Quantum Toposophy

Proefschrift

ter verkrijging van de graad van doctor
aan de Radboud Universiteit Nijmegen
op gezag van de rector magnificus prof. mr. S.C.J.J. Kortmann,
volgens besluit van het college van decanen
in het openbaar te verdedigen op donderdag 3 oktober 2013
om 12:30 uur precies

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geboren op 17 februari 1980
te Zevenaar

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To my brother, Martin.

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Introduction

This thesis is concerned with reformulations of the mathematical framework of quantum physics, using ideas from the branch of mathematics called topos theory. More specific, we are concerned with topos theoretic approaches to quantum physics which are, either directly or indirectly, influenced by the work of Chris Isham. We hope that such reformulations assist in solving conceptual problems in the foundations of quantum physics. Solving these problems is interesting in its own right, as well as in connection to the search of a quantum theory of gravity. This dissertation is founded in the belief that such a 'quantum topos' programme can only succeed if there is a strong dialectic between the mathematical framework and the physical motivation. As an example, the presheaf topos model to quantum physics claims to resemble the formalism of classical physics more closely than does the familiar Hilbert space formulation of quantum theory. In this thesis we back this claim up in the mathematically precise sense of the internal language of the topos at hand. As another example, the copresheaf topos model to quantum physics derives truth values using internal reasoning of that topos. We describe these truth values externally to the topos, and show how to interpret these truth values physically.

At the time of writing it is not yet clear whether or not these 'quantum topos' models have what it takes to live up to their ambitions regarding the conceptual problems of quantum theory. There is still a gap to bridge. Even so, taking into account the original motivation for these models, as well as the interplay between the mathematics and the physical motivation described in this thesis, the author would say that studying the interplay between quantum physics and topos theory (or quantum physics

and sheaf theory) is worth a fair shot.
As indicated before, a central theme is to match the mathematical framework to the physical motivation in the known models. This leads, as described in Chapters 2-4, to a description of the presheaf model of Butterfield, Isham and Döring which is close to the copresheaf model of Heunen, Landsman and Spitters. This allows, as is done in e.g. Chapter 5, to work these two models using the same language. Apart from investigating known models, we seek to extend these models to the setting of (algebraic) quantum field theory.

1.1 Historical Introduction

1.1.1 Butterfield and Isham

As far as the author knows, the oldest application of topos theory to quantum mechanics is due to Adelman and Corbett [5], but apparently it has not influenced subsequent authors, and indeed it will play no role in what follows. Below, we restrict our attention to applications of topos theory to quantum physics, inspired by the work of Butterfield and Isham. In a series of four papers [16, 17, 18, 19], Jeremy Butterfield and Chris Isham demonstrated that in studying foundations of quantum physics, in particular the Kochen–Specker Theorem, structures from topos theory show up in a natural way (see also [54]).
We sketch some of the ideas which led to considering the application of topos theory to quantum mechanics. For our presentation we use the more recent constructions by Döring and Isham, rather than the original presentation by Butterfield and Isham, as we use the Döring–Isham version throughout the text. The starting point in this approach is the operator algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space, associated to some quantum system of interest. More generally, instead of using $\mathcal{B}(\mathcal{H})$ we can use an arbitrary von Neumann algebra A. A **von Neumann algebra** is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, that is closed with respect to the weak operator topology [59].
Consider the Kochen-Specker Theorem [61] of quantum theory. If $A = \mathcal{B}(\mathcal{H})$, where $dim(\mathcal{H}) > 2$, this theorem amounts to the non-existence of valuations V of the following sort. Let A_{sa} denote the self-adjoint elements of A, and let $V : A_{sa} \to \mathbb{R}$ map each element $a \in A_{sa}$ to an element[1] $V(a) \in \sigma(a) \subseteq \mathbb{R}$ of the spectrum of a. If $a, b \in A_{sa}$ are related

[1] If $dim(\mathcal{H}) < \infty$, $V(a) \in \sigma(a)$ holds for each $a \in A_{sa}$ if we demand:

by a Borel function $f : \sigma(a) \to \mathbb{R}$ as $b = f(a)$, we finally demand that $V(b) = f(V(a))$. The non-existence of such valuations V is relevant to foundations of quantum physics, as it prohibits a naive realist interpretation of the theory.

In the work of Butterfield and Isham, the Kochen–Specker Theorem is restated as follows. First consider the notion of a classical **context**, or classical snapshot, represented by an Abelian von Neumann subalgebra of A. We only consider von Neumann subalgebras C where the unit of C is the unit of A. Typically, in the work of Isham et.al., the trivial algebra $\mathbb{C}1$ is excluded as a context. The classical contexts form a poset $\mathcal{C} \equiv \mathcal{C}(A)$, where the partial order is given by inclusion. Next, we consider the category $[\mathcal{C}^{op}, \mathbf{Set}]$ of contravariant functors, also called presheaves, from $\mathcal{C}$ to **Set**. Working with this functor category allows one to work with all classical contexts at the same time, whilst keeping track of relations between the different contexts. The category $[\mathcal{C}^{op}, \mathbf{Set}]$ is an example of a *topos*.

Of particular interest is the **spectral presheaf**, i.e., the contravariant functor

$$\underline{\Sigma} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}, \quad \underline{\Sigma}(C) = \Sigma_C, \tag{1.1}$$

where Σ_C is the Gelfand spectrum of the abelian von Neumann algebra $C \in \mathcal{C}$. Recall that the elements $\lambda \in \Sigma_C$ can be identified with nonzero multiplicative linear functionals $\lambda : C \to \mathbb{C}$. The operator algebra C is isomorphic to the operator algebra of continuous complex-valued functions on the compact Hausdorff space Σ_C. If $D \subseteq C$ in $\mathcal{C}$, then the corresponding arrow in the category $\mathcal{C}$ is mapped by $\underline{\Sigma}$ to the continuous map $\rho_{CD} : \Sigma_C \to \Sigma_D$, corresponding by Gelfand duality to the embedding $D \hookrightarrow C$. Note that if we see $\lambda \in \Sigma_C$ as a map $C \to \mathbb{C}$, then $\rho_{CD}(\lambda) = \lambda|_D$, the restriction of the functional λ to D.

The Kochen–Specker theorem then turns out to be equivalent to the statement that for $A = \mathcal{B}(\mathcal{H})$ the spectral presheaf has no global points, i.e., there exist no natural transformations $\underline{1} \to \underline{\Sigma}$. By itself this observation need not imply that topos theory is relevant to the foundations of quantum theory; it merely suggests that the language of presheaves might

- For each $a \in A_{sa}$, $V(a)^2 = V(a^2)$,
- $V(1) = 1$,
- For all commuting $a, b \in A_{sa}$ and $x \in \mathbb{R}$, $V(a + x \cdot b) = V(a) + x \cdot V(b)$.

be helpful. To connect to topos theory, we discuss the idea of **coarse-graining**.

In quantum theory, propositions about the system are represented by projection operators. The proposition[2] $[a \in \Delta]$, where a is a self-adjoint operator, and Δ a Borel subset of the spectrum of a, is represented by a spectral projection $\chi_{\Delta}(a)$. Restricting valuations V to the propositions (i.e. the projection operators), Butterfield and Isham consider the following alternative. Instead of trying to assign to each proposition (represented by some projection p) either **true** ($V(p) = 1$) or **false** ($V(p) = 0$), use a *contextual* and *multi-valued* logic. In this multi-valued logic, a valuation V, for example associated to a (preparation) state of the physical system, assigns to a pair (p, C), consisting of a proposition p (a projection of some von Neumann algebra A) and a context C, a certain set $V_C(p)$ of contexts. These contexts are coarser than C in the sense that if $D \in V_C(p)$, then $D \subseteq C$. The proposition p is true at stage C if $V_C(p) = (\downarrow C)$, the set of all contexts coarser than C, including C itself. The truth value gives a list of coarser contexts that express the extent to which the property p holds. If $D \in V_C(p)$ and $D' \subseteq D$ is an even coarser context, then $D' \in V_C(p)$. The bottom line is that the contextual multivalued quantum logic of Butterfield and Isham suggested using the *subobject classifier* Ω of the topos $[\mathcal{C}^{op}, \mathbf{Set}]$, and thus introduced topos theory to the foundations of quantum physics.

1.1.2 Döring and Isham

In a second series of papers [33, 34, 35, 36], Chris Isham, now working together with Andreas Döring, shows greater ambition in applying topos theory to physics. A central idea in these papers is that any theory of physics, at least in its mathematical formulation, should share certain structures [33]. These structures are assumed as they assist in giving some, hopefully non-naive, realist account of the theory. Aside from putting restrictions of the shape of the mathematical framework of physical theories, freedom is added in that we may use other topoi than the category of sets. Following Isham, we will refer to this idea as **neorealism**. The motivating example is the presheaf model of Butter-

[2] We would like to think of $[a \in \Delta]$ as the proposition stating that the physical quantity a takes only values in Δ, but in orthodox quantum theory (with the Copenhagen interpretation) this picture is dismissed as being naive realist.

field and Isham, further developed by Isham and Döring. We shall call this presheaf model of quantum physics, the **contravariant model** or **contravariant approach**.

To be a bit more precise, neorealism assigns a formal language to a physical system, and a theory is a representation of this language in a topos. This representation includes an object Σ that plays the role of a state space (in the contravariant approach it is the spectral presheaf $\underline{\Sigma}$), as well as an object $\mathcal{R}$, in which the physical quantities take their values. Physical quantities are represented by arrows $\Sigma \to \mathcal{R}$, and propositions about the system are represented by subobjects of Σ. For a complete discussion, see [37].

This notion of neorealism raises several question. For example, what makes a topos a good model? A key hope is that the topos formulation of a physical theory should resemble classical physics more than e.g. orthodox quantum physics does. The formal similarity to classical physics is seen as desirable as classical physics can be interpreted in a realist way. But in what way is a topos model, in all its abstraction, closer to classical physics? One way of making the claim that the model 'resembles classical physics' mathematically precise would be to use the internal language of the topos. This brings us to an alternative topos model, proposed by Heunen, Landsman and Spitters in [48], which was inspired by the work of Butterfield and Isham, and which indeed resembles classical physics in this internal sense.

Before we discuss the HLS topos model however, we briefly consider **daseinisation**, a technique introduced by Döring and Isham to give mathematical shape to the idea of coarse-graining. Daseinisation, and in particular outer daseinisation of projections, associates to a projection operator p of A and a context C, a projection operator $\delta^o(p)_C$ in C. The projection $\delta^o(p)_C$ is the smallest projection operator q of C satisfying $q \geq p$. Recall that for projection operators the partial order $p \leq q$ is defined as $p \cdot q = p$. Let $|\psi\rangle \in \mathcal{H}$ be a unit vector, and V the contextual multivalued valuation (as introduced by Butterfield and Isham) associated to this vector. For any $D \subseteq C$ in $\mathcal{C}$ and projection p of A, we have $D \in V_C(p)$ iff

$$\langle\psi|\delta^o(p)_D|\psi\rangle = 1.$$

Daseinisation is also used in the representation of self-adjoint operators a_{sa} as arrows $\underline{\Sigma} \to \mathcal{R}$, as we shall discuss in Chapter 3.

1.1.3 Heunen, Landsman and Spitters

The topos model of Heunen, Landsman and Spitters uses a topos of copresheaves and is closely related to the presheaf model of Butterfield, Döring and Isham [80]. We will refer to the HLS model of copresheaves as the **covariant model** or **covariant approach**.

The covariant approach is inspired by algebraic quantum theory [42], insofar as the system under investigation is described by a C*-algebra A, which we assume to be unital. A C*-algebra is, up to isomorphism, a norm closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. As the norm topology on $\mathcal{B}(\mathcal{H})$ is finer than the weak operator topology, any von Neumann algebra is a unital C*-algebra, but the converse does not hold. In the covariant approach the larger class of unital C*-algebras is used because it gives greater generality and enables to use the notion of an internal locale as a state space. Consequently there is less emphasis on daseinisation when compared with the contravariant approach. since C*-algebras may not have sufficiently many projection operators to allow the daseinisation techniques to be used.

A second ingredient of the covariant approach is *Bohr's doctrine of classical concepts* [13], or rather a particular mathematical interpretation of this principle. This principle states that we can only look at a quantum system from the point of view of some classical context. The classical contexts are represented by unital[3] commutative C*-subalgebras of A.[4] These classical contexts, partially ordered by inclusion, form a poset $\mathcal{C}$, also denoted as $\mathcal{C}_A$.

The covariant approach uses the topos $[\mathcal{C}, \mathbf{Set}]$ of covariant functors $\mathcal{C} \to$ **Set** and their natural transformations. The key object of this model is the covariant functor

$$\underline{A} : \mathcal{C} \to \mathbf{Set}, \quad \underline{A}(C) = C. \tag{1.2}$$

If $D \subseteq C$, then the corresponding arrow in $\mathcal{C}$ is mapped by $\underline{A}$ to the inclusion $D \hookrightarrow C$. The object $\underline{A}$, also called the **Bohrification** of A, is interesting because, from the internal perspective of the topos $[\mathcal{C}, \mathbf{Set}]$ it is a *commutative* unital C*-algebra. There is a version of Gelfand duality which is valid in any (Grothendieck) topos [9, 22]. Therefore there exists a Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ in $[\mathcal{C}, \mathbf{Set}]$ such that $\underline{A}$ is, up to isomorphism

[3] The unit is included for technical reasons.

[4] As in the contravariant approach, we demand that the unit of the context C is equal to the unit of A.

of C*-algebras, the algebra of continuous complex-valued functions on $\underline{A}$. However, $\underline{\Sigma}_{\underline{A}}$ is not a compact Hausdorff space, but a compact completely regular locale.
By 'internal perspective' of the topos, we mean looking at the topos using the internal language associated to that topos. Indeed, any topos has an associated internal language [14]. Using this language, the topos becomes a universe of mathematical discourse, resembling set theory. In the covariant approach, self-adjoint elements of A are represented internally as locale maps $\underline{\Sigma}_{\underline{A}} \to \underline{\mathbb{IR}}$, where $\underline{\mathbb{IR}}$ is the interval domain in $[\mathcal{C}, \mathbf{Set}]$. States, in the sense of positive normalised linear functionals $\psi : A \to \mathbb{C}$, are represented internally as probability valuations $\underline{\mu} : \mathcal{O}\underline{\Sigma}_{\underline{A}} \to \underline{[0,1]}_l$. When viewed from the internal language of the topos, states and operators therefore resemble classical physics.

1.2 Outline

We proceed to give an overview of the material in this thesis. With the exception of Chapter 6 all chapters are concerned with the contravariant approach of Butterfield, Isham en Döring as well as the covariant approach of Heunen, Landsman and Spitters. Chapter 6 deals almost exclusively with the covariant approach. A general theme is that the covariant and contravariant topos models resemble each other closely when we look at these models from the internal perspective ot the topoi at hand. The resemblence of these models to classical physics is also studied from the internal perspective. However, connecting neorealism to the internal language of topoi is not a goal in itself;we are rather interested in finding connections between the physical motivation of the topos models and the mathematics of topos theory. For only a strong dialectic between the mathematics of topos theory and the motivation from physics offers a chance of gaining new insights in the foundations of physics from the mathematical reformulations used in the topos models.

The contents of this thesis is as follows:

- **Chapter 2**: For the covariant approach, we provide an external description of the internal Gelfand spacetrum $\underline{\Sigma}_{\underline{A}}$ of the Bohrification $\underline{A}$. This external description is in the form of a bundle $\pi : \Sigma_{\uparrow} \to \mathcal{C}_{\uparrow}$. In the contravariant approach, if the spectral presheaf $\underline{\Sigma}_A$ is viewed as an internal topological space, with topology generated by the clopen subobjects, this internal space is externally described by a bundle $\pi : \Sigma_\downarrow \to \mathcal{C}_\downarrow$, closely related to the bundle of the covariant approach.

- **Chapter 3**: In both topos models, daseinisation of self-adjoint operators allows these operators to be presented as internal continuous maps from state spaces of the topos models to spaces of internal real numbers. Viewed externally, these continuous maps are defined in exactly the same way for both topos models. The only difference lies in the topologies of the spaces considered. In this chapter we also seek relations between these continuous maps and the elementary propositions, as used in the topos models.

- **Chapter 4**: This chapter completes the analysis of the previous two chapters. In both topos models states are represented as internal probability valuations on the state spaces. After discussing states and the truth values that these states provide when combined with elementary propositions, we are in a position to analyse the logics provided by the two topos models. We consider the 'quantum logics' of the complete Heyting algebras $\mathcal{O}\Sigma_\downarrow$ and $\mathcal{O}\Sigma_\uparrow$, by studying the truth values that opens of these frames produce when combined with states.

- **Chapter 5**: To a C*-algebra we can either associate a topos and an internal commutative C*-algebra, as in the covariant approach, or a topos and an internal topological space, as in the contravariant approach (duly reformulated). In this chapter we study how $*$-homomorphisms between C*-algebras induce geometric morphisms of the associated topoi, as well as internal $*$-homomorphism between the internal C*-algebras, or internal continuous maps between the internal topological spaces. In particular, we concentrate on $*$-automorphisms, and study how elementary propositions, states, and truth values transform under the action of such a morphism.

- **Chapter 6**: In this chapter we restrict ourselves to the *covariant* approach. Given a net of unital C*-algebras, as in algebraic quantum field theory, we can view the net as a contravariant functor from a category of regions of spacetime to a category of topoi with internal C*-algebras. Using a natural covering relation on the spacetime regions we can ask whether this functor is a sheaf. This corresponding sheaf condition is shown to be closely related to a known (kinematical) independence condition on the net called *C*-independence*.

- **Chapter 7**: In this short epilogue, we briefly reflect on some of the consquences of connecting neorealism to the internal language of topoi. Notably, we consider the role of the axiom of choice and the law of excluded middle.

- **Appendix A**: In the appendix we discuss parts of topos theory relevant to this thesis. In particular, geometric morphisms, locales, the internal language of a topos, presheaf semantics and geometric logic are discussed.

The results in this thesis originate from the following papers and preprints:

1. The joint work [73] with Spitters and Vickers is used in sections 2.1, 2.2 and 2.5-2.7.

2. Results from [80] are used in sections 2.2, 2.3, 2.5, 3.1-3.4 and 4.2.

3. Results from [81] are used in sections 2.4, 3.5, 3.6, 4.3-4.5, as well as Chapters 5 and 7.

4. Chapter 6 is based on joint work [43] with Hans Halvorson.

Concerning prerequisites, we assume that the reader is familiar with von Neumann algebras and C*-algebras, and knows the basics of category theory. Experience with topos theory is highly recommended. There is an appendix providing background material on topos theory, but this material is not self-contained and is mostly intended to provide further references and to fix notation.

Finally, one word of caution. We use the same symbol $\mathcal{C}$ to denote the set of contexts in the von Neumann algebraic as well as in the C*-algebraic setting. Usually, it is clear from the context (no pun intended) which version we use. If daseinisation is mentioned, we need spectral resolutions of self-adjoint operators or spectral projections. In that case the von Neumann algebraic version is assumed, and both the algebra A as well as the contexts C are taken to be von Neumann algebras. In other cases, we can choose C*-algebras, and A as well as the contexts C are taken to be unital C*-algebras.

2

State Spaces

In this chapter, we discuss the spectral presheaf $\underline{\Sigma}_A$ of the contravariant approach, as well as the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ of the covariant approach (note the difference in notation). The first three sections describe the locale $\underline{\Sigma}_{\underline{A}}$ of the covariant approach. In Section 2.1 unital commutative C*-algebras in topoi are discussed, with emphasis on functor categories. Subsequently, in Section 2.2 we provide an external presentation of the Gelfand spectrum of the Bohrification $\underline{A}$. Section 2.3 completes this discussion by dealing with the Gelfand dual of $\underline{A}$.

Next, in Section 2.4 we turn to the contravariant approach and describe the spectral presheaf as an internal topological space, equipped with the topology generated by the clopen subobjects. With the external presentations of the (contravariant) spectral presheaf and the (covariant) spectral locale by topological spaces thus obtained, we investigate the possible sobriety of these spaces in Section 2.5.

The final two sections are connected to the covariant approach. In Section 2.6 a general result on exponentiability is presented, which in particular entails local compactness of the external description of the spectral locale. Finally, Section 2.7 discusses the Gelfand spectrum in the setting of an extension of the covariant approach to algebraic quantum field theory.

2.1 Internal C*-algebras

In this section we describe C-algebras internal to topoi with a natural numbers object. We show that if the topos is a functor category, then an* *internal C*-algebra is equivalent to a functor mapping into the category of C*-algebras and $*$-homomorphisms.*

We start by discussing C*-algebras in topoi. Although such a discussion can be found in e.g. [9, 48], we include it here to make the text more self-contained. Let $\mathcal{E}$ be a topos with natural number object, and $\underline{A}$ an object of this topos. In addition, let $\underline{\mathbb{Q}[i]}$ denote the complexified rational numbers of $\mathcal{E}$. In the definition of a C*-algebra in a topos we make use of the field $\underline{\mathbb{Q}[i]}$ for scalar multiplication instead of $\underline{\mathbb{C}}$, the complexified Dedekind real numbers. This is because $\underline{\mathbb{Q}[i]}$ is preserved under the action of inverse image functors, whereas $\underline{\mathbb{C}}$ generally is not[1].
In what follows we use shorthand notation such as $\forall a, b \in \underline{A}$ for $\forall a \in \underline{A}$, $\forall b \in \underline{A}$. We can now start with the definition of a C*-algebra in $\mathcal{E}$, based on [9]. First of all, $\underline{A}$ is a $\underline{\mathbb{Q}[i]}$-vector space. This means that there are arrows

$$+ : \underline{A} \times \underline{A} \to \underline{A}, \quad \cdot : \underline{\mathbb{Q}[i]} \times \underline{A} \to \underline{A}, \quad 0 : \underline{1} \to \underline{A},$$

defining addition, scalar multiplication and the constant 0. With respect to the internal language of the topos, these maps should satisfy the usual axioms for a vector space such as:

$$\forall a, b, c \in \underline{A} \ \ ((a+b)+c = a+(b+c))\,;$$

$$\forall a \in \underline{A} \ a + 0 = a.$$

In addition, there is a multiplication map $\cdot : \underline{A} \times \underline{A} \to \underline{A}$ satisfying the axioms expressing that $\underline{A}$ is a $\underline{\mathbb{Q}[i]}$-algebra. We used the notation $\cdot$ for multiplication as well as scalar multiplication, hoping that this will not lead to confusion.
There is an arrow $* : \underline{A} \to \underline{A}$, which is involutive,

$$\forall a \in \underline{A} \ (a^*)^* = a,$$

and conjugate linear,

$$\forall a, b \in \underline{A} \ (a+b)^* = a^* + b^*,$$

$$\forall a \in \underline{A}, \ \forall x \in \underline{\mathbb{Q}[i]} \ (x \cdot a)^* = \bar{x} \cdot a^*,$$

[1] As a C*-algebra is norm-complete by definition, there seems to be no harm in restricting to $\underline{\mathbb{Q}[i]}$.

where $\bar{(\cdot)} : \underline{\mathbb{Q}[i]} \to \underline{\mathbb{Q}[i]}$ is the conjugation map $x + iy \mapsto x - iy$. The involution is antimultiplicative:

$$\forall a, b \in \underline{A} \ (a \cdot b)^* = b^* \cdot a^*.$$

In the topos **Set**, the norm is defined as a map $\| \cdot \| : A \to [0, \infty)$. Equivalently, it can be described as a subset $N \subseteq A \times \mathbb{Q}^+$, where $(a, p) \in N$ iff $\|a\| < p$. For C*-algebras in arbitrary topoi, we use the subset description as just formulated using rational numbers. A *norm* on $\underline{A}$ is a subobject $\underline{N} \subseteq \underline{A} \times \underline{\mathbb{Q}}^+$ satisfying axioms (2.1)-(2.4), (2.6) and (2.8) discussed below. The axiom

$$\forall p \in \underline{\mathbb{Q}}^+ \quad (0, p) \in \underline{N} \tag{2.1}$$

expresses $\|0\| = 0$. The fact that $\|a\| = 0$ implies $a = 0$, stating that a given semi-norm is in fact a norm, is expressed as

$$\forall a \in \underline{A} \left(\left(\forall p \in \underline{\mathbb{Q}}^+ \ (a, p) \in \underline{N} \right) \to (a = 0) \right). \tag{2.2}$$

Note that because of the second universal quantifier, this axiom does not fit within the constraints of geometric logic. The following two axioms express the idea that the norm $\underline{N}$ can be seen as a mapping $\|\cdot\| : \underline{A} \to \underline{[0, \infty]}_u$ (see e.g. [77]). The subscript u indicates that we are using *upper real numbers* here which are merely one of the various kinds of real numbers in a topos: as the internal mathematics of a topos is constructive, different ways of constructing real numbers out of the rational numbers can result in different objects [55, Section D4.7]. In particular, the lower and upper real numbers will turn out to be important to the topos approaches to quantum theory. We discuss these one-sided real numbers in the next chapter. The axioms for the norm, then, are:

$$\forall a \in \underline{A} \ \exists p \in \underline{\mathbb{Q}}^+ \quad (a, p) \in \underline{N}, \tag{2.3}$$

$$\forall a \in \underline{A} \ \forall p \in \underline{\mathbb{Q}}^+ \ (a, p) \in \underline{N} \leftrightarrow \left(\exists q \in \underline{\mathbb{Q}}^+ \ (p > q) \wedge ((a, q) \in \underline{N}) \right). \tag{2.4}$$

Note that the first axiom excludes the possibility that $\|a\|$ is equal to the upper real number ∞. The equality $\|a\| = \|a^*\|$, postulating that the $*$-involution is an isometry, follows from the involutive property of $*$ and the axiom:

$$\forall a \in \underline{A} \ \forall p \in \underline{\mathbb{Q}}^+ \ \left((a, p) \in \underline{N} \to (a^*, p) \in \underline{N} \right). \tag{2.5}$$

The triangle inequality $\|a+b\| \leq \|a\| + \|b\|$ is expressed by the axiom

$$\forall a, b \in \underline{A}\ \forall p, q \in \underline{\mathbb{Q}}^{+}\ ((a,p) \in \underline{N} \wedge (b,q) \in \underline{N}) \to (a+b, p+q) \in \underline{N}. \quad (2.6)$$

Submultiplicativity of the norm, $\|a \cdot b\| \leq \|a\| \cdot \|b\|$, is expressed by the axiom

$$\forall a, b \in \underline{A}\ \forall p, q \in \underline{\mathbb{Q}}^{+}\ \ ((a,p) \in \underline{N} \wedge (b,q) \in \underline{N}) \to (a \cdot b, p \cdot q) \in \underline{N}. \quad (2.7)$$

The property $\|x \cdot a\| = |x| \cdot \|a\|$ is expressed as

$$\begin{aligned} &\forall a \in \underline{A}\ \forall x \in \underline{\mathbb{Q}[i]}\ \forall p, q \in \underline{\mathbb{Q}}^{+} \\ &\qquad (((a,p) \in \underline{N} \wedge (|x| < q)) \to (x \cdot a, p \cdot q) \in \underline{N}), \end{aligned} \quad (2.8)$$

where we used the modulus map

$$|\cdot| : \underline{\mathbb{Q}[i]} \to \underline{\mathbb{Q}}^{+}\quad x + iy \mapsto x^2 + y^2.$$

The special C*-algebraic property $\|a\|^2 = \|a \cdot a^*\|$ is given by

$$\forall a \in \underline{A}\ \forall p \in \underline{\mathbb{Q}}^{+}\ \left((a,p) \in \underline{N} \leftrightarrow (a \cdot a^*, p^2) \in \underline{N}\right). \quad (2.9)$$

The algebra $\underline{A}$ is required to be complete with respect to the norm $\underline{N}$. This can be expressed using **Cauchy approximations**. Let $\mathcal{P}\underline{A}$ denote the power object of $\underline{A}$. A sequence $C : \underline{\mathbb{N}} \to \mathcal{P}\underline{A}$ is a Cauchy approximation if it satisfies the following two axioms:

$$\forall n \in \underline{\mathbb{N}}\ \exists a \in \underline{A}\ (a \in C(n)), \quad (2.10)$$

$$\begin{aligned} &\forall k \in \underline{\mathbb{N}}\ \exists m \in \underline{\mathbb{N}}\ \forall n, n' \geq m \\ &\qquad \left((a \in C(n)) \wedge (b \in C(n')) \to (a - b, 1/k) \in \underline{N}\right). \end{aligned} \quad (2.11)$$

Note that the first axiom simply states that each set $C(n)$ is non-empty, whereas the second axiom is the characterising property of Cauchy sequences. The difference between Cauchy sequences and Cauchy approximations is that the first uses singleton subsets of the algebras, whereas the second uses non-empty sets. Note that for the second axiom we used the shorthand notation $\forall n, n' \geq m$, meaning

$$\forall n \in \underline{\mathbb{N}}\ \forall n' \in \underline{\mathbb{N}}\ (n \geq m) \wedge (n' \geq m) \to .$$

The normed algebra $\underline{A}$ is **complete** if each Cauchy approximation converges to a unique element of $\underline{A}$. Given a Cauchy approximation C and an element $a \in \underline{A}$, we say that C converges to a iff

$$\forall k \in \underline{\mathbb{N}}\ \exists m \in \underline{\mathbb{N}}\ \forall n \geq m\ \ (b \in C(n) \to (b - a, 1/k) \in \underline{N})\,.$$

We briefly use the following notation to reduce the size of the formulae involved. Given a sequence $C : \underline{\mathbb{N}} \to \mathcal{P}\underline{A}$, let $\psi(C)$ denote the proposition that C is a Cauchy approximation (i.e. the conjunction of the two axioms given above). For a sequence C of subsets of $\underline{A}$, and $a \in \underline{A}$ let $\phi(C, a)$ denote the proposition stating that C converges to a. The normed algebra $\underline{A}$ is complete iff it satisfies

$$\forall C \in \mathcal{P}\underline{A}^{\underline{\mathbb{N}}}\ \psi(C) \to (\exists a \in \underline{A}\ \phi(C, a))\,, \tag{2.12}$$

$$\forall a, b \in \underline{A}\ \forall C \in \mathcal{P}\underline{A}^{\underline{\mathbb{N}}}\ \psi(C) \to (\phi(C, a) \wedge \phi(C, b) \to a = b)\,. \tag{2.13}$$

As in the topos **Set**, if $\underline{B}$ is any (semi)-normed $\underline{\mathbb{Q}[i]}$-algebra, we can take its Cauchy completion defined as the set of Cauchy sequences in $\underline{B}$, identifying sequences that converge to the same element [7]. This finishes the internal axiomatisation of C*-algebras in topoi.

A C*-algebra is called *commutative* if it satisfies the additional axiom

$$\forall a, b \in \underline{A}\ a \cdot b = b \cdot a.$$

A C*-algebra is called *unital* if there is a constant $1 : \underline{1} \to \underline{A}$, satisfying the axioms

$$\forall a \in \underline{A}\ \ a \cdot 1 = a = 1 \cdot a,$$

$$\forall p \in \underline{\mathbb{Q}}^{+}\ \ ((p > 1) \to (1, p) \in \underline{N})\,.$$

Definition 2.1.1. *Let $\mathcal{E}$ be a topos with natural numbers object. A unital commutative C*-algebra in $\mathcal{E}$ is a unital commutative $\underline{\mathbb{Q}[i]}$-algebra $\underline{A}$, with an involutive, conjugate linear and anti-multiplicative map $* : \underline{A} \to \underline{A}$, a norm $\underline{N} \subseteq \underline{A} \times \underline{\mathbb{Q}}^{+}$ with respect to which the $*$-involution is an isometry, is sub-multiplicative and satisfies (2.9), and such that $\underline{A}$ is complete with respect to $\underline{N}$, in the sense that it satisfies (2.12) and (2.13).*

If the topos $\mathcal{E}$ is a functor category, then the following generalisation of [48, Theorem 5] characterises all internal C*-algebras in $\mathcal{E}$.

Proposition 2.1.2. *Let $\mathcal{C}$ be any small category. An object $\underline{A}$ (with additional structure $\underline{+}, \underline{\cdot}, \underline{*}, \underline{0}$) is a C*-algebra internal to the topos $[\mathcal{C}, \mathbf{Set}]$ iff it is given by a functor $\underline{A} : \mathcal{C} \to \mathbf{CStar}$, where **CStar** is the category of C*-algebras and $*$-homomorphisms in* **Set**.
Furthermore, the internal C-algebra $\underline{A}$ is commutative iff each $\underline{A}(C)$ is commutative. The algebra $\underline{A}$ is unital iff each $\underline{A}(C)$ is unital and for each arrow $f : C \to D$, the $*$-homomorphism $\underline{A}(f) : \underline{A}(C) \to \underline{A}(D)$ preserves the unit.*

Proof. It follows from Lemma A.3.2 and the discussion in Appendix A.3 that a semi-normed $*$-algebra over $\underline{\mathbb{Q}[i]}$ in $[\mathcal{C}, \mathbf{Set}]$ is equivalent to a functor $\underline{A} : \mathcal{C} \to \mathbf{Set}$, such that each $\underline{A(C)}$ is a semi-normed $*$-algebra over $\mathbb{Q}[i]$, and, for each arrow $f : D \to C$ in $\mathcal{C}$, the function $\underline{A}(f) : \underline{A}(D) \to \underline{A}(C)$ is a $*$-homomorphism such that $\|\underline{A}(f)(a)\|_D \leq \|a\|_C$.
Here we use $\|\cdot\|_C$ to denote the semi-norm corresponding to $\underline{N}(C) \subseteq \underline{A}(C) \times \mathbb{Q}^+$. The internal semi-norm $\underline{N}$ is submultiplicative and satisfies the C*-property, iff each semi-norm $\|\cdot\|_C$ is submultiplicative and satisfies the C*-property $\|a^*a\|_C = \|a\|_C^2$.

Recall that the semi-norm $\underline{N}$ of $\underline{A}$ is defined as a subobject of $\underline{A} \times \underline{\mathbb{Q}^+}$. The internal semi-norm is connected to the external semi-norms by the identities

$$\underline{N}(C) = \{(a, q) \in \underline{A}(C) \times \mathbb{Q}^+ \mid \|a\|_C < q\}; \tag{2.14}$$

$$\|\cdot\|_C : \underline{A}(C) \to \mathbb{R}_0^+, \quad \|a\|_C = \inf\{q \in \mathbb{Q}^+ \mid (a, q) \in \underline{N}(C)\}, \tag{2.15}$$

where $\mathbb{R}_0^+$ denotes the set of non-negative real numbers. Note that the fact that $*$-homomorphisms are contractions in the sense that $\|\underline{A}(f)(a)\|_D \leq \|a\|_C$, precisely states that $\underline{N}$ defined by (2.14) is a well-defined subobject of $\underline{A} \times \underline{\mathbb{Q}^+}$. The semi-norm $\underline{N}$ is a norm iff it satisfies the axiom

$$(\forall q \in \underline{\mathbb{Q}^+} \;\; (a, q) \in \underline{N}) \Rightarrow (a = \underline{0}). \tag{2.16}$$

By the rules of presheaf semantics, externally, this axiom translates to; for every $C \in \mathcal{C}$ the semi-norm $\|\cdot\|_C$ is a norm.

Completeness can be checked in the same way as in [48], because the axiom of dependent choice is validated in any presheaf topos. To prove completeness, we thus need to check the axiom

$$\begin{aligned} \forall f \in \underline{A}^{\underline{\mathbb{N}}}(\; (\forall n \in \underline{\mathbb{N}} \; \forall m \in \underline{\mathbb{N}} \; (f(n) - f(m), 2^{-n} + 2^{-m}) \in \underline{N}) \\ \Rightarrow (\exists a \in \underline{A} \; \forall n \in \underline{\mathbb{N}}(a - f(n), 2^{-n}) \in \underline{N}) \;). \end{aligned}$$

Note that for any object $C \in \mathcal{C}$, the elements of $\underline{A}^{\underline{\mathbb{N}}}(C)$ correspond exactly to sequences $(a_n)_{n\in\mathbb{N}}$ in $\underline{A}(C)$. By presheaf semantics, the axiom for completeness holds iff for every object $C \in \mathcal{C}$ and any sequence $(a_n)_{n\in\mathbb{N}}$ in $\underline{A}(C)$, if

$$C \Vdash (\forall n \in \underline{\mathbb{N}}\ \forall m \in \underline{\mathbb{N}}\ (a_n - a_m, 2^{-n} + 2^{-m}) \in \underline{N}),$$

then

$$C \Vdash (\exists a \in \underline{A}\ \forall n \in \underline{\mathbb{N}}\ (a - a_n, 2^{-n}) \in \underline{N}).$$

This can be simplified by repeated use of presheaf semantics, and the identity

$$\|\underline{A}(f)(a_n) - \underline{A}(f)(a_m)\|_D = \|\underline{A}(f)(a_n - a_m)\|_D \leq \|a_n - a_m\|_C,$$

where $f : C \to D$ is any arrow. In the end, the axiom of completeness simplifies to the statement that given an object $C \in \mathcal{C}$ and any sequence $(a_n)_{n\in\mathbb{N}}$ in $\underline{A}(C)$ such that for any pair $n, m \in \mathbb{N}$ we have $(a_n - a_m, 2^{-n} + 2^{-m}) \in \underline{N}(C)$, there exists an element $a \in \underline{A}(C)$ such that for every $n \in \mathbb{N}$, $(a - a_n, 2^{-n}) \in \underline{N}(C)$. By definition of $\underline{N}$ this simply states that every $\underline{A}(C)$ is complete with respect to the norm $\|\cdot\|_C$. This completes the proof that C*-algebras in $[\mathcal{C}, \mathbf{Set}]$ are equivalent to functors $\mathcal{C} \to \mathbf{CStar}$. $\square$

2.2 The Spectral Locale

Here we give an external description of the internal Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ of $\underline{A}$.

The Bohrification functor $\underline{A}$ is a unital commutative C*-algebra internal to the topos $[\mathcal{C}, \mathbf{Set}]$. By the pioneering work of Banaschewski and Mulvey on Gelfand duality in topoi [7, 8, 9], there exists a compact completely regular locale $\underline{\Sigma}_{\underline{A}}$ such that $\underline{A}$ is, up to isomorphism of C*-algebras, the C*-algebra of continuous complex-valued maps on $\underline{\Sigma}_{\underline{A}}$. Following the work in [48, 49], based on the fully constructive description of the Gelfand isomorphism by Coquand [22] and Coquand and Spitters [23] we present an explicit external description of this locale. The following topological space plays a key role in that description.

Definition 2.2.1. *The space $\Sigma_\uparrow$ is the set $\Sigma = \coprod_{C\in\mathcal{C}} \Sigma_C$, where $U \subseteq \Sigma$ is open iff the following two conditions are satisfied*

1. If $\lambda \in U_C$, $C \subseteq C'$, and $\lambda' \in \Sigma_{C'}$ satisfies $\lambda'|_C = \lambda$, then $\lambda' \in U_{C'}$.

