Categorical Aspects of von Neumann Algebras and AW^*-algebras

Sander Uijlen
suijlen1337@gmail.com

Master Thesis Mathematics at the Radboud University Nijmegen,
supervised by prof. dr. N.P. Landsman,
written by Sander Uijlen, student number 0521485

Abstract

We take a look at categorical aspects of von Neumann algebras, constructing products, coproducts, and more general limits, and colimits. We shall see that exponentials and coexponentials do not exist, but there is an adjoint to the spatial tensor product, which takes the role of coexponent. We then introduce the class of AW*-algebras and try to see to what extend these categorical constructions are still valid.

Introduction

The Gelfand duality between commutative unital C^*-algebras and compact Hausdorff spaces (see Proposition 2.16) has led to the idea that we can interpret general C^*-algebras as generalized (non commutative) topological spaces. A simialar theorem is valid for von Neumann algebras; every commutative von Neumann algebra is isomorphic to the continuous functions on some hyperstonean space. This leads one to the idea that one can interpret von Neumann algebras as generalized (non commutative) measure spaces. In [5], A. Kornell studies the category of von Neumann algebras and interprets the dual category as a set-like category whose objects he calls *quantum collections*, in which quantum-mechanical computations can be made. This is inspired by the embedding of sets (seen as topological spaces with discrete topology) in the opposite of the category of von Neumann algebras via $X \mapsto {}^\circ\ell^\infty(X)$. First, the category $\mathbf{W}^*$ of von Neumann algebras and unital normal *-homomorphims is studied and this category has nice properties. It has products, coproducts, equalizers, coequalizers, and general limits and colimits. It does, however, not have exponents and coexponents. The non existence of coexponents in $\mathbf{W}^*$ is the same as the non existence of exponentials in the opposite category, so this cuts ties with **Set**, the category of sets and functions. To remedy this, it is shown that instead of coexponents (which are left adjoints to the coproduct) there does exist a construction mimicing that of a coexponent, and this is a left adjoint to the spatial tensor product, making $\mathbf{W}^*$ a closed monoidal category. A special case of this adjunction is the following formula:

$$Hom(\mathcal{M}^{*\mathcal{N}}, \mathbb{C}) \cong Hom(\mathcal{M}, \mathcal{N}),$$

which shows that any normal unital *-homomorphism between von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ comes from some homomorphic state on the *free exponentials* $\mathcal{M}^{*\mathcal{N}}$. Kornell then procedes to the category of von Neumann algebras and unital completely positive maps and shows that in this category there is a surjective natural transformation

$$Hom(\mathcal{M}^{*\mathcal{N}}, \mathbb{C}) \to Hom(\mathcal{M}, \mathcal{N}).$$

This shows that any quantum operation is induced by a state on the free exonential.

It becomes a natural question to ask if these constructions are special to von Neumann algebras, or if there is some larger class of operator algebras in which we can perform the same categorical constructions. In this paper, we try to do this for the catgory of AW^*-algebras and AW^* morphisms.
In the first chapter, we explain the basics of category theory and introduce the constructions we wish to study. We follow, in the second chapter, with the basics of operator

algebras, leading to the conclusion that von Neumann algebras and W^*-algebras are equivalent. Any reader familiar with category theory and/or operator algebras may freely skip (any of) these chapters. Any reader who wishes to learn more on these subjects can study [1] and [7] for more on category theory, and [6], [8], [**?**] and [**?**] for more on operator algebras.
The third chapter focuses on Kornell's arguments regarding the category of von Neumann algebras. We follow Kornell's reasoning and give proofs and constructions of the basic categorical constructions seen in the first chapter. We shall indeed see that von Neumann do not have coexponentials, but that it is possible to find a left adjoint with respect to the spatial tensor product, making von Neumann algebras a closed monoidal category.
In the fourth chapter, we take a look at AW^*-algebras. This is a class of algebras which closely resemble von Neumann algebras. They play a role in quantum logic, see for example [3]. Our original hope was to extend the categorical constructions valid for von Neumann algebras to these AW^*-algebras. However, since we do not have a notion of spatial theory or of tensor products for AW^*-algebras, we cannot obtain the desired results.

Acknowledgements

Finally, I would like to say that I've had a great time working on this project, and I have learned a lot. This would not have been possible without my supervisor, Klaas Landsman, who always had time for me and has helped me, in writing this, as well as in finding the next step to my scientific career. I would also like to thank Andre Kornell for his help in my understanding of his paper, and Chris Heunen for his help in my understanding of AW^*-algebras.

Sander Uijlen
August 2013

Contents

1 Category Theory

1.1 Categories

Category theory is in a certain sense a way to formalize mathematics. Many construction in different areas of mathematics, like taking direct products of sets or groups, are actually instances of a universal construction. In this chapter we shall take a look at the most regularly encountered constructions, which are also the ones we wish to obtain for von Neumann algebras later on. We begin with the very basics:

Definition 1.1. *A* category $\mathcal{C}$ *consists of*

- *a collection* objects *of* $\mathcal{C}$*,*
- *for every pair of objects* $A, B \in \mathcal{C}$ *a collection* $Hom_{\mathcal{C}}(A, B)$ *of* arrows *(or* morphisms*) from* A *to* B *such that*
  - *if* $f \in Hom_{\mathcal{C}}(A, B)$ *and* $g \in Hom_{\mathcal{C}}(B, C)$*, then there exists an arrow* $g \circ f \in Hom_{\mathcal{C}}(A, C)$*,*
  - $(f \circ g) \circ h = f \circ (g \circ h)$*,*
  - *for each object* $A \in \mathcal{C}$*, there exists a unique arrow* $id_A \in Hom_{\mathcal{C}}(A, A)$ *such that if* $f \in Hom_{\mathcal{C}}(A, B)$*, then* $f \circ id_A = f = id_B \circ f$*.*

For $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ we write $f : A \to B$ and say $A = dom(f)$, the *domain* of f and $B = cod(f)$, the *codomain* of f. We call the arrow $g \circ f$ the *composition* of f and g and the arrow id_A the *identity* of A.
We note that, because of the bijection between objects and identity arrows we could define a category in terms of arrows only, but we will not pursue this here.

Example 1.2. *(i) One of the most basic examples of a category, is the category* ***Set****, consisting of all sets as objects and functions between them as morphisms.*

(ii) Another example is given by a set on its own. A set can be considered to be a (discrete) category, where the objects are the elements of the set and only identity arrows exists.

(iii) Our last example for now is given by a poset (= partially ordered set). For two elements x, y *of such a set (which are the objects of the category), there is a unique arrow* $x \to y$ *if and only if* $x \leq y$*.*

Note that in our first example there are (usually) many morphism (=functions) between two sets, whereas in the second example there are no arrows between different objects. In the third example, if there exists an arrow between two objects, it is unique, but there does not have to be an arrow between any two objects (as is the case in **Set** with a nonempty set and the empty set).
Also note that in the definition of a category, we explicitly not require for the objects to form a set. This is because, for example in the category **Set**, the collection of all sets is not a set (at least, not in the ZermeloFraenkel set theory).

Definition 1.3. *A category is called:*

- small *if the objects and morphism actually do form a set,*
- locally small *if for any two objects, the morphisms between them form a set,*
- large *otherwise.*

We can also wish to define morphisms between categories. These are called *functors*.

Definition 1.4. *A functor is map $F : \mathcal{A} \to \mathcal{B}$ satisfies the following:*

- *F sends objects of $\mathcal{A}$ to objects of $\mathcal{B}$ and arrows of $\mathcal{A}$ to arrows of $\mathcal{B}$ in such a way that the domain of f is sent to the domain of $F(f)$, and similarly for the codomain.*
- *F respects composition and identity, i.e. $F(f \circ g) = F(f) \circ F(g)$ and $F(id_A) = id_{F(A)}$.*

Note that the composition $f \circ g$ in $F(f \circ g)$ is in the category $\mathcal{A}$, whereas the composition $F(f) \circ F(g)$ is in $\mathcal{B}$.

With these functors as morphisms we get yet another category **Cat**, consisting of categories and functors.

Definition 1.5. *Let $\mathcal{C}$ be a category. The* opposite *(or* dual*) category $\mathcal{C}^{op}$ has as its objects the objects of $\mathcal{C}$ (we shall write $^\circ A$ for the object A in $\mathcal{C}^{op}$), whilst the morphisms of $\mathcal{C}^{op}$ are precisely the morphisms of $\mathcal{C}$, only reversed.*
So for the morphisms we have $^\circ f \in Hom_{\mathcal{C}^{op}}(A, B)$ if and only if $f \in Hom_{\mathcal{C}}(B, A)$. We then have the relations $id_{^\circ A} = {^\circ}(id_A)$ and $^\circ(f \circ g) = {^\circ g} \circ {^\circ f}$ for the identity and composition.

Of course, $(\mathcal{C}^{op})^{op} = \mathcal{C}$.

Definition 1.6. *If two morphisms satisfy $f \circ g = id_{dom(g)}, g \circ f = id_{dom(f)}$, we call them* isomorphisms *and write $f = g^{-1}$ or $g = f^{-1}$.*

Note that the morphisms in question have to be part of the category. So for example, in the category of groups and group homomorphisms, the map $x \mapsto bx$ for a fixed non-identity element b is not an isomorphism, even though it is bijective.
In **Set**, this notion of an isomorphism is just that of a surjective and injective map (as we expect), but in a general category (such as a poset), the notions of injectivity and surjectivity do not make sense. There are, however, generalized such notions.

Definition 1.7.

- *We call an arrow $f : A \to B$ a* monomorphism*, if for any $g, h : C \to A$, the condition $f \circ g = f \circ h$ implies $g = h$. In this case we sometimes just say f is* mono *or f is* monic*.*

- *In a similar fashion we call an arrow $f : A \to B$ an* epimorphism*, if for any $g, h : B \to C$, the condition $g \circ f = h \circ f$ implies $g = h$. We might also say f is* epi *or f is* epic*.*

We need to check these really are generalizations, so suppose that f in **Set** is mono. Let x, x' be in A with $x \neq x'$ and let the set C be a singleton (i.e., a one-point set). Define two functions from C to A, one sending the single point in C to x, the other sending it to x'. Since f is monic, we cannot have $f(x) = f(x')$, so f is injective.
Now suppose f is injective and let $g, h : C \to A$ such that $f \circ g = f \circ h$. Then $f \circ g(x) = f \circ h(x)$, so $g(x) = h(x)$ and $g = h$.
For the corresponding assertion on epimorphisms, first suppose f is surjective and suppose we have $g \circ f = h \circ f$. Then for any $x \in dom(g)$, there is an $y \in dom(f)$ such that $x = f(y)$. Therefore we have $g(x) = g(f(y)) = h(f(y)) = h(x)$, so f is epic.
The other way around, suppose f is not surjective, so there exists an x in B which is not in the image of f. Let $g, h : B \to C$ be such that $g(y) = h(y)$ for any $y \neq x$ and $g(x) \neq h(x)$. Then we have $g \circ f = h \circ f$, but $g \neq h$, so f is not epic.

Looking at the notion of an epimorphism in the opposite category, we see that it corresponds precisely to a monomorphism in the category itself and *vice versa*. So the concept of a monomorphism is dual to that of an epimorphism.

Since an isomorphism is invertible, it is both epic and monic. The converse, however, does not hold in general. To see this, we introduce the category **Mon**.

Definition 1.8. *A* monoid *is a set M with an associative binary operation $\cdot : M \times M \to M$ (often called multiplication) and an identity u for this multiplication. Explicitly, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in M$, and $u \cdot x = x \cdot u = x$.*

Instead of $x \cdot y$ one often encounters the notation $x \times y$, $x + y$ (especially when the monoid is commutative) or just xy.
Now **Mon** is the category with objects monoids and as morphisms the unity-preserving homomorphisms. So for $(M, \cdot_M, u_M)$ and $(N, \cdot_N, u_N)$ monoids, a map $f : M \to N$ is a *monoid-morphism* if $f(u_M) = u_N$ and $f(x \cdot_M y) = f(x) \cdot_N f(y)$.

We now see that $\mathbb{N}$ and $\mathbb{Z}$ are both monoids with addition as multiplication and 0 as unit. The inclusion map $i : \mathbb{N} \hookrightarrow \mathbb{Z}$ is obviously monic, but it is also epic, whereas it is (clearly) no isomorphism. To see it is an epimorphism, let again $g, h : \mathbb{Z} \to M$ be monoid-morphisms such that $g \circ i = h \circ i$. It follows that $g(x) = h(x)$ for all $x \in \mathbb{N}$. All we need to show now is $g(-1) = h(-1)$ because then it will follow from the homomorphim property that $g = h$. To show this, we calculate

$$\begin{aligned} g(-1)u_M &= g(-1)h(0) \\ &= g(-1)h(1)h(-1) \\ &= g(-1)g(1)h(-1) \\ &= u_M h(-1). \end{aligned}$$

So indeed $g(-1) = h(-1)$ and $g = h$.

1.2 Products and Coproducts

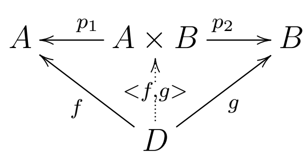
In **Set** we have the notion of a (direct or Cartesian) product of two sets. This notion can also be made categorical.

Definition 1.9. *Suppose that for two objects A and B there exists a third object C and two arrows $p_1 : C \to A$ and $p_2 : C \to B$ such that for each object D for which there are arrows $f : D \to A$ and $g : D \to B$, there exists a unique arrow $h : D \to C$, such that $p_1 \circ h = f$ and $p_2 \circ h = g$. Then C is called a* product *of A and B.*
We write $C = A \times B$ and call the functions p_1 and p_2 projections onto A and B, respectively. The unique map h is written as $< f, g >$ and called the pair or tuple of f and g.

In terms of morphisms we can characterize the product by the following rules, each of which is easily checked

- $p_1 < f, g >= f$, $p_2 < f, g >= g$,
- $< f, g > h =< fh, gh >$,
- $< p_1, p_2 >= id$.

At this point it will be very illustrative to draw a diagram. The dotted arrow indicates it is the unique arrow making the diagram commute.



The possible existence of products depends on the category. For example, products in **Set** exist (namely, the direct product) whereas they do not exist in a set when considered a discrete category.

Proposition 1.10. *If a product exists is a certain category, it must be unique up to isomorphism.*

Proof. To see this, let D in the diagram be another product. By the universal property of the product $A \times B$, there is then a unique morphism $D \to A \times B$ making the diagram commute. However, since D is also a product, there also is a unique morphisms $A \times B \to D$ making the diagram commute. The composition of these morphisms, which we shall call h, is then the unique morphism $A \times B \to A \times B$ such that $p_i = p_i h$. However, $id_{A \times B}$ (or id_D) also does this job. By uniqueness, $h = id_{A \times B}$. Replacing $A \times B$ with D in this argument shows that the two unique morphisms are mutual inverses. Therefore $A \times B$ and D are isomorphic. □

Similar arguments show that all constructions using a *universal property* are unique up to isomorphism.

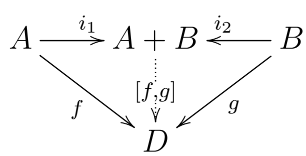
Once we have obtained the product of two objects, we could try to form a product

with a third object. We could either form $(A \times B) \times C$ or $A \times (B \times C)$, but either way, these are isomorphic (as can be seen again from drawing diagrams).
Continuing in this fashion, we can make the product of n objects ($n \geq 2$) (if they exist in the category).
An object by itself can be seen as the product of one object and we define the product of zero objects as a *terminal object*.

Definition 1.11. *A terminal object is an object* 1 *such that for each object* A*, there is a unique arrow* $A \to 1$*.*

We say the category has *finite products* if it has a terminal object and all products of n objects exists for each $n \in \mathbb{N}$.

Suppose a product exists in the opposite category. Then we obtain a structure in the original category where we have two objects A and B with morphisms i_1, i_2 going into a third object C such that, whenever there are morphisms $A \xrightarrow{f} D$ and $B \xrightarrow{g} D$, there exists a unique morphism $C \to D$ making the diagram below commute.



Here the object C is denoted $A + B$, the morphisms i_1, i_2 are called *injections*, and the unique morphism is written as $[f, g]$.

Definition 1.12. *The above object* $A + B$*, if it exists, is called a* coproduct *of* A *and* B*. Via a similar reasoning as with products, it is unique up to isomorphism.*

In terms of morphisms we characterize the coproduct by

- $[f, g]i_1 = f, [f, g]i_2 = g,$
- $h[f, g] = [hf, hg],$
- $[i_1, i_2] = id.$

Definition 1.13.

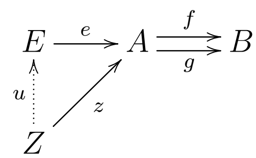
- *A* initial *object is an object* 0 *such that for every object there exists a unique arrow from* 0 *to that object.*
- *An object which is initial as well as final is called a* zero object*.*
- *We say a category has* all finite coproducts *if it has an initial object, and all coproducts of* n *objects* ($n \geq 2$) *exist.*

In sets, a coproducts are given by disjoint union.

1.3 Equalizers and Coequalizers

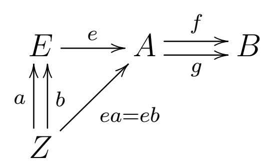
Our next categorical structure will be that of *equalizers*.

Definition 1.14. *Let $f, g : A \to B$ be two parallel arrows in a category $\mathcal{C}$. An equalizer for f and g consists of an object $E \in \mathcal{C}$ and a morphism $e : E \to A$ such that $fe = ge$ and whenever $z : Z \to A$ is such that $fz = gz$, there exists a unique arrow $u : Z \to E$ such that $eu = z$.*



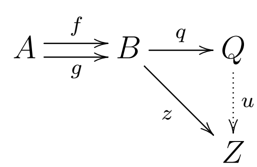
Proposition 1.15. *If e is an equalizer for a pair of arrows, it is a monomorphism.*

Proof. Let a, b be arrows such that $ea = eb$. Consider the following diagram



Since $fe = ge$ we have $fea = gea$, so by the uniqueness property of equalizers, $a = b$. □

Dual to the concept of equalizers is that of *coequalizers*. The below diagram should explain enough.

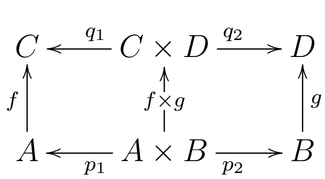


Proposition 1.16. *If q is a coequalizer for a pair of arrows, it is an epimorphism.*

Proof. A coequalizer is an equalizer in the opposite category, so it is mono in that category and hence epi in the original category. □

1.4 Exponentials and Coexponentials

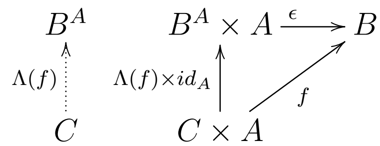
Let $C \times D$ be the product of C and D with projections q_1 and q_2 and let $A \times B$ be the product of A and B with projections p_1 and p_2. Suppose we have morphisms $f : A \to C$ and $g : B \to D$. Then we can make an arrow $f \times g : A \times B \to C \times D$ making the following diagram commute:



From this we see $f \times g =< fp_1, gp_2 >$.

Our last categorical structures for now are *exponentials* and *coexponentials*. We assume the category $\mathcal{C}$ has finite products.

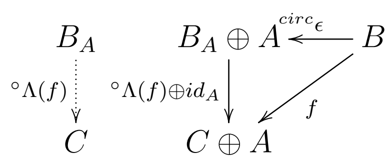
Definition 1.17. *An exponential of two objects A and B is an object B^A and a morphism $\epsilon : B^A \times A \to B$ such that for any object and arrow $f : C \times A \to B$ there exists a unique arrow $\Lambda(f) : C \to B^A$ such that $\epsilon(\Lambda(f) \times id_B) = f$.*



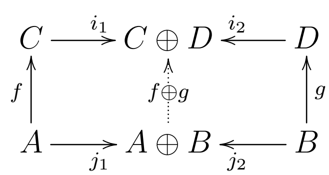
We see, from taking $C = B^A$ and $f = \epsilon$, that $\Lambda(\epsilon) = id$. Also, if we have $h : D \to C$, we have the arrow $f \circ (h \times id) : D \times A \to B$, so we have the unique arrow $\Lambda(f \circ (h \times id)) : D \to B^A$ making the diagram commute, but $\Lambda(f) \circ h$ is also such an arrow, hence $\Lambda(f \circ (h \times id)) = \Lambda(f) \circ h$.

If A and B are sets, then A^B is the set of functions from B to A, and the evaluation morphism $\epsilon : A^B \times B \to A$ is the the function sending $f \in A^B$, $x \in B$ to $f(x) \in A$.

Of course, dualizing this structure, we find a new structure called the coexponent of A and B.



Here a map of the form $f \oplus g$ is meant to be the unique map making the diagram



commute (with i_1, i_2, and j_1, j_2 the respective injections).

Definition 1.18. *A category with all finite products and exponentials is called* Cartesian closed.
A category with all finite coproducts and coexponentials is called cocartesian closed.

Sets does not have coexponentials.

Terminal objects, equalizer, and product are special cases of so-called *limits*. Likewise, initial objects, coequalizers, and coproducts are special cases of *colimits*. To define such a limit in a category $\mathcal{C}$, we begin by defining a *diagram of type J*.

Definition 1.19.
- *Let J be a category. A diagram of type J is a functor $F : J \to \mathcal{C}$.*

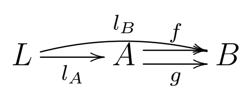
- *A* cone *to such a diagram is an object $C \in \mathcal{C}$ together with morphisms $c_i : C \to F(i)$ for every object $i \in J$, such that, if $\alpha : i \to j$ is a morphism in J, we have $c_j = F(\alpha)c_i$.*

- *A* limit *is a special cone (L, l_i), such that, for each cone (C, c_i), there exists a unique map $u : C \to L$ such that $c_i = l_i u$. In case the category J is small, we say L is a* small limit.

To see that, for example, an equalizer is a limit, let J be a category with two objects and two parallel arrows between them (and of course identity arrows). Under a functor this will be of the form

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

If L is a limit, there are two morphisms $l_A : L \to A$ and $l_B : L \to B$ such that $fl_A = l_B = gl_A$.



The universal property of the equalizer is now exactly the universal property of the limit.
A product is obtained in the special case where J has only two objects and no nontrivial arrows, and a terminal object comes from J being the empty category.

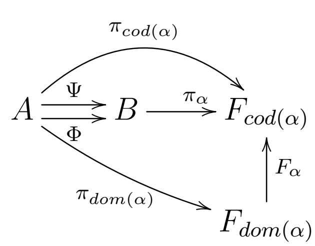
It will be clear how *cocones* and *colimits* are defined. Namely as cones, limits in the opposite category.
We have seen that equalizers and products are special cases of (finite) limits. The other way around is also true, as we have the following:

Proposition 1.20. *A category $\mathcal{C}$ has finite limits if and only if it has finite products and equalizers.*

Proof. Let $F : J \to \mathcal{C}$ be a finite diagram. We can make the product $A = \times_{i \in Obj(J)} F_i$ with projections $\pi_j : A \to F_j$ and the product $B = \times_{\alpha \in Arr(J)} F_{cod(\alpha)}$ with projections $\pi_\alpha : B \to F_{cod(\alpha)}$, where the first product is over all objects in J and the second product is over all arrows in J. We define two maps $\Psi, \Phi : A \to B$ coordinatewise by

$$\pi_\alpha \circ \Phi = \pi_{cod(\alpha)},$$

$$\pi_\alpha \circ \Psi = F_\alpha \circ \pi_{dom(\alpha)}.$$



We can take the equalizer (E, e) of Ψ and Φ and are going show that E together with the morphisms $e_i = \pi_i e$ is a limit to F.
To this end, let C be an object with morphisms $c_i : C \to F_i$. Now C, together with the c_i is a cone to F if and only if $\Psi c = \Phi c$, where $c =< c_i >$ is the product-morphism of the c_i. Indeed;

$$\begin{aligned} \Psi c = \Phi c &\Leftrightarrow \pi_\alpha \Psi c = \pi_\alpha \Phi c \\ &\Leftrightarrow F_\alpha \pi_{dom(\alpha)} c = \pi_{cod(\alpha)} c \\ &\Leftrightarrow F_\alpha c_{dom(\alpha)} = c_{cod(\alpha)}. \end{aligned}$$

This now shows that E together with the e_i is a cone and also shows that any other cone factorizes via E because it is an equalizer. □

Since we did not really use finiteness in the proof, we can conclude that a category has any type of limits if and only if it has equalizers and the same type of products.

1.5 Natural Transformations and the Yoneda Lemma

Until now, all we have done is abstractify regularly encountered constructions in certain categories. We now give a very useful result which makes it easy to see if two objects are isomorphic by looking at their homsets. This is known as the *Yoneda lemma*. First, a bit more work is needed.

Definition 1.21. *A* natural transformation *between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of maps $(\theta_C)_{C \in obj\mathcal{C}} : FC \to GC$, such that, if $f : C \to \tilde{C}$ is a morphism in $\mathcal{C}$, then the following diagram commutes.*

$$\begin{array}{ccc} FC & \xrightarrow{\theta_C} & GC \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ F\tilde{C} & \xrightarrow{\theta_{\tilde{C}}} & G\tilde{C} \end{array}$$

We write $\theta : F \to G$ for the family of morphisms (θ_C). From a categorical point of view, natural transformations are the morphisms in the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$.

