Intuitionistic quantum logic

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Bachelor's thesis
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Preface

This thesis in front of you is the result of a year of work. I have learned so much in the last year, but I could not have done so alone.

First of all, I thank Klaas Landsman, my supervisor. The idea for this project began with some notes that you wrote on a piece of paper, which I still have now. Thank you for the support and valuable feedback that you have given us, and for allowing Evert and myself to work together on this thesis. The weekly meetings with the three of us were always very productive and enjoyable.

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That is where Bert Lindenhovius comes in. Evert and I travelled to New Orleans to work with Bert on the project for two weeks. Thank you Bert for the never ending supply of ideas and support, and for showing us around the city, both during the day and at night.

Most of all, I thank Evert-Jan Hekkelman, my partner during this project. You have become an even better friend during this year. Apart from the serious academic work, you were always in for a laugh. Even though we have written separate theses, most of the ideas are shared work.

Lastly, I would like to thank my girlfriend for supporting me during the summer months. Even though times were sometimes rough, I was able to finish the thesis on time. Thank you, Vera.

Introduction

From the title of this thesis, we can already identify three main themes:

- **Intuitionism**: an approach to constructive mathematics initiated by Brouwer;
- **Quantum mechanics**: the theory of nature at the smallest level;
- **Logic**: concerns mathematical reasoning itself.

We start by explaining what we mean exactly by **logic**, in particular **classical propositional logic**. It all begins with **propositions**, such as

My name is Quinten

and

Writing this thesis was a bore

nothing being implied about the truth of these statements. For easy of notation, we will write propositions as p, q, r etc. . We can combine propositions into new ones, for example

- $p \vee q$, which is true when p or q is true (non-exclusive);
- $p \wedge q$, which is true when p and q are both true;
- $\neg p$, which is true when p is **not** true.

Starting with **axioms** and **deduction rules**, we can show that some propositions can be **proven** from other ones. We say that two propositions are equivalent if each can be proven from the other one. In classical logic, this equivalence relation is required to satisfy certain identities, such as

$$p \vee (r \wedge s) \sim (p \vee r) \wedge (p \vee s)$$

and

$$\neg(\neg p) \sim p.$$

The first is called **distributivity** and the second is the **law of the excluded middle**. This means that in classical logic, the equivalence classes of propositions form a so-called **Boolean lattice**.

However, quantum logic is **not** classical. This can be understood by the following heuristic argument. We start with one of the most basic concepts of quantum mechanics, the **Heisenberg uncertainty principle**. Consider a particle of which we measure the position x and momentum p (at the same time), with uncertainty Δx and Δp, respectively. The uncertainty principle states that

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

That is, there is an upper bound to the certainty with which we can measure the position and momentum of a particle simultaneously. $\hbar$ is a physical constant, and for simplicity's sake, let's set it equal to 2 for now. Now, let's assume we find ourselves in the following situation:

- The position of the particle is bounded between 0 and 1;
- The momentum of the particle is bounded between 0 and 1.

This means that the uncertainty in both position and momentum is 1, which is in agreement with the uncertainty principle, because

$$\Delta x \Delta p = 1.$$

We are on thin ice, however, because if we decrease the uncertainty in either position or momentum any more, the particle will no longer satisty the uncertainty principle. Next, consider the following 'quantum propositions':

$$p := \text{ if we measure position, the value will be between 0 and 1 ;}$$
$$q := \text{ if we measure momentum, the value will be between 0 and 1 .}$$

We can 'split' proposition q into two 'smaller' propositions r and s, where

$$r := \text{ if we measure momentum, the value will be between 0 and } \frac{1}{2};$$
$$s := \text{ if we measure momentum, the value will be between } \frac{1}{2} \text{ and 1.}$$

Formally,

$$q = r \vee s.$$

For the specific situtation that we are in, both p and q are true, so we conclude that

$$p \wedge q \text{ is true .}$$

However, we cannot have

$$p \wedge q = p \wedge (r \vee s) = (p \wedge r) \vee (p \wedge s). \tag{1}$$

If this were the case, either $p \wedge r$ or $p \wedge s$ would have to be true. But the uncertainty in momentum is only $\frac{1}{2}$ if r or s is true. Therefore we would break the uncertainty principle! It seems that

quantum logic is **not distributive**.

But there is another possible interpretation. We have implicitly assumed that we could split the proposition q into r and s. Because in our situation the proposition q is always true, we will write q as 1, where 1 is the proposition that is always true. Because r and s obviously cannot be true at the same time, we have

$$r \wedge s = 0,$$

where 0 is the proposition that is always false. Here's the crunch: we have assumed that

$$r \vee s = 1, \tag{2}$$

as well. In other words, r and s are **complements** of each other. 2 is also called the **law of the excluded middle**. The alternative to non-distributivity is therefore:

In quantum logic, not every proposition has a complement, or:

In quantum logic, the law of the excluded middle does not hold.

Both options are possible, and both have been studied. Dropping distributivity is, historically, the most 'popular' solution, and gave rise to the field of **orthomodular lattices**, first introduced by Birkhoff and von Neumann (see [1]). Orthomodularity is strictly weaker than distributivity, but the law of the excluded middle still holds. The alternative is to look for a logical system that **is** distributive, but in which the law of the excluded middle **does not** hold. That is where **intuitionistic** logic comes in. Intuitionistic propositional logic is in a way a 'generalization' of classical propositional logic, in the sense that the law of the excluded middle is dropped. This was done in an effort to reflect constructive human reasoning better, instead of abstract truth: if no proof of the negation of a statement is possible, this does not mean that the statement itself is not true. Intuitionistic propositional logic is modeled algebraically by **Heyting lattices**. These are therefore the intuitionistic counterparts of Boolean lattices.

In this thesis, the goal is to understand certain Heyting lattices that are associated to quantummechanical systems. These are represented mathematically by so-called C*-algebras, which are complex normed algebras with additional structure. **Commutativity** is a very important property that distinguishes different C*-algebras. The reason why the argument using the uncertainty principle worked, is that position and momentum are an example of **non-commuting observables**. This is typical of quantum mechanics, because classical physics behaves in a commutative way. Somewhat more formally, we can say that

to a commutative C*-algebra (with enough projections) we can associate a **Boolean lattice**.

Therefore, commutative C*-algebras represent classical physics. We will find that

to any C*-algebra we can associate a **Heyting lattice**.

Actually, to any C*-algebra A we can associate a topological space Σ_A, and the open sets of this space form the Heyting lattice in question.

The original aim of this project was to extend the theory of **Stone duality** to the C*-algebra setting. Stone duality has to do with Boolean lattices, and it roughly says that

any Boolean lattice B can be seen as the lattice of clopen sets of a topological space.

This topological space is called the **Stone spectrum** of B. There is an analogous theory for Heyting lattices, called **Esakia duality**. In that case, we call the associated topological space the **Esakia spectrum** of the Heyting lattice H. We can now state the goal we had in mind:

What is the Esakia spectrum of the Heyting lattice associated to a C*-algebra?

This turned out to be a very difficult question, and so far, compared to what was conjectured, mostly negative results have been found. Therefore, alternative questions were posed, these being:

Can we understand the Heyting lattice better for a certain class of C*-algebras?

What are the **categorical** properties of the assignment of a C*-algebra to a Heyting lattice?

Category theory focuses on maps between objects, in this case C*-algebras, rather than on the objects themselves. This viewpoint generalizes many distant areas of mathematics, however, its origin is quite recent ([2]).

More succes was achieved in answering these last two questions. We were able to extend a certain result on Σ_A to the class of AW*-algebras. Furthermore, functoriality and limits were explored, yielding the following results:

- The association of a C*-algebra to a Heyting lattice is functorial, when the right domain is chosen.
- The Heyting lattice of a finite-dimensional C*-algebra is the limit of the Heyting lattices corresponding to the commutative subalgebras.

Some sections are shared work between myself and Evert-Jan. This will be elaborated on at the beginning of each chapter.

Chapter 1

Operator algebras and C*-algebras

The mathematical theory of quantum mechanics is based on Hilbert spaces and C*-algebras. In this chapter we will cover bounded operators on a Hilbert space, Gelfand duality for commutative C*-algebras, and a way to extend this duality to noncommutative algebras. The material of sections 1 and 2 is based mainly on [3]. Section 3 is a heuristic summary of [4].

1.1 Hilbert spaces

Quantum mechanics was originally formulated in the framework of Hilbert spaces by John von Neumann [5]. This formalism is based upon physical postulates, for example:

1. The state of the system is given by a **wave function** ψ, which assigns to (almost) each point in space a complex number. That is, $\psi \in L^2(\mathbb{R}^3)$;
2. The value $|\psi(x)|^2$ is the probability density of the measured position x of the particle. This means that the probability to measure a particle in a measurable subset $\Delta \subseteq \mathbb{R}^3$ is equal to the integral
$$\int_\Delta |\psi(x)|^2 d^3x;$$
3. If ψ_1 and ψ_2 are states of the system, then $\psi_1 + \psi_2$ is also a state, if we normalize $\psi_1 + \psi_2$ so that the integral over $\mathbb{R}^3$ is equal to 1 (This is called the **principle of superposition**),

among others. It turns out that these postulates can be beautifully captured in the formalism of **Hilbert spaces** and operators on them. The Hilbert space represents the possible pure states the system can be in and the operators are the measurements that can be performed on it.

Before we define Hilbert spaces, we first recall the definition of a normed space.

Definition 1.1. A **normed space** is a vector space X (over $k \in \{\mathbb{R}, \mathbb{C}\}$) with a positive definite function $||\cdot|| : V \to [0, \infty)$ such that:

1. $||\lambda v|| = |\lambda|||v||$ for all $\lambda \in k, v \in X$;
2. $||v + w|| \leq ||v|| + ||w||$ for all $v, w \in X$.

Definition 1.2. A **Hilbert space** H is an inner product space over $\mathbb{C}$, that is, a $\mathbb{C}$-vector space with a positive definite sesquilinear form $\langle \cdot, \cdot \rangle$, such that H is complete in the norm induced by $\langle \cdot, \cdot \rangle$. This norm is given by
$$||x|| = \sqrt{\langle x, x \rangle}$$
for $x \in H$.

Definition 1.3. A linear map $a : H \to H$ is called **bounded** if there exists a constant $C \in \mathbb{R}$ such that $||av|| \leq C||v||$ for all $v \in H$. The infimum of all C for which this holds is denoted by $||a||$ and is called the **(operator) norm** of a.

Remark. Note that the following holds:

$$||a|| = \sup\left\{\frac{||av||}{||v||} \mid v \in H, v \neq 0\} = \sup\{||av|| \mid v \in H, ||v|| = 1\right\}.$$

Notation. The ($\mathbb{C}$-)vector space of all bounded operators on H is denoted by $B(H)$. It turns out that in addition to being a vector space, $B(H)$ is a Banach algebra (and even a C*-algebra).

Lemma 1.1. *The operator norm is indeed a norm on $B(H)$.*

Proof. Positive definiteness: a is zero on all unit vectors iff a is zero on all of H, because we can write any nonzero vector $v \in H$ as

$$v = ||v|| \frac{v}{||v||},$$

where $\frac{v}{||v||}$ is a unit vector. Suppose $v \in H, ||v|| = 1$ and $a, b \in B(H)$. Then

$$||(a+b)v|| = ||av + bv|| \leq ||av|| + ||bv|| \leq ||a|| + ||b||,$$

from which it follows that $||a+b|| \leq ||a|| + ||b||$. Part 1 of definition 1.1 is just as easy to verify. □

Lemma 1.2. *$B(H)$ is complete in the operator norm.*

Proof. Suppose $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B(H)$. Let $v \in H$ and $n, m \in \mathbb{N}$. Then $||a_n v - a_m v|| = ||(a_n - a_m)v|| \leq ||a_n - a_m|| ||v|| \to 0$ as $n, m \to \infty$. Therefore the sequence $(a_n v)$ is a Cauchy sequence in H which is complete, so it converges to a vector $av \in H$. We have to prove that the assignment $v \mapsto av$ defines a bounded operator on H and that $a_n \to a$ in the operator norm. Let $v, w \in H$ and $\lambda \in \mathbb{C}$. Then

$$||a_n(v + w) - av - aw|| = ||a_n v - av + a_n w - aw|| \leq ||a_n v - av|| + ||a_n w - aw|| \to 0,$$

whence $a_n(v + w) \to av + aw$. That is, $a(v + w) = av + aw$. Also,

$$||a_n(\lambda v) - \lambda av|| = ||\lambda a_n v - \lambda av|| \leq |\lambda| \; ||a_n v - av|| \to 0$$

so that $a(\lambda v) = \lambda(av)$. Lastly, let $\epsilon > 0$. Then there is N such that

$$\sup_{||v||=1} ||a_n v - a_m v|| < \epsilon$$

for $n, m \geq N$. Take the limit $m \to \infty$ to obtain

$$\sup_{||v||=1} ||a_n v - av|| < \epsilon. \tag{1.1}$$

We then have, for $v \in H$ with $||v|| = 1$ and $n \geq N$, that

$$\begin{aligned} ||av|| &= ||av - a_n v + a_n v|| \\ &\leq ||av - a_n v|| + ||a_n v|| \\ &< \epsilon + ||a_n||, \end{aligned}$$

which shows that a is bounded with $||a|| \leq \sup ||a_n||$. 1.1 then gives that $a_n \to a$ in the operator norm. □

We can turn $B(H)$ into an associative $\mathbb{C}$-algebra by defining muliplication as composition of linear operators: $(ab)v = a(bv)$. We then have

Lemma 1.3. *For every $a, b \in B(H)$ we have $||ab|| \leq ||a|| \; ||b||$.*

Proof. This is an easy calculation. Let $v \in H, ||v|| = 1$ and $a, b \in B(H)$. Then

$$||(ab)v|| = ||a(bv)|| \leq ||a|| \; ||bv|| \leq ||a|| \; ||b||.$$

By taking suprema we see that $||ab|| \leq ||a|| \; ||b||$. □

These properties of $B(H)$ are nicely summarized in the following definition:

Definition 1.4. A (unital) **Banach algebra** is a (unital) $\mathbb{C}$-algebra A that is also a normed space such that A is complete in its norm and multiplication satisfies $||ab|| \leq ||a||\ ||b||$ for all $a, b \in A$.

Note that $B(H)$ is always a unital Banach algebra with the identity operator $1_H : H \to H$ as the unit element. We will now consider the additional structure on $B(H)$.

Definition 1.5. A linear functional $\phi : H \to \mathbb{C}$ is called **bounded** if there is a constant $C \in \mathbb{R}$ such that $||\phi(v)|| \leq C||v||$ for all $v \in H$. The infimum of all such C is denoted by $||\phi||$.

Remark. As for linear operators, we have $||\phi|| = \sup\{|\phi(v)| \mid v \in H, ||v|| = 1\}$.

Lemma 1.4. *For any $w \in H$ the assignment $v \mapsto \langle w, v\rangle$ defines a bounded linear functional ϕ_w on H.*

Proof. This follows from the Cauchy-Schwarz inequality: let $v \in H, ||v|| = 1$, then

$$||\langle w, v\rangle|| \leq ||w||,$$

from which we see that ϕ_w is bounded with norm at most $||w||$. We have equality, since $\langle w, w\rangle = ||w||^2$. □

The converse also holds, and the result is called the **Riesz representation theorem**:

Theorem 1.1. *Every bounded linear functional $\phi : H \to \mathbb{C}$ is of the form ϕ_w for some unique $w \in H$, and $||\phi|| = ||w||$.*

Proof. For the proof we refer to [3], Theorem 1.29. □

Lemma 1.5. *For $a \in B(H), w \in H$ the assignment $v \mapsto \langle w, av\rangle$ defines a bounded linear functional $\phi_{a,w}$ on H.*

Proof. Linearity follows from linearity of the inner product and linearity of a. Boundedness follows again from the Cauchy-Schwarz inequality, because we have

$$||\langle w, av\rangle|| \leq ||w||\ ||av|| \leq ||w||\ ||a||\ ||v||.$$

Therefore, $\phi_{a,w}$ is bounded with norm at most $||w||\ ||a||$. □

Because of the Riesz representation theorem the functional in the lemma above must be given by the inner product with some fixed vector. We denote this vector by a^*w.

Lemma 1.6. *The assignment $w \mapsto a^*w$ defines a bounded operator a^* on H.*

Proof. Let $v, w, h \in H$. Then

$$\begin{aligned}\phi_{a,v+w}h = \langle v + w, ah\rangle &= \langle v, ah\rangle + \langle w, ah\rangle \\ &= \langle a^*v, h\rangle + \langle a^*w, h\rangle \\ &= \langle a^*v + a^*w, h\rangle.\end{aligned}$$

We see that the linear functional $\phi_{a,v+w}$ is given by the inner product with a^*v+a^*w as well as with $a^*(v+w)$. But this vector is unique by the Riesz representation theorem so $a^*(v+w) = a^*v+a^*w$. Part 1 of definition 1.1 follows from the observation that

$$\begin{aligned}\langle \lambda v, ah\rangle &= \overline{\lambda}\langle v, ah\rangle \\ &= \overline{\lambda}\langle a^*v, h\rangle \\ &= \langle \lambda a^*v, h\rangle.\end{aligned}$$

It follows that $a^*(\lambda v) = \lambda(a^*v)$. Lastly, by Theorem 1.1 and Lemma 1.5, we have

$$||a^*w|| = ||\phi_{a,w}|| \leq ||a||||w||,$$

so that a^* is bounded with norm at most $||a||$. □

Remark. The operator a^* is called the **adjoint** of a. It has the property that for any $v, w \in H$:

$$\langle w, av\rangle = \langle a^*w, v\rangle.$$

Proposition 1.1. *The map $a \mapsto a^*$ from $B(H)$ to itself has the following properties:*

1. $(a^*)^* = a$ *for all* $a \in B(H)$
2. $(a+b)^* = a^* + b^*$ *for all* $a, b \in B(H)$
3. $(\lambda a)^* = \overline{\lambda} a^*$ *for all* $\lambda \in \mathbb{C}, a \in B(H)$
4. $(ab)^* = b^* a^*$ *for all* $a, b \in B(H)$
5. $||a^* a|| = ||a||^2$ *for all* $a \in B(H)$

Proof. We refer to [3], Proposition 2.13 and 2.14. □

These properties lead us to the definition of a C*-algebra.

Definition 1.6. A **C*-algebra** is a Banach algebra A together with a map $^* : A \to A$ that satisfies the properties 1-5 in the proposition above.

1.2 Commutative C*-algebras

In this section we will explore Gelfand duality, which gives a characterization of commutative C*-algebras in terms of topological spaces. We start with the following.

Lemma 1.7. *For X a compact Hausdorff space the space of continuous functions $X \to \mathbb{C}$, denoted by $C(X)$, is a C*-algebra if we define addition and scalar multiplication pointwise, and furthermore*

1. $||f|| = \sup_{x \in X} |f(x)|$;
2. $f^*(x) = \overline{f(x)}$.

Note that the supremum is well-defined, because X is compact.

If we want to characterize C*-algebras we first have to define when two C*-algebras are isomorphic.

Definition 1.7. A linear map $f : A \to B$ between C*-algebras is called a ***-homomorphism** if for all $a, a' \in A$

1. $f(aa') = f(a)f(a')$
2. $f(a^*) = f(a)^*$.

A bijective *-homorphism is called a ***-isomorphism**.

It turns out that all unital commutative C*-algebras are *-isomorphic to $C(X)$ for some compact Hausdorff space X. There are many possible realizations of this space, but the easiest definition is in terms of characters, also called (nonzero) multiplicative functionals.

Definition 1.8. A **character** of a C*-algebra A is a nonzero *-homomorphism $\phi : A \to \mathbb{C}$ from A to the C*-algebra of complex numbers. We denote the set of characters of A by $\Sigma(A)$. It is called the **Gelfand spectrum** of A.

