The logic of quantum mechanics (revisited)[1]

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Dedicated to Miklos Rédei, on the occasion of his 65th birthday

Abstract

This is a personal review of the logic of quantum mechanics, with special emphasis on spatial aspects. My views originated in the topos-theoretical approach to quantum theory pioneered by Isham (initially with Butterfield and subsequently with Döring), in the form developed by Caspers, Heunen, Spitters, Wolters, and the author at Nijmegen, but the conclusion, namely that quantum logic is intuitionistic and is described by a specific Heyting lattice $Q(A)$ one may associate to each (unital) C*-algebra A, is somehow independent of this origin. Indeed, the derivation of this Heyting lattice from topos theory will only be reviewed very briefly at the end of the paper, whose emphasis is rather on the relationship between the dualities associated with the names of Stone, Birkhoff, Priestely, and Esakia (all in lattice theory), and of course Gelfand.

1 Introduction

Any new approach to some topic that has already been studied by serious people in the past comes with the obligation to explain its necessity. Quantum logic is no exception in this regard, especially since it was founded by the greatest mathematician and logician ever to have occupied himself with quantum mechanics, namely von Neumann; see von Neumann (1932) for the fundamental role of projections, and Birkhoff & von Neumann (1936) for the subsequent formalization of quantum logic in terms of specific non-Boolean lattices (initially taken to be modular lattices, later generalized to orthomodular lattices).

A noteworthy aspect of the approach of Birkhoff and von Neumann, which will also be adopted in our own theory, is its *semantic* nature: unlike traditional twentieth-century logic, which starts from syntax and subsequently moves on to semantics (i.e. model theory), they defined their quantum logic directly through its class of models. Indeed, they conceptually based their model of quantum logic on Boole's models for classical propositional logic, in which (in a physical setting) elementary propositions correspond to (measurable) subsets of phase space M (up to sets of measure zero). Birkhoff and von Neumann first recalled that such sets (or equivalence classes thereof) define a Boolean lattice under the obvious partial order $A \leq B$ iff $A \subseteq B$ (which gives rise to the lattice operations $A \vee B = A \cup B$ and $A \wedge B = A \cap B$) and the complementation $A' = A^c = M \backslash A$. In particular, this lattice is distributive and satisfies the law of the excluded middle

$$A \vee A' = \top, \tag{1}$$

where $\top$ (often called 1) is the top element of the lattice (given by M itself).

Using von Neumann's own mathematical formalism for quantum mechanics, in which each physical system is no longer associated with a phase space but with a Hilbert space

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H, and each elementary proposition is interpreted by a closed linear subspace $L \subseteq H$, Birkhoff and von Neumann observed that the set $\mathcal{L}(H)$ of all such L again forms a lattice under the natural partial ordering (i.e. inclusion), which this time gives rise to the lattice operations $L \vee M = \overline{L + M}$ (i.e. the closed linear span of L and M), and $L \wedge M = L \cap M$ (the same as in the classical case). They observed that this lattice is no longer distributive (unless $\dim(H) = 1$), but, with the obvious (ortho)complementation $L' = L^{\perp}$ (i.e. the *orthogonal* complement of L in H), it still satisfies the law of the excluded middle.[2]

All this is easy to generalize if we identify the above lattice $\mathcal{L}(H)$ of all closed subspaces of H with the lattice $\mathcal{P}(B(H))$ of all projections on H (here $B(H)$ is the algebra of all bounded operators on H, of which an element e is a projection iff $e^2 = e^* = e$); if $M \subset B(H)$ is a von Neumann algebra, then its subset of projections $\mathcal{P}(M)$ inherits the lattice structure of $\mathcal{P}(B(H)) \cong \mathcal{L}(H)$, so that each von Neumann algebra (*nomen est omen*!) defines a quantum logic in the spirit of Birkhoff and von Neumann (Rédei, 1998).

However, looking at cases like Schrödinger's Cat–at least in the naive view that it is neither alive nor dead, which view may be wrong for macroscopic objects (Landsman, 2017) but which certainly holds for microscopic ones–and also submitting that distributivity simply cannot be given up if $\wedge$ and $\vee$ are to preserve anything remotely similar to their usual logical meanings "and" and "or", one cannot avoid the impression that despite its novelty and interest, the quantum logic proposed by Birkhoff and von Neumann is:

- *too radical* in giving up distributivity (rendering it problematic to interpret the logical operations $\wedge$ and $\vee$ as conjunction and disjunction, respectively);
- *not radical enough* in keeping the law of excluded middle, which is precisely what an "intuition pump" like Schrödinger's cat challenges.

Thus it would be preferable to have a quantum logic with exactly the *opposite* features, i.e., one that remains distributive but drops the law of the excluded middle. This suggest the use of *intuitionistic logic* for quantum mechanics, and actually finding appropriate models thereof has been the main outcome of the quantum toposophy program so far.[3]

The aim of this paper is to put the intuitionistic quantum logic discovered through the topos approach in the light of the great (categorical) dualities that on the one hand deserve the name "spatial", and on the other hand are somehow related to logic, namely Gelfand duality in (commutative) C*-algebra theory, reconsidered in §2, and the dualites in lattice theory named after Stone, Birkhoff, Priestely, and Esakia, which will be reviewed in §§3, 4. In §5 we show how all of these dualities culminate in our models for intuitionistic quantum logic, which, more or less as an afterthought, are finally derived from topos theory in §6.

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[2] Birkhoff and von Neumann noted that if one works with all linear subspaces of H instead of the closed ones (in which case $L \vee M = L + M$), their lattice satisfies a weakened version of distributivity, in that $L \leq N$ implies $L \vee (M \wedge N) = (L \vee M) \wedge N$ for each M (i.e., if distributivity holds merely if $L \leq N$). This is called the *modular law*; it was later shown that their actual lattice of *closed* subspaces satisfies the modular law at least for $M = L^{\perp}$. Such lattices are called *orthomodular*; orthomodularity is equivalent to the perhaps more appealing condition that the compatibility relation $\overset{c}{\sim}$ on $\mathcal{P}(H)$ is symmetric (i.e. $L \overset{c}{\sim} M$ iff $M \overset{c}{\sim} L$), where we say that $L \overset{c}{\sim} M$ iff $L = (L \wedge M) \vee (L \wedge M^{\perp})$, i.e., the associated projections commute.

[3] Although the initial goals of the topos-theoretic approach to quantum mechanics were quite a bit more ambitious, including quantum gravity and the associated development of an entirely new language for theoretical physics–cf. the founding literature on the subject starting with Isham & Butterfield (1998) and ending with the review by Döring & Isham (2010)–in my view topos theory is best (and more modestly) seen as a tool providing a new approach to quantum logic. See Caspers, Heunen, Landsman, & Spitters (2009), Heunen, Landsman, & Spitters (2009; 2012), Heunen, Landsman, Spitters, & Wolters (2012), Landsman (2017a), Hekkelman (2018), and Rutgers (2018) for our side of the program, and Wolters (2013ab) for a comparison between the 'contravariant' approach of Isham et al and the 'covariant' Nijmegen approach.

2 Gelfand duality revisited

Our approach to Gelfand duality (as well as to all other topics treated in this paper) will be constructive, which not only means that proofs by *reductio ad absurdum*, the law of the excluded middle, and the Axiom of Choice are disabled, but also that the use of points is eschewed; instead, one relies on open sets as much as possible.[4] To this end, recall that a ***frame*** is a complete lattice L that is "infinitely distributive" in that

$$x \wedge \bigvee S = \bigvee \{x \wedge y, y \in S\}, \tag{2}$$

for arbitrary subsets $S \subset L$. Frame homomorphism by definition preserve finite infima and arbitrary suprema. This defines the category Frm of frames, whose opposite category is called the category Loc of *locales*. Thus a locale is the same thing as a frame, seen however as an object in the opposite category.[5] The motivating example of a frame is the topology $\mathcal{O}(X)$ of a space X, partially ordered by set-theoretic inclusion. Not all frames are toplogies, though (see also below), and this fact makes the following notation used in constructive mathematics pretty confusing: *any* frame is denoted by $\mathcal{O}(X)$ and the corresponding locale is called X *whether or nor the given frame is a topology, and despite the fact that even if it is, the locale is actually* $\mathcal{O}(X)$ *rather than the space* X. Oh well!

A simple frame is $2 = \{0,1\} \equiv \{\perp, \top\}$, with order $0 \leq 1$; this is just the topology $\mathcal{O}(1)$ of a singleton 1. A frame map $p^{-1} : \mathcal{O}(X) \to 2$ is the same as a locale map $p : 1 \to X$ and defines a ***point*** of the locale X. We denote the set of points of X by $\mathrm{Pt}(X)$. If $\mathcal{O}(X)$ is the topology of some space X, then each point $x \in X$ corresponds to a map

$$p_x : 1 \to X, \ \ p_x(1) = x; \tag{3}$$

whose inverse image map $p_x^{-1} : \mathcal{O}(X) \to \underline{2}$ is frame map and hence defines a point in the above sense. Conversely, if X is sober (see below), each point of $\mathcal{O}(X)$ arise in that way. The set $\mathrm{Pt}(X)$ has a natural topology, with opens

$$\mathrm{Pt}(U) = \{p \in \mathrm{Pt}(X) \mid p(1) \in U\}, \tag{4}$$

where $U \in \mathcal{O}(X)$; here $p(1) \in U$ really means $p^{-1}(U) = 1$. This gives a frame map

$$\mathcal{O}(X) \to \mathrm{Pt}(X); \tag{5}$$
$$U \mapsto \mathrm{Pt}(U). \tag{6}$$

A frame $\mathcal{O}(X)$ (or the corresponding locale X) is called ***spatial*** if this map is an isomorphism. Spatial frames are topologies, but this does not mean that any topology $\mathcal{O}(X)$ is isomorphic (as a frame) to $\mathcal{O}(\mathrm{Pt}(X))$, since $\mathrm{Pt}(X)$ may not be homeomorphic to X. Spaces X for which this *is* the case are called ***sober***; more precisely, in that case the map

$$X \to \mathrm{Pt}(X); \tag{7}$$
$$x \mapsto p_x, \tag{8}$$

is a homeomorphism. Thus a sober space X may be reconstructed (up to homeomorphism) from its topology $\mathcal{O}(X)$. The category Frm has a full subcategory Spat of spatial frames,

[4] See Johnstone (1983) for motivation, and also Mac Lane & Moerdijk (1992) for some of what follows.
[5] See Johnstone (1982) and Picado & Pultr (2012).

likewise the category Top of topological spaces has a full subcategory Sob of sober spaces, and it is well known (cf. e.g. Mac Lane & Moerdijk, 1992, §IX.3, Corollary 4) that

$$\mathsf{Spat} \simeq \mathsf{Sob}^{\mathrm{op}}, \tag{9}$$

i.e., the categories Spat and Sob are dual (here C^{op} is the opposite category C): if X is a sober space, then $\mathcal{O}(X)$ is a spatial frame, and if $\mathcal{O}(X)$ is a spatial frame, then $\mathrm{Pt}(X)$ is a sober space (with the obvious choices of maps making these associations functorial).

