. Because of the Fock nature of $\mathcal{H}_\omega$, the creation and annihilation operators $\beta^{(\dagger)}(E,\sigma^2,\theta,\phi)$ do the job. Here the β's annihilate $|0(\beta)\rangle$ and are affiliated[5] to $\mathfrak{M}\cup\mathfrak{M}'$ [cf. (i) above]. [β is called $a(\beta)$ in Refs. 2 and 7.] Since $(\hat{H})^\sim=-\hat{H}$ we have $\tilde{\beta}^{(\dagger)}(E,\dots)=\beta^{(\dagger)}(-E,\dots)$, so that we can set $E\geq 0$ in operators without a tilde. Subsequently, $\beta^{(\dagger)}$ and $\tilde{\beta}^{(\dagger)}$ are Bogoliubov transformed[2,7] into operators $a^{(\dagger)}$ and $\tilde{a}^{(\dagger)}$ that are affiliated to $\mathfrak{M}$ and $\mathfrak{M}'$, respectively. These operators are then used to construct covariant operator fields.[15] For $n\neq 0$ certain technical complications arise,[14] which are similar to those encountered in the construction of Poincaré-covariant massless fields, and so here I shall just give the result for a scalar field ($E>0$):

$$\phi_{E,\sigma}(x)=E^{-1/2}\sigma^{-2}(2\pi)^{-3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}-iEt}\delta(|\mathbf{p}|-\sigma)a(E,\mathbf{p})+\mathrm{H.c.}, \qquad (3)$$

where σ, θ, and ϕ are assembled in $\mathbf{p}$. We see that ϕ is a field with two parameters σ and E; as such it forms a generalization of the so-called one-parameter Licht field[16] that has been employed to describe unstable particles at $T=0$.[17]

By construction, the action of $\hat{H}$ on a dense set in $\mathcal{H}_\omega$ coincides with that of

$$\hat{H}_0(\phi)=\tfrac{1}{2}\int d^3x\int_0^\infty (dE/2\pi)\int d\sigma_1\, d\sigma_2\sigma_1^2\sigma_2^2[\dot{\phi}_{E,\sigma_1}(x)\dot{\phi}+E^2\phi_{E,\sigma_1}(x)\phi_{E,\sigma_2}(x)-\text{t.c.}], \tag{4}$$

where t.c. means tilde conjugate; clearly $\hat{H}_0$ governs the time evolution of the $\phi_{E,\sigma}(x)$.

For the derivation of an analog of the Gell-Mann–Low (GML) formula and thence of diagrammatic perturbation theory one needs a 'free" Hamiltonian $\hat{H}_0(A)$ expressed in terms of the Heisenberg-picture fields $A(x)$, and an asymptotic condition in the Lehmann-Symanzik-Zimmermann (LSZ) sense $A(x)\to\phi(x)$. If $\hat{H}_0(A)$ is such[7] that $\hat{H}_0(\phi)$ is weakly equal to the full $\hat{H}(A)$ in the representation space $\mathcal{H}_\omega$, then the GML formula can be derived[14] without use of the nonexistent "in" picture. It may be remarked that conventional perturbation theory at $T>0$ based on the naive $\hat{H}_0$ from the vacuum theory meets neither of the above criteria. Namely, the spectrum of the naive $\hat{H}_0$ differs from that of $\hat{H}$, and in addition the field $A(x)$ has a vanishing LSZ limit, as in the unstable-particle case at $T=0$.[16,17] I remedy this as follows. I introduce a function $Z(E,\sigma)$ defined by

$$\langle 0(\beta)\,|\,A(x)\,|\,p\rangle = e^{-iEt+i\mathbf{p}\cdot\mathbf{x}}Z^{1/2}(E,\sigma),$$

and then spectrally decompose the field $A(x)$ according to

$$A(x)=\int_0^\infty (dE/2\pi)d\sigma\sigma^2 Z^{1/2}(E,\sigma)A_{E,\sigma}(x). \tag{5}$$