2. For every $C \in \mathcal{C}$, U_C is open in Σ_C

Before explaining the relevance of this space, we make some obervations. It is shown in [55, Section C1.6] that for a locale X in **Set** the slice category $\mathbf{Loc}/X$ is equivalent the the category $\mathbf{Loc}(\mathrm{Sh}(X))$ of locales internal to $\mathrm{Sh}(X)$. Here $\mathbf{Loc}/X$ denotes the category that has locale maps $f : Y \to X$, for arbitrary locales Y in **Set**, as objects. Let f and g be such maps. An arrow $h : f \to g$ is given by a commuting triangle of locale maps:

$$\begin{tikzcd} Y \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Given a locale map $f : Y \to X$, a locale $\mathcal{I}(f)$ internal to $\mathrm{Sh}(X)$ is constructed as follows. First note that a locale map $f : Y \to X$ induces a geometric morphism $F : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. Let Ω_Y be the subobject classifier of $\mathrm{Sh}(Y)$. This object is an internal locale of $\mathrm{Sh}(Y)$. The direct image F_* of the geometric morphism f is cartesian and preserves internal complete posets. Hence $\mathcal{I}(f) = F_*(\Omega_Y)$ is an internal locale of $\mathrm{Sh}(X)$.
The previous observation is relevant because $[\mathcal{C}, \mathbf{Set}]$ is equivalent to a topos of the form $Sh(X)$. If P is a poset, then P can be seen as a topological space $P_\uparrow$ by equipping it with the Alexandroff (upper set) topology, defined as

$$U \in \mathcal{O}P \quad \leftrightarrow \quad \forall p \in P \;\; (p \in U) \wedge (p \leq q) \to (q \in U).$$

Identifying the elements $p \in P$ with the Alexandroff opens $(\uparrow p) \in \mathcal{O}P_\uparrow$, and noting that the opens $(\uparrow p)$ form a basis for the Alexandroff topology, the topos $Sh(P_\uparrow)$ is isomorphic to the topos $[P, \mathbf{Set}]$.
Let $f : Y \to \mathcal{C}_\uparrow$ be a continuous map of topological spaces, where $\mathcal{C}_\uparrow$ is the set of contexts with the Alexandroff topology. Such a function defines a locale map $L(f) : L(Y) \to L(\mathcal{C}_\uparrow)$, where $L(Y)$ and $L(\mathcal{C}_\uparrow)$ are the locales associated to the spaces. The locale map $L(f)$ defines a locale $\underline{Y}$ internal to the topos $Sh(L(\mathcal{C}_\uparrow)) = Sh(\mathcal{C}_\uparrow)$. In this way the map of spaces $f : Y \to \mathcal{C}_\uparrow$ defines a locale internal to $[\mathcal{C}, \mathbf{Set}]$.
Most of this section is devoted to proving the following theorem:

Theorem 2.2.2. *The projection map*

$$\pi : \Sigma_\uparrow \to \mathcal{C}_\uparrow, \quad (C, \lambda) \mapsto C,$$

is continuous and defines a locale $\underline{\Sigma}_\uparrow$ internal to $[\mathcal{C}, \mathbf{Set}]$ (in being its external description). Up to isomorphism, this locale is the internal Gelfand spectrum of $\underline{A}$. The frame associated to this locale is given by

$$\mathcal{O}\underline{\Sigma}_\uparrow : C \mapsto \mathcal{O}\Sigma|_{\uparrow C} = \{U \in \mathcal{O}\Sigma_\uparrow \mid U \subseteq \coprod_{C' \in (\uparrow C)} \Sigma_{C'}\},$$

where for $C \subseteq C'$ the map $\mathcal{O}\underline{\Sigma}_\uparrow(C) \to \mathcal{O}\underline{\Sigma}_\uparrow(C')$ is given by

$$U \mapsto \coprod_{C'' \in (\uparrow C')} U_{C''}.$$

Corollary 2.2.3. *Let $L(\pi) : L(\Sigma_\uparrow) \to L(\mathcal{C}_\uparrow)$ be the locale map associated to the bundle $\pi : \Sigma_\uparrow \to \mathcal{C}_\uparrow$. This locale map is the external description (in $\mathbf{Loc}/L(\mathcal{C}_\uparrow)$) of the spectral locale $\underline{\Sigma}_{\underline{A}}$ in $[\mathcal{C}, \mathbf{Set}] \cong Sh(L(\mathcal{C}_\uparrow))$. The locale $L(\Sigma_\uparrow)$ is spatial.*

We proceed to describe the locale $\underline{\Sigma}_{\underline{A}}$, and prove that it coincides with $\underline{\Sigma}_\uparrow$. Following [48, Appendix A], the spectrum $\underline{\Sigma}_{\underline{A}}$ can be constructed in three steps. In the first step we construct a distributive lattice $\underline{L}_{\underline{A}}$ from the positive cone of $\underline{A}$. The second step provides $\underline{L}_{\underline{A}}$ with a covering relation $\underline{\triangleleft}_{\underline{A}}$. The third and final step constructs the frame $\mathcal{O}\underline{\Sigma}_{\underline{A}}$ from the pair $(\underline{L}_{\underline{A}}, \underline{\triangleleft}_{\underline{A}})$ as the frame of ideals of $\underline{L}_{\underline{A}}$ which are closed under $\underline{\triangleleft}_{\underline{A}}$. We briefly consider these steps, which hold for the Gelfand spectrum of any unital commutative C*-algebra in a topos. Details can be found in [48, Appendix A]

Definition 2.2.4. *Let A be a unital commutative C^*-algebra in a topos $\mathcal{E}$, and define*

$$A^+ = \{a \in C_{sa} \mid a \geq 0\} = \{a \in A \mid \exists b \in C, a = b^*b\}.$$

Now define the following relation on A^+: $a \precsim b$ whenever there is an $n \in \mathbb{N}$ such that $a \leq nb$. Define the equivalence relation $a \approx b$ whenever $a \precsim b$ and $b \precsim a$. Let L_A denote the set of equivalence classes.

The lattice operations on A_{sa} (with respect to the partial order $a \leq b$ iff $(b - a) \in A^+$) respect the equivalence relation of the definition, turning L_A into a distributive lattice.

Next, we supply the lattice L_A with the covering relation $\lhd \subseteq L_A \times \mathcal{P}L_A$, defined as

$$\forall [a] \in L_A,\ \forall U \in \mathcal{P}L_A,\ [a] \lhd U,\ \text{iff}\ \forall q \in \mathbb{Q}^+\ \exists W \in \mathcal{F}U,\ [a-q] \leq \bigvee W,$$

where $\mathcal{F}U$ denotes the set of (Kuratowski) finite subsets of U. Note that in particular $[a] \lhd U$ iff for every $q \in \mathbb{Q}^+$, $[a-q] \lhd U$. The frame generated by the pair $(L_A, \lhd)$ consists of ideals of L_A, that are closed under the covering relation $\lhd$, in the sense that for such an ideal I, $[a]$ is covered by elements of I iff $[a] \in I$. This frame is the Gelfand spectrum of A.
We proceed to present two proofs of Theorem 2.2.2.

2.2.1 Direct Proof of Theorem 2.2.2

Let $\mathcal{C}$ be any small category, and $\underline{A}$ be a unital commutative C*-algebra in $[\mathcal{C}, \mathbf{Set}]$. From Proposition 2.1.2 we know that $\underline{A}$ is a functor mapping into the category of unital commutative C*-algebras and unit-preserving $*$-homomorphisms. The first step in calculating the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ is the construction of the distributive lattice $\underline{L}_{\underline{A}}$, which we will simply denote by $\underline{L}$.

Consider for a moment a unital commutative C*-algebra A in an arbitrary topos $\mathcal{E}$. Recall that the positive cone $A^+ = \{a \in A \mid a \geq 0\}$ is given by those $a \in A$, such that $\exists b \in A,\ a = b^*b$. As this condition is defined within the restrictions given by geometric logic, if $F : \mathcal{F} \to \mathcal{E}$ is a geometric morphism, and $B := F^*(A)$, then $B^+ \cong F^*(A^+)$, see Appendix A.3 for a discussion of geometric logic. Define an equivalence relation on A^+ by taking $a \sim b$ iff there exist natural numbers $n, m \in \mathbb{N}$ such that $nb - a \in A^+$ and $ma - b \in A^+$. The relation is defined within the confines of geometric logic and F^* preserves coequalizers, so we conclude $(B^+/\sim) \cong F^*(A^+/\sim)$. The set A^+ is a distributive lattice with respect to $\leq$, and this lattice structure descends to $A^+/\sim$. The lattice L_A is simply $A^+/\sim$ with this lattice structure. As distributive lattices, $L_B \cong F^*(L_A)$.

Returning to the C*-algebra $\underline{A}$ in the functor category $[\mathcal{C}, \mathbf{Set}]$, from the previous discussion we can derive $\underline{L}$:

$$\underline{L} : \mathcal{C} \to \mathbf{Set},\quad \underline{L}(C) = L_{\underline{A}(C)}, \tag{2.17}$$

$$\underline{L}(f) : L_{\underline{A}(C)} \to L_{\underline{A}(D)} \quad \underline{L}(f)([a]_C) = [\underline{A}(f)(a)]_D,$$

where $L_{\underline{A}(C)}$ is the distributive lattice for the commutative C*-algebra $\underline{A}(C)$ in **Set** and $[a]_C$ denotes the equivalence class of $a \in \underline{A}(C)$ in $L_{\underline{A}(C)}$. Note that $\underline{L}(f)$ is well defined. By definition $a \sim b$ in $\underline{A}(C)$ iff $a \leq nb$ and $b \leq ma$ for some natural numbers n and m. As $\underline{A}(f)$ is a $*$-homomorphism, it is a positive map, and hence

$$\underline{A}(f)(nb - a) = n\underline{A}(f)(b) - \underline{A}(f)(a) \geq 0,$$

and analogously for $ma - b$. We conclude that $\underline{A}(f)(a) \sim \underline{A}(f)(b)$.

Remark 2.2.5. *Restricting to using only expressions of geometric logic, we cannot show that L_A is a lattice, as this relies on completeness of A with respect to the norm. However, the construction of L_A out of A is expressible within geometric logic, and if L_A happens to be a distributive lattice, then so is F^*L_A.*

Internally, the spectrum $\underline{\Sigma}_{\underline{A}}$, or $\underline{\Sigma}$ for short, is the frame $\mathrm{RIdl}(\underline{L})$ of regular ideals of $\underline{L}$; i.e., ideals of $\underline{L}$ satisfying the additional condition

$$\forall \underline{U} \in \mathrm{RIdl}(\underline{L}), \quad [a] \in \underline{U} \;\leftrightarrow\; \left(\forall q \in \underline{\mathbb{Q}}^+ \; [a - q] \in \underline{U}\right).$$

Consider the following generalisation of the space $\Sigma_\uparrow$:

Definition 2.2.6. *Define the set $\Sigma = \coprod_{f:D\to C} \Sigma_{\underline{A}(C)}$, where the coproduct is taken over all the arrows f in $\mathcal{C}$. Equip Σ with the following topology, where $U \in \mathcal{O}\Sigma$ iff the following two conditions are satisfied:*

- *For each arrow $f : D \to C$, $U_f \in \mathcal{O}\Sigma_{\underline{A}(C)}$.*
- *For arrows $f : D \to C$ and $g : C \to E$, let the continuous map $\Sigma(g) : \Sigma_{\underline{A}(E)} \to \Sigma_{\underline{A}(C)}$ be the Gelfand dual of the $*$-homomorphism $\underline{A}(g) : \underline{A}(C) \to \underline{A}(E)$. We require $\Sigma(g)^{-1}(U_f) \subseteq U_{g\circ f}$.*

Theorem 2.2.7. *Up to isomorphism, the frame of $\underline{\Sigma}$ is given by the functor*

$$\mathcal{O}\underline{\Sigma} : \mathcal{C} \to \mathbf{Set}, \quad \mathcal{O}\underline{\Sigma}(C) = \mathcal{O}\Sigma|_C,$$

where $\Sigma|_C$ is $\coprod_{f:C\to D} \Sigma_{\underline{A}(D)}$, in which the coproduct is taken over all arrows with codomain C, and is equipped with the relative topology of Definition 2.2.6. In the arrow part of $\mathcal{O}\underline{\Sigma}$, transition maps are given by truncation.

Proof. Let $\underline{U} \in \mathcal{O}\underline{\Sigma}(D)$, i.e., let $\underline{U} \in \mathrm{RIdl}(\underline{L})(D)$. If $k(D)$ denotes the covariant hom-functor $Hom_{\mathcal{C}}(D, -)$, then $\underline{U}(D)$ is a subobject of $k(D) \times \underline{L}$, internally satisfying the conditions of an ideal of the lattice $\underline{L}$, as well as

$$D \Vdash \forall [a] \in \underline{L} \left([a] \in \underline{U} \Leftrightarrow \left(\forall q \in \underline{\mathbb{Q}}^+ \; [a - q] \in \underline{U} \right) \right) \tag{2.18}$$

For an arrow $f : D \to C$, define

$$U_f = \{[a] \in \underline{L}(C) \mid (f, [a]) \in \underline{U}(C)\}.$$

By a straightforward exercise in presheaf semantics, $\underline{U}$ satisfies (2.18) and the axioms of an ideal iff each U_f is an ideal of $\underline{L}(D) = L_{\underline{A}(D)}$ satisfying

$$[a] \in U_f \leftrightarrow \left(\forall q \in \mathbb{Q}^+ \;\; [a - q] \in U_f \right).$$

By Gelfand duality we identify U_f with an open $V_f \in \mathcal{O}\Sigma_{\underline{A}(C)}$. For $a \in \underline{A}(C)_{sa}$, define

$$D_a^C \in \mathcal{O}\Sigma_{\underline{A}(C)}, \quad D_a^C = \{\lambda \in \Sigma_{\underline{A}(C)} \mid \langle \lambda, a \rangle > 0\}.$$

Under the identification of U_f with V_f we have $[a] \in U_f$ iff $D_a^C \subseteq V_f$. As $\underline{U}$ is a subfunctor of $k(D) \times \underline{L}$, the condition $[a] \in \underline{U}_f$ implies that for $f : D \to C$, one has $[\underline{A}(f)(a)] \in U_{g \circ f}$. What does this imply for $V_{g \circ f}$ relative to V_f? Let $D_a^C \subseteq V_f$ (i.e. $[a] \in \; U_f$). Then

$$\begin{aligned} \Sigma(f)^{-1}(D_a^C) &= \{\lambda \in \Sigma_{\underline{A}(D)} \mid \langle \Sigma(f)(\lambda), a \rangle > 0\} \\ &= \{\lambda \in \Sigma_{\underline{A}(D)} \mid \langle \lambda, \underline{A}(f)(a) \rangle > 0\} \\ &= D^D_{\underline{A}(f)(a)}. \end{aligned}$$

As $[a] \in U_f$, we know that $[\underline{A}(f)(a)] \in U_{g \circ f}$. This implies that $D^D_{\underline{A}(f)(a)} \subseteq V_{g \circ f}$. If $D_a^C \subseteq V_f$, the calculation shows $\Sigma(f)^{-1}(D_a^C) \subseteq V_{g \circ f}$. As the sets D_a^C, with varying $a \in \underline{A}(C)_{sa}$, form a basis for $\Sigma_{\underline{A}(C)}$, we can conclude $\Sigma(f)^{-1}(V_f) \subseteq V_{g \circ f}$.
Through the correspondence $U_f \leftrightarrow V_f$, each $\underline{U} \in \mathrm{RIdl}(\underline{L})$ gives an open of Σ. This correspondence induces an isomorphism of posets, and hence an isomorphism of frames, proving the theorem. □

Note that Theorem 2.2.2 is a special case of Theorem 2.2.7

2.2.2 Proof of Theorem 2.2.2 Using Internal Sheaves

We set out to prove Theorem 2.2.2 again, now using internal sheaf topoi and iterated forcing. The idea of the proof in this subsection goes back to [72], and may be of interest when more advanced Grothendieck topoi are to be used as topos models for physics.

In the language of [55], the pair $(L_A, \lhd)$, generating the Gelfand spectrum Σ_A of a unital commutative C*-algebra A, defines a site (L_A, T) in $\mathcal{E}$, where T is a sifted, i.e. down-closed, coverage. In this coverage T, an element $[a] \in L_A$ is covered by $C([a])$, where

$$C([a]) := \{[b] \in L_A \mid \exists q \in \mathbb{Q}^+ \quad [b] \leq [a - q]\}. \tag{2.19}$$

Note that $C([a])$ is simply the down-closure of the set $\{[a - q] \mid q \in \mathbb{Q}^+\}$ in L_A. We consider the down-closure because we assume the coverage to be sifted. We wish to consider the topos of sheaves over $(\underline{L_A}, \underline{T})$, internal to $[\mathcal{C}, \mathbf{Set}]$.

Thus we consider the construction of a topos within another topos (other than **Set**). A **site** in a topos is a pair $(\underline{\mathcal{C}}, \underline{T})$, consisting of an internal (small) category $\underline{\mathcal{C}}$ and an internal coverage $\underline{T}$ on this category. For readers unfamiliar with categories of sheaves in a topos: In [55, Section B2.3] category theory internal to any topos is discussed and in [55, Section C2.4] Grothendieck topologies for small categories internal to toposes are treated (actually, a more liberal notion of coverage is treated), leading to the definition of a site in a topos.

Let $\mathbb{D}$ be a site in the topos $\mathrm{Sh}_{\mathcal{S}}(\mathbb{C})$ (where $\mathbb{C}$ is a site in the ambient topos $\mathcal{S}$), then we let $\mathrm{Sh}_{\mathrm{Sh}(\mathbb{C})}(\mathbb{D})$ denote the internal category of sheaves over $\mathbb{D}$. The category $\mathrm{Sh}_{\mathrm{Sh}(\mathbb{C})}(\mathbb{D})$ is a subtopos of the internal diagram category $[\mathbb{C}^{op}, \mathcal{S}]$ ([55, Section B2.3]).

For our purposes, the ambient topos is $\mathcal{S} = \mathbf{Set}$, the site $\mathbb{C}$ is given by the space $\mathcal{C}_\uparrow$ of contexts (with the open cover topology), and the internal site $\mathbb{D}$ in $Sh(\mathcal{C}_\uparrow)$ is defined by the locale $\underline{\Sigma}$ as follows. The frame $\mathcal{O}\underline{\Sigma}$ is generated by the lattice $\underline{L_A}$, given by the functor $\underline{L_A}(C) = L_C$, equipped with a covering relation $\underline{\lhd} \subseteq \underline{L_A} \times \mathcal{P}\underline{L_A}$.

Recall from [48] (or derive from from presheaf semantics), that the covering relation $\underline{\lhd}$ can be defined locally in the following sense: $C \Vdash [a]\underline{\lhd}\underline{U}$ iff $[a]_C \lhd_C \underline{U}(C)$. Here the latter means that for each rational $q > 0$ there exists a finite $U_0 \subseteq \underline{U}(C)$ such that $[a - q] \leq \bigvee U_0$. This is well-defined, as every L_C is a lattice. Whenever we speak of $\underline{\Sigma}$ as a site in $Sh(\mathcal{C}_\uparrow)$, think of $\underline{\Sigma}$ as the internal poset $\underline{L_A}$ together with the internal coverage defined by $\underline{\lhd}$.

In this way the locale $\underline{\Sigma}$, seen as a site, defines the unique (localic) geometric morphism $\mathrm{Sh}_{Sh(\mathcal{C}_\uparrow)}(\underline{\Sigma}) \to Sh(\mathcal{C}_\uparrow)$. At the level of locales, the composition of geometric morphisms

$$\mathrm{Sh}_{Sh(\mathcal{C}_\uparrow)}(\underline{\Sigma}) \to Sh(\mathcal{C}_\uparrow) \to \mathbf{Set} \tag{2.20}$$

corresponds to the composition of locale maps

$$\Sigma \xrightarrow{\pi} \mathcal{C}_\uparrow \xrightarrow{!} 1 = \Sigma \xrightarrow{!} 1. \tag{2.21}$$

This expression is valid only when $\mathcal{C}_\uparrow$ is sober. However, in Section 2.5 we see that the space $\mathcal{C}_\uparrow$ is in general not sober. Hence, the locale $L(\mathcal{C}_\uparrow)$ may have points that do not arise from points of the space $\mathcal{C}_\uparrow$. We could have avoided this by replacing $\mathcal{C}_\uparrow$ by its sobrification. We describe the sobrified version of the spectral bundle in Section 2.5. For the moment we ignore the sobrification. If this makes the reader squirrelly, he or she may want to restrict to finite-dimensional C*-algebras A, for which $\mathcal{C}_\uparrow$ is indeed sober.

By (2.21) the localic geometric morphism (2.20) is just Σ, the external description of the spectrum. Next, following [72], we recall some theory from [65] that will help in calculating Σ (and is interesting in its own right). Return to the more general situation where $\mathcal{S}$ is a topos, $\mathbb{C}$ a site in $\mathcal{S}$ and $\mathbb{D}$ a site in $\mathrm{Sh}_{\mathcal{S}}(\mathbb{D})$. We can construct a site $\mathbb{C} \ltimes \mathbb{D}$ in $\mathcal{S}$ such that

$$\mathrm{Sh}_{\mathrm{Sh}(\mathbb{C})}(\mathbb{D}) \cong \mathrm{Sh}_{\mathcal{S}}(\mathbb{C} \ltimes \mathbb{D}). \tag{2.22}$$

In our case this means that we can construct a posite (i.e., a site coming from a poset) $\mathcal{C} \ltimes \underline{\Sigma}$ in $\mathbf{Set}$, such that the locale it generates is Σ, the external description of the spectrum (up to isomorphism of locales).

The objects of $\mathbb{C} \ltimes \mathbb{D}$ are pairs (C, D) with C an object of $\mathbb{C}$ and $D \in \mathbb{D}_0(C)$, where $\mathbb{D}_0 : \mathbb{C}^{op} \to \mathcal{S}$ is the object of objects of the site $\mathbb{D}$. Under the identification $[\mathcal{C}, \mathbf{Set}] \cong Sh(\mathcal{C}_\uparrow)$, the objects of $\mathcal{C} \ltimes \underline{\Sigma}$ are pairs $(C, [a]_C)$, with $C \in \mathcal{C}$ and $[a]_C \in L_C$. An arrow $(f, g) : (C, D) \to (C', D')$ in $\mathbb{C} \ltimes \mathbb{D}$ is given by an arrow $f : C \to C'$ in $\mathbb{C}$ and an arrow $g \in \mathbb{D}_1(C)$, $g : D \to D'|_f$. For $\mathcal{C} \ltimes \underline{\Sigma}$ there exists a unique arrow $(C, [a]_C) \to (C', [b]_{C'})$ iff $C' \subseteq C$ and $[a]_C \leq [b]_C$ in L_C.

A collection of arrows $(f_i, g_i) : (C_i, D_i) \to (C, D)$ covers (C, D) in $\mathbb{C} \ltimes \mathbb{D}$ if the subsheaf $\underline{S}$ of $\mathbb{D}_1$, generated by the conditions $C_i \Vdash g_i \in \underline{S}$ satisfies $C \Vdash$ '$\underline{S}$ covers D'. Note that the Grothendieck topology of $\mathbb{C}$ also matters here as $\underline{S}$ is a sheaf w.r.t. this topology. For the (poset) case $\mathcal{C} \ltimes \underline{\Sigma}$, a set $U = \{(C_i, [a_i]_{C_i})\}_{i \in I}$ with $C \subseteq C_i$ and $[a_i]_{C_i} \leq [a]_{C_i}$ in L_{C_i}, covers

$(C, [a]_C)$, or $(C, [a]_C) \blacktriangleleft U$ for short, if the subsheaf $\underline{S}$ of $\underline{L_A}$ generated by the conditions $C_i \Vdash [a_i] \in \underline{S}$ satisfies $C \Vdash [a] \underline{\lhd}\ \underline{S}$, or equivalently $[a]_C \lhd_C \underline{S}(C)$. We use a black triangle $\blacktriangleleft$ for the covering relation on $\mathcal{C} \ltimes \underline{\Sigma}$ so that it will be confused neither with the covering relation $\underline{\lhd}$ on $\underline{L_A}$ nor with the covering relation $\lhd_C$ on L_C.
In what follows it is convenient to identify $\mathcal{C} \ltimes \underline{\Sigma}$ with $\coprod_{C \in \mathcal{C}} L_C$ as sets, and to write U_C for $U \cap L_C$ where $U \subseteq \mathcal{C} \ltimes \underline{\Sigma}$. The poset $\mathcal{C} \ltimes \underline{\Sigma}$ together with the covering relation $\blacktriangleleft$ generates a locale (see for example [4, Definition 14]), which we know to be Σ, [48]. The opens of Σ are obtained as follows. Take any downward closed $X \subseteq \mathcal{C} \ltimes \underline{\Sigma}$, then the corresponding open $U \in \mathcal{O}\Sigma$ is given by $U = \{(C, [a]_C) \mid (C, [a]_C) \blacktriangleleft X\}$. Then U being downward closed simply means that if $[a]_C \in U_C$, $C \subseteq C'$ and $[b]_{C'} \leq [a]_{C'}$ in $L_{C'}$, then $[b]_{C'} \in U_{C'}$.
Suppose that $(C, [a]_C) \blacktriangleleft X$. As a sheaf on $\mathcal{C}_\uparrow$ is equivalent to a set-valued functor on $\mathcal{C}$, we find for the sheaf $\underline{S}$ generated by the conditions $D \Vdash [b] \in \underline{S}$, where $(D, [b]_D) \in X$ is simply the functor $\underline{S}(C) = X_C$. Note that we used the fact that X is a downwards closed in order to get a well-defined functor. We thus find that $[a]_C \in U_C$ iff $(C, [a]_C) \blacktriangleleft X$ iff $[a]_C \lhd_C X_C$.
Note that for any downward closed set X in $\mathcal{C} \ltimes \underline{\Sigma}$ and any $C \in \mathcal{C}$, X_C is downward closed in L_C. If Σ_C is the Gelfand spectrum of C, then by constructive Gelfand duality the opens of Σ_C are generated as

$$U = \{[a]_C \in L_C \mid [a]_C \lhd_C X\},$$

where X ranges over the downward closed subsets of L_C. We can therefore identify an open of Σ as giving for every $C \in \mathcal{C}$ an open $U_C \in \mathcal{O}\Sigma_C$. The condition that $[a]_C \in X_C$ and $C \subseteq C'$, implies $[a]_{C'} \in X_{C'}$, then translates as follows. If $C \subseteq C'$ and $\rho_{C'C} : \Sigma_{C'} \to \Sigma_C$ is the restriction map, then $\rho^*_{C'C}(U_C) \subseteq U_{C'}$. We have once again shown Theorem 2.2.2. $\square$

2.2.3 Properties of the Spectral Bundle

According to the general theory of Banaschewski and Mulvey [9], internal Gelfand spectra of commutative unital C*-algebras are compact completely regular locales. Using the external presentation $\pi : \Sigma_\uparrow \to \mathcal{C}_\uparrow$, we now check this for the particular case of $\underline{\Sigma}_A$.

Definition 2.2.8. *Let L be a locale. Then L is **compact** if for any $S \subseteq L$ such that $1_L = \bigvee S$, there is a finite $F \subseteq S$ such that $1_L = \bigvee F$. Here 1_L*

denotes the top element of L. Equivalently, one can say that L is compact if for each ideal I of L such that $\bigvee I = 1_L$, we have $1_L \in I$.

The following definition and lemma help to show that $\underline{\Sigma}_\uparrow$ is compact.

Definition 2.2.9. *A continuous map of spaces $f : Y \to X$ is called **perfect** if the following two conditions are satisfied:*

1. *f has compact fibres: if $x \in X$ then $f^{-1}(x)$ is compact in Y.*
2. *f is closed: if C is closed in Y, then $f(C)$ is closed in X.*

Lemma 2.2.10. *([56], Proposition 1.1) Let $f : Y \to X$ be continuous. If f is perfect, then the internal locale $\mathcal{I}(f) = F_*(\Omega_{\mathrm{Sh}(Y)})$ in $\mathrm{Sh}(X)$ is compact.*

In the previous lemma, F_* denotes the direct image part of the geometric morphism associated to f, and $\Omega_{\mathrm{Sh}(Y)}$ denotes the subobject classifier of $\mathrm{Sh}(Y)$.

Definition 2.2.11. *([58], III.1, 1.1) Let L be a locale and $x, y \in L$. Then x is **well inside** y, denoted by $x \lessdot y$, if there exists a $z \in L$ such that $z \wedge x = 0_L$ and $z \vee y = 1_L$. A locale L is called **regular** if every $x \in L$ satisfies*

$$x = \bigvee \{y \in L | y \lessdot x\}.$$

Regularity of the internal locale $\underline{\Sigma}_\uparrow$ can conveniently be checked from its external description π, as shown by the following lemma.

Lemma 2.2.12. *([57] Lemma 1.2) Let $f : Y \to X$ be continuous. Then $F_*(\Omega_{\mathrm{Sh}(Y)})$ is regular iff for any open $U \in \mathcal{O}Y$ and $y \in U$ there is a neigborhood N of $f(y)$ in X, and there exist opens $V, W \in \mathcal{O}Y$ such that $y \in V$, $V \cap W = \emptyset$ and $f^{-1}(N) \subseteq U \cup W$.*

Corollary 2.2.13. *The internal locale $\underline{\Sigma}_\uparrow$ is compact and completely regular.*

Proof. We already knew this from constructive Gelfand duality, which establishes a duality between unital commutative C*-algebras and compact completely regular locales[2] [7, 8, 9]. However, Theorem 2.2.2 presents a way to check compactness and complete regularity directly. Indeed, Lemma 2.2.10 and Lemma 2.2.12 applied to $\pi : \Sigma_\uparrow \to \mathcal{C}_\uparrow$ prove the corollary. $\square$

[2] In **Set** completely regular locales are equivalent to compact Hausdorff spaces (using the axiom of choice).

Consider the spectrum $\underline{\Sigma}_\uparrow = \underline{\Sigma}_{\underline{A}}$ for an n-level system $A = M_n(\mathbb{C})$. For each $C \in \mathcal{C}$ the Gelfand spectrum $\mathcal{O}\Sigma_C$ is isomorphic to $\mathcal{P}(C)$ as a frame, where $\mathcal{P}(C)$ is the set of projection operators in C, partially ordered as $p \leq q$ if $p\mathbb{C}^n \subseteq q\mathbb{C}^n$. Let $C \subseteq C'$ in $\mathcal{C}$. Take $U_C \in \mathcal{O}\Sigma_C$ corresponding to the projection operator $P_C \in C$ and $U_{C'} \in \mathcal{O}\Sigma_{C'}$ corresponding to the projection operator $P_{C'} \in C'$. We have $\rho^{-1}_{C'C}(U_C) \subseteq U_{C'}$ if and only if $P_{C'} \geq P_C$. This demonstrates that for an n-level system there is a bijection

$$\mathcal{O}\Sigma_\uparrow \cong \{S : \mathcal{C} \to \mathcal{P}(A) \mid S(C) \in \mathcal{P}(C),\ C \subseteq C' \Rightarrow S(C) \leq S(C')\}.$$

This description in terms of maps S is exactly the externalization of $\mathcal{O}\underline{\Sigma}_{\underline{A}}$ for an n-level system given in [20]. It is a straightforward exercise to verify that the Heyting algebra structure given in [20] coincides with the Heyting algebra structure of $\mathcal{O}\Sigma_\uparrow$.

2.3 Gelfand Transform

Using the external description of the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ of $\underline{A}$ found in the previous section, we present the externalized Gelfand transform of $\underline{A}$ (given by (2.25), (2.26)).

By constructive Gelfand duality, the internal commutative C*-algebra $\underline{A}$ with internal spectrum $\underline{\Sigma}_\uparrow$ is isomorphic to the internal commutative C*-algebra of continuous maps $\underline{C}(\underline{\Sigma}_\uparrow, \underline{\mathbb{C}})$ (which is the object of frame maps $\mathcal{O}\underline{\mathbb{C}} \to \mathcal{O}\underline{\Sigma}_\uparrow$). Here $\underline{\mathbb{C}}$ denotes the internal locale of complex numbers, given explicitly by the external description $\pi_1 : \mathcal{C} \times \mathbb{C} \to \mathcal{C}$ (see e.g. [9]). Let $\underline{A}_{sa}$ be the self-adjoint part of $\underline{A}$, defined by the functor $\underline{A}_{sa}(C) = C_{sa}$. Then $\underline{A}_{sa}$ is naturally isomorphic to the object $\underline{C}(\underline{\Sigma}_\uparrow, \underline{\mathbb{R}})$, where $\underline{\mathbb{R}}$ is the internal locale of Dedekind real numbers. The object $\underline{C}(\underline{\Sigma}_\uparrow, \underline{\mathbb{R}})$ is the object of internal frame maps $\underline{\mathrm{Frm}}(\mathcal{O}\underline{\mathbb{R}}, \mathcal{O}\underline{\Sigma}_\uparrow)$. For $C \in \mathcal{C}$ we have

$$\underline{\mathrm{Frm}}(\mathcal{O}\underline{\mathbb{R}}, \mathcal{O}\underline{\Sigma}_\uparrow)(C) = \mathrm{Nat}_{\mathrm{Frm}}(\mathcal{O}\underline{\mathbb{R}}|_{\uparrow C}, \mathcal{O}\underline{\Sigma}_\uparrow|_{\uparrow C}). \tag{2.23}$$

The external description of $\mathcal{O}\underline{\mathbb{R}}|_{\uparrow C}$ is the frame map

$$\pi_{\mathbb{R}}^{-1} : \mathcal{O}(\uparrow C) \to \mathcal{O}(\uparrow C \times \mathbb{R}),$$

which is the inverse image of the continuous map $\pi_{\mathbb{R}} : (\uparrow C) \times \mathbb{R} \to (\uparrow C)$, the projection on the first coordinate. Here $(\uparrow C)$ has the Alexandroff topology and $(\uparrow C) \times \mathbb{R}$ carries the product topology. In [20, Section 5] the right hand side of Equation 2.23 is shown to be equal to the set of frame maps

$$\phi_C^* : \mathcal{O}(\uparrow C \times \mathbb{R}) \to \mathcal{O}\Sigma_\uparrow|_{\uparrow C}$$

that satisfy the property that for every $C' \supseteq C$,

$$\phi_C^*(\uparrow C' \times \mathbb{R}) = \Sigma_\uparrow|_{\uparrow C'} = \coprod_{C'' \in \uparrow C'} \Sigma_{C''}.$$

We denote the set of frame maps satisfying this property by

$$\mathrm{Frm}'(\mathcal{O}(\uparrow C \times \mathbb{R}), \mathcal{O}\Sigma_\uparrow|_{\uparrow C}).$$

Under this identification, the Gelfand transformation becomes the natural isomorphism

$$\tilde{\mathcal{G}} : \underline{A}_{sa} \xrightarrow{\cong} \mathrm{Frm}'(\mathcal{O}(\uparrow - \times \mathbb{R}), \mathcal{O}\Sigma_\uparrow|_{\uparrow -}),$$

defined by

$$\hat{a}_C^{-1} := \tilde{\mathcal{G}}_C(a) : \mathcal{O}(\uparrow C \times \mathbb{R}) \to \mathcal{O}\Sigma_\uparrow|_{\uparrow C},$$

$$\begin{aligned} \hat{a}_C^{-1}(\uparrow C' \times (p,q)) &= \{(C'', \lambda'') \mid C'' \in \uparrow C',\ \lambda''(a) \in (p,q)\} \\ &= \coprod_{C'' \in \uparrow C'} (\hat{a}^{(C'')})^{-1}(p,q), \end{aligned}$$

where $a \in C_{sa}$, and $\hat{a}^{(C'')}$ denotes the (classical) Gelfand transform of a, seen as element of $C'' \supseteq C$. This frame map is the inverse image of the continuous map

$$\hat{a}_C : \Sigma_\uparrow|_{\uparrow C} \to (\uparrow C \times \mathbb{R}), \quad (C', \lambda') \mapsto (C', \lambda'(a)). \tag{2.24}$$

Note that continuous maps $f : \Sigma_\uparrow|_{\uparrow C} \to (\uparrow C \times \mathbb{R})$ such that $\pi_1 \circ f = \pi$ correspond bijectively to continuous maps $f : \Sigma_\uparrow|_{\uparrow C} \to \mathbb{R}$. The Gelfand isomorphism $\tilde{\mathcal{G}}$ therefore induces the natural isomorphism

$$\mathcal{G} : \underline{A}_{sa} \xrightarrow{\cong} C(\Sigma_\uparrow|_{\uparrow -}, \mathbb{R}), \tag{2.25}$$

$$\mathcal{G}_C(a) = \hat{a}_C : \Sigma_\uparrow|_{\uparrow C} \to \mathbb{R}, \quad \hat{a}_C(C', \lambda') = \lambda'(a). \tag{2.26}$$

Note that in particular we obtain a natural isomorphism

$$\underline{C}(\underline{\Sigma}_\uparrow, \underline{\mathbb{R}}_d) \cong C(\Sigma_\uparrow|_{\uparrow -}, \mathbb{R}),$$

where the subscript d refers to the Dedekind real numbers. This may look surprising at first glance, but in fact a continuous map $f : \Sigma_{\uparrow}|_{\uparrow C} \to \mathbb{R}$ is determined by $f|_{\Sigma_C}$; this is because continuity implies that $f(C', \lambda') = f(C, \lambda'|_C)$, giving a bijection $C(\Sigma_C, \mathbb{R}) \simeq C(\Sigma_{\uparrow}|_{\uparrow C}, \mathbb{R})$. Next, note that by Equation 2.24, $\hat{a}_C|_{\Sigma_{C'}} = \hat{a}^{(C')}$. If we are using (classical) Gelfand duality to identify $C \simeq C(\Sigma_C, \mathbb{R})$ and subsequently identify $C(\Sigma_C, \mathbb{R}) \simeq C(\Sigma_{\uparrow}|_{\uparrow C}, \mathbb{R})$, we recover (2.26). We conclude that the internal Gelfand transformation of $\underline{A}_{sa}$, looked upon externally, combines the Gelfand transformations of all contexts $C \in \mathcal{C}$ into a single functor. This was already pointed out in [51].

2.4 The Spectral Presheaf as an Internal Space

The spectral presheaf of the contravariant approach and its clopen subobjects are presented as a topological space internal to the presheaf topos.

In the contravariant approach, the object $\underline{\Sigma}$ of the topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is thought of as a state space, in analogy with classical physics. But is there any mathematical justification for calling $\underline{\Sigma}$ a space? We can think of $\underline{\Sigma}$ either as an object in an abstract category, or as a functor taking values in the category of topological spaces and continuous maps. In the internal language of $[\mathcal{C}^{op}, \mathbf{Set}]$, $\underline{\Sigma}$ is just a set, and we can always consider a set as a discrete space. However, we can do better than that. Below we describe $\underline{\Sigma}$ as a topological space internal to the topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, in such a way that states on A (in the sense of normalised positive functionals) and (daseinised) self-adjoint operators have a clear internal perspective. This internal perspective strengthens the analogy with both classical physics, and the covariant approach.

Given a set X (in the internal sense) in a topos $\mathcal{E}$, a topology $\mathcal{O}X$ on X is defined in a straightforward way: it is a subset $\mathcal{O}X \subseteq \mathcal{P}X$ of the powerset of X, satisfying the condition

$$\begin{gathered}\Vdash (X \in \mathcal{O}X) \wedge (\emptyset \in \mathcal{O}X) \wedge (U, V \in \mathcal{O}X \to U \cap V \in \mathcal{O}X) \\ \wedge\, (Y \subseteq \mathcal{O}X \to \bigcup Y \in \mathcal{O}X).\end{gathered}$$

For a topos of the kind $\mathcal{E} = Sh(T)$, with T a topological space, there is a useful external description of internal topologies on a sheaf X, as explained in [64]. This is relevant because the topos $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is equivalent to the topos $Sh(\mathcal{C}_{\downarrow})$, where the space $\mathcal{C}_{\downarrow}$ is the set $\mathcal{C}$ equipped with the

downwards Alexandroff topology. With respect to this topology, $U \subseteq \mathcal{C}$ is open iff it is downwards closed in the sense that if $C \subseteq C' \in U$, then $C \in U$. We will write $\mathcal{C}_A^{\downarrow}$ for $\mathcal{C}_{\downarrow}$ if we want to emphasise that $\mathcal{C}$ comes from A. This will become important when we consider $*$-homomorphisms as in Chapter 5.
For the external description of internal topologies (i.e., a description in **Set**), first recall that the category $Sh(T)$ is equivalent to the category $\textbf{Étale}(T)$, of étale bundles over T (as explained in detail in [63, Chapter II]). An étale bundle over T is a continuous map $p : X \to T$ such that each $x \in X$ has an open neighbourhood U_x, satisfying the condition that $p|_{U_x} : U_x \to T$ is a homeomorphism onto its image.
Under the identifications $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \cong Sh(\mathcal{C}_{\downarrow}) \cong \textbf{Étale}(\mathcal{C}_{\downarrow})$, the spectral presheaf $\underline{\Sigma}$ corresponds to the étale bundle $\pi : \Sigma_e \to \mathcal{C}_{\downarrow}$, where the set Σ_e is the disjoint union of Gelfand spectra $\coprod_{C \in \mathcal{C}} \Sigma_C$. This set is equipped with the following (étale) topology: for any non-empty $U \subseteq \Sigma_e$, we have $U \in \mathcal{O}\Sigma_e$ iff the following condition holds. If $(C, \lambda) \in U$ (with $\lambda \in \Sigma_C$), and $D \subseteq C$, then $(D, \lambda|_D) \in U$. Here we use the notation $\lambda|_D$ for the restriction $\rho_{CD}(\lambda)$. The function π is simply given by the projection $(C, \lambda) \mapsto C$.