For later use (in Gelfand duality), we mention that a frame $\mathcal{O}(X)$ with top element $\top$ (which exists because $\mathcal{O}(X)$ is a complete lattice, whence $\top = \bigvee \mathcal{O}(X)$) is called ***compact*** if every subset $S \subset \mathcal{O}(X)$ with $\bigvee S = \top$ has a finite subset $F \subset S$ with $\bigvee F = \top$. Furthermore, $\mathcal{O}(X)$ is ***regular*** if each $V \in \mathcal{O}(X)$ satisfies

$$V = \bigvee \{U \in \mathcal{O}(X) \mid U \ll V\}, \tag{10}$$

where $U \ll V$ iff there exists W such that $U \wedge W = \bot$ and $V \vee W = \top$.[6] If some frame $\mathcal{O}(X)$ is a topology, then $\mathcal{O}(X)$ is compact and regular iff X is compact and Hausdorff.

Gelfand duality, at last, states, in its simplest form,[7] that one has a duality

$$\mathsf{CCA}_1 \simeq \mathsf{CH}^{\mathrm{op}}, \tag{11}$$

where CCA_1 is the category of commutative unital C*-algebras and unital homomorphims (by which we mean *-homomorphisms), CH is the category of compact Hausdorff spaces and continuous maps, and $\simeq$ denotes equivalence of categories. The idea of the proof is to map a unital C*-algebra A into its Gelfand spectrum $\Sigma(A)$, which consists of all nonzero multiplicative linear functionals $A \to \mathbb{C}$ (or, equivalently, of all pure states on A), equipped with the topology of pointwise convergence (in which $\Sigma(A)$ is compact and Hausdorff); in the opposite direction, a compact Hausdorff space X is sent to the algebra $C(X)$ of continuous functions $X \to \mathbb{C}$ with pointwise operations and the supremum-norm (in which $C(X)$ is a commutative unital C*-algebra). Functorially, any unital homomorphism $\varphi : A \to B$ induces a pullback $\varphi^* : \Sigma(B) \to \Sigma(A)$, and similarly any continuous map $f : X \to Y$ induces a pullback $f^* : C(Y) \to C(X)$. In particular, eq. (11) implies

$$A \cong C(\Sigma(A)), \quad a \mapsto \hat{a}; \tag{12}$$

$$X \cong \Sigma(C(X)), \quad x \mapsto \mathrm{ev_x}, \tag{13}$$

where $\hat{a} : \Sigma(A) \to \mathbb{C}$ is the ***Gelfand transform*** of a, neatly defined by $\hat{a}(\omega) = \omega(a)$, and $\mathrm{ev_x} : C(X) \to \mathbb{C}$ is the evaluation map at $x \in X$, i.e. $\mathrm{ev_x}(f) = f(x)$.

All (known) proofs of Gelfand duality are non-constructive, typically relying on either Zorn's Lemma (in realizations of $\Sigma(A)$ through maximal ideals, as in Gelfand's original approach) or on the (equivalent) Hahn–Banach Theorem (in the above definition of $\Sigma(A)$). Constructive versions of Gelfand duality therefore change the statement of the theorem.

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[6] Note that $U \ll V$ implies $U \leq V$, since $U = U \wedge (V \vee W) = (U \wedge V) \vee (U \wedge W) = U \wedge V \leq V$.

[7] Less elementary forms of Gelfand duality refer to the non-unital/non-compact case. One version is $\mathsf{CCAn} \simeq \mathsf{LCHp}^{\mathrm{op}}$, where CCAn is the category of commutative C*-algebras with nondegenerate homomorphisms and LCHp is the category of locally compact Hausdorff spaces and proper continuous maps. This easily follows from unitization, i.e. adding a formal unit to a C*-algebra without one, see e.g. Landsman (2017), §C.6. Another, due to An Huef, Raeburn, & Williams (2010), is $\mathsf{CCAm} \simeq \mathsf{LCH}^{\mathrm{op}}$, where CCAm is the category of commutative C*-algebras with nondegenerate homomorphims into the multiplier algebra as arrows and LCH is the category of locally compact Hausdorff spaces and continuous maps. As far as I know, the explicit categorical perspective on Gelfand duality goes back to Negrepontis (1969).

In the most radical approach (Henry, 2014ab) both sides of the duality are changed: instead of C*-algebras one uses so-called *localic* C*-algebras, whilst compact Hausdorff spaces are replaced by compact regular *locales*. It is enough for our purposes to make the second change but not the first; this slightly less radical approach to Gelfand duality goes back to Banaschewski & Mulvey (2006) and was continued by Coquand & Spitters (2009).

Constructive Gelfand duality, then, states that CCA_1 is dual to the category of compact regular locales (i.e., equivalent to the category of compact regular frames). Of course, the point is to define the constructive Gelfand spectrum $\mathcal{O}(\Sigma(A))$ directly from A as a frame (or locale), rather than as the topology of the underlying space $\Sigma(A)$.[8] This may be done as follows.[9] A ***hereditary subalgebra*** of a C*-algebra A is a C^*-subalgebra H of A with the property that $a \leq b$ for $b \in H^+$ and $a \in A^+$ implies $a \in H^+$.[10] The set of all hereditary subalgebras of A is denoted by $H(A)$. Similarly, the set of all closed left (right) ideals in A is called $L(A)$ ($R(A)$), and the closed two-sided ideals are denoted by $I(A)$. It is easy to show that there are bijective correspondences between hereditary subalgebras H of A, closed left ideals L of A, and closed right ideals R of A, given by:

$$L = \{a \in A \mid a^*a \in H^+\}; \tag{14}$$
$$R = \{a \in A \mid aa^* \in H^+\}; \tag{15}$$
$$H = L \cap L^* = R \cap R^*. \tag{16}$$

The set $H(A)$ is a complete lattice in the partial order given by set-theoretic inclusion, with inf and sup of any subset $S \subset H(A)$ given by

$$\bigwedge S = \bigcap S; \tag{17}$$
$$\bigvee S = \bigcap \{I \in H(A) \mid I \supseteq J \text{ for all } J \in S\}. \tag{18}$$

If A is commutative, with Gelfand spectrum $\Sigma(A)$, then $H(A)$ is a frame, and one has

$$\mathcal{O}(\Sigma(A)) \cong H(A), \tag{19}$$

as a frame isomorphism. Moreover, in that case $L^* = L$, $R^* = R$, and $L = R = H$, so

$$H(A) = I(A) = L(A) = R(A). \tag{20}$$

In the usual description, where $\Sigma(A)$ is a space, the map $U \mapsto C_0(U)$ provides an isomorphism (where $U \in \mathcal{O}(\Sigma(A))$, i.e. $U \subset \Sigma(A)$ is open), but constructively it is best to simply *define* the constructive Gelfand spectrum $\mathcal{O}(\Sigma(A))$ as $H(A)$. If this is taken as the starting point (and it will), then the connection with the usual theory is as follows:[11]

[8]Indeed, in most toposes different from the topos of sets (cf. Mac Lane & Moerdijk, 1992) the classical Gelfand spectrum does not even exist.

[9]The following construction of $\mathcal{O}(\Sigma(A))$ is taken from Landsman (2017a), §C.11, inspired by Akemann & Bice (2014). In the references cited in footnote 3 we used a much more complicated construction, adopted from Coquand & Spitters (2009). I did not redo our computation of $\mathcal{O}(\Sigma(A))$ in terms of $H(A)$, but the result should be the same. It should be mentioned, though, that the *proof* of the constructive formulation of Gelfand duality by Coquand & Spitters (2009) is in fact constructive, whereas my proof of (19) is not.

[10]Here A^+ is the positive cone in A, defined for example as $A^+ = \{a^*a \mid a \in A\}$, and for self-adjoint a and b we say that $a \leq b$ iff $b - a \in A^+$ (so that in particular $b \geq 0$ iff $b \in A^+$).

[11]A ***prime element*** $P \in \mathcal{O}(X)$ of some frame $\mathcal{O}(X)$ is an element $P \neq \top$ such that $U \wedge V \leq P$ iff $U \leq P$ or $V \leq P$. For a point $p^{-1} : \mathcal{O}(X) \to \underline{2}$, we write $\ker(p^{-1})$ for $\{U \in \mathcal{O}(X) \mid p^{-1}(U) = 0\}$. For any frame $\mathcal{O}(X)$ (i.e. locale X), there is a bijective correspondence between points $p^{-1} : \mathcal{O}(X) \to \underline{2}$ of X and prime

1. The frame $H(A)$ of hereditary subalgebras of a commutative C*-algebra A is spatial, with $\mathrm{Pt}(H(A)) \cong \Sigma(A)$ as topological spaces.
2. The prime elements of $H(A)$ are the maximal ideals of A, so that, equipping the set $\mathcal{M}(A)$ of maximal ideals of A with the Zariski topology, we have $\mathcal{M}(A) \cong \Sigma(A)$.
3. The Gelfand isomorphism (12) of the classical theory is replaced by

$$A \cong \mathrm{Frm}(\mathcal{O}(\mathbb{C}), H(A)), \tag{21}$$

where we refrain from using the notation $\mathcal{O}(H(A))$ for the frame $H(A)$, as the underlying locale will not occur. In general, $\mathrm{Frm}(\mathcal{O}(Y), \mathcal{O}(X)) = \mathrm{Hom}_{\mathsf{Frm}}(\mathcal{O}(Y), \mathcal{O}(X))$ denotes the set of frame maps $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$, often written as $\mathrm{Loc}(X, Y)$ or $\mathrm{Hom}_{\mathsf{Loc}}(X, Y)$ or (confusingly) even as $C(X, Y)$, since in the spatial case these are precisely the continuous maps $f : X \to Y$; with inverse image maps f^{-1} as above. Because of this, eqs. (19) and (21) recover the classical Gelfand isomorphism (12).