Definition 1.22. *A natural transformation is a* natural isomorphism *if there is an inverse natural transformation.*

Proposition 1.23. *A natural transformation $\theta : F \to G$ is a natural isomorphism if and only if every component θ_C is an isomorphism.*

Proof. If every component θ_C is an isomorphism, then an inverse for θ is easily defined by taking the componentwise inverse of the θ_C. It remains to prove this resulting θ^{-1} is natural. So we have to show the following diagram commutes.

$$\begin{array}{ccc} FC & \xleftarrow[\theta_X^{-1}]{} & GC \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FD & \xleftarrow[\theta_Y^{-1}]{} & GD \end{array}$$

From the naturality of θ, it follows that

$$\begin{aligned}\theta_D \circ F(f) \circ \theta_C^{-1} &= G(f) \circ \theta_C \circ \theta_C^{-1} \\ &= G(f) \\ &= \theta_D \circ \theta_D^{-1} G(f).\end{aligned}$$

Multiplying both side from the left with θ_D^{-1} gives the desired equality.
The other way around, suppose there exists $\theta^{-1} : G \to F$ such that $\theta \circ \theta^{-1} = id_G$ and $\theta^{-1} \circ \theta = id_F$. Then $\theta_C^{-1} \circ \theta_C = (\theta^{-1} \circ \theta)_C = id_C$. Likewise $\theta_C \circ \theta_C^{-1} = id_C$. □

The concept of isomorphisms between categories is straightforward.

Definition 1.24. *Two categories $\mathcal{A}$ and $\mathcal{B}$ are* isomorphic *if there exists functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ that satisfy $F \circ G = id_{\mathcal{B}}$ and $G \circ F = id_{\mathcal{A}}$.*

However, being an isomorphism is often too strong a condition. The concept of natural isomorphisms allows us to weaken this notion.

Definition 1.25. *If $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ are functors, then $\mathcal{C}$ and $\mathcal{D}$ are called* equivalent *if there are natural isomorphisms*

$$\eta : 1_{\mathcal{C}} \to G \circ F,$$

and

$$\rho : 1_{\mathcal{D}} \to F \circ G.$$

An important instance of equivalence is the following:

Definition 1.26. *Two categories $\mathcal{A}$ and $\mathcal{B}$ are called* dual *to each other if there is an equivalence between $\mathcal{A}$ and $\mathcal{B}^{op}$.*

Before we go on to the Yoneda Lemma, we make one last observation. We saw, in the definition of the exponential, that in a Cartesian closed category, any morphism $A \times B \to C$ corresponds to a morphism $A^C \to B$. This is a special case of the following:

Definition 1.27. *Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be functors. F is* left adjoint *to G if there is an isomorphism, natural in X and Y,*

$$Hom_{\mathcal{C}}(FX, Y) \cong Hom_{\mathcal{D}}(X, GY),$$

for $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. In this case, we also say G is right adjoint *to F.*

For $\mathcal{C}$ and $\mathcal{D}$ locally small categories, we denote by $\mathcal{D}^{\mathcal{C}}$ the *functor category*, whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations. In particular, we can look at $\mathbf{Sets}^{\mathcal{C}^{op}}$, which is called the *category of presheaves on $\mathcal{C}$*.

Definition 1.28. *The* Yoneda embedding *is a functor $y : \mathcal{C} \to \boldsymbol{Sets}^{\mathcal{C}^{op}}$, sending C to*

$$yC = Hom(-, C) : \mathcal{C}^{op} \to \boldsymbol{Sets},$$

and a morphism $f : C \to D$ to

$$yf = Hom(-, f) : Hom(-, C) \to Hom(-, D),$$

where $Hom(-, f)$ is composition with f.

Lemma 1.29 (Yoneda). *For each object $C \in \mathcal{C}$ and functor $F \in \boldsymbol{Sets}^{\mathcal{C}^{op}}$ there is an isomorphism*

$$Hom(yC, F) \cong FC.$$

This isomorphism is natural in C as well as in F.

Proof. We will only give a very rough sketch of the proof (details can be found in the literature [1], [7]).
On the one hand, there is a map $\phi_{C,F} : \mathrm{Hom}(yC, F) \to FC$, for which a natural transformation $\theta : yC \to F$ is sent to $\theta_C(1_C)$. Indeed, $\theta_C : yC(C) = \mathrm{Hom}(C, C) \to FC$.
On the other hand, given $a \in FC$ we define a natural transformation $\theta_a : yC \to F$ by defining it componentwise as $(\theta_a)_{C'} : \mathrm{Hom}(C', C) \to FC'$, $(\theta_a)_{C'}(h) = F(h)(a)$.
The rest of the proof now consists of showing these transformations are indeed natural and are mutually inverse to each other. □

Here, we just write yC instead of $y(C)$ as we will do more often for functors.

Definition 1.30. *A functor $F : \mathcal{C} \to \mathcal{D}$ induces a function $F_{C,C'} : Hom_{\mathcal{C}}(C, C') \to Hom_{\mathcal{D}}(FC, FC')$. We say F is*

- full *if $F_{C,C'}$ is injective, for all $C, C' \in \mathcal{C}$,*
- faithful *if $F_{C,C'}$ is surjective, for all $C, C' \in \mathcal{C}$,*
- fully faithful *if F is full and faithful.*

Theorem 1.31. *The Yoneda embedding $\mathcal{C} \to Sets^{\mathcal{C}^{op}}$ is fully faithful.*

Proof. By the previous lemma, for $C, D \in \mathcal{C}$, we have the isomorphism

$$\mathrm{Hom}(C, D) = yD(C) \cong \mathrm{Hom}(yC, yD).$$

We still have to show that this isomorphism is induced by the Yoneda embedding y. So let $h : C \to D$. Then, as in the previous lemma, we have the natural transformation $\theta_h : yC \to yD$ for which the components $(\theta_h)_{C'}$ act on $f : C' \to C$ as

$$\begin{aligned} (\theta_h)_{C'}(f) &= yD(f)(h) \\ &= \mathrm{Hom}(f, D)(h) \\ &= h \circ f \\ &= (yh)_{C'}(f). \end{aligned}$$

So indeed $\theta_h = yh$. □

The importance of this theorem is that if $yA \cong yB$, then $A \cong B$. That is to say, if $\mathrm{Hom}(X, A) \cong \mathrm{Hom}(X, B)$ for all objects X, then $A \cong B$.

We can do the same is a covariant setting, where we look at the functors $\mathrm{Hom}(C, -)$. We then find that if $\mathrm{Hom}(X, A) \cong \mathrm{Hom}(Y, A)$, for all objects A, then $X \cong Y$.

2 W^*-algebras and von Neumann algebras

2.1 Basics of Operator algebras

In order to study the category of von Neumann algebras, it is necessary to have the basic definitions and proposition regarding this subject. We will give these here, while omitting the proofs. We by no means claim to give a full overview of the theory, we only consider the bare essentials. We refer to [11], [8], [6], or any other book on Operator Algebras for a detailed account of the theory.

Definition 2.1.

- *A* Banach space *is a complete normed vector space over* $\mathbb{R}$ *or* $\mathbb{C}$.
- *A* Hilbert space *is an inner product space over* $\mathbb{R}$ *or* $\mathbb{C}$*, which is complete in the norm derived from this inner product.*
- *A* Banach algebra *is a Banach space* $\mathcal{X}$ *with an associative multiplication satisfying* $\|ab\| \leq \|a\|\|b\|$ *for all* $a, b \in \mathcal{X}$.
- *A* C^*-algebra *is a Banach algebra* $\mathcal{A}$ *with involution and the additional condition that* $\|a^*a\| = \|a\|^2$ *for all* $a \in \mathcal{A}$.

We note that while a C^*-algebra does not have to contain a unit, the operator spaces we are interested in here are unital. That is why we mostly consider unital C^*-algebras here. Most of the theory in this section can be done for non-unital C^*-algebras as well.

As a vector space, a Hilbert space always has a basis. We say a basis $\{e_i\}$ for H is *orthonormal* if $\langle e_i, e_j\rangle = \delta_{i,j}$, ($\delta_{i,j} = 1$ if $i = j$ and 0 otherwise). Whenever we pick a basis for a Hilbert space, we shall always mean an orthonormal basis. Whenever we have an orthonormal basis, we have *Parseval's identity*:

$$\langle f, g\rangle = \sum_i \langle f, e_i\rangle\langle e_i, g\rangle.$$

Given a Hilbert space H, we can consider all bounded linear maps $a : H \to H$, together with the operator norm

$$\|a\| = \sup\{\|ah\| \mid h \in H, \|h\| \leq 1\}.$$

Upon this space we have an involution $a \mapsto a^*$ where $\langle f, ah\rangle = \langle a^*f, h\rangle$, for $f, h \in H$. We call this space $B(H)$, the *bounded operators on* H.
We note at this point that we take our inner product to be linear in the second entry and anti-linear in the first.

Proposition 2.2. $B(H)$ *is a* C^**-algebra.*

Definition 2.3. *A subset* $S \subset B(H)$ *is called* self-adjoint *is* $a^* \in S$ *whenever* $a \in S$.

It is clear that every norm-closed linear self-adjoint subspace of $B(H)$ is also a C^*-algebra.

Let $S \subset B(H)$ be a subset. The *commutant* S' of S is the space

$$S' = \{b \in B(H) \mid bs = sb, \text{ for all } s \in S\}.$$

We write $S'' = (S')'$ for the *bicommutant* of S and continue in this fashion, e.g., $S''' = (S'')'$. We can now sum up some properties of the commutant. The proofs of these are trivial.

Proposition 2.4. *Let S and R be subsets of $B(H)$.*

- *If $S \subset R$, then $R' \subset S'$.*
- *$S \subset S''$.*
- *$S' = S'''$.*

Definition 2.5. *A* von Neumann algebra *is a C^*-algebra $\mathcal{M}$ in $B(H)$ such that $\mathcal{M} = \mathcal{M}''$.*

This definition of a von Neumann algebra is algebraic, in the sense that commutators depend only on the multiplication in $B(H)$. There are also topological conditions on a *-subalgebra of $B(H)$ to be a von Neumann algebra. To this end, we first introduce other topologies besides the one given by the norm.

Definition 2.6.

- *A net a_i of operators in $B(H)$* converges weakly *to some operator a if*

$$\langle v, (a_i - a)w \rangle \to 0,$$

for all $v, w \in H$.

- *A net a_i of operators in $B(H)$* converges strongly *to some operator a if*

$$\|(a_i - a)v\| \to 0,$$

for all $v \in H$.

- *A net a_i of operators in $B(H)$* converges σ-weakly *to some operator a if*

$$tr(\rho(a_i - a)) \to 0,$$

for all $\rho \in B_1(H)$.

Here, tr is the trace, and $B_1(H)$ are the traceclass operators on H. We come back on this in more detail later. Each of these notions of convergence endows $B(H)$ (and therefore any subalgebra) with a topology. These are called the strong topology*,* weak topology*, and* σ-weak topology*, respectively.*

This next theorem is the well known and important *double commutant theorem* of von Neumann. It links the algebraic condition on a von Neumann algebra to topological conditions. It states the following:

Theorem 2.7. *Let $\mathcal{A} \subset BH$ be a *-subalgebra such that $1_H \in \mathcal{A}$. Then the following are equivalent:*

- *$\mathcal{A}$ is a von Neumann algebra, i.e. $\mathcal{A} = \mathcal{A}''$.*
- *$\mathcal{A}$ is closed in the weak topology.*
- *$\mathcal{A}$ is closed in the strong topology.*
- *$\mathcal{A}$ is closed in the σ-weak topology.*

We proceed by looking at single operators in a C^*-algebra (and in particular a von Neumann algebra).

Definition 2.8. *Let $\mathcal{C}$ be a unital C^*-algebra and $a \in \mathcal{C}$. The* spectrum *of a is*

$$spec(a) = \{\lambda \in \mathbb{C} \mid a - \lambda \text{ is not invertible}\}.$$

It turns out that the spectrum of an operator is a compact, non-empty subset of $\mathcal{C}$.

Proposition 2.9. *If $a = a^*$, then $spec(a) \subset \mathbb{R}$.*

Definition 2.10. *An element a is* positive *if $spec(a) \subset \mathbb{R}_{\geq 0}$. In this case we write $a \geq 0$.*

This then induces an order on positive operators via $a \leq b \leftrightarrow 0 \leq b - a$.

Let $\mathcal{A}$ be a unital C^*-algebra and suppose $\mathcal{B}$ is a C^*-subalgebra, such that $\mathcal{B}$ also contains the unit. Then, if a is in $\mathcal{B} \subset \mathcal{A}$ we can calculate the spectrum of a in $\mathcal{B}$, as well as in $\mathcal{A}$. Luckily, under these circumstances we have:

Proposition 2.11. *The spectrum of a is the same in $\mathcal{A}$ as in $\mathcal{B}$.*

In particular, for any operator a, we can look at the C^*-algebra generated by a and 1. This can be realized as the norm closure of all polynomial expressions in a, a^*, and 1. Whenever a commutes with a^*, this algebra is commutative and hence can be identified with the continuous functions on some compact Hausdorff space X. If a is positive, a, now seen as function X, is a positive function, because the spectrum of a continuous function is just the closure of the range of that function. Since in $C(X)$ any positive function has a unique continuous positive square root, we find that for any positive operator, there exists a unique positive square root $a^{\frac{1}{2}}$ such that $(a^{\frac{1}{2}})^2 = a$.

The following proposition gives a nice and often used result on positivity.

Proposition 2.12. *Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$. Then $a \geq 0$ if and only if there exists a $b \in \mathcal{A}$ such that $a = b^*b$.*

With this, if a is an arbitrary element in a C^*-algebra, then a^*a is positive, so there exists a unique positive element, denoted by $|a|$, such that $|a|^2 = a^*a$. $|a|$ is called the *absolute value* of a.

This may remind us of the decomposition of a complex number $z = re^{i\theta}$, where $r = |z|$. In fact, this is indeed the case.

Proposition 2.13. *Let a be an operator on a Hilbert space H. Then there exits a unique decomposition $a = u|a|$, where u is a* partial isometry.

Here, a partial isometry is a map between Hilbert spaces $u : H_1 \to H_2$, such that u is isometric on $\ker(u)^\perp$. This is equivalent to u^*u being a projection and equivalent to uu^* being a projection.

Definition 2.14. *A bounded operator a on a Hilbert space is* finite rank *if the image of a is finite dimensional.*

A special case of a finite rank operator is the one dimensional operator $|f\rangle\langle g|$ for $f, g \in H$ mapping $x \mapsto \langle g, x\rangle f$. Any finite rank operator is a finite sum of these one dimensional operators. However, finite rank operators are not closed in the norm. For example, if $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis in a Hilbert space H, the limit of the elements $\sum_{i=1}^n 2^{-n}|e_i\rangle\langle e_i|$ is not finite rank.

Definition 2.15. *An operator is called* compact *if it is the norm limit of finite rank operators.*

Finally, we say some words on *Gelfand duality*, which may be considered to be the grandfather of this paper.

Let $\mathbf{comC}_1^*$ be the category of commutative C^*-algebras with unit and unit-preserving *-homomorphisms, and let **cHTop** be the category of compact Hausdorff spaces and continuous maps. Then, there is a functor

$$C : \mathbf{cHTop} \to \mathbf{comC}_1^*,$$

sending a compact Hausdorff space X to $C(X)$, the continuous functions on X, and sending a continuous map $f : X \to Y$ between compact Hausdorff spaces to the map $C(f) : C(Y) \to C(X)$, given by $C(f)(g)(x) = g(f(x))$, for $g \in C(Y)$ and $x \in X$. This is a contravariant functor, as we easily see from $C(g \circ h)(f)(x) = f(g(h(x))) = C(h) \circ C(g)(f)(x)$, whenever g and h are composable. There is also a functor

$$sp : \mathbf{comC}_1^* \to \mathbf{cHTop},$$

sending a commutative unital C^*-algebra $\mathcal{A}$ to the space of all non-zero *-homomorphisms $\mathcal{A} \to \mathbb{C}$, and a unital *-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ to the map $sp(\phi) : sp(\mathcal{B}) \to sp(\mathcal{A})$, given by $sp(\phi)(\tau)(a) = \tau(\phi(a))$, for τ in $sp(\mathcal{B}$ and $a \in \mathcal{A}$. This is again a contravariant functor by a similar calculation. The topology on $sp(\mathcal{A})$ is the one induced by $\rho_i \to \rho$ if $\rho_i(a) \to \rho(a)$ for all $a \in \mathcal{A}$.

Proposition 2.16. **cHTop** *is dual to* $\mathbf{comC}_1^*$.

Proof. Composing the above functors, we obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\delta_X} & sp(C(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle sp(C(f))} \\ Y & \xrightarrow[\delta_Y]{} & sp(C(Y)) \end{array}$$

Here, δ_X is given by $\delta_X(x) = \delta_x$, where $\delta_x(f) = f(x)$. Using this, we find $\delta_Y \circ f(x) = \delta_{f(x)}$, while $sp(C(f)) \circ \delta_X(x) = sp(C(f))(\delta_x)$. Now, for $g \in C(Y)$, we have

$$\begin{aligned} sp(C(f))(\delta_x)(g) &= \delta_x(C(f)g) \\ &= \delta_x(g \circ f) \\ &= g(f(x)) \\ &= \delta_{f(x)}(g). \end{aligned}$$

So the diagram is indeed commutative.
Composing the other way around, we obtain

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\theta_{\mathcal{M}}} & C(sp(\mathcal{M})) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle C(sp(f))} \\ \mathcal{N} & \xrightarrow[\theta_{\mathcal{N}}]{} & C(sp(\mathcal{N})) \end{array}$$

Here, $\theta_{\mathcal{M}}$ is given by $a \mapsto \hat{a}$, where $\hat{a}(\phi) = \phi(a)$, for $\phi \in sp(\mathcal{A})$. Now $\theta_{\mathcal{N}} \circ f(a) = \hat{f(a)}$, while for $\tau \in sp(\mathcal{N})$

$$\begin{aligned} C(sp(f)) \circ \theta_{\mathcal{M}}(a)(\tau) &= C(sp(f))\hat{a}(\tau) \\ &= \hat{a}(sp(f)\tau) \\ &= \tau(f(a)) \\ &= \hat{f(a)}(\tau). \end{aligned}$$

So this diagam also commutes. The theorem now follows from the fact that for unital C^*-algebras, the map $a \mapsto \hat{a}$ isometric isomorphism, and that all non-zero *-homomorphisms on $C(X)$ are of the form δ_x. □

The conclusion that **cHTop** is dual to $\mathbf{comC}_1^*$ has led to the idea that the dual of the category of general C^*-algebras can be interpreted as a a category of *noncommutative topological spaces.*

As a special case we can look at two particularly easy examples.

Example 2.17.

- *For the C^*-algebra 0, there are, trivially, no non-zero homomorphisms to any other algebra. Therefore, $sp(0) = \emptyset$.*
- *For the C^*-algebra $\mathbb{C}$, there is only a unique unitial *-homomorphism to any other unital C^*-algebra, given by $z \mapsto z \cdot 1$. Therefore, $sp(\mathbb{C})$ is a singleton.*

2.2 The Trace and the Predual on $B(H)$

A von Neumann algebra is explicitly defined acting on some Hilbert space. We now introduce a certain class of operator algebras that do not have this property.

Definition 2.18. *A W^*-algebra $\mathcal{N}$ is a C^*-algebra $\mathcal{N}$ that, as a Banach space, is the dual of some Banach space $\mathcal{N}_*$, called a predual.*

This section, and the next, are devoted to showing that these concepts coincide. That is, every von Neumann algebra is the dual of some Banach space, and every W^*-algebra has a faithful representation on a Hilbert space which makes it a von Neumann algebra.
We will begin by showing that, for H a Hilbert space, the von Neumann algebra $B(H)$ has a predual. To this end, we first develop the theory of the *trace* on a Hilbert space. Throughout this section, let H be a fixed Hilbert space with an orthonormal basis $\{e_i\}_i$.

Definition 2.19. *Let* $0 \leq T \in B(H)$*. The* trace *of* T *is*

$$tr(T) = \sum_i \langle e_i, Te_i \rangle \in [0, \infty].$$

This definition might seem somewhat strange; there appears to be a dependency on the choice of basis and the trace might not be finite. Later on, we will see that the trace becomes finite after restricting to a special class of operators. The dependency on the basis will be dealt with after the following lemma.

Lemma 2.20. $tr(T^*T) = tr(TT^*)$ *for* $T \in B(H)$*.*

Proof. First of all, for all i, j, we have

$$\begin{aligned} 0 &\leq |\langle e_i, T^* e_j \rangle|^2 \\ &= \langle e_i, T^* e_j \rangle \langle T^* e_j, e_i \rangle \\ &= \langle Te_i, e_j \rangle \langle e_j, Te_i \rangle. \end{aligned}$$

Now, using Parseval's identity,

$$\begin{aligned} \sum_j \langle Te_i, e_j \rangle \langle e_j, Te_i \rangle &= \langle Te_i, Te_i \rangle \\ &= \langle e_i, T^*Te_i \rangle, \end{aligned}$$

and

$$\begin{aligned} \sum_i \langle e_i, T^* e_j \rangle \langle T^* e_j, e_i \rangle &= \langle T^* e_j, T^* e_j \rangle \\ &= \langle e_j, TT^* e_j \rangle. \end{aligned}$$

From this, we find

$$\sum_j \sum_i \langle e_i, T^* e_j \rangle \langle T^* e_j, e_i \rangle = tr(TT^*),$$

$$\sum_i \sum_j \langle Te_i, e_j \rangle \langle e_j, Te_i \rangle = tr(T^*T).$$

The desired equality now follows because the terms in the sums are equal and positive, so we may change the order of summation. □

Theorem 2.21. *The trace of an operator* $T \geq 0$ *does not depend on the choice of basis.*

Proof. Let $T \geq 0$ and U unitary. Then, since there is an $X \in B(H)$ such that $T = X^*X$, we have, using the previous lemma twice,

$$\begin{aligned} tr(U^*TU) &= tr(U^*X^*XU) \\ &= tr(XUU^*X^*) \\ &= tr(XX^*) \\ &= tr(X^*X) \\ &= tr(T). \end{aligned}$$

□

From this we then find that for a bounded positive operator T and unitary operator U:

$$\begin{aligned} tr(T) &= \sum_i \langle e_i, Te_i \rangle \\ &= \sum_i \langle Ue_i, TUe_i \rangle. \end{aligned}$$

So, by definition of the operator norm, we conclude that for positive T:

$$tr(T) \geq \|T\|.$$

Lemma 2.22. *Let $T \in B(H)$ and suppose $tr(|T|) < \infty$. Then T is a compact operator.*

Proof. For an orthonormal basis $\{e_j\}_{j \in J}$ and $\epsilon > 0$, since $\sum_j \langle e_j, |T|e_j \rangle < \infty$, there exists a finite subset $I \subset J$ such that

$$\sum_{j \notin I} \langle e_j, |T|e_j \rangle < \epsilon.$$

Let P_I be the projection corresponding to the span of $\{e_j\}_{j \in I}$. Now

$$\begin{aligned} \||T|^{\frac{1}{2}}(1 - P_I)\|^2 &= \|(1 - P_I)|T|(1 - P_I)\| \\ &\leq tr((1 - P_I)|T|(1 - p_I)) \\ &< \epsilon. \end{aligned}$$

Letting $\epsilon \to 0$, we see that

$$|T|^{\frac{1}{2}} P_I \to |T|^{\frac{1}{2}}.$$

Now P_I is a finite rank operator, so $|T|^{\frac{1}{2}} P_I$ is also of finite rank, since the finite rank operators form an ideal in $B(H)$. Therefore $|T|^{\frac{1}{2}}$ is the norm-limit of finite rank operators and so it is compact. Since the compact operators also form an ideal in $B(H)$, we find that $|T| = |T|^{\frac{1}{2}}|T|^{\frac{1}{2}}$ is also compact, therefor, by the polar decomposition, $T = U|T|$ is compact too. □

Since any operator T is a linear combination of (up to) four positive operators,

$$T = \sum_{k=0}^{3} i^k T_k, \text{ with } T_k \geq 0,$$

there is no need to restrict the trace to positive operators; we can just set

$$tr(T) = \sum_{k=0}^{3} i^k Tr(T_k).$$

So, letting $K(H)$ denote the compact operators, we define the *trace class operators* as

$$B_1(H) = \{T \in K(H) \mid tr(T) < \infty\}.$$

In what follows it will be useful to have the following equations for operators in $B(H)$ (these are proven by just writing them out).

$$(S+T)^*(S+T) + (S-T)^*(S-T) = 2(S^*S + T^*T), \qquad \text{(parallelogram law)}$$

$$4T^*S = \sum_{k=0}^{3} i^k (S + i^k T)^*(S + i^k T). \qquad \text{(polarization identity)}$$

We also note that $tr(T^*) = \overline{tr(T)}$, and that for $T \geq 0$ (so that $T = X^*X$) we have

$$\begin{aligned} tr(T) &= tr(X^*X) \\ &= \sum_i \langle e_i, X^*Xe_i\rangle \\ &= \sum_i \langle Xe_i, Xe_i\rangle \\ &\geq 0. \end{aligned}$$

Therefore, $tr(\cdot)$ is a positive linear functional on $B_1(H)$.