Definition 2.4.1. *A topology on Σ_e is called a π-topology if it is coarser than the étale topology and with respect to which π is continuous.*

Proposition 2.4.2. *([64]) There is a bijection between topologies on $\underline{\Sigma}$, internal to $[\mathcal{C}^{op}, \mathbf{Set}]$, and π-topologies on Σ_e.*

Note that the étale topology itself qualifies as a π-topology, and this corresponds to the discrete topology on $\underline{\Sigma}$. It is not hard to see that the étale opens of Σ correspond to subobjects of $\underline{\Sigma}$. In the contravariant topos approach one is typically only interested in subobjects of $\underline{\Sigma}$ of a certain kind, the clopen subobjects. Recall that a subobject of $\underline{U} \subseteq \underline{\Sigma}$ is a clopen subobject iff for each $C \in \mathcal{C}$ the subset $\underline{U}(C) \subseteq \Sigma_C$ is clopen with respect to the topology on the Gelfand spectrum Σ_C. In the external description $\pi : \Sigma_e \to \mathcal{C}_{\downarrow}$, the clopen subobjects correspond to the étale opens U of Σ_e satisfying the condition that for each $C \in \mathcal{C}$, the set $U_C := U \cap \Sigma_C$ is clopen in Σ_C. These étale opens are not closed under infinite unions and therefore do not form a topology. However, they do form a basis for a topology. Note that, since we are working with von Neumann algebras, each Σ_C has a basis of clopen subsets [75]. By this observation, the internal topology of $\underline{\Sigma}$ generated by the clopen subobjects can be presented externally as follows.

Definition 2.4.3. *The space $\Sigma_{\downarrow}$ is the set Σ_e, where $U \subseteq \Sigma_e$ is open iff the following two conditions are satisfied:*

1. *If $\lambda \in U_C$ and $C' \subseteq C$, then $\lambda|_{C'} \in U_{C'}$.*
2. *For every $C \in \mathcal{C}$, U_C is open in Σ_C*

Proposition 2.4.4. *The topology $\mathcal{O}\Sigma_{\downarrow}$ is a π-topology, and thus defines an internal topology on $\underline{\Sigma}$, which is the topology generated by the closed open subobjects.*

In what follows, we write $\Sigma^{\downarrow}_{A}$ for $\Sigma_{\downarrow}$ whenever we want to emphasise A. We write $\underline{\Sigma}_{\downarrow}$ for $\underline{\Sigma}$ with the internal topology generated by the clopen subobjects.
As shown in [55, Section C1.6], the category $\mathbf{Loc}_{Sh(T)}$, of locales in $Sh(T)$, is equivalent to the category $\mathbf{Loc}/T$, of locales (in **Set**) over T. In particular, a topological space internal to $Sh(T)$ corresponds to a locale in $Sh(T)$ as such a space is described by a bundle over T. Externally, passing from topological spaces to locales means that we forget that we are working with topologies that are coarser than the étale topology of some sheaf. Internally, passing from topological spaces to locales means that we forget about the set of points that we topologised.
If A is of the form $\mathcal{B}(\mathcal{H})$, the bounded linear operators on a Hilbert space where $dim(\mathcal{H}) > 2$, then the Kochen–Specker Theorem applies, which is equivalent to the claim that the spectral presheaf has no global sections. The following theorem demonstrates that this entails that the spectral presheaf, seen as an internal locale (with the frame generated by the clopen subobjects) has no global points.

Theorem 2.4.5. *The Kochen–Specker Theorem is equivalent to the following claim: If $A = \mathcal{B}(\mathcal{H})$, satisfying $dim(\mathcal{H}) > 2$, then the internal locale $\underline{\Sigma}_{\downarrow}$ corresponding to the bundle $\pi : \Sigma_{\downarrow} \to \mathcal{C}_{\downarrow}$ does not have any global points.*

Note the similarity with the covariant approach; it was shown in [48] that the Kochen–Specker Theorem is equivalent to the spectral locale $\underline{\Sigma}_{\underline{A}}$ having no global points for $A = \mathcal{B}(\mathcal{H})$ whenever $dim(\mathcal{H}) > 2$.

Proof. First, assume that $\mathcal{C}_{\downarrow}$ is sober. In that case, a global point of the internal locale corresponds to a global section $s : \mathcal{C}_{\downarrow} \to \Sigma_{\downarrow}$ of the bundle π. For convenience, we write $s(C) = (C, r(C))$, with $r(C) \in \Sigma_C$. We will show that if $D \subseteq C$, then $r(D) = r(C)|_D$. Once we know this,

we can conclude that s is also a global section of the étale bundle, and the non-existence of such sections is equivalent to the Kochen–Specker Theorem [16]. Let $\lambda \in \Sigma_D$ be different from $r(C)|_D$. Choose an open neighbourhood U of $r(C)|_D$ in Σ_D such that $\lambda \notin U$. Next, consider $V_C = \rho_{CD}^{-1}(U)$, an open neighbourhood of $r(C)$ in Σ_C. Define the open V in $\Sigma_\downarrow$ as follows. If $D' \subseteq C$, then $V_{D'} = \rho_{CD'}(V_C)$. If D' is not below C, then $V_{D'} = \emptyset$. Note that we used the fact that the restriction maps ρ_{CD} are open maps, in order for V to be open in $\Sigma_\downarrow$. By construction $C \in s^{-1}(V)$. It follows that $D \in s^{-1}(V)$, as s is continuous. We conclude that $\lambda \neq r(D)$. As λ was an arbitrary element of Σ_D such that $\lambda \neq r(C)|_D$, we conclude $r(D) = r(C)|_D$. This proves the theorem for sober $\mathcal{C}_\downarrow$.
Next, we no longer assume that $\mathcal{C}_\downarrow$ is a sober space. The sobrification of $\mathcal{C}_\downarrow$ can be identified with the set $\mathcal{F}$ of filters of $\mathcal{C}$, equipped with the Scott topology. Recall that a subset $F \subseteq \mathcal{C}$ is a filter iff it is non-empty, upward closed, and downward directed. A subset $W \subseteq \mathcal{F}$ is Scott open iff it is upward closed with respect to the inclusion relation, and if for any directed family of filters $(F_i)_{i\in I}$, satisfying $\bigcup_{i\in I} F_i \in W$, implies that there exists an $i_0 \in I$, such that $F_{i_0} \in W$. The Scott topology on $\mathcal{F}$ is generated by the basis

$$W_C = \{F \in \mathcal{F} \mid C \in F\}, \quad C \in \mathcal{C}.$$

The continuous map

$$i : \mathcal{C}_\downarrow \to \mathcal{F}, \quad i(C) = \uparrow C := \{E \in \mathcal{C} \mid E \supseteq C\},$$

defines, through its inverse image, an isomorphism of frames

$$i^{-1} : \mathcal{O}\mathcal{F} \to \mathcal{O}\mathcal{C}, \quad i^{-1}(W_C) = \downarrow C.$$

Using this frame isomorphism we identify $Sh(\mathcal{C}_\downarrow)$ with $Sh(\mathcal{F})$. Let $\pi_{\mathcal{F}} : \Sigma_{\mathcal{F}} \to \mathcal{F}$ be the étale bundle corresponding to the spectral presheaf. Using the observation that for the principal filter $(\uparrow C)$, the smallest Scott open neighborhood is W_C, we identify the fibre $\pi_{\mathcal{F}}^{-1}(\uparrow C)$ with $\underline{\Sigma}(\downarrow C) = \Sigma_C$. Using the basis W_C of $\mathcal{F}$ and the definition of the étale topology, it follows that the injective map

$$j : \Sigma_e \to \Sigma_{\mathcal{F}}, \quad j(C, \lambda) = (\uparrow C, \lambda),$$

defines an isomorphism of frames on the corresponding topologies. Recall that Σ_e denotes the total space of the étale bundle of the spectral

presheaf, viewed as an object in $Sh(\mathcal{C}_\downarrow)$. We recognize the space $\Sigma_{\mathcal{F}}$ as the sobrification of Σ_e. By the isomorphism j^{-1}, the coarser (than étale) topology $\mathcal{O}\Sigma_\downarrow$ on Σ corresponds to a coarser than étale topology $\mathcal{O}\Sigma_{\mathcal{F}}^{\downarrow}$ on $\Sigma_{\mathcal{F}}$. It is straightforward to verify that with respect to this topology the map $\pi_{\mathcal{F}} : \Sigma_{\mathcal{F}}^{\downarrow} \to \mathcal{F}$ is continuous, and the internal locale associated to this bundle is the spectral presheaf with the topology generated by clopen subobjects, viewed as an object of $Sh(\mathcal{F})$.
From the previous discussion we conclude that a continuous section $\mathcal{F} \to \Sigma_{\mathcal{F}}^{\downarrow}$ of $\pi_{\mathcal{F}}$ restricts to a continuous section $\mathcal{C}_\downarrow \to \Sigma_\downarrow$. The theorem now follows from the case where we assumed $\mathcal{C}_\downarrow$ to be sober. □

The spectral presheaf viewed as a locale resembles the spectral locale of the covariant approach. Can this locale be a Gelfand spectrum of some internal C*-algebra? In other words, is it a compact completely regular locale?

Proposition 2.4.6. *The locale $\underline{\Sigma}_\downarrow$ in $[\mathcal{C}^{op}, \mathbf{Set}]$ associated to the bundle $\pi : \Sigma_\downarrow \to \mathcal{C}_\downarrow$ is compact.*

Proof. If we can show that $\pi : \Sigma_\downarrow \to \mathcal{C}_\downarrow$ is a closed map that has compact fibres, internal compactness follows from Lemma 2.2.10. The fact that π has compact fibres is evident. Let F be closed in $\Sigma_\downarrow$ and U be the set-theoretic complement of F. If $(D, \lambda) \in F$ and $C \supseteq D$, by surjectivity of ρ_{CD} there exists a $\lambda' \in \Sigma_C$ such that $\lambda'|_D = \lambda$. As $(D, \lambda) \notin U$, by the definition of $\mathcal{O}\Sigma_\downarrow$, $(C, \lambda') \notin U$, and $(C, \lambda') \in F$. We conclude that $\pi(F)$ is upwards closed in $\mathcal{C}$, which is equivalent to $\pi(F)$ being closed in $\mathcal{C}_\downarrow$. □

Up to this point it did not matter wether we excluded the trivial algebra $\mathbb{C}1$ from the set of contexts or not. For the discussion of regularity that follows, it does matter, so we need to be precise about it. Usually the trivial algebra is excluded in the contravariant approach. However, in discussions of composite systems in the contravariant approach (see e.g. Section 11 of [37]) the trivial context is included. For the moment, we will include the trivial subalgebra as a context.

Proposition 2.4.7. *Let A be a von Neumann algebra such that $\mathcal{C} \neq \{\mathbb{C} \cdot 1\}$. Then the locale $\underline{\Sigma}_\downarrow$ in $[\mathcal{C}^{op}, \mathbf{Set}]$ is not regular.*

Proof. Consider the following open subsets of $\Sigma_\downarrow$:

$$B_{C,u} = \{(D, \lambda|_D) \mid D \subseteq C, \lambda \in u\}, \quad C \in \mathcal{C}, u \in \mathcal{O}\Sigma_C.$$

Note that openness of $B_{C,u}$ relies on the restriction maps ρ_{CD} being open. By Lemma 2.2.12, $\underline{\Sigma}$ is regular iff for any $U \in \mathcal{O}\Sigma_\downarrow$ and any $(C,\lambda) \in U$ there exist opens $V, W \in \mathcal{O}\Sigma_\downarrow$ such that $(C,\lambda) \in V$, $V \cap W = \emptyset$ and $B_{C,\Sigma_C} \subseteq U \cup W$. By assumption, there exists a context C such that Σ_C has at least two elements. This follows from the Gelfand-Mazur Theorem, which implies that if Σ_C is a singleton, then $C \cong \mathbb{C}$. Take any two distinct $\lambda_1, \lambda_2 \in \Sigma_C$. We have $(C,\lambda_1) \in U := B_{C,\Sigma_C \setminus \{\lambda_2\}}$. If $\underline{\Sigma}$ is regular, there are $V, W \in \mathcal{O}\Sigma_\downarrow$ such that $(C,\lambda_1) \in V$, $(C,\lambda_2) \in W$ and $V \cap W = \emptyset$. In particular, for every $D \subseteq C$ we find that $\lambda_1|_D \neq \lambda_2|_D$. For $D = \mathbb{C} \cdot 1$ this condition is not satisfied, so that the compact locale $\underline{\Sigma}_\downarrow$ is not regular. □

Hence the space $\Sigma_\downarrow$ is not regular, so it cannot be the Gelfand spectrum of a commutative C*-algebra. However, it does satisfy the T_0-axiom. If we leave out the trivial context, Proposition 2.4.7 becomes slightly weaker. For example, the locale $\underline{\Sigma}_\downarrow$ associated to the von Neumann algebra $A = M_2(\mathbb{C})$ does happen to be regular (the space $\Sigma_\downarrow$ has the discrete topology in this case). In general, though, the locale $\underline{\Sigma}_\downarrow$ is not regular. For example, using the proof of Proposition 2.4.7 it is not hard to show that the locale $\underline{\Sigma}_\downarrow$ is not regular for $A = M_n(\mathbb{C})$, for any $n > 2$.

2.5 Sobriety

Sobriety of the spaces $\Sigma_\uparrow$ and $\Sigma_\downarrow$ is investigated. In addition, the external description of the spectral locale $\underline{\Sigma}_{\underline{A}}$ of the covariant approach is described as a bundle of sober spaces.

Recall the space $\Sigma_\downarrow$, introduced in Section 2.4 as the total space of the bundle $\pi : \Sigma_\downarrow \to \mathcal{C}_\downarrow$, the external description of the spectral presheaf $\underline{\Sigma}_A$, equipped with the topology generated by the clopen subobjects.

Definition 2.5.1. *Let $\Sigma = \{(C,\lambda) \mid C \in \mathcal{C}, \lambda \in \Sigma_C\}$. Then $U \in \mathcal{O}\Sigma_\downarrow$ iff*

1. *$\forall C \in \mathcal{C}\ U_C \in \mathcal{O}\Sigma_C$.*
2. *If $\lambda \in U_C$ and $C' \subseteq C$, then $\lambda|_{C'} \in U_{C'}$.*

Recall the space $\Sigma_\uparrow$, introduced in Section 2.2 as the total space of the bundle $\pi : \Sigma_\uparrow \to \mathcal{C}_\uparrow$, the external descritpion of the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ of the Bohrification functor $\underline{A}$.

Definition 2.5.2. *Let* $\Sigma = \{(C,\lambda) \mid C \in \mathcal{C}, \lambda \in \Sigma_C\}$. *Then* $U \in \mathcal{O}\Sigma_\uparrow$ *iff*

1. $\forall C \in \mathcal{C}\ U_C \in \mathcal{O}\Sigma_C$.
2. *If* $\lambda \in U_C$ *and* $C \subseteq C'$, *then* $\lambda' \in U_{C'}$ *whenever* $\lambda'|_C \in U_C$.

Before we start with the investigation of the spaces $\Sigma_\uparrow$, and $\Sigma_\downarrow$, we first consider the internal space $\underline{\Sigma}_\downarrow$ of the previous section. Even though $\underline{\Sigma}_\downarrow$ was shown to be not regular in general, it is always sober.

Proposition 2.5.3. *The internal space* $\underline{\Sigma}_\downarrow$, *associated to the bundle* $\pi_\downarrow : \Sigma_\downarrow \to \mathcal{C}_\downarrow$ *of Proposition 2.4.4 is sober (internally).*

Proof. First assume that $\mathcal{C}_\downarrow$ is a sober space. Let $\pi_e : \Sigma_e \to \mathcal{C}_\downarrow$, and $\pi_\downarrow : \Sigma_\downarrow \to \mathcal{C}_\downarrow$ denote the bundles associated to $\underline{\Sigma}$ and $\underline{\Sigma}_\downarrow$. Note that the étale space Σ_e is sober because $\mathcal{C}_\downarrow$ is sober. Let $j : \Sigma_e \to \Sigma_\downarrow$ be the function $(C,\lambda) \mapsto (C,\lambda)$, corresponding to the inclusion $j^{-1} : \mathcal{O}\Sigma_\downarrow \hookrightarrow \mathcal{O}\Sigma_e$. By [64, Corollary 3.2], the space $\underline{\Sigma}_\downarrow$ is sober internally iff the function $s \mapsto s \circ j$, mapping continuous sections of the bundle π_e to continuous sections of the bundle $\pi_\downarrow$ is a bijection. It was already demonstrated in the proof of Theorem 2.4.5 that each continuous section of $\pi_\downarrow$ is a continuous section of π_e, proving the proposition for sober $\mathcal{C}_\downarrow$.
Next, drop the assumption that $\mathcal{C}_\downarrow$ is sober. Let $\mathcal{F}$, $\Sigma_\mathcal{F}$ and $\Sigma_\mathcal{F}^\downarrow$ be as in the proof of Theorem 2.4.5. If we see $\lambda \in \Sigma_C$ as an element of $\lambda \in \underline{\Sigma}(W_C)$, and F is any filter in $\mathcal{C}$ containing C, let $[\lambda]_F$ denote the germ of λ in F. Note that the (étale) topology on $\Sigma_\mathcal{F}$ is generated by the basis

$$B_{C,\lambda} = \{(F,[\lambda]_F) \in \Sigma_\mathcal{F} \mid F \in W_C\} \quad C \in \mathcal{C}, \quad \lambda \in \Sigma_C,$$

whereas the topology on $\Sigma_\mathcal{F}^\downarrow$ is generated by the coarser basis

$$B_{C,u} = \{(F,[\lambda]_F \in \Sigma_\mathcal{F} \mid \ F \in W_C, \quad \lambda \in u\}, \quad C \in \mathcal{C}, \quad u \in \mathcal{O}\Sigma_C.$$

Using the same reasoning as for sober $\mathcal{C}_\downarrow$, any continuous section of the bundle $\Sigma_\mathcal{F}^\downarrow \to \mathcal{F}$ is a continuous section of $\Sigma_\mathcal{F} \to \mathcal{F}$. $\square$

We now investigate the sobriety of the spaces $\Sigma_\downarrow$ and $\Sigma_\uparrow$. We start with the space $\Sigma_\uparrow$ of the covariant approach. A point of $\Sigma_\uparrow$ by definition corresponds to a frame map $p : \mathcal{O}\Sigma_\uparrow \to \underline{2}$. If we define U to be the union of all $V \in \mathcal{O}\Sigma_\uparrow$ such that $p(V) = 0$, then $U \in \mathcal{O}\Sigma_\uparrow$ is the largest open set mapped to 0 by p. This can be translated to the following condition: if there are $U_1, U_2 \in \mathcal{O}\Sigma_\uparrow$ such that $U = U_1 \cap U_2$, then either

$U_1 = U$ or $U_2 = U$. Switching to complements, one can equivalently look at irreducible closed sets, i.e., sets F that are closed with respect to $\mathcal{O}\Sigma_\uparrow$ such that if there exist closed sets F_1 and F_2 with the property $F = F_1 \cup F_2$, then either $F = F_1$ or $F = F_2$.

Lemma 2.5.4. *Let F be closed in $\Sigma_\uparrow$. Then F is irreducible if and only if the following two conditions are satisfied:*

1. *$\forall C \in \mathcal{C}$: if $F_C \neq \emptyset$, then F_C is a singleton.*
2. *$\forall C_1, C_2 \in \mathcal{C}$: if F_{C_1} and F_{C_2} are both nonempty, then there exists a $C_3 \in \mathcal{C}$ such that $C_1, C_2 \subseteq C_3$ and F_{C_3} is nonempty.*

Proof. By definition of $\mathcal{O}\Sigma_\uparrow$, a set F is closed iff the following two conditions are satisfied. First, for every $C \in \mathcal{C}$ the set F_C is closed in Σ_C. Second, if $\lambda \in F_C$ and $D \subseteq C$, then $\lambda|_D \in F_D$.
Conversely, assume that there is a $C \in \mathcal{C}$ such that F_C has more than one element. The set F_C is reducible in Σ_C, so there are closed $F_{1C}, F_{2C} \subset F_C$ with the property $F_{1C} \cup F_{2C} = F_C$. Define the sets F_i, $i = 1, 2$ as follows. For any $C' \supseteq C$ take $(F_i)_{C'} = \rho^{-1}_{C'C}(F_{iC}) \cap F_{C'}$. For all other $C' \in \mathcal{C}$ take $(F_i)_{C'} = F_{C'}$. It is easily verified that the sets F_i are closed in $\Sigma_\uparrow$, that $F_i \subset F$, and that $F_1 \cup F_2 = F$. Hence the first condition of the lemma is a necessary condition for irreducibility.
Assume that there are contexts $C_1, C_2 \in \mathcal{C}$ such that F_{C_1} and F_{C_2} are nonempty and that for each $C' \in \mathcal{C}$ with the property $C_1, C_2 \subseteq C'$ we have $F_{C'} = \emptyset$. In that case, define F_i with $i \in \{1, 2\}$, as follows. If $C' \supseteq C_i$ then $(F_i)_{C'} = \emptyset$. For all other $C' \in \mathcal{C}$ take $(F_i)_{C'} = F_{C'}$. Again this produces closed sets $F_1, F_2 \subset F$ such that $F = F_1 \cup F_2$. Thus the second condition in the lemma has also been shown to be necessary.
Assume that F satisfies both conditions of the lemma. Let $F = F_1 \cup F_2$ and $F \neq F_2$. Then there is a $\lambda \in F_C$ such that $\lambda \in (F_1)_C$ and $\lambda \notin (F_2)_C$. Pick any $\lambda' \in F_{C'}$. By assumption, there is a context $C'' \in \mathcal{C}$ such that $\lambda'' \in F_{C''}$ and $C, C' \subseteq C''$. Evidently, $\lambda = \lambda''|_C$ and $\lambda' = \lambda''|_{C'}$. As $\lambda \notin (F_2)_C$ and F_2 is closed, we find $\lambda'' \notin (F_2)_{C''}$. As $F = F_1 \cup F_2$, one has $\lambda'' \in (F_1)_{C''}$. Using that F_1 is closed, we find that $\lambda' \in (F_1)_{C'}$. Thus $F \subseteq F_1$, proving irreducibility. □

Theorem 2.5.5. *Let $\mathcal{C}$ satisfy the following* ***ascending chain property:*** *every chain of contexts*

$$C_1 \subseteq C_2 \subseteq C_3 \subseteq ...,$$

stabilizes, in the sense that there exists an $n \in \mathbb{N}$ such that for all $m \geq n$ we have $C_{m+1} = C_m$. Then the space $\Sigma_A^{\uparrow}$ is sober. In particular, if A is finite-dimensional, then $\Sigma_A^{\uparrow}$ is sober.

Proof. Take any totally ordered subset of $\mathcal{Q}_B = \{C \in \mathcal{C} \mid F_C \neq \emptyset\}$, where the order is given by inclusion. Then the ascending chain condition ensures that there is an upper bound. An application of Zorn's Lemma tells us that $\mathcal{Q}_B$ has a maximal element. By Lemma 2.5.4(2), the set $\mathcal{Q}_B$ is upward directed so this maximal element must be unique. If C is this maximal element and $F_C = \{\lambda\}$, then we recognize F as the closure of (C, λ). For C*-algebras where the ascending chain condition applies, such as n-level systems, and assuming the axiom of choice, the points of the locale $\Sigma_{\uparrow}$ correspond to the points of the topological space $\Sigma_{\uparrow}$. $\square$

Next, we consider the points $\Sigma_{\downarrow}$.

Lemma 2.5.6. *Let F be an irreducible closed subset of $\Sigma_{\downarrow}$. Suppose there is a context $C \in \mathcal{C}$ such that for all $D \subset C$ we have $F_D = \emptyset$, while $F_C \neq \emptyset$. Then there is a unique $\lambda \in \Sigma_C$ such that F is the closure of (C, λ).*

Proof. By definition of $\mathcal{O}\Sigma_{\downarrow}$, a set F is closed iff the following two conditions are satisfied:

1. For every $C \in \mathcal{C}$ the set F_C is closed in Σ_C,
2. If $\lambda \in F_C$, $C \subseteq C'$ and $\lambda' \in \rho_{C'C}^{-1}(\lambda)$ then $\lambda' \in F_{C'}$.

Define F_1 as follows: for each $C' \neq C$ we take $(F_1)_{C'} = F_{C'}$, and at the context C we take $(F_1)_C = \emptyset$. It is easily checked that $F_1 \subset F$ and that F_1 is closed. Define F_2 as follows: if $C' \supseteq C$, then $(F_2)_{C'} = \rho_{C'C}^{-1}(F_C)$. For all other $C' \in \mathcal{C}$, define $(F_2)_{C'} = \emptyset$. The set F_2 is closed and $F = F_1 \cup F_2$. By irreducibility of F it follows that $F = F_2$.
Suppose that F_C has more than one element. In that case F_C is reducible in Σ_C and we find two proper closed subsets $F_{1C}, F_{2C} \subset F$ such that $F_{1C} \cup F_{2C} = F_C$. Define the sets F_i', for $i = 1, 2$, as follows. If $C' \supseteq C$, then $(F_i')_{C'} = \rho_{C'C}^{-1}(F_{iC})$. For all other $C' \in \mathcal{C}$ take $(F_i')_{C'} = \emptyset$. Again, $F_i' \subset F$, the F_i' are closed, and $F = F_1' \cup F_2'$. As F is irreducible, F_C must be a singleton. If $F_C = \{\lambda\}$, then F is clearly the closure of (C, λ). $\square$

Proposition 2.5.7. *Let $\mathcal{C}$ satify the following* ***descending chain property:*** *every chain of contexts*

$$... \subseteq C_3 \subseteq C_2 \subseteq C_1,$$

stabilizes in the sense that there exists an $n \in \mathbb{N}$ such that for all $m \geq n$ we have $C_{m+1} = C_m$. Then the space $\Sigma_A^{\downarrow}$ is sober. In particular, if A is finite-dimensional, then $\Sigma_A^{\downarrow}$ is sober.

Proof. Take any totally ordered subset of $\mathcal{Q}_I = \{C \in \mathcal{C} | F_C \neq \emptyset\}$, where the order is now given by reversed inclusion. Then the descending chain condition ensures that there is an upper bound. An application of Zorn's Lemma tells us that $\mathcal{Q}_I$ has a maximal element, which is a minimal context C such that $F_C \neq \emptyset$. It follows from Lemma 2.5.6 that this minimal context must be unique. For C*-algebras where the descending chain condition applies, such as n-level systems, and assuming the axiom of choice, the points of the locale $\Sigma_{\downarrow}$ correspond to the points of the topological space $\Sigma_{\downarrow}$. □

If $\epsilon \in \{\uparrow, \downarrow\}$, by the previous discussion sobriety of Σ_ϵ depends strongly on sobriety of $\mathcal{C}_\epsilon$. Recall that in Section 2.4, when we wanted to work with sober spaces, the space $\mathcal{C}_{\downarrow}$ was replaced by its sobrification $\mathcal{F}$, the set of filters in $\mathcal{C}$, equipped with the Scott topology. In the same way, the sobrification of $\mathcal{C}_{\uparrow}$ can be identified with $\mathcal{I}_A$, the set of ideals of $\mathcal{C}$, equipped with the Scott topology. A subset $I \subseteq \mathcal{C}$ is an ideal iff it is non-empty, downwards closed, and upward directed. A subset $W \subseteq \mathcal{I}$ is Scott open iff it is upwards closed with respect to the inclusion relation, and if for any directed family of ideals $(I_i)_{i \in J}$, satisfying $\bigcup_{i \in J} I_i \in W$, implies that there exists an $i_0 \in J$, such that $I_{i_0} \in W$. The elements of $\mathcal{C}$ are identified with the principal ideals $(\downarrow C)$ of $\mathcal{I}_A$.
The Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ is a locale in $Sh(\mathcal{I}_A)$, which is equivalent to a locale map $Y \to \mathcal{I}_A$, where we see the sober space $\mathcal{I}_A$ as a locale. By the following lemma, taken from [67], the locale Y is spatial.

Lemma 2.5.8. *Let $f : A \to B$ be a map of locales which corresponds to a locally compact internal locale in* $\mathrm{Sh}(B)$*. If B is spatial, then so is A.*

The internal locale $\underline{\Sigma}_{\underline{A}}$, being a Gelfand spectrum, is compact completely regular and hence stably locally compact [56, Cor VII.3.5]. The locale $\mathcal{I}_A$ is defined by a topology so it is trivially spatial. Spatiality of Y follows immediately from the lemma. Externally, the spectrum $\underline{\Sigma}_{\underline{A}}$ corresponds to a unique bundle of sober spaces $Y \to \mathcal{I}_A$. We already know how to find this bundle. Start with the bundle of spaces $\pi : \Sigma_{\uparrow} \to \mathcal{C}_{\uparrow}$. If $\Sigma_{\mathcal{I}}$ denotes the sobrification of $\Sigma_{\uparrow}$, the bundle π defines a maps of sober spaces $\pi_{\mathcal{I}} : \Sigma_{\mathcal{I}} \to \mathcal{I}_A$. By construction of $\pi_{\mathcal{I}}$, the internal frame associated

to this bundle is the frame associated to the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$, so we recognize $\pi_{\mathcal{I}}$ as the external description of $\underline{\Sigma}_{\underline{A}}$.
We close this section by describing the space $\Sigma_{\mathcal{I}} := pt(\Sigma_{\uparrow})$.

By Lemma 2.5.4 a point of $L(\Sigma_{\uparrow})$ (the locale associated to the space $\Sigma_{\uparrow}$) can be described as a pair $\sigma = (I, \overrightarrow{\lambda})$, where $I \in \mathcal{I}_A$, and $\overrightarrow{\lambda} = (\lambda_C)_{C\in I}$ gives, for each $C \in I$ an element $\lambda_C \in \Sigma_C$ with the property that if $C_1 \subseteq C_2$ are both in I then $\lambda_{C_2}|_{C_1} = \lambda_{C_1}$. These points were first described in [72], where they are called consistent ideals of measurement outcomes. Expressed in terms of the contravariant approach, a **point** labelled by I is simply a **section** of $\underline{\Sigma}|_I$, where $\underline{\Sigma}$ is the spectral presheaf. If the ideal I is a principal ideal $(\downarrow C)$, then such a point corresponds to an element $\lambda \in \Sigma_C$. Without the assumption that $I \in \mathcal{I}_A$ is principal, the set I is still directed, and therefore elements of the $C \in I$ mutually commute. The ideal has a least upper bound $C_I = \bigvee_{C\in I} C$.

Proposition 2.5.9. *Let $\pi_{\mathcal{I}} : \Sigma_{\mathcal{I}} \to \mathcal{I}_A$ be the external description of the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ by sober spaces as described above. The fibre of $\pi_{\mathcal{I}}$ over the ideal $I \in \mathcal{I}_A$ is Σ_{C_I}. In particular, points of the locale $L(\Sigma_{\uparrow})$ correspond bijectively to elements of the set $\coprod_{I\in\mathcal{I}(A)} \Sigma_{C_I}$.*

Proof. If $D \in I$, then D is a subalgebra of C_I, and we can relate $\Sigma_I := \Sigma_{C_I}$ to Σ_D as follows. Define the equivalence relation on Σ_I by $\lambda_1 \sim_D \lambda_2$ iff for every $a \in D$, $\langle\lambda_1, a\rangle = \langle\lambda_2, a\rangle$. For each $D \in I$ this equivalence relation gives a partition of Σ_I in closed subsets. For $D \in I$, the point σ provides an element $\lambda_D \in \Sigma_D$. Such a λ_D corresponds to a unique equivalence class z_D with respect to the relation $\sim_D$. If $C_1 \subseteq C_2$ in I, then $\lambda_{C_2}|_{C_1} = \lambda_{C_1}$, and so $z_{C_2} \subseteq z_{C_1}$. Next, consider the intersection $Z := \bigcap_{D\in I} z_D$. We will show that this set is a singleton. Note that if it contains an element, then that element must be unique. For any two $\lambda_1, \lambda_2 \in \Sigma_I$, there is an $a \in C_{sa}$ that separates them. As the commutative C*-subalgebras $D \in I$ form a generating set of C_I, there must be some $D_0 \in I$ that separates λ_1 and λ_2. For this commutative C*-subalgebra $\lambda_1 \notin z_{D_0}$ or $\lambda_2 \notin z_{D_0}$. We conclude that Z has at most one element. Now assume that $Z = \emptyset$. Then by taking complements, the $u_D := z_D^c$ give an open cover of the compact space Σ_I, which has a finite subcover. There exist $D_1, ..., D_n \in I$ such that $Z_n = \bigcap_{i=1}^{i=n} z_i = \emptyset$. As I is an ideal, we know that $D = \bigvee_{i=1}^{i=n} D_i \in I$. The point σ provides an element $\lambda_D \in \Sigma_D$, such that the nonempty set $z_D \subseteq Z_n$, contradicting Z being empty. We conclude that Z is a singleton $\{\lambda\}$. For every $D \in I$ we have $\lambda_D = \lambda|_D$. □

2.6 A result on exponentiability

We demonstrate a result on exponentiability, which implies that for a locally compact locale X, and a locally compact locale $\underline{Y}$ in $Sh(X)$, with external description $Y \to X$, the locale Y is locallly compact.

Internally, the spectrum $\underline{\Sigma}_{\underline{A}}$ of $\underline{A}$ in $[\mathcal{C}, \mathbf{Set}]$ is a compact and completely regular locale. Externally, the locale is described by the bundle $\pi : \Sigma_{\uparrow} \to \mathcal{C}_{\uparrow}$ of Theorem 2.2.2. A simple check reveals that the space $\Sigma_{\uparrow}$ of the bundle is locally compact, hence the frame is a continuous lattice ([58, Lemma VII.4.2]) and $\Sigma_{\uparrow}$, seen as a locale, is locally compact as well. Recall:

Definition 2.6.1. *Let L be a locale with associated frame $\mathcal{O}L$. Define the* way below relation $\ll$ *on $\mathcal{O}L$ as: for $a, b \in \mathcal{O}L$, $b \ll a$ when for each ideal I in $\mathcal{O}L$, if $\bigvee I \geq a$, then $b \in I$. The frame $\mathcal{O}L$ is called a* continuous lattice *iff*

$$\forall a \in \mathcal{O}L, \quad a = \bigvee \{ b \in \mathcal{O}L \mid b \ll a \}.$$

A locale L is locally compact *if $\mathcal{O}L$ is a continuous lattice.*

Lemma 2.6.2. *The space $\Sigma_{\uparrow}$ from Theorem 2.2.2 is locally compact.*

Proof. To be precise, we call $\Sigma_{\uparrow}$ locally compact if for each point $(C, \lambda) \in \Sigma$ and each open neighborhood U of (C, λ), there is a compact neighborhood K of (C, λ), contained in U.
Let V be an open neighborhood of (C, λ). Consider the following basic open neighborhood U of the same point, which is contained in V. If $C' \in (\uparrow C)$, then $U_{C'} = \rho^{-1}_{C'C}(V_C)$. For every other $C' \in \mathcal{C}$ define $U_{C'} = \emptyset$. By definition of the topology on Σ, the set U_C is an open neighborhood of λ in Σ_C. The space Σ_C is compact Hausdorff, and hence locally compact. There exists a compact neighborhood K_C of λ such that $K_C \subseteq U_C$.
Define $K \subseteq \Sigma$ as follows. If $C' \in (\uparrow C)$ then $K_{C'} = \rho^{-1}_{C'C}(K_C)$ and $C' \notin (\uparrow C)$, then $K_{C'} = \emptyset$. Clearly $K \subseteq U$ and $(C, \lambda) \in K$. It remains to show that K is compact in Σ. Let $\{U_i\}_{i \in I}$ be an open cover of K. Then $\{(U_i)_C\}_{i \in I}$ gives an open cover of K_C. As K_C is compact, there is a finite subcover $\{(U_j)_C\}_{j=1}^n$. The opens $\{U_j\}_{j=1}^n$ cover K. To see this, note that

$$\bigcup_j (U_j)_{C'} \supseteq \bigcup_j \rho^{-1}_{C'C}((U_j)_C) = \rho^{-1}_{C'C}\left(\bigcup_j (U_j)_C \right) = \rho^{-1}_{C'C}(K_C) = K_{C'}.$$

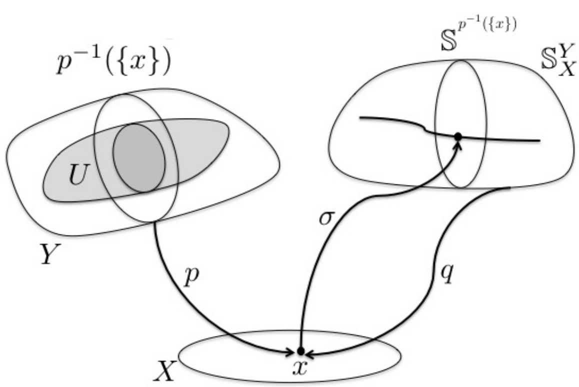
As the U_i cover K, we conclude that $K_{C'} = \bigcup_j (U_j)_{C'}$ for every $C' \in (\uparrow C)$. $\square$

The following theorem [52] holds for locales in any topos.

Theorem 2.6.3. *Let X be a locale, then the following are equivalent.*

1. *X is locally compact.*
2. *The functor $(-) \times X : \mathbf{Loc} \to \mathbf{Loc}$ has a right adjoint $(-)^X$.*
3. *The exponential $\mathbb{S}^X$ exists, where $\mathbb{S}$ denotes the Sierpiński locale.*

This section presents a general result on exponentiability, given by Theorem 2.6.4. Combined with Theorem 2.6.3, this theorem implies that the external description $\Sigma_{\underline{A}}$ of the spectrum is locally compact. More generally, let X be a locally compact locale, and $\underline{Y}$ a locally compact locale in $\mathrm{Sh}(X)$, with external description $p : Y \to X$, then by the theorem Y is locally compact. In this setting, in order to prove local compactness of Y it suffices to show that the exponential $\mathbb{S}^Y$ exists. Suppose for a moment that it does. A point of $\mathbb{S}^Y$ is equivalent to a map $Y \to \mathbb{S}$, which corresponds to an open $U \in \mathcal{O}Y$. By assumption $\underline{Y}$ is locally compact in $\mathrm{Sh}(X)$, so the exponential $\underline{\mathbb{S}}^{\underline{Y}}$ exists in $\mathrm{Sh}(X)$, where $\underline{\mathbb{S}}$ denotes the internal Sierpiński locale. Using the fact that $\mathbf{Loc}_{\mathrm{Sh}(X)}$ is equivalent to $\mathbf{Loc}/X$, the locale $\underline{\mathbb{S}}^{\underline{Y}}$ has an external description by a locale map $q : \mathbb{S}^Y_X \to X$, for some locale $\mathbb{S}^Y_X$. The external description of $\underline{\mathbb{S}}$ is the projection $\pi_2 : \mathbb{S} \times X \to X$. Exponentiation of locales can be described in terms of geometric logic as shown in [76, Sec 10], implying that the fibre $q^*(\{x\})$ over a point x in X is given by $\mathbb{S}^{Y_x}$, where we write $Y_x := p^*(\{x\})$.