For example, if A is finite-dimensional (and still commutative), so that $A \cong \mathbb{C}^n$, we have

$$\mathcal{P}(A) \xrightarrow{\cong} H(A); \tag{22}$$

$$e \mapsto eA = \{a \in A \mid ea = a\}. \tag{23}$$

Indeed, if $A = \mathbb{C}^n$, so that $a \in A$ is an n-tuple $(a_0, \ldots, a_{n-1})$ with $a_k \in \mathbb{C}$, then each projection $e = (e_0, \ldots, e_{n-1}) \in \mathcal{P}(\mathbb{C}^n)$ is an n-tuple whose only entries are 0 and 1; the pertinent isomorphism $\mathcal{P}(\mathbb{C}^n) \to H(\mathbb{C}^n)$ then maps e to the ideal $e \cdot \mathbb{C}^n$ consisting of all $(a_0, \ldots, a_{n-1}) \in \mathbb{C}^n$ such that $a_k = 0$ if $e_k = 0$ $(k = 0, \ldots, n-1)$. Equivalently,

$$\mathcal{P}(\mathbb{C}^n) \cong P(n); \tag{24}$$

$$H(\mathbb{C}^n) \cong P(n), \tag{25}$$

where the natural number n is seen (à la von Neumann) as the set $\{0, 1, \ldots, n-1\}$, and $P(n)$ is its power set (partially ordered, as always, by inclusion). The (frame) isomorphism (24) comes from the bijection $P(n) \to \mathcal{P}(\mathbb{C}^n)$ that maps $s \in P(n)$ to the projection e with $e_k = 1$ iff $k \in s$ (and hence $e_k = 0$ iff $k \notin s$), whilst (25) is the bijection $P(n) \to H(\mathbb{C}^n)$ that maps $s \subset n$ to the ideal $I_s = \{a \in \mathbb{C}^n \mid a_k = 0 \,\forall\, k \notin s\}$. Similarly,

$$\mathcal{P}(A) \xrightarrow{\cong} \mathcal{O}(\Sigma(A)) = P(\Sigma(A)); \tag{26}$$

$$e \mapsto \{\varphi \in \Sigma(A) \mid \varphi(e) = 1\}. \tag{27}$$

It is enough to prove this for the special case $A = \mathbb{C}^n$, where $\Sigma(\mathbb{C}^n) \cong n$ under the bijection $n \to \Sigma(\mathbb{C}^n)$ given by $k \mapsto \varphi_k$, where $\varphi_k(a) = a_k$ $(k \in n)$, and hence $\mathcal{O}(\Sigma(\mathbb{C}^n)) \cong P(n)$, i.e., the discrete topology on n. For one thing, together with (24) this reproduces (26) - (27). Furthermore, we obtain the classical Gelfand isomorphism $A \to C(n)$ as $a \mapsto \hat{a}$ with $\hat{a}(k) = a_k$, as well as the constructive Gelfand isomorphism (21) as $a \mapsto \tilde{a}$, with

$$\tilde{a} : \mathcal{O}(\mathbb{C}) \to P(n); \tag{28}$$

$$U \mapsto \{k \in n \mid a_k \in U\}. \tag{29}$$

elements $P \in \mathcal{O}(X)$, given by $P = \bigvee \ker(p^{-1})$ and $p^{-1}(U) = 0$ iff $U \leq P$. Under this correspondence, the topology on $\mathrm{Pt}(X)$ is given by the ***Zariski topology***, whose *closed* sets F_P consist of all $Q \supseteq P$, where P is some prime element of $\mathcal{O}(X)$. The prime elements of $H(A)$, where A is a commutative C*-algebra, are the ***prime ideals*** in A, i.e., the proper ideals $J \subset A$ such that $J_1 J_2 \subset J$ iff $J_1 \subseteq A$ or $J_2 \subseteq A$, for any ideals J_1, J_2 of A (closed by definition, like J); note that $J_1 J_2 = J_1 \cap J_2$. the topology on $\mathrm{Pt}(X)$ is given by the ***Zariski topology***, whose *closed* sets F_P consist of all $Q \supseteq P$, where P is some prime element of $\mathcal{O}(X)$. A proof of the three claims in the main text may be found in Landsman (2017a), Theorem C.86.

3 Stone duality and its relatives

We now turn to Stone duality, once again starting with its classical (i.e. spatial) form. A space X is called ***totally disconnected*** if it has no other connected subspaces than its points (so any larger subspace $\neq X$ is the union of two proper clopen sets). A ***Stone space*** is a totally disconnected compact Hausdorff space, and we have a full subcategory St of CH whose objects are Stone spaces. At the other side of the duality we have the category BL of Boolean lattices (i.e. distributive orthocomplemented lattices) with homomorphisms of orthocomplemented lattices as arrows.[12] Like the power set $P(X)$ of any set, the poset $\text{Clopen}(X)$ of all clopen subsets of some Stone space X, (partially ordered by set-theoretic inclusion, so that suprema are unions and infima are intersections), is a Boolean lattice. Conversely, a Boolean lattice L is isomorphic to $\text{Clopen}(X)$ for some Stone space $X = \mathcal{S}(L)$, called the ***Stone spectrum*** of L, which is uniquely determined by L up to homeomorphism (in a manner reviewed below). This gives ***Stone duality***:

$$\mathsf{BL} \simeq \mathsf{St}^{\text{op}}. \tag{30}$$

The Stone spectrum $\mathcal{S}(L)$ of a Boolean lattice L has a canonical realization resembling the set of points of a frame just discussed. This time, we regard $2 = \{0, 1\} = P(1)$ as a Boolean lattice, and define $\text{Pt}(L)$ as the set of all homomorphisms $\varphi : L \to 2$, with topology generated by the basic opens U_x defined in (32) below, where $x \in L$, and their set-theoretic complements U_x^c. Then $\text{Pt}(L)$ is a Stone space, and

$$L \xrightarrow{\cong} \text{Clopen}(\text{Pt}(L)); \tag{31}$$

$$x \mapsto U_x = \{\varphi \in \text{Pt}(L) \mid \varphi(x) = 1\}, \tag{32}$$

is an isomorphism of Boolean lattices. The (contravariant) functorial nature of the Stone spectrum comes out particularly clearly from the above description: given a homomorphism $h : L \to L'$, one immediately obtains a map $h^* : \text{Pt}(L') \to \text{Pt}(L)$ by pullback. Conversely, a continuous map $f : X \to Y$ induces the inverse image map $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$, as above, which restricts to $f^{-1} : \text{Clopen}(Y) \to \text{Clopen}(X)$. The duality (30) then implies

$$L \cong \text{Clopen}(\mathcal{S}(L)); \tag{33}$$

$$X \cong \mathcal{S}(\text{Clopen}(X)). \tag{34}$$

A key example of this construction comes from classical propositional logic. Let $S = \{p_1, p_2, \ldots\}$ be an alphabet of atomic propositions, with associated set $\text{wff}(S)$ of well-formed formulae over S according to the rules of classical propositional logic.[13] Because of the recursive definition of $\text{wff}(S)$, any map $v : S \to 2$ (where $2 = \{0, 1\}$) has a unique extension $v : \text{wff}(S) \to 2$ (called a ***valuation***) subject to the rules (with abuse of notation):

$$v(\perp) = 0; \tag{35}$$

$$v(\neg\alpha) = \neg v(\alpha); \tag{36}$$

$$v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta); \tag{37}$$

$$v(\alpha \vee \beta) = v(\alpha) \vee (\beta); \tag{38}$$

$$v(\alpha \to \beta) = v(\alpha) \to (\beta), \tag{39}$$

[12] An ***orthocomplementation*** on a lattice L with 0 and 1 is a map $\perp: L \to L$, $x \to x^\perp$, that satisfies $x^{\perp\perp} = x$, $x \leq y$ iff $y^\perp \leq x^\perp$, $x \wedge x^\perp = 0$, and $x \vee x^\perp = 1$. A lattice (poset) with an orthocomplementation is called ***orthocomplemented***. A ***homomorphism*** of orthocomplemented lattices is an lattice morphism that also preserves the orthocomplementation, as well as 0 or 1.

[13] See e.g. Givant & Halmos (2009) or van Dalen (2013).

where the expressions on the right-hand side are determined by the usual truth tables.

Let $\mathcal{T}$ be some theory, i.e. a subset of wff(S), with associated Lindenbaum algebra

$$L(S, \mathcal{T}) = \mathrm{wff}(S)/\sim_{\mathcal{T}}, \tag{40}$$

where, for any $\psi, \varphi \in \mathrm{wff}(S)$, we say that $\psi \sim_{\mathcal{T}} \varphi$ if $\mathcal{T} \vdash \psi \leftrightarrow \varphi$, i.e., $\psi \leftrightarrow \varphi$ (which abbreviates $(\psi \to \varphi) \wedge (\varphi \to \psi)$) is provable from $\mathcal{T}$. Unlike wff(S), the set $L(S, \mathcal{T})$ is a Boolean lattice in the partial order defined by $[\psi] \leq [\varphi]$ whenever $\mathcal{T} \vdash \psi \to \varphi$, and the orthocomplementation defined by $[\psi]' = [\neg\psi]$; suprema and infima are given (with some abuse of notation) by $[\psi] \vee [\varphi] = [\psi \vee \varphi]$ and $[\psi] \wedge [\varphi] = [\psi \wedge \varphi]$, respectively.