Lemma 1.8. *Let $\phi : A \to \mathbb{C}$ be a character. Then ϕ is bounded with $||\phi|| = 1$.*

Proof. This is [3], Theorem 5.20. □

We can turn $\Sigma(A)$ into a topological space by putting the **weak-* topology** on it. This is the initial topology on $\Sigma(A)$ with respect to the maps

$$\hat{a} : \Sigma(A) \to \mathbb{C}, \ \hat{a}(\phi) = \phi(a),$$

where $a \in A$. This topology can be characterized by its convergent nets, namely $\phi_\lambda \to \phi$ in $\Sigma(A)$ iff $\phi_\lambda(a) \to \phi(a)$ in $\mathbb{C}$ for all $a \in A$. This is why the weak-* topology is also sometimes called the **topology of pointwise convergence**. Another common name is **Gelfand topology**.

Lemma 1.9. *Let A be a unital commutative C*-algebra. Then $\Sigma(A)$ is a compact Hausdorff space in the weak-* topology.*

Proof. [3], Theorem 5.42. □

Example. If A is finite-dimensional with dimension n, then its Gelfand spectrum $\Sigma(A)$ is the discrete space with n points.

We now have a map $\hat{}: A \to C(\Sigma(A))$, defined by

$$a \mapsto \hat{a}, \ \hat{a}(\phi) = \phi(a). \tag{1.2}$$

This is called the **Gelfand representation** of A.

Theorem 1.2. *For A a unital commutative C*-algebra the Gelfand representation 1.2 is an isometric *-isomorphism between A and $C(\Sigma(A))$.*

Proof. [3], Theorem 5.44. □

In order to make Gelfand duality a true duality of categories (see appendix A) we need to consider morphisms.

Definition 1.9. The category of unital commutative C*-algebra **CCStar** has

1. Unital commutative C*-algebras as objects;
2. *-homomorphisms as morphisms.

Definition 1.10. The category of compact Hausdorff spaces **CptHaus** has

1. Compact Hausdorff spaces as objects;
2. Continuous maps as morphisms.

Lemma 1.10. *Gelfand spectrum Σ is a contravariant functor* **CCStar** $\to$ **CptHaus**.

Proof. We have already established that for a unital commutative C*-algebra A, $\Sigma(A)$ is a compact Hausdorff space. Now consider a *-homomorphism $f : A \to B$. We obtain a map

$$\Sigma(f) : \Sigma(B) \to \Sigma(A),$$
$$\Sigma(f)(\phi) = \phi \circ f.$$

We claim that this map is continuous. Let $\phi_\lambda \to \phi$ be a convergent net in $\Sigma(B)$. This is equivalent to $\phi_\lambda(b) \to \phi(b)$ in $\mathbb{C}$, for all $b \in B$. If $a \in A$, then $\phi_\lambda(f(a)) \to \phi(f(a))$, that is

$$\Sigma(f)(\phi_\lambda)(a) \to \Sigma(f)(\phi)(a).$$

Since this is true for all $a \in A$, we have $\Sigma(f)(\phi_\lambda) \to \Sigma(f)(\phi)$, which establishes the continuity of $\Sigma(f)$. The functoriality is easy. $\Sigma(\mathrm{id}_A)(\phi) = \phi \circ \mathrm{id}_A = \phi$ so $\Sigma(\mathrm{id}_A) = \mathrm{id}_{\Sigma(A)}$. If $f : A \to B$ and $g : B \to C$, then

$$\Sigma(f)(\Sigma(g)(\phi)) = (\phi \circ g) \circ f = \phi \circ (g \circ f) = \Sigma(g \circ f)(\phi).$$ □

Lemma 1.11. *$C(\bullet)$ is a contravariant functor* **CptHaus** $\to$ **CCStar**.

Proof. We know that $C(X)$ is a unital commutative C*-algebra. Let $\phi : X \to Y$ be a continuous map. We obtain $C(\phi) : C(Y) \to C(X)$ by $C(\phi)(f) = f \circ \phi$. This is a *-homomorphism, because all operations are defined pointwise. The proof of functoriality is exactly the same as in 1.10. □

Lemma 1.12. *Let X be a compact Hausdorff space. Then the map $X \to \Sigma(C(X))$ given by $x \mapsto \mathrm{ev}_x$ is a homeomorphism.*

Theorem 1.3. *The categories* **CCStar** *and* **CptHaus***are dual.*

Proof. We need to prove that $C \circ \Sigma \cong \mathrm{id}_{\mathbf{CCStar}}$ and $\Sigma \circ C \cong \mathrm{id}_{\mathbf{CptHaus}}$. For the first, take a *-homomorphism $f : A \to B$ and consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ C(\Sigma(A)) & \xrightarrow{C(\Sigma(f))} & C(\Sigma(B)). \end{array}$$

Let $a \in A$. We then have

$$C(\Sigma(f))(\hat{a})(\phi) = (\hat{a} \circ \Sigma(f))(\phi) = \hat{a}(\phi \circ f) = (\phi \circ f)(a) = \phi(f(a)) = \widehat{f(a)}(\phi),$$

so that $C(\Sigma(f))(\hat{a}) = \widehat{f(a)}$. Now take a continuous map $\phi : X \to Y$ and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow\wr & & \downarrow\wr \\ \Sigma(C(X)) & \xrightarrow{\Sigma(C(\phi))} & \Sigma(C(Y)). \end{array}$$

Let $x \in X$. We then have

$$\Sigma(C(\phi))(\mathrm{ev}_x)(f) = (\mathrm{ev}_x \circ C(\phi))(f) = \mathrm{ev}_x(C(\phi)(f)) = \mathrm{ev}_x(f \circ \phi) = f(\phi(x)) = \mathrm{ev}_{\phi(x)}(f),$$

so that

$$\Sigma(C(\phi))(\mathrm{ev}_x) = \mathrm{ev}_{\phi(x)}.$$ □

1.3 Noncommutative C*-algebras and quantum toposophy

If one interprets a C*-algebra as representing the observables of a certain quantum system, then a **noncommutative** C*-algebra signifies true quantum behaviour. An example is the noncommutativity of measurement of position and momentum in the Heisenberg relation. However, the position of Bohr was that one could only reason about a quantum system in classical terms, and measurements could only be described with classical quantities. To bridge the gap between quantum and classical we can look at so called **'classical contexts'** of a C*-algebra A. These are **commutative** subalgebras $C \subseteq A$, i.e. subsets of A that are commutative C*-algebras with the structure inherited from A. This gives us

Definition 1.11. For a unital C*-algebra A we define

$$\mathcal{C}(A) = \{C \subseteq A \mid C \text{ is a unital commutative subalgebra}\}$$

Remark. $\mathcal{C}(A)$ is naturally partially ordered by set-theoretic inclusion, and it is a so-called meet-semilattice in this order. However, it usually does not have any more interesting structure unless we know more about A.

Gelfand duality can in principle only be used for commutative C*-algebras, but by using $\mathcal{C}(A)$ we can bridge the gap to noncommutative C*-algebras. For this, we need to use the framework of **topos theory**. A topos is a category that has certain nice properties so that it can be used as an alternative, in a sense, to set theory (i.e. the category **Sets**). In particular, in some topoi the term C*-algebra has a specific meaning, as does Gelfand duality. These are then called **internal** C*-algebras (as well as internal Gelfand duality). To get the whole thing going we consider the following functor category:

$$\mathbf{Sets}^{\mathcal{C}(A)}$$

where $\mathcal{C}(A)$ is seen as a posetal category. This is the category of **co-presheaves** of sets on $\mathcal{C}(A)$. It turns out that this category is in fact a topos. In this functor category we have a very special functor denoted by $\underline{A}$:

$$\underline{A}(C) = C,$$

which assigns to each commutative subalgebra $C \subseteq A$ its underlying set and to each inclusion $C \subseteq D$ the inclusion map in **Sets**. It can be shown that in the topos $\mathbf{Sets}^{\mathcal{C}(A)}$ the object $\underline{A}$ is a **commutative** internal C*-algebra. Because of this, we can take its internal Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$. However, this is an **internal** object in the topos $\mathbf{Sets}^{\mathcal{C}(A)}$, and these can be hard to deal with. We therefore want a so-called **external description** of this internal object. That is, we want a topological structure in the category **Sets** that corresponds to $\underline{\Sigma}_{\underline{A}}$. It turns out that an internal 'topological' space (actually a so-called pointfree space) $\underline{Y}$ in $\mathbf{Sets}^{\mathcal{C}(A)}$ may be identified with a continuous map

$$\pi : Y \to \mathcal{C}(A),$$

where $\mathcal{C}(A)$ is given the Alexandrov topology. This is the topology in which all upsets are open. The space that will correspond to $\underline{\Sigma}_A$ can be described as follows: The underlying set is given by the disjoint union of all the regular Gelfand spectra $\Sigma(C)$ for $C \in \mathcal{C}(A)$:

$$\Sigma_A = \coprod_{C \in \mathcal{C}(A)} \Sigma(C).$$

We then have the following result:

Theorem 1.4. *The external description of the pointfree Gelfand spectrum $\underline{\Sigma}_A$ may be identified with the canonical projection*

$$\pi : \Sigma_A \to \mathcal{C}(A).$$

Proof. This is [4], Theorem 2. □

The topology on Σ_A can be described as follows. A subset $\mathcal{U} \subseteq \Sigma_A$ is open if and only if

1. For each $C \in \mathcal{C}(A)$ the set $\mathcal{U}_C := \mathcal{U} \cap \Sigma(C)$ is open in $\Sigma(C)$.
2. Suppose $C \subseteq D$ in $\mathcal{C}(A)$. If $\lambda \in \Sigma(D)$ with $\lambda|_C \in \mathcal{U}_C$ then $\lambda \in \mathcal{U}_D$.

It turns out that this is the weakest topology on Σ_A making the canonical projection $\pi : \Sigma_A \to \mathcal{C}(A)$ continuous.

Remark. We can reformulate the second condition in terms of the Gelfand functor $\Sigma(\bullet)$. Suppose $C \subseteq D$ in $\mathcal{C}(A)$. Then we have an inclusion map $j : C \hookrightarrow D$. The Gelfand functor then gives us a restriction map $\Sigma(j) : \Sigma(D) \to \Sigma(C)$. The condition then translates to: if $\lambda \in \Sigma(D)$ and $\Sigma(j)(\lambda) \in \mathcal{U}_C$, then $\lambda \in \mathcal{U}_D$, i.e.

$$\Sigma(j)^{-1}[\mathcal{U}_C] \subseteq \mathcal{U}_D. \tag{1.3}$$

Chapter 2

Lattices and frames

This chapter covers order theory and lattice theory. Lattices form the algebraic structures of mathematical logic, and therefore are the basis also of quantum logic. The material in this chapter is based on [6] and [7].

2.1 Partial orders and lattices

Definition 2.1. A **partially ordered set (poset)** is a set X equipped with a partial order $\leq$. Equivalently, it is a category $\mathbf{C}$ in which $|\mathbf{C}(A, B)| \leq 1$ for all objects $A, B \in \mathbf{C}$.

Definition 2.2. A **lattice** is a partially ordered set L so that for all $x, y \in L$ there exist:

1. a smallest upper bound $x \vee y$ for x and y, called the **join** or **supremum** of x and y;
2. a greatest lower bound $x \wedge y$ for x and y, called the **meet** or **infimum** of x and y.

Equivalently, L is a posetal category with all finite products and coproducts. A lattice is called **complete** if every subset $S \subseteq L$ has a supremum $\bigvee S$ and an infimum $\bigwedge S$.

Example. An example of a lattice is the power set $\mathcal{P}(X)$ of a set X, which is ordered by inclusion and where the join and meet are given by union and intersection, respectively.

The following properties of lattices are of great importance.

Definition 2.3. 1. A **bounded** lattice is a lattice with a greatest element 1 and a smallest element 0.

2. A **distributive** lattice is a lattice L where for all $x, y, z \in L$,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

and

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

These two equalities are equivalent ([6], Lemma 4.3).

Remark. In a bounded lattice the elements 0 and 1 are unique. A **finite** lattice is automatically bounded, with greatest element given by $\bigvee_{x \in L} x$ and lowest element by $\bigwedge_{x \in L} x$.

Definition 2.4. A bounded distributive lattice B is called **Boolean** if it has a **complementation**: for every $x \in B$ there is $\neg x \in B$ such that $x \wedge \neg x = 0$ and $x \vee \neg x = 1$.

Remark. Boolean lattices are important for classical logic, because they are models for classical propositional logic. This also means that the law of the excluded middle holds in these lattices:

$$\neg(\neg x) = x \text{ for all } x \in B.$$

Example. The open sets $\mathcal{O}(X)$ of a topological space X always form a bounded distributive lattice under set-theoretic operations. But $\mathcal{O}(X)$ has more structure, because for any collection $\{U_i\}_{i\in I}$ of open sets the union $\bigcup_{i\in I} U_i$ is also open. Furthermore, the law of complete distributivity holds: for any open set V we have

$$V \cap \Big(\bigcup_{i\in I} U_i\Big) = \bigcup_{i\in I}(V \cap U_i).$$

This example is the motivation for the following definition.

Definition 2.5. A **frame** is a bounded distributive lattice F such that any collection $\{f_i\}_{i\in I}$ of elements of F has a smallest upper bound $\bigvee_{i\in I} f_i$ and such that the law of infinite distributivity holds: for any $g \in F$ we have

$$g \wedge \Big(\bigvee_{i\in I} f_i\Big) = \bigvee_{i\in I}(g \wedge f_i). \tag{2.1}$$

Remark. This implies that any collection $\{f_i\}_{i\in I}$ also has a lower bound $\bigwedge_{i\in I} f_i$. So see this, let L be the set of lower bounds of the collection $\{f_i\}_{i\in I}$. Then L has a supremum $\bigvee L$, and

$$\bigwedge_{i\in I} f_i = \bigvee L.$$

Therefore, any frame F is a complete lattice. However, the analagous infinite distributivity law may **not** hold for the infimum, generally.

2.2 Lattice homomorphisms and filters

Now that we have defined our objects, namely lattices, we can discuss the morphisms between them.

Definition 2.6. A map $f : X \to Y$ between posets X, Y is called **order preserving** if $f(x) \leq f(x')$ in Y whenever $x \leq x'$ in X.

We need the following definition in sections 2.3 and 2.4.

Definition 2.7. Two order preserving maps $f : X \to Y$, $g : Y \to X$ form a **Galois connection** if, for all $x \in X, y \in Y$,

$$f(x) \leq y \text{ if and only if } x \leq g(y).$$

f is called the **lower adjoint** (or **left adjoint**) of g, and g is called the **upper adjoint** (or **right adjoint**) of f.

Definition 2.8. An order preserving map $g : L \to K$ between lattices L, K is called a **lattice homomorphism** if for all $l, l' \in L$:

1. $g(l \vee l') = g(l) \vee g(l')$;
2. $g(l \wedge l') = g(l) \wedge g(l')$.

If L, K are bounded we also require that $g(0) = 0$ and $g(1) = 1$.

Definition 2.9. If $g : L \to K$ is a lattice homomorphism where K is bounded, the **kernel** of g is defined as

$$\operatorname{Ker}(g) = \{l \in L \mid g(l) = 1\}$$

The kernel has the following properties:

Lemma 2.1. *Let $g : L \to K$ be a lattice homomorphism where K is bounded. Then:*

1. *If L is also bounded then* $\operatorname{Ker}(g)$ *is a proper nonempty subset of L;*
2. *If $l \in \operatorname{Ker(g)}$ and $l' \geq l$ then $l' \in \operatorname{Ker}(g)$;*
3. *If $l, l' \in \operatorname{Ker}(g)$ then $l \wedge l' \in \operatorname{Ker}(g)$.*

Proof. For the first claim, we have defined $g(1) = 1$, so that the kernel is nonempty, and $g(0) = 0$, so that it is proper. The second item is clear, since g is order preserving and 1 is the greatest element of K. For the third property, suppose $l, l' \in \mathrm{Ker}(g)$. Then $g(l \wedge l') = g(l) \wedge g(l') = 1 \wedge 1 = 1$ so $l \wedge l' \in \mathrm{Ker}(g)$. □

This leads us to the following definition.

Definition 2.10. A subset $F \subseteq L$ of a bounded lattice L is called a **filter** when

1. F is proper nonempty subset of L;
2. If $f \in F$ and $f' \geq f$ then $f' \in F$;
3. If $f, f' \in F$ then $f \wedge f' \in F$.

A filter F is called **proper** if $F \neq L$.

A very important class of lattice homomorphisms are those into the two-element (bounded distributive) lattice $\underline{2} := \{0, 1\}$. The kernels of these homomorphisms have an additional property.

Lemma 2.2. *Let $f : L \to \underline{2}$ be a lattice homomorphism. If $l \vee l' \in \mathrm{Ker}(f)$, then either $l \in \mathrm{Ker}(f)$ or $l' \in \mathrm{Ker}(f)$.*

Proof. Suppose not, then $f(l) = f(l') = 0$, since there are only two possible elements for the image. But then $f(l \vee l') = f(l) \vee f(l') = 0 \vee 0 = 0$, which is a contradiction. □

Definition 2.11. A filter $F \subseteq L$ is called **prime** if whenever $f, f' \in L$ with $f \vee f' \in F$, then either $f \in F$ or $f' \in F$.

Notation. The set of all prime filters of a lattice L is an important object. It is denoted by $\mathcal{PF}(L)$ and will be used in the theory of Stone and Priestley duality. Note that it naturally has the structure of a poset, with order given by set-theoretic inclusion.

2.3 Heyting lattices and frames

Definition 2.12. A **Heyting lattice** is a bounded distributive lattice H with for every $a, b \in H$ an element $a \to b \in H$ such that: for all $x \in H$,

$$x \leq a \to b \text{ if and only if } x \wedge a \leq b.$$

Alternatively, the map $\to$ satisfies the following algebraic properties:

1. $a \to a = 1$
2. $a \wedge (a \to b) = a \wedge b$
3. $b \wedge (a \to b) = b$
4. $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$

The map $\to$ is called the **(Heyting) implication**.

Remark. Note that in a Heyting lattice H, for any $a \in H$, the maps

$$x \mapsto x \wedge a$$

and

$$x \mapsto a \to x$$

form a Galois connection.

In a Heyting lattice H we can define the following operation: for $x \in H$ we can consider $\neg x := x \to 0$ where 0 is the smallest element of H. As opposed to a Boolean lattice, it is then generally not true that $\neg(\neg x) = x$. Heyting lattices are the algebraic models of **intuitionistic** propositional logic, which is therefore different from classical logic because the law of the excluded middle does not hold, generally.

There is a strong connection between frames and **complete** Heyting lattices. In fact, they turn out to be the same thing.

Proposition 2.1. *Any frame F can be given the structure of a complete Heyting lattice by defining*

$$a \to b = \bigvee \{y \in F \mid y \wedge a \leq b\}.$$

Proof. This supremum always exists because F is a frame. We have to prove that this operation satisfies

$$x \leq a \to b \text{ if and only if } x \wedge a \leq b.$$

If $x \wedge a \leq b$ then trivially

$$x \leq \bigvee \{y \in F \mid y \wedge a \leq b\} = a \to b,$$

because this supremum is an upper bound for x. Conversely, suppose that $x \leq a \to b$. Then, by using the infinite distribituvity law 2.1,

$$a \wedge x \leq a \wedge (a \to b) = a \wedge \bigvee \{y \in F \mid y \wedge a \leq b\} = \bigvee \{a \wedge y \mid y \wedge a \leq b\} \leq b.$$

Therefore, $x \wedge a \leq b$. Lastly, the Heyting lattice constructed from F is complete because F is complete. □

Conversely, if H is a complete Heyting lattice, in the sense that all suprema and infima exist, then it is automatically a frame, because the infinite distributive law holds in this case. To prove this, we first need a lemma.