An open $U \in \mathcal{O}Y$ and a point $x \in X$ give an open U_x in the fibre Y_x, as in the figure below. This in turn is equivalent to a map $Y_x \to \mathbb{S}$, which is an element of $q^*(\{x\})$. This suggests that the global points of $\mathbb{S}^Y$ correspond exactly to the global sections of the bundle $q : \mathbb{S}^Y_X \to X$. That is, global points of $\mathbb{S}^Y$ correspond to maps $\sigma : X \to \mathbb{S}^Y_X$ such that $q \circ \sigma = id_X$.

Theorem 2.6.4. *Let $\mathcal{C}$ be a category with finite limits, and let X be an exponentiable object in $\mathcal{C}$. Let $p : Y \to X$ be an object $\underline{Y}$ of $\mathcal{C}/X$, let Z be an object of $\mathcal{C}$, and suppose the exponential $\underline{Z}^{\underline{Y}}$ exists in $\mathcal{C}/X$, written as Z^Y_X. Then Z^Y exists in $\mathcal{C}$.*

Proof. By the considerations above, we arrive at the following candidate for the exponential Z^Y. Take the equalizer

$$\begin{tikzcd} E \arrow[r, hook, "eq"] & (Z^Y_X)^X \arrow[r, shift left, "q^X"] \arrow[r, shift right, "\ulcorner X \urcorner \circ !"'] & X^X , \end{tikzcd}$$

where $\ulcorner X \urcorner : 1 \to X^X$ denotes the transpose of the identity arrow of X. Note that the exponentials $(Z^Y_X)^X$ and X^X exist in $\mathcal{C}$ by exponentiability of X. Also note that the global points of E are exactly the global sections of q. Next we need to find a suitable evaluation map $ev : E \times Y \to Z$. For the definition of ev we will make use of the internal evaluation arrow $\underline{Z}^{\underline{Y}} \times \underline{Y} \to \underline{Z}$. Externally this gives the following commuting triangle:

$$\begin{tikzcd} Z^Y_X \times_X Y \arrow[rr, "ev"] \arrow[dr] & & Z \times X \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

With some abuse of notation we will denote the map $\pi_1 \circ ev : Z^Y_X \times_X Y \to Z$ again by ev. For the next step in defining the evaluation map, the diagram given below is commutative by definition of E.

The evaluation maps and exponentials in this diagram exist because of exponentiability of X. By the universal property of pullbacks this diagrams yields an arrow $E \times Y \to Z^Y_X \times_X Y$. Taking the composition with $ev : Z^Y_X \times_X Y \to Z$ coming from the internal exponential $\underline{Z}^{\underline{Y}}$ then gives the desired evaluation map $ev : E \times Y \to Z$. It remains to check that this map satisfies the desired universal property.

$$\begin{tikzcd}
E \times Y \arrow[r, "E\times p"] \arrow[ddd, "\pi_2"'] & E \times X \arrow[rr, "eq\times X"] \arrow[ddddrrrr, phantom] \arrow[dddrrrr, bend right, "\pi_2"'] & & (Z^Y_X)^X \times X \arrow[rr, "ev"] \arrow[dl, "!\times X"'] \arrow[ddr, "q^X\times X"] & & Z^Y_X \arrow[ddd, "q"] \\
& & 1 \times X \arrow[ddrr, phantom] \arrow[drr, "\ulcorner X \urcorner\times X"] & & & \\
& & & & X^X \times X \arrow[dr, "ev"] & \\
Y \arrow[rrrrr, "p"'] & & & & & X
\end{tikzcd}$$

Maps $Z \times Y \to Z$ correspond bijectively with maps $(Z \times X) \times_X Y \to Z \times X$ over X. By the existence of the internal exponent $\underline{Z}^{\underline{Y}}$, the latter maps correspond bijectively with maps $Z \times X \to Z^Y_X$ over X. Using exponentiability of X, maps $Z \times X \to Z^Y_X$ correspond bijectively with maps $Z \to (Z^Y_X)^X$. The maps $Z \times X \to Z^Y_X$ that are maps over X precisely correspond to the maps $Z \to (Z^Y_X)^X$ that factor through E. This proves that E is indeed an exponential Z^Y. $\square$

Corollary 2.6.5. *Let X be a locally compact locale, and $\underline{Y}$ a locally compact locale in* $\mathrm{Sh}(X)$*, with external description $p : Y \to X$, then Y is a locally compact locale.*

Proof. By Theorem 2.6.3, X is exponentiable in **Loc**, and the exponential $\underline{\mathbb{S}}^{\underline{Y}}$ exists in **Loc**$/X$. Theorem 2.6.4 implies that the exponential $\mathbb{S}^Y$ exists. By Theorem 2.6.3, Y is a locally compact locale. $\square$

2.7 Algebraic Quantum Field Theory

We consider an extension of the covariant approach to algebraic quantum field theory and compute the points of the state space in this setting.

The covariant approach is based on algebraic quantum theory in the sense that in this approach quantum theory is described using abstract C*-algebras. In this section we seek to extend the covariant approach, as already suggested in [48], to the Haag–Kastler formalism, which is an algebraic approach to quantum field theory. Introductions to the Haag–Kastler formalism, or algebraic quantum field theory (AQFT), can be found in [6, 42]. In this formalism (where, for the sake of simplicity we consider Minkowski spacetime $\mathcal{M}$) the physical content of a quantum field theory is described by a net of C*-algebras $O \to \mathfrak{A}(O)$, where O ranges over certain (open connected causally complete) regions of spacetime. This means that we associate to each region O of spacetime of interest, a C*-algebra $\mathfrak{A}(O)$. We think of the self-adjoint elements of $\mathfrak{A}(O)$ as the observables that can be measured in the region O. With this in mind, we can make the assumption that if $O_1 \subseteq O_2$, then $\mathfrak{A}(O_1) \subseteq \mathfrak{A}(O_2)$. If $\mathcal{K}(\mathcal{M})$ denotes the set of the spacetime regions of interest, partially ordered by inclusion, then a net of C*-algebras defines a covariant functor $\mathfrak{A} : \mathcal{K}(\mathcal{M}) \to \mathbf{CStar}$. We assume that the algebras $\mathfrak{A}(O)$ are unital for convenience.

By Proposition 2.1.2 an AQFT is a C*-algebra $\mathfrak{A}$ internal to $[\mathcal{K}(\mathcal{M}), \mathbf{Set}]$. Note that $\mathfrak{A}$ is in general not commutative. As for the copresheaf approach we can Bohrify the C*-algebra $\mathfrak{A}$. This means that we make it commutative by considering it as a copresheaf over the poset of commutative subalgebras. The difference with the copresheaf approach is that the Bohrification takes place internal in the topos $[\mathcal{K}(\mathcal{M}), \mathbf{Set}]$, instead of the topos **Set**. We obtain a commutative C*-algebra internal to a topos (which in turn is internal to a functor category), and, using the same reasoning as in Section 2.2, we describe the points of the Gelfand spectrum of this commutative C*-algebra.

Instead of the Haag–Kastler formalism, we could have considered the more general and more recent locally covariant quantum field theories [15]. This amounts to replacing the poset $\mathcal{K}(\mathcal{M})$ by a more complicated category of manifolds and embeddings (which is no longer a poset). Although Bohrification of the locally covariant field theories can be described using the same techniques, we stick with the Haag–Kastler formalism, as this makes the presentation easier.

Note that an internal unital commutative C*-subalgebra of $\mathfrak{A}$ is simply a subobject $\mathfrak{C}$ of $\mathfrak{A}$ such that for each $O \in \mathcal{K}(\mathcal{M})$, $\mathfrak{C}(O)$ is a commutative unital C*-algebra in **Set**. These internally defined commutative C*-subalgebras form a poset $\mathcal{C}(\mathfrak{A})$ in $[\mathcal{K}(\mathcal{M}), \mathbf{Set}]$ and we can consider the (internal) functor category over this poset. Using the techniques of Subsection 2.2.2, in particular (2.22), we can describe this functor category within a functor category using a single topos over **Set**, namely by constructing the site $\mathcal{K}(\mathcal{M}) \ltimes \mathcal{C}(\mathfrak{A})$. In this (composite) topos, the Bohrified net is given by the functor $(O, \mathfrak{C}) \mapsto \mathfrak{C}(O)$, where $O \in \mathcal{K}(\mathcal{M})$ and $\mathfrak{C}$ is a commutative unital C*-subalgebra of $\mathfrak{A}|_{\uparrow O}$. Using Theorem 2.2.7

we could give an explicit description of the spectrum of the Bohrified net $(O,\mathfrak{C}) \mapsto \mathfrak{C}(O)$. However, we first simplify the topos in which we are working. Instead of labeling the objects of the base category by subalgebras $\mathfrak{C}$ of $\mathfrak{A}|_{\uparrow O}$, we only concentrate on the part $\mathfrak{C}(O)$. This motivates using the topos $[\mathcal{P}, \mathbf{Set}]$, where the poset $\mathcal{P}$ is defined as follows: an element $(O,C) \in \mathcal{P}$, consists of an $O \in \mathcal{V}(M)$ and a $C \in \mathcal{C}_O := \mathcal{C}(\mathfrak{A}(O))$, and the order relation is given by

$$(O_1, C_1) \leq (O_2, C_2) \;\text{ iff }\; O_1 \subseteq O_2, \;\; C_1 \subseteq C_2.$$

We are interested in the unital commutative C*-algebra $\underline{A} : \mathcal{P} \to \mathbf{Set}$, $(O,C) \mapsto C$ in the topos $[\mathcal{P}, \mathbf{Set}]$. Note that $\underline{A}((O_1,C_1) \leq (O_2,C_2))$ is the inclusion map $C_1 \hookrightarrow C_2$. By Theorem 2.2.7 we can describe the spectrum $\underline{\Sigma}$ of $\underline{A}$ by a space Σ. The space Σ is given by $\Sigma = \coprod_{(O,C)\in\mathcal{P}} \Sigma_C$, where $U \in \mathcal{O}\Sigma$ iff

- For each $O \in \mathcal{K}(\mathcal{M})$ and $C \in \mathcal{C}_O$, $U_{(O,C)} \in \mathcal{O}\Sigma_C$.
- If $O_1 \subseteq O_2$ in $\mathcal{K}(\mathcal{M})$, and if $C_1 \in \mathcal{C}_{O_1}$ and $C_2 \in \mathcal{C}_{O_2}$ satisfy $C_1 \subseteq C_2$, then
$$\rho^{-1}_{C_2,C_1}(U_{(O_1,C_1)}) \subseteq U_{(O_2,C_2)},$$
where ρ_{C_2,C_1} denotes the restriction map corresponding by Gelfand duality to the inclusion $C_1 \hookrightarrow C_2$.

Note that this space Σ, where the coproduct is indexed by the elements $p \in \mathcal{P}$, is smaller than the space of Definition 2.2.6, where the coproduct is indexed by all inequalities $p \leq q$ in $\mathcal{P}$. However, both spaces lead to the same internal frame $p \to \mathcal{O}\underline{\Sigma}(p)$, so we use the simpler description.

Next, we want to compute the points of this external locale Σ. In order to accomplish this, we will use the same reasoning as in [72]. As a category, the topos $[\mathcal{P}, \mathbf{Set}]$ is equivalent to the topos $\mathrm{Sh}(\mathcal{P}_{\uparrow})$. As in Subsection 2.2.2 we can find a site $\mathcal{P} \ltimes \underline{\Sigma}$ such that $\mathrm{Sh}_{\mathrm{Sh}(\mathcal{P})}(\underline{\Sigma}) \cong \mathrm{Sh}(\mathcal{P} \ltimes \underline{\Sigma})$. The locale Σ is the locale generated by the posite $\mathcal{P} \ltimes \underline{\Sigma}$. We use the posite description $\mathcal{P} \ltimes \underline{\Sigma}$ in order to find the points. We now use the functor $\underline{L}$, which is the distributive lattice object in $[\mathcal{P}, \mathbf{Set}]$, given by

$$\underline{L} : \mathcal{P} \to \mathbf{Set}, \;\; \underline{L}(O,C) = L_C,$$

$$\underline{L}((O_1,C_1) \leq (O_2,C_2)) : L_{C_1} \to L_{C_2}, \; [a]_{C_1} \mapsto [a]_{C_2}.$$

The elements of $\mathcal{P} \ltimes \underline{\Sigma}$ are triples $(O, C, [a]_C)$, where $O \in \mathcal{V}(M)$, $C \in \mathcal{C}_O$ and $[a]_C \in L_C$. The order of this poset is given by

$$(O_1, C_1, [a_1]_{C_1}) \leq (O_2, C_2, [a_2]_{C_2}), \text{ iff } O_2 \subseteq O_1, \; C_2 \subseteq C_1, \; [a_1]_{C_1} \leq [a_2]_{C_1}.$$

The poset $\mathcal{P} \ltimes \underline{\Sigma}$ is equipped with the following covering relation $\lhd$, which is inherited from the covering relation $\unlhd$, exploiting the fact that we are working over $\mathcal{P}$. We have a covering $(O, C, [a]_C) \lhd W$ iff for $W_0 = \{[b]_C \in L_C \mid (O, C, [b]_C) \in W\}$, the condition $[a]_C \lhd W_0$ holds in L_C. Note that the covering relation on $\mathcal{P} \ltimes \underline{\Sigma}$ is completely described in terms of covering relations on the L_C.
A point σ of the external spectrum Σ corresponds to a completely prime filter of $\mathcal{P} \ltimes \underline{\Sigma}$. Recall that a filter σ is a nonempty, upward closed and lower directed subset of $\mathcal{P} \ltimes \underline{\Sigma}$, and that σ is completely prime if it satisfies

$$(O, C, [a]_C) \in \sigma \;\text{ and }\; (O, C, [a]_C) \lhd W, \;\text{ implies }\; U \cap \sigma \neq \emptyset.$$

Let σ be a point of Σ. It is straightforward to show that

$$\mathcal{R} = \{O \in \mathcal{V}(M) \mid \exists C \in \mathcal{C}_O, \exists [a]_C \in L_C, \text{ s.t. } (O, C, [a]_C) \in \sigma\}$$

is an ideal of $\mathcal{V}(M)$. Fix any $O \in \mathcal{R}$ and consider the set

$$\mathcal{I}_O = \{C \in \mathcal{C}_O \mid \exists [a]_C \in L_C \text{ s.t. } (O, C, [a]_C) \in \sigma\}.$$

For any $O \in \mathcal{R}$, $\mathcal{I}_O$ is an ideal of $\mathcal{C}_O$. For a pair $O \in \mathcal{R}$ and $C \in \mathcal{I}_O$, define

$$\sigma_{O,C} := \{[a]_C \in L_C \mid (O, C, [a]_C) \in \sigma\}.$$

As in [72], it can be shown that $\sigma_{O,C}$ is a completely prime filter of L_C. A completely prime filter $\sigma_{O,C}$ on L_C corresponds to a unique point $\lambda(O, C)$ of the Gelfand spectrum Σ_C.
Next, we show how for different $O \in \mathcal{R}$, $C \in \mathcal{I}_O$, the $\lambda(O, C) \in \Sigma_C$ are related. Let, for some fixed $O \in \mathcal{R}$, $D \subset C$ in $\mathcal{C}_O$. Let $a \in D^+$ and assume that $[a]_C \in \sigma_{O,C}$. By the order on $\mathcal{P} \ltimes \underline{\Sigma}$,

$$(O, C, [a]_C) \leq (O, D, [a]_D) \in \sigma,$$

where we used that σ is a filter, and therefore it is upward closed. For any $a \in D^+$, if $[a]_C \in \sigma_{O,C}$, then $[a]_D \in \sigma_{O,D}$. The filter $\sigma_{O,D}$ can be viewed as a frame map $\sigma_{O,D} : \mathcal{O}\Sigma_C \to \underline{2}$ mapping the open $X_a^D = \{\lambda \in \Sigma_D \mid \langle \lambda, a \rangle > 0\}$ to 1 iff $\lambda(O, D) \in X_a^D$, iff $[a]_D \in \sigma_{O,D}$. If $\rho_{CD} : \Sigma_C \to \Sigma_D$

is the restriction map, then $\sigma_{O,C} \circ \rho_{CD}^{-1} : \mathcal{O}\Sigma_D \to 2$ corresponds to the point $\lambda(O,C)|_D$. At the level of points of Σ_D, the implication

$$\forall a \in D^+, \quad [a]_C \in \sigma_{O,C} \Rightarrow [a]_D \in \sigma_{O,D}$$

translates to:

$$\forall a \in D^+, \quad \big(\sigma_{O,C}(X_a^C) = 1\big) \Rightarrow \big(\sigma_{O,D}(X_a^D) = 1\big)\,.$$

As the X_a^D form a basis of the Hausdorff space Σ_D, and $\rho^{-1}(X_a^D) = X_a^C$, this can only mean that $\sigma_{O,D} = \sigma_{O,C} \circ \rho_{DC}^{-1}$. In other words, whenever $D \subseteq C$, one has $\lambda(O,D) = \lambda(O,C)|_D$.
Assume that $O' \subset O$ in $\mathcal{V}(M)$ and that $C \in \mathcal{C}_{O'}$. In $\mathcal{P} \ltimes \underline{\Sigma}$,

$$\forall [a]_C \in L_C, \quad (O, C, [a]_C) \leq (O', C, [a]_C).$$

If $[a]_C \in \sigma_{O,C}$, then by the filter property of σ, $[a]_C \in \sigma_{O',C}$. We conclude that if $O' \subseteq O$ in $\mathcal{R}$ and $C \in \mathcal{C}_{O'}$, then $\lambda(O',C) = \lambda(O,C)$. Hence:

Theorem 2.7.1. *A point σ of Σ is described by a triple $(\mathcal{R}, \mathcal{I}_\mathcal{R}, \lambda_{\mathcal{R},\mathcal{I}})$, where:*

- *$\mathcal{R}$ is an ideal in $\mathcal{V}(M)$.*
- *The function $\mathcal{I}_\mathcal{R}$ associates to each $O \in \mathcal{R}$, an ideal $\mathcal{I}_O$ of $\mathcal{C}_O$ satisfying two conditions. Firstly, if $O_1 \subseteq O_2$, then $\mathcal{I}_{O_2} \cap \mathcal{C}_{O_1} \subseteq \mathcal{I}_{O_1}$. Secondly, if $C_i \in \mathcal{I}_{O_i}$, where $i \in \{1,2\}$, then there is an $O \in \mathcal{R}$ and a $C \in \mathcal{I}_O$ such that $O_i \subseteq O$ and $C_i \subseteq C$.*
- *The function $\lambda_{\mathcal{R},\mathcal{I}}$ associates to each $O \in \mathcal{R}$ and $C \in \mathcal{I}_O$, an element $\lambda_{O,C} \in \Sigma_C$, such that if $O_1 \subseteq O_2$ and $C_1 \subseteq C_2$, then $\lambda_{O_1,C_1} = \lambda_{O_2,C_2}|_{C_1}$.*

The two conditions in the second bullet point are included to ensure that the set

$$\mathcal{I} = \{(O,C) \in \mathcal{P} \mid O \in \mathcal{R}, C \in \mathcal{I}_O\}$$

is an ideal of $\mathcal{P}$. Mathematically, the theorem would look more elegant if it were formulated in terms of ideals of $\mathcal{P}$ instead of using pairs $(\mathcal{R}, \mathcal{I}_\mathcal{R})$, but that description would miss an important physical point. Namely, a spacetime point $x \in M$ corresponds to a specific filter of $\mathcal{V}(M)$, consisting of all $O \in \mathcal{V}(M)$ containing x. However, a point σ of Σ is labelled by an ideal $\mathcal{R}$ of $\mathcal{V}(M)$ and not by a filter. With this observation in mind, it might be interesting to look at the contravariant functor $\underline{\Sigma} : \mathcal{P}^{op} \to \mathbf{Set}$, $(O,C) \mapsto \Sigma_C$. This functor is also closer to the work of Nuiten [68], as we shall see in Subsection 6.3.2.

3

Daseinisation

In the contravariant approach propositions regarding the system of interest, expressing the idea that a physical quantity takes certain values, are represented as clopen subobjects of the spectral presheaf $\underline{\Sigma}_A$. In the covariant approach they correspond to opens of the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$. Similarly, in the contravariant approach, physical quantities, associated to self-adjoint elements $a \in A_{sa}$, are represented as arrows $\underline{\Sigma}_A \to \underline{\mathbb{R}}^{\leftrightarrow}$, whereas in the covariant approach they correspond to locale maps $\underline{\Sigma}_{\underline{A}} \to \underline{\mathbb{IR}}$. These constructions rely on daseinisation techniques, which approximate operators in A using elements from a fixed context $C \in \mathcal{C}_A$. In this chapter we investigate all such constructions.

The first section reviews the basics of daseinisation of self-adjoint operators, as used in the contravariant approach. Section 3.2 applies these constructions to the covariant approach. The two sections that follow look at some of the consequences of this covariant daseinisation arrow. Section 3.5 discusses various kinds of real numbers in the functor categories used in the topos approaches. This material is put to use in Section 3.6 where, in both approaches, daseinised self-adjoint operators define continuous maps from the state space $\underline{\Sigma}$ to a space of values, defined using internal real numbers. In particular, emphasis will be put on relating daseinisation of self-adjoint operators to daseinisation of the spectral projections of these operators.

3.1 Daseinisation of Self-Adjoint Operators

Inner and outer daseinisation of both projections and more generally self-adjoint operators, as well as the definition of elementary propositions in the contravariant approach are reviewed.

We start with elementary propositions and daseinisation of selfadjoint operators in the contravariant approach. The reader who is already familiar with daseinisation can skip this subsection, as it contains no new material. An extensive discussion of daseinisation in the contravariant approach can be found in the paper [29] by Döring. First we deal with outer daseinisation of projection operators, as we need these to define elementary propositions. In order to motivate outer daseinisation, let $a \in A_{sa}$ and $\Delta \in \mathcal{O}\mathbb{R}$. In quantum logic à la von Neumann, the elementary proposition "$a \in \Delta$" is represented by a projection operator $p = \chi_\Delta(a)$, where χ_Δ is defined by Borel functional calculus (or, equivalently, by the Spectral Theorem). In the contravariant approach a proposition is a clopen subobject of the spectral presheaf $\underline{S} \rightarrowtail \underline{\Sigma}$. Therefore, to each context $C \in \mathcal{C}$ we want to associate a clopen subset $\underline{S}(C)$ of the spectrum Σ_C in such a way that these choices combine to give a presheaf. If $p \in C$, then the natural choice would be

$$\underline{S}(C) = \{\lambda \in \Sigma_C \mid \lambda(p) = 1\},$$

but what about the other contexts? Let $C \in \mathcal{C}$ be any context. Following [34], we approximate the projection operator p using the projection operators available in C as follows:

$$\delta^o(p)_C = \bigwedge \{q \in \mathcal{P}(C) \mid q \geq p\}, \tag{3.1}$$

where $\mathcal{P}(C)$ is the lattice of projections in C. Hence $\delta^o(p)_C$ is the smallest projection operator C that is larger than p. Note that if $p \in C$, then $\delta^o(p)_C = p$. Also note that $\delta^o(p)_C$ must be an element of C, since the projections in a von Neumann algebra form a *complete* lattice [59].[1] Next, define

$$\underline{\delta^o}(p)(C) = \{\lambda \in \Sigma_C \mid \lambda(\delta^o(p)_C) = 1\}. \tag{3.2}$$

[1]If the context C is a commutative unital C*-algebra, then it could very well be that $\delta^o(p)_C \notin C$, but for abelian von Neumann algebras or for the larger class of commutative AW*-algebras the daseinisation operation works.

This is a closed open subset of Σ_C, because the Gelfand transform of $\delta^o(p)_C$ is a continuous function on Σ_C. Noting that for $C \subseteq C'$ we have $\delta^o(p)_C \geq \delta^o(p)_{C'}$, it is easy to check that $\underline{\delta}^o(p)$ defines a clopen subobject of the spectral presheaf. The elementary proposition $\underline{[a \in \Delta]} \rightarrowtail \underline{\Sigma}$ is defined as

$$\underline{[a \in \Delta]}(C) := \underline{\delta}^o(\chi_\Delta(a))(C) = \{\lambda \in \Sigma_C \mid \lambda(\delta^o(\chi_\Delta(a))_C) = 1\}, \quad (3.3)$$

where $\chi_\Delta(a)$ denotes the spectral projection operator associated to "$a \in \Delta$". Note that because for $C \subseteq C'$ we have $\delta^o(p)_C \geq \delta^o(p)_{C'}$, the definition of elementary propositions fits very well with the coarse-graining philosophy.

In addition to the daseinisation of projection operators given by (3.1), which we call **outer daseinisation**, we also consider **inner daseinisation**. Inner daseinisation approximates a projection operator p by taking, in each context, the largest projection operator in C that is smaller than p. In other words:

$$\delta^i(p)_C = \bigvee \{q \in \mathcal{P}(C) \mid q \leq p\}.$$

Note that if $p \in C$, we have $\delta^i(p)_C = p$ and that if $C \subseteq C'$, then $\delta^i(p)_C \leq \delta^i(p)_{C'}$. Inner daseinisation does not yield propositions in the same way as outer daseinisation does, but it remains an important construction. For example, it is needed for the definition of the outer daseinisation of self-adjoint operators, and we shall also use it to define elementary propositions in the covariant approach.

Next, we turn our attention to daseinisation of self-adjoint operators. By the spectral theorem [59], each self-adjoint element $a \in A$ has a spectral resolution $\{e^a_\lambda\}_{\lambda \in \mathbb{R}}$, where $e^a_\lambda = \chi_{(-\infty,\lambda]}(a)$. The daseinisation of a self-adjoint operator a can be defined from the daseinisation of projection operators, applied to the spectral resolution of a.

Thus far, we only made use of the partial order $\leq$ on self-adjoint operators, where $a \leq b$ means that $b - a$ is a positive operator. In what follows, we will in addition use a different partial order on A_{sa}, which was first considered in [69]. Let $a, b \in A_{sa}$, with spectral resolutions $\{e^a_\lambda\}$ and $\{e^b_{\lambda'}\}$. Then a is below b in the spectral order, denoted $a \leq_s b$, if for every $\lambda \in \mathbb{R}$ we have $e^a_\lambda \geq e^b_\lambda$.[2] The spectral order is coarser than the linear

[2]Equivalently, for positive operators a and b, $a \leq_s b$ iff $\forall_{n \in \mathbb{N}}\ a^n \leq b^n$ [69, Theorem 3].

order in the sense that $a \leq_s b$ implies $a \leq b$, while the converse need not hold in general. However, let p be a projection operator in A. Then the spectral resolution of p is given by

$$e^p_\lambda = \begin{cases} 0 & \text{if } \lambda \in (-\infty, 0); \\ 1 - p & \text{if } \lambda \in [0, 1); \\ 1 & \text{if } \lambda \in [1, \infty). \end{cases}$$

From this it follows that if p and q are projections in A, then $p \leq_s q$ iff $p \leq q$. Also, if $a, b \in A_{sa}$ such that $[a, b] = 0$, then similarly $a \leq_s b$ iff $a \leq b$. So in each context the spectral order $\leq_s$ reduces to the usual order $\leq$. The proof of this last claim and more information on the spectral order can be found in [41].

Definition 3.1.1. *Let $a \in A_{sa}$. Define the outer and inner daseinisations of a at context $C \in \mathcal{C}$ by,*

$$\delta^o(a)_C = \bigwedge \{b \in C_{sa} \mid b \geq_s a\},$$

$$\delta^i(a)_C = \bigvee \{b \in C_{sa} \mid b \leq_s a\},$$

respectively.

The self-adjoint operators $\delta^o(a)_C$ and $\delta^i(a)_C$ are elements of C because, if A (and C) is a von Neumann algebra, C_{sa} is a boundedly complete lattice with respect to the spectral order. The daseinisation of self-adjoint operators can be described through the daseinisation of the projections in their spectral resolution. Let $\lambda \mapsto e_\lambda$ be the spectral resolution of a self-adjoint bounded operator a. Then

$$\lambda \mapsto \bigwedge_{\mu > \lambda} \delta^o(e_\mu)_C,$$

$$\lambda \mapsto \delta^i(e_\lambda)_C,$$

are also spectral resolutions of self-adjoint bounded operators [37, 41].

Lemma 3.1.2. *Let $a \in A_{sa}$. Then the spectral resolutions of the outer and inner daseinisations of a at context C are*

$$\delta^o(a)_C = \int \lambda d(\delta^i(e^a_\lambda)_C);$$

$$\delta^i(a)_C = \int \lambda d(\bigwedge_{\mu > \lambda} \delta^o(e^a_\mu)_C).$$

Note that the *outer* daseinisation of a uses the *inner* daseinisation of the spectral resolution $\lambda \mapsto e^a_\lambda$, and vice versa. It also follows from the definition that for any $D, C \in \mathcal{C}$ with $D \subseteq C$ we have

$$\delta^i(a)_D \leq_s \delta^i(a)_C \leq_s a \leq_s \delta^o(a)_C \leq_s \delta^o(a)_D. \tag{3.4}$$

If $a \in C$, then $a = \delta^i(a)_C = \delta^o(a)_C$. Let p be a projection operator. Then the outer daseinisation of p as a self-adjoint operator, as in Definition 3.1.1, coincides with the outer daseinisation of p as a projection, as in (3.1). For inner daseinisation we have a similar situation.

For a projection operator p, (3.4) implies that if we move from a context C to a coarser context D then outer daseinisation approximates p by a larger projection operator in the coarser context D. Hence a coarser context means a weaker proposition, fitting well with the idea of coarse-graining. For inner daseinisation, moving to a coarser context amounts to taking a smaller projection operator. This does not seem to fit with the idea of coarse-graining.

We can consider a different view that does fit with coarse-graining, and involves both inner and outer daseinisation. By Gelfand duality, for any $a \in A_{sa}$, we can see $\delta^i(a)_C$ and $\delta^o(a)_C$ as real-valued continuous functions on the spectrum Σ_C. Given a local state $\lambda \in \Sigma_C$, we cannot assign a sharp value of a to that state (except in the special case $a \in C_{sa}$). However, we can assign the closed interval $[\lambda(\delta^i(a)_C), \lambda(\delta^o(a)_C)] \subset \mathbb{R}$ to a and state λ. If we restrict the state to a coarser context D, then (3.4) tells us that we associate a larger interval $[\lambda|_D(\delta^i(a)_D), \lambda|_D(\delta^o(a)_D)]$ to a. As contexts become coarser, the associated values become less sharp. At a heuristic level this two-sided daseinisation fits with coarse-graining.

The reader might wonder why the spectral order $\leq_s$ is used, instead of the natural order $\leq$. For example, we could define an inner daseinisation by

$$\delta_i(a)_C = \bigvee \{b \in C_{sa} \mid b \leq a\}.$$

Indeed, this supremum $\delta_i(a)_C$ exists and is an element of C, because the spectral order $\leq_s$ and the usual order $\leq$ coincide on C. However, $\delta_i(a)_C \leq a$ may not hold, as is shown in the following example using $A = M_2(\mathbb{C})$. Define

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad b_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} -1/4 & 0 \\ 0 & -3 \end{pmatrix}.$$

For any $\underline{v} = (v_1, v_2)^t \in \mathbb{C}^2$ it is easily seen that

$$(\underline{v}, (a - b_1)\underline{v}) \geq (|v_1| - |v_2|)^2 \geq 0,$$

$$(\underline{v}, (a - b_2)\underline{v}) \geq (1/4|v_1| - 4|v_2|)^2 \geq 0.$$

We find $b_1, b_2 \leq a$. But $b_1 \vee b_2 \not\leq a$, which follows from

$$b_1 \vee b_2 = \begin{pmatrix} -1/4 & 0 \\ 0 & 0 \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} -i \\ i \end{pmatrix},$$

$$(\underline{w}, (a - b_1 \vee b_2)\underline{w}) = -3/4.$$

It is because of the spectral order that the daseinisation of an operator can be compared with the operator itself, as in (3.4).

Note that we can view inner and outer daseinisation as (categorical) adjunctions. If we see both C_{sa} and A_{sa} as posets with respect to the spectral order, then the inner and outer daseinisation form right and left adjoints to the inclusion map $i_C : C_{sa} \hookrightarrow A_{sa}$. Let $a \in A_{sa}$ and $b \in C_{sa}$. Assume that $b \leq_s \delta^i(a)_C$. As $\delta^i(a)_C \leq_s a$, we conclude that $i_C(b) \leq_s a$. Conversely, assume that $b \leq_s a$. As $\delta^i(a)_C$ is by definition the join of $b \in C_{sa}$ satisfying $b \leq_s a$ we conclude that $b \leq_s \delta^i(a)_C$. By an analogous reasoning, $a \leq_s i(b)$ iff $\delta^o(a)_C \leq_s b$. We conclude that[3]

$$\delta^o(-)_C \dashv i_C \dashv \delta^i(-)_C : C_{sa} \to A_{sa}.$$

We close this section with a simple proposition which tells us how the norm $\|a\|$, of any self-adjoint element a, can be found from the daseinisation of a.

Proposition 3.1.3. *For any $a \in A_{sa}$*

$$\|a\| = max\{|\delta^i(a)_\mathbb{C}|, |\delta^o(a)_\mathbb{C}|\}.$$

Proof. Recall that

$$\delta^i(a)_\mathbb{C} = \bigvee\{x \in \mathbb{R} = \mathbb{C}_{sa} \mid x \leq a\},$$

where $x \leq_s a$ has been replaced by $x \leq a$, since $[x, a] = 0$. If $\sigma(a)$ denotes the spectrum of a, then we deduce that $\delta^i(a)_\mathbb{C} = \inf(\sigma(a))$. Analogously, we can deduce $\delta^o(a)_\mathbb{C} = \sup(\sigma(a))$. □

[3]For a pair (F, G) of adjoint functors, we write $F \dashv G$ to denote that F is a left adjoint to G, or, equivalently, G is a right adjoint to F.

3.2 Daseinisation in the Covariant Approach

A new daseinisation map, with corresponding elementary propositions, is introduced in the covariant approach. This map uses the daseinisation techniques of the contravariant approach. The new daseinisation map is compared to the original covariant daseinisation map.

In this subsection we investigate the covariant version of the daseinisation map. The original daseinisation arrow of the covariant approach was first introduced in [48], where all the details of its construction can be found. We will from the start present a different definition of the daseinisation arrow, which we regard as an improvement or at least as a simplification of the original one. Subsequently we recall the original definition [48] and compare it with this new definition.
Before we can define the daseinisation arrow, a discussion of Scott's interval domain is in order [2]. As a set, the interval domain $\mathbb{IR}$ consists of all compact $[a, b]$ with $a, b \in \mathbb{R}$, $a \leq b$. This includes the singletons $[a, a] = \{a\}$. The elements of $\mathbb{IR}$ are ordered by reverse inclusion.[4] The interval domain is equipped with the so-called Scott topology, in which a set $U \subseteq \mathbb{IR}$ is (Scott) closed if it satisfies the following two conditions: firstly, it is downward closed in the sense that if $[a, b] \in U$ and $[a, b] \subseteq [a', b']$ then $[a', b'] \in U$, and secondly, it is closed under suprema of upwards directed subsets. What is important for us is that the collection

$$(p, q)_S := \{[r, s] \mid p < r \leq s < q\}, \quad p, q \in \mathbb{Q}, \quad p < q,$$

defines a basis for the Scott topology $\mathcal{O}\mathbb{IR}$. We also need the interval domain $\underline{\mathbb{IR}}$, internal to $[\mathcal{C}, \mathbf{Set}]$. This is an internal locale, whose associated frame $\mathcal{O}\underline{\mathbb{IR}}$ has external description

$$\pi_1^{-1} : \mathcal{O}(\mathcal{C}_\uparrow) \to \mathcal{O}(\mathcal{C}_\uparrow \times \mathbb{IR}),$$

where π_1^{-1} is the inverse image of the continuous projection

$$\pi_1 : \mathcal{C}_\uparrow \times \mathbb{IR} \to \mathcal{C}_\uparrow, \quad (C, [a, b]) \mapsto C.$$

Next, we would like to use the daseinisation of self-adjoint operators introduced in Subsection 3.1, but we are immediately faced with a problem:

[4] We might think of elements of $\mathbb{IR}$ as approximations of real numbers (this idea goes back to L.E.J. Brouwer). A smaller set provides more information about the real number it approximates than a larger interval. The smaller interval is higher in the information order.

these constructions do not work for arbitrary C*-algebras, because these generally do not have enough projections. For the remainder of this subsection, also in the covariant approach we will therefore use the context category $\mathcal{C}$ of abelian von Neumann subalgebras. Von Neumann algebras have the advantage that a covariant daseinisation arrow can be given explicitly, in terms of the daseinisation maps of the contravariant approach. This also makes it easier to compare the two topos approaches.
Without further ado we now define the covariant daseinisation map.

Definition 3.2.1. *The **covariant daseinisation map** is the function*

$$\delta : A_{sa} \to C(\Sigma_{\uparrow}, \mathbb{IR}), \quad \delta(a) : (C, \lambda) \mapsto [\lambda(\delta^i(a)_C), \lambda(\delta^o(a)_C)]. \tag{3.5}$$

In $[\mathcal{C}, \mathbf{Set}]$, *define the arrow* $\underline{\delta}(a)^{-1} : \mathcal{O}\underline{\mathbb{IR}} \to \mathcal{O}\underline{\Sigma}_{\uparrow}$ *by*

$$\underline{\delta}(a)_C^{-1}(\uparrow C' \times (p,q)_S) = \delta(a)^{-1}(p,q)_S \cap \Sigma_{\uparrow}|_{\uparrow C'}, \tag{3.6}$$

where $\Sigma_{\uparrow}|_{\uparrow C'} = \coprod_{C'' \in \uparrow C'} \Sigma_{C''}$, *and* $(\uparrow C') \times (p,q)$ *denotes the basic open subset* $\{(C, [r,s]) \mid C \in \mathcal{C}, C \supseteq C', p < r \leq s < q\}$ *of* $\mathcal{C} \times \mathbb{IR}$.

The map δ is well defined, which requires some checking.

Proposition 3.2.2. *For each* $a \in A_{sa}$, *the map* $\delta(a) : \Sigma_{\uparrow} \to \mathbb{IR}$ *is continuous. Furthermore,* $\underline{\delta}(a)^{-1}$ *is a frame map in* $[\mathcal{C}, \mathbf{Set}]$, *and hence defines a locale map* $\underline{\delta}(a) : \underline{\Sigma}_{\uparrow} \to \underline{\mathbb{IR}}$.

Proof. In order to prove continuity, note that

$$\begin{aligned}(\delta(a)^{-1}(p,q)_S)_C &= \{\lambda \in \Sigma_C \mid \lambda(\delta^i(a)_C) > p\} \cap \{\lambda \in \Sigma_C \mid \lambda(\delta^o(a)_C) < q\} \\ &= X^C_{\delta^i(a)_C - p} \cap X^C_{q - \delta^o(a)_C} = X^C_{(\delta^i(a)_C - p) \wedge (q - \delta^o(a)_C)},\end{aligned}$$

where we used the notation

$$X_b^C = \{\lambda \in \Sigma_C \mid \langle \lambda, b \rangle > 0\}, \quad C \in \mathcal{C}, \quad b \in C_{sa}.$$

Therefore, $\delta(a)^{-1}(p,q)_S$ satisfies the first condition on opens of $\Sigma_{\uparrow}$ given in Definition 2.5.2. The second condition follows from (3.4).
The map $\delta(a)$ defines an internal locale map $\underline{\delta}(a)$, with external description simply given by the commutative triangle of continuous maps

$$\begin{tikzcd} \Sigma_{\uparrow} \arrow[r, "\langle \pi, \delta(a) \rangle"] \arrow[d, "\pi"'] & \mathcal{C} \times \mathbb{IR} \arrow[dl, "\pi_1"] \\ \mathcal{C} & \end{tikzcd} \qquad \begin{tikzcd} (C, \lambda) \arrow[r, mapsto] \arrow[d, mapsto] & (C, \delta(a)(C, \lambda)) \arrow[dl, mapsto] \\ C & \end{tikzcd}$$

$\square$

We may use the daseinisation map $\delta : A_{sa} \to C(\Sigma_{\uparrow}, \mathbb{IR})$ to define elementary propositions.