Define $\mathrm{Mod}_2(S, \mathcal{T})$ as the set of binary models of $\mathcal{T}$, i.e., the set of all valuations $v : \mathrm{wff}(S) \to 2$ that satisfy $v(\alpha) = 1$ for each axiom $\alpha \in \mathcal{T}$. Then any $v \in \mathrm{Mod}_2(S, \mathcal{T})$ descends to a homomorphism $v' : L(S, \mathcal{T}) \to 2$ of Boolean lattices, and *vice versa*, each such homomorphism v' comes from a unique binary model $v \in \mathrm{Mod}_2(S, \mathcal{T}))$. Hence the Stone spectrum of the Boolean lattice $L = L(S, \mathcal{T})$ (realized as explained earlier) is just

$$\mathcal{S}(L(S, \mathcal{T})) = \mathrm{Mod}_2(S, \mathcal{T})), \tag{41}$$

topologized as explained above (31), and the isomorphism (31) - (32) is neatly given by

$$L(S, \mathcal{T}) \xrightarrow{\cong} \mathrm{Clopen}(\mathrm{Mod}_2(S, \mathcal{T})); \tag{42}$$

$$[\psi] \mapsto \{V \in \mathrm{Mod}_2(S, \mathcal{T})) \mid V(\psi) = 1\}. \tag{43}$$

Thus Stone duality maps the logical equivalence class $[\psi]$ of some wff $\psi \in \mathrm{wff}(S)$ with respect to the given theory $\mathcal{T}$ (*syntax*) to the set of all models of $\mathcal{T}$ in which ψ is true (*semantics*). Alas, this equaivalence cannot be achieved in intuitionistic quantum logic.

For a pointfree or constructive description of Stone duality, we note that the topology $\mathcal{O}(\mathcal{S}(L))$ of the Stone spectrum $\mathcal{S}(L)$ of a Boolean lattice L may be given directly as

$$\mathcal{O}(\mathcal{S}(L)) \cong \mathrm{Idl}(L), \tag{44}$$

where Idl(L) is the set of all ideals in L, partially ordered by inclusion.[14] Indeed, for any lattice L, the poset Idl(L) is a frame, whose points are its prime elements, and it is well known that the set $\mathcal{U}(L)$ of ultrafilters of L (which in a Boolean lattice coincide with the prime filters of L), topologized by declaring the sets

$$U'_x = \{F \in \mathcal{U}(L) \mid x \in F\} \quad (x \in L), \tag{45}$$

to be a basis of the topology, is a model of the Stone spectrum of L, too: for any $\varphi \in \mathrm{Pt}(L)$, the set $\varphi^{-1}(\{1\})$ is an ultrafilter in L (and $\varphi^{-1}(\{0\})$ is a maximal ideal).[15] Unfortunately, the constructive Stone spectrum (44) of a Boolean lattice is less useful than the constructive Gelfand spectrum $H(A)$ of a commutative C*-algebra given by (19), since the Gelfand

[14]An ***ideal*** in a lattice L is a subset $I \subseteq L$ such that $x, y \in I$ implies $x \vee y \in I$, and $y \leq x \in I$ implies $y \in I$. An ideal I is ***proper*** if $I \neq L$. A ***maximal ideal*** is an ideal that is maximal in the set of all proper ideals, ordered by inclusion. In a Boolean lattice, maximal ideals coincide with ***prime ideals***, which are ideals I that do not contain the top element 1, and where $x \wedge y \in I$ implies $x \in I$ or $y \in I$. ***Filters*** in L are defined dually, i.e. as nonempty subsets $F \subset L$ such that $x, y \in F$ implies $x \wedge y \in F$, and $y \geq x \in F$ implies $y \in F$. The (set-theoretic) complement of a maximal ideal is a maximal filter (i.e. an ultrafilter), so that an ideal I in a Boolean lattice is maximal (i.e. prime) iff for any $x \in L$ either $x \in I$ or $x' \in I$.

[15]This was even Stone's original description of his spectrum of a Boolean lattice!

isomorphism (21) actually involves $H(A)$, whereas the Stone isomorphism (33) uses the Boolean lattice $\mathrm{Clopen}(\mathcal{S}(L))$ rather than the (non-Boolean) frame $\mathcal{O}(\mathcal{S}(L))$.

Comparing (11) and (30) and noting that by definition St is a full subcategory of CH, there must be a relationship between Gelfand duality and Stone duality, which is subtle:

1. Commutative C*-algebras are not the same things as Boolean algebras; this difference will be overcome by looking at projections.[16] The set of all projections in a C*-algebra A is denoted by $\mathcal{P}(A)$, and if A is commutative and has a unit 1_A, then $\mathcal{P}(A)$ is a Boolean lattice in the partial order $e \leq f$ iff $ef = e$, with orthocomplementation $e' = 1_A - e$; infima are simply given by $e \wedge f = ef$, and suprema are most easily stated through De Morgan's Law, i.e. $e \vee f = (e' \wedge f')'$. Without any further assumptions on A (i.e. beyond commutativity and unitality), we then have

$$\mathcal{P}(A) \cong \mathrm{Clopen}(\Sigma(A)). \tag{46}$$

2. One needs conditions on a commutative unital C*-algebra A that make its Gelfand spectrum $\Sigma(A)$ a Stone space. This turns out to be the case iff A has *real rank zero*, written $\mathrm{rr}(A) = 0$,[17] or, equivalently (given that A is commutative), iff A is an approximately finite-dimensional or AF C*-algebra.[18] We then have:[19]

$$\begin{aligned} \Sigma(A) &\cong \mathcal{S}(\mathcal{P}(A)); &(47)\\ H(A) &\cong \mathrm{Idl}(\mathcal{P}(A)); &(48)\\ A &\cong C(\mathcal{S}(\mathcal{P}(A))), &(49) \end{aligned}$$

as topological spaces, frames, and C*-algebras, respectively. Conversely, for any Boolean lattice L the C*-algebra $C(\mathcal{S}(L))$ is AF (and has real rank zero), and

$$L \cong \mathcal{P}(C(\mathcal{S}(L))). \tag{50}$$

The case $A = \mathbb{C}^n$ remains instructive: eq. (46) reproduces our earlier isomorphisms $\mathcal{P}(\mathbb{C}^n) \cong P(n)$ and $\Sigma(\mathbb{C}^n) \cong n$ with discrete topology, so that $\mathrm{Clopen}(n) = P(n)$. The Stone spectrum $\mathcal{S}(P(n))$ consists of all homomorphisms $\varphi : P(n) \to 2$ of Boolean lattices; since each subset $s \subset n$ is the supremum of its elements, φ is determined by its values on each $\{k\} \subset n$, where $k \in n$, and if $\varphi(k) = 1$, then the condition that φ be a homomorphism enforces $\varphi(l) = 1 - \varphi(n\backslash\{l\}) = 1 - 1 = 0$ for each $l \neq k$, since $k \in n\backslash\{l\}$ and hence $\varphi(n\backslash\{l\}) = 1$). Therefore, $\mathcal{S}(P(n)) \cong n$ under the map $n \mapsto \mathcal{S}(P(n))$ defined by $k \mapsto \varphi_k$ with $\varphi_k(s) = 1$ iff $k \in s$. This gives (47). Since $P(n)$ is finite, we have $I = \downarrow(\bigvee I)$ for any $I \in \mathrm{Idl}(P(n))$, and hence $\mathrm{Idl}(P(n)) \cong P(n)$ under the bijection $P(n) \to \mathrm{Idl}(P(n))$ given by $s \mapsto \downarrow s = \{t \in P(n) \mid t \subset s\}$, which gives (48). Eq. (49) is the classical Gelfand isomorphism $\mathbb{C}^n \cong C(n)$ discussed in §2, and finally, for $L = P(n)$ the isomorphism (50) follows by unfolding: $\mathcal{S}(L) \cong n$, $C(n) \cong \mathbb{C}^n$, and $\mathcal{P}(\mathbb{C}^n) \cong P(n)$ as above.

[16] Just like the case $A = B(H)$, an element $e \in A$ of any C*-algebra A is called a projection if $e^2 = e^* = e$. This implies that projections are positive and in particular self-adjoint elements, and the natural partial order on projections, i.e. $e \leq f$ iff $ef = e$, is a special case of the order defined in footnote 10.

[17] This is the case iff the invertible self-adjoint elements of A are dense in all self-adjoint elements of A.

[18] This means that A is the norm-closure of the union of some (not necessarily countable) directed set of finite-dimensional C*-subalgebras (which in turn are necessarily direct sums of matrix algebras).

[19] Cf. Landsman (2017a), Theorem C.168 for further details and a proof.

4 Priestley duality and Esakia duality

Eqs. (46) - (49) relate classical *propositional* logic to *commutative* C*-algebras, at least in so far as the semantic side of the former is concerned.[20] Towards *intuitionistic quantum* logic, we need to move to *intuitionistic* logic as well as to *non-commutative* C*-algebras. In support of the first move, we first review the well-known concept of a Heyting lattice (or Heyting algebra), which plays the role of a Boolean lattice in intuitionistic propositional logic. A ***Heyting lattice*** is a lattice L with top $\top$ and bottom $\bot$, equipped with a (necessarily unique) map $\to: L \times L \to L$, called ***(material) implication***, that satisfies

$$a \leq (b \to c) \text{ iff } (a \wedge b) \leq c. \tag{51}$$

A Heyting algebra is automatically distributive. Negation (which in a Boolean lattice is orthocomplementation and belongs to the primary structure) is *derived* from $\to$ by

$$\neg a \equiv (a \to \bot); \tag{52}$$

in classical logic this is a tautology (which may be used to eliminate negation), but in intuitionstic logic it is a definition. A Heyting algebra is ***complete*** when it is complete as a lattice, in that arbitrary suprema (and hence also arbitrary infima) exist. In that case, condition (2) is satisfied, so that a complete Heyting algebra is a frame. Conversely, a frame becomes a complete Heyting algebra if we define the implication arrow $\to$ by

$$b \to c = \bigvee \{a \in L \mid a \wedge b \leq c\}. \tag{53}$$

However, frames and complete Heyting algebras drift apart as soon as morphisms are concerned, for although in both cases one requires maps to preserve the partial order, maps between Heyting algebras must preserve $\to$ rather than $\bigvee$ (which in the case of incomplete Heyting lattices would not even be defined). This defines a category HL of (not necessarily complete) Heyting lattices, for which we would like to find a natural dual category of spaces, analogous to St, with an ensuing generalization of Stone duality.[21]

Let DL be the category of bounded distributive lattices (possessing $\bot \equiv 0$ and $\top \equiv 1$) with bounded lattice homomorphisms (i.e. maps preserving 0, 1, $\vee$ and $\wedge$) as arrows. For each bounded distributive lattice L, the associated poset $\mathcal{I}_p(L)$ of prime ideals in L is both a topological space and a poset. The topology is generated by all sets

$$U_a = \{I \in \mathcal{I}_p(L) \mid a \notin I\} \quad (a \in L), \tag{54}$$

and their set-theoretic complements U_x^c, and this makes $\mathcal{I}_p(L)$ a Stone space. The partial order on $\mathcal{I}_p(L)$ is simply given by set-theoretic inclusion.[22] The topology and the order satisfy a compatibility condition called the ***Priestley separation axiom***:[23]

[20] Commutative von Neumann algebras are a special case of commutative AF-algebras, with the special property that their spectrum is *Stonean*, which adds a measure-theoretic property to the Stone condition. So from a logical perspective von Neumann algebras do not form a particularly natural class of C*-algebras.