Lemma 2.3. *Let X, Y be posets with a Galois connection that has $f : X \to Y$ as upper adjoint and $g : Y \to X$ as lower adjoint. If a subset $S \subseteq Y$ has a supremum $\bigvee S$, then the image $g(S)$ also has a supremum, and*

$$\bigvee g(S) = g\left(\bigvee S\right).$$

Proof. If $x \in g(S)$, then there is $y \in S$ with $g(y) = x$. But $y \leq \bigvee S$, and therefore

$$x = g(y) \leq g\left(\bigvee S\right),$$

because g is order preserving. So $g(\bigvee S)$ is an upper bound for $g(S)$. We need to prove that it is the least upper bound. Let b be any upper bound for $g(S)$. For all $y \in S$, we have

$$g(y) \leq b \iff y \leq f(b).$$

Therefore,

$$\bigvee S \leq f(b) \iff g(\bigvee S) \leq b.$$

We conclude that $g(\bigvee S)$ is the supremum of $g(S)$. □

Proposition 2.2. *The infinite distributivity law 2.1 holds in a complete Heyting lattice H.*

Proof. Because the map

$$x \mapsto a \wedge x$$

is the lower adjoint of a Galois connection, we can use the previous lemma. If $S \subseteq H$, then it has a supremum $\bigvee S$, since H is complete. Furthermore, by the lemma,

$$\bigvee \{a \wedge s \mid s \in S\} = a \wedge \left(\bigvee S\right),$$

which is exactly the infinite distributivity law. □

We now consider morphisms.

Definition 2.13. A lattice homomorphism $f : H \to K$ between Heyting lattices H, K is called a **Heyting lattice homomorphism** if for all $a, b \in H$, $f(a \to b) = f(a) \to f(b)$.

Definition 2.14. A lattice homomorphism $h : F \to G$ between frames F, G is called a **frame homomorphism** if h preserves infinite suprema, that is, for every collection $\{f_i\}_{i \in I}$ in F we have

$$h\left(\bigvee_i f_i\right) = \bigvee_i h(f_i).$$

Definition 2.15. 1. The category **Frm** has frames as objects and frame homomorphisms as morphisms.

2. The category **CHeyt** has complete Heyting lattices as objects, and as morphisms it has frame homomorphisms that are also Heyting lattice homomorphisms.

Remark. By Propositions 2.1 and 2.2 the categories **Frm** and **CHeyt** have the same objects. However, the morphisms in **CHeyt** are required to preserve the operation of Heyting implication.

Example. To show that **CHeyt** and **Frm** really are different categories, we will give an example of a frame homomorphism that is not a Heyting lattice homomorphism. Let F be a frame, and suppose that $a \in F$ is **not** complemented. That is,

$$\neg(\neg a) \neq a \text{ or } a \vee \neg a \neq 1.$$

Consider the map

$$a \vee - : F \to \uparrow a.$$

This is a surjective frame homomorphism. Suppose it preserves the Heyting implication, that is,

$$a \vee (x \to y) = (a \vee x) \to (a \vee y),$$

for all $x, y \in F$. If we take $x = a$ and $y = 0$, then

$$a \vee (a \to 0) = (a \vee a) \to (a \vee 0) = a \to a = 1.$$

But $a \to 0 = \neg a$, giving a contradiction. This example was taken from [8].

2.4 Locales and nuclei

We can make $\mathcal{O}$ into a contravariant functor **Top** $\to$ **Frm**, by associating to a continuous map $f : X \to Y$ the frame homomorphism

$$\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X),$$
$$U \mapsto f^{-1}(U).$$

If we define the category **Loc** of **locales** as

$$\mathbf{Loc} := \mathbf{Frm}^{\mathrm{op}},$$

then we have a **covariant** functor

$$\mathcal{O} : \mathbf{Top} \to \mathbf{Loc}.$$

We can therefore think of locales as 'generalized' topological spaces. The generalization of the notion of a topological subspace, is the notion of a **sublocale**. This has a categorical definition, namely a regular quotient in **Frm**.

Definition 2.16. If F, G are locales, then G is called a **sublocale** of F if, when we see F and G as frames, there is a surjective frame homomorphism

$$p : F \to G.$$

However, they can be described more easily by using **nuclei**.

Definition 2.17. Let F be a frame. A **nucleus** on F is a function $j : F \to F$ that satisfies

1. $j(a \wedge b) = j(a) \wedge j(b)$
2. $a \leq j(a)$
3. $j(j(a)) \leq j(a)$

for all $a, b \in F$.

Example. For any frame the function j_a defined by $j_a(b) = b \vee a$ is a nucleus:

1. $j_a(b \wedge c) = (b \wedge c) \vee a = (b \vee a) \wedge (c \vee a) = j_a(b) \wedge j_a(c)$ by distributivity.
2. $b \leq b \vee a$ because $b \vee a$ is an upper bound for a and b.
3. $(b \vee a) \vee a = b \vee a$ because $a \leq b \vee a$.

Definition 2.18. If $j : F \to F$ is a nucleus on a frame F, then we define the set F/j of j**-closed elements** of F as

$$F/j := \{a \in F \mid j(a) = a\}.$$

Example. The set of closed elements corresponding to j_a is the set of those b that satisfy $b \vee a = b$. This is true if and only if $a \leq b$, so this set is the **upset** of a.

The correspondence between sublocales and nuclei is as follows:

- If j is a nucleus on F, then F/j is a frame, and we can view j as a surjective frame homomorphism

  $$j^* : F \to F/j.$$

  Therefore, F/j is a sublocale of F.

- Suppose G is a sublocale of F, that is, there is a surjective frame homomorphism $p : F \to G$. Then p has a lower adjoint $p_* : G \to F$. It can be shown that the map

  $$j := p_* \circ p : F \to F$$

  is a nucleus on F, and p_* is an order embedding with image G. This means that

  $$p_*(x) \leq p_*(y) \iff x \leq y.$$

For more details, see [9], section II.2.

Definition 2.19. A sublocale $G \subseteq F$ is called **dense**, if for the corresponding nucleus j, we have

$$j(0) = 0.$$

Chapter 3

Quantum logic

Now that Hilbert spaces, C*-algebras and lattices have been covered separately, we can combine them into quantum logic. Firstly, quantum logic in its original form is discussed, and its connection with projections is explained. This is then generalized to C*-algebras. Lastly, the alternative, intuitionistic approach to quantum logic is presented.

3.1 Original quantum logic

Quantum logic originated with the paper of Birkhoff and Von Neumann [1]. Their logic was based on the closed linear subspaces of a Hilbert space H, denoted by $P(H)$, and ordered by inclusion. If C, D are closed linear subspaces of H, we can define

1. $C \wedge D := C \cap D$
2. $C \vee D := \overline{\mathrm{span}(C + D)}$
3. $C^{\perp} := \{v \in H \mid \langle v, w\rangle = 0 \ \forall w \in C\}$

Lemma 3.1. *$C \wedge D$, $C \vee D$ and $C^{\perp}$ are closed linear subspaces of H.*

Proposition 3.1. *These operations satisfy:*

1. *$\wedge$ and $\vee$ make $P(H)$ into a lattice;*
2. *$P(H)$ has a greatest and lowest element 0 and 1;*
3. *$C \wedge C^{\perp} = 0$, $C \vee C^{\perp} = 1$;*
4. *$(C^{\perp})^{\perp} = C$;*
5. *If $C \subseteq D$ then $D^{\perp} \subseteq C^{\perp}$;*
6. *If $C \subseteq D$ then $D = C \vee (D \wedge C^{\perp})$.*

Remark. (2)-(5) say that $(-)^{\perp}$ is an **orthocomplementation** on $P(H)$. Note that (4) is the law of the excluded middle in this context. (6) is called the **orthomodular law** and therefore $P(H)$ is an **orthomodular** lattice. It is even **complete** in the sense that all **infinite** infima and suprema exist.

This lattice of closed subspaces of H is closely related to the **projections** in $B(H)$. The following theorem clarifies this.

Theorem 3.1. *There is an bijection between:*

1. *$P(H)$: the closed linear subspaces of a Hilbert space H; and*
2. $\mathrm{Proj}(B(H)) = \{e \in B(H) \mid e^2 = e^* = e\}$*: the projections in $B(H)$*

This bijection is given by associating to a closed linear subspace C the orthogononal projection e_C onto C. Conversely, to a projection e we associate its image eH, which is a closed linear subspace of H.

This shows that the set of projections in $B(H)$ can be given the structure of a lattice inherited from $P(H)$. The order is given by

$$e \leq f \text{ if and only if } eH \subseteq fH.$$

If $(e_i)_{i \in I}$ is a collection of projections then we can form the closed linear subspaces

- $\bigvee e_i H$: the closure of the subspace generated by the $e_i H$
- $\bigwedge e_i H = \bigcap e_i H$.

To these closed linear subspaces then correspond projections $\bigvee e_i$ and $\bigwedge e_i$, which are the supremum and infimum of the collection $(e_i)_{i \in I}$, respectively. This shows that the projections in $B(H)$ always form a complete orthomodular lattice. The orthocomplementation associates to a projection e_C onto C, the projection $e_C^\perp$ onto the orthogonal complement, $C^\perp$, of C. It can be shown that $e_C^\perp = 1 - e_C$.

The quantum logic of Birkhoff and von Neumann can be characterised as follows:

1. The logic is **not** distributive, but only the (weaker) orthomodular law (3.1.6) holds. This law is weaker then distributivity, because if we set $x = C$, $y = D$, $z = C^\perp$ in Definition 2.3, point 2, then we obtain
$$C \vee (D \wedge C^\perp) = (C \vee D) \wedge (C \vee C^\perp) = D \wedge 1 = D.$$
2. The law of the excluded middle (3.1.4) **does** hold.

As noted in [10] we view this approach as too radical for dropping distributivity.

3.2 Projections in C*-algebras

For a C*-algebra A we can define **projections** in A, but it turns out that these do not always form an orthomodular lattice, as in the case of $B(H)$. For this we need extra assumptions on A.

Definition 3.1. Let A be a C*-algebra. An element $e \in A$ is called a **projection** if $e^2 = e^* = e$ That is, e is a self-adjoint idempotent element.

Notation. The set of projections of A is denoted by $\mathrm{Proj}(A)$.

We can turn $\mathrm{Proj}(A)$ into a poset by defining

$$e \leq f \text{ if and only if } ef = e$$

Lemma 3.2. $\mathrm{Proj}(A)$ *with* $\leq$ *defined above is a bounded poset.*

Proof.

- $e \leq e$ since $e^2 = e$.
- If $e \leq f$ and $f \leq g$ then $eg = efg = ef = e$, so that $e \leq g$.
- If $e \leq f$ and $f \leq e$, we have $ef = e$ and $fe = f$. Therefore $e = e^* = (ef)^* = f^* e^* = fe = f$, so that $e = f$.
- The smallest and greatest elements are given by 1 and 0: for any projection e we have $e \cdot 1 = e$, so that $e \leq 1$, and $0 \cdot e = 0$, so that $0 \leq e$.

□

The problem is that for general A, two arbitrary projections $e, f \in A$ might not have a supremum or infimum. However, they do exist if the projections commute.

Lemma 3.3. *Suppose* $e, f \in \mathrm{Proj}(A)$ *commute. Then they have an infimum and a supremum given by*

- $e \wedge f = ef$ *and*
- $e \vee f = e + f - ef$ *,*

respectively.

Proof. For the infimum, we have that

$$(ef)e = e^2 f = ef$$

and

$$(ef)f = ef^2 = ef,$$

so that $ef \leq e$ and $ef \leq f$. Now suppose that there is $g \in \mathrm{Proj}(A)$ with $g \leq e$ and $g \leq f$. This means that $ge = g$ and $gf = g$. But then

$$g(ef) = (ge)f = gf = g$$

which implies that $g \leq ef$. We conclude that ef is the greatest lower bound of e and f.

For the supremum the proof is similar. We have

$$e(e + f - ef) = e^2 + ef - ef = e$$

and

$$f(e + f - ef) = fe + f^2 - fef = f,$$

so that $e \leq e + f - ef$ and $f \leq e + f - ef$. If $e \leq g$ and $f \leq g$ then $eg = e$, $fg = f$ and

$$(e + f - ef)g = eg + fg - efg = e + f - ef.$$

We see that $e + f - ef \leq g$ and $e + f - ef$ is the lowest upper bound of e and f. □

For $e \in \mathrm{Proj}(A)$, we define

$$e^\perp = 1 - e.$$

We compute that

$$(1 - e)^2 = 1^2 - 2e + e^2 = 1 - 2e + e = 1 - e \text{ and}$$
$$(1 - e)^* = 1^* - e^* = 1 - e,$$

which shows that $e^\perp \in \mathrm{Proj}(A)$. We then have the following:

Proposition 3.2. *Let* $e, f \in \mathrm{Proj}(A)$. *Then*

1. $(e^\perp)^\perp = e$;
2. *If* $e \leq f$, *then* $f^\perp \leq e^\perp$;
3. e *and* $e^\perp$ *commute;*
4. $e \wedge e^\perp = 0$;
5. $e \vee e^\perp = 1$.

Proof. We calculate

$$(e^\perp)^\perp = 1 - (1 - e) = 1 - 1 + e = e,$$

which is (1). Now, if $e \leq f$ then $ef = e$, and

$$(1 - f)(1 - e) = 1^2 - e - f + ef = 1 - e - f + e = 1 - f.$$

That is, $f^\perp \leq e^\perp$ which proves (2). Next,

$$e(1 - e) = (1 - e)e = e - e^2 = e - e = 0,$$

which shows that e and $e^\perp$ commute and that their product, which is also their infimum, is 0. Lastly, we have

$$e + (1 - e) - e(1 - e) = 1 - 0 = 1.$$

□

Clauses 1-2 and 4-5 show that the map $e \mapsto e^\perp$ gives $\mathrm{Proj}(A)$ the structure of an **orthoposet**.

Definition 3.2. We say that two projections $e, f \in \mathrm{Proj}(A)$ are **orthogonal**, which we denote by $e \perp f$, if $e \leq f^\perp$.

Remark. From Proposition 3.2 it follows that $f \leq e^\perp$ as well. Furthermore, we can calculate that the condition $e \leq f^\perp$ is equivalent to

$$e \leq f^\perp \iff ef^\perp = e \iff e(1-f) = e \iff e - ef = e \iff ef = 0.$$

Since we also have $f \leq e^\perp$, e and f commute, $ef = fe = 0$, $e \wedge f = 0$ and $e \vee f = e + f$.

Proposition 3.3. *Let $e, f \in \mathrm{Proj}(A)$ such that $e \leq f$. Then $e^\perp \wedge f$ exists, $e \vee (e^\perp \wedge f)$ exists and*

$$e \vee (e^\perp \wedge f) = f. \tag{3.1}$$

Proof. First, we make the observation that, if $e \leq f = (f^\perp)^\perp$, then $e \perp f^\perp$. This means that $e \vee f^\perp$ exists, and is given by

$$e \vee f^\perp = e + f^\perp = 1 + e - f.$$

But by de Morgan's laws ([11], Lemma B.4.2), $e^\perp \wedge f$ also exists and is given by

$$e^\perp \wedge f = (e \vee f^\perp)^\perp = 1 - (1 + e - f) = f - e.$$

Furthermore, $e^\perp \wedge f \leq e^\perp$, so that $e \perp e^\perp \wedge f$ and $e \vee (e^\perp \wedge f)$ exists. Finally, we calculate that

$$e \vee (e^\perp \wedge f) = e + (e^\perp \wedge f) = e + (f - e) = f. \qquad \square$$

Again, 3.1 is called the orthomodular law. We conclude that for an arbitrary C*-algebra A

$\mathrm{Proj}(A)$ is an **orthomodular poset**.

Even more is true if we assume that A is commutative.

Proposition 3.4. *Let A be a commutative C*-algebra. Then $\mathrm{Proj}(A)$ is a Boolean lattice.*

Proof. Since A is commutative, all infima and suprema in $\mathrm{Proj}(A)$ exist by Lemma 3.3, so that $\mathrm{Proj}(A)$ is a lattice. Since we already know that $(-)^\perp$ is an orthocomplemntation, we only have to check distributivity. To this end, let $e, f, g \in \mathrm{Proj}(A)$. We calculate

$$\begin{aligned} e \wedge (f \vee g) &= e(f + g - fg) = ef + eg - efg \\ &= ef + eg - e^2 fg = ef + eg - (ef)(eg) \\ &= ef \vee eg = (e \wedge f) \vee (e \wedge g). \end{aligned} \qquad \square$$

Theorem 3.2. *Let A be a finite-dimensional C*-algebra. Then $\mathrm{Proj}(A)$ is a complete orthomodular lattice. If A is also commutative, then $\mathrm{Proj}(A)$ is a finite (hence complete) Boolean lattice.*

3.3 Intuitionistic approach

In order to overcome the problems presented in the first section we will present an intuitionistic approach to quantum logic based on the topological space Σ_A. The open sets $\mathcal{O}(\Sigma_A)$ of this space form a Heyting lattice, as explained in chapter 2. Using Heyting lattices instead of orthomodular lattices for quantum logic seems to be what we need. These lattices are distributive but the law of the excluded middle does not hold in general.

To understand the lattice $\mathcal{O}(\Sigma_A)$ better we will describe it in simpler terms for finite dimensional C*-algebras. We will use the following theorem.

Theorem 3.3. *If A is a finite dimensional commutative C*-algebra, there is an isomorphism of complete Boolean lattices*

$$\beta_A : \mathrm{Proj}(A) \to \mathcal{O}(\Sigma(A)), \tag{3.2}$$

given by

$$p \mapsto (\gamma_A e)^{-1}[\{1\}],$$

where γ_A is the Gelfand transform $\gamma_A : A \to C(\Sigma(A))$.

Proof. See [12], 2.4 . □

We note that $\mathcal{O}(\Sigma(A)) = \mathcal{P}(\Sigma(A))$, the power set of $\Sigma(A)$, since $\Sigma(A)$ has the discrete topology. This means that $\mathcal{O}(\Sigma(A))$ is indeed a complete Boolean lattice, because all power sets are.

Lemma 3.4. *β is a natural transformation* $\mathrm{Proj} \to \mathcal{O} \circ \Sigma$.

Proof. We use naturality of the Gelfand transform $\gamma : \mathrm{Id} \to C \circ \Sigma$ on C*-algebras. For a morphism $f : A \to B$ we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B} \\ C(\Sigma(A)) & \xrightarrow{C(\Sigma(f))} & C(\Sigma(B)), \end{array}$$

which means that

$$\gamma_B \circ f = C(\Sigma(f)) \circ \gamma_A.$$

The morphism f restricts to a morphism

$$f : \mathrm{Proj}(A) \to \mathrm{Proj}(B).$$

We then obtain the diagram

$$\begin{array}{ccc} \mathrm{Proj}(A) & \xrightarrow{f} & \mathrm{Proj}(B) \\ {\scriptstyle \beta_A}\downarrow & & \downarrow{\scriptstyle \beta_B} \\ \mathcal{O}(\Sigma(A)) & \xrightarrow{\mathcal{O}(\Sigma(f))} & \mathcal{O}(\Sigma(B)). \end{array}$$

We claim that this diagram commutes. To this end, take $p \in \mathrm{Proj}(A)$. It is mapped one way to

$$\beta_B(f(p)) = (\gamma_B f(p))^{-1}[\{1\}]$$

and the other way to

$$\mathcal{O}(\Sigma(f))(\beta_A(p)) = \Sigma(f)^{-1}((\gamma_A p)^{-1}[\{1\}]) = (\gamma_A p \circ \Sigma(f))^{-1}[\{1\}].$$

But

$$\gamma_B f(p) = C(\Sigma(f))(\gamma_A p) = \gamma_A(p) \circ \Sigma(f).$$ □

This isomorphism can be used to construct an isomorphism between $\mathcal{O}(\Sigma_A)$ and a Heyting lattice, which is easier to conceptualize. An open set $\mathcal{U} \subseteq \mathcal{O}(\Sigma_A)$ is given by

$$\mathcal{U} = \coprod_{C \in \mathcal{C}(A)} \mathcal{U}_C.$$

For each $C \in \mathcal{C}(A)$ we can take the projection $e \in \mathrm{Proj}(C)$ given by $e = \beta_C^{-1}(\mathcal{U}_C)$.