Definition 3.2.3. *Let $a \in A_{sa}$ and $(p,q) \in \mathcal{O}\mathbb{R}$. Then the* ***covariant elementary proposition*** $[a \in (p,q)] \in \mathcal{O}\Sigma_{\uparrow}$ *is defined by*

$$[a \in (p,q)] = \delta(a)^{-1}(p,q)_S \tag{3.7}$$

$$= \coprod_{C \in \mathcal{C}} \{\lambda \in \Sigma_C \mid [\lambda(\delta^i(a)_C), \lambda(\delta^o(a)_C)] \in (p,q)_S\}. \tag{3.8}$$

Each elementary proposition $[a \in (p,q)]$ defines an open of the spectrum of $\underline{A}$ by

$$\underline{[a \in (p,q)]} : \underline{1} \to \mathcal{O}\underline{\Sigma}_{\uparrow}, \ \ \underline{[a \in (p,q)]}_C(*) = \coprod_{C' \in \uparrow C} [a \in (p,q)]_{C'}.$$

If we define

$$\underline{(p,q)} : \underline{1} \to \mathcal{O}\underline{\mathbb{IR}}, \quad \underline{(p,q)}_C(*) = \uparrow C \times (p,q)_S,$$

then

$$\underline{[a \in (p,q)]} = \underline{\delta}(a)^{-1} \circ \underline{(p,q)} : \underline{1} \to \mathcal{O}\underline{\Sigma}_{\uparrow}. \tag{3.9}$$

Compare the covariant elementary proposition (3.9) with the contravariant elementary propostions which, when viewed as elements of $\mathcal{O}\Sigma_{\downarrow}$, are given by

$$[a \in (p,q)] = \coprod_{C \in \mathcal{C}} \{\lambda \in \Sigma_C \mid \lambda(\delta^o(\chi_{(p,q)}(a))_C) = 1\}.$$

This clearly differs from the covariant version. In the contravariant approach, which is motivated by coarse-graining, the spectral projection associated to "$a \in (p,q)$" as a whole is approximated, whereas in the covariant approach the operator a itself is approximated, which in turn implies the formula (3.7) for $[a \in (p,q)]$. The covariant approach uses both inner and outer daseinisation, whereas the contravariant approach only uses outer daseinisation. We could have chosen to define covariant elementary propositions in a different way such that it more closely mirrors the contravariant version. Consider

$$[a \in (p,q)] = \coprod_{C \in \mathcal{C}} \{\lambda \in \Sigma_C \mid \lambda(\delta^i(\chi_{(p,q)}(a))_C) = 1\}. \tag{3.10}$$

This subset of $\Sigma_{\uparrow}$ is open because it is equal to $\delta(\chi_{(p,q)}(a))^{-1}(1-\epsilon, 1+\epsilon)$ (for any positive ϵ smaller than 1). In Section 3.6 the covariant elementary proposition of Definition 3.2.3 gives rise to a natural counterpart in the contravariant approach. In the next chapter we see how the covariant elementary proposition of Definition 3.2.3 is related to (3.10).

Next, we compare the covariant daseinisation arrow with the Gelfand transform $\mathcal{G}$ of Subsection 2.3. Let $i : \mathbb{R} \to \mathbb{IR}$, $x \mapsto [x, x]$, be the inclusion map, and let $\delta(a)|_{\uparrow C(a)}$ denote the restriction of $\delta(a) : \Sigma_{\uparrow} \to \mathbb{IR}$ to $\Sigma_{\uparrow}|_{\uparrow C(a)}$, where $C(a)$ is the context generated by a. Then we have the following commutative triangle:

$$\begin{tikzcd} \Sigma_{\uparrow}|_{\uparrow C(a)} \arrow[r, "\delta(a)|_{\uparrow C(a)}"] \arrow[dr, "\mathcal{G}_{C(a)}"'] & \mathbb{IR} \\ & \mathbb{R} \arrow[u, hook, "i"'] \end{tikzcd} .$$

Hence, on the open $\Sigma_{\uparrow}|_{\uparrow C(a)} \in \mathcal{O}\Sigma_{\uparrow}$, the daseinisation of a coincides with the Gelfand transform of a, formulated as a locale map.

Finally, we show how our new covariant daseinisation arrow of Definition 3.2.1 is related to the original covariant daseinisation arrow of [48]. The covariant daseinisation map of [48] is a function $\delta : A_{sa} \to C(\underline{\Sigma}_{\underline{A}}, \underline{\mathbb{IR}})$, which for each $a \in A_{sa}$, gives a locale map $\underline{\delta}(a) : \underline{\Sigma}_{\underline{A}} \to \underline{\mathbb{IR}}$ internal to $[\mathcal{C}, \mathbf{Set}]$. The inverse image of this daseinisation map, i.e. $\underline{\delta}(a)^{-1} : \mathcal{O}\underline{\mathbb{IR}} \to \mathcal{O}\underline{\Sigma}_{\underline{A}}$, is given by the frame maps

$$\underline{\delta}(a)_C^{-1} : \mathcal{O}(\uparrow C \times \mathbb{IR}) \to \mathcal{O}\underline{\Sigma}_{\underline{A}}(C),$$

$$\underline{\delta}(a)_C^{-1}(\uparrow C' \times (p,q)_S) : C'' \to \left\{ \begin{array}{rl} \emptyset & \text{if } C'' \not\supseteq C' \\ Y_{C''}(p,q,a) & \text{if } C'' \supseteq C'. \end{array} \right. \tag{3.11}$$

In (3.11), $Y_{C''}(p,q,a) \subseteq L_{C''}$ is defined as $D_b^{C''} \in Y_{C''}(p,q,a)$ iff

$$D_b^{C''} \lhd_{C''} \{D^{C''}_{(a_0 - r)\wedge(s-a_1)} \mid a_0, a_1 \in C''_{sa},\ a_0 \leq a \leq a_1,\ [r,s] \in (p,q)_S\}, \tag{3.12}$$

where we used the notation D_a^C for the element $[a]_C$ of the distributive lattice L_C (we use this alternative notation here to connect better with [48]). Recall that the covering relation $\lhd_C$ was defined by: $D_a^C \lhd_C U$ iff

for each rational number $q > 0$, there exists a finite subset $U_0 \subseteq U$, such that $D^C_{a-q} \leq \bigvee U_0$ in L_C.
In order to relate $\underline{\delta}(a)$ to the daseinisation arrow of Definition 3.2.1, we replace the C*-algebras in the context category $\mathcal{C}$ by abelian von Neumann algebras, and replace $a_0 \leq a \leq a_1$ in (3.12) by $a_0 \leq_s a \leq_s a_1$, where $\leq_s$ is the spectral order.

Lemma 3.2.4. *Define $\omega = (\delta^i(a)_C - p) \wedge (q - \delta^o(a)_C)$, where $\delta^i(a)_C$ and $\delta^o(a)_C$ are the daseinisations of a, as in Subsection 3.1. Then $D^C_b \lhd_C \{D^C_\omega\}$ iff D^C_b is an element of*

$$\{D^C_{(a_0-r)\wedge(s-a_1)} \mid a_0, a_1 \in C_{sa},\ a_0 \leq_s a \leq_s a_1,\ [r,s] \in (p,q)_S\}. \quad (3.13)$$

Proof. Call the set in (3.13) X for convenience. If $a_0 \leq_s a$, then by definition $a_0 \leq_s \delta^i(a)_C$, and if $a \leq_s a_1$ then $\delta^o(a)_C \leq_s a_1$. If $[r,s] \in (p,q)_S$ then $D^C_{(a_0-r)\wedge(s-a_1)} \leq D^C_\omega$. This proves that $X \lhd_C \{D^C_\omega\}$. In order to prove that $D^C_\omega \lhd_C X$, it suffices to show that for $\epsilon \in \mathbb{Q}^+$ small enough, there is a $[r,s] \in (p,q)$ such that

$$D^C_{\omega-\epsilon} \leq D^C_{(\delta^i(a)_C - r)\wedge(s-\delta^o(a)_C)}.$$

Take ϵ such that $p+\epsilon < q-\epsilon$. Taking $r = p+\epsilon$ and $s = q-\epsilon$ gives the desired inequality. $\square$

This lemma may be used to simplify the covariant daseinisation map of [48]. For $C \subseteq C' \subseteq C''$ we have

$$D^{C''}_b \in \underline{\delta}(a)^{-1}_C(\uparrow C' \times (p,q)_S)(C'') \text{ iff } D^{C''}_b \lhd_C \{D^{C''}_{(\delta^i(a)_{C''}-p)\wedge(q-\delta^o(a)_{C''})}\}.$$

Identifying $\underline{\Sigma}_{\underline{A}}$ with $\underline{\Sigma}_\uparrow$ by Theorem 2.2.2, $\underline{\delta}(a)^{-1}_C(\uparrow C' \times (p,q)_S)$ corresponds to the following open set of $\mathcal{O}\Sigma_\uparrow|_{\uparrow C}$:

$$\{(C'',\lambda'') \in \Sigma \mid C'' \supseteq C',\ \lambda''(\delta^i(a)_{C''}) > p,\ \lambda''(\delta^o(a)_{C''}) < q\}. \quad (3.14)$$

As $\delta^i(a)_{C''} \leq_s \delta^o(a)_{C''}$ and the spectral order is coarser than the order $\leq$, the set in (3.14) is equal to

$$\{(C'',\lambda'') \in \Sigma \mid C'' \supseteq C',\ \ [\lambda''(\delta^i(a)_{C''}), \lambda''(\delta^o(a)_{C''})] \in (p,q)_S\}.$$

This is exactly the inverse image of $\uparrow C' \times (p,q)_S$ of the continuous function

$$\delta(a) : \Sigma_\uparrow \to \mathcal{C} \times \mathbb{IR},\quad (C,\lambda) \to (C, [\lambda(\delta^i(a)_C), \lambda(\delta^o(a)_C)]),$$

We recognize this as the external description of the daseinisation arrow $\underline{\delta}(a) : \underline{\Sigma}_{\uparrow} \to \underline{\mathbb{IR}}$ given in Proposition 3.2.2. Hence, by the observations above and Lemma 3.2.4, the daseinisation arrow Definition 3.2.1 simply follows from the original covariant daseinisation arrow of [48] by replacing the partial order $\leq$ by the spectral order $\leq_s$ (and replacing C*-algebras by von Neumann algebras accordingly). Looking at Section 5.2 of [48], and in particular at equation (54) and footnote 20, such a close relationship is not surprising.

3.3 Observable and Antonymous Functions

We discuss some technical points regarding so-called observable and antonymous functions. Of particular importance are the identities (3.18) and (3.19), which are used in later sections to relate daseinisation of self-adjoint operators to daseinisation of the spectral projections of these operators.

In this subsection we investigate the connection between the observable functions and the antonymous functions [29, 37] on the one hand, and the covariant daseinisation map of Definition 3.2.1 on the other. These functions were introduced in the contravariant approach in [35], and are based on work of de Groote [40]. Let A be a von Neumann algebra, and $C \in \mathcal{C}$. Let $\mathcal{F}_C$ denote the set of filters in $\mathcal{P}(C)$.[5] We give $\mathcal{F}_C$ a topology $\mathcal{OF}_C$, by taking the following sets as a basis:

$$Ext(p) = \{F \in \mathcal{F}_C \mid p \in F\}, \quad p \in \mathcal{P}(C).$$

We combine the filter spaces into one ambient space, just like we did for the Gelfand spectra.[6]

Definition 3.3.1. *Define the set $\mathcal{F} = \coprod_{C \in \mathcal{C}} \mathcal{F}(C)$. Then $\mathcal{F}$ is given the topology $\mathcal{OF}$, where U is open iff the following two conditions are satisfied:*

1. $\forall C \in \mathcal{C}, \quad U_C \in \mathcal{OF}_C$.

[5]Recall that $F \subset \mathcal{P}(C)$ is a filter if the following three conditions are satisfied. Firstly, $0 \notin F$. Secondly, if $p, q \in F$, then $p \wedge q \in F$. Thirdly, if $p \in F$, and $q \geq p$, then $q \in F$.

[6]Note that the filters spaces $\mathcal{F}_C$ define a presheaf $\mathcal{F} : \mathcal{C}^{op} \to \mathbf{Set}$, by $\mathcal{F}(C) = \mathcal{F}_C$ and for $C \subseteq C'$ the function $\mathcal{F}(i_{CC'}) : \mathcal{F}_{C'} \to \mathcal{F}_C$ is defined as $F' \mapsto F' \cap \mathcal{P}(C)$.

2. *If $C \subseteq C'$, $F \in U_C$, and $F' \in \mathcal{F}_{C'}$, such that $F' \cap \mathcal{P}(C) = F$, then $F' \in U_{C'}$.*

The projection $\pi : \mathcal{F} \to \mathcal{C}$ defines the locale $\underline{\mathcal{F}}$ in $[\mathcal{C}, \mathbf{Set}]$ that has the projection π as its external description.

We should be careful not to confuse the space $\mathcal{F}$, defined from filters in the projection lattices $\mathcal{P}(C)$, with the sobrification of $\mathcal{C}_\downarrow$, which was considered in the previous chapter, and has also been denoted by $\mathcal{F}$.

Lemma 3.3.2. *The map*

$$J : \Sigma_\uparrow \to \mathcal{F}, \quad (C, \lambda) \mapsto (C, F_\lambda),$$

where $F_\lambda = \{p \in \mathcal{P}(C) \mid \lambda(p) = 1\}$, is continuous and injective, and hence it defines an injective locale map $\underline{\Sigma}_\uparrow \to \underline{\mathcal{F}}$ in $[\mathcal{C}, \mathbf{Set}]$.

Proof. We only prove continuity of J, leaving the rest to the reader. Take $U \in \mathcal{O}\mathcal{F}$. We need to check that $J^{-1}(U)_C \in \mathcal{O}\Sigma_C$. First note that, $J^{-1}(U)_C = J^{-1}(U_C)$. Without loss of generality, we assume that $U_C = Ext(p)$. We find

$$J^{-1}(Ext(p))_C = \{\lambda \in \Sigma_C \mid \lambda(p) = 1\},$$

which is open in Σ_C. Next, assume that $\lambda \in J^{-1}(U)_C$, $C \subseteq C'$, and $\lambda' \in \Sigma_{C'}$ such that $\lambda'|_C = \lambda$. From $\lambda \in J^{-1}(U)_C$ it follows that $F_\lambda \in U_C$. From $\lambda'|_C = \lambda$ it follows that $F_{\lambda'} \cap \mathcal{P}(C) = F_\lambda$. By definition of $\mathcal{O}\mathcal{F}$, we find $F_{\lambda'} \in U_{C'}$. We conclude that $\lambda' \in J^{-1}(U)_{C'}$, proving that J is continuous. □

Now we introduce the *antonymous* functions and the *observable* functions. Let $\mathcal{N}$ be a von Neumann algebra (read A or C for $\mathcal{N}$), and let $\mathcal{F}(\mathcal{N})$ denote the set of all filters in the projection lattice $\mathcal{P}(\mathcal{N})$. Let $a \in \mathcal{N}_{sa}$, with spectrum $\sigma(a)$ and spectral resolution $\{e^a_r\}_{r \in \mathbb{R}}$. Then the **antonymous function** $g^{\mathcal{N}}_a$ is defined by [29]

$$g^{\mathcal{N}}_a : \mathcal{F}(\mathcal{N}) \to \sigma(a), \quad F \mapsto \sup\{r \in \mathbb{R} \mid 1 - e^a_r \in F\}.$$

The **observable function** $f^{\mathcal{N}}_a$ is defined by [29]

$$f^{\mathcal{N}}_a : \mathcal{F}(\mathcal{N}) \to \sigma(a), \quad F \mapsto \inf\{r \in \mathbb{R} \mid e^a_r \in F\}.$$

Proposition 3.3.3. *Define the map*

$$h(a) : \mathcal{F} \to \mathbb{IR}, \quad h(a)(C, F) = [g^C_{\delta^i(a)_C}(F), f^C_{\delta^o(a)_C}(F)].$$

This map is continuous and defines a locale map $\underline{\mathcal{F}} \to \underline{\mathbb{IR}}$ *in* $[\mathcal{C}, \mathbf{Set}]$.

Proof. We use the shorthand notation h for $h(a)$. For $(r, s) \in \mathcal{O}\mathbb{IR}$ we need to show that $h^{-1}(r, s)$ is open in $\mathcal{F}$. If $p \in \mathcal{P}(C)$, then $(\uparrow p) = \{q \in \mathcal{P}(C) \mid q \geq p\}$ is the smallest filter $\mathcal{P}(C)$ containing p. If $F \subseteq F'$, then it follows from the definition of h that $h(F) \leq_{\mathbb{IR}} h(F')$. Consequently, if $(\uparrow p) \in h^{-1}(r, s)_C$ then $Ext(p) \subseteq h^{-1}(r, s)$. We conclude that

$$\bigcup_{(\uparrow p) \in h^{-1}(r,s)} Ext(p) \subseteq h^{-1}(r, s). \tag{3.15}$$

The next step is to show that if $F \in h^{-1}(r, s)$, then there exists a $p \in F$ with the property $(\uparrow p) \in h^{-1}(r, s)_C$. Once this has been shown, the inclusion of (3.15) becomes an equality. If $F \in h^{-1}(r, s)$, then $r < g^C_{\delta^i(a)_C}(F) \leq f^C_{\delta^o(a)_C}(F) < s$. Define $x = g^C_{\delta^i(a)_C}$ and $y = f^C_{\delta^o(a)_C}$ and $\epsilon = 1/2\text{min}(x - r, s - y)$. By definition, $y = \inf\{t \in \mathbb{R} \mid e^a_t \in F\}$. Choose any $\epsilon_1 \leq \epsilon$ such that $e^a_{y+\epsilon_1} \in F$. Similarly, choose an $\epsilon_0 \leq \epsilon$ such that $1 - e^a_{x-\epsilon_0} \in F$. Define $p = e^a_{y+\epsilon_1} \wedge (1 - e^a_{x-\epsilon_0})$. Note that

$$f^C_{\delta^o(a)_C}(\uparrow e^a_{y+\epsilon_1}) = y + \epsilon_1 < s,$$

$$g^C_{\delta^i(a)_C}(\uparrow (1 - e^a_{x-\epsilon_0})) = x - \epsilon_0 > r,$$

$$h(\uparrow e^a_{y+\epsilon_1}) \leq_{\mathbb{IR}}, \; h(\uparrow p), \quad h(\uparrow (1 - e^a_{x-\epsilon_0})) \leq_{\mathbb{IR}} h(\uparrow p).$$

We conclude that $(\uparrow p) \in h^{-1}(r, s)$ and that $F \in \bigcup_{(\uparrow p) \in h^{-1}(r,s)} Ext(p)$. Thus we have shown that $h^{-1}(r, s)_C$ is open in $\mathcal{F}_C$. It remains to show that if $F \in h^{-1}(r, s)_C$, $C \subseteq C'$ and $F \in \mathcal{F}_{C'}$ is such that $F' \cap \mathcal{P}(C) = F$, then $F' \in h^{-1}(r, s)_{C'}$. This is easily checked and will be left to the reader. $\square$

Theorem 3.3.4. *The covariant daseinisation map* $\delta(a)$ *factors through* $\mathcal{F}$. *In other words, the following triangle is commutative:*

$$\begin{tikzcd} \Sigma_\uparrow \arrow[r, "J"] \arrow[dr, "\delta(a)"'] & \mathcal{F} \arrow[d, "h(a)"] \\ & \mathbb{IR} \end{tikzcd} .$$

Proof. This follows from the identities

$$\lambda(\delta^i(a)_C) = g^C_{\delta^i(a)_C}(F_\lambda), \quad \lambda(\delta^o(a)_C) = f^C_{\delta^o(a)_C}(F_\lambda). \tag{3.16}$$

A proof of these identities is given in [29, Corollary 7, Corollary 9]. □

Now that we have shown the close relationship between the covariant daseinisation map on one hand, and the observable functions and antonymous functions on the other, we conclude by showing how the observable and antonymous functions can be of help in calculating the daseinisation arrow.
It is shown in [29] that

$$g^C_{\delta^i(a)_C}(F_\lambda) = g^A_a(\uparrow_A F_\lambda), \quad f^C_{\delta^o(a)_C}(F_\lambda) = f^A_a(\uparrow_A F_\lambda), \tag{3.17}$$

where $\uparrow_A F_\lambda = \{p \in \mathcal{P}(A) \mid \exists q \in F_\lambda,\ p \geq q\}$. This identity also follows from continuity of h^a and the observation $(\uparrow_A F_\lambda) \cap \mathcal{P}(C) = F_\lambda$.
Combining (3.16) and (3.17), we obtain useful identities for calculating the daseinisation arrow, viz.

$$\lambda(\delta^i(a)_C) = \sup\{r \in \mathbb{R} \mid 1 - e^a_r \in (\uparrow_A F_\lambda)\}, \tag{3.18}$$

$$\lambda(\delta^o(a)_C) = \inf\{r \in \mathbb{R} \mid e^a_r \in (\uparrow_A F_\lambda)\}. \tag{3.19}$$

We will use the identities (3.18) - (3.19) in the proof of Lemma 3.4.1.

3.4 Physical Interpretation of Propositions

Interpreting the covariant topos model for quantum physics in a way similair to a Kripke model for intuitionistic logic, we briefly discuss a physical interpretation for the covariant elementary propositions. The covariant approach is subsequently compared to the intuitionistic quantum logic of Coecke. A technical lemma which relates daseinisation of self-adjoint operators to daseinisation of the associated spectral projections is proven.

In Section 3.2 we proposed two different versions of covariant elementary propositions. We repeat these for convenience, and add labels in order to distinguish between them:

$$[a \in (p,q)]_1 = \coprod_{C \in \mathcal{C}} \{\lambda \in \Sigma_C \mid [\lambda(\delta^i(a)_C), \lambda(\delta^o(a)_C)] \in (p,q)_S\}, \tag{3.20}$$

$$[a \in (p,q)]_2 = \coprod_{C \in \mathcal{C}} \{\lambda \in \Sigma_C \mid \lambda(\delta^i(\chi_{(p,q)}(a))_C) = 1\}, \tag{3.21}$$

where the first identity coincides with (3.7) and the second with (3.10). The elementary propositions $[a \in (p,q)]_1$, are closest to the covariant elementary propositions in [48]. We first investigate if and how these covariant elementary propositions fit in the neorealism scheme of Isham and Döring.

In their setting, a physical quantity is represented by an arrow $\underline{a} : \underline{S} \to \underline{R}$, where $\underline{S}$ is the state object of the topos and $\underline{R}$ is the value object. In the covariant setting we have a locale map $\underline{\delta}(a) : \underline{\Sigma}_{\uparrow} \to \underline{\mathbb{IR}}$, which means that we have an internal frame morphism $\underline{\delta}(a)^{-1} : \mathcal{O}\underline{\mathbb{IR}} \to \mathcal{O}\underline{\Sigma}_{\uparrow}$, which is however **not** an arrow from a state object to a value object. So we redefine $\underline{\Sigma}_{\uparrow}$ and $\underline{\mathbb{IR}}$ a little. Define the covariant functors $\underline{S}, \underline{R} : \mathcal{C} \to \mathbf{Set}$ by $\underline{S}(C) = \Sigma_{\uparrow}|_{\uparrow C}$ and $\underline{R}(C) = (\uparrow C) \times \mathbb{IR}$, using truncation for the transition maps. Next, for $a \in A_{sa}$ we rewrite the locale map $\underline{\delta}(a)$ as the natural transformation

$$\underline{a} : \underline{S} \to \underline{R}, \quad \underline{a}_C(C', \lambda) = (C', [\lambda(\delta^i(a)_{C'}), \lambda(\delta^o(a)_{C'})]), \tag{3.22}$$

where $C' \in (\uparrow C)$ and $\lambda \in \Sigma_{C'}$. The open $(p,q)_S \in \mathcal{O}\mathbb{IR}$ can be seen as a subobject $\underline{(p,q)}$ of $\underline{R}$ in a natural way. In the neorealist formalism of Döring and Isham one not only considers arrows $\underline{a} : \underline{S} \to \underline{R}$, representing physical quantities, but also the subobjects[7] $\{\tilde{s} \mid \underline{a}(\tilde{s}) \in \tilde{\Delta}\}$ of $\underline{S}$, where $\tilde{s}$ is a variable of type $\underline{S}$, represented by $id : \underline{S} \to \underline{S}$, and $\tilde{\Delta}$ is a variable of type $\mathcal{P}\underline{R}$, represented by the identity $id : \mathcal{P}\underline{R} \to \mathcal{P}\underline{R}$ (e.g. Section 4.2. of [37]).

If we consider the object $\{\tilde{s} \mid \underline{a}(\tilde{s}) \in \underline{(p,q)}\}$ instead, it is easy to prove that for $\underline{a}$ given by (3.22), we have

$$\{\tilde{s} \mid \underline{a}(\tilde{s}) \in \underline{(p,q)}\}(C) = [a \in (p,q)]_1|_{\uparrow C}.$$

The elementary propositions $[a \in (p,q)]_1$ fit well into the Döring–Isham formalism, because this formalism uses the internal 'set theory' of the topos. But aside from the mathematical formalism, do these elementary propositions make sense *physically*?

It was noted in Subsection 3.1 that at least at a heuristic level, the maps $(C, \lambda) \mapsto [\lambda(\delta^i(a)_C), \lambda(\delta^o(a)_C)]$ fit well into the coarse-graining philosophy of the contravariant approach, as the value assigned to a becomes less

[7] With some abuse of notation. Strictly speaking, one considers the term $\{\tilde{s} \mid a(\tilde{s}) \in \tilde{\Delta}\}$ in the local language $\mathcal{L}(A)$ of the system S, where a is the linguistic precursor of $\underline{a}$. The object $\{\tilde{s} \mid \underline{a}(\tilde{s}) \in \tilde{\Delta}\}$ that we consider is the representation of this term in the functor topos.

sharp when we move from a context C to a coarser context D. Conversely, we can say that if we move from a context C to a more refined context C', the values assigned to quantities become sharper. At a heuristic level, this matches with the view of the covariant approach as a Kripke model. Indeed, the Kripke–Joyal semantics (in particular presheaf semantics) for the functor topos $[\mathcal{C}, \mathbf{Set}]$ is the same as that of a Kripke model for intuitionistic logic. We can use this observation to give a physical interpretation for the covariant approach. Let ϕ denote a proposition in the internal language of $[\mathcal{C}, \mathbf{Set}]$. Typically, ϕ will express that relative to some state, an elementary proposition is true. If ϕ holds relative to context C in the internal language, i.e. $C \Vdash \phi$, then we interpret this as follows. By only making use of the measurements corresponding to C we can verify that the claim made by ϕ holds. Note that the 'information order' of this Kripke model agrees with physical intuition in the following sense. If $C' \subset C$ in $\mathcal{C}$, then C' is lower than C in the 'information order' of the Kripke model, and from the physics point of view one can describe fewer physical observations from C' (compared to C). In the next chapter we compute $C \Vdash \phi$. Only when this is done, we can see whether or not it is a good idea to interpret truth in the covariant approach at context C as verification of the claim ϕ by the allowed measurements of C.

The covariant elementary propositions $[a \in (p,q)]_2$ also fit very well with this Kripke model perspective. To each context C, these elementary propositions assign, the largest available projection in C that is smaller than the spectral projection $\chi_{(p,q)}(a)$. Heuristically, for each context C, we take the weakest proposition that can be investigated by the means of C, such that verification of this proposition entails that $a \in (p,q)$ holds. But what do we mean by: "$a \in (p,q)$" holds? We could read it in an instumentalist way by saying that a measurement of a yields a value in (p,q) with certainty. However, if we wish to avoid instrumentalist notions, we can consider the following, different way in which $a \in (p,q)$ holds. First note that from (3.4), it follows that $\delta^i(\chi_{(p,q)}(a))_C \leq \delta^o(\chi_{(p,q)}(a))_{C'}$ for each $(p,q) \in \mathcal{O}\mathbb{IR}$, $a \in A_{sa}$ and $C, C' \in \mathcal{C}$. We then interpret $[a \in (p,q)]_2$ at context C as the weakest proposition that (i) can be investigated by the means of C *and* (ii) implies that the proposition $a \in (p,q)$ is true in the sense of the contravariant approach[8].

[8]The elementary propositions $[a \in (p,q)]_2$ are defined in terms of the inner daseinisation map. In the Kripke model interpretation, the knowledge that can be gained about the system from a context plays an important role. The name *daseinisation*, which (with capital D) is a reference to Heidegger, seems somewhat misplaced in this

We return to the Kripke model interpretation of the covariant approach once we have discussed states in the next chapter. The following lemma shows the way the two covariant elementary propositions are related.

Lemma 3.4.1. *For any $a \in A_{sa}$, $(p, q) \in \mathcal{O}\mathbb{IR}$, $C \in \mathcal{C}$, and $\lambda \in \Sigma_C$, the following holds:*

1. *If $\lambda(\delta^i(a)_C) > p$ and $\lambda(\delta^o(a)_C) < q$, then $\lambda(\delta^i(\chi_{(p,q)}(a))_C) = 1$. In short, $[a \in (p, q)]_1 \subseteq [a \in (p, q)]_2$.*
2. *If $\lambda(\delta^i(\chi_{(p,q)}(a))_C) = 1$, i.e. $(C, \lambda) \in [a \in (p, q)]_2$, then $\lambda(\delta^i(a)_C) \geq p$ and $\lambda(\delta^o(a)_C) \leq q$.*

Proof. First rewrite the identities (3.18) and (3.19) of the previous subsection as

$$\lambda(\delta^i(a)_C) = \sup\{r \in \mathbb{R} \mid \exists p \in \mathcal{P}(C),\ \lambda(p) = 1,\ p \leq \chi_{[r,\infty)}(a)\}, \quad (3.23)$$

$$\lambda(\delta^o(a)_C) = \inf\{r \in \mathbb{R} \mid \exists p \in \mathcal{P}(C),\ \lambda(p) = 1,\ p \leq \chi_{(-\infty,r)}(a)\}. \quad (3.24)$$

In the remainder of the proof we denote $(p, q)_S \in \mathcal{O}\mathbb{IR}$ by $(r, s)_S$ instead, as we already use the letters p and q to denote projections of C. If $\lambda(\delta^i(a)_C) > r$, then by (3.23), for $\epsilon > 0$ small enough there exists a projection $p_i \in \mathcal{P}(C)$ such that $\lambda(p_i) = 1$ and $p_i \leq \chi_{[r+\epsilon,\infty)}(a)$. If $\lambda(\delta^o(a)_C) < s$, then by (3.24) there exists a projection $p_o \in \mathcal{P}(C)$ such that $\lambda(p_o) = 1$ and $p_o \leq \chi_{(-\infty,s)}(a)$.
Defining $p = p_i \cdot p_o$, we obtain a projection $p \in \mathcal{P}(C)$ such that $\lambda(p) = 1$ and $p \leq \chi_{[r+\epsilon,s)}(a) \leq \chi_{(r,s)}(a)$. From the definition of the inner daseinisation map we now conclude $p \leq \delta^i(a)_C$, and subsequently $\lambda(\delta^i(\chi_{(r,s)}(a))_C) = 1$, proving the first claim of the lemma.
For the second claim, assume that $\lambda(\delta^i(\chi_{(r,s)}(a))_C) = 1$. Noting that $\delta^i(\chi_{(r,s)}(a))_C) \leq \chi_{(-\infty,s)}(a)$, the claim $\lambda(\delta^o(a)_C) \leq s$ follows from (3.24). Using $\delta^i(\chi_{(r,s)}(a))_C \leq \chi_{[r,\infty)}(a)$, the claim $\lambda(\delta^i(a)_C) \geq r$ follows from (3.23). $\square$

Which version of covariant elementary proposition is to be preferred? We will consider this question in the next chapter, where we will see how the elementary propositions pair with states.

We close this section with a short comparison between the *covariant* approach and the work of Coecke on intuitionistic quantum logic [21]. In

interpretation.

prepatation, first consider the *contravariant* approach, where one often considers the map $\delta^o : \mathcal{P}(A) \to \mathcal{O}_{cl}\underline{\Sigma}$, where $\mathcal{O}_{cl}\underline{\Sigma}$ denotes the complete Heyting algebra of closed open subobjects of the spectral presheaf (e.g. Section 4 of [29]). In order to compare this with the covariant construction, we see δ^o as a map

$$\delta^o : \mathcal{P}(A) \to \mathcal{O}\Sigma_{\downarrow}, \quad \delta^o(p) = \coprod_{C \in \mathcal{C}} \{\lambda \in \Sigma_C \mid \lambda(\delta^o(p)_C) = 1\}. \tag{3.25}$$

This map is injective, and preserves all joins of the projection lattice $\mathcal{P}(A)$, but not the meets. It cannot preserve both, as $\mathcal{O}\Sigma_{\downarrow}$ (and likewise $\mathcal{O}_{cl}\underline{\Sigma}$) is distributive, whereas $\mathcal{P}(A)$ is nondistributive. Dually, for the covariant approach we define

$$\delta^i : \mathcal{P}(A) \to \mathcal{O}\Sigma_{\uparrow}, \quad \delta^i(p) = \coprod_{C \in \mathcal{C}} \{\lambda \in \Sigma_C \mid \lambda(\delta^i(p)_C) = 1\}. \tag{3.26}$$

This injective map preserves all meets of $\mathcal{P}(A)$, but generally it does not preserve the joins[9]: thinking of $\mathcal{P}(A)$ as a property lattice, the map δ^i is in fact a balanced inf-embedding (the relevant definitions can be found in [21]). For each property $p \in \mathcal{P}(A)$, the open $\delta^i(p) \in \mathcal{O}\Sigma_{\uparrow}$ can be thought of as the set of pure states in context (C, λ), such that the property p is actual in that state. Thinking of the elements of $\mathcal{P}(A)$ in operational terms and thinking of conjunction and disjunction intuitionistically (just like in the Kripke model interpretation of the covariant approach), the meets of $\mathcal{P}(A)$ should be preserved by δ^i, as these coincide with conjunctions. The joins of $\mathcal{P}(A)$ need not be preserved, as these do not coincide with disjunctions because of the possibility of superposition[10].

Typically, in Coecke's approach one considers a balanced inf-embedding $\mu : L \to H$, with L the property lattice and H a complete Heyting algebra which is the injective hull of L (obtained using the so-called Bruns–Lakser construction). Although μ need not preserve all joins, it should preserve all distributive joins. A join $p_1 \vee p_2$ in L, is called *distributive* if for each $q \in L$ we have $q \wedge (p_1 \vee p_2) = (q \wedge p_1) \vee (q \wedge p_2)$. The frame $\mathcal{O}\Sigma_{\uparrow}$ is the injective hull of $\mathcal{P}(A)$ iff the following two conditions are satisfied:

1. $\delta^i(\mathcal{P}(A))$ is join dense in $\mathcal{O}\Sigma_{\uparrow}$:

   If $U \in \mathcal{O}\Sigma_{\uparrow}$, then $U = \bigcup\{\delta^i(p) \mid \delta^i(p) \subseteq U\}$.

[9] This can be shown in the same way as for the dual properties of δ^o.

[10] For two properties $p_1, p_2 \in \mathcal{P}(A)$, it may be the case that the property $p_1 \vee p_2$ is actual for a given state, while neither property p_1 nor property p_2 is actual for that same state.

2. δ^i preserves distributive joins.

In general, neither condition is satisfied, as we can see from the simple example $A = M_3(\mathbb{C})$. Consider the singleton $U = \{(C, \lambda)\} \in \mathcal{O}\Sigma_{\uparrow}$, where C is any maximal context and λ any element of its spectrum. The only set $\delta^i(p)$ that is a subset of U, is $\delta^i(0) = \emptyset$. This is not a real problem. The reader can check that for any von Neumann algebra A, the opens of the form $\delta^i(p)$ form a basis for a topology on Σ, and if we restrict $\mathcal{O}\Sigma_{\uparrow}$ to the topology generated by the opens $\delta^i(p)$, then the first condition is satisfied.
For the $A = M_3(\mathbb{C})$ example, we show δ^i fails to preserve distributive joins. Consider the set $X \subset \mathcal{P}(M_3(\mathbb{C}))$ consisting of all rank 1 projections. The reader can check that X has a distributive join. For each context $C \in \mathcal{C}$, $\delta^i(\bigvee X)_C = \delta^i(1)_C = \Sigma_C$. For the trivial context $\mathbb{C}1$,[11] $\bigvee_{p \in X} \delta^i(p)_{\mathbb{C}1} = \bigvee_{p \in X} \emptyset = \emptyset$. Hence, the distributive join is not preserved.
Distributive joins are not preserved because of contextuality. If $\lambda \in \Sigma_C$ is a state in context C, then for any property $p \in \mathcal{P}(A)$ we can only say that p is certain for state λ in context C if there is a property $q \in \mathcal{P}(A)$ which can be invesitgated from the context C, i.e. $q \in \mathcal{P}(C)$, and which implies p, i.e. $q \leq p$. In the example above one has $\bigvee_{p \in X} \delta^i(p)_{\mathbb{C}1} = \emptyset$ because the context $\mathbb{C}1$ is so coarse that only trvial properties such as $1 = \bigvee X$ can be inferred from it.
As distributive joins are not preserved by δ^i, the intuitionistic quantum logic of Coecke appears to differ from the Heyting algebra structure of $\mathcal{O}\Sigma_{\uparrow}$.

3.5 Spaces of Values

Lower, upper and Dedekind reals in the functor categories of the topos approaches are discussed. The value object of the contravariant approach is derived from an internal point of view.

In Section 2.4 the state space object of the contravariant approach, i.e., the spectral presheaf $\underline{\Sigma}$, has been given the structure of an internal topological space. In this subsection we concentrate on the *value object* of the contravariant approach. The value object is thought of as the universal space of values for physical quantities. This object need not be the real numbers (insofar as one can even speak of **the** real numbers in a topos).

[11] In fact, any nonmaximal context can be used for this counterexample.

Indeed, as sketched in e.g. [53], one of the aims of these topos models is to investigate alternative spaces of values, e.g. because relying on real numbers may turn out to be problematic for theories of quantum gravity. In this subsection, we see how the value object of the contravariant approach is related to internal real numbers. The results of this section are used in the next section, where we describe daseinisation of self-adjoint operators as continuous real-valued maps in both topos approaches, in a uniform way.

Let $\mathbb{R}$ denote the real numbers in the topos **Set**, and let P be a poset. In what follows $\mathrm{OP}(P, \mathbb{R})$ will denote the set of order-preserving functions $r : P \to \mathbb{R}$ and $\mathrm{OR}(P, \mathbb{R})$ will denote the set of order-reversing functions $s : P \to \mathbb{R}$. We write $r \leq s$ if $r(p) \leq s(p)$ for all $p \in P$. The standard choice for the value object in the contravariant approach is the functor $\underline{\mathbb{R}}^{\leftrightarrow} : \mathcal{C}^{op} \to \mathbf{Set}$, defined by

$$\underline{\mathbb{R}}^{\leftrightarrow}(C) = \{(r, s) \in \mathrm{OP}(\downarrow C, \mathbb{R}) \times \mathrm{OR}(\downarrow C, \mathbb{R}) \mid r \leq s\}, \tag{3.27}$$

where the restriction map corresponding to the inclusion $D \subseteq C$ maps (r, s) to $(r|_{\downarrow D}, s|_{\downarrow D})$. This object is closely related to two different kinds of real numbers in the topos $[\mathcal{C}^{op}, \mathbf{Set}]$. Using the natural numbers $\underline{\mathbb{N}}$ of this topos, we can construct real numbers as we would in the topos **Set**. However, the axiom of choice and law of excluded middle are not validated in the presheaf topos $[\mathcal{C}^{op}, \mathbf{Set}]$. This entails that constructions that yield the same set of real numbers in the topos **Set**, may yield different objects in the topos $[\mathcal{C}^{op}, \mathbf{Set}]$. In particular, we are interested in the three versions of real numbers in the following definition.