[21] The following result originates with Esakia (1985), which is in Russian (which I could not read). I am indebted to Nick Bezhanishvili for this reference, and also for Morandi (2005), from which I learnt it. More general duality results for distributive lattices go back to Birkhoff (Jr.) and Priestley, cf. Davey & Priestley (2002) and the special issue of *Studia Logica* (no. 56, vols. 1–2, 1996) dedicated to such results.

[22] The same analysis may be carried out using prime filters: the set-theoretic complement of a prime ideal in L is a prime filter. The appropriate partial order then of course changes direction.

[23] A ***down-set*** in any poset $(P, \leq)$ is a subset $D \subset P$ such that $x \leq y \in P$ implies $x \in P$. So an ideal in a lattice is a down-set that is closed under finite suprema (a filter is an up-set closed under finite infima).

If $b \nleq a$, there is a clopen down-set $U \subset \mathcal{I}_p(L)$ such that $a \in U$ and $b \notin U$.

A partially ordered Stone space satisfying the Priestley separation axiom is called a ***Priestley space***. Such spaces form a category Pr with continuous order-preserving maps as arrows. This category has been invented to yield ***Priestley duality***, stating that

$$\mathsf{DL} \simeq \mathsf{Pr}^{\mathrm{op}}; \tag{55}$$

- a bounded distributive lattice L yields a Priestley space $\mathcal{P}r(L) = \mathcal{I}_p(L)$;
- a Priestley space X gives rise to the poset $\mathrm{Clopen}_{\downarrow}(X)$ of *clopen down-sets* of X (ordered by set-theoretic inclusion), which form a bounded distributive lattice.

Functorially, a bounded lattice homomorphism $\varphi : L \to M$ gives rise to a continuous order morphism $\varphi^{-1} : \mathcal{I}_p(M) \to \mathcal{I}_p(L)$ (i.e. the inverse image map), and a continuous order morphism $f : X \to Y$ similarly induces a pull-back $f^{-1} : \mathrm{Clopen}_{\downarrow}(Y) \to \mathrm{Clopen}_{\downarrow}(X)$. In particular, for any bounded distributive lattice L and any Priestley space X we have

$$L \cong \mathrm{Clopen}_{\downarrow}(\mathcal{I}_p(L)), \quad a \mapsto U_a; \tag{56}$$

$$X \cong \mathcal{I}_p(\mathrm{Clopen}_{\downarrow}(X)), \quad x \mapsto \{U \in \mathrm{Clopen}_{\downarrow}(X) \mid x \notin U\}. \tag{57}$$

Two special cases may clarify this result. We call $0 \neq a \in L$ ***join-irreducible*** if $a = b \vee c$ implies $a = b$ or $a = c$ (equivalently, $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$). Let $\mathcal{J}(L)$ be the set of join-irreducible elements in L, which is a poset in the partial order inherited from L.

- If L is finite, we have an order isomorphism (Davey & Priestley, 2002, Lemma 10.8)

  $$\mathcal{J}(L) \xrightarrow{\cong} \mathcal{I}_p(L); \tag{58}$$

  $$a \mapsto L \backslash \uparrow a, \tag{59}$$

  which maps each down-set $\downarrow a \subset \mathcal{J}(L)$, where $a \in \mathcal{J}(L)$, into the (clopen) subset $U_a \subset \mathcal{I}_p(L)$. Consequently, Priestley duality reduces to ***Birkhoff duality*** between finite distributive lattices L and finite posets P, according to which we have

  $$L \cong \mathrm{Down}(\mathcal{J}(L)), \quad a \mapsto (\downarrow a) \cap \mathcal{J}(L); \tag{60}$$

  $$P \cong \mathcal{J}(\mathrm{Down}(P)), \quad p \mapsto \downarrow p, \tag{61}$$

  where $\mathrm{Down}(P))$ is the lattice of all down-sets in P, partially ordered by set-theoretic inclusion. In this case the *topology* on $\mathcal{I}_p(L)$ is trivial (discrete) and plays no role.

- If L is Boolean we recover Stone duality. Thus for Boolean lattices the *partial order* on $\mathcal{I}_p(L)$ is trivial and drops out. If L is Boolean and finite, $\mathcal{J}(L)$ coincides with the set $\mathcal{A}(L)$ of atoms in L, and Birkhoff duality reduces to $L \cong P(\mathcal{A}(L))$.

Our goal lies in Heyting lattices L. An ***Esakia space*** is a Priestley space X such that:

For any open set $U \in \mathcal{O}(X)$, the corresponding down-set $\downarrow U$ (defined as the smallest down-set containing U, i.e. the intersection of all down-sets containing U) is open, too.[24]

[24] This may equivalently be stated in terms of clopen sets. Esakia spaces are alternatively called ***Heyting spaces***, much as Stones spaces are someties called *Boolean spaces*.

The appropriate arrows $f : X \to Y$ between Esakia spaces X, Y are called ***p-morphisms***, which not only preserve order and topology (i.e. are continuous), but in addition satisfy:

If $y \geq f(x)$, there is $x' \geq x$ such that $f(x') = y$ (for all $x \in X$, $y \in Y$).

If E is the category of Esakia spaces with p-morphisms, the desired duality is given by

$$\mathsf{HL} \simeq \mathsf{E}^{\mathrm{op}}, \tag{62}$$

where the pertinent functors are the restrictions of those just stated for Priestley duality. In particular, the Esakia spectrum $\mathcal{E}(L) = \mathcal{I}_p(L)$ of a Heyting lattice L is the same as the associated Priestley spectrum $\mathcal{P}r(L)$ of L (merely seen as a bounded distributive lattice).[25] Unfortunately, towards the applications to logic we are after, Esakia duality has a drawback compared to Stone duality, in that there seems to be no neat intuitionistic analogue of (42) - (43). Indeed, the Lindenbaum algebra $L(S, \mathcal{T})$ of an intuitionistic propositional theory remains perfectly well defined and duly yields a Heyting algebra, but the realization of its Esakia spectrum $\mathcal{E}(L(S, \mathcal{T}))$ in terms of binary models of $\mathcal{T}$ is meaningless in intuitionistic logic (since the law of the excluded middle, which intuitionistic logic denies, is automatically valid in binary models). Furthermore, Gödel (1932) proved that there cannot be a single *finite* Heyting lattice replacing the Boolean lattice 2 in providing a complete semantics of intuitionistic propositional logic, and Bezhanishvili *et al* (2010) extended this no-go result to arbitrary Heyting lattices. One therefore needs some family of finite Heyting lattices to obtain a complete semantics of intuitionistic propositional logic, such as the well-known ***Kripke models***.[26] For any (finite) poset P, the set $\mathrm{Up}(P)$ of all up-sets U of P (i.e. $y \geq x \in U$ implies $y \in U$), is a complete Heyting algebra in the partial order defined by inclusion, with $\vee = \cup$, $\wedge = \cap$, and implication

$$U \to V = \{x \in P \mid (\uparrow x) \cap U \subseteq V\}. \tag{63}$$

This is actually a special case of (53), since the up-sets form a topology on any poset P, called the ***Alexandrov topology*** (i.e. $\mathrm{Up}(P) = \mathcal{O}(P)$ in this topology), which has the principal up-sets $\uparrow x = \{y \in P \mid y \geq x\}$ as a basis ($x \in P$). In intuitionistic mathematics, elements $x \in P$ are typically interpreted as states of knowledge or information, so that $x \leq y$ means that y carries more knowledge than x (perhaps because y is 'later' than x).

As in the classical case, one has a set $\mathrm{wff}(S)$ of well-formed formulae over some alphabet S, and once again it follows from the recursive definition of $\mathrm{wff}(S)$ that any map $v : S \to \mathrm{Up}(P)$ uniquely extends to a valuation $v : \mathrm{wff}(S) \to \mathrm{Up}(P)$ subject to the rules (35) - (39), where this time the expressions on the right-hand sides are defined in the Heyting lattice $\mathrm{Up}(P)$ (with $\neg U \equiv U \to \bot$), rather than in the Boolean lattice 2. We say that $\varphi \in \mathrm{wff}(S)$ is ***valid*** with respect to v if $v(\varphi) = 1$ (i.e. the top element $1 = P$ of $\mathrm{Up}(P)$). For any $x \in P$ and $\varphi \in \mathrm{wff}(S)$ we write $x \Vdash \varphi$ iff $x \in v(\varphi)$, and say that x ***forces*** φ.

[25]This means that the constructive description (44) of the Priestley or Esakia spectrum remains valid, but it seems less useful here since it carries no information about the order (which is defined on the points).

[26]See e.g. van Dalen (2013) and Dummett (2000). Also, Palmgren (2009) is a useful summary.