Lemma 3.5. *For an open $\mathcal{U} \in \mathcal{O}(\Sigma_A)$, the map $S : \mathcal{C}(A) \to \mathrm{Proj}(A)$, $C \mapsto \beta_C^{-1}(\mathcal{U}_C)$ is order preserving and $S(C) \in \mathrm{Proj}(C)$ for all $C \in \mathcal{C}(A)$.*

Proof. Because $\mathcal{U}_C$ is open in $\Sigma(C)$, $S(C) = \beta_C^{-1}(\mathcal{U}_C) \in \mathrm{Proj}(C)$. Furthermore, if $C \subseteq D$ in $\mathcal{C}(A)$, then, by 1.3,

$$\Sigma(j)^{-1}[\mathcal{U}_C] \subseteq \mathcal{U}_D,$$

where $j : C \hookrightarrow D$ is the inclusion map. Therefore, by naturality of β,

$$S(C) = \beta_C^{-1}(\mathcal{U}_C) = \beta_D^{-1}(\Sigma(j)^{-1}[\mathcal{U}_C]) \leq \beta_D^{-1}(\mathcal{U}_D) = S(D),$$

because β_D is order preserving. □

This leads us to the following definition.

Definition 3.3. For a finite-dimensional C*-algebra A we define

$$Q(A) = \{S : \mathcal{C}(A) \to \mathrm{Proj}(A) \mid S(C) \in \mathrm{Proj}(C),\ S \text{ order preserving}\}.$$

We order $Q(A)$ by

$$S \leq T \text{ if and only if } S(C) \leq T(C) \text{ for all } C \in \mathcal{C}(A).$$

The lattice structure is given by

$$(S \wedge T)(C) = S(C) \wedge T(C);$$
$$(S \vee T)(C) = S(C) \vee T(C),$$

and the Heyting implication is given by

$$(S \to T)(C) = \bigvee\{e \in \mathrm{Proj}(C) \mid e \leq S(D) \vee T(D)^{\perp} \forall\ D \supseteq C\}.$$

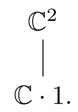
Actually $Q(A)$ is a **frame** because the projections of any $C \in \mathcal{C}(A)$ form a complete lattice. This means that in addition to taking binary (and finitary) suprema we can take arbitrary suprema.

Proposition 3.5. *For a finite-dimensional C*-algebra there is an isomorphism of complete Heyting algebras* $\mathcal{O}(\Sigma_A) \cong Q(A)$.

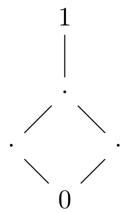
Remark. We will prove a version of this proposition later in a more general setting than finite-dimensional algebras.

3.4 Examples: $\mathbb{C}^2$ and $M_2(\mathbb{C})$

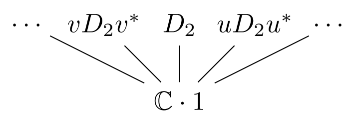
Every unital C*-algebra has a unique one-dimensional unital commutative subalgebra, namely $\mathbb{C}\cdot 1$. Since $\mathbb{C}^2$ is two-dimensional and commutative, the whole algebra is the only two-dimensional unital commutative subalgebra. Therefore, $\mathcal{C}(\mathbb{C}^2)$ has the following structure:



Furthermore, $\mathrm{Proj}(\mathbb{C}^2)$ is the four-element Boolean lattice and $\mathrm{Proj}(\mathbb{C}\cdot 1)$ is the two-element Boolean lattice consisting of the top and bottom elements of $\mathrm{Proj}(\mathbb{C}^2)$. Let $S \in Q(\mathbb{C}^2)$ and suppose $S(\mathbb{C}\cdot 1) = 1$. Then also $S(\mathbb{C}^2) = 1$ since S is order preserving. If $S(\mathbb{C}\cdot 1) = 0$, there are no restrictions on $S(\mathbb{C}^2)$. We conclude that $Q(\mathbb{C}^2)$ looks like



Every maximal unital commutative subalgebra of $M_2(\mathbb{C})$ is of the form uD_2u^*, where D_2 is the subalgebra of $M_2(\mathbb{C})$ consisting of the diagonal matrices, and $u \in \mathrm{U}(2)$. Therefore, $\mathcal{C}(M_2(\mathbb{C}))$ is



However, sometimes $u, v \in \mathrm{U}(2)$ generate the same subalgebra. To remove this problem, we can also describe $\mathcal{C}(M_2(\mathbb{C}))$ in terms of projections. If we let $M_2(\mathbb{C})$ act on $\mathbb{C}^2$, then we say that a projection $e \in M_2(\mathbb{C})$ is **one-dimensional** if its image $e\mathbb{C}^2$ is one-dimensional. The set of one-dimensional projections in $M_2(\mathbb{C})$ is denoted by

$$\mathrm{Proj}_1(\mathbb{C}^2).$$

Any maximal commutative subalgebra of $M_2(\mathbb{C})$ is then generated by one such one-dimensional projection and the identity matrix. Two projections generate the same subalgebra if and only if they are complements. It turns out that, by a parametrization theorem, that

$$\mathrm{Proj}_1(\mathbb{C}^2) \cong S^2,$$

where complements in $\mathrm{Proj}_1(\mathbb{C}^2)$ correspond to antipodal points. Therefore, by identifying complements, we obtain

$$\max \mathcal{C}(M_2(\mathbb{C})) \cong \mathrm{Proj}_1(\mathbb{C}^2)/\sim \cong S^2/\sim \cong \mathbb{RP}^2.$$

We will now calculate $Q(M_2(\mathbb{C}))$. We first note that the projections of any maximal commutative subalgebra are isomorphic to $\mathrm{Proj}(\mathbb{C}^2)$, which is the four-element Boolean lattice, call it B. Again, if $S(\mathbb{C}\cdot 1) = 1$, then $S(C) = 1$ for all commutative subalgebras C. If $S(\mathbb{C}\cdot 1) = 0$, then we can associate any projection to a maximal commutative subalgebra. This corresponds to a function $\mathbb{RP}^2 \to B$. The lattice structure in $Q(M_2(\mathbb{C}))$ then corresponds to pointwise operations on these functions. We conclude that $Q(M_2(\mathbb{C}))$ has the following structure:

$$\begin{array}{c} 1 \\ | \\ \text{Functions } \mathbb{RP}^2 \to B. \end{array}$$

Chapter 4

Lattice duality

This chapter covers the theory of lattice duality. These are categorial dualities between classes of lattices and classes of topological spaces. Specifically, we will cover the duality between Boolean lattices and Stone spaces, distributive lattices and Priestley spaces, and Heyting lattices and Esakia spaces. Furthermore, we will give some (negative) results on the Esakia space related to $Q(A)$. The first three section are an adaptation of [13], and were written by Evert-Jan. Some proofs are omitted, and can be found in [14].

4.1 Stone Duality

Stone duality concerns Boolean lattices and Stone spaces, which are defined as follows:

Definition 4.1. A **Stone space** X is a topological space that is compact and Hausdorff, and in which all the clopen subsets in X form a basis of the topology. (This means any open set U can be written as a union of clopen subsets of X.)

The category of Boolean lattices is denoted by **BL**, in which the morphisms are lattice homomorphisms that preserve complementation, i.e. $f(\neg x) = \neg f(x)$ which we will call Boolean homomorphisms. The category of Stone spaces is denoted by **Stone**, and its morphisms are continuous maps.

To prove the duality between these categories, some steps are needed. Firstly, a construction is needed of the partially ordered space $\mathcal{PF}(L)$ for a bounded distributive lattice. This is defined as the set of prime filters in L ordered by inclusion, with a topology generated by the sets $\phi(a) = \{F \in \mathcal{PF}(L) \mid a \in F\}$ and their complements.

If we denote the opens in this topology on $\mathcal{PF}(L)$ by $\mathcal{O}(\mathcal{PF}(L))$, then this ϕ can be seen as a lattice homomorphism.

Lemma 4.1. *The map*

$$\begin{aligned}\phi : L &\to \mathcal{O}(\mathcal{PF}(L));\\ a &\mapsto \{F \in \mathcal{PF}(L) \mid x \in F\},\end{aligned}$$

is an injective lattice homomorphism.

With this lemma we can now also conclude that the collection of sets $\phi(a) \cap \phi(b)^c$ forms a basis of the topology on $\mathcal{PF}(L)$.

Lemma 4.2. *If L is a bounded distributive lattice, then $\mathcal{PF}(L)$ is a Stone space.*

It will come in handy to simplify the basis of the topology on $\mathcal{PF}(L)$ in the case that L is a Boolean lattice.

Lemma 4.3. *If B is a Boolean lattice, then $\phi : B \to \mathcal{O}(\mathcal{PF}(B))$ is an injective Boolean homomorphism.*

Proof. Due to lemma 4.1 we already know that ϕ is an injective lattice homomorphism. Since B is a Boolean lattice, any prime filter F is also a maximal filter: given some element a which is not in

F, we know that $a \vee \neg a = 1$, and 1 is in F. Since F is prime, it follows that $\neg a$ is in F. Note that a and $\neg a$ can never both be in a prime filter together, since then it would follow that $a \wedge \neg a = 0$ is in the filter, and thus the prime filter would be B itself, which cannot be. Hence, for any prime filter F, a is in F if and only if $\neg a$ is not in F. Thus $\phi(a)^c = \phi(\neg a)$. □

With this lemma, the basis for the topology on $\mathcal{PF}(B)$ can simply be written as the collection $\phi(a)$, for a in B.

It is now also possible to define the functors from **BL** to **Stone** and vice-versa. For the functor from **BL** to **Stone**, we send a Boolean lattice B to $\mathcal{PF}(\mathcal{B})$. This mapping is well defined thanks to the lemma above. For the morphisms, given a Boolean homomorphism $f : B \to C$, we define $\mathcal{PF}(f) : \mathcal{PF}(C) \to \mathcal{PF}(B)$ by sending a filter Q to $f^{-1}(Q)$. This is again a prime filter. Defined this way, the functor $\mathcal{PF} : \mathbf{BL} \to \mathbf{Stone}$ is contravariant. To show that $\mathcal{PF}(f)$ is continuous, note that

$$\begin{aligned}
\mathcal{PF}(f)^{-1}(U_b) &= \left\{Q \in \mathcal{PF}(C) \mid b \in f^{-1}(Q)\right\} \\
&= \{Q \in \mathcal{PF}(C) \mid f(b) \in Q\} \\
&= \phi(f(b)),
\end{aligned}$$

which means that $\mathcal{PF}(f)$ is indeed a continuous map. It is easily checked that $\mathcal{PF}$ defines a functor.

For the other way around, let X be a Stone space. Then $\mathcal{CP}(X)$, defined as the set of all clopens in X, is a Boolean lattice ordered by inclusion. This gives the functor the other way around, $\mathcal{CP} : \mathbf{Stone} \to \mathbf{BL}$. For the morphisms, if $f : X \to Y$ is a continuous map, define $\mathcal{CP}(f) : \mathcal{CP}(Y) \to \mathcal{CP}(X)$ by sending an open U to $f^{-1}(U)$. Since f is continuous, $f^{-1}(U)$ is again clopen. It is again easily verified that this is a Boolean homomorphism, and that $\mathcal{CP}$ is a contravariant functor.

To show that the functors $\mathcal{PF}$ and $\mathcal{CP}$ provide a duality between **BL** and **Stone**, we first show that $B \cong \mathcal{CP}(\mathcal{PF}(B))$, and then that $X \cong \mathcal{PF}(\mathcal{CP}(X))$.

Lemma 4.4. *Let B be a Boolean lattice. Then the map $F_B : B \to \mathcal{CP}(\mathcal{PF}(B))$, defined by $F_B(b) = \phi(b)$, is an isomorphism of Boolean lattices.*

Proof. $\mathcal{CP}(\mathcal{PF}(B))$ is a Boolean lattice, and necessarily a Boolean subalgebra of the power set of $\mathcal{PF}(B)$ by construction. F_B is a well-defined map and an injective Boolean homomorphism due to lemma 4.3. It still needs to be shown to be surjective. Let C be a clopen subset of $\mathcal{PF}(B)$. Then C is open, so $C = \bigcup \phi(x_i)$ for some collection x_i in B. Since C is also a closed subset of a compact space, it is compact itself. Therefore, since ϕ is a Boolean homomorphism, $C = \bigcup_{i=1}^n \phi(x_i) = \phi(a)$, where $a = \bigvee_{i=1}^n x_i$. □

Lemma 4.5. *Let X be a Stone space. Then the map*

$$\begin{aligned}
G_X : X &\to \mathcal{PF}(\mathcal{CP}(X)); \\
x &\mapsto \{U \in \mathcal{CP}(X) \mid x \in U\},
\end{aligned}$$

is a homeomorphism.

Proof. First it needs to be checked if $G_X(x)$ is indeed a prime filter in $\mathcal{CP}(X)$. It is already clear that $G_X(x)$ is an upset in $\mathcal{CP}(X)$. Now if U, V are in $G_X(x)$, then x is in both U and V, and so $U \cap V$ is in $G_X(x)$. If x is in $U \cup V$, then clearly x is in U or in V. Therefore, U or V is in $G_X(x)$, and thus $G_X(x)$ is a prime filter.

Now we check if $G_X(x)$ is continuous. Let $U \in \mathcal{CP}(X)$, and consider the basic clopen set $V = \{P \in \mathcal{PF}(\mathcal{CP}(X)) \mid U \in P\}$. Then

$$\begin{aligned}
G_X^{-1}(V) &= \{x \in X \mid G_X(x) \in V\} \\
&= \{x \in X \mid U \in G_X(x)\} \\
&= \{x \in X \mid x \in U\} \\
&= U,
\end{aligned}$$

and therefore G_X is continuous.

Next, we note that $\{z\} = \bigcap G_X(z)$ for any $z \in X$. This follows because X is Hausdorff, and the basis of the topology on X consists of clopens. Therefore, if $G_X(x) = G_X(y)$, then $x = y$, and hence G_X is injective. It is also surjective; if P is a prime filter in $\mathcal{CP}(X)$, consider $\bigcap P$. This is a collection of closed subsets of the compact set X, which implies it has the finite intersection property [15]. Furthermore, for any finite collection $F_1, ..., F_N \in P$, their intersection $F_1 \cap ... \cap F_N$ is also in P and therefore non-empty (as $\emptyset \notin P$). Thus $\bigcap P$ is non-empty. If $\bigcap P$ contains distinct points x and y, then there is a clopen set U with $x \in U$ and $y \in U^c$. Moreover, either U is in P, or U^c is in P. Without loss of generality, assume that U is in P. Then y cannot be in $\bigcap P$. Thus $\bigcap P = \{x\}$ for some $x \in X$ and so $P \subseteq G_X(x)$. However, P and $G_X(x)$ are both prime filters and therefore maximal filters in $\mathcal{CP}(X)$, and hence $P = G_X(x)$. We have now shown that G_X is a bijective continuous map. To show that it is a homeomorphism, note that G_X is a map between Stone spaces, which are compact and Hausdorff by definition. Therefore, if $A \subseteq X$ is closed, it is then also compact. The set $G_X(A)$ then also has to be compact, and as a compact subset of a Hausdorff space it is closed. Thus G_X is a closed map, and we can conclude it is a homeomorphism. □

Theorem 4.1. *The functors $\mathcal{PF}$ and $\mathcal{CP}$ give a duality between the categories* **BL** *and* **Stone***.*

Proof. To show that $\mathcal{PF}$ and $\mathcal{CP}$ yield an equivalence of categories, we define a natural isomorphism $F : \mathrm{id}_{\mathbf{BL}} \to \mathcal{CP} \circ \mathcal{PF}$. For a Boolean lattice B, define $F_B : B \to \mathcal{CP}(\mathcal{PF}(B))$ like before, $F_B(b) = \phi(b)$. As we have seen in Lemma 4.4, F_B is an isomorphism of Boolean lattices. Now it can be seen that F is a natural transformation, since if $f : A \to B$ is a Boolean homomorphism, the diagram

$$\begin{array}{ccc} A & \xrightarrow{F_A} & \mathcal{CP}(\mathcal{PF}(A)) \\ f\downarrow & & \downarrow \mathcal{CP}(\mathcal{PF}(f)) \\ B & \xrightarrow{F_B} & \mathcal{CP}(\mathcal{PF}(B)) \end{array}$$

commutes, since if $a \in A$, then

$$\begin{aligned} \mathcal{CP}(\mathcal{PF}(f))(F_A(a)) &= \mathcal{CP}(\mathcal{PF}(f))(\phi(a)) \\ &= \mathcal{PF}(f)^{-1}(\phi(a)) \\ &= \left\{Q \in \mathcal{PF}(\mathcal{B}) \mid f^{-1}(Q) \in \phi(a)\right\} \\ &= \left\{Q \in \mathcal{PF}(B) \mid a \in f^{-1}(Q)\right\} \\ &= \{Q \in \mathcal{PF}(B) \mid f(a) \in Q\} \\ &= F_B(f(a)). \end{aligned}$$

Next, define $G : \mathrm{id}_{\mathbf{Stone}} \to \mathcal{PF} \circ \mathcal{CP}$ for a Stone space X like before, by $G_X(x) = \{U \in \mathcal{CP}(X) \mid x \in U\}$. From lemma 4.5 we already know that G_X is a homeomorphism. Moreover, G is a natural transformation, since if $g : X \to Y$ is continuous, then the diagram

$$\begin{array}{ccc} X & \xrightarrow{G_X} & \mathcal{PF}(\mathcal{CP}(X)) \\ g\downarrow & & \downarrow \mathcal{PF}(\mathcal{CP}((g)) \\ Y & \xrightarrow{G_Y} & \mathcal{PF}(\mathcal{CP}(Y)) \end{array}$$

is commutative. This can be seen since if $x \in X$, then

$$G_Y(g(x)) = \{V \in \mathcal{CP}(Y) \mid g(x) \in V\},$$

and therefore

$$\begin{aligned}\mathcal{PF}(\mathcal{CP}(g))(G_X(x)) &= \mathcal{PF}(\mathcal{CP}(g))(\{U \in \mathcal{CP}(X) \mid x \in U\}) \\ &= \mathcal{CP}(g)^{-1}(\{U \in \mathcal{CP}(X) \mid x \in U\}) \\ &= \left\{V \in \mathcal{CP}(Y) \mid x \in g^{-1}(V)\right\} \\ &= \{V \in \mathcal{CP}(Y) \mid g(x) \in V\} \\ &= G_Y(g(x)).\end{aligned}$$

□

4.2 Priestley Duality

Now we extend Stone duality to the case of bounded distributive lattices. This is often called Priestley duality. If L is a bounded distributive lattice, $\mathcal{CP}(\mathcal{PF}(L))$ is a Boolean lattice. Therefore, we need to determine how to recover L from $\mathcal{PF}(L)$. If a is in L and P is in $\phi(a)$, then for any prime filter Q with $P \subseteq Q$, Q is also in $\phi(a)$. Inclusion is of course a partial order on $\mathcal{PF}(L)$, and so we see that $\phi(a)$ is a clopen upper set of $\mathcal{PF}(L)$ for any a in L.

Definition 4.2. $(X, \leq)$ is called a **Priestley space** if it is a Stone space with a partial order satisfying the *Priestley separation axiom*: for all x and y in X with $x \not\leq y$, there is a clopen upset U with $x \in U$ and $y \notin U$.

The category **Pries** consists of Priestley spaces where the maps are continuous and order preserving. We will show that this category is dually equivalent to the category of bounded distributive lattices, **BDL**, where the maps are lattice homomorphisms.