Definition 3.5.1. *Consider the following versions of real numbers:*

- *The* ***lower real numbers****,* $\underline{\mathbb{R}}_l$*, are the rounded down-closed subsets of* $\underline{\mathbb{Q}}$*, where* $\underline{x} \subseteq \underline{\mathbb{Q}}$ *is called* rounded *if* $p \in \underline{x}$ *implies that there exists a* $p < q \in \underline{\mathbb{Q}}$ *such that* $q \in \underline{x}$*, and* $\underline{x} \subseteq \underline{\mathbb{Q}}$ *is called down-closed if* $p < q \in \underline{x}$ *implies that* $p \in \underline{x}$*. If* $\underline{x} \in \underline{\mathbb{R}}_l$ *and* $q \in \underline{\mathbb{Q}}$*, then we write* $q < \underline{x}$ *whenever* q *is in* $\underline{x}$*.*

- *The* ***upper real numbers****,* $\underline{\mathbb{R}}_u$ *, are the rounded up-closed subsets of* $\underline{\mathbb{Q}}$*. In this case rounded means that if* $p \in \bar{x}$ *then there exists a* $q < p$ *such that* $q \in \bar{x}$*. If* $\bar{x} \in \underline{\mathbb{R}}_u$ *and* $q \in \underline{\mathbb{Q}}$*, then we write* $\bar{x} < q$ *whenever* q *is in* $\bar{x}$*.*

- *The **Dedekind real numbers**, $\mathbb{R}_d$, are pairs $\langle \underline{x}, \bar{x} \rangle$, where $\underline{x} \in \mathbb{R}_l$ is non-empty, $\bar{x} \in \mathbb{R}_u$ is non-empty, $\underline{x} \cap \bar{x} = \emptyset$, and $\underline{x}$ and $\bar{x}$ are arbitrarily close, in that if $q, r \in \mathbb{Q}$, with $q < r$, then either $q < \underline{x}$ or $\bar{x} > r$.*

Note that by the above definition the sets $\mathbb{Q}$ and $\emptyset$ are lower and upper real numbers. If we exclude $\mathbb{Q}$ and $\emptyset$, we note that in the topos **Set**, any lower real can be identified with its supremum, and any upper real can be identified with its infimum. Therefore, in **Set** all three versions of real numbers given above can be identified with each other and with $\mathbb{R}$ (or with $\mathbb{R}$ extended with $\{-\infty, +\infty\}$, if we want to include $\mathbb{Q}$ and $\emptyset$). The definitions given above make sense internally to every topos that has a natural numbers object, and hence in particular to every Grothendieck topos. In such a topos $\mathcal{E}$, these constructively different notions of real numbers need not correspond to the same object, as they do in **Set**.
Even though the sets $\mathbb{R}_l$, $\mathbb{R}_u$ and $\mathbb{R}_d$ coincide in **Set**, the natural topologies on these sets differ. The topology on $\mathbb{R}_l$ is the topology generated by upper half intervals $(y, +\infty]$, $y \in \mathbb{R}$. The topology on $\mathbb{R}_u$ is the topology generated by half open intervals $[-\infty, y)$, with $y \in \mathbb{R}$. The topology on $\mathbb{R}_d$ is the familiar Hausdorff topology on $\mathbb{R}$ generated by the open intervals (x, y), with $x, y \in \mathbb{R}$.
The previous statement requires some clarification. In what sense are these topologies natural? Each of the real numbers of the definition can be captured by a propositional theory, within the constraints of geometric logic [78, 55]. To such a theory we can associate a frame, just like a Lindenbaum algebra associated to a classical propositional theory. The points of this frame are the (standard) models of the theory, which, in our case, are the real numbers. The topologies that we consider are the (Lindenbaum) frames of the corresponding theories.
What do the lower, upper and Dedekind reals look like in the topos $[\mathcal{C}^{op}, \mathbf{Set}]$? As these reals are defined by a geometric propositional theory, we can view them as either locales (whose frame is constructed like the Lindenbaum algebra of a classical propositional theory) or as sets (the set models of the theory). We will also describe them as internal topological spaces, which will be convenient when we consider daseinised self-adjoint operators.
The fact that these real numbers are described by propositional geometric theories also makes it easy to find the external description of their frames. Under the identification of the category of locales internal to $[\mathcal{C}^{op}, \mathbf{Set}]$ with the category of locales over $\mathcal{C}_\downarrow$ with the Alexandroff down

set topology, the different kinds of real numbers are given by the following bundles.

Lemma 3.5.2. *The external description of the locales of lower, upper and Dedekind real numbers in* $[\mathcal{C}^{op}, \mathbf{Set}]$ *is given by the bundles*

$$\pi_1 : \mathcal{C}_{\downarrow} \times \mathbb{R}_{\alpha} \to \mathcal{C}_{\downarrow}, \quad (C, x) \mapsto C, \tag{3.28}$$

where for α *we may take* l*,* u *or* d*, and* $\mathbb{R}_{\alpha}$ *is viewed as a topological space in* $\mathbf{Set}$ *with the topologies given above.*

A discussion why (3.28) gives the right description can be found in[12] of [55, Section D4.7]. The bundle (3.28), with $\alpha = l$ describes the lower reals as a locale. The corresponding internal set of lower reals in $[\mathcal{C}^{op}, \mathbf{Set}]$ (the set of points of the locale) is given by the functor

$$\underline{\mathbb{R}}_l : \mathcal{C}^{op} \to \mathbf{Set}, \quad \underline{\mathbb{R}}_l(C) = C((\downarrow C), \mathbb{R}_l),$$

the presheaf of (Alexandroff) continuous functions taking values in $\mathbb{R}_l$. For any topological space X, a function $\mu : X \to \mathbb{R}_l$ is continuous iff it is lower semicontinuous, when seen as a function $\mu : X \to \mathbb{R}$. By definition of the down set topology on $\mathcal{C}$ the function μ is lower semicontinuous iff it is order reversing. In the contravariant approach the following presheaf plays an important rôle, see for example [37, Definition 8.2]:

$$\underline{\mathbb{R}}^{\preceq} : \mathcal{C}^{op} \to \mathbf{Set}, \quad \mathbb{R}^{\preceq}(C) = \mathrm{OR}((\downarrow C), \mathbb{R}).$$

We recognise the presheaf $\underline{\mathbb{R}}^{\preceq}$ as the presheaf of lower real numbers $\underline{\mathbb{R}}_l$. In the same way the sets of upper and Dedekind real numbers in $[\mathcal{C}^{op}, \mathbf{Set}]$ can be described.

Lemma 3.5.3. *Externally, the set of lower real numbers in* $[\mathcal{C}^{op}, \mathbf{Set}]$ *is the presheaf*

$$\underline{\mathbb{R}}_l : \mathcal{C}^{op} \to \mathbf{Set}, \quad \underline{\mathbb{R}}_l(C) = OR((\downarrow C), \mathbb{R}).$$

Externally, the set of upper real numbers in $[\mathcal{C}^{op}, \mathbf{Set}]$ *is the presheaf*

$$\underline{\mathbb{R}}_u : \mathcal{C}^{op} \to \mathbf{Set}, \quad \underline{\mathbb{R}}_u(C) = OP((\downarrow C), \mathbb{R}).$$

The set of Dedekind real numbers $\underline{\mathbb{R}}_d$ *of* $[\mathcal{C}^{op}, \mathbf{Set}]$ *is externally given by the constant functor* $\Delta(\mathbb{R})$*.*

[12] Actually, in [55] it is assumed that we are working over a sober space. We could consider the sobrification of $\mathcal{C}_A^{\downarrow}$ and consider the upper, lower and Dedekind reals over this space. However, this leads to the same frames we are using.

The next corollary describes the value object of the contravariant model internally. It uses the following notation:

$$\forall \underline{x} \in \underline{\mathbb{R}}_l \ \forall \epsilon \in \underline{\mathbb{Q}} \ \ \underline{x} + \epsilon := \{q + r \mid q < \underline{x}, \ r < \epsilon\};$$

$$\forall \overline{x} \in \underline{\mathbb{R}}_u \ \forall \epsilon \in \underline{\mathbb{Q}} \ \ \overline{x} + \epsilon := \{q + r \mid q > \underline{x}, \ r > \epsilon\}.$$

If we view a rational number ϵ as an upper or lower real number, i.e.,

$$\underline{\epsilon} = \{q \in \mathbb{Q} \mid q < \epsilon\}, \ \ \overline{\epsilon} = \{q \in \mathbb{Q} \mid q > \epsilon\},$$

then $\underline{x} + \epsilon$ coincides with the sum $\underline{x} + \underline{\epsilon}$, and $\overline{x} - \epsilon$ coincides with both $\overline{x} + (\overline{-\epsilon})$, and $\overline{x} - \underline{\epsilon}$, where addition and subtraction are defined as in [76]

Corollary 3.5.4. *The presheaf (3.27) is the external description of the internal set*

$$\{(\overline{x}, \underline{x}) \in \underline{\mathbb{R}}_u \times \underline{\mathbb{R}}_l \mid \forall \epsilon \in \underline{\mathbb{Q}}^+ \ \overline{x} - \epsilon < \underline{x} + \epsilon\},$$

where, $\overline{x} - \epsilon < \underline{x} + \epsilon$ *means that* $(\overline{x} - \epsilon) \cap (\underline{x} + \epsilon)$ *contains a rational number.*

In what follows, we would like to view the upper and lower real numbers as internal spaces. Consider $\underline{\mathbb{R}}_l$, the internal set of real numbers in $[\mathcal{C}_A^{op}, \mathbf{Set}]$. The corresponding étale bundle is given by $\pi_l : \mathcal{R}_{l,A}^{\downarrow} \to \mathcal{C}_A^{\downarrow}$, where

$$\mathcal{R}_{l,A}^{\downarrow} = \{(C, s) \mid C \in \mathcal{C}, s \in OR((\downarrow C), \mathbb{R})\},$$

and $U \subseteq \mathcal{R}_A^{\downarrow}$ is open with respect to the étale topology iff the following implication holds:

$$\text{If} \ \ (C, s) \in U \ \ \text{and} \ \ D \subseteq C \ \ \text{then} \ \ (D, s|_{\downarrow D}) \in U.$$

Provide $\mathcal{R}_{l,A}^{\downarrow}$ with the coarser topology generated by the étale opens

$$U_{x,C} = \{(D, s) \in \mathcal{R}_{l,A}^{\downarrow} \mid D \in (\downarrow C), \ s(D) > x\}, \quad C \in \mathcal{C}_A, x \in \mathbb{R}. \quad (3.29)$$

Note that with respect to this topology, the function

$$j : \mathcal{R}_{l,A}^{\downarrow} \to \mathcal{C}_A^{\downarrow} \times \mathbb{R}_l, \quad (C, s) \mapsto (C, s(C))$$

is a continuous map over $\mathcal{C}_{l,A}^{\downarrow}$, whilst the inverse image map j^{-1} is an isomorphism of frames on the topologies. Whenever we want to see the internal lower reals as a topological space rather than a locale, we can use the bundle $\pi_l : \mathcal{R}^{\downarrow}_{l,A} \to \mathcal{C}^{\downarrow}_{A}$, where $\mathcal{R}^{\downarrow}_{l,A}$ has the topology generated by (3.29). The same can be done for the upper reals using a topological space $\mathcal{R}^{\downarrow}_{u,A}$.

Having studied both the state spaces and the value spaces, we turn to describing physical quantities as continuous maps from the state space to the value space.

3.6 Physical Quantities as Continuous Maps

In both topos approaches, inner and outer daseinisation of self-adjoint operators are described as continuous maps from the pertinent state space to a certain space of pairs of internal real numbers. Subsequently, these maps are related to the elementary propositions of the approaches.

In this subsection we take an internal perspective on daseinised selfadjoint operators, by thinking of them as continuous functions from the space of states to the space of values. We have two reasons for this. The obvious one is that we are investigating the interplay between the internal language of the topoi and neorealism, i.e. formal proximity to classical structures. The second reason is that we want to investigate to what extent the elementary propositions $[a \in \Delta]$ can be obtained in an internal way. These propositions are labelled by opens $\Delta \in \mathcal{O}\mathbb{R}$ and bear no obvious relation to the internal value object $\mathcal{R}$. Ideally, we would like to relate opens $\Delta \in \mathcal{O}\mathbb{R}$ to subobjects $\underline{\Delta} \subseteq \mathcal{R}$, such that for an operator a, represented by an arrow $\delta(a) : \Sigma \to \mathcal{R}$, the elementary open $[a \in \Delta]$ is obtained internally as $\delta(a)^{-1}(\underline{\Delta})$.

Before we can start, we first need to consider continuous maps between internal topological spaces [64]. We encountered internal topological spaces in Section 2.4. Let $(\underline{X}, \mathcal{O}\underline{X})$ and $(\underline{Y}, \mathcal{O}\underline{Y})$ be two topological spaces in $Sh(T)$ externally described by bundles $p : X \to T$, and $q : Y \to T$ respectively. A continuous map $f : (\underline{X}, \mathcal{O}\underline{X}) \to (\underline{Y}, \mathcal{O}\underline{Y})$ is a sheaf morphism $f : \underline{X} \to \underline{Y}$ satisfying

$$\Vdash \forall U \in \mathcal{P}\underline{Y} \;\; (U \in \mathcal{O}\underline{Y}) \Rightarrow (f^{-1}(U) \in \mathcal{O}\underline{X}).$$

The sheaf morphism $f^{-1} : \mathcal{P}\underline{Y} \to \mathcal{P}\underline{X}$ used in this condition is described in [63, Section IV.1], where it is aptly called $\mathcal{P}f$. Under the identification of $Sh(T)$ with **Étale**(T), sheaf morphisms $\underline{f} : \underline{X} \to \underline{Y}$ correspond to commuting triangles of continuous functions

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & T & \end{tikzcd}$$

Here the map f is continuous with respect to the étale topologies on X and Y. Such a map f corresponds to an internal continuous map $\underline{f} : (\underline{X}, \mathcal{O}\underline{X}) \to (\underline{Y}, \mathcal{O}\underline{Y})$ iff, in addition, f is continuous with respect to the coarser topologies on X and Y coming from the internal topologies $\mathcal{O}\underline{X}$. and $\mathcal{O}\underline{Y}$.

We proceed to show how outer and inner daseinisation define continuous maps in the above sense. For any $a \in A_{sa}$ and $C \in \mathcal{C}$, outer daseinisation provides an element $\delta^o(a)_C \in C_{sa}$. By Gelfand duality, we can see this as a continuous map

$$\widehat{\delta^o(a)_C} : \Sigma_C \to \mathbb{R}.$$

If $D \subseteq C$, then $\delta^o(a)_D \geq \delta^o(a)_C$ by definition of outer daseinisation. This entails

$$\forall a \in A_{sa} \;\; \forall C \in \mathcal{C} \;\; \forall \lambda \in \Sigma_C \;\; \widehat{\delta^o(a)_C}(\lambda) \leq \widehat{\delta^o(a)_D}(\lambda|_D),$$

as this follows straight from

$$\lambda(\delta^o(a)_C) = \langle \delta^o(a)_C, \lambda \rangle \leq \langle \delta^o(a)_D, \lambda \rangle = \langle \delta^o(a)_D, \lambda|_D \rangle = \lambda|_D(\delta^o(a)_D).$$

For a fixed $a \in A_{sa}$, and varying $C \in \mathcal{C}$, we can combine these maps into a single arrow $\underline{\delta^o(a)} : \underline{\Sigma}_\downarrow \to \underline{\mathbb{R}}^{\preceq}$. So internally, $\underline{\delta^o(a)}$ defines a function from the spectral presheaf to the set of lower real numbers. The arrow is given by

$$\underline{\delta^o(a)}_C : \Sigma_C \to \mathrm{OR}((\downarrow C), \mathbb{R}), \quad \underline{\delta^o(a)}_C(\lambda)(D) = \langle \lambda, \delta^o(a)_D \rangle.$$

Proposition 3.6.1. *The function $\underline{\delta^o(a)} : \underline{\Sigma}_\downarrow \to \underline{\mathbb{R}}_l$ is a continuous map of internal topological spaces.*

Proof. At the level of étale bundles, the natural transformation $\underline{\delta^o(a)}$: $\underline{\Sigma}_\downarrow \to \underline{\mathbb{R}}_l$ is given by

$$\begin{tikzcd} \Sigma_A^\downarrow \arrow[rr, "\delta^o(a)"] \arrow[dr, "\pi"'] & & \mathcal{R}_{l,A}^\downarrow \arrow[dl, "\pi_1"] \\ & \mathcal{C}_A^\downarrow & \end{tikzcd}$$

where

$$\delta^o(a)(C,\lambda) = (C, D \mapsto \langle \lambda, \delta^o(a)_D \rangle).$$

The function $\delta^o(a)$ is continuous with respect to the étale topologies, simply because it comes from a natural transformation, but we need to check that it is also continuous with respect to the coarser topologies, corresponding to the internal topologies. Consider the basic open $U_{x,C}$ of $\mathcal{R}_{l,A}^\downarrow$. Then

$$\delta^o(a)^{-1}(U_{x,C})_D = \begin{cases} \widehat{\delta^o(a)_D}^{-1}(x, +\infty) & \text{if } D \subseteq C \\ \emptyset & \text{if } D \nsubseteq C. \end{cases} \tag{3.30}$$

From (3.30) it is clear that for each $D \in \mathcal{C}_A$, the set $\delta^o(a)^{-1}(U_{x,C})_D$ is open in Σ_D. Also, if $(D,\lambda) \in \delta^o(a)^{-1}(U_{x,C})$ and $D' \subseteq D$, then

$$\langle \lambda|_{D'}, \delta^o(a)_{D'} \rangle \geq \langle \lambda, \delta^o(a)_D \rangle > x,$$

so that $(D', \lambda|_{D'}) \in \delta^o(a)^{-1}(U_{x,C})$. We conclude that $\delta^o(a)^{-1}(U_{x,C})$ is open in $\Sigma_A^\downarrow$ with respect to the topology generated by the closed open subobjects. $\square$

Instead of continuous maps of spaces, we can view $\delta^o(a)$ as an internal map of locales by considering the commutative triangle

$$\begin{tikzcd} \Sigma_A^\downarrow \arrow[rr, "\delta^o(a)"] \arrow[dr, "\pi"'] & & \mathcal{C}_A^\downarrow \times \mathbb{R}_l \arrow[dl, "\pi_1"] \\ & \mathcal{C}_A^\downarrow & \end{tikzcd}$$

where the map $\delta^o(a) : \Sigma_\downarrow \to \mathcal{C}_\downarrow \times \mathbb{R}_l$ is given by $(C,\lambda) \mapsto (C, \langle \lambda, \delta^o(a)_C \rangle)$. Under the identification of the category of locales in $[\mathcal{C}^{op}, \mathbf{Set}]$ with the

category of locales over $\mathcal{C}_{\downarrow}$, the triangle of locale maps over $\mathcal{C}_{\downarrow}$ corresponds to an internal locale map $\underline{\delta^o(a)} : \underline{\Sigma}_{\downarrow} \to \underline{\mathbb{R}}_l$.
Just like the presheaf of order-reversing functions, we can define the presheaf of order-preserving function $\underline{\mathbb{R}}^{\succeq}$. This presheaf coincides with presheaf of upper real numbers $\underline{\mathbb{R}}_u$. Inner daseinisation of a self-adjoint operator defines a natural transformation $\underline{\delta^i(a)} : \underline{\Sigma}_{\downarrow} \to \underline{\mathbb{R}}^{\succeq}$. We leave it to the reader to prove the following analogue of the previous proposition:

Proposition 3.6.2. *The function $\underline{\delta^i(a)} : \underline{\Sigma}_{\downarrow} \to \underline{\mathbb{R}}_u$ is a continuous map of internal topological spaces.*

3.6.1 Covariant Version

Before we connect the continuous daseinised operators to the elementary propositions, we first look at the way this works in the covariant version of the topos approach. Here we exploit the fact that the topos $[\mathcal{C}_A, \mathbf{Set}]$ is equivalent (even isomorphic) to the topos of sheaves over $\mathcal{C}_A^{\uparrow}$, the set $\mathcal{C}_A$ equipped with the upset Alexandroff topology [48].

Lemma 3.6.3. *In $[\mathcal{C}, \mathbf{Set}]$, the internal lower and upper reals (as sets) are externally given by the functors*

$$\underline{\mathbb{R}}_l : \mathcal{C}_A \to \mathbf{Set}, \quad \underline{\mathbb{R}}_l(C) = OP((\uparrow C), \mathbb{R}),$$
$$\underline{\mathbb{R}}_u : \mathcal{C}_A \to \mathbf{Set}, \quad \underline{\mathbb{R}}_u(C) = OR((\uparrow C), \mathbb{R}).$$

Note that with respect to the contravariant version the roles of order-preserving and order-reversing functions have been interchanged. In the covariant model, the role of the spectral presheaf as a state space is played by the internal Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ of $\underline{A}$. Using the identification $[\mathcal{C}, \mathbf{Set}] \cong Sh(\mathcal{C}_{\uparrow})$, as well as the observation that locales in $Sh(\mathcal{C}_{\uparrow})$ correspond to locale maps over $L(\mathcal{C}_{\uparrow})$, we describe the spectrum as a continuous map $\pi_A : \Sigma_{\underline{A}}^{\uparrow} \to \mathcal{C}_A^{\uparrow}$ of topological spaces, where $\Sigma_{\underline{A}}^{\uparrow}$ was introduced in Definition 2.2.1. In the covariant version the inner and outer daseinised operators define locale maps:

Proposition 3.6.4. *Outer daseinisation defines a commutative triangle of continuous maps*

$$\begin{tikzcd} \Sigma_{\underline{A}}^{\uparrow} \arrow[rr, "\delta^o(a)"] \arrow[dr, "\pi_A"'] & & \mathcal{C}_A^{\uparrow} \times \mathbb{R}_u \arrow[dl, "\pi_1"] \\ & \mathcal{C}_A^{\uparrow} & \end{tikzcd}$$

for which we denote the corresponding internal locale map by

$$\underline{\delta^o(a)} : \underline{\Sigma}_{\underline{A}} \to \underline{\mathbb{R}}_u.$$

In the same way, inner daseinisation defines a locale map

$$\underline{\delta^i(a)} : \underline{\Sigma}_{\underline{A}} \to \underline{\mathbb{R}}_l.$$

At the level of sets and functions, this is the same triangle as for the contravariant version; the difference is only in the topologies. The same holds for inner daseinisation.

We can pair the two daseinisation maps together as the locale map

$$\underline{\delta(a)} = \langle \underline{\delta^i(a)}, \underline{\delta^o(a)} \rangle : \underline{\Sigma}_{\underline{A}} \to \underline{\mathbb{R}}_l \times \underline{\mathbb{R}}_u,$$

which externally is described by

$$\begin{tikzcd}
\Sigma^{\uparrow}_{\underline{A}} \arrow[rr, "\delta(a)"] \arrow[dr, "\pi"'] & & \mathcal{C}^{\uparrow}_A \times \mathbb{R}_l \times \mathbb{R}_u \arrow[dl, "\pi_1"] \\
& \mathcal{C}^{\uparrow}_A &
\end{tikzcd}$$

where $\delta(a)(C,\lambda) = (C, \langle \lambda, \delta^i(a)_C \rangle, \langle \lambda, \delta^o(a)_C \rangle)$, and we used the identification

$$(\mathcal{C}^{\uparrow}_A \times \mathbb{R}_l) \times_{\mathcal{C}^{\uparrow}_A} (\mathcal{C}^{\uparrow}_A \times \mathbb{R}_u) \cong \mathcal{C}^{\uparrow}_A \times \mathbb{R}_l \times \mathbb{R}_u.$$

In Section 3.2 the interval domain $\mathbb{IR}$ was used in considering two-sided daseinisation of self-adjoint operators. Consider the injective function

$$j : \mathbb{IR} \to \mathbb{R}_l \times \mathbb{R}_u, \quad j([x,y]) = (x,y).$$

This function is continuous because $j^{-1}((r,+\infty] \times [-\infty, s))$ is equal to $(r,s) \in \mathcal{O}\mathbb{IR}$ if $r < s$, and equals the empty set if $r \geq s$. Note that for each context $C \in \mathcal{C}$ and any $\lambda \in \Sigma_C$, we have the inequality $\langle \lambda, \delta^i(a)_C \rangle \leq \langle \lambda, \delta^o(a)_C \rangle$, so the map $\delta(a)$ factors through the interval domain as

$$\begin{tikzcd}
\Sigma^{\uparrow}_{\underline{A}} \arrow[r, "\delta(a)"] \arrow[dr] & \mathcal{C}^{\uparrow}_A \times \mathbb{R}_l \times \mathbb{R}_u \\
& \mathcal{C}^{\uparrow}_A \times \mathbb{IR} \arrow[u, "\mathcal{C} \times j"']
\end{tikzcd}$$

Note that this is a commutative triangle in $\mathbf{Loc}/\mathcal{C}_A^{\uparrow}$, where $\mathcal{C}_A^{\uparrow} \times \mathbb{IR}$ is seen as the total space of a bundle over $\mathcal{C}_A^{\uparrow}$ by projecting on the first coordinate. This bundle $\pi_1 : \mathcal{C}_A^{\uparrow} \times \mathbb{IR} \to \mathcal{C}_A^{\uparrow}$ is the external description of the interval domain $\underline{\mathbb{IR}}$ in $[\mathcal{C}_A, \mathbf{Set}]$, and the factorised map $\Sigma_{\underline{A}}^{\uparrow} \to \mathcal{C}_A^{\uparrow} \times \mathbb{IR}$ is the external description of the daseinisation map $\underline{\Sigma}_{\underline{A}} \to \underline{\mathbb{IR}}$ introduced in Section 3.2.
Now we can connect this daseinisation map to the elementary propositions, at least for the case where we consider an open interval $\Delta = (r, s)$ in the set of (Dedekind) real numbers. We can translate this to an open subset of $\mathcal{C}_A^{\uparrow} \times \mathbb{IR}$ (or an open subset of $\mathcal{C}_A^{\uparrow} \times \mathbb{R}_l \times \mathbb{R}_u$) by

$$\hat{\Delta} = \{(C, [x, y]) \in \mathcal{C}_A^{\uparrow} \times \mathbb{IR} \mid C \in \mathcal{C}, [x, y] \in (r, s)\},$$

where we view (r, s) as an open of $\mathbb{IR}$. In addition, for any real number $\epsilon > 0$ define

$$\hat{\Delta} + \epsilon = \{(C, [x, y]) \in \mathcal{C}_A^{\uparrow} \times \mathbb{IR} \mid C \in \mathcal{C}, [x, y] \in (r - \epsilon, s + \epsilon)\}.$$

For $\hat{\Delta}$, we obtain the corresponding open of $\Sigma_{\underline{A}}^{\uparrow}$, given by

$$\delta(a)^{-1}(\hat{\Delta}) = \{(C, \lambda) \in \Sigma \mid r < \langle \lambda, \delta^i(a)_C \rangle \leq \langle \lambda, \delta^o(a)_C \rangle < s\}. \quad (3.31)$$

Define the elementary proposition $[a \in \Delta]$, viewed externally as an open of $\Sigma_{\uparrow}$, as

$$[a \in \Delta] = \{(C, \lambda) \in \Sigma \mid \langle \lambda, \delta^i(\chi_\Delta(a))_C \rangle = 1\}, \quad (3.32)$$

where $\chi_\Delta(a)$ is the spectral projection of a, associated with Δ. Note that this elementary proposition (3.32) was introduced to mimic the contravariant elementary proposition. By Lemma 3.4.1, the opens (3.31) and (3.32) are related as:

Theorem 3.6.5. *Let $r < s$ in $\mathbb{R}$, and $\epsilon > 0$. Then*

$$\delta(a)^{-1}(\hat{\Delta}) \subseteq [a \in \Delta] \subseteq \delta(a)^{-1}(\hat{\Delta} + \epsilon).$$

This theorem establishes the relations between elementary propositions, described by inner daseinised projections, and two-sided daseinised self-adjoint operators. Furthermore, through this correspondence the external space of real numbers is linked to the internal value object in a straightforward way. Recall that Lemma 3.4.1 was based on the identities

$$\langle \lambda, \delta^i(a)_C \rangle = \sup\{r \in \mathbb{R} \mid \exists p \in \mathcal{P}(C),\ \langle \lambda, p \rangle = 1,\ \ p \leq 1 - \chi_{[-\infty, r)}(a)\}, \quad (3.33)$$

$$\langle \lambda, \delta^o(a)_C \rangle = \inf\{r \in \mathbb{R} \mid \exists p \in \mathcal{P}(C),\ \langle \lambda, p \rangle = 1, \quad p \leq \chi_{[-\infty,r)}(a)\}, \tag{3.34}$$

where $\mathcal{P}(C)$ is the Boolean algebra of projection operators of C. From these identities the connection to inner daseinisation of spectral projections of a becomes clear. Note that for each projection operator $q \in \mathcal{P}(A)$ there exists a $p \in \mathcal{P}(C)$ with the properties $\langle \lambda, p \rangle = 1$ and $p \leq q$, iff $\langle \lambda, \delta^i(q)_C \rangle = 1$. So, for example, (3.34) can be written as

$$\langle \lambda, \delta^o(a)_C \rangle = \inf\{r \in \mathbb{R} \mid \langle \lambda, \delta^i(\chi_{[-\infty,r)}(a))_C \rangle = 1\}.$$

3.6.2 Contravariant Version

As in the previous subsection, we can combine the two daseinisation maps into a single map $\mathcal{C}_A^{\downarrow}$, given by

$$\delta(a) : \Sigma_A^{\downarrow} \to \mathcal{C}_A^{\downarrow} \times \mathbb{R}_u \times \mathbb{R}_l, \quad (C, \lambda) \mapsto (C, \langle \lambda, \delta^i(a)_C \rangle, \langle \lambda, \delta^o(a)_C \rangle),$$

which is the external description of the internal continuous map

$$\underline{\delta(a)} = \langle \underline{\delta^i(a)}, \underline{\delta^o(a)} \rangle : \underline{\Sigma}_A \to \underline{\mathbb{R}}_u \times \underline{\mathbb{R}}_l.$$

Given $\Delta = (s, r)$, with $r, s \in \mathbb{R}$ such that $s < r$, we consider the open

$$\hat{\Delta} = \mathcal{C}_A \times [-\infty, r) \times (s, +\infty] \in \mathcal{O}(\mathcal{C}_A^{\downarrow} \times \mathbb{R}_u \times \mathbb{R}_l).$$

Likewise, if $\epsilon > 0$, then $\hat{\Delta} + \epsilon$ is defined as $\mathcal{C}_A \times [-\infty, r + \epsilon) \times (s - \epsilon, +\infty]$.

$$\begin{aligned} \delta(a)^{-1}(\hat{\Delta}) &= \{(C, \lambda) \in \Sigma \mid r > \langle \lambda, \delta^i(a)_C \rangle \leq \langle \lambda, \delta^o(a)_C \rangle > s\}, \\ &= \delta^i(a)^{-1}([-\infty, r)) \cap \delta^o(a)^{-1}((s, +\infty]). \end{aligned}$$

Theorem 3.6.6. *Let $a \in A_{sa}$, $r, s \in \mathbb{R}$, $r < s$, and $\epsilon > 0$. Then*

$$\delta(a)^{-1}(\hat{\Delta}) \subseteq [a < r] \cap [a > s] \subseteq \delta(a)^{-1}(\hat{\Delta} + \epsilon).$$

For the proof we start by considering outer and inner daseinisation separately. First, recall that the elementary propositions of the contravariant model are given by

$$[a \in \Delta] = \{(C, \lambda) \in \Sigma \mid \langle \lambda, \delta^o(\chi_\Delta(a))_C \rangle = 1\}, \tag{3.35}$$

where $a \in A_{sa}$, $\Delta \in \mathcal{O}\mathbb{R}$, and $\chi_\Delta(a)$ is the spectral projection associated to this pair. For half-intervals we use the notation $[a < r] := [a \in (-\infty, r)]$ and $[a > s] := [a \in (s, +\infty)]$.

Lemma 3.6.7. *For $r \in \mathbb{R}$, $a \in A_{sa}$, let $\delta^i(a) : \Sigma^{\downarrow}_A \to \mathcal{C}^{\downarrow}_A \times \mathbb{R}_u$ be the corresponding (continuous) inner daseinised map. If we identify the half-interval $(-\infty, r)$ of the real numbers $\mathbb{R}$ with the open*

$$\mathcal{C}^{\downarrow}_A \times [-\infty, r) = \{(C, x) \in \mathcal{C}_A \times \mathbb{R}_u \mid x < r\} \in \mathcal{O}(\mathcal{C}^{\downarrow}_A \times \mathbb{R}_u),$$

and write this open as $[-\infty, r)$ (with some abuse of notation), then, for each $\epsilon > 0$,

$$\delta^i(a)^{-1}([-\infty, r)) \subseteq [a < r] \subseteq \delta^i(a)^{-1}([\infty, r + \epsilon)). \tag{3.36}$$

Proof. Assume that $\lambda \in [a < r]_C$. By definition, this is equivalent to

$$\langle \lambda, \delta^o(\chi_{[-\infty,r)}(a))_C \rangle = 1. \tag{3.37}$$

By definition of outer daseinisation of projections, this is in turn equivalent to

$$\forall p \in \mathcal{P}(C), \quad p \geq \chi_{[-\infty,r)}(a) \quad \to \quad \langle \lambda, p \rangle = 1.$$

Switching to $\neg p = 1 - p$, this is equivalent to

$$\forall p \in \mathcal{P}(C) \quad p \leq 1 - \chi_{[-\infty,r)}(a) \quad \to \quad \langle \lambda, p \rangle = 0.$$

For any $x \geq r$,

$$1 - \chi_{[-\infty,x)}(a) \leq 1 - \chi_{[-\infty,r)}(a).$$

Assume that for some projection $p \in \mathcal{P}(C)$, one has $p \leq 1 - \chi_{[-\infty,x)}(a)$. Then $p \leq 1 - \chi_{[-\infty,r)}(a)$, and by assumption $\langle \lambda, p \rangle = 0$. We conclude that if $x \in \mathbb{R}$ is an element of the set

$$\{y \in \mathbb{R} \mid \exists\, p \in \mathcal{P}(C) \quad p \leq 1 - \chi_{[-\infty,y)}(a) \ \wedge \ \langle \lambda, p \rangle = 1\},$$

then $x < r$. As $\langle \lambda, \delta^i(a)_C \rangle$ is the supremum of such $x \in \mathbb{R}$ by (3.33), we know that $\langle \lambda, \delta^i(a)_C \rangle \leq r$. By definition, for each $\epsilon > 0$, $\lambda \in \delta^i(a)^{-1}([-\infty, r + \epsilon))_C$. Hence we have shown that

$$[a < r] \subseteq \delta^i(a)^{-1}([-\infty, r + \epsilon)).$$

Next, assume that $\lambda \in \delta^i(a)^{-1}([-\infty, r))_C$. From (3.33) we deduce that

$$\forall p \in \mathcal{P}(C) \quad p \geq \chi_{[-\infty,x)}(a) \ \wedge \ \langle \lambda, p \rangle = 0 \quad \to x < r.$$

If $p \geq \chi_{[\infty,r)}(a)$, then $\langle \lambda, p \rangle = 1$. This is equivalent to (3.37). We conclude that $\lambda \in [a < r]_C$, completing the proof of the lemma. $\square$

Lemma 3.6.8. *For $s \in \mathbb{R}$, $a \in A_{sa}$, let $\delta^o(a) : \Sigma^{\downarrow}_A \to \mathcal{C}^{\downarrow}_A \times \mathbb{R}_l$ be the corresponding (continuous) outer daseinised map. If we identify the half-interval $(s, +\infty)$ of the real numbers $\mathbb{R}$ with the open*

$$\mathcal{C}^{\downarrow}_A \times (s, +\infty] = \{(C, x) \in \mathcal{C}_A \times \mathbb{R}_l \mid x > s\} \in \mathcal{O}(\mathcal{C}^{\downarrow}_A \times \mathbb{R}_l),$$

and write this open as $(s, +\infty]$ (with some abuse of notation). Then, for each $\epsilon > 0$,

$$\delta^o(a)^{-1}((s, +\infty]) \subseteq [a > s] \subseteq \delta^o(a)^{-1}((s - \epsilon, +\infty]). \tag{3.38}$$

Proof. Assume that $\lambda \in \delta^o(a)^{-1}((s, +\infty])_C$, implying $\langle \lambda, \delta^o(a)_C \rangle > s$. Define

$$\epsilon_0 = \frac{1}{2}(\langle \lambda, \delta^o(a)_C \rangle - s).$$

Let $p \in \mathcal{P}(C)$ satisfy $p \geq \chi_{(s,+\infty]}(a)$. Using

$$p \geq \chi_{(s,+\infty]}(a) \geq 1 - \chi_{[-\infty, s+\epsilon_0)}(a),$$

as well as (3.34), which tells us that

$$\inf\{x \in \mathbb{R} \mid \exists p \in \mathcal{P}(C) \;\; p \geq 1 - \chi_{[-\infty,x)}(a) \;\wedge\; \langle \lambda, p \rangle = 0\} > s + \epsilon_0,$$

we conclude that $\langle \lambda, p \rangle = 1$. This implies that $\lambda \in [a > s]_C$, proving the left inclusion of (3.38). For the right inclusion, assume that $\lambda \in [a > s]_C$. Let $x < s$, and assume that $p \geq 1 - \chi_{[-\infty,x)}(a)$. As $x < s$, we know that

$$p \geq 1 - \chi_{[-\infty,x)}(a) \geq \chi_{(s,+\infty]}(a).$$

By assumption, this implies that $\langle \lambda, p \rangle = 1$. This, in turn, implies

$$\inf\{x \in \mathbb{R} \mid \exists p \in \mathcal{P}(C) \;\; p \geq 1 - \chi_{[-\infty,x)}(a) \;\wedge\; \langle \lambda, p \rangle = 0\} \geq s.$$

By (3.34) $\langle \lambda, \delta^o(a) \rangle \geq s$, completing the proof of the lemma. □

Theorem 3.6.6 follows from the previous two lemmas. Note that the 'contravariant' inclusions

$$\delta(a)^{-1}(\hat{\Delta}) \subseteq [a < r] \cap [a > s] \subseteq \delta(a)^{-1}(\hat{\Delta} + \epsilon).$$

take the same shape as for the covariant version, given by Theorem 3.6.5, especially when we note that in the covariant version one has

$$[a \in (s, r)] = [a < r] \cap [a > s];$$

this equality is consequence of the fact that inner daseinisation of projections preserves meets. Outer daseinisation does not preserves meets though (it does preserve joins), so in the contravariant case we only have the inclusion

$$[a \in (s, r)] \subseteq [a < r] \cap [a > s], \tag{3.39}$$

where typically the inequality is strict.

Theorem 3.6.6 tells us that at least in some cases, such as $\underline{[a < r]}$ and $\underline{[a > s]}$, where a has a discrete spectrum, elementary propositions can be obtained internally as $\underline{\delta(a)}^{-1}(\underline{\Delta})$, where $\underline{\Delta}$ is suitably chosen open of the value space.