Then obviously $v(\varphi) = 1$ iff $x \Vdash \varphi$ for all $x \in P$, and we have the ***forcing rules***:[27]

$$x \Vdash \varphi \quad \text{and } y \geq x \text{ imply } y \Vdash \varphi; \tag{64}$$
$$x \Vdash \bot \quad \text{for no } x \in P; \tag{65}$$
$$x \Vdash \varphi \wedge \psi \quad \text{iff } x \Vdash \varphi \text{ and } x \Vdash \psi; \tag{66}$$
$$x \Vdash \varphi \vee \psi \quad \text{iff } x \Vdash \varphi \text{ or } x \Vdash \psi; \tag{67}$$
$$x \Vdash \varphi \rightarrow \psi \quad \text{iff for all } y \geq x,\ y \Vdash \varphi \text{ implies } y \Vdash \psi; \tag{68}$$
$$x \Vdash \neg\varphi \quad \text{iff for all } y \geq x,\ y \Vdash \varphi \text{ is false.} \tag{69}$$

For any theory $\mathcal{T} \subset \mathrm{wff}(S)$, the associated Lindenbaum agebra $L(S, \mathcal{T})$ differs from its classical counterpart, since intuitionistic logic has fewer derivation rules than classical logic (in particular, intuitionistic propositional logic lacks the *reductio ad absurdum* (RAA) rule). This difference makes $L(S, \mathcal{T})$ merely a Heyting lattice (rather than a Boolean one), and, similarly to the classical case, any valuation $v : \mathrm{wff}(S) \to \mathrm{Up}(P)$ that satisfies $v(\alpha) = 1$ for each $\alpha \in \mathcal{T}$ (i.e. any model of $\mathcal{T}$ in $\mathrm{Up}(P)$, aptly called a *Kripke model*) descends to a Heyting lattice homomorphism $v' : L(S, \mathcal{T}) \to \mathrm{Up}(P)$. Conversely, any such homomorphism comes from a valuation. What seems missing here is a realization of the Esakia spectrum of $L(S, \mathcal{T})$ in terms of Kripke models, but we will come close in the next section, at least for intuitionistic 'quantum' logics defined by C*-algebras.

5 Intuitionistic quantum logic

We now move straight to intuitionistic quantum logic, explaining its origin in topos theory in the next section. The idea is to associate a Heyting lattice $Q(A)$ to any unital C*-algebra A, in contrast with the Birkhoff–von Neumann idea of associating the (orthomodular) projection lattice $\mathcal{P}(A)$ to A (which, unlike our procedure, only makes sense if A has sufficiently many projections, for example if it is a von Neumann algebra). An important role will be played by the poset $\mathcal{C}(A)$ of all unital commutative C*-subalgebras of A, ordered by set-theoretic inclusion,[28] so we will say a few things about this poset first.[29] The poset $\mathcal{C}(A)$ has a bottom element, namely $\bot = \mathbb{C} \cdot 1_A$, but no top element unless A is commutative, in which case $\top = A$. Similarly, $\mathcal{C}(A)$ has arbitrary infima (i.e. meets), given by intersection, but it only has suprema (i.e. joins) of families of elements that mutually commute. Indeed, it is easy to show that $\mathcal{C}(A)$ is a complete lattice iff A is commutative. In that case, using the Gelfand isomorphism, $\mathcal{C}(A)$ has a purely topological description, as follows.[30] Let $A = C(X)$. Any $C \in \mathcal{C}(A)$ induces an equivalence relation $\sim_C$ on X by

$$x \sim_C y \text{ iff } f(x) = f(y)\ \forall f \in C. \tag{70}$$

This, in turn, defines a partition $X = \bigsqcup_\lambda K_\lambda$ of X (henceforth called π), whose blocks $K_\lambda \subset X$ are the equivalence classes of $\sim_C$. This partition is ***upper semicontinuous***:

[27] These rules easily follow from the construction of a valuation $v : \mathrm{wff}(S) \to \mathrm{Up}(P)$. Originally (i.e. in the work of Kripke and his followers), eqs. (64) - (68), which imply (69), were taken to be *axioms* extending a binary "forcing" relation $x \Vdash p$ on $P \times \mathcal{T}$ to $P \times I_{\mathcal{T}}$.

[28] One may think of this poset as a mathematical home for Bohr's notion of ***complementarity***, in that each $C \in \mathcal{C}(A)$ represents some classical or experimental context, which has been decoupled from the others, *except for the inclusion relations, which relate* compatible *experiments* (in general there seem to be no preferred *pairs* of complementary subalgebras $C, C' \in \mathcal{C}(A)$ that jointly generate A, although Bohr typically seems to have had such pairs in mind, e.g. position and momentum). See Landsman (2017b).

[29] See also Heunen (2014), Lindenhovius (2016), and Landsman (2017a).

[30] This description follows from Firby (1973), cf. Lindenhovius (2016), Ch. 4 or Landsman (2017a), §9.1.

- Each block K_λ of the partition π is closed;
- For each block K_λ of π, if $K_\lambda \subseteq U$ for some open $U \in \mathcal{O}(X)$, then there is $V \in \mathcal{O}(X)$ such that $K_\lambda \subseteq V \subseteq U$ and V is a union of blocks of π (in other words, if K is such a block, then $V \cap K = \emptyset$ implies $K = \emptyset$).

Conversely, any upper semicontinuous partition π of X defines some $C \in \mathcal{C}(C(X))$ by

$$C = \bigcap_{K_\lambda \in \pi} \dot{I}_{K_\lambda}, \tag{71}$$

where $I_K = \{f \in C(X) \mid f(x) = 0 \,\forall\, x \in K\}$ and $\dot{I}_{K_\lambda}$ is its unitization. Therefore, the poset $\mathcal{C}(C(X))$ is anti-isomorphic to the poset $\mathrm{USC}(X)$ of all upper semicontinuous decompositions of X in the ordering $\leq$ in which $\pi \leq \pi'$ if π is finer than π', and both posets are actually lattices. If X is finite and hence $A \cong \mathbb{C}^n$, then $\mathcal{C}(\mathbb{C}^n)$ is anti-isomorphic to the partition lattice Π_n of n, such that a partition $n = \bigsqcup_\lambda s_\lambda$ (i.e. $\pi = \{s_\lambda\}$, $s_\lambda \subset n$) corresponds to the set of all $(a_1, \ldots, a_n)$ in $\mathbb{C}^n$ for which $a_i = a_j$ whenever $i, j \in s_\lambda$.[31]

We return to the general case (in which A may be non-commutative). Although the poset $\mathcal{C}(A)$ is not itself our intuitionistic quantum logic, we may nonetheless compare it with the projection lattice $\mathcal{P}(A)$ of traditional quantum logic. For $A = B(H)$, the C*-algebra of all bounded operators on some (not necessarily finite-dimensional) Hilbert space H, the projection lattice $\mathcal{P}(A)$ is already a powerful invariant of A in the following sense: if $\dim(H) > 2$, any order isomorphism $\mathsf{N} : \mathcal{P}(B(H)) \to \mathcal{P}(B(H))$ preserving orthocomplementation (i.e. $\mathsf{N}(1_H - e) = 1_H - \mathsf{N}(e)$ for each $e \in \mathcal{P}(B(H))$, where 1_H is the unit operator on H) takes the form $\mathsf{N}(e) = ueu^*$, where the operator u is either unitary or anti-unitary, and is uniquely determined by N up to a phase. This is a corollary of Wigner's Theorem in quantum mechanics, cf. Landsman (2017a), Theorem 5.4. Similarly, any order isomorphism $\mathsf{B} : \mathcal{C}(B(H)) \to \mathcal{C}(B(H))$ takes the form $\mathsf{B}(C) = uCu^*$ (etc.).

More generally, if A and B are unital C*-algebras, we define a ***weak Jordan isomorphism*** of A and B as an invertible map $\mathsf{J} : A \to B$ whose restriction to each $C \in \mathcal{C}(A)$, is a unital homomorphism (of commutative C*-algebras) onto its image, and which also satisfies $\mathsf{J}(a + ib) = \mathsf{J}(a) + i\mathsf{J}(b)$ for all self-adjoint $a, b \in A$. ***Hamhalter's Theorem*** then states that any order isomorphism $\mathsf{B} : \mathcal{C}(A) \to \mathcal{C}(B)$ is implemented by a weak Jordan isomorphism $\mathsf{J} : A \to B$ (whose restrictions to all $C \in \mathcal{C}(A)$ define a map $\mathcal{C}(A) \to \mathcal{C}(B)$). If A is isomorphic to neither $\mathbb{C}^2$ nor $M_2(\mathbb{C})$, then J is uniquely determined by B.[32] The proof of this theorem also gives an explicit reconstruction of A from $\mathcal{C}(A)$, though, of course, as a Jordan algebra rather than as a C*-algebra.[33] In order to recover A as a C*-algebra one needs to endow the poset $\mathcal{C}(A)$ with additional structure, see e.g. Heunen & Reyes (2014) for AW*-algebras and Döring (2014) for von Neumann algebras.[34]

We now define our Heyting lattices. If A is finite-dimensional,[35] then $Q(A)$ is given by

$$Q(A) = \{S : \mathcal{C}(A) \to \mathcal{P}(A) \mid S(C) \in \mathcal{P}(C),\, S(C) \leq S(D) \text{ if } C \subseteq D\}. \tag{72}$$

[31] The ordering on Π_n has $\pi' \leq \pi$ iff π' is *finer* than π, i.e. iff any $s' \in \pi'$ is contained is some $s \in \pi$.

[32] See Hamhalter (2011), Lindenhovius (2016), §4.7, or Landsman (2017a), Theorem 9.4.

[33] Connes (1975) produced a C*-algebra that is not anti-isomorphic to itself. See also Phillips (2001).

[34] A similar problem arises of one wants to reconstruct A from its pure state space (seen as a set with a transition probability), cf. Landsman (1998), or from its state space (seen as a compact convex set), see Alfsen & Shultz (2003).

[35] As shown by Hekkelman (2018) and Rutgers (2018), eq. (72) also makes sense for AW*-algebras.