Lemma 2.23. *$B_1(H)$ is a self-adjoint ideal in $B(H)$.*

Proof. Let $T \in B_1(H)$ and $S \in B(H)$. Without loss of generality we may assume $T \geq 0$. Then by the polarization identity

$$\begin{aligned} 4TS &= 4T^{\frac{1}{2}}(T^{\frac{1}{2}}S) \\ &= \sum_{k=0}^{3} i^k (T^{\frac{1}{2}}S + i^k T^{\frac{1}{2}})^*(T^{\frac{1}{2}}S + i^k T^{\frac{1}{2}}) \\ &= \sum_{k=0}^{3} i^k (S^*T^{\frac{1}{2}} + i^{-k}T^{\frac{1}{2}})(T^{\frac{1}{2}}S + i^k T^{\frac{1}{2}}) \\ &= \sum_{k=0}^{3} i^k (S^* + i^{-k})T(S + i^k) \\ &= \sum_{k=0}^{3} i^k (S + i^k)^* T(S + i^k). \end{aligned}$$

For fixed k, write $V = (S + i^k)$. We then calculate

$$\begin{aligned} tr(V^*TV) &= tr(V^*T^{\frac{1}{2}}T^{\frac{1}{2}}V) \\ &= tr(T^{\frac{1}{2}}VV^*T^{\frac{1}{2}}) \\ &= \sum_j \langle e_j, T^{\frac{1}{2}}VV^*T^{\frac{1}{2}}e_j\rangle \\ &= \sum_j \langle V^*T^{\frac{1}{2}}e_j, V^*T^{\frac{1}{2}}e_j\rangle \\ &\leq \sum_j \|V\|^2\|T^{\frac{1}{2}}e_j\|^2 \\ &= \|V\|^2 tr(T). \end{aligned}$$

Combining these results, we find

$$|tr(TS)| \leq \left|\frac{1}{4}\sum_{k=0}^{3} i^k\|S + i^k\|^2\right| tr(T) < \infty.$$

Therefore, $B_1(H)$ is a right ideal and since it is obviously self-adjoint, it is a two sided ideal. □

As a consequence, we can give another characterization of the trace-class operators. If $T \in B_1(H)$, then, by the polar decomposition, so is $UT = |T|$. If $|T| \in B_1(H)$, then, again by the polar decomposition, so is $T = U^*|T|$. Therefore, we have

$$B_1(H) = \{T \in B(H) \mid tr(|T|) < \infty\}.$$

Lemma 2.24. *Let $T \in B_1(H)$ and $S \in B(H)$. Then*

$$|tr(ST)| \leq \|S\|\, tr(|T|).$$

Proof. The expression $(S,T)_{tr} = tr(TS^*)$ is a sesquilinear form $B(H) \times B_1(H) \to \mathbb{C}$. As such, we have the Cauchy-Schwarz inequality. If $T = U|T|$, then

$$\begin{aligned} |tr(ST)|^2 &= |tr(SU|T|^{\frac{1}{2}}|T|^{\frac{1}{2}})|^2 \\ &= |(|T|^{\frac{1}{2}}, SU|T|^{\frac{1}{2}})_{tr}|^2 \\ &\leq (|T|^{\frac{1}{2}}, |T|^{\frac{1}{2}})_{tr}(SU|T|^{\frac{1}{2}}, SU|T|^{\frac{1}{2}})_{tr} \\ &= tr(|T|)tr(SU|T|^{\frac{1}{2}}(SU|T|^{\frac{1}{2}})^*) \\ &= tr(|T|)tr(|T|^{\frac{1}{2}}U^*S^*SU|T|^{\frac{1}{2}}) \\ &= tr(|T|)\sum_j \langle SU|T|^{\frac{1}{2}}e_j, SU|T|^{\frac{1}{2}}e_j\rangle \\ &= tr(|T|)\sum_j \|SU|T|^{\frac{1}{2}}e_j\|^2 \\ &\leq tr(|T|)\sum_j \|SU\|^2\||T|^{\frac{1}{2}}e_j\|^2 \\ &\leq tr(|T|)\|S\|^2\sum_j \langle |T|^{\frac{1}{2}}e_j, |T|^{\frac{1}{2}}e_j\rangle \\ &= \|S\|^2 tr(|T|)^2. \end{aligned}$$

□

Before we continue, we need to take a little detour and define the *Hilbert-Schmidt* operators, denoted $B_2(H)$.

Definition 2.25.

$$B_2(H) = \{T \in B(H) \mid tr(T^*T) < \infty\}.$$

Lemma 2.26. *The Hilbert-Schmidt operators form a self adjoint ideal in $B(H)$.*

Proof. The earlier parallelogram law implies

$$(S+T)^*(S+T) \leq 2(S^*S + T^*T),$$

which shows that $B_2(H)$ is a linear subspace. From the fact that $tr(T^*T) = tr(TT^*)$, we then find that $B_2(H)$ is self-adjoint. It is clear that $T^*T \in B_1(H)$ whenever $T \in B_2(H)$. Therefore, we find that $tr(S^*T^*TS) < \infty$ whenever $T \in B_2(H)$, showing that $TS \in B_2(H)$ whenever T is. Since we already know $B_2(H)$ is self-adjoint, we are done. □

Lemma 2.27. *If $T, S \in B_2(H)$, or if $T \in B_1(H)$ and $S \in B(H)$, then*

$$tr(TS) = tr(ST).$$

Proof. Straightforward calculation using the polarization formula gives for $S, T \in B_2(H)$:

$$\begin{aligned}
4\,tr(T^*S) &= \sum_{k=0}^{3} i^k tr((S + i^kT)^*(S + i^kT) \\
&= \sum_{k=0}^{3} i^k tr((S + i^kT)(S + i^kT)^*) \\
&= \sum_{k=0}^{3} i^k tr(i^{-k}(S^* + i^{-k}T^*)^* i^k(S^* + i^{-k}T^*)) \\
&= \sum_{k=0}^{3} i^k tr((T^* + i^kS^*)^*(T^* + i^kS^*)) \\
&= 4\,tr(ST^*).
\end{aligned}$$

Now let $T \in B_1(H)$ and $S \in B(H)$. Then, using the previous result, the polar decomposition and the fact that $|T|^{\frac{1}{2}} \in B_2(H)$, we find

$$\begin{aligned}
tr(TS) &= tr(U|T|S) \\
&= tr((U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}S)) \\
&= tr(|T|^{\frac{1}{2}}(SU|T|^{\frac{1}{2}})) \\
&= tr(ST).
\end{aligned}$$

□

This next proposition is only partially relevant to our goal (i.e., the existence of a predual), since we only ask for a predual to be a Banach *space*. However, the full result is too nice to omit here.

Proposition 2.28. *$B_1(H)$ is a Banach algebra (and in particular a Banach space) under the trace-norm $\|\cdot\|_1 = tr(|\cdot|)$.*

Proof. First of all we need to show that $\|\cdot\|_1 = tr(|\cdot|)$ is indeed a norm. Homogeneity is clear and positivity follows from the fact that $\|\cdot\|_1 \geq \|\cdot\|$. The only non-trivial property is the triangle inequality. Let U be the partial isometry from the polar decomposition of $S+T$. Then

$$\begin{aligned} \|S+T\|_1 &= tr(|S+T|) \\ &= tr(U^*(S+T)) \\ &= tr(U^*S) + tr(U^*T) \\ &\leq \|U^*\| tr(|S|) + \|U^*\| tr(|T|) \\ &= \|S\|_1 + \|T\|_1. \end{aligned}$$

Next is the Banach algebra norm-estimate. Let V be the partial isometry from the polar decomposition of ST. Then

$$\begin{aligned} \|ST\|_1 &= tr(V^*ST) \\ &\leq \|V^*S\| tr(|T|) \\ &\leq \||S|\| tr(|T|) \\ &\leq \|S\|_1 \|T\|_1. \end{aligned}$$

Finally, we need to show $B_1(H)$ is complete. So let T_m be a Cauchy sequence with respect to $\|\cdot\|_1$. Then $\|T_n - T_m\| \leq \|T_n - T_m\|_1$, so the T_m converge in norm to some T, which is therefore compact. We now would like to say something about $\|T - T_n\|_1$, but we do not know if this exists. Therefore, let P be a finite rank projection. Since $tr(\cdot)$ is continuous we have

$$\begin{aligned} tr(P|T - T_n|) &= \lim_m tr(P|T_m - T_n|) \\ &\leq \limsup_m \|P\| \|T_m - T_n\|_1 \\ &\leq \limsup_m \|T_n - T_m\|_1. \end{aligned}$$

Since this holds for any finite rank projection P, we find

$$\|T - T_n\|_1 \leq \limsup_m \|T_n - T_m\|_1 \to 0.$$

So $T \in B_1(H)$, which is therefore a Banach algebra. □

Now we are finally ready to show that $B(H)$ has a predual. Given all the work we have done so far, it will come as no surprise that this predual is precisely the space $B_1(H)$ of trace-class operators.

Proposition 2.29. *There is an isometric isomorphism between $B(H)$ and $B_1(H)^*$.*

Proof. Given $S \in B(H)$, we define $\psi_S \in B_1(H)^*$ as $\psi_S(T) = tr(TS)$. The map $S \mapsto \psi_S$ is isometric. Indeed, we have

$$\begin{aligned} \|\psi_S\| &= \sup\{|tr(ST)| \mid T \in B_1(H), \|T\|_1 \leq 1\} \\ &\leq \sup\{\|S\|\|T\|_1 \mid T \in B_1(H), \|T\|_1 \leq 1\} \\ &= \|S\|. \end{aligned}$$

While we also have, since the projections P_h (projecting on $h \in H$) are trace class,

$$\begin{aligned}
\|\psi_S\| &= \sup\{|tr(ST)| \mid T \in B_1(H), \|T\|_1 \leq 1\} \\
&\geq \sup\{|tr(SP_h)| \mid \|h\| \leq 1\} \\
&= \sup\{|\langle h, Sh\rangle| \mid \|h\| \leq 1\} \\
&= \|S\|.
\end{aligned}$$

The map $S \mapsto \psi_S$ is also injective, since if $\psi_S = \psi_{\tilde{S}}$, then again, using the projections P_h, we have $\langle h, Sh\rangle = \langle h, \tilde{S}h\rangle$ for all $h \in H$, so $S = \tilde{S}$.
To show this map is also surjective, let $\psi \in B_1(H)^*$. We need to find an operator $S \in B(H)$ such that $\psi = \psi_S$. Now, any $\psi \in B_1(H)^*$ gives rise to a sesquilinear form B_ψ via $B_\psi(x, y) = \psi(|y\rangle\langle x|)$, for $x, y \in H$. There is also an isometric bijective correspondence between sesquilinear forms B on H and elements S_B in $B(H)$, given by $\langle x, S_B y\rangle = B(x, y)$. So, from ψ, we can form the element $S_{B_\psi} \in B(H)$. We then find that

$$\begin{aligned}
\psi_{S_{B_\psi}}(T) &= tr(S_{B_\psi}T) \\
&= \sum_i \langle e_i, S_{B_\psi}Te_i\rangle \\
&= \sum_i \langle e_i, S_{B_\psi}\left(\sum_{j,k} \lambda_{j,k}|e_j\rangle\langle e_k|\right) e_i\rangle \\
&= \sum_{j,k} \lambda_{j,k}\langle e_k, S_{B_\psi}e_j\rangle \\
&= \sum_{j,k} \lambda_{j,k}\psi(|e_j\rangle\langle e_k|) \\
&= \psi(T).
\end{aligned}$$

□

2.3 Predual of a von Neumann algebra

Now that we know that $B(H)$ has a predual, we are going to show that each von Neumann algebra has a unique predual. In other words, we will show that every von Neumann algebra is a W^*-algebra. Since the case $B(H)$ is already known, we will, when convenient, assume that the von Neumann algebras below are proper subalgebras of $B(H)$.
The relation $B_1(H)^* \cong B(H)$ allows us to define the concept of *annihilators*. Given a subset $\mathcal{A} \subset B(H)$, we define a subset $\mathcal{A}^\perp \subset B_1(H)$ as

$$\mathcal{A}^\perp = \{T \in B_1(H) \mid tr(Ta) = 0 \, \forall a \in \mathcal{A}\}.$$

Likewise, given a subset $\mathcal{B} \subset B_1(H)$ we define a subset $\mathcal{B}^\perp \subset B(H)$ as

$$\mathcal{B}^\perp = \{x \in B(H) \mid tr(xb) = 0 \, \forall b \in \mathcal{B}\}.$$

These spaces are called the *annihilator* of $\mathcal{A}$, $\mathcal{B}$, respectively. We are interested in the setting where $\mathcal{A} \subset B(H)$ is a von Neumann algebra and $\mathcal{B} = \mathcal{A}^\perp \subset B_1(H)$. So from now on we assume this (even though it is not necessary for all of the arguments).

Lemma 2.30. *$\mathcal{A}^\perp$ is norm-closed in $B_1(H)$ with respect to the norm $\|\cdot\|_1$.*

Proof. Let T_n be Cauchy in $\mathcal{A}^\perp$. Then, as a subset of the compact operators, we have $T_n \to T$ for some compact T. We now calculate, for $a \in \mathcal{A}$

$$\begin{aligned} |tr(Ta)| &= |tr(Ta - T_n a + T_n a)| \\ &= |tr((T - T_n)a) + 0| \\ &\leq \|a\| tr(|T - T_n|) \to 0. \end{aligned}$$

□

Corollary 2.31. *$B_1(H)/\mathcal{A}^\perp$ is again a Banach space.*

Lemma 2.32. *Let $\mathcal{A}$ be a von Neumann algebra. Then $\mathcal{A} = (\mathcal{A}^\perp)^\perp$ (which we just denote by $\mathcal{A}^{\perp\perp}$).*

Proof. First of all, it is clear that $\mathcal{A} \subset (\mathcal{A}^\perp)^\perp$. Now let $a \notin \mathcal{A}$. Since $\mathcal{A}$ is σ-weakly closed, its complement $B(H)\backslash\mathcal{A}$ is σ-weakly open. Therefore, by the definition of the σ-weak topology, we can find $\phi \in B_1(H)$ and $\epsilon > 0$ such that

$$\{b \in B(H) \mid |\phi(a) - \phi(b)| < \epsilon\} \subset B(H) - \mathcal{A}.$$

This means that if $b \in \mathcal{A}$, then $|\phi(a) - \phi(b)| \geq \epsilon$. What's more, since $\mathcal{A}$ is a linear space, $|\phi(a) - \lambda\phi(b)| \geq \epsilon$ for each $\lambda \in \mathbb{C}$. This is, of course, only possible if $\phi(b) = 0$, which shows that $\phi \in \mathcal{A}^\perp$. It also shows that $|\phi(a)| \geq \epsilon$, so that $a \notin \mathcal{A}^{\perp\perp}$. □

In what follows, if A, B are Banach algebras and $f : A \to B$ is a bounded linear map of Banach spaces, we denote by f^* the Banach space adjoint map $f^* : B^* \to A^*$ such that

$$< x, f^*\phi > = < fx, \phi >,$$

where $x \in A$, $\phi \in B^*$ and $< \cdot, \cdot >$ is the pairing between a Banach space and its dual. We shall often just write $\phi(b) = < b, \phi >$ to denote such a pairing, so that the equation for the Banach space adjoint map becomes

$$f^*\phi(x) = \phi(fx).$$

Lemma 2.33. *For $\mathcal{A} \subset B(H)$ a von Neumann algebra, let Q denote the canonical quotient map*

$$Q : B_1(H) \to B_1(H)/\mathcal{A}^\perp.$$

Then we may identify Q^* with the inclusion map of $\mathcal{A}$ into $B(H)$.

Proof. Let $\phi \in (B_1(H)/\mathcal{A}^\perp)^*$, then for any $T \in \mathcal{A}^\perp$ we have

$$Q^*\psi(T) = \psi(QT) = 0.$$

Therefore, $Q^*\psi \in \mathcal{A}^{\perp\perp} = \mathcal{A}$. Now Q is a quotient map and hence it maps the open unit ball of $B_1(H)$ onto the open unit ball of $B_1(H)/\mathcal{A}^\perp$. Therefore, we find that Q^* is isometric (and hence also injective), since

$$\begin{aligned} \|Q^*\psi\| &= \sup\{|Q^*\psi(S)| | S \in B_1(H), \|S\|_1 < 1\} \\ &= \sup\{|\psi(QS)| | S \in B_1(H), \|S\|_1 < 1\} \\ &= \|\psi\|. \end{aligned}$$

Furthermore, $a \in \mathcal{A}^{\perp\perp} = \mathcal{A}$ gives rise to a map $\phi_a \in (B_1(H)/\mathcal{A}^\perp)^*$ given by

$$\phi_a([T]) = tr(Ta),$$

where $[T]$ is the class of T in $B_1(H)/\mathcal{A}^\perp$. This map is easily seen to be well defined. Furthermore, it satisfies

$$< T, a >= tr(Ta) = \phi_a(QT) =< QT, \phi_a >,$$

for $T \in B_1(H)$. From this which we conclude that $a = Q^*\psi_a$, so Q^* is surjective. □

The above proposition states that we can identify the Banach adjoint map

$$Q^* : (B_1(H)/\mathcal{A}^\perp)^* \to B_1(H)^* = B(H)$$

with the inclusion map $\mathcal{A} \hookrightarrow B(H)$. From this we arrive at our desired conclusion:

Proposition 2.34. *Every von Neumann algebra $\mathcal{A}$ in $B(H)$ has a predual $\mathcal{A}_*$, which is given by*

$$\mathcal{A}_* = B_1(H)/\mathcal{A}^\perp.$$

Thus the predual of $\mathcal{A}$ is denoted by $\mathcal{A}_*$, so that $\mathcal{A} \cong (\mathcal{A}_*)^* = \mathcal{A}_*{}^*$. Under the identification of a class of trace class operators $[T]$ in $B_1(H)/\mathcal{A}^\perp$ and the corresponding linear functional $tr(T\cdot)$, $\mathcal{A}_*$ is a subspace of the Banach dual space $\mathcal{A}^*$ of $\mathcal{A}$. In fact, we now show that $\mathcal{A}_* \subset \mathcal{A}^*$ consists precisely of the ultraweakly continuous linear functionals on $\mathcal{A}$.

Proposition 2.35. *Let $\mathcal{A}$ be a von Neumann algebra with predual $\mathcal{A}_* \subset \mathcal{A}^*$ equipped with the associated ultraweak topology. This is the topology defined by $a_i \to a$ if $\phi(a_i) \to \phi(a)$ for all $\phi \in \mathcal{A}_*$. Then the dual of $\mathcal{A}$ (in this topology) is precisely $\mathcal{A}_*$.*

Proof. Of course, $\mathcal{A}_*$ is contained in the dual of $\mathcal{A}$ with respect to the ultraweak topology. Now let f be a linear functional, continuous with respect to the ultraweak topology on $\mathcal{A}$. Then there are $\phi_1, \dots, \phi_n \in \mathcal{A}_*$, such that

$$f(a) \leq \sum_{i=1}^{n} |\phi_i(a)|,$$

for all $a \in \mathcal{A}$ (if this were not the case, then f would not be bounded, hence it could not be continuous). In particular:

$$\bigcap_i \ker(\phi_i) \subset ker(f).$$

We can take the ϕ_i to be linearly independent, so there are elements $a_i \in \mathcal{A}$ such that $\phi_j(a_i) = \delta_{i,j}$. Now let $x \in \mathcal{A}$ and define $y = x - \sum_i \phi_i(x)a_i$. Then $\phi_j(y) = 0$ for all j, so,

$$\begin{aligned} 0 &= f(y) \\ &= f(x) - \sum_i \phi_i(x) f(a_i). \end{aligned}$$

So f is a linear combination of the ϕ_i and $f \in \mathcal{A}_*$. □

In particular, we never used the explicit form of the predual we have found for a von Neumann algebra. This means that any predual is isomorphic to the space of all ultraweakly continuous linear functionals on $\mathcal{A}$. In particular:

Proposition 2.36. *The predual of a von Neumann algebra is unique up to isomorphism.*

So we can justly speak of *the* predual, instead of just *a* predual.

2.4 Equivalence of von Neumann algebras and W^*-algebras

We have now seen that every von Neumann algebra is in fact a W^*-algebra. To see the converse, i.e., that every W^* algebra is a von Neumann algebra, we need to find, given a W^* algebra $\mathcal{A}$, some Hilbert space H such that $\mathcal{A}$ acts on this space as a von Neumann algebra. So we need to find a representation $\pi : \mathcal{A} \to B(H)$ that is faithful and such that $\pi(\mathcal{A})$ is a von Neumann algebra. Any such representation will do, the point is that we need to make sure such a representation does indeed always exist. We will now give a sketch of this and refer to [6], [11] for more details.

Definition 2.37. *Let $\mathcal{A}$ be a unital C^*-algebra. A linear functional on $\phi : \mathcal{A} \to \mathbb{C}$ on $\mathcal{A}$ is called*

- positive *if $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$,*
- normalized *if $\phi(1) = 1$,*
- *a* state *it it is both positive and normalized,*
- faithful *if $\phi(a^*a) = 0$ implies $a = 0$,*
- normal *if for any bounded increasing net op positive operators a_i, we have $\phi(sup_i a_i) = sup_i \phi(a_i)$,*
- completely additive *if for any orthogonal family of projections p_i we have $\phi(\sum_i p_i) = \sum_i \phi(p_i)$.*

It turns out that for a von Neumann algebra, the concepts of being normal and being completely additive coincide, and in fact are both equivalent to being ultraweakly continuous. Any state which satisfies any (and hence all) of these conditions will be referred to as a *normal state.*
A state in $\mathcal{A}_*$ can be used to construct a cyclic representation of $\mathcal{A}$ via the so-called *GNS-construction* as follows.
Let ϕ be a non-zero state in $\mathcal{A}_*$ and let $\phi^\perp$ be the subspace of $\mathcal{A}$ given by

$$\phi^\perp = \{a \in \mathcal{A} \mid \phi(a^*a) = 0\}.$$

Then $\mathcal{A}/\phi^\perp$ is a pre-Hilbert space under the inner product defined by

$$\langle [x], [y] \rangle = \phi(x^*y).$$

Then $\mathcal{A}$ acts on this pre-Hilbert space by left multiplication, i.e., $a[x] = [ax]$, where of course $[\cdot]$ denotes the class of a given element. Now we can complete this pre-Hilbert

space in the norm given by this inner product. We have then obtained a representation of $\mathcal{A}$ which we denote $\{\pi_\phi, H_\phi\}$.
This representation always has a *cyclic vector*, i.e., a vector Ω such that $\pi_\phi(\mathcal{A})\Omega$ is dense in H_ϕ. With respect to this vector we have

$$\phi(a) = \langle \Omega, \pi_\phi(a)\Omega \rangle.$$

The existence of such an Ω is easy if $\mathcal{A}$ is unital (which is the case since $\mathcal{A}$ is a von Neumann algebra), since we can just take Ω to be the class of the unit. For a general C^*-algebra, we can take any linear functional to perform this construction and Ω will be the class of an approximate unit.

For $0 \neq a \in \mathcal{A}$, since $\mathcal{A}$ is the dual space of its predual, there exists a $\phi_a \in \mathcal{A}_*$ such that $\phi_a(a) \neq 0$. In fact, we can always take ϕ_a to be a state. Taking the direct sum of all the representations $\{\pi_{\phi_a}, H_{\phi_a}\}$ corresponding to such ϕ_a gives a representation $\{\pi, H\}$, in which $\mathcal{A}$ acts faithfully by construction. Likewise we could take the direct sum of all representations derived from all states.
The final thing we need to know is that $\mathcal{A}$ acts as a von Neumann algebra. This is the case if π is normal, which in turn is the case if each π_ϕ is normal. So this amounts to the question whether π_ϕ is normal if ϕ is normal. Indeed, if pi_ϕ is normal, it preserves all ultraweak limits, so the resulting space is ultraweakly closed and therefore a von Neumann algebra by the double commutant theorem.