4

States and Truth Values

In this chapter, we consider states and the truth values obtained by these states when combined with elementary propositions. The first two sections review states in the contravariant and covariant approaches, respectively. Motivated by the covariant states, in Section 4.3 contravariant states are described as internal probability valuations on the spectral presheaf. Note that if we think of the spectral presheaf or spectral locale as a (quantum) phase space, the description of states as probability valuations on this phase space reminds of classical physics. The last two sections investigate the Heyting algebra structure of the frames $\mathcal{O}\Sigma_{\downarrow}$ and $\mathcal{O}\Sigma_{\uparrow}$ by means of the truth values that the opens produce when they are paired with states.

4.1 Contravariant Approach

We review various descriptions of states in the contravariant approach, as well as the truth values these states yield when combined with elementary propositions[1].

We start with state-related objects in the contravariant approach. We first discuss pseudo-states and truth objects. After that we treat the more recent measures introduced by Döring [30, 31]. By the Kochen–Specker theorem, the spectral presheaf typically does not have global

[1]None of the material presented in this subsection is new, but it has been included for completeness and its role of forming a complement to our description of the covariant approach.

points, so that global points $\underline{1} \to \underline{\Sigma}$ do not give a fruitful concept of state. Let $A = \mathcal{B}(\mathcal{H})$, and let $|\psi\rangle \in \mathcal{H}$ be a unit vector. In the contravariant approach one associates two closely related objects to the vector $|\psi\rangle$, namely the truth object $\underline{\mathbb{T}}^{|\psi\rangle}$ and the pseudo-state $\underline{\mathfrak{w}}^{|\psi\rangle}$. A more complete discussion of these objects may be found in [37, Section 6], [31].
In order to define $\underline{\mathbb{T}}^{|\psi\rangle}$ it is convenient to first introduce the so-called **outer presheaf** $\underline{O} : \mathcal{C}^{op} \to \mathbf{Set}$. For $C \in \mathcal{C}$ we have $\underline{O}(C) = \mathcal{P}(C)$, i.e. the set of projection operators in C. If $C \subseteq C'$ we have $\underline{O}(i) : \mathcal{P}(C') \to \mathcal{P}(C)$ given by $P \mapsto \delta^o(p)_C$. Each projection operator $p \in \mathcal{P}(\mathcal{H})$ defines a global point $\underline{1} \to \underline{O}$ of the outer presheaf by outer daseinisation $\underline{\delta}^o(p)$, which at stage C picks the projection operator $\delta^o(p)_C$. The truth object $\underline{\mathbb{T}}^{|\psi\rangle}$ is a subobject of the outer presheaf, given by

$$\underline{\mathbb{T}}_C^{|\psi\rangle} = \{p \in \mathcal{P}(C) \mid \langle\psi|p|\psi\rangle = 1\} = \{p \in \mathcal{P}(C) \mid p \geq |\psi\rangle\langle\psi|\}.$$

It is shown in Subsection 6.5.2 of [37] that there is a monic arrow $\underline{O} \rightarrowtail \mathcal{O}_{cl}\underline{\Sigma}$. Hence, the truth object $\underline{\mathbb{T}}^{|\psi\rangle}$ can be seen as a subobject of $\mathcal{O}_{cl}\underline{\Sigma}$, or, equivalently, as a point of $P\mathcal{O}_{cl}\underline{\Sigma}$. The truth object has been defined for a vector state $|\psi\rangle$, but there is also a generalization for mixed states, which we are going to discuss at the end of this subsection.
The point $\underline{\delta}^o(p) : \underline{1} \to \underline{O}$ can also be viewed as a clopen subobject of the spectral presheaf, as we have seen in Subsection 3.1. Thus it represents a proposition in the contravariant approach. Together with the truth object $\underline{\mathbb{T}}^{|\psi\rangle}$, it forms the sentence $\underline{\delta}^o(p) \in \underline{\mathbb{T}}^{|\psi\rangle}$ in the language of $[\mathcal{C}^{op}, \mathbf{Set}]$. A sentence is represented by a subobject of the terminal object $\underline{1}$ and hence is equivalent to a truth value $\underline{1} \to \underline{\Omega}$. Recall that $\underline{\Omega}(C)$ is the set of sieves on C. At context C, the truth value of $\underline{\delta}^o(p) \in \underline{\mathbb{T}}^{|\psi\rangle}$ is given by

$$\nu(\underline{\delta}^0(p) \in \underline{\mathbb{T}}^{|\psi\rangle})_C = \{C' \in (\downarrow C) \mid \langle\psi|\delta^o(p)_{C'}|\psi\rangle = 1\}.$$

The second state-related object is the **pseudo-state** $\underline{\mathfrak{w}}^{|\psi\rangle}$. This is a subobject of the spectral presheaf, defined by

$$\underline{\mathfrak{w}}_C^{|\psi\rangle} = \underline{\delta}^o(|\psi\rangle\langle\psi|)_C = \{\lambda \in \Sigma_C \mid \lambda(\delta^o(|\psi\rangle\langle\psi|)_C) = 1\}, \qquad (4.1)$$

where $|\psi\rangle\langle\psi|$ denotes the projection onto the ray $\mathbb{C}|\psi\rangle$. Once again, consider $\underline{\delta}^o(p)_C$. Rather than as a point of the outer presheaf, it is now seen as a subobject of the spectral presheaf, as in (3.2). We form the sentence $\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta}^o(p)$, whose associated truth value is

$$\nu(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta}^o(p))_C = \{C' \in (\downarrow C) \mid \delta^o(|\psi\rangle\langle\psi|)_{C'} \leq \delta^o(p)_{C'}\}.$$

Proposition 4.1.1. *(Section 6.4.2 [37]) Let $A = \mathcal{B}(\mathcal{H})$, and let $|\psi\rangle \in \mathcal{H}$ be a unit vector. Then,*

$$\forall C \in \mathcal{C} \quad \nu(\underline{\delta}^0(p) \in \underline{\mathbb{T}}^{|\psi\rangle})_C = \nu(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta}^o(p))_C.$$

Proof. The observation to make is that $\langle\psi|\delta^o(p)_C|\psi\rangle = 1$ iff $|\psi\rangle\langle\psi| \leq_s \delta^o(p)_C$, [37]. Suppose that $C' \in \nu(\underline{\delta}^0(p) \in \underline{\mathbb{T}}^{|\psi\rangle})_C$, which is equivalent to $C' \Vdash \underline{\delta}^0(p) \in \underline{\mathbb{T}}^{|\psi\rangle}$. This implies that $\delta^o(p)_{C'} \geq_s |\psi\rangle\langle\psi|$. By definition, $\delta^0(|\psi\rangle\langle\psi|)_{C'}$ is the smallest projection operator in C' that is greater than $|\psi\rangle\langle\psi|$. It follows that $\delta^o(p)_{C'} \geq \delta^0(|\psi\rangle\langle\psi|)_{C'}$. Thus $C' \in \nu(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta}^o(p))_C$. Conversely, assume that $C' \in \nu(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta}^o(p))_C$, which is equivalent to $C' \Vdash \underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta}^o(p)$. Then $\delta^o(p)_{C'} \geq \delta^0(|\psi\rangle\langle\psi|)_{C'} \geq |\psi\rangle\langle\psi|$. It is immediate that $C' \in \nu(\underline{\delta}^0(p) \in \underline{\mathbb{T}}^{|\psi\rangle})_C$. □

In [30] and [31], Döring uses measures of closed open subobjects of the spectral presheaf in order to describe states. This description of states has the advantage that it generalises to mixed states.

Definition 4.1.2. *A measure on the spectral presheaf is a function $\mu : \mathcal{O}_{cl}\underline{\Sigma} \to OR(\mathcal{C}, [0,1])$, such that for every $C \in \mathcal{C}$ and $\forall \underline{S}_1, \underline{S}_2 \in \mathcal{O}_{cl}\underline{\Sigma}$,*

- $\mu(\underline{\Sigma})(C) = 1$;
- $\mu(\underline{S}_1)(C) + \mu(\underline{S}_2)(C) = \mu(\underline{S}_1 \wedge \underline{S}_2)(C) + \mu(\underline{S}_1 \vee \underline{S}_2)(C)$.
- *For any fixed $C \in \mathcal{C}$, the function $\mu^C := \mu(-)(C) : \mathcal{O}_{cl}\underline{\Sigma} \to [0,1]$, $\underline{S} \mapsto \mu(\underline{S})(C)$ depends only on $\underline{S}_C$. We write $\mu^C(\underline{S}) = \mu^C(\underline{S}_C)$ with slight abuse of notation.*

Any state, in the guise of a normalised positive linear functional $\rho : A \to \mathbb{C}$, defines a measure by $\mu_\rho(\underline{S})(C) = \rho(p_{\underline{S}(C)})$, where $p_{\underline{S}(C)}$ denotes the projection corresponding to the closed open subset $\underline{S}(C)$ of Σ_C. In order to see that these measures in fact generalize pseudo-states and truth objects, we use the internal structure of the topos. For example, the lower reals $\underline{[0,1]}_l$ in the topos $[\mathcal{C}^{op}, \mathbf{Set}]$ are given by the presheaf $\underline{[0,1]}_l(C) = OR(\downarrow C, [0,1])$ (in order to show this, recall that $[\mathcal{C}^{op}, \mathbf{Set}]$ is equivalent to the topos of sheaves on $\mathcal{C}_\downarrow$). Any measure as in Definition 4.1.2 defines a natural transformation

$$\underline{\mu} : \mathcal{O}\underline{\Sigma}_{cl} \to \underline{[0,1]}_l, \quad (\underline{\mu})_C(\underline{S}) = \mu(\underline{S})|_{\downarrow C}, \tag{4.2}$$

where $\mathcal{O}\underline{\Sigma}_{cl}$ is the presheaf $\mathcal{O}\underline{\Sigma}_{cl}(C) = \mathcal{O}_{cl}\underline{\Sigma}|_{\downarrow C}$, which is the set of clopen subobjects of the spectral presheaf $\underline{\Sigma}$, restricted to the contexts $(\downarrow C)$.

The function $\underline{1}_l : \mathcal{C} \to [0,1]$ that is constantly 1 can be seen as a global point $\underline{1}_l : \underline{1} \to \underline{[0,1]}_l$. A measure $\underline{\mu}$ as in (4.2) together with a subobject $\underline{S} \in \mathrm{Sub}_{cl}\underline{\Sigma}$, define (by means of the language of the topos) a truth value $[\underline{\mu}(\underline{S}) = \underline{1}_l] : \underline{1} \to \underline{\Omega}$. If $\underline{\mu}$ comes from a vector state ψ, then

$$(\underline{\mu}_{\psi})_C(\underline{S})(C) = \langle \psi | p_{\underline{S}(C)} | \psi \rangle.$$

Taking $\underline{S} = \underline{\delta}^o(p)$, we then find

$$\nu(\underline{\mu}_{\psi}(\underline{\delta}^o(p)) = \underline{1}_l)_C = \{C' \in (\downarrow C) \mid \langle \psi | \delta^o(p)_C | \psi \rangle = 1\}.$$

The measures of Definition 4.1.2 that come from vector states yield exactly the same truth values as pseudo-states and truth objects paired with propositions. In this sense, the measures of Definition 4.1.2 are a generalization of both pseudo-states and truth objects.

4.2 Covariant Approach

We show that internal probability valuations on the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$ correspond to quasi-states on A.

Next, we consider covariant states and how these combine with elementary propositions. We only present a short discussion of the subject. A more complete treatment can be found in [48, Section 4].
In the covariant approach, a state is described by a probability valuation on the spectrum $\underline{\Sigma}_{\underline{A}}$.

Definition 4.2.1. *Let X be a locale in any topos $\mathcal{E}$, and let $[0,1]_l$ be the set of lower reals between 0 and 1. A* ***probability valuation*** *on X is a function $\mu : \mathcal{O}X \to [0,1]_l$ satisfying the following conditions. Let $U, V \in \mathcal{O}X$ and $\{U_\lambda\}_{\lambda \in I} \subseteq \mathcal{O}X$ be a directed subset. Then:*

- *μ is monotone; i.e., if $U \leq V$, then $\mu(U) \leq \mu(V)$.*
- *$\mu(\bot) = 0$, $\mu(\top) = 1$, where $\bot$ and $\top$ are respectively the bottom and top element of $\mathcal{O}X$.*
- *$\mu(U) + \mu(V) = \mu(U \wedge V) + \mu(U \vee V)$.*
- *$\mu\left(\bigvee_{\lambda \in I} U_\lambda\right) = \bigvee_{\lambda \in I} \mu(U_\lambda)$.*

We can think of a probability valuation on a locale (or in particular on a topological space) as a probability measure that is defined only on the opens instead of on the Borel algebra generated by the opens.
Assume for convenience that the C*–algebra A is a subalgebra of some $\mathcal{B}(\mathcal{H})$. A unit vector $|\psi\rangle \in \mathcal{H}$ defines a state on A (in the sense of a positive normalized linear functional) by $\rho_\psi : A \to \mathbb{C}$, $\rho_\psi(a) = \langle\psi|a|\psi\rangle$. A state $\rho_\psi : A \to \mathbb{C}$ defines a probability integral $I_\psi : \underline{A}_{sa} \to \underline{\mathbb{R}}$, on the Bohrification $\underline{A}$ [48, Definition 10, Theorem 14]. By the generalized Riesz-Markov Theorem, [24, 48], probability integrals $I : \underline{A}_{sa} \to \underline{\mathbb{R}}$ correspond to probability valuations $\underline{\mu} : \mathcal{O}\underline{\Sigma}_{\underline{A}} \to \underline{[0,1]}_l$. In this way, any unit vector $|\psi\rangle \in \mathcal{H}$ gives rise to a probability valuation $\underline{\mu}_\psi$ on $\underline{\Sigma}_{\underline{A}}$.
Before we explore what probability valuations on $\underline{\Sigma}_{\underline{A}}$ look like externally, we first explain how these valuations combine with propositions, so as to give truth values. As before, identify the internal spectrum $\underline{\Sigma}_{\underline{A}}$ with the locale $\underline{\Sigma}_\uparrow$. The lower reals $\underline{[0,1]}_l$ in $[\mathcal{C}, \mathbf{Set}]$ are given by $\underline{[0,1]}_l(C) = L(\uparrow C, [0,1])$, [20, Appendix A.3], where the right-hand side stands for the set of lower semicontinuous functions $(\uparrow C) \to [0,1]$. A function $f : (\uparrow C) \to [0,1]$ is lower semicontinuous iff it is order-preserving: if $C \subseteq C'$, then $f(C) \leq f(C')$.
Let $1_C : \uparrow C \to [0,1]$ denote the function that is constantly 1. Define

$$\underline{1}_l : \mathcal{O}\underline{\Sigma}_\uparrow \to \underline{[0,1]}_l, \quad (\underline{1}_{l,C})(U) = 1_C.$$

Let $\underline{\mu} : \mathcal{O}\underline{\Sigma}_\uparrow \to \underline{[0,1]}_l$ be a probability valuation on $\underline{\Sigma}_\uparrow$. Using the internal language of $[\mathcal{C}, \mathbf{Set}]$, we form the arrow

$$[\underline{\mu} = \underline{1}_l] : \mathcal{O}\underline{\Sigma}_\uparrow \to \underline{\Omega}.$$

Any open $U \in \mathcal{O}\Sigma_\uparrow$ yields a point $\underline{U} : \underline{1} \to \mathcal{O}\underline{\Sigma}_\uparrow$. For any probability valuation $\underline{\mu}$ on the spectrum of $\underline{A}$ and any proposition $U \in \mathcal{O}\Sigma_\uparrow$, we obtain a truth value

$$[\underline{\mu}(\underline{U}) = \underline{1}_l] = [\underline{\mu} = \underline{1}_l] \circ \underline{U} : \underline{1} \to \underline{\Omega}.$$

We return to describing the probability valuations of the covariant approach. By the generalised (topos valid) Riesz–Markov theorem these correspond bijectively to quasi-states [24, 48], defined as follows

Definition 4.2.2. *A function $\psi : A \to \mathbb{C}$ is a **quasi-state** if it satisfies*

- *ψ is positive; for each $a \in A$, $\psi(a^*a) \geq 0$.*

- *ψ normalised; $\psi(1) = 1$.*
- *ψ is quasi-linear; for each $C \in \mathcal{C}_A$, $\psi|_C$ is linear.*
- *If $a, b \in A_{sa}$, then $\psi(a + ib) = \psi(a) + i\psi(b)$.*

We demonstrate that the quasi-states indeed correspond to probability valuations on the Gelfand spectrum $\underline{\Sigma}_{\underline{A}}$. Instead of using the topos valid version of the Riesz–Markov theorem, we use presheaf semantics and the Riesz–Markov theorem of **Set**. This proof seems more helpful when, in the next section, we ask if in the contravariant approach quasi-states on A correspond to probability valuations on $\underline{\Sigma}_{\downarrow}$.

Proposition 4.2.3. *In the covariant approach, probability valuations on $\underline{\Sigma}_{\underline{A}}$ correspond bijectively to quasi-states on A.*

Proof. If $\underline{\mu}$ is a probability valuation on the spectral locale, then for each $C \in \mathcal{C}$ it gives a function

$$\mu_C : \mathcal{O}\Sigma^{\uparrow}_{C} \to \mathrm{OP}(\uparrow C, [0, 1]). \tag{4.3}$$

If C is a maximal context, then (4.3) can be seen as a function

$$\mu_C : \mathcal{O}\Sigma_C \to [0, 1].$$

As $\underline{\mu}$ is a probability valuation, each such μ_C also satisfies the conditions for a probability valuation of Σ_C. This means that μ_C corresponds to a unique state $\psi_C : C \to \mathbb{C}$. These local states combine to a single quasi-state iff, given $D \subseteq C_1, C_2$, $\psi_{C_1}|_D = \psi_{C_2}|_D$. We proceed to show that this is indeed the case. If $D \subseteq C$, and $U \in \mathcal{O}\Sigma^{\uparrow}_{D}$, then

$$\mu_D(U)(C) = \mu_C(U \cap \Sigma^{\uparrow}_{C})(C). \tag{4.4}$$

Let $p \in D$ be a projection operator, corresponding to the closed open subset $S \subseteq \Sigma_D$. Define $(\uparrow S)$, an open of $\Sigma_{\uparrow}$, by taking $(\uparrow S)_C = \rho^{-1}_{CD}(S)$ if $C \supseteq D$ and $(\uparrow S)_C = \emptyset$ if $C \not\supseteq D$. The set-theoretic complement S^c of S in Σ_D is open and closed in Σ_D and also defines an open $\uparrow S^c$. Note that $(\uparrow S) \cap (\uparrow S^c) = \emptyset$ and $(\uparrow S) \cup (\uparrow S^c) = \Sigma^{\uparrow}_{D}$. This implies that for any $D \subseteq C$,

$$\mu_D(\uparrow S)(D) + \mu_D(\uparrow S^c)(D) = 1 = \mu_D(\uparrow S)(C) + \mu_D(\uparrow S^c)(C). \tag{4.5}$$

As $\mu_D(\uparrow S)$ and $\mu_D(\uparrow S^c)$ are both order preserving with respect to $\uparrow D$, we conclude

$$\mu_D(\uparrow S)(C) = \mu_D(\uparrow S)(D). \tag{4.6}$$

Return to the situation $D \subseteq C_1, C_2$ with C_i maximal, and recall that $p \in D$ is the projection operator corresponding to S in Σ_D. Consequently, p corresponds to $\rho^{-1}_{C_iD}(S)$ in Σ_{C_i}. We now compute

$$\begin{aligned}
\psi_{C_2}(p) &= \mu_{C_2}(\rho^{-1}_{C_2D}(S))(C_2) \\
&= \mu_D(\uparrow S)(C_2) \\
&= \mu_D(\uparrow S)(D) \\
&= \mu_D(\uparrow S)(C_1) \\
&= \mu_{C_1}(\rho^{-1}_{C_1D}(S))(C_1) = \psi_{C_1}(p),
\end{aligned}$$

where we used (4.4) for the second and fifth equalities, and (4.6) for the third and fourth equalities. This proves that $\psi_{C_2}|_D = \psi_{C_1}|_D$, and demonstrates how internal valuations on the spectral locale can be identified with quasi-states. □

4.3 Contravariant States as Valuations

In this subsection we investigate the connection between states, and probability valuations on the spectral presheaf $\underline{\Sigma}_A$, viewed internally to $[\mathcal{C}^{op}, \mathbf{Set}]$ as an internal topological space.

In the covariant approach states are described as internal probability valuations on the spectral locale $\underline{\Sigma}_{\underline{A}}$. Probability valuations on $\underline{\Sigma}_{\underline{A}}$ correspond bijectively with quasi-states on A [48]. Is it true for the contravariant approach that internal valuations on the spectral presheaf $\underline{\Sigma}_{\downarrow}$ correspond bijectively to quasi-states on A? We should be careful not to confuse internal probability valuations with the measures of Definition 4.1.2. For these measures it is not hard to see that they correspond bijectively to quasi-states. A measure μ in the sense of Definition 4.1.2 gives, for each $C \in \mathcal{C}$, a finitely additive measure $\mu^C : \mathcal{O}_{cl}\Sigma_C \to [0,1]$. Such a local measure can be extended to a unique state $\psi_C : C \to \mathbb{C}$. These local states $(\psi_C)_{C\in\mathcal{C}}$ combine to a single quasi-state ψ iff whenever $D \subseteq C$ in $\mathcal{C}$, $\psi_D = \psi_C|_D$. If $p \in \mathcal{P}(D)$ is any projection, then $\psi_D(p) \geq \psi_C(p)$ and $\psi_D(1-p) \geq \psi_C(1-p)$ because μ is order-reversing. But in addition:

$$\psi_D(p) + \psi_D(1-p) = 1 = \psi_C(p) + \psi_C(1-p).$$

We conclude that $\psi_C(p) = \psi_D(p)$ for all projection operators p in D. As we are working with von Neumann algebras, this is enough to conclude that $\psi_C|_D = \psi_D$. As a consequence, the measures of Definition 4.1.2 correspond to quasi-states. We return to internal probability valuations on the spectral presheaf.

Lemma 4.3.1. *A quasi-state $\psi : A \to \mathbb{C}$ defines a probability valuation μ_ψ on the spectral presheaf $\underline{\Sigma}_A$.*

Proof. Using presheaf semantics we can describe what a probability valuation on $\underline{\Sigma}_A$ comes down to externally for the topos $[\mathcal{C}_A^{op}, \mathbf{Set}]$. Recall that

$$\underline{[0,1]}_l(C) \cong \mathrm{OR}(\downarrow C, [0,1]).$$

An internal probability valuation $\underline{\mu} : \mathcal{O}\underline{\Sigma}_A \to \underline{[0,1]}_l$ is externally described by giving, for each $C \in \mathcal{C}$, a function

$$\mu_C : \mathcal{O}\Sigma_C^{\downarrow} \to \mathrm{OR}(\downarrow C, [0,1]),$$

such that, if $C \in \mathcal{C}$, and $U \in \mathcal{O}\Sigma_C^{\downarrow}$, then

$$\forall D \in (\downarrow C) \;\; \mu_C(U)(D) = \mu_D(U \cap \Sigma_D^{\downarrow})(D).$$

We use the notation $\Sigma_C^{\downarrow}$ to denote $\coprod_{D\in(\downarrow C)} \Sigma_D$, equipped with the relative topology of $\Sigma_A^{\downarrow}$. The four axioms of Definition 4.2.1 translate externally to the following four conditions. For each $C \in \mathcal{C}$, and $D \in (\downarrow C)$,

- If $U \subseteq V$ in $\mathcal{O}\Sigma_C^{\downarrow}$, then $\mu_C(U)(D) \leq \mu_C(V)(D)$.

- $\mu_C(\Sigma_C^{\downarrow})(D) = 1$ and $\mu_C(\emptyset)(D) = 0$.

- If $U, V \in \mathcal{O}\Sigma_C^{\downarrow}$, then

$$\mu_C(U)(D) + \mu_C(V)(D) = \mu_C(U \cap V)(D) + \mu_C(U \cup V)(D).$$

- If $\{U_\lambda\}_{\lambda\in\Lambda}$ is a directed subset of $\mathcal{O}\Sigma_C^{\downarrow}$, then

$$\mu_C\left(\bigcup_\lambda U_\lambda\right)(D) = \sup_\lambda\left(\mu_C(U_\lambda)(D)\right).$$

Let ψ be a positive normalised linear functional on A, then ψ defines such a valuation $\{\mu_C\}_{C\in\mathcal{C}}$ as follows. Restricting ψ to $C \in \mathcal{C}$ gives a positive normalised linear functional $\psi|_C : C \to \mathbb{C}$. By the Riesz–Markov theorem this is equivalent to a probability valuation $\mu_\psi^{(C)} : \mathcal{O}\Sigma_C \to [0,1]$. Define

$$(\mu_\psi)_C : \mathcal{O}\Sigma_C^{\downarrow} \to \mathrm{OR}(\downarrow C, [0,1]), \quad (\mu_\psi)_C(U)(D) = \mu_\psi^{(D)}(U_D),$$

where $U_D = U \cap \Sigma_D$. It is straightforward to verify that this definition satisfies all conditions required to define an internal probability valuation, and we leave this to the reader. $\square$

Are there any other internal probability valuation than those arising from quasi-states? In the covariant topos model, the probability valuations on the spectral locale $\underline{\Sigma}_A$ correspond bijectively with quasi-states on A. In particular, for von Neumann algebras without a type I_2 summand, probability valuations correspond to the states on A. This correspondence follows straight from the topos-valid version of the Riesz–Markov Theorem. However, in the contravariant case this theorem cannot be used. In Proposition 4.2.3 we derived the correspondence of the covariant case using presheaf semantics. Note, however, that this proof cannot be directly applied to the contravariant model for probability valuations on the spectral presheaf. One obstacle is that restricting the valuation to a maximal context C does not yield a probability valuation on Σ_C (it is defined on $\mathcal{O}\Sigma_C^{\downarrow}$). Another obstacle is that for any given closed open S of Σ_D, in general $(\downarrow S) \cap (\downarrow S^c) \neq \emptyset$. As far as the author knows, it is an open question whether there exist probability valuations on the spectral presheaf that do not arise from quasi-states[2].

A state ψ induces a probability valuation $\mu_\psi : \mathcal{O}\underline{\Sigma}_A \to \underline{[0,1]}_l$ and a property P is represented by an open of the space $\underline{\Sigma}_A$, $\underline{P} : \underline{1} \to \mathcal{O}\underline{\Sigma}_A$. For any $x \in [0,1]$, we can consider the proposition $\mu_\psi(\underline{P}) \geq \underline{x}$. Here, $\underline{x} : \underline{1} \to \underline{[0,1]}_l$ is the constant function

$$\underline{x}_C : (\downarrow C) \to [0,1], \quad \underline{x}_C(D) = x.$$

[2]If we represent states as internal probability valuations, we obtain the same truth values as normally used in the contravariant approach. One of the goals of the neorealism program was to get rid of probabilities altogether, by replacing them by generalised topos-theoretic truth values. We will not follow this interesting idea. More information can be found in [32].

Note that in this way we represent x not only as a lower real number, but as a Dedekind real number as well, since the function $\underline{x}$ is constant. The truth value of this proposition is given by the sieve

$$[\mu_\psi(\underline{P}) \geq \underline{x}] = \{C \in \mathcal{C} \mid \psi(p_C) \geq x\}. \tag{4.7}$$

In particular, for $x = 1$ we get the same truth value as for the measures of Definition 4.1.2.

4.4 Contravariant Quantum Logic

We describe the Heyting algebra structure of $\mathcal{O}\Sigma_\downarrow$ by means of the truth values the opens produce when these are paired with states.

In the two topos models, properties of the system under investigation, such as $[a \in \Delta]$, are represented by opens of $\mathcal{O}\Sigma_\downarrow$, or $\mathcal{O}\Sigma_\uparrow$. These two frames can be viewed as complete Heyting algebras. With this Heyting algebra structure, both $\mathcal{O}\Sigma_\downarrow$ and $\mathcal{O}\Sigma_\uparrow$ produce alternatives to the quantum logic of Birkhoff and von Neumann. At first glance these alternatives offered by the topos approaches look promising. In orthodox quantum logic, the lattice is non-distributive, making it hard to interpret $\wedge$ as *and*, and $\vee$ as *or*. Heyting algebras, on the other hand, are always distributive. Another point is that orthodox quantum logic lacks a satisfactory implication operator, whereas a Heyting algebra has an implication by definition. In this respect the logics produced by $\mathcal{O}\Sigma_\downarrow$ and $\mathcal{O}\Sigma_\uparrow$ look good. However, we should realise that it is not *a priori* clear that the operations of these Heyting algebras $(\wedge, \vee, \neg, \rightarrow)$, have any physical significance. Consider the following simple example, which shows that recovering distributivity is not an achievement by itself.
Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{PH}$ be the power set of this space. Just as any power object in any topos defines a complete Heyting algebra, $\mathcal{PH}$ defines a complete Heyting algebra, when ordered by inclusion. As we are working in **Set** it is even a complete Boolean algebra. Consider a proposition $[a \in \Delta]$. We associate to this proposition a projection operator $\chi_\Delta(a)$. Such a projection operator can be identified with a subset of $\mathcal{H}$ (which happens to be a closed subspace). In this way we represent elementary propositions $[a \in \Delta]$ as elements of a complete Boolean algebra, but the algebra $\mathcal{PH}$ is hardly an interesting quantum logic.
We don't expect the topos models to perform as badly as the logic $\mathcal{PH}$, which completely ignores the linear structure of quantum theory. Even so,

we need to investigate the Heyting algebras of the topos models. Below, we try to understand the Heyting algebra structures of $\mathcal{O}\Sigma_\downarrow$, and $\mathcal{O}\Sigma_\uparrow$ by looking at the truth values these operations produce, when combined with states.
In this subsection and the next one, we *assume that A is of the form $M_n(\mathbb{C})$*. This will make it easier to deal with the negation operation explicitly. It also implies that $\mathcal{O}\Sigma_\downarrow$ coincides with $\mathcal{O}_{cl}(\underline{\Sigma})$, the complete Heyting algebra typically considered in the contravariant model.
We start with the Heyting algebra $\mathcal{O}\Sigma_\downarrow$ of the contravariant model, and treat the covariant version $\mathcal{O}\Sigma_\uparrow$ in the next section.

4.4.1 Single Proposition

Consider an elementary proposition $[a \in \Delta]$. We represent such a proposition as an open of $\mathcal{O}\Sigma_\downarrow$ by taking the outer daseinisation of the spectral projection $\chi_\Delta(a)$. If Δ is an open half-interval, then by Theorem 3.6.6, this $[a \in \Delta]$ is equal to $\delta(a)^{-1}(\hat{\Delta})$, for a suitably chosen $\hat{\Delta}$.
Let ψ be a state on A, and let $\underline{\mu} : \mathcal{O}\underline{\Sigma} \to \underline{[0,1]}_l$ be the internal probability valuations associated to it as in Subsection 4.3. The elementary proposition $[a \in \Delta]$ defines an open $\underline{[a \in \Delta]} : \underline{1} \to \mathcal{O}\underline{\Sigma}$. Consider the internal proposition (in the sense of a closed formula)

$$\underline{\mu}(\underline{[a \in \Delta]}) = \underline{1}.$$

By (4.7), this proposition is true at stage C iff

$$\psi(\delta^o(\chi_\Delta(a))_C) = 1,$$

which, by spelling out the definition of outer daseinisation of projections, is

$$\psi\left(\bigwedge\{p \in \mathcal{P}(C) \mid p \geq \chi_\Delta(a)\right) = 1.$$

This leads to the following proposition.

Proposition 4.4.1. *For any $a \in A_{sa}$, $\Delta \in \mathcal{O}\mathbb{R}$, and state ψ, the following two conditions are equivalent:*

1. $C \Vdash \underline{\mu}((\underline{[a \in \Delta]}) = \underline{1})$;
2. $\forall p \in \mathcal{P}(C) \;\; p \geq \chi_\Delta(a) \;\; \to \psi(p) = 1.$

In other words, the proposition $[a \in \Delta]$ is true, relative to a state ψ, and in context C, iff by performing only the measurements allowed by C, it is impossible to refute, i.e., given the system, prepared in the state ψ, that a measurement of a yields a value in Δ with certainty. Of course, if we want to completely avoid operationalist notions, then this does not yield a satisfactory account of truth.

4.4.2 Disjunction

Let $a_1, a_2 \in A_{sa}$, and $\Delta_1, \Delta_2 \in \mathcal{O}\mathbb{R}$. In order to obtain an understanding of the clopen subobject

$$\underline{[a_1 \in \Delta_1]} \vee \underline{[a_2 \in \Delta_2]},$$

we pick an arbitrary state ψ and consider the truth value of the proposition

$$\underline{\mu}(\underline{[a_1 \in \Delta_1]} \vee \underline{[a_2 \in \Delta_2]}) = \underline{1},$$

where $\underline{\mu} = \underline{\mu}_{\psi}$ is internal probability valuation corresponding to ψ. This proposition is true at stage $C \in \mathcal{C}$ (equivalently, the sieve of the truth value of the proposition contains C) iff

$$\mu_C(\underline{[a_1 \in \Delta_1]}_C \cup \underline{[a_2 \in \Delta_2]}_C) = 1.$$

By definition of the local valuation μ_C, this simply states that,

$$\psi(\delta^o(\chi_{\Delta_1}(a_1))_C \vee \delta^o(\chi_{\Delta_2}(a_2))_C) = 1.$$

Recall that outer daseinisation of projections respects $\vee$, giving the simplification

$$\psi(\delta^o(\chi_{\Delta_1}(a_1) \vee \chi_{\Delta_2}(a_2))_C) = 1.$$

Spelling out the definition of outer daseinisation of projections, this is equivalent to

$$\forall p \in \mathcal{P}(C) \;\; p \geq \chi_{\Delta_1}(a_1) \vee \chi_{\Delta_2}(a_2) \;\; \to \psi(p) = 1.$$

We collect this result in the following proposition.

Proposition 4.4.2. *If we define $\underline{[a \in \Delta]}$ using $\delta^o(\chi_\Delta(a))$ as in (3.3), then, in the contravariant model, the following two conditions are equivalent:*

1. *$C \Vdash \ \underline{\mu}(\underline{[a_1 \in \Delta_1]} \vee \underline{[a_2 \in \Delta_2]}) = \underline{1}$;*
2. *If $p \in \mathcal{P}(C)$ satisfies $p \geq \chi_{\Delta_1}(a_1)$ and $p \geq \chi_{\Delta_2}(a_2)$, then $\psi(p) = 1$.*

We could interpret the result of this proposition in the following way: the internal proposition $\underline{[a_1 \in \Delta_1]} \vee \underline{[a_2 \in \Delta_2]}$ is true at context C iff by using a single measurement allowed by C it is impossible to refute both claims: for the system in state ψ, a measurement of a_i yields a value in Δ_i with certainty, where $i \in \{1, 2\}$.

4.4.3 Conjunction

At least on the mathematical level, the truth values from Proposition 4.4.2 take on a simple form. This is a consequence of the fact that outer daseinisation respects joins of projection operators. How do the conjunctions fare? Consider the truth value of the proposition

$$\underline{\mu}(\underline{[a_1 \in \Delta_1]} \wedge \underline{[a_2 \in \Delta_2]}) = \underline{1}.$$

This proposition is true at stage C iff

$$\psi(\delta^o(\chi_{\Delta_1}(a_1))_C \wedge \delta^o(\chi_{\Delta_2}(a_2))_C) = 1. \tag{4.8}$$

Spelling out the definition of outer daseinisation, and using distributivity of meets, this is equivalent to

$$\psi\left(\bigwedge\{p \in \mathcal{P}(C) \mid p \geq \chi_{\Delta_1}(a_1) \text{ or } p \geq \chi_{\Delta_2}(a_2)\}\right) = 1.$$

Note that this identity implies

$$\forall p \in \mathcal{P}(C)\ p \geq \chi_{\Delta_1}(a_1) \text{ or } p \geq \chi_{\Delta_2}(a_2) \ \rightarrow \ \psi(p) = 1. \tag{4.9}$$

Next, assume (4.9). It follows that $\psi(\delta^o(\chi_{\Delta_i}(a_i))_C) = 1$, where $i \in \{1, 2\}$. Let the clopens S_i of Σ_C correspond to the projections $\delta^o(\chi_{\Delta_i}(a_i))_C$, and let $\mu : \mathcal{O}\Sigma_C \to [0, 1]$ denote the probability valuation corresponding to the state $\psi|_C$. By assumption $\mu(S_i) = 1$, for $i \in \{1, 2\}$. The modular law implies

$$\mu(S_1 \cap S_2) = \mu(S_1) + \mu(S_2) - \mu(S_1 \cup S_2) = 1,$$

which in turn implies (4.8).

Proposition 4.4.3. *If we define $\underline{[a \in \Delta]}$ using $\delta^o(\chi_\Delta(a))$ as in (3.3), then, in the contravariant model, the following two conditions are equivalent :*

1. *$C \Vdash \underline{\mu}(\underline{[a_1 \in \Delta_1]} \wedge \underline{[a_2 \in \Delta_2]}) = \underline{1}$;*
2. *If $p \in \mathcal{P}(C)$ satisfies $p \geq \chi_{\Delta_1}(a_1)$ or $p \geq \chi_{\Delta_2}(a_2)$, then $\psi(p) = 1$.*

We could interpret the result of this proposition in the following way: the internal proposition $\underline{[a_1 \in \Delta_1]} \wedge \underline{[a_2 \in \Delta_2]}$ is true at context C iff using only measurements of C, we cannot refute either claim, in that a measurement of a_1 yields a value in Δ_1 with certainty, and, a measurement of a_2 yields a value in Δ_2 with certainty.

4.4.4 Negation

Negation in $\mathcal{O}\Sigma_{\downarrow}$ is more complicated than either conjunction or disjunction. In order to describe it, we use the following notation. If $p \in C$ is a projection operator, then S_p^C denotes the corresponding clopen subset of Σ_C (the superscript C is added to distinguish between S_p^D and S_p^C, whenever $p \in D \subseteq C$).
The negation of $\mathcal{O}\Sigma^{\downarrow}$ is given by

$$(\neg\underline{[a \in \Delta]})_C = \{\lambda \in \Sigma_C \mid \forall D \subseteq C \quad \lambda|_D \notin \underline{[a \in \Delta]}_D\}.$$

This is more conveniently written as

$$(\neg\underline{[a \in \Delta]})_C = \bigcap_{D \subseteq C} \rho_{CD}^{-1}(S^D_{\delta^o(\chi_\Delta(a))_D})^{co},$$

where the superscript *co* denotes the set-theoretic complement. For any $p \in D \subseteq C$, we have $\rho_{CD}^{-1}(S_p^D) = S_p^C$. By this observation, the previous expression simplifies to

$$(\neg\underline{[a \in \Delta]})_C = \bigcap_{D \subseteq C} (S^C_{\delta^o(\chi_\Delta(a))_D})^{co}.$$

We can get rid of the set-theoretic complement by using the relation

$$\forall C \in \mathcal{C} \; \forall p \in \mathrm{Proj}(A) \quad \delta^i(1-p)_C = 1 - \delta^o(p)_C;$$

see e.g. [37, (5.59)]. At the level of the Gelfand spectra, this translates to

$$(S^C_{\delta^o(p)_D})^{co} = S^C_{\delta^i(1-p)_D}.$$

We deduce

$$(\neg\underline{[a \in \Delta]})_C = \bigcap_{D \subseteq C} S^C_{\delta^i(1-\chi_\Delta(a))_D} = S^C_{\bigwedge_D \delta^i(1-\chi_\Delta(a))_D}.$$

Using this identity, we find that the condition

$$C \Vdash \underline{\mu}(\neg \underline{[a \in \Delta]}) = \underline{1}$$

is equivalent to

$$\psi\left(\bigwedge_{D \in (\downarrow C)} \delta^i(1 - \chi_\Delta(a))_D\right) = 1. \tag{4.10}$$

Assuming that the intersection of Δ with the set of eigenvalues of a is non empty (i.e., $\chi_\Delta(a) \neq 0$), condition (4.10) cannot be satisfied. This simply follows from the observation that the inner daseinisation of $1 - \chi_\Delta(a)$ with respect to the trivial context $\mathbb{C}$ is equal to 0. However, if we remove the bottom element $\mathbb{C}$ from $\mathcal{C}$, things become more interesting, at least mathematically. As we shall see, in this setting, the context C needs to satisfy strong conditions in order for (4.10) to hold. In what follows we use the notation $q := 1 - \chi_\Delta(a)$.