As stated before, the partial order on $\mathcal{C}(A)$ is here given by set-theoretic inclusion and the one on $\mathcal{P}(A)$ is $e \leq f$ iff $ef = e$. The partial order $\leqslant$ on $Q(A)$ is defined by $S \leqslant T$ iff $S(C) \leq T(C)$ for all $C \in \mathcal{C}(A)$, and in this order $Q(A)$ is a Heyting lattice, with operations

$$(S \wedge T)(C) = S(C) \wedge T(C); \tag{73}$$

$$(S \vee T)(C) = S(C) \vee T(C); \tag{74}$$

$$(S \to T)(C) = \bigwedge_{D \supseteq C}^{\mathcal{P}(C)} S(D)^{\perp} \vee T(D), \tag{75}$$

where the right-hand side of (75) is shorthand for

$$\bigwedge_{D \supseteq C}^{\mathcal{P}(C)} S(D)^{\perp} \vee T(D) \equiv \bigvee \{e \in \mathcal{P}(C) \mid e \leq S(D)^{\perp} \vee T(D) \, \forall \, D \supseteq C\}. \tag{76}$$

In contrast to traditional quantum logic, both logical connectives $\wedge$ and $\vee$ on $Q(A)$ are physically meaningful, as they only involve *local* conjunctions $S(C) \wedge T(C)$ and disjunctions $S(C) \vee T(C)$, for which $S(C) \in \mathcal{P}(C)$ and $T(C) \in \mathcal{P}(C)$ commute. With similar notation in (77) - (78) below, the derived operations $\neg$ and $\neg\neg$ are then given by

$$(\neg S)(C) = \bigwedge_{D \supseteq C}^{\mathcal{P}(C)} S(D)^{\perp}; \tag{77}$$

$$(\neg\neg S)(C) = \bigwedge_{D \supseteq C}^{\mathcal{P}(C)} \bigvee_{E \supseteq D}^{\mathcal{P}(D)} S(E). \tag{78}$$

A Heyting algebra is Boolean iff $\neg\neg S = S$ for each S, and one sees from (78) that (at least if $n > 1$) the property $\neg\neg S = S$ only holds iff S is either $\top$ or $\perp$, so that already for $A = M_2(\mathbb{C})$ the Heyting algebra $Q(A)$ is non-Boolean and hence properly intuitionistic.

In the Birkhoff–Neumann approach each projection $e \in \mathcal{P}(A)$ defines an elementary proposition, whereas in ours (where the 'classical context' C is crucial) an elementary proposition is a *pair* (C, e), where $e \in \mathcal{P}(C)$; this is supposed to incorporate Bohr's insight that every proposition in quantum theory ought to be accompanied by the (experimental) context in which it is measured. If for each such pair (C, e) we define

$$S_{(C,e)} : \mathcal{C}(A) \to \mathcal{P}(A); \tag{79}$$

$$D \mapsto e \quad (C \subseteq D); \tag{80}$$

$$D \mapsto \perp \quad \text{otherwise}, \tag{81}$$

we see that each pair (C, e) injectively defines an element of $Q(A)$. As pointed out by Hermens (2016), each element $S \in Q(A)$ is a disjunction over such elementary propositions:

$$S = \bigvee_{C \in \mathcal{C}(A)} S_{(C, S(e))}. \tag{82}$$

In the finite-dimensional commutative case $A \cong \mathbb{C}^n$ it is straightforward to compute $Q(A)$, since we already know that $\mathcal{C}(A) \cong \Pi_n$ (i.e. the partition lattice of n, partially ordered by the *opposite* of the usual refinement order), and $\mathcal{P}(A) \cong P(n)$. Hence

$$Q(\mathbb{C}^n) \cong \tilde{Q}(\mathbb{C}^n) = \{\tilde{S} : \Pi_n \to P(n) \mid \pi \leq \{\tilde{S}(\pi), n \backslash \{\tilde{S}(\pi)\}, \tilde{S}(\pi) \subseteq \tilde{S}(\pi') \text{ if } \pi' \leq \pi\}. \tag{83}$$

Here the first condition after the bar means that, for any $\pi \in \Pi_n$, any cell $s \in \pi$ must be contained in either $\tilde{S}(\pi) \subset n$ or in its complement, and the second condition simply states that $\tilde{S}$ is (opposite) order-preserving. Let us initially ignore the first condition, however, and compute the poset $\tilde{Q}'(\mathbb{C}^n) \subset \tilde{Q}(\mathbb{C}^n)$ defined by

$$\tilde{Q}'(\mathbb{C}^n) = \{\tilde{S} : \Pi_n \to P(n) \mid \tilde{S}(\pi) \subseteq \tilde{S}(\pi') \text{ if } \pi' \leq \pi\}. \tag{84}$$

For any poset $(X, \leq)$, the Hom-set of homomorphisms of posets from X to $P(n)$ is

$$\text{Hom}(X, P(n)) \cong \text{Hom}(X, 2^n) \cong (\text{Hom}(X, 2))^n \cong (\text{Up}(X))^n, \tag{85}$$

and so $\tilde{Q}'(\mathbb{C}^n)$ is isomorphic (as poset and even as a Heyting lattice) to

$$\tilde{Q}'(\mathbb{C}^n) \cong (\text{Down}(\Pi_n))^n, \tag{86}$$

where we have $\text{Down}(\Pi_n)$ instead of $\text{Up}(\Pi_n)$ in view of the opposite order. Therefore,[36]

$$Q(\mathbb{C}^n) \cong \{(U_1, \ldots, U_n) \in (\text{Down}(\Pi_n))^n \mid \forall_{\pi \in \Pi_n} \forall_{s \in \pi}((\forall_{k \in s} \pi \in U_k) \vee (\forall_{k \in s} \pi \notin U_k))\}. \tag{87}$$

Here, like in (85), the partial order is given by inclusion, and the condition after the bar may equivalently be stated as $(k \sim_\pi l) \to (\pi \in U_k \leftrightarrow \pi \in U_l)$, where $k \sim_\pi l$ means that k and l lie in the same cell s of the partition π. See also the description (98) - (99) below.

It is an open question what the Esakia spectrum $\mathcal{E}(Q(A))$ of the Heyting algebra $Q(A)$ is. The closest approximation to the classical case would be to replace the value set $\{0, 1\}$, seen as the discrete topology of a singleton, by the Alexandrov topology of the poset $P = \mathcal{C}(A)$, and hence to replace (41) by the set $\text{Mod}_{\mathcal{C}(A)}(Q(A))$ of all Heyting lattice homomorphisms from $Q(A)$ to $\text{Up}(\mathcal{C}(A))$. Indeed,[37] any state ω on A defines a function

$$V_\omega : Q(A) \to \text{Up}(\mathcal{C}(A)); \tag{88}$$

$$V_\omega(S) = \{C \in \mathcal{C}(A) \mid \omega(S(C)) = 1\}. \tag{89}$$

If we say that $S \in Q(A)$ is ***true*** in a state ω provided $V_\omega(S) = \mathcal{C}(A)$ (i.e. the top element of the frame $\text{Up}(\mathcal{C}(A))$), and call S ***false*** if $V_\omega(S) = \emptyset$ (i.e. the bottom element of $\text{Up}(\mathcal{C}(A))$, then $\neg S$ is true iff S is false, and $S \vee T$ is true iff either S or T is true.[38] Consequently, (89) simply lists the contexts C in which $S(C)$ is true, and we have $C \Vdash S$ iff $\omega(S(C)) = 1$.

The problem, however, is that the Kochen–Specker Theorem implies that for reasonably non-commutative A (and $A = M_n(\mathbb{C})$ for $n > 1$ is already a case in point) the set of Heyting lattice homomorphisms from $Q(A)$ to $\text{Up}(\mathcal{C}(A))$ is empty.[39] The ensuing disappointment is only limited, since, as already pointed out in the text following (62), the poset $\mathcal{C}(A)$ would not be able to do the job on its own in any case. Nonetheless, it would be desirable to map propositions in $Q(A)$ to the (clopen down-) sets of ($\mathcal{C}(A)$-valued Kripke) models (replacing binary models) in which they are true, as in the classical case.

[36] I am indebted to Nick Bezhanishvili, Guram Bezhanishvili, David Gabelaia, and Mamuka Jibladze for help with the computation (87)), in response to an erroneous conjecture in an earlier draft of this paper.

[37] Note that $V_\omega(S)$ indeed defines an up-set in $\mathcal{C}(A)$, for if $C \subseteq D$ then $S(C) \leq S(D)$, so that $\omega(S(C)) \leq \omega(S(D))$ by positivity of states, and hence $\omega(S(D)) = 1$ whenever $\omega(S(C)) = 1$ (given that $\omega(S(D)) \leq 1$, which is true since $0 \leq \omega(e) \leq 1$ for any projection e).

[38] Since $V_\omega(S) = \mathcal{C}(A)$ iff $S(\mathbb{C} \cdot 1) = 1$, which forces $S(C) = 1$ for all C.

[39] See e.g. Landsman (2017), §12.5, based on Heunen, Landsman, and Spitters (2009, 2012). I am very grateful to my students Evert-Jan Hekkelman and Quinten Rutgers for reminding me of our own result; their B.Sc. theses Hekkelman (2018) and Rutgers (2018) contain many interesting results in this direction.

This analysis can be generalized to any unital C*-algebra A.[40] First, we define the set

$$\Sigma_A = \bigsqcup_{C\in\mathcal{C}(A)} \Sigma(C), \tag{90}$$

i.e. the disjoint union over all Gelfand spectra $\Sigma(C)$, where $C \in \mathcal{C}(A)$. We then equip Σ_A with the weakest topology making the canonical projection

$$\pi : \Sigma_A \to \mathcal{C}(A); \tag{91}$$
$$\pi(\sigma) = C, \tag{92}$$

where $\sigma \in \Sigma(C) \subset \Sigma_A$, continuous with respect to the Alexandrov topology on $\mathcal{C}(A)$. To be more specific, note that any $\mathcal{U} \subset \Sigma_A$ takes the form

$$\mathcal{U} = \bigsqcup_{C\in\mathcal{C}(A)} \mathcal{U}_C; \tag{93}$$
$$\mathcal{U}_C = \mathcal{U} \cap \Sigma(C). \tag{94}$$

Then $\mathcal{U}$ is open iff the following two conditions are satisfied for each $C \in \mathcal{C}(A)$:

1. $\mathcal{U}_C \in \mathcal{O}(\Sigma(C))$.
2. For all $D \supseteq C$, if $\lambda \in \mathcal{U}_C$ and $\lambda' \in \Sigma(D)$ such that $\lambda'_{|C} = \lambda$, then $\lambda' \in \mathcal{U}_D$.

Being a frame, the topology $\mathcal{O}(\Sigma_A)$ is a Heyting lattice, which generalizes our earlier Heyting lattice $Q(A)$ in (72) to arbitrary unital C*-algebras A. To see that $Q(A)$ is indeed a special case of $\mathcal{O}(\Sigma_A)$, where A is taken to be finite-dimensional (e.g. $A = M_n(\mathbb{C})$), we use (26) - (27), where A is now replaced by $C \in \mathcal{C}(A)$, so that we have an isomorphism (of Boolean lattices) $\beta : \mathcal{P}(C) \to \mathcal{O}(\Sigma(C))$. We then obtain a Heyting lattice isomorphism

$$\mathcal{O}(\Sigma_A) \xrightarrow{\cong} Q(A); \tag{95}$$
$$\mathcal{U} \mapsto S_{\mathcal{U}}; \tag{96}$$
$$S_{\mathcal{U}}(C) = \beta^{-1}(\mathcal{U}_C). \tag{97}$$

Conversely, each $S \in Q(A)$ defines $\mathcal{U} \in \mathcal{O}(\Sigma(C))$ by (93) with $\mathcal{U}_C = \beta(S(C))$.