Lemma 2.38. *A map $\pi : \mathcal{A} \to \mathcal{B}$ of W^*-algebras is normal if and only if its dual $\pi^* : \mathcal{B}^* \to \mathcal{A}^*$ maps $\mathcal{B}_*$ into $\mathcal{A}_*$.*

Proof. If we denote a limit in the ultraweak topology by σ-lim, then for any $\phi \in B_*$ we have

$$\begin{aligned}
\phi(\sigma\text{-}\lim_i \pi(a_i)) &= \sigma\text{-}\lim_i \phi(\pi(a_i)) \\
&= \sigma\text{-}\lim_i \pi^*(\phi)(a_i) \\
&= \pi^*(\phi)(\sigma\text{-}\lim_i a_i) \\
&= \phi(\pi(\sigma\text{-}\lim_i a_i)).
\end{aligned}$$

So $\pi(\sigma\text{-}\lim_i a_i) = \sigma\text{-}\lim_i \pi(a_i)$. Reading the calculation in another order proves the converse. □

Proposition 2.39. *The cyclic representation π_ϕ associated to a normal state ϕ is normal.*

Proof. Let ϕ be a normal state on $\mathcal{A}$ and let π_ϕ be the corresponding cyclic representation. Let $\rho_{f,g}$ be the functional on $\pi_\phi(\mathcal{A})''$ defined by $\rho_{f,g}(a) = \langle f, \pi_\phi(a)g \rangle$, for f, g in H_ϕ. Then, since we can approximate any vector by $\pi_\phi(a_n)\Omega$ for certain elements $a_n \in \mathcal{A}$, we have for the Banach space dual π_ϕ^* and for $x \in \mathcal{A}$, that

$$\begin{aligned}
\pi_\phi^*(\rho_{f,g})(x) &= \rho_{f,g}(\pi_\phi(x)) \\
&= \langle f, \pi_\phi(x)g \rangle \\
&= \lim_n \langle \pi_\phi(a_n)\Omega, \pi_\phi(x)\pi_\phi(b_n)\Omega \rangle \\
&= \lim_n \pi_\phi(b_n^*)\phi\pi_\phi(a_n)(x),
\end{aligned}$$

where $a\phi b$ is is the functional $a\phi b(x) = \phi(bxa)$, which is normal since multiplication is ultra-weakly continuous. So π_ϕ^* maps normal states to normal states. That is to say, it maps the predual of $\pi_\phi(\mathcal{A})''$ into the predual of $\mathcal{A}$. From this and the previous lemma we conclude that π_ϕ is normal. □

We conclude that indeed every von Neumann algebra is a W^*-algebra, and *vice versa*. From a categorical point of view we even have the following:

Proposition 2.40. *The category of von Neumann algebras and the category of W^*-algebras are equivalent.*

Proof. Let us explicitly distinguish the category of von Neumann algebras **vNA**, and the category of W^*-algebras $\mathbf{W}^*$. We can then define two functors:

$$U : \mathbf{vNA} \to \mathbf{W}^*,\ \mathcal{M} \subset B(H) \mapsto \mathcal{M},$$

forgetting the concrete Hilbert space representation, and

$$R : \mathbf{W}^* \to \mathbf{vNA},\ \mathcal{N} \mapsto \mathcal{N} \subset B(\oplus_\phi H_\phi),$$

representing $\mathcal{N}$ on the universal representation. Then $U \circ R = id_{\mathbf{W}^*}$, while for $R \circ U$ we have, if $f : \mathcal{M} \subset B(H) \to \mathcal{N} \subset B(K)$,

$$\begin{array}{ccc} \mathcal{M} \subset B(H) & \xrightarrow{RU_\mathcal{M}} & \mathcal{M} \subset B(\oplus_\phi H_\phi) \\ \downarrow f & & \downarrow f \\ \mathcal{N} \subset B(K) & \xrightarrow[RU_\mathcal{N}]{} & \mathcal{N} \subset B(\oplus_\xi H_\xi) \end{array}$$

Per construction, the Hilbert space structure on the right side of the diagram is precisely the algebraic structure on the left side of the diagram. Therefore the maps $RU_\mathcal{M}$ and $RU_\mathcal{N}$ are just the identity on $\mathcal{M}$ and $\mathcal{N}$, respectively, so that the diagram commutes. This shows that **vNA** and $\mathbf{W}^*$ are equivalent categories. □

Because of this, we will often interchange (and have interchanged) the terms W^*-algebra and von Neumann algebra.

3 The Category of von Neumann Algebras

We now study the category $\mathbf{W}^*$ of von Neumann algebras, where the objects are von Neumann algebras and the morphisms are unital normal *-homomorphisms. We now try to construct most of the basic categorical constructions known for sets in $\mathbf{W}^*$.

3.1 Products

Before looking at products we are going to see if $\mathbf{W}^*$ has a terminal object.

Proposition 3.1. $\mathbf{W}^*$ *has a terminal object.*

Proof. A little thought shows that the only possibility is the von Neumann algebra 0 acting on $B(0)$. Then for any von Neumann algebra $\mathcal{M}$ we have the map $0 : \mathcal{M} \to 0$, $m \mapsto 0$, which is obviously unique. □

When we look at the dual category, we find (since 0 is abelian) that it corresponds to the set of non-zero homomorphisms from 0 to $\mathbb{C}$, which is empty. This is precisely what we expect, since the empty set is the initial object in the category of compact Hausdorff spaces.
One might be inclined to think that 0 is also initial via the map $0 \mapsto 0 \in \mathcal{M}$, but since 0 is also the unit ($0 \cdot 0 = 0$), it should also map to $1 \in \mathcal{M}$. Therefore the only W^*-morphism going from 0 is its identity.
Our next aim will be the construction of (small) products, but before we do so, we should first say something about uncountable sums. Whenever I is an uncountable index set, we want to give meaning to the expression $\sum_{i\in I} a_i$, for $a_i \in \mathbb{C}$.

Definition 3.2. *Let I be any (in particular an uncountable) set and $\{a_i\}_{i\in I}$ non-negative real numbers. Then their sum is*

$$\sum_{i\in I} a_i := \sup_{S\subset I} \sum_{i\in S} a_i,$$

where the supremum is taken over all finite sets $S \subset I$.

Lemma 3.3. *This sum can only be finite if the number of non-zero elements is at most countable.*

Proof. The number of elements $x_i \geq 1$ should obviously be finite. Now for any $n \geq 1$ the set $\{i \in I \mid \frac{1}{n+1} \leq x_i < \frac{1}{n}\}$ should also be finite for the sum to be finite. The union of all these sets is exactly the set of all elements greater than zero and it is a countable union of finite sets and therefore countable. □

For arbitrary complex numbers a_i, we have four sets of non-negative real numbers consisting of the positive real parts, negative real parts, positive imaginary parts and negative imaginary parts of the a_i.

Definition 3.4. *Let I be any set, and $a_i, i \in I$ complex numbers. If $\sum_i |a_i|$ converges (in the sense of Definition 3.2), then $\sum_i a_i$ is defined as the sum of the four sums over the above mentioned sets.*

If $\sum_i |a_i|$ does not converge, the original sum has no meaning.

Proposition 3.5. *The category* $\mathbf{W}^*$ *has small products.*

Proof. Let some family $\{\mathcal{M}_\alpha\}_{\alpha\in I}$ of von Neumann algebras be given, together with their respective preduals $\mathcal{M}_{\alpha *}$, for some index set I. Define

$$\bigoplus_\alpha \mathcal{M}_\alpha = \left\{ m : I \to \bigcup_\alpha \mathcal{M}_\alpha \;\middle|\; m(\alpha) \in \mathcal{M}_\alpha,\ \sup_\alpha \|m(\alpha)\| < \infty \right\},$$

and

$$\left(\bigoplus_\alpha \mathcal{M}_\alpha\right)_* = \left\{ \mu : I \to \bigcup_\alpha (\mathcal{M}_\alpha)_* \;\middle|\; \mu(\alpha) \in (\mathcal{M}_\alpha)_*,\ \sum_\alpha \|\mu(\alpha)\| < \infty \right\}.$$

Our claim is now that $(\bigoplus_\alpha \mathcal{M}_\alpha)_*$ is the predual of the von Neumann algebra $\bigoplus_\alpha \mathcal{M}_\alpha$. First of all, we check that $\bigoplus_\alpha \mathcal{M}_\alpha$ is a C^*-algebra.
Define a norm $\|m\| = \sup_\alpha \|m(\alpha)\|$ and pointwise multiplication, addition and involution. We then find

$$\|mn\| = \sup_\alpha \|m(\alpha)n(\alpha)\| \le \sup_\alpha \|m(\alpha)\|\|n(\alpha)\| \le \|m\|\|n\|,$$

and

$$\|m^*m\| = \sup_\alpha \|m(\alpha)^*m(\alpha)\| = \sup_\alpha \|m(\alpha)\|^2 = \|m\|^2.$$

Now let m_i be a Cauchy sequence in $\bigoplus_\alpha \mathcal{M}_\alpha$. Then

$$\sup_\alpha \|m_i(\alpha) - m_j(\alpha)\| \to 0$$

implies

$$\|m_i(\alpha) - m_j(\alpha)\| \to 0$$

for every α. Since every $\mathcal{M}_\alpha$ is complete, each such Cauchy sequence has a limit, which we denote by $m(\alpha)$. This defines an element $m \in \bigoplus_\alpha \mathcal{M}_\alpha$. Now for any $\epsilon > 0$ we can find an $i \in I$ such that $\|m(\alpha)\| \le \|m_i(\alpha)\| + \epsilon$. Therefore $\|m\| < \infty$ and $\bigoplus_\alpha \mathcal{M}_\alpha$ is a C^*-algebra.
Because of the calculation

$$m(\mu) = \sum_\alpha m(\alpha)\big(\mu(\alpha)\big) \le \sup_\alpha \|m(\alpha)\| \sum_\alpha \|\mu(\alpha)\| < \infty,$$

we find

$$\bigoplus_\alpha \mathcal{M}_\alpha \subset (\bigoplus_\alpha \mathcal{M}_\alpha)_*{}^*.$$

Now let $\phi \in (\bigoplus_\alpha \mathcal{M}_\alpha)_*{}^*$. For a $\mu \in (\bigoplus_\alpha \mathcal{M}_\alpha)_*$, we know by the previous lemma that there are at most a countable number of α such that $\mu(\alpha) \neq 0$. We write $\phi(\mu) = \sum_{\mu(\alpha)\neq 0} \phi(\mu_\alpha)$ where $\mu_\alpha(\beta) = 0$ if $\alpha \neq \beta$ and $\mu_\alpha(\alpha) = \mu(\alpha)$. So μ_α can be identified with $\mu(\alpha) \in (\mathcal{M}_\alpha)_*$. Under this identification ϕ decomposes as a direct sum of elements in $(\mathcal{M}_\alpha)_*{}^* = \mathcal{M}_\alpha$. So indeed

$$\bigoplus_\alpha \mathcal{M}_\alpha = (\bigoplus_\alpha \mathcal{M}_\alpha)_*{}^*.$$

The projections are the obvious ones

$$p_\alpha : \begin{array}{ll} \bigoplus_\alpha & \mathcal{M}_\alpha \to \mathcal{M}_\alpha, \\ m & \mapsto m(\alpha). \end{array}$$

To see they are w^*-continuous, suppose $m_i \xrightarrow{w^*} m$, so that $\phi(m_i) \to \phi(m)$ for all ϕ in the predual. We want to show $P_\alpha(w_i) \xrightarrow{w^*} P_\alpha(w)$. That is to say, $w_i(\alpha) \xrightarrow{w^*} w(\alpha)$. This is easily seen by using the elements μ_α as before. The final thing to show is that $\bigoplus_\alpha \mathcal{M}_\alpha$ indeed satisfies the categorical property of a product. So let $\mathcal{W}$ be a W^*-algebra and $q_\alpha : \mathcal{W} \to \mathcal{M}_\alpha$ a set of morphisms. We define $Q : \mathcal{W} \to \bigoplus_\alpha \mathcal{M}_\alpha$ as $Q(w) = m_w$, where $m_w(\alpha) = q_\alpha(w)$. Then we certainly have $p_\alpha Q = q_\alpha$, and this property fixes Q, so it is unique. If $w_i \xrightarrow{w^*} w$, we have

$$f_{w_i}(\mu) = \sum_\alpha f_{w_i}(\alpha)(\mu(\alpha)) \to \sum_\alpha f_w(\alpha)(\mu(\alpha)) = f_w(\mu).$$

Finally, since Q is obviously unital, it is indeed a W^*-morphism. □

If each $\mathcal{M}_\alpha$ acts on some Hilbert space H_α, $\alpha \in I$, there is another way to see that the direct sum of these von Neumann algebras is again a von Neumann algebra. To this end we form the direct sum Hilbert space

$$\bigoplus_\alpha H_\alpha = \left\{ (\psi_i)_{i \in I} \,\middle|\, \psi_i \in H_i, \sum_i \|\psi_i\|^2 < \infty \right\},$$

with componentwise addition, scalar multiplication and inner product. Now $\bigoplus_\alpha \mathcal{M}_\alpha$ acts on this Hilbert space via $m((\psi_i)_{i\in I}) = (m(i)(\psi_i))_{i \in I}$. Then $\bigoplus_\alpha \mathcal{M}_\alpha$ is a (unital) subset of $B(\bigoplus_\alpha H_\alpha)$. Let X be a operator in the strong closure of $\bigoplus_\alpha \mathcal{M}_\alpha$. Let P_i be the projection of $\bigoplus_\alpha H_\alpha$ onto H_i, so $P_i : I \to \bigcup_\alpha \mathcal{M}_\alpha$, $P_i(\alpha) = 1$ if $\alpha = i$ and $P_i(\alpha) = 0$ otherwise. We see that for each $i \in I$, P_i is in the center of $\bigoplus_\alpha \mathcal{M}_\alpha$. By the double commutant theorem, X then commutes with P_i for all i. Now let X_i be some sequence in $\bigoplus_\alpha \mathcal{M}_\alpha$ converging strongly to X. Then we find

$$\|(X_i P_j - X P_j)\psi\| = \|(X_i - X) P_j \psi\| \to 0,$$

so XP_j is also an element of the strong closure of $\bigoplus_\alpha \mathcal{M}_\alpha$. But XP_j is just the restriction X_j of X to H_j, so since $\mathcal{M}_j$ is strongly closed, we conclude that X_j is in $\mathcal{M}_j$ for all j. So X is in $\bigoplus_\alpha \mathcal{M}_\alpha$, which is therefore strongly closed and hence a von Neumann algebra.

3.2 Coproducts

First of all, we note that the von Neumann algebra $\mathbb{C}$ is initial in $\mathbf{W}^*$, since the only *-homomorphisms from $\mathbb{C}$ to a general von Neumann algebra can be the map $z \mapsto z1$.[1]

[1] We exclude the empty set as being a von Neumann algebra.

Another way to see this is to note that $\mathbb{C}$, as an von Neumann algebra, is dually equivalent to a compact Hausdorff space consisting of one point, which is final in the category of compact Hausdorff spaces with continuous functions.

To construct the coproduct of a family of von Neumann algebras, we will look at its universal property. Therefore, let $(\mathcal{M}_\alpha)_{\alpha\in I}$ be such a family and let $(\rho_\alpha)_{\alpha\in I}$ be a family of morphisms $\rho_\alpha : \mathcal{M}_\alpha \to \mathcal{R}$ into some von Neumann algebra $\mathcal{R}$. Then there exists a von Neumann algebra $\mathcal{N} \subset \mathcal{R}$ such that $\mathcal{N}$ is generated by the images of the ρ_α. If now $\mathcal{M}$ is a coproduct, $\mathcal{N}$ should, in some sense, be part of $\mathcal{M}$. This will be the way we construct the coproduct, but before we do that, we need to be sure that $\mathcal{M}$ will not have "too many parts". The following lemma will be the key argument for this.

Lemma 3.6. *A von Neumann algebra $\mathcal{M}$ generated by κ many elements, for some cardinal number κ, has a faithful representation on a Hilbert space of dimension at most $2^{\aleph_0\kappa}$.*

Proof. Let $\mathcal{M}$ be a von Neumann algebra generated by κ elements. This is the smallest von Neumann algebra containing all finite products of these generators and their adjoints, in other words, it is the σ-weak closure of all words over these generators and their adjoints. This set of words has cardinality $\aleph_0\kappa$. Now any normal state $\mathcal{M} \to \mathbb{C}$ is precisely determined by its action on these words. So there are at most $|\mathbb{C}|^{\aleph_0\kappa} = 2^{\aleph_0\cdot\aleph_0\kappa} = 2^{\aleph_0\kappa}$ normal states. The Hilbert space associated with the GNS representation of such a normal state is the closed linear span of the $\aleph_0\kappa$ words and therefore has dimension at most $\aleph_0\kappa$. If we now take the direct sum of all these normal GNS representations, we find a faithful representation on a Hilbert space of dimension at most $\aleph_0\kappa \cdot 2^{\aleph_0\kappa} = 2^{\aleph_0\kappa}$ □

Theorem 3.7. $\mathbf{W}^*$ *has all small coproducts.*

Proof. Let $(\mathcal{M}_\alpha)_{\alpha\in I}$ again be a family of W^*-algebras. If $i_\alpha : \mathcal{M}_\alpha \to \mathcal{N}$ is a cocone, we call it *generating* if $\mathcal{N}$ is generated by the images of the i_α. By the above lemma, each such generating cocone $\mathcal{N}$ can be represented faithfully on a Hilbert space of dimension at most $\lambda = 2^{\aleph_0 \sum \kappa_\alpha}$, where κ_α is the cardinality of the generators of $\mathcal{M}_\alpha$. Since Hilbert spaces of the same dimension are isomorphic, we may restrict our attention to those cocones for which the codomain is represented on a fixed Hilbert space H of dimension no greater than λ. There is a set of algebras $\mathcal{M}_\alpha$ and for each $\mathcal{M}_\alpha$ we have a set $Hom(\mathcal{M}_\alpha, B(H))$ of morphisms. Therefore, there is a set of generating cocones. We call this set S and write $s = \{i^s_\alpha : \mathcal{M}_\alpha \to \mathcal{N}_s\}$ for such a generating cocone.
Now consider the morphisms $j_\alpha : \mathcal{M}_\alpha \to \bigoplus_{s\in S} \mathcal{N}_s$ defined by $j_\alpha(m) = \oplus_{s\in S} i^s_\alpha(m)$ for $m \in \mathcal{M}_\alpha$. Finally we define $\mathcal{M}$ to be the W^*-algebra generated by the images of the j_α in $\bigoplus_{s\in S} \mathcal{N}_s$. We now show that this is indeed a coproduct for the $\mathcal{M}_\alpha$.
Let $\{\rho_\alpha\}_{\alpha\in I}$ be any family of morphisms $\rho_\alpha : \mathcal{M}_\alpha \to \mathcal{R}$ for some W^*-algebra $\mathcal{R}$. As mentioned before, $\mathcal{R}$ contains a subalgebra $\mathcal{N}$ generated by the images of the ρ_α. These morphisms ρ_α, together with $\mathcal{N}$, form a generating cocone for the $\mathcal{M}_\alpha$. Therefore, there is an $s \in S$ such that $s = \{\rho_\alpha\}$. Let π_s be the projection of $\bigoplus_{t\in S} \mathcal{N}_t$ onto $\mathcal{N}_s = \mathcal{N}$. Then we see that $\pi_s j_\alpha = i^s_\alpha = \rho_\alpha$.
Now suppose there is a $\pi : \mathcal{M} \to \mathcal{R}$ such that $\pi j_\alpha = \rho_\alpha$, then of course π coincides with π_s on the images of the j_α. Since these images generate $\mathcal{M}$ and π is a W^*-morphism, π must equal π_s. □

Before we go on, we note that the coproduct of two commutative von Neumann algebras is in general not commutative any more. For this we shall look at $\mathbb{C}^2 * \mathbb{C}^2$ ($\mathbb{C}^2$ being the functions on a two-point set). Now $\mathbb{C}^2$ is generated by a single element, which we can take to be $(1,2)$. So any map $f : \mathbb{C}^2 \to \mathcal{M}$ is determined if we know the image of $(1,2)$. Since our morphisms are unital, the image of $(1,1)$ is already fixed and so the map only depends on the image of $(0,1)$, which is a projection, so the image has to be a projection as well.
By construction of the coproduct, we take a direct sum over "all" images of morphisms from $\mathbb{C}^2$, so in particular it contains a subalgebra of, for example, the 2×2 matrices, generated by two noncommuting projections.

3.3 Limits and Colimits

Now we can look at general limits (colimits). Since W^* has all products (coproducts), by Proposition 1.20 (and its dual statement), it suffices to construct equalizers (co-equalizers).

Proposition 3.8. $\mathbf{W}^*$ *has equalizers.*

Proof. Let $f, g : \mathcal{M} \to \mathcal{N}$ be given. Define $E = \{m \in \mathcal{M} \mid f(m) = g(m)\}$ and let i be the inclusion of E in $\mathcal{M}$. By construction, we have $fi = gi$. If Z is any von Neumann algebra with a morphism h such that $fh = gh$, then $h(z) \in E$ for any $z \in Z$ and $ih = h$, as it should. Since E contains the unit of $\mathcal{M}$, i is unital, and since i is an inclusion map, $h : Z \to E$ is unique. Finally, we need to show E is a von Neumann algebra. Indeed, let $e_i \xrightarrow{w^*} e$, with all $e_i \in E$. Then, since f and g are W^*-morhisms, they preserve ultraweak limits, so

$$f(e) = \sigma\text{-}\lim_i f(e_i) = \sigma\text{-}\lim_i g(e_i) = g(e).$$

□

Proposition 3.9. $\mathbf{W}^*$ *has coequalizers.*

Proof. Let $f, g : \mathcal{M} \to \mathcal{N}$ be given. We can look at the set $\{f(m) - g(m) \mid m \in \mathcal{M}\} \subset \mathcal{N}$. Let $\mathcal{I}$ be the ideal generated by this set. We define $q : \mathcal{N} \to Q = \mathcal{N}/\mathcal{I}$ as the quotient map to the quotient algebra. Now by construction we have $qf = qg$.
If $h : \mathcal{N} \to Z$ is any morphism such that $hf = hg$, then let $\bar{q} : Q \to Z$ be the map $\bar{q} : q([n]) \mapsto h(n)$, where $[n]$ is the class of n in Q. This map is well defined since Q is the quotient algebra. Furthermore, since $hf = hg$, h must vanish on $\mathcal{I}$ and therefore it must take the same values on the quotient classes, showing that $\bar{q}$ is unique. □

We have thus proven:

Proposition 3.10. $\mathbf{W}^*$ *has all general limits and colimits.*

However, $\mathbf{W}^*$ is not (co)cartesian closed. Indeed:

Proposition 3.11. $\mathbf{W}^*$ *does not have all coexponentials.*

Proof. Suppose $\mathbf{W}^*$ would have all coexponentials. Then the opposite category $^\circ\mathbf{W}^*$, would be cartesian closed, meaning that

$$Hom(A \oplus B, C) \cong Hom(A, C^B).$$

So in $\mathbf{W}^*$ we would have

$$Hom(\mathcal{C}, \mathcal{A} * \mathcal{B}) \cong Hom(\mathcal{C}^{\mathcal{B}}, \mathcal{A})$$

and therefore taking the coproduct would be a right adjoint to forming the coexponent. Now right adjoints preserve limits, so in particular we would have

$$(\mathcal{A} \oplus \mathcal{A}) * \mathcal{B} \cong (\mathcal{A} * \mathcal{B}) \oplus (\mathcal{A} * \mathcal{B}).$$

However, taking $\mathcal{A} = \mathbb{C}$ and $\mathcal{B} = \mathbb{C}^2$ would yield

$$\mathbb{C}^2 * \mathbb{C}^2 \cong \mathbb{C}^2 \oplus \mathbb{C}^2,$$

because $\mathbb{C}$ is the initial object. But $\mathbb{C}^2 \oplus \mathbb{C}^2$ is commutative and $\mathbb{C}^2 * \mathbb{C}^2$ is not, as we have seen in the previous section. □

Similarly:

Proposition 3.12. $\mathbf{W}^*$ *does not have all exponentials.*

Proof. As in the previous proof, we suppose that exponentials do exist. Then the product would be a left adjoint to forming the exponent. Now left adjoints preserve colimits, so we would have the relation

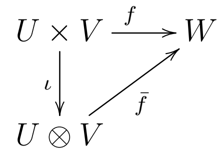
$$(A * B) \oplus C \cong (A \oplus C) * (B \oplus C).$$

However, taking $A = B = 0$ we would have $C \cong C * C$ for all von Neumann algebras C, but this fails already for $C = \mathbb{C}^2$. □

Our conclusion that $\mathbf{W}^*$ has no exponentials comes as no surprise. The opposite of $\mathbf{W}^*$ should be a category which reminds us of Sets, but **Set** does not have coexponentials. However, the fact that $\mathbf{W}^*$ does not have coexponents is quite a surprise, by the same reasoning. To remedy this situation, we introduce the spatial tensor product of von Neumann algebras and show that, just like in **Set** the exponential is right adjoint to the product, this tensor product has a left adjoint (remember that we work in a dual setting here).

3.4 Spatial Tensor Product of von Neumann algebras

In this section we define the spatial tensor product of W^*-algebras and see how this fits into a more general categorical framework. To this end we first define a tensor product for Hilbert spaces.
Let U, V and W be vector spaces over some field. It is well known that there exists a vector space $U \otimes V$ and a bilinear map $\iota : U \times V \to U \otimes V$ such that every bilinear map $f : U \times V \to W$ corresponds to a linear map $\bar{f} : U \otimes V \to W$ via $f = \bar{f}\iota$.



This can be seen by taking $U \otimes V$ to be the free vector space over the *set* $U \times V$ quotiented by the subspace generated by

$$\{(u_1 + u_2, v) - (u_1, v) - (u_2, v), \quad (u, v_1 + v_2) - (u, v_1) - (u, v_2),$$
$$(\lambda u, v) - \lambda(u, v), \quad (u, \lambda v) - \lambda(u, v)\},$$

where λ is from the field, and then letting $\iota(u, v)$ be the class of (u, v) in $U \otimes V$ which will be denoted $u \otimes v$.

Definition 3.13. *The space $U \otimes V$ is called the* (algebraic) tensor product *of the vector spaces U and V. Elements of the form $u \otimes v$ are called* elementary tensors.