Lemma 4.6. *If L is a bounded distributive lattice, then $(\mathcal{PF}(L), \subseteq)$ is a Priestley space.*

Now we recover L from $\mathcal{PF}(L)$:

Lemma 4.7. *The clopen upsets of $\mathcal{PF}(L)$ are precisely the sets $\phi(a)$, for a in L.*

If $(X, \leq)$ is a Priestley space, we denote the clopen upsets of $(X, \leq)$ by $\mathcal{CU}(X, \leq)$. Then we can define one contravariant functor by $\mathcal{PF}$: **BDL** $\to$ **Pries** which turns a lattice homomorphism $f : L \to M$ into a Priestley homomorphism $\mathcal{PF}(f) : \mathcal{PF}(M) \to \mathcal{PF}(L)$, defined by $\mathcal{PF}(f)(Q) = f^{-1}(Q)$. For the other way around, we have $\mathcal{CU}$: **Pries** $\to$ **BDL** which turns a Priestley homomorphism $g : X \to Y$ into a lattice homomorphism $\mathcal{CU}(g) : \mathcal{CU}(Y) \to \mathcal{CU}(X)$, defined by $\mathcal{CU}(g)(V) = g^{-1}(V)$. It is elementary to verify that these are well-defined functors.

Lemma 4.8. *If L is a distributive lattice, then the map $F_L : L \to \mathcal{CU}(\mathcal{PF}(L), \subseteq)$, defined by $F_L(a) = \phi(a)$, is a lattice isomorphism.*

Lemma 4.9. *If $(X, \leq)$ is a Priestley space, then*

$$\begin{aligned}G_X : (X, \leq) &\to \mathcal{PF}(\mathcal{CU}(X, \leq)) \\ x &\mapsto \{U \in \mathcal{CU}(X, \leq) \mid x \in U\}\end{aligned}$$

is an isomorphism of Priestley spaces.

Proof. The proof that $G_X(x)$ is indeed a prime filter is the same as in lemma 4.5. To see that G_X is order preserving, take $x \leq y$ and $U \in G_X(x)$. Then $x \in U$, and since U is an upset, $y \in U$. Thus $U \in G_X(y)$. G_X is also continuous: let V be a clopen upset in $(X, \leq)$, and consider the basic clopen set $\phi(U) = \{P \in \mathcal{PF}(\mathcal{CU}(X, \leq)) \mid U \in P\}$. Then

$$\begin{aligned}G_X^{-1}(\phi(U)) &= \{x \in X \mid G_X(x) \in \phi(U)\} \\ &= \{x \in X \mid U \in G_X(x)\} \\ &= \{x \in X \mid x \in U\} \\ &= U,\end{aligned}$$

so G_X is continuous, and G_X is indeed a valid Priestley homomorphism.

The Priestley separation axiom shows that if $z \in X$, then any point in X not above z can be separated from z by a clopen upset. Therefore, $\uparrow z = \bigcap G_X(z)$. From this, it is clear that if

$G_X(x) \subseteq G_X(y)$ then $x \leq y$, so G_X is strictly order preserving and hence also injective. Moreover, we note that G_X is a closed map, since its domain is compact and its codomain is Hausdorff. To finish the proof, we only need to prove that G_X is surjective (since a continuous map that is bijective and closed is also a homeomorphism). Now note that $G_X(X)$ is closed in $\mathcal{PF}(\mathcal{CU}(X, \leq))$. If G_X is not surjective, there is some prime filter P in $\mathcal{CU}(X, \leq)$ not contained in $G_X(X)$. Therefore, there must be some basic open set $V = \phi(U_1) \cup \phi(U_2)^c$ containing P but disjoint from $G_X(X)$, for some U_1, U_2 in $\mathcal{CU}(X, \leq)$. Now, $\emptyset = G_X^{-1}(V) = G_X^{-1}(\phi(U_1)) \cap G_X^{-1}(\phi(U_2))^c$. We have already seen above that $G_X^{-1}(\phi(U)) = U$. Therefore, $\emptyset = U_1 \cap U_2^c$, implying that $U_1 \subseteq U_2$. But then $V = \phi(U_1) \cap \phi(U_2)^c = \emptyset$. This contradiction shows that G_X is surjective. □

With these lemmas we are in the position to prove Priestley duality in full categorical glory!

Theorem 4.2. *The functors $\mathcal{CU}$ and $\mathcal{PF}$ give a duality of categories between* **BDL** *and* **Pries**.

Proof. Define the natural transformation $F : \mathrm{id}_{\mathbf{BDL}} \to \mathcal{CU} \circ \mathcal{PF}$ where, for a bounded distributive lattice L, the map $F_L : L \to \mathcal{CU}(\mathcal{PF}(L))$ is defined by $F_L(a) = \phi(a)$. Then F_L is a lattice isomorphism as we have seen before in lemma 4.8. To see that the diagram

$$\begin{array}{ccc} L & \xrightarrow{F_L} & \mathcal{CU}(\mathcal{PF}(L)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{CU}(\mathcal{PF}(f))} \\ M & \xrightarrow{F_M} & \mathcal{CU}(\mathcal{PF}(M)) \end{array}$$

commutes, let $l \in L$. Then

$$\begin{aligned} \mathcal{CU}(\mathcal{PF}(f))(F_L(a)) &= \mathcal{CU}(\mathcal{PF}(f))(\phi(a)) \\ &= \mathcal{PF}(f)^{-1}(\phi(a)) \\ &= \left\{Q \in \mathcal{PF}(\mathcal{M}) \mid f^{-1}(Q) \in \phi(a)\right\} \\ &= \left\{Q \in \mathcal{PF}(M) \mid a \in f^{-1}(Q)\right\} \\ &= \{Q \in \mathcal{PF}(M) \mid f(a) \in Q\} \\ &= F_M(f(a)). \end{aligned}$$

Next, for a Priestley space $(X, \leq)$, define $G : \mathrm{id}_{\mathbf{Pries}} \to \mathcal{PF} \circ \mathcal{CU}$ as before by $G_X(x) = \{U \in \mathcal{CU}(X, \leq) \mid x \in U\}$. From lemma 4.9 we already know that G_X is a Priestley isomorphism. Moreover, G is a natural transformation, since if $g : (X, \leq) \to (Y, \preceq)$ is continuous, then the diagram

$$\begin{array}{ccc} (X, \leq) & \xrightarrow{G_X} & \mathcal{PF}(\mathcal{CU}(X, \leq)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathcal{PF}(\mathcal{CU}((g))} \\ (Y, \preceq) & \xrightarrow{G_Y} & \mathcal{PF}(\mathcal{CU}(Y, \preceq)) \end{array}$$

commutes. Indeed, if $x \in X$, then

$$G_Y(g(x)) = \{V \in \mathcal{CU}(Y) \mid g(x) \in V\},$$

hence

$$\begin{aligned} \mathcal{PF}(\mathcal{CU}(g))(G_X(x)) &= \mathcal{PF}(\mathcal{CU}(g))(\{U \in \mathcal{CU}(X) \mid x \in U\}) \\ &= \mathcal{CU}(g)^{-1}(\{U \in \mathcal{CU}(X) \mid x \in U\}) \\ &= \left\{V \in \mathcal{CU}(Y) \mid x \in g^{-1}(V)\right\} \\ &= \{V \in \mathcal{CU}(Y) \mid g(x) \in V\} \\ &= G_Y(g(x)). \end{aligned}$$

Therefore, F and G yield a duality between **BDL** and **Pries**. □

4.3 Esakia Duality

In this section we specialize Priestley duality from bounded distributive lattices to the category **HA** of Heyting algebras. If we wish to restrict Priestley duality to this category, we need to determine which Priestley spaces are duals of Heyting algebras, and which morphisms of such spaces are dual to Heyting morphisms.

Definition 4.3. Let $g : (X, \leq) \to (Y, \preceq)$ be a morphism of posets. We say that g is a ***p*-morphism** if for every $x \in X$ and $y \in Y$ with $g(x) \preceq z$ there is an $x' \in X$ with $x \leq x'$ and $g(x') = z$.

Definition 4.4. An **Esakia space** is a Priestley space $(X, \leq)$ such that if U is clopen, then so is $\downarrow U$.

We denote the category of Esakia spaces by **Esa**, where the morphisms are continuous p-morphisms. In this section we see that Priestley duality restricts to a duality between **HA** and **Esa**. We start with some preliminary lemmas.

Lemma 4.10. *Let $(X, \leq)$ be a Priestley space.*

- *The relation $\leq$ is closed, i.e. the set $R = \{(x, y) \in X \times X \mid x \leq y\}$ is closed in $X \times X$.*
- *If C is closed in X, then so are $\uparrow C$ and $\downarrow C$.*

Lemma 4.11. *Let H be a Heyting algebra. If $a, b \in H$, then*

$$\downarrow(\phi(a) \cap \phi(b)^c) = \phi(a \to b)^c.$$

Lemma 4.12. *Let $f : (X, \leq) \to (Y, \preceq)$ be a poset morphism. Then the following conditions are equivalent:*

1. *f is a p-morphism,*
2. *$f^{-1}(\downarrow A) = \downarrow f^{-1}(A)$ for every subset $A \subseteq Y$,*
3. *$f^{-1}(\downarrow y) = \downarrow f^{-1}(\{y\})$ for every $y \in Y$.*

We now consider the functor $\mathcal{PF} : \mathbf{HA} \to \mathbf{Pries}$ defined by the restriction of $\mathcal{PF} : \mathbf{BDL} \to \mathbf{Pries}$.

Lemma 4.13. *If H is a Heyting algebra, then $(\mathcal{PF}(H), \subseteq)$ is an Esakia space.*

Proof. We already know that $(\mathcal{PF}(H), \subseteq)$ is a Priestley space. Let U be a clopen set in $\mathcal{PF}(H)$. Then $U = \bigcup_{i=1}^n \phi(a_i) \cap \phi(b_i)^c$ for some a_i, b_i in H. By lemma 4.11, we have

$$\downarrow U = \bigcup_{i=1}^n \downarrow(\phi(a_i) \cap \phi(b_i)^c) = \bigcup_{i=1}^n \phi(a_i \to b_i)^c,$$

a clopen set. Therefore, $(\mathcal{PF}(H), \subseteq)$ is an Esakia space. □

Lemma 4.14. *Let $f : H \to H'$ be a Heyting morphism. Then $\mathcal{PF}(f) : \mathcal{PF}(H') \to \mathcal{PF}(H)$ is a p-morphism.*

Proof. Let $Q \in \mathcal{PF}(H')$ and $P \in \mathcal{PF}(H)$ with $f^{-1}(Q) \subseteq P$. For notational convenience, we write $\mathcal{PF}(f) = g$. Let C be a clopen set in $\mathcal{PF}(H)$ containing P. Then C is a finite union of sets of the form $\phi(a) \cap \phi(b)^c$ with $a \in P$ and $b \notin P$. We have

$$\begin{aligned} g^{-1}(\downarrow(\phi(a) \cap \phi(b)^c)) &= g^{-1}(\phi(a \to b)^c) \\ &= g^{-1}(\phi(a \to b))^c \\ &= \phi(f(a \to b))^c \\ &= \phi(f(a) \to f(b))^c \\ &= \downarrow(\phi(f(a)) \cap \phi(f(b))^c). \end{aligned}$$

By considering finite unions, we then see that $g^{-1}(\downarrow C) = \downarrow g^{-1}(C)$ for any clopen set. Since $g(Q) \subseteq P$, we see that $Q \in g^{-1}(\downarrow C) = \downarrow g^{-1}(C)$ for any clopen C containing P. Thus $\uparrow Q \cap g^{-1}(C) \neq \emptyset$. Since the set of clopens containing P is closed under finite intersections, compactness implies that $\bigcap(\uparrow Q \cap g^{-1}(C)) \neq \emptyset$, where the intersection is over all clopens C containing P. This yields $\uparrow Q \cap \bigcap g^{-1}(C) \neq \emptyset$, so $\uparrow Q \cap g^{-1}(\{P\}) \neq \emptyset$. Therefore, there is some Q' with $Q \subseteq Q'$ and $g(Q') = P$. This proves that $g = \mathcal{PF}(f)$ is a p-morphism. □

The previous two lemmas show that $\mathcal{PF}$ is a functor from **HA** to **Esa**. We now consider the functor $\mathcal{CU}$: **Pries** $\to$ **BDL** restricted to **Esa**.

Lemma 4.15. *Let $(X, \leq)$ be an Esakia space. Then $\mathcal{CU}(X, \leq)$ is a Heyting algebra, where implication is defined by $U \to V = (\downarrow(U \cap V^c))^c$.*

Proof. We already know that $\mathcal{CU}(X, \leq)$ is a bounded distributive lattice. Now let U and V be clopen upsets. Then $U \cap V^c$ is clopen and since $(X, \leq)$ is an Esakia space, $\downarrow(U \cap V^c)$ is clopen. Then $(\downarrow(U \cap V^c))^c$ is a clopen upset, so we define

$$U \to V = (\downarrow(U \cap V^c))^c.$$

To see that this is a Heyting implication, we need to check that for any clopen upset W, we have $U \cap W \subseteq V$ if and only if $W \subseteq U \to V$.

Since $U \to V \subseteq (U \cap V^c)^c$, we have

$$U \cap (U \to V) \subseteq U \cap (U \cap V^c)^c = U \cap (U^c \cup V) = V.$$

Therefore, if $W \subseteq U \to V$, then

$$U \cap W \subseteq U \cap (U \to V) \subseteq V.$$

Suppose that $U \cap W \subseteq V$. Then $U \cap V^c \subseteq W^c$. Since W^c is a downset, we obtain $\downarrow(U \cap V^c) \subseteq W^c$. Thus $W \subseteq (\downarrow(U \cap V^c))^c = U \to V$. □

Lemma 4.16. *Let $g : (X, \leq) \to (Y, \preceq)$ be a morphism of Esakia spaces. Then the map*

$$\begin{aligned} \mathcal{CU}(g) : \mathcal{CU}(Y, \preceq) &\to \mathcal{CU}(X, \leq); \\ U &\mapsto g^{-1}(U), \end{aligned}$$

is a Heyting morphism.

Proof. We know that $\mathcal{CU}(g)$ is a lattice homomorphism, so we only need to show that it preserves implication. Let U, V be clopen upsets of Y. Since

$$g^{-1}(U) \cap g^{-1}(U \to V) = g^{-1}(U \cap (U \to V)) \subseteq g^{-1}(V),$$

we see that $g^{-1}(U \to V) \subseteq g^{-1}(U) \to g^{-1}(V)$. For the reverse inclusion, suppose that $x \notin g^{-1}(U \to V)$. Since $U \to V = (\downarrow(U \cap V^c))^c$, we have $x \in g^{-1}(\downarrow(U \cap V^c))$, so $g(x) \in \downarrow(U \cap V^c)$. Therefore, there is a $y \in U \cap V^c$ with $g(x) \preceq y$. Since g is a p-morphism, there is a $z \in X$ with $x \leq z$ and $y = g(z)$. Then $z \in g^{-1}(U \cap V^c) = g^{-1}(U) \cap g^{-1}(V)^c$. Thus $x \in \downarrow(g^{-1}(U) \cap g^{-1}(V)^c)$, and so $x \notin g^{-1}(U) \to g^{-1}(V)$. This proves the reverse inclusion. Therefore,

$$g^{-1}(U \to V) = g^{-1}(U) \to g^{-1}(V),$$

so $\mathcal{CU}(g)$ is a Heyting morphism. □

We have shown that $\mathcal{CU}$ is a functor from **Esa** to **HA**. To prove that these categories are dual to each other, we have little work left to do.

Lemma 4.17. *Let H be a Heyting algebra. Then the map*

$$\begin{aligned} F_H : H &\to \mathcal{CU}(\mathcal{PF}(H)); \\ a &\mapsto \phi(a), \end{aligned}$$

is a Heyting isomorphism.

Proof. We have seen in lemma 4.8 that F_H is an isomorphism of bounded distributive lattices. Therefore, we only need to check if F_H preserves implication. Let $a, b \in H$. Then by lemmas 4.11 and 4.15,

$$F_H(a \to b) = \phi(a \to b) = (\downarrow(\phi(a) \cap \phi(b)^c))^c = \phi(a) \to \phi(b).$$

Thus F_H is an isomorphism of Heyting algebras. □

Lemma 4.18. *Let $(X, \leq)$ be an Esakia space. Then*

$$G_X : (X, \leq) \to \mathcal{PF}(\mathcal{CU}(X, \leq));$$
$$x \mapsto \{U \in \mathcal{CU}(X, \leq) \mid x \in U\},$$

is an Esakia isomorphism.

Proof. We have seen in lemma 4.9 that G_X is an isomorphism of Priestley spaces. Since G_X and G_X^{-1} are then in particular poset isomorphisms, they are both p-morphisms. Thus G_X is an Esakia isomorphism. □

Theorem 4.3. *The functors $\mathcal{CU}$ and $\mathcal{CU}$ give a duality of categories between* **HA** *and* **Esa**.

4.4 Dual of $Q(A)$

For a Boolean lattice B, its associated Stone space $S(B)$ can also be realised as the space of Boolean lattice homomorphisms from B to the two-element Boolean lattice $\underline{2}$. That is, we have a homeomorphism

$$S(B) \cong \mathrm{Hom}(B, \underline{2}). \tag{4.1}$$

We will attempt to generalize this to the Heyting lattice $Q(A)$. Consider a finite-dimensional C*-algebra A and a state ω on A. Then we can define a function V_ω, the 'valuation' associated to ω:

$$V_\omega : Q(A) \to \mathrm{Up}(\mathcal{C}(A)),\ V_\omega(S) = \{C \in \mathcal{C}(A) \mid \omega(S(C)) = 1\}$$

The question arises if these valuations are Heyting morphisms. Suppose that they were, then this would give a motivation to look at the space

$$\mathrm{Mod}_{\mathcal{C}(A)}(Q(A)) = \{\text{Heyting morphisms } Q(A) \to \mathrm{Up}(\mathcal{C}(A))\}$$

We conjecture, like in classical logic (where $\mathrm{Up}(\mathcal{C}(A))$ is replaced by $\underline{2}$), that $\mathrm{Mod}_{\mathcal{C}(A)}(Q(A))$ is an Esakia space and that the Esakia space $E(Q(A))$ associated to $Q(A)$ is isomorphic to it.

However, multiple things go wrong:

1. The functions V_ω are not, in general, Heyting morphisms (we will shortly prove a counterexample even for $A = \mathbb{C}^2$).
2. The set $\mathrm{Mod}_{\mathcal{C}(A)}(Q(A))$ is not, in general, an Esakia space and for some A it is even **empty**.

Example. Let $A = \mathbb{C}^2$ and consider the state $\omega : \mathbb{C}^2 \to \mathbb{C}$ given by $(x, y)^T \mapsto x$. Define $S, T \in Q(A)$ by $S(\mathbb{C} \cdot 1) = T(\mathbb{C} \cdot 1) = 0$ and $S(\mathbb{C}^2) = (1, 0)^T$, $T(\mathbb{C}^2) = (0, 1)^T$. Then

$$V_\omega(S \to T) = V_\omega(T) = \emptyset.$$

However,

$$V_\omega(S) \to V_\omega(S) = \emptyset \to \emptyset = \mathcal{C}(A).$$

Therefore, V_ω is not a Heyting morphism.

The following proposition explains the second point.

Proposition 4.1. *Let $A = M_n(\mathbb{C})$ for some $n > 2$. Then $Mod_{\mathcal{C}(A)}(Q(A)) = \emptyset$.*

Proof. This is explained in [16], page 8. □

This means that in our quantum logic there is, unfortunately, no similar result as 4.1 in classical logic.

Chapter 5

Extension of $Q(A)$

In chapter 3 we have defined the Heyting lattice $Q(A)$ for finite-dimensional C*-algebras in order to obtain an isomorphism $\mathcal{O}(\Sigma_A) \cong Q(A)$. The goal of this chapter will be to extend the definition of $Q(A)$ to so-called AW*-algebras. However, it turns out that instead of them being isomorphic, $Q(A)$ will be a **dense sublocale** of $\mathcal{O}(\Sigma_A)$. The material on AW*-algebras is based on [11], Section 2.4 and Chapter 8.