Here the $A_{E,\sigma}(x)$ must be chosen such that $\langle 0(\beta)\times|A_{E,\sigma}(x)|p\rangle\neq 0$ for $|p\rangle=|E,\sigma^2,\theta,\phi\rangle\in\mathcal{H}^{(1)}$; that (5) makes sense follows from the completeness of the $\phi_{E,\sigma}(x)$.[14] An analogous construction using Licht fields $A_s(x)$ has been performed at $T=0$, where it can be shown that in contrast to $A(x)$ the $A_s(x)$ do have an LSZ limit in the distributional sense.[16,17] At $T>0$ the existence of an asymptotic limit is hard to prove because of poor cluster properties, and so we have to postulate it. If there is no such limit, the following is void of content. If there is such a limit, the normalization factor Z in (5) ascertains that the $\phi_{E,\sigma}(x)$ can be chosen so as to be equal to the LSZ limit of $A_{E,\sigma}(x)$ for $t\to-\infty$.

According to the preceding remarks one can then derive the GML formula for n-point functions of the $A_{E,\sigma}(x)$, and hence of $A(x)$ by (5), by choosing $\hat{H}_0=\hat{H}_0(A)$, which is simply (4) with $\phi_{E,\sigma}$ replaced by $A_{E,\sigma}$. This leads[14] to the free propagator (in thermal-doublet notation[7,2])

$$\int d^4x\, e^{ipx}\langle 0(\beta)\,|\,TA^\mu(x)A^\nu(0)\,|\,0(\beta)\rangle_0 = \int_0^\infty dE\, Z(E,|\mathbf{p}|)\tilde{D}^{\mu\nu}(p_0,E). \tag{6}$$

Here $\tilde{D}^{\mu\nu}(p_0,E)$ is the well-known[2] free thermal propagator matrix with ω_p replaced by E. Notice that the illegal choice $Z=\delta(E-\omega_p)$ reproduces the conventional formalism. My particular choice for $\hat{H}_0$ also induces a counterterm, because $\hat{H}_0$ does not occur already in $\hat{H}$. The naive $\hat{H}_0$ from the vacuum theory emerges as a counterterm as well; the complete Feynman rules are given in Ref. 14. An important property is that perturbation theory with the propagator (6) identically satisfies the KMS condition.[2]

Finally, the function $Z(E,\sigma)$ should be specified. Clearly $Z(E,\sigma)$ is a continuous generalization of the wave-function-renormalization constant Z that appears in the case of a single asymptotic field. The determination of this constant [by the demand that the full propagator of $A(x)$ has a pole with residue Z] cannot be trivially generalized, because the thermal two-point function has no pole in p_0. However, the "bare-bones propagator"

$$\langle 0(\beta)\,|\,TA_{E,\sigma}(x)A_{E',\sigma'}(0)\,|\,0(\beta)\rangle$$

in momentum space does have a pole in $p_0=E$ by construction. This may be used to derive[14] a self-consistent equation for $Z(E,\sigma)$, which can be solved with use of perturbation theory for the ordinary self-energy $\Sigma^{\mu\nu}(p)$,[2,7] but now computed with internal lines given by (6) which itself contains Z. The solution will be a function with two free parameters μ and κ which determines the bare constants m_0 and λ_0 in $\hat{H}$ in terms of μ, κ, T, and a cutoff. Lacking a direct physical interpretation and being arbitrary, μ and κ may subsequently be exchanged for, e.g., the heat capacity c_v and a static transport coefficient η plus a set of renormalization conditions ("conditions of fit"). So ultimately one obtains a completely renormalized perturbation expansion in terms of thermal observables c_v and η rather than the contextually meaningless $T=0$ parameters m and λ. One can follow this algorithm even without actually having solved $Z(E,\sigma)$ from the self-consistency equation, by making a suitable two-parameter *Ansatz*. A useful guess would be

$$Z(E,\sigma)=\kappa/\pi[(E-(\mu^2+\sigma^2)^{1/2})^2+\kappa^2]^{-1}.$$

Surprisingly enough, for $\kappa\ll\mu$ [so that the E integration in (6) can be extended to $-\infty$] this choice reproduces the "dissipative perturbation theory" proposed by Arimitsu, Umezawa, and Yamanaka,[18] including its purely imaginary counterterms,[14] in the stationary case.

Let me finally note that my formalism is highly *non*-perturbative from the point of view of naive perturbation theory. In particular, the latter does not produce the correct analytic properties of the full propagator (e.g., a cut through the entire real axis, related to the theory of H^p spaces) in any finite order, although the necessary accumulation of branch points is approached in higher orders.[19]

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