If $\tau : U \to U'$ and $\rho : V \to V'$ are linear maps, then the map $\tau \times \rho : U \times V \to U' \otimes V'$ given by $(u, v) \mapsto \tau(u) \otimes \rho(k)$ is bilinear. Therefore there exists a unique linear map $\tau \otimes \rho : U \otimes V \to U' \otimes V'$ such that $\tau \otimes \rho(u \otimes v) = \tau(u) \otimes \rho(k)$.
If $\alpha : U \to U'$ and $\beta : V \to V'$ are conjugate linear maps, then there still exists a unique conjugate linear map $\alpha \otimes \beta : U \otimes V \to U' \otimes V'$ such that $\alpha \otimes \beta(u \otimes v) = \alpha(u) \otimes \beta(v)$. To see this, note that $u \times v \mapsto \bar{\alpha}(u) \otimes \bar{\beta}(v)$ is linear, so there exists a unique linear map $\bar{\alpha} \otimes \bar{\beta}$ such that $\bar{\alpha} \otimes \bar{\beta}(u \otimes v) = \bar{\alpha}(u) \otimes \bar{\beta}(v)$. We now set $\alpha \otimes \beta = \overline{\bar{\alpha} \otimes \bar{\beta}}$.
Now let H, K be Hilbert Spaces with algebraic tensor product $H \otimes K$. We want to put an inner product on this tensor product to make it a pre Hilbert space and such that this inner product reflects the inner products on the original spaces. This last property would translate to

$$\langle h \otimes k, h' \otimes k' \rangle = \langle h, h' \rangle \langle k, k' \rangle,$$

for elementary tensors. It turns out that if we require this property, the inner product is fixed.

Proposition 3.14. *For Hilbert spaces H, K there exists a unique inner product on the tensor product such that $\langle h \otimes k, h' \otimes k' \rangle = \langle h, h' \rangle \langle k, k' \rangle$.*

Proof. For $h \in H$, the map $\tau_h : H \to \mathbb{C}$, $\tau_h(h') = \langle h', h \rangle$ is conjugate linear, as is the similarly defined map τ_k for $k \in K$. The map $(h', k') \mapsto \tau_h(h')\tau_k(k')$ is therefore biconjugate linear, so we have seen there exists a unique conjugate linear map $\tau_h \otimes \tau_k : H \otimes K \to \mathbb{C}$ such that $\tau_h \otimes \tau_k(h' \otimes k') = \tau_h(h')\tau_k(k')$.
Let X be the space of all conjugate linear functionals on $H \otimes K$. The map $(h, k) \mapsto \tau_h \otimes \tau_k$ from $H \times K$ to X is bilinear. Therefore, there exists a unique linear map $T : H \otimes K \to X$ such that $T(h \otimes k) = \tau_h \otimes \tau_k$.
Now $\langle \cdot, \cdot \rangle : (H \otimes K) \times (H \otimes K) \to \mathbb{C}$ given by $\langle z, z' \rangle = T(z')(z)$ is a sesquilinear form. If $z = h \otimes k$, $z' = h' \otimes k'$, we have

$$\langle z, z' \rangle = T(z')(z) = \tau_{h'} \otimes \tau_{k'}(h \otimes k) = \langle h, h' \rangle \langle k, k' \rangle.$$

Since any element of $H \otimes K$ is a sum over elementary tensors, and the maps T and $T(z)$ are linear and conjugate linear, respectively, this form is unique.

We do, however, still have to show this form really is an inner product. So, let $z = \sum_{i=1}^{n} h_i \otimes k_i$ and let $\{e_i\}$ be a basis for the linear span of the k_i. Because of the bilinearity of the tensor product, we can write $z = \sum_{i=1}^{m} \tilde{h}_i \otimes e_i$ for certain $\tilde{h}_i \in H$. Now

$$\begin{aligned}
\langle z, z\rangle &= \sum_{i,j=1}^{m} \langle \tilde{h}_i \otimes e_i, \tilde{h}_j \otimes e_j\rangle \\
&= \sum_{i,j=1}^{m} \langle \tilde{h}_i, \tilde{h}_j\rangle\langle e_i, e_j\rangle \\
&= \sum_{i=1}^{m} \langle \tilde{h}_i, \tilde{h}_i\rangle \\
&= \sum_{i=1}^{m} \|\tilde{h}_i\|^2 \geq 0.
\end{aligned}$$

This also shows that if $\langle z, z\rangle = 0$, then the $\tilde{h}_i$ are zero, and so $z = 0$. □

The algebraic tensor product $H \otimes K$ is now a pre-Hilbert space and so we can take the completion to obtain a Hilbert space which we also denote by $H \otimes K$ and refer to as the *Hilbert space tensor product.*
The norm on elementary tensors is particularly easy:

$$\|h \otimes k\| = \sqrt{\langle h \otimes k, h \otimes k\rangle} = \sqrt{\langle h, h\rangle\langle k, k\rangle} = \|h\|\|k\|.$$

If $\{e_i \mid i \in I\}$ is an orthonormal basis for H and $\{f_j \mid j \in J\}$ is an orthonormal basis for K, for some index sets I, J, then $\{e_i \otimes f_j \mid i \in I, j \in J\}$ is an orthonormal basis $H \otimes K$.
If $x \in B(H), y \in B(K)$, we denote by $x \otimes y$ the (unique) operator on the algebraic tensor product such that $x \otimes y(h \otimes k) = x(h) \otimes y(k)$. Since any z in the algebraic tensor product can be written as $\sum_{i=1}^{m} h_i \otimes e_i$ with the $e_i \in K$ orthonormal we find

$$\begin{aligned}
\|x \otimes 1_K(\sum_{i=1}^{m} h_i \otimes e_i)\|^2 &= \|\sum_{i=1}^{m} x(h_i) \otimes e_i\|^2 \\
&= \sum_{i=1}^{m} \|x(h_i)\|^2 \\
&\leq \|x\|^2 \sum_{i=1}^{m} \|h_i\|^2 \\
&= \|x\|^2 \|\sum_{i=1}^{m} h_i \otimes e_i\|^2.
\end{aligned}$$

So $x \otimes 1$ is bounded, in particular $\|x \otimes 1\| \leq \|x\|$, and likewise $1 \otimes y$ is bounded, with bound $\|y\|$. Now

$$\begin{aligned}
\|x \otimes y\| &= \|(x \otimes 1)(1 \otimes y)\| \\
&\leq \|x \otimes 1\|\|1 \otimes y\| \\
&\leq \|x\|\|y\|,
\end{aligned}$$

showing $x \otimes y$ is bounded on the algebraic tensor product and therefore it extends to a bounded operator on $H \otimes K$, also denoted $x \otimes y$.
For $\epsilon > 0$ we can find unit vectors $h \in H$ and $k \in K$ such that $\|x(h)\| > \|x\| - \epsilon > 0$ and $\|y(k)\| > \|y\| - \epsilon > 0$. Then

$$\|x \otimes y(h \otimes k)\| = \|x(h)\| \|y(k)\| > (\|x\| - \epsilon)(\|y\| - \epsilon).$$

Letting $\epsilon \to 0$ we find

$$\|x \otimes y\| \geq \|x\| \|y\|.$$

Therefore we conclude

$$\|x \otimes y\| = \|x\| \|y\|.$$

Definition 3.15. *Let $\mathcal{M} \subset B(H)$ and $\mathcal{N} \subset B(K)$ be von Neumann algebras. The von Neumann algebra generated by elements of the from $x \otimes y \in B(H \otimes K)$ is called the* spatial tensor product *of $\mathcal{M}$ and $\mathcal{N}$, denoted by $\mathcal{M} \overline{\otimes} \mathcal{N}$ i.e., $\mathcal{M} \overline{\otimes} \mathcal{N} = (\mathcal{M} \otimes \mathcal{N})''$.*

3.5 Monoidal Structures

The above section has a categorical generalization; that of monoidal structures. Here we introduce this subject and then show that $\mathbf{W}^*$ indeed is a (symmetric) monoidal category.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition 3.16. *The* product category $\mathcal{C} \times \mathcal{D}$ *of $\mathcal{C}$ and $\mathcal{D}$ is the category which has*

- *pairs of objects (A, B), $A \in \mathcal{C}, B \in \mathcal{D}$ as objects,*
- *pairs of morphims (f, g), $f : A \to C, g : B \to D$ as morphims from (A, B) to (C, D),*
- *composition defined as $(f, g) \circ (h, k) = (f \circ h, g \circ k)$ whenever f, h and g, k are composable,*
- *The morphims $1_{(A,B)} = (1_A, 1_B)$ as the identity of (A, B).*

Definition 3.17. *A* bifunctor *is a functor T which has a product category as its domain, i.e., $T : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, for some category $\mathcal{E}$.*
Let us spell out this definition. Given $C \in \mathcal{C}, D \in \mathcal{D}$, there is an object $T(C, D) \in \mathcal{E}$. Given morphism $(f : A \to C) \in \mathcal{C}, (g : B \to D) \in \mathcal{D}$ we obtain a morphism $T(f, g) : T(A, B) \to T(C, D)$ in $\mathcal{E}$ such that, if $(h : C \to E) \in \mathcal{C}, (k : D \to F) \in \mathcal{D}$ are morphisms, we have $T((h, k) \circ (f, g)) = T(h \circ f, k \circ g) = T(h, k) \circ T(f, g)$.

In what follows, $\otimes$ shall be a bifunctor and we denote $\otimes(A, B)$ as $A \otimes B$, doing the same for morphisms. The composition identity will then be $(h \circ f) \otimes (k \circ g) = (h \otimes k) \circ (f \otimes g)$.

Definition 3.18. *A* monoidal category *is a category $\mathcal{C}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an* identity object *$I$ and three natural isomorphisms*

- $\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$,

- $\lambda_A : I \otimes A \xrightarrow{\cong} A$ *and*
- $\rho_A : A \otimes I \xrightarrow{\cong} A$*,*

that satisfy the triangle identity

$$\begin{array}{ccc} (A\otimes I)\otimes B & \xrightarrow{\alpha_{A,I,B}} & A\otimes(I\otimes B) \\ & {\scriptstyle \rho_A\otimes 1}\searrow \qquad \swarrow{\scriptstyle 1\otimes\lambda_B} & \\ & A\otimes B & \end{array}$$

and the pentagon identity

$$\begin{array}{ccccc} ((A\otimes B)\otimes C)\otimes D & \xrightarrow{\alpha_{A,B,C}\otimes 1} & (A\otimes(B\otimes C))\otimes D & \xrightarrow{\alpha_{A,B\otimes C,D}} & A\otimes((B\otimes C)\otimes D) \\ \downarrow{\scriptstyle \alpha_{A\otimes B,C,D}} & & & & \downarrow{\scriptstyle 1\otimes\alpha_{B,C,D}} \\ (A\otimes B)\otimes(C\otimes D) & & \xrightarrow{\alpha_{A,B,C\otimes D}} & & A\otimes(B\otimes(C\otimes D)) \end{array}$$

meaning that both diagrams commute.

If a category has products, it is automatically monoidal. We see this by taking

- the product to be the bifunctor in question,
- the terminal object as the identity object,
- the natural isomorphisms as the canonical ones.

The same is true if the category has coproducts, with the only difference that we then take the initial object as the identity object.

Definition 3.19. *Let $\mathcal{C}$ be a monoidal category. Suppose we have a natural isomorphism with components $\beta_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$ such that*

$$\begin{array}{ccccc} & A\otimes(B\otimes C) & \xrightarrow{\beta_{A,B\otimes C}} & (B\otimes C)\otimes A & \\ {\scriptstyle \alpha_{A,B,C}}\nearrow & & & & \searrow{\scriptstyle \alpha_{B,C,A}} \\ (A\otimes B)\otimes C & & & & B\otimes(C\otimes A) \\ {\scriptstyle \beta_{A,B}\otimes 1}\searrow & & & & \nearrow{\scriptstyle 1\otimes\beta_{A,C}} \\ & (B\otimes A)\otimes C & \xrightarrow{\alpha_{B,A,C}} & B\otimes(A\otimes C) & \end{array}$$

and

$$\begin{array}{ccccc} & (A\otimes B)\otimes C & \xrightarrow{\beta_{A\otimes B,C}} & C\otimes(A\otimes B) & \\ {\scriptstyle \alpha^{-1}_{A,B,C}}\nearrow & & & & \searrow{\scriptstyle \alpha^{-1}_{C,A,B}} \\ A\otimes(B\otimes C) & & & & (C\otimes A)\otimes B \\ {\scriptstyle 1\otimes\beta_{B,C}}\searrow & & & & \nearrow{\scriptstyle \beta_{A,C}\otimes 1} \\ & A\otimes(C\otimes B) & \xrightarrow{\alpha^{-1}_{A,C,B}} & (A\otimes C)\otimes B & \end{array}$$

commute. Then $\mathcal{C}$ is called a braided monoidal category.

The first diagram says that

$$\alpha\beta\alpha = (1 \otimes \beta)\alpha(\beta \otimes 1),$$

while the second gives the equality

$$\alpha^{-1}\beta\alpha^{-1} = (\beta \otimes 1)\alpha^{-1}(1 \otimes \beta).$$

Since α and β are isomorphisms, we can take the inverse of this second equality to give

$$\alpha\beta^{-1}\alpha = (1 \otimes \beta^{-1})\alpha(\beta^{-1} \otimes 1).$$

This is precisely the first equality, only with β^{-1} instead of β. This is because if we go from $A \otimes B \xrightarrow{\beta_{A,B}} B \otimes A$ we can go back with either $\beta_{B,A}$, or with $\beta^{-1}_{A,B}$ and these morphisms do not have to coincide.

Definition 3.20. *Whenever $\beta_{B,A} \circ \beta_{A,B} = 1$ for all A, B, that is to say $\beta_{A,B} = \beta^{-1}_{B,A}$, we call the braided monoidal category* symmetric *and just speak of a* symmetric monoidal category.

Of course, we would now like to know if and how this works for the category $\mathbf{W}^*$. So let $\mathcal{M} \subset B(H), \mathcal{N} \subset B(K)$ and $\mathcal{R} \subset B(L)$ be von Neumann algebras.

Proposition 3.21. *The category $\mathbf{W}^*$ is monoidal.*

Proof. We will just work out the checklist for a monoidal category.

- The bifunctor will be the spatial tensor product.
- The identity object will be $\mathbb{C}$.
- The map $\alpha_{\mathcal{M},\mathcal{N},\mathcal{R}}$ will be the linear extension of the map $(m \otimes n) \otimes r \mapsto m \otimes (n \otimes r)$, for $m \in \mathcal{M}, n \in \mathcal{N}$ and $r \in \mathcal{R}$.
- The maps $\lambda_{\mathcal{M}}$ and $\rho_{\mathcal{M}}$ wil be the linear extension of the maps $m \otimes \lambda \mapsto \lambda m$ and $\lambda \otimes m \mapsto \lambda m$, respectively, for $m \in \mathcal{M}, \lambda \in \mathbb{C}$.

It is obvious that the maps α, λ and ρ satisfy the right equations for $\mathbf{W}^*$ to be a monoidal category and that they are natural isomorphisms. The only part that might require some explanation is the functoriality of the spatial tensor product.
So let $-\overline{\otimes}- : \mathbf{W}^* \times \mathbf{W}^* \to \mathbf{W}^*$ be as follows:

- For $\mathcal{M}, \mathcal{N}$ von Neumann algebras, let $\mathcal{M}\overline{\otimes}\mathcal{N}$ be the von Neumann algebra as in definition 3.15.
- If $f : \mathcal{M} \to \mathcal{A}, g : \mathcal{N} \to \mathcal{B}$ are morphisms, we obtain a morphism $f\overline{\otimes}g : \mathcal{M}\overline{\otimes}\mathcal{N} \to \mathcal{A}\overline{\otimes}\mathcal{B}$ defined via the linear extension of $f\overline{\otimes}g(m \otimes n) = f(m) \otimes g(n)$. Functorality in both entries follows from the simple observation that $(f \circ h)\overline{\otimes}g = (f\overline{\otimes}1) \circ (h\overline{\otimes}g)$ and $f\overline{\otimes}(g \circ k) = (f\overline{\otimes}g) \circ (1\overline{\otimes}k)$, for h, k appropriate morphisms.

□

It will come as no surprise that we are in fact dealing with a braided category and that the braiding is even symmetric.

Proposition 3.22. *The category* $\mathbf{W}^*$ *is a symmetric monoidal category.*

Proof. The braiding is of course given by the linear extension of the map $\beta_{\mathcal{M},\mathcal{N}} : n \otimes m \mapsto m \otimes n$, which obviously satisfies the required equations and is a natural isomorphism. We also directly see that $\beta_{\mathcal{M},\mathcal{N}} \circ \beta_{\mathcal{N},\mathcal{M}} = 1$, which shows that the braiding is symmetric. □

3.6 Exponentials in $^\circ\mathbf{W}^*$

We know that $^\circ\mathbf{W}^*$ is not Cartesian closed, since the category $\mathbf{W}^*$ does not have all coexponentials. As mentioned, we can, however, hope for $^\circ\mathbf{W}^*$ to be a closed monoidal category with respect to another form of product. We will see this is indeed the case with respect to the spatial tensor product $\overline{\otimes}$. To this end we define a coexponent like structure in $\mathbf{W}^*$ and then show it is indeed in adjunction to the tensor product.

Lemma 3.23. *Let* $\mathcal{M} \subset B(H)$ *be a von Neumann algebra. Then* $\mathcal{M}$ *is generated by a subset of cardinality bounded by* $\dim(H)$.

Proof. If H is finite dimensional, $\mathcal{M}$ is isomorphic to a direct sum of matrix algebras, for which the proposition is true. We show this first. Let $e_{i,j}$ be the $n \times n$ matrix which has a 1 on entry (i,j) and zero everywhere else. These matrices satisfy $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$. Now consider the $n-1$ matrices $e_{i,i+1}, i = 1, \dots, n-1$. Then $e_{i,i+1}e_{i+1,i+2} = e_{i,i+2}$, $i = 1, \dots, n-2$, which has a 1 on the second off-diagonal entry. Continuing in this fashion, we obtain all $e_{i,j}$ with $j > i$ and taking adjoints, we also obtain all $e_{i,j}$ with $j < i$. Finally, we obtain the diagonal via $e_{i,i} = e_{i,k}e_{k,i}$.
So now we focus on the case where $\dim(H)$ is infinite.
We can take a dense subset of H of cardinality $\dim(H)$ (consisting, for example, of all elements of the form $\sum_{i=1}^n q_i e_i$, where $n \in \mathbb{N}, q_i \in \mathbb{Q}$ and e_i basis vectors). This then gives a dense subset X in $\mathcal{M}_*$ (with respect to the norm on this predual) via the correspondence

$$h \mapsto \omega_h \text{ with } \omega_h(x) = \langle h, xh \rangle,$$

for $x \in \mathcal{M}$. We now consider $\mathcal{M}_1$, the unit ball of $\mathcal{M}$, with two topologies on it, namely the weak-* topology and the topology induced by X, i.e., the coarsest (also called weakest or smallest) topology on $\mathcal{M}_1$ such that all elements of X (seen as maps $\mathcal{M}_1 \to \mathbb{C}$) are continuous. We denote the latter topology by τ. By the Banach-Alaoglu theorem, $\mathcal{M}_1$ is compact in the weak-* topology. Also, the topology τ is Hausdorff, since if we have two distinct elements $m_1, m_2 \in \mathcal{M}$, there is some element $\phi \in \mathcal{M}_*$ such that $m_1(\phi) \neq m_2(\phi)$ (under the identification $\mathcal{M} = \mathcal{M}_*{}^*$). Since X is dense, m_1 and m_2 cannot be the same on each $x \in X$. Now let $a = m_1(x) \neq m_2(x) = b$. Then we take open, disjoint sets A, B containing a, b respectively. By definition of τ, the sets $x^{-1}(A), x^{-1}(B)$ are open disjoint sets containing m_1, m_2 respectively.
Since the weak-* topology is such that it makes every element of $\mathcal{M}_*$ continuous, it is of course finer than τ (τ is coarser than the weak-* topology). The identity map from $(\mathcal{M}_1, \text{weak-}^*)$ to $(\mathcal{M}_1, \tau)$ is now a homeomorphism, since it is a continuous map from

a compact Hausdorff space to a Hausdorff space. Therefore, the weak-* topology and τ are the same.
A basis for the topology on $\mathbb{C}$ is given by open rectangles of the form $(q_1, r_1) \times (iq_2, ir_2)$ with $q_1, q_2, r_1, r_2 \in \mathbb{Q}$. Taking the pre-image of these sets with respect to all $x \in X$, we find a basis of cardinality no more than $\aleph_0 \cdot card(X) = card(X) = \dim(H)$ for τ and hence also for the weak-* topology. Taking an operator from each of these base sets, we obtain a dense subset of $\mathcal{M}_1$ with cardinality no greater than $\dim(H)$. Multiplying every such element of this dense subset with $\mathbb{Q}$ we obtain a dense subset for $\mathcal{M}$ with the same cardinality. □

Once we know this, we can continue to the key lemma in the construction of the coexponent like structure, but first we need some notation. For a Hilbert space H, and a vector $\psi \in H$, let $\hat{\psi}$ be the map $\hat{\psi} : \mathbb{C} \to H$ given by $\hat{\psi}(\lambda) = \lambda\psi$. This map is obviously linear and determined by its value on 1. Its adjoint must satisfy

$$\langle \hat{\psi}(\lambda), \phi \rangle = \langle \lambda, \hat{\psi}^*(\phi) \rangle,$$

from which we find $\hat{\psi}^*(\phi) = \langle \psi, \phi \rangle$.

Lemma 3.24. *Let K, L be Hilbert spaces and $\mathcal{N} \subset B(K), \mathcal{M} \subset B(L)\overline{\otimes}\mathcal{N}$ von Neumann algebras. Then there exists a smallest von Neumann algebra $\mathcal{S} \subset B(L)$ such that $\mathcal{M} \subset \mathcal{S}\overline{\otimes}\mathcal{N}$. Furthermore, if $\mathcal{M}$ has a faithful representation on a Hilbert space H, then $\mathcal{S}$ has a faithful representation on a Hilbert space of dimension no more than $2^{\aleph_0 \dim(H) \dim(K)}$.*

Proof. We have $\mathcal{M} \subset B(L)\overline{\otimes}\mathcal{N} \subset B(L)\overline{\otimes}B(K) \cong B(L \otimes K)$. By the previous lemma, there is a generating subset $M \subset \mathcal{M}$ of cardinality no more than $\dim(L \otimes K) = \dim(L)\dim(K)$. Let $\{e_i\}_{i \in I}$ be an orthonormal basis for K. First we take $\mathcal{N} = B(K)$. For a generator $m \in M$ and basis vectors e_a, e_b, e_c we have the following relation:

$$[(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b^*)] \otimes (\hat{e}_c\hat{e}_c^*) = (1 \otimes \hat{e}_c\hat{e}_a^*)m(1 \otimes \hat{e}_b\hat{e}_c^*).$$

We show this now. Let $k \in K, l \in L$. Then

$$[(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b^*)] \otimes (\hat{e}_c\hat{e}_c^*)(l \otimes k) = [(1 \otimes \hat{e}_a^*)m(l \otimes e_b)] \otimes \langle e_c, k \rangle e_c,$$

while

$$\begin{aligned} (1 \otimes \hat{e}_c\hat{e}_a^*)m(1 \otimes \hat{e}_b\hat{e}_c^*)(l \otimes k) &= (1 \otimes \hat{e}_c\hat{e}_a^*)m(l \otimes \langle e_c, k \rangle e_b) \\ &= (1 \otimes \hat{e}_c)(1 \otimes \hat{e}_a^*)m(l \otimes \langle e_c, k \rangle e_b) \\ &= (1 \otimes \hat{e}_a^*)m(l \otimes \langle e_c, k \rangle e_b) \otimes e_c. \end{aligned}$$

Therefore, if $\mathcal{M} \subset \tilde{\mathcal{S}}\overline{\otimes}B(K)$ for some von Neumann algebra $\tilde{\mathcal{S}} \subset B(L)$, then the operators $[(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b^*)] \otimes (\hat{e}_c\hat{e}_c^*) = (1 \otimes \hat{e}_c\hat{e}_a^*)m(1 \otimes \hat{e}_b\hat{e}_c^*)$ are elements of $\tilde{\mathcal{S}}\overline{\otimes}B(K)$. Taking the sum

$$\sum_c [(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b^*)] \otimes (\hat{e}_c\hat{e}_c^*) = [(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b^*)] \otimes 1,$$

then shows that $(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b^*) \in \tilde{\mathcal{S}}$. So any von Neumann algebra $\tilde{\mathcal{S}}$ such that $\mathcal{M} \subset \tilde{\mathcal{S}}\overline{\otimes}B(K)$ contains the elements $(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b^*)$. Now let $\mathcal{S}$ be the smallest von Neumann algebra generated by the elements $(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b^*)$. We claim that $\mathcal{M} \subset \mathcal{S}\overline{\otimes}B(K)$. Indeed, for $m \in M \subset \mathcal{M}$ we have

$$\begin{aligned} m &= \sum_{a,b}(1 \otimes \hat{e}_a\hat{e}_a^*)m(1 \otimes \hat{e}_b\hat{e}_b^*) \\ &= \sum_{a,b}(1 \otimes \hat{e}_a)(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b)(1 \otimes \hat{e}_b^*) \\ &= \sum_{a,b}(1 \otimes \hat{e}_a^*)m(1 \otimes \hat{e}_b) \otimes \hat{e}_a\hat{e}_b^*, \end{aligned}$$

and all the operators in this last sum are obviously elements of $\mathcal{S}\overline{\otimes}B(K)$.
Now we go back to the general case, where $\mathcal{N} \subset B(K)$ is any von Neumann algebra. The $\mathcal{S}$ we have found still is the smallest von Neumann algebra such that $\mathcal{M} \subset \mathcal{S}\overline{\otimes}B(K)$, but we also know that $\mathcal{M} \subset B(L)\overline{\otimes}\mathcal{N}$. Therefore, $\mathcal{M} \subset \mathcal{S}\overline{\otimes}\mathcal{N}$ and $\mathcal{S}$ is still minimal, since for any $\tilde{\mathcal{S}}$ such that $\mathcal{M} \subset \tilde{\mathcal{S}}\overline{\otimes}\mathcal{N}$ we have $\mathcal{M} \subset \tilde{\mathcal{S}}\overline{\otimes}B(K)$.
The statement about a faithful representation of $\mathcal{S}$ now follows directly from Lemma 3.6 and from knowing the number of generators of $\mathcal{S}$, which we know can be taken to be $\dim(K)\mathrm{card}(M)\dim(K) = \dim(H)\dim(K)$. □

Note that we explicitly found a set of generators of the von Neumann algebra $\mathcal{S}$.