5.1 AW*-algebras and Stonean spaces

In order to extend the definition of $Q(A)$ to more general C*-algebras than just finite-dimensional ones, we want $\mathrm{Proj}(A)$ to have a 'nice' structure. A natural choice is the class of **AW*-algebras**. There are many equivalent definitions of AW*-algebras, but since we only need the structure of the projections, we will also define them in terms of projections.

Definition 5.1. A (unital) C*-algebra A is called an **AW*-algebra** if

- every maximal commutative subalgebra of A is generated by its projections;
- $\mathrm{Proj}(A)$ is a complete orthomodular lattice.

Remark. This implies that if A is a **commutative** AW*-algebra, then $\mathrm{Proj}(A)$ is a complete **Boolean** lattice by Proposition 3.4.

Example. For H a Hilbert space, $B(H)$ is an AW*-algebra ([11], Proposition 2.4.16). More generally, any von Neumann algebra is an AW*-algebra. In fact, it is hard to find examples of AW*-algebras that are not von Neumann algebras. Also, every finite-dimensional C*-algebra is an AW*-algebra. We will show this in section 5.3.

We also need to define the notions of AW*-homomorphism and AW*-subalgebra.

Definition 5.2. Let $\phi : A \to B$ be a *-homomorphism between AW*-algebras. Then ϕ is called an **AW*-homomorphism** if the map

$$\phi|_{\mathrm{Proj}(A)} : \mathrm{Proj}(A) \to \mathrm{Proj}(B)$$

preserves all suprema.

Definition 5.3. Let A be an AW*-algebra. Then a C*-subalgebra B of A is called an **AW*-subalgebra** if

- B is an AW*-algebra;
- If $E \subseteq \mathrm{Proj}(B)$ is a collection of projections in B, its supremum $\bigvee_{e \in E} e$, **calculated in** A, is an element of B.

Equivalently, B is an AW*-algebra and the inclusion $i : B \hookrightarrow A$ is an AW*-homomorphism.

Since any commutative C*-algebra A is *-isomorphic to $C(X)$ for some compact Hausdorff space X (Theorem 1.2) , we would like a criterion on X that determines if A is an AW*-algebra.

Definition 5.4. A topological space X is called **extremally disconnected** if, for any open $U \in \mathcal{O}(X)$, its closure $\overline{U}$ is also open. An extremally disconnected space that is also compact Hausdorff is called a **Stonean** space.

Theorem 5.1. *Let A be an AW*-algebra. Then A is commutative if and only if its Gelfand spectrum $\Sigma(A)$ is a Stonean space. That is, $A \cong C(X)$ for X a Stonean space.*

Proof. [11], Corollary 2.4.6. □

We need some results on Stonean spaces in the following sections. They are proven here, but are recalled later when they are used.

Lemma 5.1. *If X is a topological space, U, V open, and $U \cap V = \emptyset$, then*

$$\overline{U} \cap V = \emptyset.$$

Proof. Let $x \in \overline{U}$. Then for all open neighborhoods O of x we have $O \cap U \neq \emptyset$. Therefore $U \cap V = \emptyset$ implies that $V \notin \mathcal{O}(x)$, and $x \notin V$. □

Lemma 5.2. *If X is a topological space, and $\{U_i\}_{i\in I}$ is a collection of open sets in X, then*

$$\overline{\bigcup_{i\in I} U_i} = \overline{\bigcup_{i\in I} \overline{U_i}}.$$

Proof. It is clear that

$$\overline{\bigcup_{i\in I} U_i} \subseteq \overline{\bigcup_{i\in I} \overline{U_i}}.$$

Now let $x \notin \overline{\bigcup_{i\in I} U_i}$. Then there is an open neighborhood O of x such that

$$O \cap \bigcup_{i\in I} U_i = \emptyset.$$

But

$$O \cap \bigcup_{i\in I} U_i = \bigcup_{i\in I} O \cap U_i,$$

which implies that $O \cap U_i = \emptyset$ for all $i \in I$. By the previous Lemma, we then have

$$O \cap \overline{U_i} = \emptyset \text{ for all } i \in I.$$

Therefore,

$$O \cap \bigcup_{i\in I} \overline{U_i} = \emptyset,$$

which means that

$$x \notin \overline{\bigcup_{i\in I} \overline{U_i}}.$$

□

Lemma 5.3. *If X is a Stonean space and $U, V \in \mathcal{O}(X)$, then*

$$\overline{U \cap V} = \overline{U} \cap \overline{V}.$$

Proof. Now let $x \notin \overline{U \cap V}$. Then there is an open neighborhood O of x such that

$$U \cap V \cap O = \emptyset.$$

Therefore, by the Lemma 5.1,

$$\overline{U} \cap V \cap O = \emptyset.$$

But X is Stonean, so $\overline{U} \cap O$ is open, so that we have, again by Lemma 5.1,

$$\overline{U} \cap \overline{V} \cap O = \emptyset,$$

which shows that that $x \notin \overline{\overline{U} \cap \overline{V}} \subseteq \overline{U} \cap \overline{V}$. Hence,

$$\overline{U} \cap \overline{V} \subseteq \overline{U \cap V}.$$

Since we always have $\overline{U \cap V} \subseteq \overline{U} \cap \overline{V}$, we conclude that

$$\overline{U \cap V} = \overline{U} \cap \overline{V}.$$

□

Lemma 5.4. *If X is a Stonean space then the collection* Clopen(X) *of clopen subsets of X is a complete Boolean lattice.*

Proof. Since every Stonean space is a Stone space, the collection of clopen subsets is a Boolean lattice (Chapter 4). We need only prove that it is complete. If $\{U_i\}_{i \in I}$ is a collection of clopen sets in X, then the union $\bigcup_{i \in I} U_i$ is open. Therefore, the set

$$V = \overline{\bigcup_{i \in I} U_i}$$

is clopen. We will prove that V is the supremum of the collection $\{U_i\}$. It is definitely an upper bound, so let W, a clopen set, be an arbitrary upper bound of the collection $\{U_i\}$. This means that $U_i \subseteq W$ for all $i \in I$, implying that

$$\bigcup_{i \in I} U_i \subseteq W.$$

But W is closed, so

$$\overline{\bigcup_{i \in I} U_i} \subseteq W.$$

Therefore, $\overline{\bigcup_{i \in I} U_i}$ is the smallest upper bound of the U_i. □

5.2 $Q(A)$ and $\mathcal{O}(\Sigma_A)$ for AW*-algebras

The main modification to the definition of $Q(A)$ and $\mathcal{O}(\Sigma_A)$ will be to replace $\mathcal{C}(A)$ by

$$\mathcal{A}(A) = \{C \in \mathcal{C}(A) \mid C \text{ is an AW}^*\text{-subalgebra of A}\}.$$

This will allow us to use the properties of AW*-algebras not only for A itself, but also for all the commutative subalgebras that we consider.

Proposition 5.1. *For a commutative AW*-algebra C there is an isomorphism of complete Boolean lattices $\beta_C : \mathrm{Proj}(C) \to \mathrm{Clopen}(\Sigma(C))$.*

Proof. [11], Proposition C.3.6. □

Remark. This results holds in particular for finite-dimensional algebras, which are always AW*-algebras. Then this is just 3.2, because in this case $\Sigma(C)$ has the discrete topology. Also, this isomorphism is natural and the proof is the same as for 3.2.

Notation. For an AW*-algebra A we will write $\widehat{Q(A)}$ and $\widehat{\mathcal{O}(\Sigma_A)}$ if we use $\mathcal{A}(A)$ instead of $\mathcal{C}(A)$.

Our goal will be to prove the following theorem:

Theorem 5.2. *Let A be an AW*-algebra. Then $\widehat{Q(A)}$ is a dense sublocale of $\widehat{\mathcal{O}(\Sigma_A)}$.*

We refer to Section 2.4 for the definitions of locales, sublocales and nuclei. To prove the theorem, we first define the following maps:

$$\Psi : \widehat{Q(A)} \to \widehat{\mathcal{O}(\Sigma_A)}$$
$$S \mapsto \coprod_{C \in \mathcal{A}(A)} \beta_C(S(C)),$$

and

$$\Phi : \widehat{\mathcal{O}(\Sigma_A)} \to \widehat{Q(A)}$$
$$U \mapsto S_{\mathcal{U}}, S_{\mathcal{U}}(C) = \beta_C^{-1}(\overline{\mathcal{U}_C}).$$

Lemma 5.5. *The map Ψ is well defined.*

Proof. It is clear that $\beta_C(S(C)) \in \mathcal{O}(\Sigma(C))$ for all $C \in \mathcal{A}(A)$, since clopens are open. For $C \subseteq D$ we have the inclusion $j : C \hookrightarrow D$. Now suppose $\mu \in \Sigma(D)$, $\mu|_C \in \mathcal{U}_C$ but

$$\mu \notin \mathcal{U}_D = \beta_D(S(D)) \supseteq \beta_D(S(C)) = \Sigma(j)^{-1}(\beta_C(S(C)) = \Sigma(j)^{-1}(\mathcal{U}_C).$$

This means that $\mu \circ j \notin \mathcal{U}_C$ i.e. $\mu|_C \notin \mathcal{U}_C$. □

Lemma 5.6. *The map Φ is well defined.*

Proof. Since $\overline{\mathcal{U}_C}$ is clopen, $S_{\mathcal{U}}(C)$ does indeed define a projection in C by the properties of β_C. It remains to prove that $C \subseteq D$ implies that $S_{\mathcal{U}}(C) \leq S_{\mathcal{U}}(D)$, i.e. $\beta_C^{-1}(\overline{\mathcal{U}_C}) \leq \beta_D^{-1}(\overline{\mathcal{U}_D})$. We have $\beta_C^{-1}(\overline{\mathcal{U}_C}) = \beta_D^{-1}(\Sigma(j)^{-1}(\overline{\mathcal{U}_C}))$. If we prove that $\Sigma(j)^{-1}(\overline{\mathcal{U}_C}) = \overline{\Sigma(j)^{-1}(\mathcal{U}_C)}$, then

$$\beta_D^{-1}(\overline{\Sigma(j)^{-1}(\mathcal{U}_C)}) \leq \beta_D^{-1}(\overline{\mathcal{U}_D}),$$

since $\Sigma(j)^{-1}(\mathcal{U}_C) \subseteq \mathcal{U}_D$ and β_D is order preserving. □

Proposition 5.2. *The following properties hold for the maps Ψ and Φ:*

1. *Ψ is order preserving.*
2. *Φ is a frame morphism.*
3. *The map $\Phi \circ \Psi$ is the identity on $\widehat{Q(A)}$.*
4. *The map $\Psi \circ \Phi$ is a nucleus on $\widehat{\mathcal{O}(\Sigma_A)}$.*

Proof. 1. Suppose $S \leq T$. Then for all $C \in \mathcal{A}(A)$, $S(C) \leq T(C)$ and therefore

$$\beta_C(S(C)) \subseteq \beta_C(T(C)),$$

because β_C is order preserving. But this means that $\Psi(S) \subseteq \Psi(T)$.

2. Let $\mathcal{U}, \mathcal{V}$ be opens in $\widehat{\mathcal{O}(\Sigma_A)}$. In a Stonean space we have

$$\overline{U \cap V} = \overline{U} \cap \overline{V}$$

by Lemma 5.3. Therefore, for $C \in \mathcal{A}(A)$,

$$\begin{aligned} S_{\mathcal{U}\cap\mathcal{V}}(C) &= \beta_C^{-1}(\overline{\mathcal{U}_C \cap \mathcal{V}_C}) \\ &= \beta_C^{-1}(\overline{\mathcal{U}_C} \cap \overline{\mathcal{V}_C}) \\ &= \beta_C^{-1}(\overline{\mathcal{U}_C}) \cap \beta_C^{-1}(\overline{\mathcal{V}_C}) \\ &= S_{\mathcal{U}}(C) \cap S_{\mathcal{V}}(C) \\ &= (S_{\mathcal{U}} \cap S_{\mathcal{V}})(C). \end{aligned}$$

Here we have used that β_C^{-1} is a lattice homomorphism. So we have

$$\Phi(\mathcal{U} \cap \mathcal{V}) = \Phi(\mathcal{U}) \cap \Phi(\mathcal{V}).$$

Now let $\{\mathcal{U}_i\}_{i\in I}$ be open in $\widehat{\mathcal{O}(\Sigma_A)}$. Then, by Lemma 5.2,

$$\begin{aligned}
S_{\cup \mathcal{U}_i}(C) &= \beta_C^{-1}\Big(\overline{\bigcup_{i\in I} \mathcal{U}_{i,C}}\Big)\\
&= \beta_C^{-1}\Big(\overline{\bigcup_{i\in I} \overline{\mathcal{U}_{i,C}}}\Big)\\
&= \beta_C^{-1}\Big(\bigvee_{i\in I} \overline{\mathcal{U}_{i,C}}\Big)\\
&= \bigvee_{i\in I} \beta_C^{-1}(\overline{\mathcal{U}_{i,C}})\\
&= \bigvee_{i\in I} S_{\mathcal{U}_i}(C)\\
&= \Big(\bigvee_{i\in I} S_{\mathcal{U}_i}\Big)(C),
\end{aligned}$$

where we use that β_C^{-1} preserves suprema. We conclude that

$$\Phi\Big(\bigcup_{i\in I} \mathcal{U}_i\Big) = \bigvee_{i\in I} \Phi(\mathcal{U}_i).$$

3. Let $S \in \widehat{Q(A)}$. Then

$$(\Phi \circ \Psi(S))(C) = \beta_C^{-1}(\overline{\beta_C(S(C))}).$$

But $\beta_C(S(C))$ is clopen in $\Sigma(C)$, so

$$(\Phi \circ \Psi(S))(C) = \beta_C^{-1}(\beta_C(S(C))) = S(C),$$

since β_C is an isomorphism.

4. Let $\mathcal{U} \in \widehat{\mathcal{O}(\Sigma_A)}$. Then

$$(\Psi \circ \Phi)(\mathcal{U}) = \coprod_{C\in\mathcal{A}(A)} \beta_C(\beta_C^{-1}(\overline{\mathcal{U}_C})) = \coprod_{C\in\mathcal{A}(A)} \overline{\mathcal{U}_C}.$$

Denote $j := \Psi \circ \Phi$. We have

$$\begin{aligned}
j(\mathcal{U} \cap \mathcal{V}) &= \coprod_{C\in\mathcal{A}(A)} \overline{\mathcal{U}_C \cap \mathcal{V}_C}\\
&= \coprod_{C\in\mathcal{A}(A)} \overline{\mathcal{U}_C} \cap \overline{\mathcal{V}_C}\\
&= \coprod_{C\in\mathcal{A}(A)} \overline{\mathcal{U}_C} \cap \coprod_{C\in\mathcal{A}(A)} \overline{\mathcal{V}_C}\\
&= j(\mathcal{U}) \cap j(\mathcal{V}).
\end{aligned}$$

Furthermore, it is clear that $\mathcal{U} \subseteq j(\mathcal{U})$ always holds and since j is idempotent, so does $j(j(\mathcal{U})) = j(\mathcal{U})$. Therefore, j is a nucleus on $\widehat{\mathcal{O}(\Sigma_A)}$.

□

Proof of Theorem 5.2:

Proof. Since $\Phi \circ \Psi$ is the identity on $\widehat{Q(A)}$, Φ is a surjective frame homomorphism. Furthermore, $\Phi \circ \Psi$ is a nucleus on $\widehat{\mathcal{O}(\Sigma_A)}$. If we prove that Ψ is a right adjoint to Φ, and that Ψ is an order embedding, then we will have proven that $\widehat{Q(A)}$ is a sublocale of $\widehat{\mathcal{O}(\Sigma_A)}$. Let $\mathcal{U} \in \widehat{\mathcal{O}(\Sigma_A)}$ and $S \in \widehat{Q(A)}$. Then

$$\begin{aligned}
\Phi(\mathcal{U}) \le S &\iff S_{\mathcal{U}}(C) \le S(C)\ \forall C \in \mathcal{A}(A)\\
&\iff \beta_C^{-1}(\overline{\mathcal{U}_C}) \le S(C)\ \forall C \in \mathcal{A}(A)\\
&\iff \overline{\mathcal{U}_C} \subseteq \beta_C(S(C))\ \forall C \in \mathcal{A}(A)\\
&\iff \mathcal{U}_C \subseteq \beta_C(S(C))\ \forall C \in \mathcal{A}(A)\\
&\iff \mathcal{U} \subseteq \Phi(S),
\end{aligned}$$

where in the third step we have used that β_C is an isomorphism, and in the fourth step we have used that $\beta_C(S(C))$ is clopen. Therefore, Ψ is a right adjoint to Φ. Furthermore, if $S, T \in \widehat{Q(A)}$, then we have

$$\begin{aligned}\Psi(S) \subseteq \Psi(T) &\iff \beta_C(S(C)) \subseteq \beta_C(T(C))\ \forall C \in \mathcal{A}(A)\\ &\iff \beta_C^{-1}(\beta_C(S(C))) \subseteq \beta_C^{-1}(\beta_C(T(C)))\ \forall C \in \mathcal{A}(A)\\ &\iff S(C) \subseteq T(C)\ \forall C \in \mathcal{A}(A)\\ &\iff S \leq T.\end{aligned}$$

This means that Ψ is an order embedding. Lastly, $\widehat{Q(A)}$ is a **dense** sublocale of $\widehat{\mathcal{O}(\Sigma_A)}$, because for the associated nucleus we have

$$j(\emptyset) = \coprod_{C \in \mathcal{A}(A)} \overline{\emptyset_C} = \coprod_{C \in \mathcal{A}(A)} \emptyset = \emptyset.$$

□

5.3 Finite-dimensional case

If A is finite-dimensional, the situation is easier. First of all we need to show that every finite-dimensional C*-algebra is an AW*-algebra. A detailed proof can be found in [11], Proposition 2.4.28. We will give a summary.

First of all, by the Artin-Wedderburn theorem, any finite-dimensional C*-algebra can be written as a direct sum of matrix algebras of the form

$$M_n(\mathbb{C}),$$

where $n \in \mathbb{N}$. Because

$$M_n(\mathbb{C}) \cong B(\mathbb{C}^n),$$

these matrix algebras are all AW*-algebras. Lastly, a direct sum of AW*-algebras is itself an AW*-algebra. Therefore, any finite-dimensional C*-algebra is an AW*-algebra.

In the finite-dimensional case we have $\mathcal{A}(A) = \mathcal{C}(A)$ by [11], Corollary 2.4.29. From this it follows that

$$\widehat{Q(A)} = Q(A),$$

and

$$\widehat{\mathcal{O}(\Sigma_A)} = \mathcal{O}(\Sigma_A).$$

Furthermore, Ψ is also a frame morphism, and $\Psi \circ \Phi$ is the identity on $\mathcal{O}(\Sigma_A)$. This is easily seen by noting that $\Sigma(C)$ has the discrete topology for all $C \in \mathcal{C}(A)$. Therefore, if $\mathcal{U} \in \mathcal{O}(\Sigma_A)$, we have

$$\overline{\mathcal{U}_C} = \mathcal{U}_C.$$

It follows that

$$Q(A) \cong \mathcal{O}(\Sigma_A)$$

as frames.

Chapter 6

Functoriality

In this chapter we will consider the functoriality of $\widehat{Q(A)}$ for AW*-algebras, with $Q(A)$ for A finite-dimensional as a special case, as well as of Σ_A for general C*-algebras. We will have to choose the right domain in order to establish the functoriality of these constructions.