In our running example $A \cong \mathbb{C}^n$ this leads to the description

$$\Sigma_{\mathbb{C}^n} \cong \bigsqcup_{\pi\in\Pi_n} \pi, \tag{98}$$

whose elements we denote by pairs (π, s) with $s \in \pi$ (and hence $s \subset n$). The topology $\mathcal{O}(\Sigma_{\mathbb{C}^n})$ is then given by all subsets $\mathcal{U} \subset \Sigma_{\mathbb{C}^n}$ such that if $(\pi, s) \in \mathcal{U}$, then $(\pi', s') \in \mathcal{U}$ whenever $\pi' \leq \pi$ (i.e. π' is finer than π) and $s' \subset s$ (where $\subset$ is the same as $\subseteq$). The previous description (87) is then recovered through

$$U_k = \{\pi \in \Pi_n \mid \exists_{s\subset n}(((\pi, s) \in \mathcal{U}) \wedge (k \in s))\}. \tag{99}$$

The topological condition on $\mathcal{U}$ then precisely gives rise to the condition on the U_k in (87).

[40] The discovery that $Q(A)$ is a topology, which is far from obvious even in de special case (72), is due to Wolters (2013ab). See also Heunen, Landsman, Spitters, & Wolters (2012) and Landsman (2017a), §12.4.

Also in the general case we may write $\mathcal{U}$ as a disjunction à la (82), viz.

$$\mathcal{U} = \bigvee_{C \in \mathcal{C}(A)} \mathcal{U}_C, \tag{100}$$

which is even almost trivial, since $\vee = \cup$ in the frame $\mathcal{O}(\Sigma_A)$. The elementary propositions $\mathcal{U}_C \subset \Sigma(C)$ (which in the finite-dimensional case may be identified with projections in C and hence in A, as we have seen) are open subsets of the 'classical phase spaces' $\Sigma(C)$, which, in the spirit of Bohr, carry the contextual label C. Consequently, the intuitionistic quantum logic of (unital) C*-algebras would be largely understood at the level of propositional logic, except for the possible functoriality of the map $A \mapsto \mathcal{O}(\Sigma_A)$, i.e. of $A \mapsto Q(A)$ for finite-dimensional C*-algebra A.[41] As soon as this is solved, we would have a new duality between *arbitrary* (unital) C*-algebras and a particular class of Heyting lattices, which is meant to replace Gelfand duality for *commutative* C*-algebras.

What remains is an extension to first-order logic, which we suggest to be the internal logic of the sheaf topos $\mathsf{Sh}(\Sigma_A)$. To understand this suggestion, we now briefly review the topos-theoretic background of the above construction of the Heyting lattice $Q(A)$, which framework (as pointed out before) we now regard as a means rather than as an end.

6 Epilogue: from topos theory to quantum logic

Let A be a unital C*-algebra, with associated poset $\mathcal{C}(A)$ of all unital commutative C*-subalgebras of A, as before. Regarding $\mathcal{C}(A)$ as a (posetal) category, in which there is a unique arrow $C \to D$ iff $C \subseteq D$ and there are no other arrows, we obtain the topos $\mathsf{T}(A)$ of *covariant* functors $\underline{F} : \mathcal{C}(A) \to \mathsf{Sets}$ from $\mathcal{C}(A)$ into the category Sets of sets, i.e.,[42]

$$\mathsf{T}(A) = [\mathcal{C}(A), \mathsf{Sets}]. \tag{101}$$

Since for any poset X we have an equivalence (even an isomorphism) of categories $[X, \mathsf{Sets}] \simeq \mathsf{Sh}(X)$, where X is endowed with the Alexandrov topology,[43] we may alternatively write

$$\mathsf{T}(A) \simeq \mathsf{Sh}(\mathcal{C}(A)). \tag{102}$$

This category is a ***topos***,[44] which makes it a "universe of discourse" in which to do mathematics, replacing set theory.[45] One major difference with set theory is that the logic in most toposes (including $\mathsf{T}(A)$) the logic is intuitionistic.[46] Nonetheless, (Dedekind) real numbers and C*-algebras can be defined in toposes (with a natural numbers object), and one even has various constructive versions of Gelfand duality.[47] All we need is the fact that each *commutative* C*-algebra $\underline{A}$ in a topos T has a constructive Gelfand spectrum $\underline{\Sigma}(\underline{A})$,

[41]This problem is highly non-trivial, cf. van den Berg & Heunen (2012) and Döring (2012).

[42]One usually works with presheaves on a given category C, i.e. *contravariant* functors $\mathsf{C} \to \mathsf{Sets}$. Thus $\mathsf{T}(A)$ consists of presheaves on $\mathcal{C}(A)^{\mathrm{op}}$, in which the order on $\mathcal{C}(A)$ is reversed.

[43]This isomorphism maps a functor $\underline{F} : X \to \mathsf{Sets}$ to a sheaf $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathsf{Sets}$, by defining the latter on a basis of the Alexandrov topology as $F(\uparrow x) = \underline{F}(x)$ extended to general Alexandrov opens by the sheaf property. *Vice versa*, a sheaf F on X defines $\underline{F}$ by reading the previous equation from right to left.

[44]A topos is a cartesian closed category (i.e., having a terminal object, binary products, and function spaces) with pullbacks and a subobject classifier. See Mac Lane & Moerdijk (1992) or Johnstone (2002).

[45]Although in our approach set theory remains the metamathematics in which topos theory is studied.

[46]This implies in particular that all definitions and proofs have to be constructive, in that *reduction ad absurdum*, the law of the excluded middle, and the Axiom of Choice are not enabled.

[47]See Banaschewski & Mulvey (2006), Coquand & Spitters (2009), and Henry (2014ab).

defined as a locale in T, and an associated Gelfand isomorphism à la (21), also within T, where $\underline{\Sigma}(\underline{A})$ may be either defined or computed as $\underline{H}(\underline{A})$, the lattice of hereditary C*-subalgebras of $\underline{A}$. Here we underline objects in T, especially the internal C*-algebra $\underline{A}$, in order to distinguish (underlined) constructions internal to T from constructions in Sets, like the given C*-algebra A, on which our reasoning in T ultimately relies. The entire argument hinges on the following C*-algebra $\underline{A}$ in our topos $\mathsf{T}(A)$:[48]

$$\underline{A} : \mathcal{C}(A) \to \mathsf{Sets}; \tag{103}$$
$$C \mapsto C; \tag{104}$$
$$(C \subseteq D) \mapsto (C \hookrightarrow D), \tag{105}$$

where, despite the identical notation, on the left-hand side of (104) C is an element of $\mathcal{C}(A)$, whereas on the right-hand side it is the corresponding C*-subalgebra of A seen as a set, and in (105) the notation $C \subseteq D$ denotes the unique arrow in $\mathsf{T}(A)$ from C to D, which the functor $\underline{A}$ maps to the inclusion map $\hookrightarrow$ from C into D in the category of sets.

The point is that $\underline{A}$ is a *commutative* C*-algebra in $\mathsf{T}(A)$ under pointwise operations, called the ***Bohrification*** of A (which may be as non-commutative as one desires). Its constructive Gelfand spectrum $\underline{\Sigma}(\underline{A})$ has been explicitly computed within $\mathsf{T}(A)$,[49] but one of the virtues of toposes of sheaves $\mathsf{Sh}(X)$ (at least for beginners, like the author) is that any locale $\underline{Y}$ in $\mathsf{Sh}(X)$ has a so-called *external* description in set theory,[50] namely as a locale map $\pi : Y \to X$ (in set theory), or, more precisely, as a frame map

$$\pi^{-1} : \mathcal{O}(X) \to \mathcal{O}(Y). \tag{106}$$

Here $\mathcal{O}(Y) = \overline{\mathcal{O}(\underline{Y})}(X)$, which is a frame in Sets. Applied to the Gelfand spectrum $\underline{\Sigma}(\underline{A})$ in $\mathsf{T}(A)$, where $X = \mathcal{C}(A)$ in the Alexandrov topology, and $\underline{Y} = \underline{\Sigma}(\underline{A})$, it turns out that $\mathcal{O}(Y)$ is spatial, so that the associated frame $\mathcal{O}(Y)$ is the topology of a genuine space and (106) is the inverse image map of a continuous map $\pi : Y \to X$, which is given by (91).

This may be seen as a 'derivation' of our intuitionistic quantum logic from topos theory, but of course this derivation is based on certain categorical input that may be hardly more convincing than just postulating the Heyting lattice $\mathcal{O}(\Sigma_A)$ or its finite-dimensional case $Q(A)$. Whichever way one looks at its origin, the advantages of $\mathcal{O}(\Sigma_A)$ or $Q(A)$ over the Birkhoff-von Neumann lattice $\mathcal{P}(A)$ are impressive: the logic is distributive (which is needed in order to interpret the lattice operations $\wedge$ and $\vee$ as "and" and "or", respectively, even in quantum theory), lacks the excluded middle third property (which indeed is highly questionable in quantum theory), and it is spatial in two different senses of the word:

- Seen as a frame, the lattice $\mathcal{O}(\Sigma_A)$ is spatial;
- Seen as a Heyting lattice, $\mathcal{O}(\Sigma_A)$ has an associated Esakia spectrum $\mathcal{E}(\mathcal{O}(\Sigma_A))$.

[48] This C*-algebra was introduced in Heunen, Landsman, & Spitters (2009). The proof that $\underline{A}$ is actually a C*-algebra in $\mathsf{T}(A)$ is nontrivial and is somewhat incomplete in the above reference. An improved version may be found in Landsman (2017a), §12.1, relying on details in Banaschewski & Mulvey (2006).

[49] See Heunen, Landsman, & Spitters (2009), Wolters (2013), and Landsman (2017), §12.4.

[50] See Joyal & Tierney (1984) and Johnstone (2002), §C1.6, summarized in Landsman (2017a), §E.4.

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