We now are finally ready to get to the heart of this section: the construction of ”coexponentials” in $\mathbf{W}^*$.

Proposition 3.25. *Let $\mathcal{M} \subset B(H), \mathcal{N} \subset B(K)$ be von Neumann algebras. There exists a von Neumann algebra $\mathcal{M}^{*\mathcal{N}}$ and a morphism $\epsilon : \mathcal{M} \to \mathcal{M}^{*\mathcal{N}}\overline{\otimes}\mathcal{N}$ such that for any von Neumann algebra $\mathcal{R}$ and morphism $\pi : \mathcal{M} \to \mathcal{R}\overline{\otimes}\mathcal{N}$ there is a unique $\rho : \mathcal{M}^{*\mathcal{N}} \to \mathcal{R}$ such that*

$$\begin{array}{ccc} \mathcal{M}^{*\mathcal{N}}\overline{\otimes}\mathcal{N} & \xrightarrow{\rho\overline{\otimes}1} & \mathcal{R}\overline{\otimes}\mathcal{N} \\ \uparrow{\scriptstyle \epsilon} & \nearrow{\scriptstyle \pi} & \\ \mathcal{M} & & \end{array}$$

commutes, i.e., $\pi = (\rho\overline{\otimes}1)\epsilon$.

Proof. Let $\kappa = 2^{\aleph_0 \dim(H)\dim(K)}$. We consider all morphisms $\sigma : \mathcal{M} \to B(l^2(\kappa))\overline{\otimes}\mathcal{N}$. For each such morphism there is, by the previous lemma, a smallest von Neumann algebra $\mathcal{S}_\sigma$ such that $\sigma(\mathcal{M}) \subset \mathcal{S}_\sigma\overline{\otimes}\mathcal{N}$.
We now define $\epsilon : \mathcal{M} \to \bigoplus_\sigma(\mathcal{S}_\sigma\overline{\otimes}\mathcal{N}) = (\bigoplus_\sigma \mathcal{S}_\sigma)\overline{\otimes}\mathcal{N}$ as $\epsilon(m) = \oplus_\sigma\sigma(m)$. Now let $\mathcal{M}^{*\mathcal{N}}$ be the smallest von Neumann algebra in $\bigoplus_\sigma \mathcal{S}_\sigma$ such that $\epsilon(\mathcal{M}) \subset \mathcal{M}^{*\mathcal{N}}\overline{\otimes}\mathcal{N}$. We claim this satisfies the properties given in the proposition.
So let $\mathcal{R}$ be any von Neumann algebra and let $\pi : \mathcal{M} \to \mathcal{R}\overline{\otimes}\mathcal{N}$ be any morphism. Again by the previous lemma, there is a minimal von Neumann algebra $\mathcal{S} \subset \mathcal{R}$ such that $\pi(\mathcal{M}) \subset \mathcal{S}\overline{\otimes}\mathcal{N}$. By construction of $\mathcal{M}^{*\mathcal{N}}$, this $\mathcal{S}$ is a summand of $\mathcal{M}^{*\mathcal{N}}$, so let ρ be the projection onto this summand. Then, of course,

$$\pi = (\rho\overline{\otimes}1)\epsilon : \mathcal{M} \to \mathcal{S}\overline{\otimes}\mathcal{N} \subset \mathcal{R}\overline{\otimes}\mathcal{N}.$$

The last thing to show is that this morphism ρ is unique with respect to this property. So let $\tilde{\rho} : \mathcal{M}^{*\mathcal{N}} \to \mathcal{R}$ also satisfy $\pi = (\tilde{\rho}\overline{\otimes}1)\epsilon$. We know that elements of the form $(1 \otimes \hat{k}_1^*)\epsilon(m)(1 \otimes \hat{k}_2)$ generate $\mathcal{M}^{*\mathcal{N}}$ for $k_1, k_2 \in K$ and $m \in \mathcal{M}$. So we take such an element

$$(1 \otimes \hat{k}_1^*)\epsilon(m)(1 \otimes \hat{k}_2) = (1 \otimes \hat{k}_1^*)(\oplus_\sigma \sigma(m))(1 \otimes \hat{k}_2).$$

Now every $\sigma(m)$ is an element of $\mathcal{S}_\sigma\overline{\otimes}\mathcal{N}$ and therefore is of the form

$$\sigma(m) = \sum_i s_{\sigma(m),i} \otimes n_{\sigma(m),i}.$$

From this we find that the map $(1 \otimes \hat{k}_1^*)\epsilon(m)(1 \otimes \hat{k}_2)$ acts on a vector ψ in $\mathcal{M}^{*\mathcal{N}}$ as

$$\psi \mapsto \oplus_\sigma \sum_i \langle k_1, n_{\sigma(m),i} k_2 \rangle \cdot s_{\sigma(m),i}\psi.$$

So, for a linear map f we find

$$f((1 \otimes \hat{k}_1^*)\epsilon(m)(1 \otimes \hat{k}_2)) = (1 \otimes \hat{k}_1^*)(f\overline{\otimes}1)(\epsilon(m))(1 \otimes \hat{k}_2).$$

In particular, we now have a chain of equalities

$$\begin{aligned}
\rho((1 \otimes \hat{k}_1^*)\epsilon(m)(1 \otimes \hat{k}_2)) &= (1 \otimes \hat{k}_1^*)(\rho\overline{\otimes}1)(\epsilon(m))(1 \otimes \hat{k}_2) \\
&= (1 \otimes \hat{k}_1^*)\pi(m)(1 \otimes \hat{k}_2) \\
&= (1 \otimes \hat{k}_1^*)(\tilde{\rho}\overline{\otimes}1)(\epsilon(m))(1 \otimes \hat{k}_2) \\
&= \tilde{\rho}((1 \otimes \hat{k}_1^*)(\epsilon(m))(1 \otimes \hat{k}_2)),
\end{aligned}$$

showing that ρ is indeed unique. □

Definition 3.26. *The von Neumann algebra $\mathcal{M}^{*\mathcal{N}}$ of Proposition 3.25, together with the morphism $\epsilon : \mathcal{M} \to \mathcal{M}^{*\mathcal{N}}\overline{\otimes}\mathcal{N}$, is called the* free exponential *of $\mathcal{M}$ and $\mathcal{N}$. By Proposition 3.25, it is unique up to isomorphism.*

Now for fixed $\mathcal{N}$ we have the map of objects $\mathcal{M} \mapsto \mathcal{M}^{*\mathcal{N}}$. We wish this map to be a functor, so we need to define it on morphisms, too. Let $f : \mathcal{M}_0 \to \mathcal{M}_1$ be a morphism and consider the following (part of a) diagram, where ϵ_0, ϵ_1 are the maps defined in the previous proposition:

$$\begin{array}{ccc}
\mathcal{M}_0 & \xrightarrow{f} & \mathcal{M}_1 \\
\downarrow{\scriptstyle \epsilon_0} & & \downarrow{\scriptstyle \epsilon_1} \\
\mathcal{M}_0^{*\mathcal{N}}\overline{\otimes}\mathcal{N} & & \mathcal{M}_1^{*\mathcal{N}}\overline{\otimes}\mathcal{N}
\end{array}$$

Now the morphism $\epsilon_1 \circ f$ is a map $\mathcal{M}_0 \to \mathcal{M}_1^{*\mathcal{N}}\overline{\otimes}\mathcal{N}$, so by the universal property of the free coexponent, there is a unique map $f^{*\mathcal{N}} : \mathcal{M}_0^{*\mathcal{N}} \to \mathcal{M}_1^{*\mathcal{N}}$ such that $f^{*\mathcal{N}}\overline{\otimes}1$ completes the above diagram, i.e., $(f^{*\mathcal{N}}\overline{\otimes}1) \circ \epsilon_0 = \epsilon_1 \circ f$.
The fact that this is indeed a functor is now easy to prove. For the identity $id : \mathcal{M} \to \mathcal{M}$ we find a morphism such that $\epsilon = ((id)^{*\mathcal{N}}\overline{\otimes}1) \circ \epsilon$. But the identity of $\mathcal{M}^{*\mathcal{N}}$ also accomplishes this, so by uniqueness they are equal.

Given $\mathcal{M}_0 \xrightarrow{f} \mathcal{M}_1 \xrightarrow{g} \mathcal{M}_2$, with associated morphisms $\epsilon_0, \epsilon_1, \epsilon_2$, we find a morphism $(gf)^{*\mathcal{N}}$, such that

$$\begin{aligned} ((gf)^{*\mathcal{N}}\overline{\otimes}1)\epsilon_0 &= \epsilon_2 gf \\ &= (g^{*\mathcal{N}}\overline{\otimes}1)\epsilon_1 f \\ &= (g^{*\mathcal{N}}\overline{\otimes}1)(f^{*\mathcal{N}}\overline{\otimes}1)\epsilon_0 \\ &= (g^{*\mathcal{N}} f^{*\mathcal{N}}\overline{\otimes}1)\epsilon_0. \end{aligned}$$

So again by uniqueness, $(gf)^{*\mathcal{N}} = g^{*\mathcal{N}} f^{*\mathcal{N}}$.

What we were looking for was an adjunction between a product like structure and an exponential like structure in $^\circ\mathbf{W}^*$. This next theorem shows that we have indeed found such an adjunction.

Theorem 3.27. *Let $\mathcal{N}$ be any von Neumann algebra. The functor $(-)^{*\mathcal{N}}$ is left adjoint to the functor $-\overline{\otimes}\mathcal{N}$.*

Proof. We recall the universal property of ϵ as in Proposition 3.25. For $\mathcal{M}$ and $\mathcal{R}$ von Neumann algebras and $\pi : \mathcal{M} \to \mathcal{R}\overline{\otimes}\mathcal{N}$ a morphism, there exists a unique morphism $\rho : \mathcal{M}^{*\mathcal{N}} \to \mathcal{R}$ such that

$$\pi = (\rho\overline{\otimes}1)\epsilon.$$

Because of this, the function $\phi_{\mathcal{M},\mathcal{R}} : \mathrm{Hom}(\mathcal{M}^{*\mathcal{N}}, \mathcal{R}) \to \mathrm{Hom}(\mathcal{M}, \mathcal{R}\overline{\otimes}\mathcal{N})$ given by $\phi_{\mathcal{M},\mathcal{R}}(\rho) = (\rho\overline{\otimes}1)\epsilon$ is a bijection. Indeed, the existence part of the universal property of ϵ shows surjectivity, whereas the uniqueness part shows injectivity. What remains is to show that ϕ is natural in $\mathcal{M}$ and $\mathcal{R}$. To this end, let $f : \mathcal{M}_1 \to \mathcal{M}_0$ and $g : \mathcal{R}_0 \to \mathcal{R}_1$ be morphisms. We have to show the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}({\mathcal{M}_0}^{*\mathcal{N}}, \mathcal{R}_0) & \xrightarrow{\phi_{\mathcal{M}_0,\mathcal{R}_0}} & \mathrm{Hom}(\mathcal{M}_0, \mathcal{R}_0\overline{\otimes}\mathcal{N}) \\ {\scriptstyle \mathrm{Hom}(f^{*\mathcal{N}},g)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(f,g\overline{\otimes}\mathcal{N})} \\ \mathrm{Hom}({\mathcal{M}_1}^{*\mathcal{N}}, \mathcal{R}_1) & \xrightarrow{\phi_{\mathcal{M}_1,\mathcal{R}_1}} & \mathrm{Hom}(\mathcal{M}_1, \mathcal{R}_1\overline{\otimes}\mathcal{N}) \end{array}$$

Starting with some morphism ρ and going via the top we have

$$\rho \mapsto (\rho\overline{\otimes}1)\epsilon_0 \mapsto (g\overline{\otimes}1)(\rho\overline{\otimes}1)\epsilon_0 f = (g\rho\overline{\otimes}1)\epsilon_0 f,$$

whilst going via the bottom we have

$$\rho \mapsto g\rho f^{*\mathcal{N}} \mapsto (g\rho f^{*\mathcal{N}}\overline{\otimes}1)\epsilon_1 = (g\rho\overline{\otimes}1)(f^{*\mathcal{N}}\overline{\otimes}1)\epsilon_1,$$

and these are equal since $(g^{*\mathcal{N}}\overline{\otimes}1)\epsilon_1 = \epsilon_0 g$. □

We can now obtain numerous of results by just using basic category theory. For example, since the functor $-\overline{\otimes}\mathcal{N}$ is a right adjoint, it preserves limits. Likewise, since the functor $(-)^{*\mathcal{N}}$ is a left adjoint, it preserves colimits. In particular, this applies to the the product and coproduct, proving the following:

Proposition 3.28. *Let $\mathcal{M}_\alpha$ and $\mathcal{N}$ be a W^* algebras. then*

$$(*_\alpha\mathcal{M}_\alpha)^{*\mathcal{N}} \cong *_\alpha\mathcal{M}_\alpha^{*\mathcal{N}},$$

and

$$(\bigoplus_\alpha \mathcal{M}_\alpha)\overline{\otimes}\mathcal{N} \cong \bigoplus_\alpha \mathcal{M}_\alpha\overline{\otimes}\mathcal{N}.$$

Using the Yoneda lemma, we can make some more statements about the behavior of the exponent with respect to other constructions.

Proposition 3.29. *For $\mathcal{A}, \mathcal{B}, \mathcal{M}$ W^*-algebras, we have*

$$\mathcal{M}^{*(\mathcal{A}\oplus\mathcal{B})} \cong \mathcal{M}^{*\mathcal{A}} * \mathcal{M}^{*\mathcal{B}}.$$

Proof. As mentioned, we use use the Yoneda lemma. We then find

$$\begin{aligned} \mathrm{Hom}(\mathcal{M}^{*(\mathcal{A}\oplus\mathcal{B})}, \mathcal{N}) &\cong \mathrm{Hom}(\mathcal{M}, \mathcal{N}\overline{\otimes}(\mathcal{A}\oplus\mathcal{B})) \\ &\cong \mathrm{Hom}(\mathcal{M}, \mathcal{N}\overline{\otimes}\mathcal{A} \oplus \mathcal{N}\overline{\otimes}\mathcal{B}) \\ &\cong \mathrm{Hom}(\mathcal{M}, \mathcal{N}\overline{\otimes}\mathcal{A}) \times \mathrm{Hom}(\mathcal{M}, \mathcal{N}\overline{\otimes}\mathcal{B}) \\ &\cong \mathrm{Hom}(\mathcal{M}^{*\mathcal{A}}, \mathcal{N}) \times \mathrm{Hom}(\mathcal{M}^{*\mathcal{B}}, \mathcal{N}) \\ &\cong \mathrm{Hom}(\mathcal{M}^{*\mathcal{A}} * \mathcal{M}^{*\mathcal{B}}, \mathcal{N})). \end{aligned}$$

□

Proposition 3.30. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$ be W^*-algebras. Then we have*

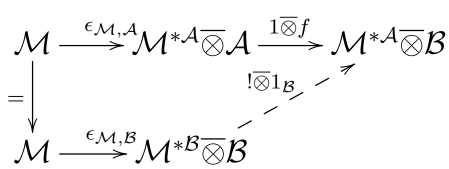
$$(\mathcal{M}^{*\mathcal{A}})^{*\mathcal{B}} \cong \mathcal{M}^{*(\mathcal{B}\overline{\otimes}\mathcal{A})}.$$

Proof.

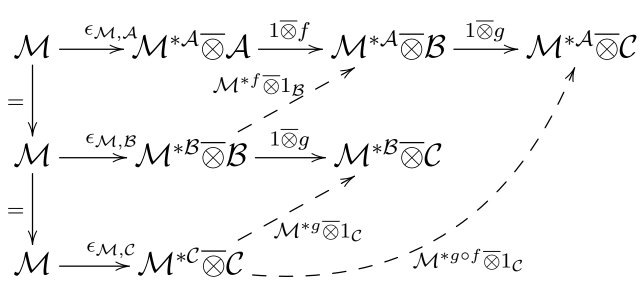
$$\begin{aligned} \mathrm{Hom}((\mathcal{M}^{*\mathcal{A}})^{*\mathcal{B}}, \mathcal{N}) &\cong \mathrm{Hom}(\mathcal{M}^{*\mathcal{A}}, \mathcal{N}\overline{\otimes}\mathcal{B}) \\ &\cong \mathrm{Hom}(\mathcal{M}, \mathcal{N}\overline{\otimes}\mathcal{B}\overline{\otimes}\mathcal{A}) \\ &\cong \mathrm{Hom}(\mathcal{M}^{*(\mathcal{B}\overline{\otimes}\mathcal{A})}, \mathcal{N}). \end{aligned}$$

□

The exponential $\mathcal{M}^{*\mathcal{N}}$ also gives rise to the assignment $\mathcal{N} \mapsto \mathcal{M}^{*\mathcal{N}}$. We will show that this also defines a functor $\mathcal{M}^{*-}$, so we need to show how this works on morphisms. Let $f : \mathcal{A} \to \mathcal{B}$. Then we have



Here the arrow ! is the unique arrow $\mathcal{M}^{*\mathcal{B}} \to \mathcal{M}^{*\mathcal{A}}$ that follows from the universal property of $\epsilon_{\mathcal{M},\mathcal{B}}$. We shall write $! = \mathcal{M}^{*f}$ and this will be how $\mathcal{M}^{*-}$ acts on morphisms. If $\mathcal{M}^{*-}$ is indeed a functor, we see from this that it is contravariant. We check the composition property explicitly using the following diagram:



Following this diagram, we find this chain of equations:

$$\begin{aligned}(\mathcal{M}^{*g\circ f}\overline{\otimes}1)\circ\epsilon_{\mathcal{M},\mathcal{C}} &= (1\overline{\otimes}g)(1\overline{\otimes}f)\circ\epsilon_{\mathcal{M},\mathcal{A}} \\ &= (1\overline{\otimes}g)(\mathcal{M}^{*f}\overline{\otimes}1)\circ\epsilon_{\mathcal{M},\mathcal{B}} \\ &= (\mathcal{M}^{*f}\overline{\otimes}1)(1\overline{\otimes}g)\circ\epsilon_{\mathcal{M},\mathcal{B}} \\ &= (\mathcal{M}^{*f}\overline{\otimes}1)(\mathcal{M}^{*g}\overline{\otimes}1)\circ\epsilon_{\mathcal{M},\mathcal{C}}.\end{aligned}$$

By uniqueness we have $\mathcal{M}^{*g\circ f} = \mathcal{M}^{*f}\circ\mathcal{M}^{*g}$ and preservation of the identity is easily seen, so we indeed have a (contravariant) functor.
For contravariant functors, the notion of adjunction is slightly different from the notion of adjunction for covariant functors. Two contravariant functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ between categories $\mathcal{C}$ and $\mathcal{D}$ can be in left or right adjunction via $\mathrm{Hom}_{\mathcal{D}}(FC, D) \cong \mathrm{Hom}_{\mathcal{C}}(GD, C)$ or $\mathrm{Hom}_{\mathcal{D}}(D, FC) \cong \mathrm{Hom}_{\mathcal{C}}(C, GD)$, respectively. We can see this, for example, via the identification of a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ with a covariant functor ${}^\circ F : \mathcal{C}^{op} \to \mathcal{D}$. Now we have $\mathrm{Hom}_{\mathcal{D}}({}^\circ FC, D) \cong \mathrm{Hom}_{\mathcal{C}^{op}}(C, {}^\circ GD) \cong \mathrm{Hom}_{\mathcal{C}}({}^\circ GD, C)$.

Proposition 3.31. *The functor $\mathcal{M}^{*-}$ is its own adjoint on the left.*

Proof. We just check

$$\begin{aligned}\mathrm{Hom}(\mathcal{M}^{*\mathcal{N}}, \mathcal{R}) &\cong \mathrm{Hom}(\mathcal{M}, \mathcal{R}\overline{\otimes}\mathcal{N}) \\ &\cong \mathrm{Hom}(\mathcal{M}, \mathcal{N}\overline{\otimes}\mathcal{R}) \\ &\cong \mathrm{Hom}(\mathcal{M}^{*\mathcal{R}}, \mathcal{N}).\end{aligned}$$

□

Proposition 3.32. *The functor $\mathcal{M}^{*-}$ sends all limits to corresponding colimits. That is to say, if $\mathcal{A}_i$ is some diagram in $\mathbf{W}^*$ and $\lim_i \mathcal{A}_i$ is its limit (we remove the morphisms from the notation), then*

$$\mathcal{M}^{*\lim_i \mathcal{A}_i} \cong \operatorname{co}\lim_i \mathcal{M}^{*\mathcal{A}_i}.$$

Proof. Once again, we use the Yoneda lemma and find

$$\begin{aligned}\mathrm{Hom}(\mathcal{M}^{*\lim_i \mathcal{A}_i}, \mathcal{R}) &\cong \mathrm{Hom}(\mathcal{M}, \mathcal{R}\overline{\otimes}\lim_i \mathcal{A}_i) \\ &\cong \mathrm{Hom}(\mathcal{M}, \lim_i \mathcal{R}\overline{\otimes}\mathcal{A}_i) \\ &\cong \lim_i \mathrm{Hom}(\mathcal{M}, \mathcal{R}\overline{\otimes}\mathcal{A}_i) \\ &\cong \lim_i \mathrm{Hom}(\mathcal{M}^{*\mathcal{A}_i}, \mathcal{R}) \\ &\cong \mathrm{Hom}(\operatorname{co}\lim_i \mathcal{M}^{*\mathcal{A}_i}, \mathcal{R}).\end{aligned}$$

So all limits are sent to corresponding colimits. □

As an example, we have already found the isomorphism $\mathcal{M}^{*\mathcal{A}\oplus\mathcal{B}} \cong \mathcal{M}^{*\mathcal{A}} * \mathcal{M}^{*\mathcal{B}}$ in Proposition 3.29.
We note here that if the adjunction would have been on the right, then all colimits would have been sent to corresponding limits.

Another special case is that of the terminal object 0, which is mapped to the initial object, $\mathbb{C}$, so we find

$$\mathcal{M}^{*0} \cong \mathbb{C}.$$

The universal property of ϵ in the special case of $\mathcal{M} = 0$ tells us there is a unique morphism $\epsilon_{0,\mathcal{N}} : 0 \to 0^{*\mathcal{N}} \overline{\otimes} \mathcal{N}$. Now the only morphism with domain 0 is its identity function, so $0^{*\mathcal{N}} \overline{\otimes} \mathcal{N} = 0$. If $\mathcal{N} \neq 0$, this implies that $0^{*\mathcal{N}} = 0$. For $\mathcal{N} = 0$, we have already found the answer $0^{*0} = \mathbb{C}$. Going to the opposite category and interpreting $^{\circ}\mathcal{M}^{*\mathcal{N}}$ as the functions from $^{\circ}\mathcal{N}$ to $^{\circ}\mathcal{M}$, we indeed find 0^{*0} to correspond with the set of functions from $\emptyset$ to $\emptyset$. There is exactly one such function, so it is a singleton, which corresponds to $\mathbb{C}$ in $\mathbf{W}^*$. There are no morphisms from $\emptyset$ to any other object, so the formula $0^{*\mathcal{N}} = 0$ makes sense in the same way. We also note the similarity with natural numbers, where $0^n = 0$ for $n \neq 0$ and $0^0 = 1$.

3.7 Conclusion

In conclusion, $\mathbf{W}^*$ is a nicely behaved category which has products, coproducts, general limits and colimits, but is not Cartesian closed. We have seen we can circumvent this problem. The role of coproduct in the original definition of coexponential is replaced by the spatial tensor product. To this end we can think of $^{\circ}\mathcal{M}^{*\mathcal{N}} \overline{\otimes} \mathcal{N}$ as a Cartesian product of the space of functions from $^{\circ}\mathcal{N}$ to $^{\circ}\mathcal{M}$ with $^{\circ}\mathcal{N}$. The map $^{\circ}\epsilon_{\mathcal{M},\mathcal{N}}$ is then thought of as evaluation.