6.1 Functoriality of $Q(\bullet)$

We consider the category of **AWStar-Inj** of AW*-algebras with injective AW*-homomorphisms as morphisms. For an AW*-algebra A we have previously defined $\widehat{Q(A)}$. We will now consider the morphisms. Suppose $\phi : A \to B$ is an injective AW*-homomorphism. By [11], Theorem 8.2.1, we have a map

$$\mathcal{A}(\phi)_* : \mathcal{A}(B) \to \mathcal{A}(A),$$
$$D \mapsto \phi^{-1}[D],$$

which is upper adjoint to the map

$$\mathcal{A}(\phi) : \mathcal{A}(A) \to \mathcal{A}(B),$$
$$C \mapsto \phi[C].$$

That is, if D is a commutative AW*-subalgebra of B, then $\phi^{-1}[D]$ is a commutative AW*-subalgebra of A. Now suppose we have $S \in \widehat{Q(A)}$, $S : \mathcal{A}(A) \to \text{Proj}(A)$. We will use the map $\mathcal{A}(\phi)_*$ to define $\widehat{Q(\phi)}(S) \in \widehat{Q(B)}$ by demanding that the following diagram be commutative:

$$\begin{array}{ccc} \mathcal{A}(A) & \xleftarrow{\mathcal{A}(\phi)_*} & \mathcal{A}(B) \\ \downarrow S & & \downarrow \widehat{Q(\phi)}(S) \\ \text{Proj}(A) & \xrightarrow{\phi|_{\text{Proj}(A)}} & \text{Proj}(B). \end{array}$$

That is, $\widehat{Q(\phi)}(S) = \phi|_{\text{Proj}(A)} \circ S \circ \mathcal{A}(\phi)_*$. First we need the following lemma.

Lemma 6.1. *Let $\phi : A \to B$ be an AW*-homomorphism between commutative AW*-algebras. Then $\phi|_{\text{Proj}(A)} : \text{Proj}(A) \to \text{Proj}(B)$ is a homomorphism of complete Boolean algebras.*

Proof. This follows from Lemma 3.3, since the supremum and infimum of two projections e, f in A are given by polynomial expressions in e and f, which are preserved by ϕ. Furthermore, the complement of e is $1 - e$, which is also preserved by ϕ. Lastly, ϕ preserves arbitrary suprema by definition. □

In our case A, B are not commutative, but ϕ is injective. Then for any $D \in \mathcal{A}(B)$ the map

$$\phi|_{\phi^{-1}[D]} : \phi^{-1}[D] \to D$$

is an AW*-homomorphism between commutative AW*-algebras. This means that

$$\phi|_{\text{Proj}(\phi^{-1}[D])} : \text{Proj}(\phi^{-1}[D]) \to \text{Proj}(D)$$

is a homomorphism of complete Boolean algebras by 6.1 .

Lemma 6.2. *$\widehat{Q(\phi)}$ is well defined.*

Proof. We first have to check that for $S \in \widehat{Q(A)}$ the map $S_\phi := \widehat{Q(\phi)}(S)$ lies in $\widehat{Q(B)}$. First, let $D \in \mathcal{A}(B)$. Then

$$S_\phi(D) = \phi(S(\mathcal{A}(\phi)_*(D))) = \phi(S(\phi^{-1}[D]).$$

Since $S \in \widehat{Q(A)}$, $e := S(\phi^{-1}[D]) \in \phi^{-1}[D]$. Therefore, $\phi(e) \in D$, which implies that

$$S_\phi(D) \in \mathrm{Proj}(D).$$

Second, suppose $E \subseteq D$ in $\mathcal{A}(B)$. Then $\mathcal{A}(\phi)_*(E) \subseteq \mathcal{A}(\phi)_*(D)$ in $\mathcal{A}(A)$. Since $S \in \widehat{Q(A)}$, it follows that $S(\mathcal{A}(\phi)_*(E)) \leq S(\mathcal{A}(\phi)_*(D))$. But $\phi|_{\mathrm{Proj}(\phi^{-1}[D])}$ is order preserving, so

$$S_\phi(E) \leq S_\phi(D).$$

□

Lemma 6.3. *$\widehat{Q(\phi)}$ is a frame homomorphism.*

Proof.

- $\widehat{Q(\phi)}$ is order preserving:
  Suppose $S \leq T$ in $\widehat{Q(A)}$ i.e. $S(C) \leq T(C)\ \forall C \in \mathcal{A}(A)$. Let $D \in \mathcal{A}(B)$. Since $S \leq T$, $S(\mathcal{A}(\phi)_*(D)) \leq T(\mathcal{A}(\phi)_*(D))$ so $S_\phi(D) \leq T_\phi(D)$ since $\phi|_{\mathrm{Proj}(\phi^{-1}[D])}$ is order preserving.
- For $D \in \mathcal{A}(B)$,
  $$0_\phi(D) = \phi(0(\mathcal{A}(\phi)_*(D))) = \phi(0) = 0,$$
  so $\widehat{Q(\phi)}(0) = 0$.
- Let $S, T \in \widehat{Q(A)}$. Then
  $$\begin{aligned} \widehat{Q(\phi)}(S \wedge T)(D) &= \phi(S \wedge T(\phi^{-1}[D])) \\ &= \phi(S(\phi^{-1}[D]) \wedge T(\phi^{-1}[D])) \\ &= \phi(S(\phi^{-1}[D])) \wedge \phi(T(\phi^{-1}[D])) \\ &= Q(\phi)(S) \wedge Q(\phi)(T) \end{aligned}$$
  since $\phi|_{\mathrm{Proj}(\phi^{-1}[D])}$ preserves meets.

A similar argument shows that $\widehat{Q(\phi)}$ preserves arbitrary suprema. Here we use that $\phi|_{\mathrm{Proj}(\phi^{-1}[D])}$ preserves arbitrary suprema. □

Theorem 6.1. *$\widehat{Q(\bullet)}$ is a covariant functor* **AWStar-Inj** → **Frm**

Proof. We have already proven that for A an AW*-algebra, $\widehat{Q(A)}$ is a frame. From the previous lemmas, $\widehat{Q(\phi)} : \widehat{Q(A)} \to \widehat{Q(B)}$ is a frame homomorphism if $\phi : A \to B$ is an injective AW*-homomorphism. It remains to prove that $\widehat{Q(\mathrm{id}_A)} = \mathrm{id}_{\widehat{Q(A)}}$ and $\widehat{Q(\psi \circ \phi)} = \widehat{Q(\psi)} \circ \widehat{Q(\phi)}$. The first is clear, since both $\mathcal{A}(\mathrm{id}_A)_*$ and $\mathrm{id}_A|_{\mathrm{Proj}(A)}$ are the identity on their respective domains. Now consider the following diagrams:

$$A \xrightarrow{\ \phi\ } B \xrightarrow{\ \psi\ } C$$

$$\begin{array}{ccccc} \mathcal{A}(A) & \xleftarrow{\mathcal{A}(\phi)_*} & \mathcal{A}(B) & \xleftarrow{\mathcal{A}(\psi)_*} & \mathcal{A}(C) \\ \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle Q(\phi)(S)} & & \downarrow \\ \mathrm{Proj}(A) & \xrightarrow{\phi|_{\mathrm{Proj}(A)}} & \mathrm{Proj}(B) & \xrightarrow{\psi|_{\mathrm{Proj}(B)}} & \mathrm{Proj}(C) \end{array}$$

We can define the map on the right in two ways. We will prove that they coincide:

$$\begin{aligned} \widehat{Q(\psi)}(\widehat{Q(\phi)}(S)) &= \widehat{Q(\psi)}(\phi|_{\mathrm{Proj}(A)} \circ S \circ \mathcal{A}(\phi)_*) \\ &= \psi|_{\mathrm{Proj}(B)} \circ (\phi|_{\mathrm{Proj}(A)} \circ S \circ \mathcal{A}(\phi)_*) \circ \mathcal{A}(\psi)_* \\ &= (\psi \circ \phi)|_{\mathrm{Proj}(A)} \circ S \circ (\mathcal{A}(\psi \circ \phi)_*) \\ &= \widehat{Q(\psi \circ \phi)}(S). \end{aligned}$$

□

Remark. For this result we only need $\mathcal{A}(\bullet)$ to be a contravariant functor. We chose the domain **AWStar-Inj** to accomplish this, but using the category **CAWStar** of commutative AW*-algebras with **all** AW*-homomorphims also yields a similar result.

If A is finite-dimensional, we have seen that

$$\widehat{Q(A)} = Q(A).$$

Furthermore, any injective *-homomorphism between commutative finite-dimensional C*-algebras is automatically an AW*-homomorphism, because the lattices of projections are finite in that case. Therefore, we can restrict $\widehat{Q(\bullet)}$ to the category **FDC*-Inj** of finite-dimensional C*-algebras and injective *-homomorphisms to obtain a functor

$$Q(\bullet) : \mathbf{FDC^*\text{-}Inj} \to \mathbf{Frm}.$$

6.2 Functoriality of $\Sigma_\bullet$

We recall the definition of Σ_A for a C*-algebra A. It is a topological space with underlying set

$$\Sigma_A = \coprod_{C \in \mathcal{C}(A)} \Sigma(C),$$

where $\Sigma(C)$ is the Gelfand spectrum of C. We will denote elements of Σ_A by (C, λ), where $C \in \mathcal{C}(A)$ and $\lambda \in \Sigma(C)$. We define $\mathcal{U} \subseteq \Sigma_A$ to be open if the following two conditions are satisfied:

1. For each $C \in \mathcal{C}(A)$ the component $\mathcal{U}_C := \mathcal{U} \cap \Sigma(C)$ is open in $\Sigma(C)$.

2. For all inclusions $C \subseteq D$ in $\mathcal{C}(A)$, if $\lambda \in \Sigma(D)$ and $\lambda|_C \in \mathcal{U}_C$, then $\lambda \in \mathcal{U}_D$.

Lemma 6.4. *Σ_A is a topological space.*

Proof.

- $\emptyset$ is open since $\emptyset \cap \Sigma(C) = \emptyset$ which is open. The second condition is void.

- Σ_A is open since $\Sigma_A \cap \Sigma(C) = \Sigma(C)$ which is open. The second condition is also satisfied since $\mathcal{U}_D = \Sigma(D)$.

- Suppose $\mathcal{U}$, $\mathcal{V}$ open. For $C \in \mathcal{C}(A)$, $(\mathcal{U} \cap \mathcal{V})_C = \mathcal{U}_C \cap \mathcal{V}_C$ which is open. Furthermore, if $j : C \hookrightarrow D$, then

$$\Sigma(j)^{-1}[\mathcal{U}_C \cap \mathcal{V}_C] = \Sigma(j)^{-1}[\mathcal{U}_C] \cap \Sigma(j)^{-1}[\mathcal{V}_C] \subseteq \mathcal{U}_D \cap \mathcal{V}_D.$$

A similar argument proves that an arbitrary union of opens is open. □

Let $\phi : A \hookrightarrow B$ be an injective $*$-homomorphism between C*-algebras. We obtain a map

$$\Sigma_\phi : \Sigma_B \to \Sigma_A, (C, \lambda) \mapsto (\phi^{-1}[C], \lambda \circ \phi)$$

Lemma 6.5. *Σ_ϕ is continuous.*

Proof. Let $\mathcal{U} \subseteq \Sigma_A$ be open. Define $\mathcal{V} := \Sigma_\phi^{-1}[\mathcal{U}] = \{(C, \lambda) \in \Sigma_B \mid (\phi^{-1}[C], \lambda \circ \phi) \in \mathcal{U}\}$.

1. $\mathcal{V}_C \subseteq \Sigma(C)$ open for all $C \in \mathcal{C}(A)$: We have

$$\begin{aligned} \mathcal{V}_C &= \{\lambda \in \Sigma(C) \mid (\phi^{-1}[C], \lambda \circ \phi) \in \mathcal{U}\} \\ &= \{\lambda \in \Sigma(C) \mid \lambda \circ \phi \in \mathcal{U}_{\phi^{-1}[C]}\} \\ &= \Sigma(\phi|_{\phi^{-1}[C]})^{-1}[\mathcal{U}_{\phi^{-1}[C]}], \end{aligned}$$

which is open since $\Sigma(\phi|_{\phi^{-1}[C]})$ is continuous.

2. Suppose $C \subseteq D$ in $\mathcal{C}(B)$ with inclusion map $j : C \hookrightarrow D$. Then

$$\begin{aligned}\Sigma(j)^{-1}[\mathcal{V}_C] &= \Sigma(j)^{-1}[\Sigma(\phi|_{\phi^{-1}[C]})^{-1}[\mathcal{U}_{\phi^{-1}[C]}]] \\ &= (\Sigma(\phi|_{\phi^{-1}[C]}) \circ \Sigma(j))^{-1}[\mathcal{U}_{\phi^{-1}[C]}] \\ &= \Sigma(j \circ \phi|_{\phi^{-1}[C]})^{-1}[\mathcal{U}_{\phi^{-1}[C]}] \\ &= \Sigma(\phi|_{\phi^{-1}[D]} \circ \phi_j)^{-1}[\mathcal{U}_{\phi^{-1}[C]}] \\ &= \Sigma(\phi|_{\phi^{-1}[D]})^{-1}[\Sigma(\phi_j)^{-1}[\mathcal{U}_{\phi^{-1}[C]}]] \\ &\subseteq \Sigma(\phi|_{\phi^{-1}[D]})^{-1}[\mathcal{U}_{\phi^{-1}[D]}] = \mathcal{V}_D.\end{aligned}$$

where in the fourth step we have used the commutativity of the diagram

$$\begin{array}{ccc} C & \xhookrightarrow{\;j\;} & D \\ {\scriptstyle \phi|_{\phi^{-1}[C]}}\uparrow & & \uparrow{\scriptstyle \phi|_{\phi^{-1}[D]}} \\ \phi^{-1}[C] & \xhookrightarrow{\;\phi_j\;} & \phi^{-1}[D] \end{array}$$

and ϕ_j is the inclusion map induced by j.

We have proven that $\mathcal{V}$ is open in Σ_B, so Σ_ϕ is continuous. □

Theorem 6.2. $\Sigma_\bullet$ *is a contravariant functor* **CStar-Inj** $\to$ **Top**.

Proof. We have proven that Σ_A is a topological space for a C*-algebra A, and that an injective $*$-homomorphism $\phi : A \hookrightarrow B$ induces a continuous map $\Sigma_\phi : \Sigma_B \to \Sigma_A$. It is clear that $\Sigma_{\mathrm{id}_A} = \mathrm{id}_{\Sigma_A}$. Furthermore, if we have maps $A \overset{\phi}{\hookrightarrow} B \overset{\psi}{\hookrightarrow} C$ they induce maps $\Sigma_C \xrightarrow{\Sigma_\psi} \Sigma_B \xrightarrow{\Sigma_\phi} \Sigma_A$.
Let $(E, \lambda) \in \Sigma_C$. Then

$$\begin{aligned}(\Sigma_\phi \circ \Sigma_\psi)(E, \lambda) &= (\phi^{-1}[\psi^{-1}[E]], (\lambda \circ \psi) \circ \phi) \\ &= ((\psi \circ \phi)^{-1}[E], \lambda \circ (\psi \circ \phi)) = \Sigma_{\psi\circ\phi}(E, \lambda).\end{aligned}$$ □

Remark. Similar to the previous section, we could also use the category **CCStar** of commutative C*-algebras as the domain of $\Sigma_\bullet$ to obtain a functor **CCStar** $\to$ **Top**. Furthermore, if we compose the contravariant functor $\Sigma_\bullet$ with the contravariant functor $\mathcal{O}(\bullet)$: **Top** $\to$ **Frm**, we obtain a covariant functor $\mathcal{O}(\Sigma_\bullet)$.

Chapter 7

Limits

The space Σ_A is constructed by gluing together the Gelfand spectra of the commutative subalgebras of A. It is therefore natural to ask if the frame $Q(A)$ can in some way be 'approximated' using the $C \in \mathcal{C}(A)$. For this we will use the categorical framework of limits (see section A.4). We will recover $Q(A)$ as a limit of the $Q(C)$ for $C \in \mathcal{C}(A)$ in two different ways. In this chapter, we will only consider finite-dimensional C*-algebras.

7.1 $Q(A)$ as an equalizer

First of all, we will try to recover $Q(A)$ as a certain equalizer. We make the observation that we can restrict a map $S : \mathcal{C}(A) \to \mathrm{Proj}(A)$, $S \in Q(A)$ to any $\mathcal{C}(C)$ for $C \in \mathcal{C}(A)$ to obtain a map $S|_{\mathcal{C}(C)} : \mathcal{C}(C) \to \mathrm{Proj}(C)$ which is in $Q(C)$. Furthermore, if we have a collection of maps $S_C \in Q(C)$ for all $C \in \mathcal{C}(A)$, then we can make two maps for every pair C, D of commutative subalgebras. We can restrict the map S_C to the intersection $C \cap D$ or we can take S_D instead. This leads to the following diagram in the category **Frm**:

$$Q(A) \xrightarrow{e} \prod_{C \in \mathcal{C}(A)} Q(C) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{C,D \in \mathcal{C}(A)} Q(C \cap D) \tag{7.1}$$

where the maps are given by

$$S \overset{e}{\longmapsto} (S|_{\mathcal{C}(C)})_{C \in \mathcal{C}(A)}$$

and

$$(S_E)_{E \in \mathcal{C}(A)} \overset{p}{\longmapsto} (S_C|_{\mathcal{C}(C \cap D)})_{C,D \in \mathcal{C}(A)}$$

$$(S_E)_{E \in \mathcal{C}(A)} \overset{q}{\longmapsto} (S_D|_{\mathcal{C}(C \cap D)})_{C,D \in \mathcal{C}(A)}$$

Lemma 7.1. *In the diagram above we have $p \circ e = q \circ e$.*

Proof. Let $S \in Q(A)$. If we first restrict S to C and then to $C \cap D$ we obtain the same map as if we restrict to D and then to $C \cap D$. Both are equal to the restriction of S to $C \cap D$. Therefore $p(e(S)) = q(e(S))$. □

For $Q(A)$ to be the equalizer of the diagram we need that e is the 'universal' map with $p \circ e = q \circ e$. We will prove this in the next proposition.

$$\begin{array}{ccccc}
F & \searrow^{f} & & & \\
\Big\downarrow{\scriptstyle \exists! g} & & & & \\
Q(A) & \xrightarrow{e} & \prod_{C \in \mathcal{C}(A)} Q(C) & \underset{q}{\overset{p}{\rightrightarrows}} & \prod_{C,D \in \mathcal{C}(A)} Q(C \cap D)
\end{array}$$

Proposition 7.1. *$Q(A)$ (together with e) is the equalizer of the diagram 7.1.*

Proof. Suppose we have a frame F and a frame homomorphism $F \xrightarrow{f} \prod_{C\in\mathcal{C}(A)} Q(C)$ such that $p \circ f = q \circ f$. We have to prove that there is a unique frame homomorphism $F \xrightarrow{g} Q(A)$ such that $f = e \circ g$. For all $x \in F$ we have $p(f(x)) = q(f(x))$. Suppose $f(x) = (S_C)_{C\in\mathcal{C}(A)}$. Then $S_C|_{\mathcal{C}(C\cap D)} = S_D|_{\mathcal{C}(C\cap D)}$ for all $C, D \in \mathcal{C}(A)$. We will define $g(x)(C) = S_C(C)$. Then

1. $g(x) \in Q(A)$: $S_C \in \mathcal{C}(C)$, so $S_C(C) \in \text{Proj}(C)$. If $C \subseteq D$ then

$$S_C(C) = S_D|_{\mathcal{C}(C)}(C) = S_D(C) \leq S_D(D).$$

2. g is a frame homomorphsim: let $x, y \in F$, $f(x) = (S_C)_{C\in\mathcal{C}(A)}$ and $f(y) = (T_C)_{C\in\mathcal{C}(A)}$. Then

$$f(x \wedge y) = f(x) \wedge f(y) = (S_C \wedge T_C)_{C\in\mathcal{C}(A)}.$$

   Now,

$$\begin{aligned} g(x \wedge y)(C) &= (S_C \wedge T_C)(C) \\ &= S_C(C) \wedge T_C(C) \\ &= g(x)(C) \wedge g(y)(C) \\ &= (g(x) \wedge g(y))(C), \end{aligned}$$

   so $g(x \wedge y) = g(x) \wedge g(y)$.