The fact that we think of $^{\circ}\mathbf{W}^*$ as a category resembling **Set** comes from the fact that for any set X, we have the von Neumann algebra $\ell^{\infty}(X)$ of bounded functions on X equiped with the discrete topology.

Kornell then proceeds by looking at the category of von Neumann algebras with completely positive *-homomorphisms[2] as morphisms. We do not follow approach, but instead look to generalize the construction valid in $\mathbf{W}^*$ to some other category of operator algebras.

[2] A map $\psi : \mathcal{M} \to \mathcal{N}$ is completely positive if all maps $\psi \otimes id_{M_n} : \mathcal{M} \overline{\otimes} M_n \to \mathcal{N} \overline{\otimes} M_n$ are positive for all $n \in \mathbb{N}$, where M_n are the $n \times n$ matrices over $\mathbb{C}$.

4 AW^*-algebras

In this section we define the notion of AW^*-algebras, which historically were invented by I. Kaplansky to be an algebraic generalization of von Neumann algebras. Our goal is to construct the categorical constructions valid for von Neumann algebras. This will first lead us to take a look at rings.

4.1 Rickart *-rings and Baer *-rings

Definition 4.1. *Let A be a ring and let $S \subset A$ be a non-empty subset of A. We define the* right annihilator[3] *of S as*

$$R(S) = \{x \in A \mid sx = 0 \, \forall s \in S\}.$$

Similarly, we define the left annihilator *of S as*

$$L(S) = \{x \in A \mid xs = 0 \, \forall s \in S\}.$$

We now state some very simple, but important, properties of these annihilators.

Proposition 4.2. *Let A be a ring and $S \subset A$ a non-empty subset of A. Then the following properties hold:*

- *$R(S)$ is a right ideal in A. Likewise, $L(S)$ is a left ideal in A.*
- *$S \subset L(R(S))$. Likewise, $S \subset R(L(S))$.*
- *If $B \subset S$ then $R(S) \subset R(B)$ and $L(S) \subset L(B)$.*
- *$L(S) = L(R(L(S)))$ and $R(S) = R(L(R(S)))$.*

Proof.

- If $r \in R(S)$ and $s \in S$, then of course $sra = 0$ for all $a \in A$.
- For $r \in R(S)$, we have $sr = 0$ for all $s \in S$ so $s \in L(R(S))$.
- For $r \in R(S)$ and $b \in B$ we have $br = 0$ since $b \in B \subset S$.
- Combine the above two items.

□

Interesting things happen when the ring A has more structure, like an involution to make it a *-ring, and/or a structure making it an algebra. If so, we can add the following to our list of properties:

Proposition 4.3. *Let A be a ring and $S \subset A$ a non-empty subset of A.*

- *If A is also an algebra, then $R(S)$ and $L(S)$ are linear subspaces of A.*
- *If A is a *-ring, then $L(S^*)^* = R(S)$. Likewise, $L(S) = R(S^*)^*$.*

[3] Note that these annihilators are different from those introduced in the first section.

Proof.

- It is clear that any multiple of an element in $R(S)$ is still in $R(S)$ and the same applies to sums of elements in $R(S)$.
- We calculate

$$\begin{aligned} L(S^*)^* &= \{x^* \in A \mid xs^* = 0\,\forall s \in S\} \\ &= \{x \in A \mid (sx)^* = 0\,\forall s \in S\} \\ &= R(S). \end{aligned}$$

□

Definition 4.4. *A* Rickart *-ring *is a *-ring A such that the right annihilator of any singleton is of the form eA with e a projection ($e^2 = e = e^*$) in A, i.e.,*

$$R(\{x\}) = eA = \{ea \mid a \in A\}$$

is the right ideal generated by e.

Proposition 4.5. *Let A be a Rickart *-ring and $x \in A$. Then the projection generating $R(\{x\})$ is unique. That is, if $R(\{x\}) = eA = fA$ with e, f projections, then $e = f$.*

Proof. Suppose $eA = fA$, then $e = ee \in eA = fA$. Say $e = fb$. Then

$$fe = ffb = fb = e,$$

so $e \leq f$. By the same argument, $f \leq e$. Therefore $f = e$. □

The condition that any right annihilator of a singleton is a right ideal generated by a single projection implies that the left annihilators of singletons are left ideals generated by a single projection, making the definition of a Rickart *-ring left-right symmetric. Indeed,

$$L(\{x\}) = R(\{x^*\})^* = (hA)^* = Ah$$

where h is the projection generating the right annihilator of $\{x^*\}$.

Proposition 4.6. *Every Rickart *-ring A has a unique unit element.*

Proof. We look at $R(\{0\})$. On the one hand this is of the form eA for a projection e. On the other hand, we know it is the whole of A so, since e is a projection, e is a left unit. Since we also have $A = A^* = (eA)^* = Ae$, we find that e is a unit. Now if 1 is also a unit, we find $e = e1 = 1$. □

Proposition 4.7. *Let A be a Rickart *-ring. If $x^*x = 0$, then $x = 0$. In other words, the involution is* proper.

Proof. Suppose $x^*x = 0$. Then we know that

$$x \in R(\{x^*\}) = hA,$$

for some projection $h \in A$. Therefore, $hx = x$ (since h acts trivially on $R(\{x^*\}) = hA$). Taking adjoints we then find $x^* = x^*h$. Since A has a unit we have that $h \in R(\{x^*\})$, so

$$0 = x^*h = x^*.$$

Now $x = x^{**} = 0^* = 0$. □

Proposition 4.8. *In any *-ring with proper involution (in particular in a Rickart *-ring) we have*

$$xy = 0 \Leftrightarrow x^*xy = 0.$$

In particular, we find that

$$R(\{x\}) = R(\{x^*x\}).$$

Proof. Suppose $x^*xy = 0$. Then

$$0 = y^*x^*xy = (xy)^*xy,$$

so $xy = 0$. The other way around is trivial. □

Proposition 4.9. *Let A be a Rickart *-ring and $x \in A$. Then there exists a unique projection e which satisfies the following two conditions:*

$$xe = x,$$

$$xy = 0 \Leftrightarrow ey = 0.$$

Furthermore, if h is a projection such that $xh = x$, then $e \leq h$.

Proof. Let $e = 1 - g$, where g is the unique projection such that $R(\{x\}) = gA$. We first show that e satisfies the proper equations. Indeed, we obviously have

$$xe = x(1 - g) = x.$$

Furthermore, suppose $xy = 0$, then $y = ga$ for some $a \in A$. Therefore

$$ey = ega = 0.$$

Finally, suppose $ey = 0$, then, since $x = xe$, we have

$$xy = xey = 0.$$

Now suppose h is a projection such that $x = xh$. Then $x(1 - h) = 0$, so $e(1 - h) = 0$ and $e = eh$.
Finally, to show uniqueness, suppose that the above element h in addition satisfies

$$xy = 0 \Leftrightarrow hy = 0.$$

Then we find $xe = x = xh$, so that $x(e - h) = 0$. Therefore, $h(e - h) = 0$. We conclude that $he = h$, so that $e = h$. □

Of course, a similar argument shows that if $f = 1 - q$, where q is the unique projection such that $L(\{x\}) = Aq$, then f is the unique projection such that

- $fx = x$,
- $yx = 0 \Leftrightarrow yf = 0$,
- if h is a projection such that $hx = x$, then $f \leq h$.

We call e the *right projection* of x and call f the *left projection* of x, denoted $e = RP(x), f = LP(x)$.
From now on, we write $R(x)$ instead of $R(\{x\})$ to denote the annihilator of the subset $\{x\} \subset A$.

An immediate consequence of Proposition 4.8 is that

$$RP(x^*x) = RP(x).$$

Corollary 4.10. *Let A be a Rickart *-ring and $x \in A$. Then $LP(x) = RP(x^*)$.*

Proof. Let $x \in A$. Then since $LP(x)x = x$, we have $x^*LP(x) = x^*$. Therefore, $RP(x^*) \leq LP(x)$. Since $xRP(x) = x$ we have $RP(x)x^* = x^*$. Therefore, $LP(x^*) \leq RP(x)$. Combining these results, we find

$$LP(x) \geq RP(x^*) \geq LP(x).$$

□

Theorem 4.11. *Let A be a Rickart *-ring. Then the projections of A form a lattice. Explicitly, we have:*

$$\begin{aligned} p \vee q &= q + RP[p(1-q)], \\ p \wedge q &= q - LP[q(1-p)]. \end{aligned}$$

Proof. Since $p(1-q)q = 0$, we have $RP[p(1-q)]q = 0$. Therefor, $q + RP[p(1-q)]$ is indeed a projection. Of course, we have $q \leq q + RP[p(1-q)]$. To see that $p \leq q + RP[p(1-q)]$, we calculate

$$\begin{aligned} 0 &= p(1-q)RP[p(1-q)] - p(1-q) \\ &= pRP[p(1-q)] - p(qRP[p(1-q)]) - p + pq \\ &= pRP[p(1-q)] - p(RP[p(1-q)]q)^* - p + pq \\ &= pRP[p(1-q)] - p + pq \\ &= p(q + RP[p(1-q)]) - p. \end{aligned}$$

Now suppose r is a projection such that $p \leq r$ and $q \leq r$. Then we have $p(1-q)r = p(1-q)$, from which it follows that $RP[p(1-q)] \leq r$, but then also $q + RP[p(1-q)] \leq r$. We now claim that this implies that $p \wedge q$ also exists. In fact, it equals

$$p \wedge q = 1 - (1-p) \vee (1-q).$$

For this, first note that if $x \leq y$ then $(1-y) \leq (1-x)$ for all projections x, y. We introduce the notation $p^\perp = 1 - p$ for a projection p. Now of course

$$p^\perp \vee q^\perp \geq p^\perp,$$

and

$$p^\perp \vee q^\perp \geq q^\perp.$$

So indeed

$$(p^\perp \vee q^\perp)^\perp \leq p,$$

and

$$(p^\perp \vee q^\perp)^\perp \leq q.$$

Now suppose $x \leq p, x \leq q$, then we need to show that $x \leq (p^\perp \vee q^\perp)^\perp$, or, which comes to the same,

$$p^\perp \vee q^\perp \leq x^\perp.$$

Since

$$\begin{aligned} p^\perp(1-q^\perp)x^\perp &= p^\perp x^\perp - p^\perp q^\perp x^\perp \\ &= p^\perp - p^\perp q^\perp \\ &= p^\perp(1-q^\perp), \end{aligned}$$

we find $RP[p^\perp(1-q^\perp)] \leq x^\perp$. We already have $q^\perp \leq x^\perp$, so we finally find

$$q^\perp + RP[p^\perp(1-q^\perp)] \leq x^\perp.$$

To get to our desired formula, we calculate

$$\begin{aligned} p \wedge q &= (p^\perp \vee q^\perp)^\perp \\ &= 1 - p^\perp \vee q^\perp \\ &= 1 - (q^\perp + RP[p^\perp(1-q^\perp)]) \\ &= q - RP[(1-p)q] \\ &= q - RP[(q(1-p))^*] \\ &= q - LP[q(1-p)]. \end{aligned}$$

□

If S_1 and S_2 are two subsets of a ring A, we can look at the right annihilator of their union. It is clear that

$$R(S_1 \cup S_2) = R(S_1) \cap R(S_2).$$

More generally,

$$R(\bigcup_i S_i) = \bigcap_i R(S_i),$$

for any family of subsets $S_i \subset A$. Now if A is a Rickart *-ring and $x, y \in A$, we have

$$R(x) = pA \text{ and } R(y) = qA,$$

for certain projections p and q. We then find

$$R(\{x,y\}) = pA \cap qA.$$

We claim

$$R(\{x,y\}) = (p \wedge q)A.$$

Indeed, for $a \in A$, we have

$$\begin{aligned} (p \wedge q)a &= p(p \wedge q)a \\ &= q(p \wedge q)a, \end{aligned}$$

so

$$(p \wedge q)A \subset pA \cap qA.$$

The other way around, let $x = pa = qa'$ be an element of $pA \cap qA$. Now

$$\begin{aligned}
(p \wedge q)x &= (q - LP[q(1-p)])x \\
&= qqa' - LP[q(1-p)]pa \\
&= qa' \\
&= x,
\end{aligned}$$

where we have used

$$q(1-p)p = 0 \Leftrightarrow LP[q(1-p)]p = 0.$$

So indeed,

$$x \in (p \wedge q)A.$$

More generally, we can do this for any finite number of elements. However, if we try to do this for arbitrary subsets, there is of course no guarantee that such a generating projection (which is the infimum of all generating projections of the right annihilators of the singletons) still exists, since the lattice of projections need not be complete.

Definition 4.12. *A* Baer *-ring *is a *-ring A such that for each non-empty subset $S \subset A$, the right annihilator of S is generated as an right ideal by a projection, i.e., $R(S) = qA$ for some projection q.*

Suppose A is a Baer *-ring and $S \subset A$ is some subset. Then

$$R(S) = \bigcap_{s \in S} R(S) = \bigcap_{s \in S} p_s A = qA,$$

where the p_s are the right annihilating projections generating $R(s)$ and q is a projection. Then, for each $s \in S$ there is a $a_s \in A$ such that $q = p_s a_s$. It follows that

$$p_s q = p_s p_s a = q,$$

so $q \leq p_s$ for all $s \in S$. Now, if e is a projection such that $q \leq e \leq p_s$ for all $s \in S$, then

$$qa = eqa \in eA,$$

so $qA \subset eA$. Furthermore,

$$p_s xa = xa,$$

for any $a \in A$ and $s \in S$, so

$$xA \subset \bigcap_{s \in S} p_s A = qA.$$

We have found that

$$qA = eA,$$

so in particular $e = qa$ for some $a \in A$ and so $qe = e$. Also, $q = ea'$ for some $a' \in A$, therefore

$$eq = q = e.$$

The conclusion is that q is the infimum of all right annihilators.
So suppose we have an arbitrary family of projections $\{p_i\}_i$ in a Baer *-ring A. We let $S \subset A$ be the subset $S = \{1 - p_i\}_i$. Then

$$R(S) = \bigcap_i p_i A = pA$$

where p exists by definition of a Baer *-ring. By what we have just seen, we have now proven:

Proposition 4.13. *In a Baer *-ring the infima of arbitrary families of projections exist, i.e., the projection lattice is complete.*

By the same reasoning as before we find that suprema of arbitrary families of projections also exits and are equal to

$$\sup_i p_i = 1 - \inf\{1 - p_i\}.$$

We can now also easily prove the converse to Proposition 4.13

Proposition 4.14. *If A is a Rickart *-ring in which the projection lattice is complete, then A is a Baer *-ring.*

Proof. Let S be an arbitrary subset in A. For every $s \in S$ we have $R(s) = p_s A$ for some projection p_s. Let p be the infimum of the p_s, which exists by assumption. Then as we have seen $R(S) = pA$. □

Definition 4.15. *A Baer *-ring that is also a C^*-algebra is called an* AW^*-algebra.

In particular, the projection lattice of an AW^*-algebra is complete.

4.2 Commutative AW^*-algebras

Whenever we deal with commutative C^*-algebras we know that, up to isomorphism, every such algebra is of the form $C(X)$ for some (locally) compact Hausdorff space X. In the case of a von Neumann algebra, X is hyperstonean [11]. To be complete, let us recall the definitions.

Definition 4.16.

- *A space X is called a* Stone space *if it is compact, Hausdorff and totally disconnected in the sense that the empty set and the singletons (one point sets) are the only connected subsets of X.*
- *A space X is called* Stonean *if it is Stone and the closure of any open set is open (and therefore clopen).*
- *A measure μ on X is called* normal *if for any increasing bounded net of continuous functions $\{f_i\}_i$ on X with supremum f we have*

$$\mu(f) = \sup_i \mu f_i.$$

- *A space X is called* hyperstonean *if it is Stonean and if for any nonzero positive continuous function f on X there exists a positive normal measure μ with $\mu(f) > 0$.*

So a commutative AW^*-algebra is certainly of the form $C(X)$ for some compact Hausdorff space X. However we may not expect X to be hyperstonean, since there are AW^*-algebras that are not von Neumann algebras. One might already guess where this is heading: a commutative AW^*-algebra is of the form $C(X)$ where X is a Stonean space.

Theorem 4.17. *Let T be a compact Hausdorff space. Then $C(T)$ is an AW^*-algebra if and only if T is Stonean.*

We note that if p is a projection in $C(T)$, then $p^* = p = p^2$ means that p is a continuous function taking only the values 0 and 1. This means that every projection is the characteristic function of a clopen set, and *vice versa*.

Proof. First we show that if T is Stonean, then $C(T)$ is an AW^*-algebra.
We will show $C(T)$ to be a Rickart *-ring with a complete projection lattice ($C(T)$ is obviously a C^*-algebra). So, let $f \in C(T)$ and $U = \{t \in T \mid f(t) \neq 0\}$. Let p be the characteristic function of $\overline{U}$. Since U is open, $\overline{U}$ is clopen and p is a projection such that $1 - p$ is the right annihilator of f. By Proposition 4.13, it suffices to show the projection lattice is complete. Let U_i be a family of clopen sets. We will show that $U = \overline{\bigcup_i U_i}$ is the supremum of this family. Of course every $U_i \subset U$. If V is clopen and $U_i \subset V$ for all i, then $\bigcup_i U_i \subset V$ and so $U = \overline{\bigcup_i U_i} \subset \overline{V} = V$.
The other way around, suppose $C(T)$ is an AW^*-algebra. Let U be open T, we need to show $\overline{U}$ is open. We will use the fact that clopen sets in T are basic for the topology on T. With that, we can find a family of clopen sets U_i such that $\bigcup_i U_i = U$. Since the projection lattice of $C(T)$ is complete, so is the lattice of clopen sets. Let V be the supremum of the U_i in this lattice of clopen sets. Then every $U_i \subset V$ and so $U \subset V$ and therefore $\overline{U} \subset \overline{V} = V$. We are done if $\overline{U} = V$, since then $\overline{U}$ is (cl)open. So let $W = V - \overline{U}$ and assume $W \neq \emptyset$. Then, using again the fact that the clopens form a basis for the topology, there is a clopen set $Q \subset W$. Now $Q \cap U_i = \emptyset$ for all i, so for each i we have $U_i \subset T - Q$, but then also $V \subset T - Q$. However, we now have $V \cap Q = \emptyset$, while we assumed $Q \subset W = V - \overline{U}$. So indeed, $Q = \emptyset$. □

Lemma 4.18. *If $C(T)$ is an AW^*-algebra, then the clopen sets in T form a basis for T.*

Proof. Let $x, y \in T$ be distinct points. Since T is Hausdorff, we can find $U \ni x, V \ni y$ open such that $U \cap V = \emptyset$. Now let f, g be continuous functions on T such that f satisfies $f(x) \neq 0$, $f = 0$ on $T - U$ and g satisfies $g(y) \neq 0$, $g = 0$ on $T - V$. Now $RP(f)$ corresponds to a clopen set P such that $x \in P$ and $y \notin P$ since $RP(f)g = 0$. So we can separate points in T by clopen sets.
Now let U be a open set, and $x \in U$. For each $y \in T - U$, we can find clopen sets separating x and y. Since T is compact and U is open, $T - U$ is compact and so we can find a finite number of clopen sets V_i such that $T - U \subset \bigcup_i V_i$. Now $T - \bigcup_i V_i$ is a clopen set, containing x and contained in U. □

Lemma 4.19. *Let A be an AW^*-algebra and $0 \neq x \in A$, with $x \geq 0$. Given $\epsilon > 0$, there exists an $y \in \{x\}''$, $y \geq 0$, such that*

1. *$xy = e$ for some nonzero projection e,*

2. *$\|x - xe\| < \epsilon$.*

Proof. Since $\{x\}''$ is a commutative AW^*-algebra, it is of the form $\{x\}'' = C(T)$ for some Stonean space T. The spectrum of x in $C(T)$ is the same as its spectrum in A, so as a function on T, x assumes only non-negative values.
If $\epsilon \geq \|x\|$, part two of the theorem is trivial, so we assume $0 < \epsilon < \|x\|$. Let

$$U = \{t \in T \mid x(t) > \frac{\epsilon}{2}\}.$$

Then $U = x^{-1}((\frac{\epsilon}{2}, \infty))$ is open, and since $\|x\| \in U$, it is also non-empty. Since T is Stonean, the closure $\overline{U}$ of U, is clopen. Let e be the characteristic function of $\overline{U}$, so that e is a non-zero projection in $C(T)$. Since $x(1-e) = 0$ on $\overline{U}$ and $x(1-e) \leq \frac{\epsilon}{2}$ on the complement of $\overline{U}$, we have

$$x(1-e) \leq \frac{\epsilon}{2} \cdot 1_T,$$

and therefore $\|x - xe\| < \epsilon$. Furthermore, since $\overline{U}$ is the intersection of all closed subsets X of T such that $U \subset X$, and $x^{-1}([\frac{\epsilon}{2}, \infty])$ is such a set, we have $x(t) \geq \frac{\epsilon}{2}$ on $\overline{U}$. Therefore we can define a function y on T via

$$y(t) = \begin{cases} \frac{1}{x(t)} & \text{if } t \in \overline{U}, \\ 0 & \text{if } t \notin \overline{U}. \end{cases}$$

Since $\overline{U}$ is clopen and x is continuous, y is continuous and therefore it is an element of $C(T) = \{x\}''$. Now y obviously satisfies $xy = e$ and $y \geq 0$, which finishes the proof. □

If we do not suppose $x \geq 0$ in the previous theorem, then we can still look at x^*x which is positive. Therefore there exists $y \in \{x^*x\}''$, $y \geq 0$ such that, given $\epsilon > 0$,

1. $x^*xy = e$ for some nonzero projection e,

2. $\|x^*x - x^*xe\| < \epsilon$.

Since $e = x^*xy$ and $y \in \{x^*x\}''$, we have $e \in \{x^*x\}''$. Therefore, e commutes with x^*x. Using this, we furthermore find

$$\begin{aligned} \|x - xe\|^2 &= \|(x - xe)^*(x - xe)\| \\ &= \|x^*x - ex^*x - x^*xe + ex^*xe\| \\ &= \|x^*x - x^*xe\| \\ &\leq \epsilon. \end{aligned}$$

4.3 AW^*-subalgebras

Let A be a Baer *-ring and B a *-subset in A. It could of course happen that B is again a Baer *-ring in its own right. However, when calculating something like suprema or a right projection it could matter if we view B on its own or as a subset of A. Therefore, we have the following definition.

Definition 4.20. *Let A be a Baer *-ring and B a *-subring in A. We say B is a* Baer *-subring *if*

- *whenever $x \in B$, we have $RP(x) \in B$,*
- *if S is a nonempty set of projections in B, then $\sup S \in B$.*

Lemma 4.21. *Let A be a Baer *-ring and let $e \in A$ be a projection. Then $B = eAe$ is a Baer *-subring of A.*

Proof. Let p be a projection in A. Then of course epe is a projection in B and $epe \leq e$. Let q be a projection in B. Then q is also a projection in A and $q = eqe$, so $q \leq e$. We see that the projections in B are precisely the projections in A which are $\leq e$.
Now let $\{p_i\}$ be some set of projections in B with supremum p in A. Since each p_i is in B, we have

$$p_i = ep_ie.$$

Therefore

$$pe = \sup_i (ep_ie)e = p,$$

so $p \leq e$ and $p \in B$. Furthermore, let $x \in B$, then $x = exe$, so $x(1-e) = 0$. Therefore $RP(x)(1-e) = 0$, which is to say $RP(x) \leq e$. □

Because of the contrived definition of a Baer *-subring, we do not *a priori* know that a Baer *-subring is itself a Baer *-ring. Fortunately:

Lemma 4.22. *If B is a Baer *-subring of a Baer *-ring A, then B is itself a Baer *-ring.*

Proof. The projection $e = \sup\{RP(x) \mid x \in B\}$ is an element of B and acts as a unit element on B. Now for $x \in B$, $RP(x)$ is in B, and

$$RP(x)e = RP(x),$$

so $e - RP(x)$ is a projection. For $y \in B$ we have

$$\begin{aligned} xy = 0 &\Leftrightarrow RP(x)y = 0 \\ &\Leftrightarrow (e - RP(x))y = y, \end{aligned}$$

so

$$R(x) = (e - RP(x))B.$$

This shows that B is a Rickart *-ring. By definition, the suprema of arbitrary families of projections in B are contained in B and hence, so are their infima. So B is a Rickert *-ring with complete projection lattice, and therefore a Baer *-ring. □

The following lemma shows that in the definition of a Baer *-subring it suffices to look at orthogonal projections instead of arbitrary sets of projections.