   A similar argument works for arbitrary joins, since f is a frame homomorphism.

3. $f = e \circ g$: Let $x \in F$, $f(x) = (S_C)_{C\in\mathcal{C}(A)}$. Then $e(g(x)) = (g(x)|_{\mathcal{C}(C)})_{C\in\mathcal{C}(A)}$. Let $C \in \mathcal{C}(A)$. We need that $g(x)|_{\mathcal{C}(C)} = S_C$. So suppose $D \subseteq C$, then

$$\begin{aligned} g(x)|_{\mathcal{C}(C)}(D) = g(x)(D) &= S_D(D) \\ &= S_D|_{\mathcal{C}(D)} = S_C|_{\mathcal{C}(D)} \\ &= S_C(D). \end{aligned}$$

4. g is unique: Suppose there is g' with $f = e \circ g'$. If $x \in F$ and $f(x) = (S_C)_{C\in\mathcal{C}(A)}$ then S_C is the restriction of $g'(x)$ to $\mathcal{C}(C)$ so $g'(x)(C) = S_C(C) = g(x)$. This shows that $f(x)$ completely determines $g(x)$, i.e. g is unique. □

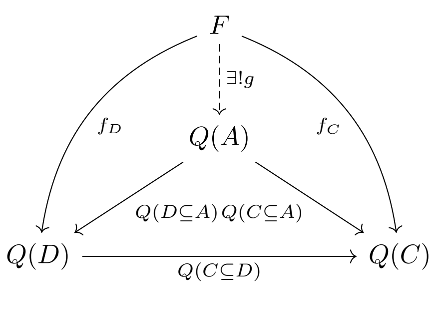
Because $Q(A)$ is the equalizer of the diagram, we know that, as sets,

$$Q(A) \cong \Big\{ (S_C)_{C\in\mathcal{C}(A)} \in \prod_{C\in\mathcal{C}(A)} Q(C) \;\Big|\; S_C|_{\mathcal{C}(C\cap D)} = S_D|_{\mathcal{C}(C\cap D)} \;\forall C, D \in \mathcal{C}(A) \Big\}.$$

In fact, it is sufficient to take the product only over the **maximal** commutative subalgebras of A. In order to prove this we first realize $Q(A)$ as a different limit.

7.2 $Q(A)$ as a direct limit

We can view $Q(\bullet)$ as a contravariant functor $\mathcal{C}(A) \to \mathbf{Frm}$ by $C \mapsto Q(C)$, and if $C \subseteq D$ we define $Q(C \subseteq D)$ to be the restriction to $\mathcal{C}(C)$. We obtain a diagram $\mathbf{J}$ of shape $\mathcal{C}(A)^{\text{op}}$ in the category $\mathbf{Frm}$. Restriction to each $C \in \mathcal{C}(A)$ gives a cone $Q(A) \to \mathbf{J}$.



It is clear that for all $C \subseteq D$ we have $Q(C \subseteq D) \circ Q(D \subseteq A) = Q(C \subseteq A)$ since it doesn't matter if we first restrict to D or not. This means that we indeed have a cone $Q(A) \to \mathbf{J}$. We will prove that this cone is universal.

Proposition 7.2. *$Q(A)$ is the limit of the diagram $Q(\bullet) : \mathcal{C}(A)^{\mathrm{op}} \to \mathbf{Frm}$.*

Proof. Suppose we have another cone $F \xrightarrow{f} \mathbf{J}$, so that for all $C \subseteq D$ we have $Q(C \subseteq D) \circ f_D = f_C$. This means $f_D(x)|_{\mathcal{C}(C)} = f_C(x)$ for all $x \in F$. Define $g(x)(C) = f_C(x)(C)$. Then

1. $g(x) \in Q(A)$: since $f_C(x) \in Q(C)$, $f_C(x)(C) \in \mathrm{Proj}(C)$. Now let $C \subseteq D$, then

$$\begin{aligned} g(x)(C) &= f_C(x)(C) \\ &= f_D(x)|_{\mathcal{C}(C)}(C) \\ &= f_D(x)(C) \leq f_D(x)(D) = g(x)(D), \end{aligned}$$

   because $f_D(x) \in Q(D)$.

2. $Q(D \subseteq A) \circ g = f_D$ for all $D \in \mathcal{C}(A)$: let $D \in \mathcal{C}(A)$. Then $Q(D \subseteq A)(g(x)) = g(x)|_{\mathcal{C}(D)}$ and for $C \in \mathcal{C}(D)$ we have

$$\begin{aligned} g(x)|_{\mathcal{C}(D)}(C) &= f_C(x)(C) \\ &= f_D|_{\mathcal{C}(C)}(C) \\ &= f_D(C), \end{aligned}$$

   and we conclude that $Q(D \subseteq A) \circ g = f_D$.

3. g is a frame homomorphism: let $x, y \in F$. Then

$$\begin{aligned} g(x \wedge y)(C) &= f_C(x \wedge y)(C) \\ &= (f_C(x) \wedge f_C(y))(C) \\ &= f_C(x)(C) \wedge f_C(y)(C) \\ &= g(x)(C) \wedge g(y)(C) \\ &= (g(x) \wedge g(y))(C). \end{aligned}$$

   A similar argument shows that g preserves arbitrary joins, since f_C is a frame homomorphism.

4. g is unique: suppose there is a frame homomorphism $g' : F \to Q(A)$ with $Q(D \subseteq A) \circ g' = f_D$ for all $D \in \mathcal{C}(A)$. Then

$$\begin{aligned} g'(x)(D) &= g'(x)|_{\mathcal{C}(D)}(D) \\ &= Q(D \subseteq A)(g'(x))(D) \\ &= f_D(x)(D) = g(x)(D). \end{aligned}$$

   This shows that g is unique. □

Consider the frame

$$Q_{\max}(A) = \Big\{ (S_M)_{M \in \max\, \mathcal{C}(A)} \in \prod_{M \in \max\, \mathcal{C}(A)} Q(M) \Big| S_M|_{\mathcal{C}(M \cap N)} = S_N|_{\mathcal{C}(M \cap N)}\ \forall M, N \in \max \mathcal{C}(A) \Big\}.$$

We will prove that this frame is still the limit of the diagram, even though we only use the maximal subalgebras.

First, we have to define the cone $Q_{\max}(A) \to \mathbf{J}$. To this end, let $C \in \mathcal{C}(A)$. By [11], Theorem 3.1.3, there is a maximal commutative subalgebra M with $C \subseteq M$. We can now define the map $\phi_C : Q_{\max}(A) \to Q(C)$. If $S = (S_M)_{M \in \max\, \mathcal{C}(A)}$ is in $Q_{\max}(A)$, then $\phi_C(S) = S_M|_{\mathcal{C}(C)}$. Now suppose that $C \subseteq N$ as well, where $N \in \max \mathcal{C}(A)$. Then $C \subseteq M \cap N$, but this means that $S_M|_{\mathcal{C}(C)}$ and $S_N|_{\mathcal{C}(C)}$ are equal, since S_M and S_N agree on $\mathcal{C}(M \cap N)$ by definition.

Lemma 7.2. *The maps $\phi_C : Q_{\max} \to Q(C)$ give a cone $\phi : Q_{\max} \to \mathbf{J}$.*

Proof. Let $C, D \in \mathcal{C}(A)$, where $C \subseteq D$, and $S \in Q_{\max}(A)$. Suppose that $C \subseteq D \subseteq M$ for $M \in \max \mathcal{C}(A)$. Then

$$\begin{aligned} Q(C \subseteq D)(\phi_D(S)) &= Q(C \subseteq D)(S_M|_{\mathcal{C}(D)}) \\ &= (S_M|_{\mathcal{C}(D)})|_{\mathcal{C}(D)} \\ &= S_M|_{\mathcal{C}(D)} = \phi_C(S), \end{aligned}$$

so that $Q(C \subseteq D) \circ \phi_D = \phi_C$. This means that ϕ is a cone. □

Proposition 7.3. *$Q_{\max}(A)$ is the limit of the diagram* **J**.

Proof. Suppose we have another cone $F \xrightarrow{f} \mathbf{J}$ given by $f_C : F \to Q(C)$ for all $C \in \mathcal{C}(A)$. Define $g : F \to Q_{\max}(A)$ by $g(x) = (f_M(x))_{M \in \max \mathcal{C}(A)}$.

1. $g(x) \in Q_{\max}(A)$ for all $x \in F$: suppose $M, N \in \max \mathcal{C}(A)$. Then

$$f_M(x)|_{\mathcal{C}(M \cap N)} = f_{M \cap N}(x) = f_N(x)|_{\mathcal{C}(M \cap N)},$$

   since f is a cone.

2. Let $C \in \mathcal{C}(A)$ and suppose $C \subseteq M$ where $M \in \max \mathcal{C}(A)$. If $x \in F$, then

$$\phi_C(g(x)) = f_M(x)|_{\mathcal{C}(C)} = f_C(x),$$

   which means that $\phi_C \circ g = f_C$.

3. Let $x, y \in F$. Then

$$\begin{aligned} g(x \wedge y) &= (f_M(x \wedge y)) = (f_M(x) \wedge f_M(y)) \\ &= (f_M(x)) \wedge (f_M(y)) = g(x) \wedge g(y). \end{aligned}$$

   A similar argument shows that g preserves arbitrary joins.

4. Suppose there is $g' : F \to Q_{\max}(A)$ with $\phi_C \circ g = f_C$ for all $C \in \mathcal{C}(A)$. Let $x \in F$ and $g'(x) = (S'_M)_{M \in \max \mathcal{C}(A)}$. Then

$$f_M(x) = \phi_M(g'(x)) = S'_M|_{\mathcal{C}(M)} = S'_M$$

   so that g is unique.

□

Theorem 7.1. *There is a frame isomorphism $Q(A) \cong Q_{\max}(A)$.*

Proof. This follows directly from Proposition A.1 and the fact that $Q(A)$ and $Q_{\max}(A)$ are both limits of the diagram **J**. □

Appendices

Appendix A

Category Theory

This appendix on category theory is needed mainly for the categorical aspects that are covered in chapter 6 and 7. We will cover the concepts of categories, functors, natural transformations and limits. This material is based on [17].

A.1 Categories

Definition A.1. A **category C** consists of

1. A collection $\text{Obj}(\mathbf{C})$ of **objects**.
2. For every $A, B \in \text{Obj}(\mathbf{C})$ a collection $\text{Hom}_{\mathbf{C}}(A, B)$ of **morphisms** between A and B.
3. For $A, B, C \in \text{Obj}(\mathbf{C})$ a map
$$\circ : \text{Hom}_{\mathbf{C}}(B, C) \times \text{Hom}_{\mathbf{C}}(A, B) \to \text{Hom}_{\mathbf{C}}(A, C) \tag{A.1}$$
called **composition** (notation: $f \circ g := \circ(f, g)$)

such that:

1. Composition is associative.
2. For every $A \in \text{Obj}(\mathbf{C})$ there is an **identity** morphism $\text{id}_A \in \text{Hom}_{\mathbf{C}}(A, A)$ such that $f \circ \text{id}_A = f$ if $f \in \text{Hom}_{\mathbf{C}}(A, C)$ and $\text{id}_A \circ g = g$ if $g \in \text{Hom}_{\mathbf{C}}(C, A)$.

Remark. We usually write $A \in \mathbf{C}$ instead of $A \in \text{Obj}(\mathbf{C})$. A morphism $f \in \text{Hom}_{\mathbf{C}}(A, B)$ is denoted by $f : A \to B$.

Example.
1. The category **Sets** which has sets as objects and functions as morphisms. Composition is ordinary composition of functions.
2. The category **CStar** with C*-algebras as objects and *-homomorphisms as morphisms.
3. Any poset (or more generally any preorder) can be regarded as a category with an arrow $x \to y$ if and only if $x \leq y$. Transitivity of the order relation is then just composition of these arrows. Such a category is called a **posetal category**.
4. If **C** is a category, we can make a new category $\mathbf{C}^{op}$ called the **opposite category** of **C**. $\mathbf{C}^{op}$ has the same objects as **C**, but a morphism $X \to Y$ in $\mathbf{C}^{op}$ is a morphism $Y \to X$ in **C**. The morphisms are 'reversed' in $\mathbf{C}^{op}$.

Remark. Most categories that we will work with are so-called **concrete** categories. This means that the objects are **sets**, possibly with extra structure, and that all morphisms are functions between these sets. Also we usually work with **locally small** categories in which each collection $\text{Hom}_{\mathbf{C}}(A, B)$ is actually a **set**.

A.2 Functors

Definition A.2. A **(covariant) functor** is a map $F : \mathbf{C} \to \mathbf{D}$ between categories $\mathbf{C}$ and $\mathbf{D}$ that associates to each $C \in \mathrm{Obj}(\mathbf{C})$ an object $F(C) \in \mathrm{Obj}(\mathbf{D})$ and to each morphism $f : A \to B$ in $\mathbf{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathbf{D}$, such that:

1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathbf{C}$.
2. $F(f \circ g) = F(f) \circ F(g)$ for all morphisms f, g for which the composite $f \circ g$ is defined.

Remark. We can also define **contravariant** functors. A contravariant functor assigs to a morphism $f : A \to B$ a morphism $F(f) : F(B) \to F(A)$ so the direction is reversed. In addition, condition no. 2 is replaced by $F(f \circ g) = F(g) \circ F(f)$.

Example. The functor $\mathcal{P}$ that assigns to each set X its powerset $\mathcal{P}(X)$ can be made into a contravariant functor. To a function $f : X \to Y$ we associate the function $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$, $A \mapsto f^{-1}(A)$.

Example. For any category $\mathbf{C}$ and object $X \in \mathbf{C}$ we have the functor $\mathrm{Hom}_{\mathbf{C}}(-, X)$ from $\mathbf{C}$ to **Sets**. To an object $Y \in \mathbf{C}$ it assigns the set of morphisms $\mathrm{Hom}_{\mathbf{C}}(Y, X)$ from Y to X. For any morphism $f : Y \to Z$ we get a morphism $f^* : \mathrm{Hom}_{\mathbf{C}}(Z, X) \to \mathrm{Hom}_{\mathbf{C}}(Y, X)$ given by $g \mapsto g \circ f$.

A.3 Natural transformations

Definition A.3. A **natural transformation** $\alpha : F \to G$ between two covariant functors $F, G : \mathbf{C} \to \mathbf{D}$ consists of morphisms $\alpha_C : F(C) \to G(C)$ for each $C \in \mathbf{C}$ such that for each morphism $f : A \to B$ in $\mathbf{C}$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

commutes. That is, $\alpha_B \circ F(f) = G(f) \circ \alpha_A$.

Example. Let **Ring** be the category of rings. We first consider the functor $\mathrm{GL}_n(-)$ that assigns to a ring R the ring $\mathrm{GL}_n(R)$ of n times n invertible matrices with coefficients in R. By applying a homomorphism $f : R \to S$ to the coefficients this becomes a functor. Next, we can also form the group of units R^* and any homomorphism $f : R \to S$ restricts to $f : R^* \to S^*$. The determinant assigns to an n by n invertible matrix A an element $\det(A) \in R^*$. This is an example of a natural transformation.

Definition A.4. A natural transformation $\alpha : F \to G$ is called a **natural isomorphism** when there is a natural transformation $\beta : G \to F$ such that $\alpha \circ \beta = \mathrm{id}_G$ and $\beta \circ \alpha = \mathrm{id}_F$.

Notation. We write $F \cong G$ if there is a natural isomorphism $F \to G$.

Remark. α is a natural isomorphism if and only if each component α_C is an isomorphism.

Example. In the category **Sets** there is a natural isomorphism

$$\mathcal{P}(X) \cong \mathrm{Hom}_{\mathbf{Sets}}(X, \underline{2}), \tag{A.2}$$

given by associating a subset $A \subseteq X$ with its characteristic function χ_A. This is natural because for any function $f : X \to Y$ and subset $B \subseteq Y$ we have $\chi_{f^{-1}(B)} = \chi_B \circ f$.

Definition A.5. Two categories $\mathbf{C}, \mathbf{D}$ are said to be **equivalent** if there are (covariant) functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F \cong \mathrm{id}_{\mathbf{C}}$ and $F \circ G \cong \mathrm{id}_{\mathbf{D}}$. The functors F and G are then said to give an **equivalence of categories**.

Remark. If the functors F and G are contravariant the categories are said to be **dual** to each other and F and G give a **duality of categories**.

Natural transformations can be thought of as ‘morphisms between functors’. This can be made precise using the concept of a **functor category**.

Definition A.6. Let **C**, **D** be categories. Then the **functor category $\mathbf{D}^{\mathbf{C}}$** of functors from **C** to **D** has

- functors $\mathbf{C} \to \mathbf{D}$ as objects;
- The morphisms $F \to G$ are the natural transformations between F and G.

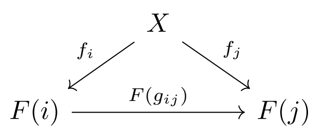
Remark. The identity natural transformation $\mathrm{id} : F \to F$, with components given by the identity $\mathrm{id}_{F(C)} : F(C) \to F(C)$ for $C \in \mathbf{C}$, serves as the identity morphism in this category. Also natural transformations can be composed componentwise, and this is associative.

A.4 Limits

Definition A.7. Let **J** be a category. A **diagram** of shape **J** in a category **C** is a functor $\mathbf{J} \to \mathbf{C}$.

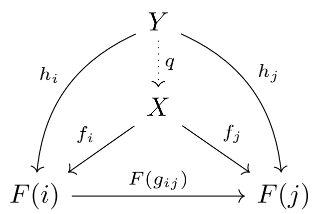
Remark. A diagram can be thought of as a collection of objects and morphisms that indexed by the category **J**.

Definition A.8. A **cone** to a diagram $F : \mathbf{J} \to \mathbf{C}$ consists of an object $X \in \mathbf{C}$ and morphisms $f_i : X \to F(i)$ such that for every morphism $g_{ij} : i \to j$ in **J**, $F(g_{ij}) \circ f_i = f_j$. This means that the following diagram



commutes for every morphism $g_{ij} : i \to j$.

Definition A.9. A **limit** of a diagram $F : \mathbf{J} \to \mathbf{C}$ is a **universal cone** to F. That is, a cone $f_i : X \to F(i)$ to F such that for every cone $h_i : Y \to F(i)$ there is a **unique** morphism $q : Y \to X$ with $h_i = f_i \circ q$ for all $i \in \mathbf{J}$.



Notation. A limit of a diagram $F : \mathbf{J} \to \mathbf{C}$ is denoted by $X = \lim_{\mathbf{J}} F$.

Example.

- A **product** is a limit of a diagram of shape

$$\bullet \qquad \bullet$$

  That is, if $A, B \in \mathbf{C}$ then a product of A and B is an object P together with morphisms $\pi_A : P \to A, \pi_B : P \to B$ satisfying: if $Y \in \mathbf{C}$ and $f_A : Y \to A$, $f_B : Y \to B$ then there is a unique morphism $q : Y \to P$ with $f_A = \pi_A \circ q$ and $f_B = \pi_B \circ q$.

- An **equalizer** is a limit of a diagram of shape

$$\bullet \rightrightarrows \bullet$$

Proposition A.1. *Let $F : \mathbf{J} \to \mathbf{C}$ be a diagram with a limit $X = lim_{\mathbf{J}} F$. Then X is unique up to unique isomorphism.*

Proof. The idea is that if there are two limits X and Y we have unique morphisms $q : X \to Y$, $r : Y \to X$ and these are mutually inverse. The details can be found in [17]. □

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