Lemma 4.23. *Let A be a Baer *-ring and $B \subset A$ be a *-subring. Then B is a Baer *-subring if and only if the following two conditions are both satisfied:*

1. *$x \in B \Rightarrow RP(x) \in B$,*

2. *if $\{e_i\}$ is a family of orthogonal projections in B, then $\sup\{e_i\} \in B$.*

Proof. A Baer *-subring obviously satisfies these conditions, so we only need to prove the converse. Let $\{p_i\}$ be any family of projections in B and set

$$p = \sup\{p_i\}.$$

Choose a maximal orthogonal family $\{e_i\}$ of projections in B such that $e_i \leq p$ for all i. Set

$$e = \sup\{e_i\},$$

then $e \in B$ and $e \leq p$. If, for any p_i, we have $p_i - p_i e \neq 0$, then

$$p_i \vee e - e = RP(p_i - p_i e),$$

which (as we see from Theorem 4.11) is a projection in B, for which

$$(p_i \vee e - e)p_j = p_j - p_j = 0,$$

for any j and

$$(p_i \vee e - e)p = p_i \vee e - e.$$

This cannot be, since the family $\{e_i\}$ was chosen to be maximal. Therefore, $p_i \leq e$ for all i, and hence also $p \leq e$ so $p = e \in B$. □

Lemma 4.24. *Let A be a AW^*-algebra. If $0 \neq x \in A$ and $\{e_i\}_i$ is a maximal orthogonal family of nonzero projections in A such that for each e_i*

$$e_i = x^* x {y_i}^* y_i,$$

for some $y_i \in \{x^* x\}''$, then

$$\sup_i \{e_i\} = RP(x).$$

Proof. Let $e = \sup_i\{e_i\}$. We calculate, for any i:

$$\begin{aligned} RP(x)e_i &= RP(x)x^* x {y_i}^* y_i \\ &= (xRP(x))^* x {y_i}^* y_i \\ &= x^* x {y_i}^* y_i \\ &= e_i. \end{aligned}$$

Therefore $RP(x) \geq e_i$ for all i, so $RP(x) \geq e$. Therefore, $RP(x) - e$ is a projection. Suppose $RP(s) - e \neq 0$, then it is orthogonal to all e_i, as we see from

$$(RP(x) - e)e_i = e_i - e_i = 0.$$

Furthermore, $RP(x) = RP(x^*x) \in \{x^*x\}''$, because $\{x^*x\}''$ is an AW^*-algebra containing x^*x. Also, $e \in \{x^*x\}''$ because it is the supremum of the e_i which are all a product of elements in $\{x^*x\}''$. So $RP(x) - e \in \{x^*x\}''$ is as in the statement of the lemma since $\{x^*x\}''$ is commutative and $RP(x) - e \geq 0$. Then, since the e_i are assumed to be maximal, $RP(x) - e$ should be one of the e_i. However, $e + e_i$ is not a projection (since $ee_i \neq 0$), whereas $RP(x)$ is a projection. So $RP(x) - e = 0$. □

Theorem 4.25. *Let A be a AW^*-algebra and $B \subset A$ a *-subring that is also an AW^*-algebra satisfying $\sup S \in B$ for any nonempty set S of orthogonal projections in B. Then*

- *$x \in B \Rightarrow RP(x) \in B$,*
- *B is a AW^*-subalgebra.*

Proof. Once we know the first of these statements, the second follows from Lemma 4.23. So let $0 \neq x \in B$. By Lemma 4.19 (or rather, its immediate consequence) there exists an $y \in \{x^*x\}''$ such that $x^*xy^*y = e$ is a nonzero projection. We take a maximal orthogonal set of such projections $\{e_i\}$. By assumption, their supremum is contained in B, but by the previous lemma this supremum is precisely $RP(x)$. □

Let A, B be AW^*-algebras. The question then arises what the "right" notion of morphisms is between these algebras.

Lemma 4.26. *Let $f : A \to B$ be a *-homomorphism between AW^*-algebras such that f preserves the suprema of orthogonal families of projections. Then the kernel of f is generated (as an ideal) by a central projection, i.e., a projection that commutes with every element in A.*

Proof. Let $\{e_i\}_i$ be a maximal orthogonal family of projections in $\ker(f)$ and let e be the supremum of the e_i. Then

$$f(e) = f(\sup e_i) = \sup f(e_i) = 0,$$

so $e \in \ker(f)$. Since f is a homomorphism, this implies that

$$eA \subset \ker(f) \supset Ae.$$

Now let $y \in \ker(f)$. We want to show that $x = y - ey = 0$, which would imply $\ker(f) = eA$. If $x \neq 0$, then $xx^* \neq 0$. By Lemma 4.19 there would exist a $z \in \{xx^*\}''$ such that $xx^*z = p$ for some non zero projection p. Then

$$ep = e(y - ey)x^*z = 0,$$

and

$$e_ip = e_i(y - ey)x^*z = (e_iy - e_iy)x^*z = 0,$$

and of course $f(p) = 0$ because $f(x) = 0$. This would imply that the family $\{e_i\}$ were not maximal, so indeed, $x = 0$ and $\ker(f) = eA$. By the same reasoning, we find $\ker(f) = Ae$ and so we conclude $\ker(f) = eAe$.
To show e is central, we take $a \in A$ and consider $ea = ea'e$ for some $a' \in A$. Right multiplication with $(1-e)$ on both sides then gives $ea(1-e) = 0$. Likewise, if we consider $ae = ea''e$ for some $a'' \in A$ and left multiply with $(1 - e)$, we find $(1 - e)ae = 0$. This finally gives $ea = eae = ae$, so indeed e is central. □

Lemma 4.27. *If e is a central projection in a Baer *-ring A, we can make the decomposition*

$$A = (1-e)A(1-e) \oplus eAe.$$

If $x \in (1-e)A(1-e)$ and $y \in eAe$, then

$$RP(x+y) = RP(x) + RP(y).$$

Proof. Since $(1-e)A(1-e)$ and eAe are Baer *-subrings of A, we have

$$RP(x) \in (1-e)A(1-e),$$

and

$$RP(y) \in eAe.$$

We check the relevant properties of the right projection:

- Because $xRP(y) = 0$ (since $xy = 0$) and likewise for $yRP(x)$, we have

$$\begin{aligned}(x+y)(RP(x)+RP(y)) &= xRP(x) + yRP(y) + xRP(y) + yRP(x) \\ &= x+y.\end{aligned}$$

- Suppose $(x+y)z = 0$. Then $-xz = yz$, and so

$$yz = eyz = -exz = 0 = -xz.$$

  Hence we find

$$(RP(x)+RP(y))z = RP(x)z + RP(y)z = 0.$$

- Suppose $(RP(x)+RP(y))z = 0$. Then

$$RP(x)z = -RP(y)z,$$

  and so

$$eRP(x)z = 0 = -RP(y)z = RP(x)z,$$

  so $yz = 0 = xz$ and obviously $(x+y)z = 0$.

□

Lemma 4.28. *Let $f : A \to B$ be a *-homomorphism between AW^*-algebras such that f preserves suprema of orthogonal families of projections. Then f preserves RP, i.e.,*

$$f(RP(x)) = RP(f(x)).$$

Proof. Let e be the projection generating the kernel of f, then

$$A = (1-e)A(1-e) \oplus eAe.$$

Let $x \in A$. We can write

$$x = \tilde{x} + x'$$

with $\tilde{x} \in (1-e)A(1-e)$ and $x' \in eAe$. Then by the previous lemma we have

$$RP(x) = RP(\tilde{x} + x') = RP(\tilde{x}) + RP(x'),$$

and therefore

$$f(RP(x)) = f(RP(\tilde{x})).$$

On the other hand,

$$f(x) = f(\tilde{x}),$$

so

$$RP(f(x)) = RP(f(\tilde{x})).$$

So in proving $RP(f(x)) = f(RP(x))$ we may assume

$$x \in (1-e)A(1-e).$$

Now, since e is central:

$$\begin{aligned} f(A) &\cong A/ker(f) \\ &= (eA \oplus (1-e)A)/eA \\ &\cong (1-e)A \\ &= (1-e)A(1-e). \end{aligned}$$

So by Lemmas 4.21 and 4.22, $f(A)$ is an AW^*-algebra. Therefore we have $RP(f(x)) \in f(A)$ and $RP(f(x))$ is the same whether we calculate it in B or in $f(A)$. Calculating in $f(A)$ and using the fact that $f : (1-e)A(1-e) \to f(A)$ is an isomorphism, we indeed find

$$f(RP(x)) = RP(f(x)).$$

□

The following theorem shows that the three most "obvious" choices of morphisms for AW^*-algebras actually coincide.

Theorem 4.29. *Let $f : A \to B$ be a *-homomorphism between AW^*-algebras A and B. Then the following are equivalent:*

(1) f preserves the right annihilating projections of arbitrary subsets of A.

(2) f preserves suprema of arbitrary families of projections.

(3) f preserves suprema of orthogonal families of projections.

Before we start the proof we note, for completeness, that (1) means that if $Y \subset A$ is a subset with $R(Y) = eA$ for some projection e, then $R(f(Y)) = f(e)B$.

Proof.

- Suppose (1) holds. Let P be any family of projections with supremum s. We have $R(P) = gA$ for some projection g. By (1) we then have

$$R(f(P)) = f(g)B.$$

Now we claim that $1-f(g)$ is a supremum for $f(P)$; let $f(p) \in f(P)$, then

$$f(p)(1-f(g)) = f(p),$$

so

$$f(p) \leq 1-f(g).$$

Now let $q \in B$ be such that $f(p) \leq q$ for all $p \in P$. Then, since

$$f(p)q - f(p) = 0,$$

we find that

$$1-q \in R(f(P)) = f(g)B.$$

So $1-q = f(g)b$ for some $b \in B$. It follows that

$$f(g)(1-q) = f(g)f(g)b = f(g)b = 1-q,$$

so

$$1-q \leq f(g),$$

which means that

$$1-f(g) \leq q.$$

So indeed $1-f(g)$ is a supremum for $f(P)$. By the same reasoning, we find that $1-g$ is a supremum for P, so

$$1-g = s,$$

and

$$f(s) = f(1-g) = 1-f(g)$$

is the supremum for $f(P)$ which shows (2).

- (2) trivially implies (3).
- We now show that (3) implies (2). By Lemma 4.26 we know that $\ker(f) = zA$ for some central projection z, and so we decompose

  $$A = zAz \oplus (1-z)A(1-z).$$

  Let $\mathcal{P}$ be an arbitrary family of projections in A. Obviously, each $p \in \mathcal{P}$ has such a decomposition as well, i.e.,

  $$p = zp + (1-z)p,$$

  and all elements in $z\mathcal{P}$ are orthogonal to all elements in $(1-z)\mathcal{P}$. Because of this, we find that

  $$f(\bigvee \mathcal{P}) = f(\bigvee z\mathcal{P}) + f(\bigvee (1-z)\mathcal{P}) = f(\bigvee (1-z)\mathcal{P}).$$

  So we can just restrict to $(1-z)A(1-z)$. The rest now follows from the proof of Lemma 4.23.

- Finally, we show that (2) implies (1). Let $Y \subset A$ be any subset, then

  $$R(Y) = \bigcap_{y\in Y} R(y) = \bigcap_{y\in Y}(1 - RP(y))A = \inf_{y\in Y}\{1 - RP(y)\}A.$$

  By assumption, f preserves arbitrary suprema and therefore it also preserves arbitrary infima. Since f also preserves RP, we find

  $$f(\inf_{y\in Y}\{1 - RP(y)\}) = \inf_{y\in Y} 1 - RP(f(y)),$$

  which we see to be precisely the right annihilating projection of $f(Y)$.

□

Thus the morphisms of AW^*-algebras are the same as the morphisms of W^*-algebras.

4.4 W^*-algebras and AW^*-algebras

We began this chapter by saying that AW^*-algebras were introduced as an algebraic generalization of von Neumann algebras. In this section we want to clarify this. We start by showing that AW^*-algebras indeed are such generalizations.

Proposition 4.30. *Let $\mathcal{A} \subset B(H)$ be a von Neumann algebra, then $\mathcal{A}$ is an AW^*-algebra.*

Proof. Let $a \in \mathcal{A}$ be an element. We claim that the right annihilator of a is the projection onto the kernel of a. Indeed, let P be the projection onto $\ker(a)$, we show $1-P$ is the right projection of a. Obviously $a(1-P) = a$. Now suppose $ax = 0$, then x maps H into the kernel of a, so $(1 - P)x = 0$. The next step is to show that $R(a) = P$ is actually in $\mathcal{A}$. We will show $P \in \mathcal{A}'' = \mathcal{A}$. To this end, let $b \in \mathcal{A}'$. If $k \in \ker(a)$, then $bak = 0 = abk$, so b maps $\ker(a)$ in $\ker(a)$.
If $b \in \mathcal{A}'$ then $b^* \in \mathcal{A}'$. Indeed, since $\mathcal{A}$ is closed under involution, $ba^* = a^*b$ for all $a \in \mathcal{A}$. Taking the star on both sides then gives $ab^* = b^*a$ for all $a \in \mathcal{A}$. In particular, b^* also maps $\ker(a)$ into itself. If $h \in \ker(a)^\perp$ and $k \in ker(a)$, then $\langle k, bh\rangle = \langle b^*k, h\rangle = 0$, so b also maps $\ker(a)^\perp$ into itself.
Now for any $h \in H$ we may write $h = h^\parallel + h^\perp$ with $h^\parallel \in \ker(a)$ and $h^\perp \in \ker(a)^\perp$. We then have $bPh = bh^\parallel$, while $Pbh = P(bh^\parallel + bh^\perp) = bh^\parallel$, showing $P \in \mathcal{A}'' = \mathcal{A}$. So any von Neumann algebra is certainly a Rickart *-ring. We need to show its projection lattice is complete. This will be done in the following Lemma. □

Lemma 4.31. *In a von Neumann algebra $\mathcal{A} \subset B(H)$, the projection lattice is complete.*

Proof. Given a family of projections $\{p_i\}$ in $\mathcal{A}$, each p_i projects on some closed subspace of H. Then $\sup_i p_i$ is the projection onto the closure of the ranges of the p_i, i.e.,

$$\sup_i p_i = P_{\overline{\bigcup_i p_i H}}.$$

The infimum of the p_i also exists, and is equal to

$$\inf_i p_i = P_{\bigcap_i p_i H}.$$

Note that all p_iH are closed, so $\bigcap_i p_iH$ is automatically closed, while $\bigcup_i p_iH$ need not be. That these expressions indeed satisfy the right properties for sup and inf is easily seen. The fact that $\sup_i p_i$ and $\inf_i p_i$ are elements of $\mathcal{A}$ follows in the same way as in the above theorem. □

We could have also finished the proof of the theorem in a more direct way. Let $S \subset \mathcal{A}$ be some subset, and set $K = \bigcap_{s\in S} \ker(s)$. As an intersection of closed subspaces, K is a closed subspace. Therefore there is a projection P_K projecting on K. By the same argument as in the theorem, P_K is an elements of $\mathcal{A}$. It is then clear that $R(S) = P_K B(H)$.

We now know that von Neumann algebras are AW^*-algebras. In fact, we can say something more, but first a lemma.

Lemma 4.32. *Let $\{e_i\}$ be a family of projections in a Rickart *-ring $\mathcal{A}$. Suppose $\sup e_i \in \mathcal{A}$ (this is in particular the case if $\mathcal{A}$ is a Baer *-ring). Then for $x \in \mathcal{A}$ we have $e_ix = 0$ for all i if and only if $ex = 0$.*

Proof. We have the following chain of equivalences:

$$\begin{aligned} e_ix = 0 &\Leftrightarrow e_iLP(x) = 0 \\ &\Leftrightarrow e_i = e_i(1 - LP(x)) \\ &\Leftrightarrow e_i \leq 1 - LP(x) \\ &\Leftrightarrow e \leq 1 - LP(x) \\ &\Leftrightarrow eLP(x) = 0 \\ &\Leftrightarrow ex = 0. \end{aligned}$$

□

Proposition 4.33. *Let $\mathcal{A} \subset B(H)$ be a von Neumann algebra. Then $\mathcal{A}$ is an AW^*-subalgebra of $B(H)$.*

Proof. We first show that $x \in \mathcal{A}$ implies $RP(x) \in \mathcal{A}$. To this end, let $b \in \mathcal{A}'$ and $x \in \mathcal{A}$. Then, since $xb = bx$ and $x = xRP(x)$ we have

$$0 = b(x - xRP(x)) = x(b - bRP(X)).$$

Therefore

$$RP(x)(b - bRP(x)) = 0,$$

so

$$RP(x)b = RP(x)bRP(x).$$

Doing the same for b^*, we find $RP(x)b^* = RP(x)b^*RP(x)$. Taking adjoints, we find $bRP(x) = RP(x)bRP(x)$. Combining these two results gives

$$bRP(x) = RP(x)bRP(x) = RP(x)b,$$

so $RP(x)$ commutes with $b \in \mathcal{A}'$ and therefore $RP(x) \in \mathcal{A}'' = \mathcal{A}$.
Now let e_i be an orthogonal family of projections in $\mathcal{A}$ and let $e = \sup_i e_i$. For $b \in \mathcal{A}'$ we have $e_ib = be_i$. Also we have $ee_i = e_ie = e_i$. It follows that

$$0 = b(e_i - e_ie) = e_i(b - be),$$

for all i. By the above Lemma, we have $e(b - be) = 0$, so $eb - ebe = 0$. Again, doing the same with b^* we find $eb = ebe = be$, so $e \in \mathcal{A}'' = \mathcal{A}$. □

The converse is also true and follows from a more general result by G. Pedersen [10]:

Proposition 4.34. *An AW^*-algebra with a separating family of completely additive states is a von Neumann algebra.*

Here, a family of states is separating if for any $a \in \mathcal{A}$, there is a state ϕ in this family, such that $\phi(a) \neq 0$.

Using this, we have:

Proposition 4.35. *If $\mathcal{A}$ is an AW^*-subalgebra of $B(H)$, then $\mathcal{A}$ is a von Neumann algebra.*

Proof. If $\mathcal{A}$ is an AW^*-subalgebra of $B(H)$, the states $\langle \psi, \cdot\, \psi \rangle$ for $\psi \in H$ are a separating family of completely additive states. □

4.5 AW^*-algebras as a Category

We now look at the category $\mathbf{AW}^*$ that has as its objects AW^*-algebras and as morphisms unital *-homomorphisms satisfying any (and hence all) of the conditions in Theorem 4.29. We note that, for example, the inclusion of an eAe in A for e a projection and A an AW^*-algebra does not preserve the unit. As in the von Neumann case, the AW^*-algebras 0 and $\mathbb{C}$ are the terminal and initial objects, respectively.
If $A_i, i \in I$ is a family of AW^*-algebras, we define their direct sum $\bigoplus_i A_i$ as

$$\{f : I \to \bigcup_i A_i | f(i) \in A_i, \sup_i \|f(i)\| < \infty\},$$

with pointwise multiplication, addition, and involution. The norm is defined as

$$\|f\| = \sup_i \|f(i)\|.$$

Instead of writing f we shall mostly just use the notation $(a_i)_i$ with $a_i \in A_i$ for an element of $\bigoplus_i A_i$.

Theorem 4.36. *Let A_i be a family of AW^*-algebras. Then $\bigoplus_i A_i$ is the categorical product, with projections*

$$\pi_j : \bigoplus_i A_i \to A_j,$$

$$(a_i)_i \mapsto a_j.$$

Proof. The fact that $\bigoplus_i A_i$ is a C^*-algebra was already established before. What we still need to show is that it is also a Baer *-ring, that the projections $\pi_i : \bigoplus_j A_j \to A_i$ are AW^*-morphisms, and that $\bigoplus_i A_i$ satisfies the appropriate universal property.
Let $Y \subset \bigoplus_i A_i$ be any subset. Then $Y = (Y_i)_i$ for certain subsets $Y_i \subset A_i$. Since each

A_i is an AW^*-algebra, we have $R(Y_i) = p_iA_i$ for some projection p_i. Now $(p_i)_i$ is a projection in $\bigoplus_i A_i$ and

$$R(Y) = (p_i)_i \bigoplus_i A_i,$$

showing $\bigoplus_i A_i$ is a Baer *-ring.
Since all operations are defined coordinatewise, we see that $RP((a_i)_i) = (RP(a_i))_i$, which under π_j maps to $RP(a_j) = RP(\pi_j(a_i)_i)$. If for each i there is a morphism $h_i : W \to A_i$ for some AW^*-algebra W, then we define $g : W \to \bigoplus_i A_i$ as $g(w) = (h_i(w))_i$. It should be clear that this is the unique AW^*-morphism with $\pi_j g = h_i$. □

To define coproducts we need to have a notion of what it means for an AW^*-algebra to be generated by a family of elements.

Lemma 4.37. *Let A be an AW^*-algebra and B_i a family of AW^*-subalgebras of A. Then $\bigcap_i B_i$ is again an AW^*-subalgebra of A.*

Proof. This is clear from the definition of an AW^*-subalgebra. Given $x \in \bigcap_i B_i$, then $x \in B_i$ for all i, therefore $RP(x) \in B_i$ for all i, and $RP(x) \in \bigcap_i B_i$. The same argument applies to suprema of projections. □

In particular, if S is any set in an AW^*-algebra M, we can look at all AW^*-subalgebras of M containing S and take their intersection. The resulting space is then of course the smallest AW^*-subalgebra containing S.

Definition 4.38. *Let M be a AW^*-algebra and let $S \subset M$ be any subset, then the smallest AW^*-subalgebra of M containing S is called the* AW^*-subalgebra generated by *S.*

Theorem 4.39. *The category of* $\mathbf{AW}^*$ *has coproducts.*

Proof. Let a family $(M_\alpha)_{\alpha \in I}$ of AW^*-algebras be given. Just as in the W^* case we call a cocone $i_\alpha : M_\alpha \to N$ *generating* if N is generated by the images of the i_α. Let S be the set of all such generating cocones. We write $s = \{i^s_\alpha : M_\alpha \to N_s\}$ for a generating cocone in S.
Now look at $j_\alpha : M_\alpha \to \bigoplus_{s \in S} N_s$, with $j_\alpha(m) = \oplus j^s_\alpha(m)$ for $m \in M_\alpha$. Let M be the AW^*-algebra generated by the images $j_\alpha(M_\alpha)$. The fact that this is the coproduct of the M_α follows in the same way as in the $\mathbf{W}^*$ case. □

Proposition 4.40. $\mathbf{AW}^*$ *has all general limits and colimits.*

Proof. The construction for equalizers and coequalizers carries over from $\mathbf{W}^*$. □

The fact that $\mathbf{AW}^*$ does not have exponentials or coexponentials also follows from exactly the same example as in the $\mathbf{W}^*$ case. So what we would like to do is, just as in $\mathbf{W}^*$, define something like the free exponential and a tensor product, which are in adjunction with each other. However, here we run into problems. Proposition 4.34 shows that any AW^*-algebra that can be embedded in $B(H)$ for some Hilbert space H automatically is a von Neumann algebra. Therefore, if we have an AW^*-algebra that is not a von Neumann algebra, we cannot embed it that way. We have to conclude there is no spatial theory for AW^*-algebras. Of course, since every AW^*-algebra is also a C^*-algebra, we can embed it in $B(H)$, as a C^*-algebra, however, the resulting algebra might not have the same lattice structure as the original AW^*-algebra (and in fact does not if the original AW^*-algebra is not W^*). We might also try to construct a free exponential, but as we see from the proof of Proposition 3.25, this construction hinges heavily on the existence of a spatial theory.

Of course, some tensor product, other than the spatial tensor product, might exist for AW^*-algebras. This should then be some closure of the algebraic tensor product of two AW^*-algebras, with respect to some norm on this algebraic tensor product, such that the resulting space is again AW^*. However, unlike von Neumann algebras, which we know are ultraweakly closed, there are, to our knowledge, no relations between the topological and the algebraic properties of AW^*-algebras. So there is no preferred norm (if any at all), in which to take the closure.
As a final thought, let us ignore the closure, and just focus on the algebraic tensor product. Let A and B be AW^*-algebras and $A \otimes B$ their algebraic tensor product. We want the canonical injections $i_A : A \to A \otimes B$, $a \mapsto a \otimes 1$ and $i_B : B \to A \otimes B$, $b \mapsto 1 \otimes B$ to be AW^* morphisms, and therefore we find

$$RP(a \otimes 1) = RP(i_A(a)) = i_A(RP(a)) = RP(a) \otimes 1,$$

and likewise $RP(1 \otimes b) = 1 \otimes RP(b)$. For a general elementary tensor $a \otimes b$, we conjecture that a right annihilating projection is given by

$$\begin{aligned} R(a \otimes b) &= R(A) \otimes 1 + 1 \otimes R(B) - R(A) \otimes R(B) \\ &= 1 \otimes 1 - (1 - R(a)) \otimes (1 - R(b)) \\ &= 1 - RP(a) \otimes RP(b). \end{aligned}$$

However, since we do not know what relation there is between right annihilators and sums, we have no idea what the right annihilator of an arbitrary tensor is.

References

[1] S. Awodey, *Category Theory*, Oxford University Press, 2010.

[2] S. Berberian, *Baer *-Rings*, Springer-Verlag, Berlin, 1972.

[3] C. Heunen, N. Landsman, B. Spitters, *Bohrification of Operator Algebras and Quantum Logic*, Synthese 186, 719-752 (2012).

[4] I. Kaplansky, *Rings of Operators*, W.A. Benjamin, Inc., 1968.

[5] A. Kornell, *Quantum Collections*, `arXiv:1202.2994v1`.

[6] Li Bing-Ren, *Introduction to Operator Algebras*, World Scientific, Singapore, 1992.

[7] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1998.

[8] G. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Inc., London, 1990.

[9] G. Pedersen, *Analysis Now*, Springer-Verlag, New York, 1989.

[10] G. Pedersen *Operator Algebras with Weakly Closed Abelian Subalgebras*, Bull. London Math. Soc., 1972.

[11] M. Takesaki, *Theory of Operator Algebras 1*, Springer-Verlag, Berlin, 2001.