Let $p \in \mathcal{P}(C)$ have the property that neither $p \leq q$, nor $1 - p \leq q$. If $D \subseteq C$ is the context generated by p, then $\delta^i(q)_D = 0$. As $\psi(0) = 0$, we conclude that a necessary condition for (4.10) to hold is

$$\forall p \in \mathcal{P}(C) \ \text{ either } \ p \leq q \ \text{ or } \ 1 - p \leq q.$$

Note that $q < 1$ by assumption, so only one of the two options can hold. Also note that this condition implies that C commutes with q. As we are working with matrix algebras $A = M_n(\mathbb{C})$ in this subsection, we can find projections $p_1, \ldots, p_k$ in C such that $p_i \cdot p_j = 0$ if $i \neq j$ and $\sum_{i=1}^k p_i = 1$. If (4.10) holds, we can sort these projections as follows. The set $L = \{p_1, \ldots, p_l\}$ consists of the p_i such that $p_i \leq q$. The set $R = \{p_{l+1}, \ldots, p_k\}$ consists of the p_j such that $1 - p_j \leq q$. Note that $L \cap R = \emptyset$, and both sets are non empty. Also note that $\delta^i(q)_C = p_1 + \ldots + p_l$.

Assume that L has at least two elements. Let D_1, and D_2 be the context generated by the projections

$$D_1 = \{p_1 + p_{l+1}, p_2, \ldots, p_l, p_{l+2}, \ldots, p_k\}'';$$

$$D_2 = \{p_1, p_2 + \ldots + p_l + p_{l+1}, p_{l+2}, \ldots, p_k\}''.$$

Then $\delta^i(q)_{D_1} = p_2 + \ldots + p_l$ and $\delta^i(q)_{D_2} = p_1$. We conclude that

$$\delta^i(q)_{D_1} \wedge \delta^i(q)_{D_2} = 0,$$

and (4.10) cannot be satisfied. So if (4.10) holds, then L is a singleton. In an analogous way it can be shown that R contains exactly one element. This implies that the projection lattice of C must be of the form $\{0, p, 1-p, 1\}$, with either $p \leq q$, or $1-p \leq q$.

Proposition 4.4.4. *If we define $\underline{[a \in \Delta]}$ using $\delta^o(\chi_\Delta(a))$, and remove $\mathbb{C}$ from $\mathcal{C}$, then the following two conditions are equivalent in the contravariant model:*

1. *$C \Vdash \underline{\mu}(\neg\underline{[a \in \Delta]}) = \underline{1}$;*
2. *There exists a projection $p \in C$, that generates C, and satisfies $p \geq \chi_\Delta(a)$, as well as $\psi(p) = 0$.*

Only the coarsest contexts that commute with $\chi_\Delta(a)$ contribute to the truth value of $\underline{\mu}(\neg\underline{[a \in \Delta]}) = \underline{1}$. This emphasis on coarser contexts makes it hard to find a physical interpretation of the negation operation, if this is possible at all. We will also encounter this problem with the more general Heyting implication. This problem may suggest that it is a mistake to seek an interpretation of the contravariant quantum logic in terms of refutation, as we did for conjunction and disjunction, but what alternatives are there? Unfortunately, it seems that the formal negation and implication, as natural as they are from a topos theoretic perspective, do not seem to have a clear physical motivation.

4.5 Covariant Quantum Logic

We describe the Heyting algebra structure of $\mathcal{O}\Sigma_\uparrow$ by means of the truth values the opens produce when these are combined with states.

We continue with the complete Heyting algebra $\mathcal{O}\Sigma_\uparrow$ of the covariant model. As in the previous subsection we restrict to matrix algebras $A = M_n(\mathbb{C})$. The elementary proposition $[a \in \Delta]$ will be represented by taking the inner daseinisation of the spectral projection $\chi_\Delta(a)$. As long as Δ is an open interval or open half-interval, $[a \in \Delta]$ is also of the form $\delta(a)^{-1}(\hat{\Delta})$, for an appropriate $\hat{\Delta}$.

4.5.1 Single Proposition

The elementary proposition $[a \in \Delta]$ defines an open $\underline{[a \in \Delta]} : \underline{1} \to \mathcal{O}\underline{\Sigma}_{\underline{A}}$. Relative to a state ψ, represented by an internal probability valuation $\underline{\mu}$, we will study the condition

$$C \Vdash \underline{\mu}(\underline{[a \in \Delta]}) = \underline{1}.$$

This is equivalent to

$$\psi(\delta^i(\chi_\Delta(a))_C) = 1.$$

Recall that $\delta^i(\chi_\Delta(a))_C$ is the largest projection of C that is smaller than $\chi_\Delta(a)$.

Proposition 4.5.1. *For any $a \in A_{sa}$ and $\Delta \in \mathcal{O}\mathbb{R}$, and state ψ the following two conditions are equivalent in the covariant model:*

1. $C \Vdash \underline{\mu}((\underline{[a \in \Delta]}) = \underline{1})$;
2. $\exists p \in \mathcal{P}(C) \;\; p \leq \chi_\Delta(a)$ *and* $\psi(p) = 1$.

Truth of $[a \in \Delta]$ relative to the state ψ and context C holds iff C provides us with a measurement with which we can affirm that the system, prepared in state ψ, upon a measurement of a yields a value in Δ with certainty.

4.5.2 Conjunction

Our treatment of the covariant conjunction operation resembles that of the contravariant disjunction. Let $a_1, a_2 \in A_{sa}$, and $\Delta_1, \Delta_2 \in \mathcal{O}\mathbb{R}$. Consider

$$C \Vdash \underline{\mu}(\underline{[a_1 \in \Delta_1]} \wedge \underline{[a_2 \in \Delta_2]}) = \underline{1},$$

where $\underline{\mu}$ is internal probability valuation corresponding to ψ. This condition is equivalent to

$$\psi(\delta^i(\chi_{\Delta_1}(a_1))_C \wedge \delta^i(\chi_{\Delta_2}(a_2))_C) = 1.$$

Recall that inner daseinisation of projections respects $\wedge$, giving the simplification

$$\psi(\delta^i(\chi_{\Delta_1}(a_1) \wedge \chi_{\Delta_2}(a_2))_C) = 1.$$

As in the single proposition case, this amounts to

$$\exists p \in \mathcal{P}(C) \;\; p \leq \chi_{\Delta_1}(a_1) \wedge \chi_{\Delta_2}(a_2) \;\; \rightarrow \psi(p) = 1.$$

Proposition 4.5.2. *If we define $\underline{[a \in \Delta]}$ using $\delta^i(\chi_\Delta(a))$ as in (3.10), then, in the covariant model, the following two conditions are equivalent:*

1. $C \Vdash \underline{\mu}(\underline{[a_1 \in \Delta_1]} \wedge \underline{[a_2 \in \Delta_2]}) = \underline{1}$;
2. $\exists p \in \mathcal{P}(C)$ *such that* $p \leq \chi_{\Delta_1}(a_1)$, $p \leq \chi_{\Delta_2}(a_2)$, *and* $\psi(p) = 1$.

Truth of the meet of elementary propositions, relative to a state ψ, and context C is therefore equivalent to: there is a measurement allowed by C, by which we can affirm that for the system, when prepared in the state ψ, a measurement of a_1 would yield a value in Δ_1 with certainty, and for such a system, a measurement of a_2 would yield a value in Δ_2 with certainty.

4.5.3 Disjunction

Our treatment of the covariant disjunction reminds us of the contravariant conjunction. Consider the forcing relation

$$C \Vdash \underline{\mu}(\underline{[a_1 \in \Delta_1]} \vee \underline{[a_2 \in \Delta_2]}) = \underline{1},$$

or, equivalently,

$$\psi(\delta^i(\chi_{\Delta_1}(a_1))_C \vee \delta^i(\chi_{\Delta_2}(a_2))_C) = 1.$$

Spelling out the definition of inner daseinisation, and using distributivity of joins, this is equivalent to

$$\psi\left(\bigvee\{p \in \mathcal{P}(C) \mid p \leq \chi_{\Delta_1}(a_1) \text{ or } p \leq \chi_{\Delta_2}(a_2)\}\right) = 1. \tag{4.11}$$

Note that this identity is implied by the proposition

$$\exists p \in \mathcal{P}(C)\ p \leq \chi_{\Delta_1}(a_1) \text{ or } p \leq \chi_{\Delta_2}(a_2) \ \text{ and } \ \psi(p) = 1. \tag{4.12}$$

Note that (4.11) and (4.12) are **not** equivalent. This is because for a pair $p_1, p_2 \in \mathcal{P}(C)$, it is possible that $\psi(p_1 \vee p_2) = 1$, whilst neither $\psi(p_1) = 1$, nor $\psi(p_2) = 1$. The forcing relation is weaker than the affirmation of one of the two claims: given the system, prepared in state ψ, a measurement of a_i yields a value in Δ_i with certainty ($i \in \{1, 2\}$).

4.5.4 Negation

For matrix algebras, the negation of $\mathcal{O}\Sigma_{\uparrow}$ was first described in [20] in terms of projections. There it was shown that the open subset $(\neg[a \in \Delta])_C$ of Σ_C corresponds to the projection

$$\bigvee\{p \in \mathcal{P}(C) \mid \forall E \in (\uparrow C) \ \ p \leq 1 - \delta^i(p)_E\}.$$

Using

$$1 - \delta^i(\chi_\Delta(a))_E = \delta^o(1 - \chi_\Delta(a))_E = \delta^o(\chi_{\mathbb{R}-\Delta}(a))_E,$$

we find

$$(\neg[a \in \Delta])_C = \bigcap_{E \supseteq C} \{\lambda \in \Sigma_C \mid \rho^{-1}_{EC}(\lambda) \subseteq S^E_{\delta^o(\chi_{\mathbb{R}-\Delta}(a))_E}\}.$$

This complicated expression makes it hard to understand the condition

$$C \Vdash \underline{\mu}(\neg\underline{[a \in \Delta]}) = \underline{1}.$$

However, if we restrict our attention to maximal contexts, then the negation simplifies considerably. The forcing relation is satisfied iff

$$\psi(\delta^o(\chi_{\mathbb{R}-\Delta}(a))_C) = 1,$$

which is equivalent to

$$\forall p \in \mathcal{P}(C) \;\; p \geq \chi_{\mathbb{R}-\Delta}(a) \;\; \rightarrow \;\; \psi(p) = 1.$$

Proposition 4.5.3. *If we define* $\underline{[a \in \Delta]}$ *using* $\delta^i(\chi_\Delta(a))$ *as in (3.10), and consider a maximal context* C *in* $\mathcal{C}$*, then, in the covariant model, the following two conditions are equivalent:*

1. $C \Vdash \underline{\mu}(\neg\underline{[a \in \Delta]}) = \underline{1}$;
2. *For each projection* $p \in C$*, if* $p \leq \chi_\Delta(a)$*, then* $\psi(p) = 0$;
3. *For each projection* $p \in C$*, if* $p \geq \chi_{\mathbb{R}-\Delta}(a)$*, then* $\psi(p) = 1$.

Hence, using only measurements allowed by the maximally refined context C we cannot refute the claim that the system, when prepared in the state ψ, upon a measurement of a yields a value outside of Δ with certainty. If C is not maximal, then $C \Vdash \underline{\mu}(\neg\underline{[a \in \Delta]}) = \underline{1}$ implies that for any refinement E of C (i.e. $E \supseteq C$) we cannot refute the aforementioned claim using only measurements from the context E. As for the contravariant model, in that case the physical content of the negation operation seems questionable.

4.5.5 Discussion

Guided by the truth values obtained from state-proposition pairs, it seems tempting to read the logic of the contravariant model as a logic of refutation and the logic of the covariant logic as one of affirmation. Note that this logic of affirmation fits well with the idea that the covariant topos approach is to be interpreted as a physical version of a Kripke model as suggested in Section 3.4. Through the correspondence $\delta^o(1-p)_C = 1-\delta^i(p)_C$, the logics of the contravariant of covariant approaches seem to be related. Even so, not all the connectives (especially the negation) received a satisfactory interpretation in this way. In addition, we might worry how such instrumentalist pictures of truth may square with a more realist perspective on quantum theory. We may avoid an instrumentalist reading of truth by avoiding it for the contravariant approach, and expressing truth in the covariant approach in terms of truth of the contravariant approach, as we did in Section 3.4.

5

Morphisms and Dynamics

We shift our attention from kinematics to dynamics. In particular, we study how the spectral presheaf $\underline{\Sigma}_A$ and the spectral locale $\underline{\Sigma}_{\underline{A}}$ transform under the action of a $*$-automorphism $h : A \to A$. Almost all of the material in this section either relies on the ideas in [68] or coincides with constructions from [27, 28]. In [68] the emphasis is on the covariant model, and constructions such as daseinisation of self-adjoint operators are not considered. In [27, 28] the emphasis is on the contravariant model, and the internal perspective of the topos is not considered. Although the ideas in these references may appear different, they turn out to be closely related. Below we treat these ideas on dynamics for both topos models, with emphasis on internal reasoning.

In Section 5.1 we see how a $*$-homomorphism between two C*-algebras induces an internal $*$-homomorphism between the associated internal commutative C*-algebras of the covariant approach. We also describe the associated locale map on the corresponding Gelfand spectra. In similair vein, for the contravariant approach, Section 5.2 describes internal continuous maps on the spectral presheaves, induced by $*$-homomorphisms. In Section 5.3 we restrict to $*$-automorphisms and show how daseinised self-adjoint operators, states and truth values in both topos approaches transform under the action of these morphisms. In this chapter, as in the previous ones, one of the guiding ideas was to let the mathematics of the topos approaches resemble classical physics when viewed from the internal language of topoi.

5.1 Covariant Model

We show how, in the covariant approach, to a $*$-homomorphism between C-algebras we can assign a corresponding transformation on the associated topoi and internal commutative C*-algebras. We also look at the corresponding transformation on the internal Gelfand spectra.*

5.1.1 C*-algebras

As remarked in the introduction, the covariant model is typically applied to unital C*-algebras instead of von Neumann algebras. For the moment we will use all unital C*-algebras. In Subsection 5.3, when daseinisation enters into the discussion, we will again restrict attention to von Neumann algebras.
In the covariant approach, given a unital C*-algebra A, we assign to it a pair $([\mathcal{C}_A, \mathbf{Set}], \underline{A})$, consisting of a topos and a unital commutative C*-algebra internal to this topos. In this section we look at the way $*$-homomorphisms $f : A \to B$ induce morphisms on the associated pairs $([\mathcal{C}_A, \mathbf{Set}], \underline{A})$, $([\mathcal{C}_B, \mathbf{Set}], \underline{B})$. We start by recalling two categories, introduced in earlier literature on topos approaches to quantum theory, and which will help in answering this question. We will subsequently show how these two categories are related. The first of the two is the category $\mathbf{cCTopos}_N$, introduced by Nuiten [68, Definition 4].

Definition 5.1.1. *The category* $\mathbf{cCTopos}_N$ *consists of the following:*

- *Objects are pairs $(\mathcal{E}, \underline{A})$, where $\mathcal{E}$ is a topos and $\underline{A}$ is a unital commutative C*-algebra internal to the topos $\mathcal{E}$.*
- *An arrow $(G, \underline{g}) : (\mathcal{E}, \underline{A}) \to (\mathcal{F}, \underline{B})$, is given by a geometric morphism $G : \mathcal{E} \to \mathcal{F}$ and a $*$-homomorphism $\underline{g} : G^*\underline{B} \to \underline{A}$ in $\mathcal{F}$.*
- *Composition of arrows is defined by $(G, \underline{g}) \circ (F, \underline{f}) = (G \circ F, \underline{f} \circ F^*\underline{g})$.*

For an arbitrary geometric morphism G, the object $G^*\underline{B}$ need not be a C*-algebra in $\mathcal{F}$. It is, at the very least, a semi-normed commutative $*$-algebra over $\underline{\mathbb{Q}[i]}$. The notion of a $*$-homomorphism, in the sense of a $*$-preserving homomorphism of $\underline{\mathbb{Q}[i]}$-algebras, still makes sense when the domain is $G^*\underline{B}$. If the geometric morphism comes from a $*$-homomorphism, as discussed below, then $G^*\underline{B}$ will always be an internal C*-algebra. Otherwise, we can take its Cauchy completion and turn it into an internal C*-algebra.

The second category of interest was introduced by Andreas Döring in [27]:

Definition 5.1.2. *Let* $\mathcal{C}$ *be any small category. The category* $\mathbf{Copresh}(\mathcal{C})$ *is defined by*

- *Objects are functors* $Q : \mathcal{J} \to \mathcal{C}$*, where* J *is any small category.*
- *An arrow* $(f, \eta) : Q_1 \to Q_2$*, where* $Q_i : \mathcal{J}_i \to \mathcal{C}$*, is given by a functor* $f : \mathcal{J}_1 \to \mathcal{J}_2$*, and a natural transformation* $\eta : Q_1 \to F^* Q_2$*. Here* F^* *denotes the inverse image functor of the essential geometric morphism associated to* f*.*

The motivating example is when $\mathcal{C}$ is equal to $\mathbf{cuC}^*$, the category of unital commutative C*-algebras and unit preserving $*$-homomorphisms. We know from Subsection 2.1 that functors $A : \mathcal{J} \to \mathbf{ucC}^*$ correspond exactly to the unital commutative C*-algebra internal to the topos $[\mathcal{J}, \mathbf{Set}]$. We can think of the objects of $\mathbf{Copresh}(\mathbf{ucC}^*)$ as pairs $(\mathcal{E}, \underline{A})$, where $\mathcal{E}$ is a topos (and in particular a functor category), and $\underline{A}$ is a unital commutative C*-algebra in $\mathcal{E}$.
For any pair of small categories $\mathcal{J}_1$, $\mathcal{J}_2$, a functor $f : \mathcal{J}_1 \to \mathcal{J}_2$ defines an essential geometric morphism $F : [\mathcal{J}_1, \mathbf{Set}] \to [\mathcal{J}_2, \mathbf{Set}]$, where *essential* means that the inverse image functor F^* (which is the left adjoint in the adjunction defining F) also has a left adjoint $F_!$. See, for example, [63, (VII.2 Theorem 2)]. The inverse image functor F^* is given by

$$\forall \underline{X} \in [\mathcal{J}_2, \mathbf{Set}], \quad \forall C \in \mathcal{J}_1, \quad F^* \underline{X}(C) = \underline{X}(f(C)).$$

The following lemma gives the converse statement.

Lemma 5.1.3. *([55] Lemma 4.1.5) Let* $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *be two small categories such that* $\mathcal{J}_2$ *is Cauchy-complete (i.e., all idempotent morphisms split). Then every essential geometric morphism* $[\mathcal{J}_1, \mathbf{Set}] \to [\mathcal{J}_2, \mathbf{Set}]$ *is induced by a functor* $\mathcal{J}_1 \to \mathcal{J}_2$ *as above.*

If the base category $\mathcal{J}_2$ is a poset category, then the only idempotent arrows are the identity morphisms. The base categories for the quantum topoi are therefore trivially Cauchy-complete. On the level of contexts, the order-preserving maps $\phi : \mathcal{C}_A \to \mathcal{C}_B$, correspond to geometric morphisms between the corresponding topoi, where the left-adjoint ϕ^* itself has a left adjoint $\phi_!$.
An arrow in $\mathbf{Copresh}(\mathbf{ucC}^*)$ can thus be seen as a pair $(F, \underline{f}) : (\mathcal{E}, \underline{A}) \to (\mathcal{F}, \underline{B})$, where the topoi $\mathcal{E}$, $\mathcal{F}$ are functor categories, $F : \mathcal{E} \to \mathcal{F}$ is an

essential geometric morphism, and $\underline{f} : \underline{A} \to F^*\underline{B}$ is a natural transformation.
We replace the category $\mathbf{Copresh}(\mathbf{ucC}^*)$ by the a related category.

Definition 5.1.4. *The category* $\mathbf{cCTopos}_D$ *is given by:*

- *Objects are pairs* $(\mathcal{E}, \underline{A})$*, where* $\mathcal{E}$ *is a topos and* $\underline{A}$ *is a unital commutative* C^**-algebra in* $\mathcal{E}$*.*
- *Arrows* $(F, \underline{f}) : (\mathcal{E}, \underline{A}) \to (\mathcal{F}, \underline{B})$ *are given by a geometric morphism* $F : \mathcal{E} \to \mathcal{F}$*, and a* $*$*-homomorphism* $\underline{f} : \underline{A} \to F^*\underline{B}$ *in* $\mathcal{E}$*.*
- *Composition of arrows is defined by* $(G, \underline{g}) \circ (F, \underline{f}) = (G \circ F, F^*\underline{g} \circ \underline{f})$*.*

Remark 5.1.5. *The category* $\mathbf{Copresh}(\mathcal{C})$ *was introduced in [27] in connection with another category* $\mathbf{Presh}(\mathcal{D})$*, to which it is dually equivalent. Here* $\mathcal{D}$ *is a category which is dually equivalent to* $\mathcal{C}$ *by assumption. When we replace* $\mathbf{Copresh}(\mathbf{ucC}^*)$ *by* $\boldsymbol{cCTopos}_D$ *this duality is lost. In the next section, where we look at the contravariant version of the topos approach, a category closely connected to* $\mathbf{Presh}(\mathcal{D})$ *is considered.*

Let $f : A \to B$ be a unit-preserving $*$-homomorphism (in $\mathbf{Set}$). Then f induces an arrow

$$(F, \underline{f}) : ([\mathcal{C}_A, \mathbf{Set}], \underline{A}) \to ([\mathcal{C}_B, \mathbf{Set}], \underline{B}) \tag{5.1}$$

in $\mathbf{cCTopos}_D$. To see this, observe that f induces an order-preserving map

$$\hat{f} : \mathcal{C}_A \to \mathcal{C}_B, \quad \hat{f}(C) = f[C], \tag{5.2}$$

which in turn induces a geometric morphism $F : [\mathcal{C}_A, \mathbf{Set}] \to [\mathcal{C}_B, \mathbf{Set}]$. The inverse image functor acting on $\underline{B}$ is given by

$$F^*\underline{B} : \mathcal{C}_A \to \mathbf{Set}, \; F^*\underline{B}(C) = \underline{B} \circ \hat{f}(C) = f[C]. \tag{5.3}$$

The internal $*$-homomorphism induced by f is now simply given by

$$\underline{f} : \underline{A} \to F^*\underline{B}, \quad \underline{f}_C : C \to f[C], \quad \underline{f}_C = f|_C. \tag{5.4}$$

Definition 5.1.6. *A unital* $*$*-homomorphism* $f : A \to B$ *is said to* ***reflect commutativity*** *if*

$$\forall a_1, a_2 \in A, \quad [f(a_1), f(a_2)] = 0 \;\Rightarrow\; [a_1, a_2] = 0.$$

Note that if f is injective, then f reflects commutativity. A unital $*$-homomorphism $f : A \to B$ that reflects commutativity defines an arrow

$$(G, \underline{g}) : ([\mathcal{C}_B, \mathbf{Set}], \underline{B}) \to ([\mathcal{C}_A, \mathbf{Set}], \underline{A}) \tag{5.5}$$

in $\mathbf{cCTopos}_N$. As f reflects commutativity, we can define the order-preserving map

$$\hat{g} : \mathcal{C}_B \to \mathcal{C}_A, \quad \hat{g}(D) = f^{-1}(D). \tag{5.6}$$

As before, this induces an essential geometric morphism $G : [\mathcal{C}_B, \mathbf{Set}] \to [\mathcal{C}_A, \mathbf{Set}]$. The associated $*$-morphism is given by

$$\underline{g} : G^*\underline{A} \to \underline{B}, \quad \underline{g}_D : f^{-1}(D) \to D, \quad \underline{g}_D = f|_{f^{-1}(D)}. \tag{5.7}$$

Note that $\hat{g}$ is a right adjoint to $\hat{f}$. As a consequence, the geometric morphisms $F^* \dashv F_*$ and $G^* \dashv G_*$ are closely related. More precisely, $G_* = F^*$. As inverse image functors preserve colimits, it is clear that in this setting $*$-homomorphisms $G^*\underline{A} \to \underline{B}$ are equivalent to $*$-homomorphisms $\underline{A} \to F^*\underline{B}$.
We end with a small summary of the material in this section.

Proposition 5.1.7. *A unital $*$-homomorphism $f : A \to B$ induces an arrow*

$$(F, \underline{f}) : ([\mathcal{C}_A, \mathbf{Set}], \underline{A}) \to ([\mathcal{C}_B, \mathbf{Set}], \underline{B}) \tag{5.8}$$

in $\mathbf{cCTopos}_D$ such that the internal $*$-homomorphism $\underline{f}$ is given by $\underline{f}_C = f|_C$. If f reflects commutativity, then it also induces an arrow

$$(G, \underline{g}) : ([\mathcal{C}_B, \mathbf{Set}], \underline{B}) \to ([\mathcal{C}_A, \mathbf{Set}], \underline{A}) \tag{5.9}$$

in $\mathbf{cCTopos}_N$ such that the internal $*$-homomorphism $\underline{g}$ is $\underline{g}_D = f|_{f^{-1}(D)}$.

5.1.2 Locales

Next, we describe the internal $*$-homomorphisms of the previous subsection at the level of the Gelfand spectra. We use the following observations. As noted before, given a locale X, in $\mathbf{Set}$, the categories $\mathbf{Loc}_{Sh(X)}$ and $\mathbf{Loc}/X$ are equivalent [55, C1.6]. In addition, a map of locales $f : X \to Y$ induces an adjunction

$$(F_* \dashv F^\sharp) \quad F_* : \mathbf{Loc}_{Sh(X)} \to \mathbf{Loc}_{Sh(Y)} : F^\sharp. \tag{5.10}$$

There is a good reason for writing the left adjoint as F_*. The continuous map f defines a geometric morphism $F : Sh(X) \to Sh(Y)$. Unlike the inverse image functor F^*, the direct image functor F_* preserves frames and morphisms of frames. In fact, this property is crucial for the equivalence of the categories $\mathbf{Loc}_{Sh(X)}$ and $\mathbf{Loc}/X$. The left adjoint F_* of (5.10) is the restriction of the direct image functor F_* to frames and frame homomorphisms. The right adjoint $F^\sharp$ is most easily described under the identification $\mathbf{Loc}_{Sh(X)} \cong \mathbf{Loc}/X$. As a functor $\mathbf{Loc}/Y \to \mathbf{Loc}/X$ it maps a bundle $Z \to Y$, to the pullback of this bundle along the map $f : X \to Y$.
In [68], in addition to $\mathbf{cCTopos}_N$, another, related category was introduced.

Definition 5.1.8. *The category* $\mathbf{spTopos}_N$ *of* spaced topoi *is given by:*

- *Objects are pairs $(\mathcal{E}, \underline{L})$, where $\mathcal{E}$ is a topos and $\underline{L}$ is a locale in $\mathcal{E}$.*
- *An arrow $(G, \underline{s}) : (\mathcal{E}, \underline{L}) \to (\mathcal{F}, \underline{M})$ is given by a geometric morphism $G : \mathcal{E} \to \mathcal{F}$ and a locale map $\underline{s} : G_*\underline{L} \to \underline{M}$ in $\mathcal{F}$.*
- *Composition of arrows is defined as $(G, \underline{t}) \circ (F, \underline{s}) = (G \circ F, \underline{t} \circ G_*\underline{s})$.*

A unital C*-algebra A defines a spaced topos $([\mathcal{C}_A, \mathbf{Set}], \underline{\Sigma}_{\underline{A}})$, where $\underline{\Sigma}_{\underline{A}}$ denotes the internal Gelfand spectrum of $\underline{A}$. A unital $*$-homomorphism $f : A \to B$ that reflects commutativity defines an arrow in $\mathbf{spTopos}_N$ as follows. We know that f induces a $*$-homomorphism $\underline{g} : G^*\underline{A} \to \underline{B}$. By Gelfand duality this defines a locale map on the spectra

$$\underline{\Sigma}(\underline{g}) : \underline{\Sigma}_{\underline{B}} \to \underline{\Sigma}_{G^*\underline{A}}.$$

Recall from Section 2.2 that the spectrum $\underline{\Sigma}_{\underline{B}}$ can be described externally as the bundle of topological spaces

$$\pi_B : \Sigma^\uparrow_{\underline{B}} \to \mathcal{C}^\uparrow_B, \quad (D, \lambda) \mapsto D. \tag{5.11}$$

Analogously, the spectrum $\underline{\Sigma}_{G^*\underline{A}}$ can be represented by the bundle

$$\Sigma^\uparrow_{G^*\underline{A}} \to \mathcal{C}^\uparrow_B, \quad (D, \lambda) \mapsto D, \tag{5.12}$$

where, as sets, $\Sigma^\uparrow_{G^*\underline{A}}$ is equal to $\coprod_{D \in \mathcal{C}_B} \Sigma_{f^{-1}(D)}$, and U is open in $\Sigma^\uparrow_{G^*\underline{A}}$ iff the following two conditions hold:

1. If $D \in \mathcal{C}_B$, then $U_D := U \cap \Sigma_{f^{-1}(D)}$ is open in $\Sigma_{f^{-1}(D)}$;

2. If $D_1 \subseteq D_2$, then $\rho^{-1}_{f^{-1}(D_2)f^{-1}(D_1)}(U_{D_1}) \subseteq U_{D_2}$, where $\rho_{f^{-1}(D_2)f^{-1}(D_1)}$: $\Sigma_{f^{-1}(D_2)} \to \Sigma_{f^{-1}(D_1)}$ is the restriction map.

A straightforward calculation (or [68, Lemma 3.4]) reveals that (5.12) is simply the bundle $\pi_A : \Sigma^{\uparrow}_{\underline{A}} \to \mathcal{C}^{\uparrow}_A$, pulled back along the order-preserving function $\hat{g} : \mathcal{C}^{\uparrow}_B \to \mathcal{C}^{\uparrow}_A$, $D \mapsto f^{-1}(D)$, seen as an Alexandroff-continuous map. Externally, the map $\underline{\Sigma}(\underline{g})$ is given by

$$\begin{tikzcd}
\Sigma^{\uparrow}_{\underline{B}} \arrow[rr, "\Sigma(\underline{g})"] \arrow[dr, "\pi_B"'] & & \hat{g}^*\Sigma^{\uparrow}_{\underline{A}} \arrow[dl, "\hat{g}^*\pi_A"] \\
& \mathcal{C}^{\uparrow}_B &
\end{tikzcd}$$

$$\Sigma(\underline{g}) : \Sigma^{\uparrow}_{\underline{B}} \to \hat{g}^*\Sigma^{\uparrow}_{\underline{A}}, \quad (D, \lambda) \mapsto (D, \lambda \circ f|_{f^{-1}(D)}). \tag{5.13}$$

Note that internally this is a locale map $\underline{\Sigma}_{\underline{B}} \to G^{\sharp}\underline{\Sigma}_{\underline{A}}$ in $[\mathcal{C}_B, \mathbf{Set}]$. This is, in turn, equivalent to a locale map $G_*\underline{\Sigma}_{\underline{B}} \to \underline{\Sigma}_{\underline{A}}$ in $[\mathcal{C}_A, \mathbf{Set}]$. In this way, the $*$-homomorphism $f : A \to B$ defines a morphism

$$([\mathcal{C}_B, \mathbf{Set}], \underline{\Sigma}_{\underline{B}}) \to ([\mathcal{C}_A, \mathbf{Set}], \underline{\Sigma}_{\underline{A}})$$

in the category $\mathbf{spTopos}_N$.
Next, drop the assumption that $f : A \to B$ reflects commutativity. From the previous subsection we know that f defines a $*$-homomorphism $\underline{f}$: $\underline{A} \to F^*\underline{B}$ in $[\mathcal{C}_A, \mathbf{Set}]$. As before, by Gelfand duality this defines a continuous map of locales on the spectra

$$\underline{\Sigma}(\underline{f}) : \underline{\Sigma}_{F^*\underline{B}} \to \underline{\Sigma}_{\underline{A}}. \tag{5.14}$$

The spectrum $\underline{\Sigma}_{F^*\underline{B}}$ can be represented by the bundle

$$\Sigma^{\uparrow}_{F^*\underline{B}} \to \mathcal{C}^{\uparrow}_A, \quad (C, \lambda) \mapsto C, \tag{5.15}$$

where, as sets, $\Sigma^{\uparrow}_{F^*\underline{B}}$ is equal to $\coprod_{C \in \mathcal{C}_A} \Sigma_{f[C]}$, and U is open in $\Sigma^{\uparrow}_{F^*\underline{B}}$ iff the following two conditions hold:

1. If $C \in \mathcal{C}_A$, then $U_C := U \cap \Sigma_{f[C]}$ is open in $\Sigma_{f[C]}$;

2. If $C_1 \subseteq C_2$, then $\rho^{-1}_{f[C_2]f[C_1]}(U_{C_1}) \subseteq U_{C_2}$, where $\rho_{f[C_2]f[C_1]} : \Sigma_{f[C_2]} \to \Sigma_{f[C_1]}$ is the restriction map.

This bundle can be identified as $\pi_B : \Sigma^{\uparrow}_{\underline{B}} \to \mathcal{C}^{\uparrow}_B$, pulled back along $\hat{f} : \mathcal{C}^{\uparrow}_A \to \mathcal{C}^{\uparrow}_B$, $C \mapsto f[C]$. Externally, the locale map $\underline{\Sigma}(\underline{f}) : \underline{\Sigma}_{F^*\underline{B}} \to \underline{\Sigma}_{\underline{A}}$ is given by the continuous function

$$\Sigma(\underline{f}) : \hat{f}^*\Sigma^{\uparrow}_{\underline{B}} \to \Sigma^{\uparrow}_{\underline{A}}, \quad (C,\lambda) \mapsto (C, \lambda \circ f|_C), \tag{5.16}$$

over $\mathcal{C}^{\uparrow}_A$. Note that in (5.16) $\lambda \in \Sigma_{f[C]}$. Internally we obtain a locale map $F^\sharp\underline{\Sigma}_{\underline{B}} \to \underline{\Sigma}_{\underline{A}}$.

For the remainder of this subsection, assume once again that f reflects commutativity. How is the locale map $\Sigma(\underline{f}) : F^\sharp\underline{\Sigma}_{\underline{B}} \to \underline{\Sigma}_{\underline{A}}$, obtained from the $*$-homomorphism $\underline{f} : \underline{A} \to F^*\underline{B}$, related to the locale map[1] $\Sigma(\underline{g}) : G_*\underline{\Sigma}_{\underline{B}} \to \underline{\Sigma}_{\underline{A}}$ obtained from the $*$-homomorphism $\underline{g} : G^*\underline{A} \to \underline{B}$? We know that $G_* = F^*$, but this does not imply that on the level of locales G_* and $F^\sharp$ are the same. In fact, $G_*\underline{\Sigma}_{\underline{B}}$ and $F^\sharp\underline{\Sigma}_{\underline{B}}$ are slightly different locales.

The locale $\underline{\Sigma}_{\underline{B}}$ corresponds to a frame object $\mathcal{O}\underline{\Sigma}_{\underline{B}}$ in the topos $[\mathcal{C}_B, \mathbf{Set}]$. For $C \in \mathcal{C}_A$,

$$G_*(\mathcal{O}\underline{\Sigma}_{\underline{B}})(C) = F^*(\mathcal{O}\underline{\Sigma}_{\underline{B}})(C) = \mathcal{O}\underline{\Sigma}_{\underline{B}}(f[C]). \tag{5.17}$$

Using the external description $\Sigma^{\uparrow}_{\underline{B}}$, the right-hand side of (5.17) is given by the subspace topology of $\Sigma^{\uparrow}_{\underline{B}}$ on the subset $\coprod_{D \in \mathcal{C}_B \cap (\uparrow f[C])} \Sigma_D$.

$$G_*(\mathcal{O}\underline{\Sigma}_{\underline{B}})(C) = \mathcal{O}\left(\coprod_{D \in \mathcal{C}_B \cap (\uparrow f[C])} \Sigma_D\right). \tag{5.18}$$

On the other hand,

$$\mathcal{O}(F^\sharp\underline{\Sigma}_{\underline{B}})(C) = \mathcal{O}\left(\coprod_{C' \in \mathcal{C}_A \cap (\uparrow C)} \Sigma_{f[C]}\right), \tag{5.19}$$

where on the right-hand side we take the subspace topology from $\hat{f}^*\Sigma^{\uparrow}_{\underline{B}}$. We can now see that the sets (5.18) and (5.19) are different. The only difference is that $G_*(\mathcal{O}\underline{\Sigma}_{\underline{B}})(C)$ considers all contexts $D \in \mathcal{C}_B$ which are above $f[C]$, whereas $\mathcal{O}(F^\sharp\underline{\Sigma}_{\underline{B}})(C)$ only considers those contexts which

[1]Using $\Sigma(\underline{g})$ for this map is a slight abuse of notation, as this name was used earlier to denote the corresponding locale map $\underline{\Sigma}_{\underline{B}} \to G^\sharp\underline{\Sigma}_{\underline{A}}$.

come from an $C' \in \mathcal{C}_A$. As the locale map $\Sigma(\underline{g})$ comes from a $*$-homomorphism in $Sh(\mathcal{C}_{\underline{B}}^{\uparrow})$, and the locale map $\Sigma(\underline{f})$ comes from a $*$-homomorphism in $Sh(\mathcal{C}_{\underline{A}}^{\uparrow})$, this slight difference was to be expected.

Proposition 5.1.9. *A unital $*$-homomorphism $f : A \to B$ induces a continuous map of locales $\underline{\Sigma}(\underline{f}) : F^{\sharp}\underline{\Sigma}_{\underline{B}} \to \underline{\Sigma}_{\underline{A}}$ in $[\mathcal{C}_A, \mathbf{Set}]$. The external description of this map is given by the continuous function*

$$\Sigma(\underline{f}) : \hat{f}^{*}\Sigma_{\underline{B}}^{\uparrow} \to \Sigma_{\underline{A}}^{\uparrow}, \quad (C, \lambda) \mapsto (C, \lambda \circ f|_C).$$

If f reflects commutativity, then there is also a locale map $\underline{\Sigma}(\underline{g}) : \underline{\Sigma}_{\underline{B}} \to G^{\sharp}\underline{\Sigma}_{\underline{A}}$ in $[\mathcal{C}_B, \mathbf{Set}]$, externally given by (5.13).

If we think of $\underline{\Sigma}_{\underline{A}}$ as an internal state space, then ideally a $*$-automorphism $h : A \to A$ induces an isomorphism of locales $\underline{\Sigma}_{\underline{A}} \to \underline{\Sigma}_{\underline{A}}$ internal to the topos. However, the automorphism h induces a map $\hat{h} : \mathcal{C}_A \to \mathcal{C}_A$, and we need to take into account how h shuffles the contexts around. Instead of an isomorphism $\underline{\Sigma}_{\underline{A}} \to \underline{\Sigma}_{\underline{A}}$, we arrived at an isomorphism of locales of the form $H^{\sharp}\underline{\Sigma}_{\underline{A}} \to \underline{\Sigma}_{\underline{A}}$.

5.2 Contravariant Version

Turning to the contravariant approach, we see how $*$-homomorphisms induce continuous maps on the associated spectral presheaves.

In the contravariant model, we associate a pair $([\mathcal{C}_A^{op}, \mathbf{Set}], \underline{\Sigma}_A)$ to a von Neumann algebra A, consisting of a topos and a topological space within this topos. Here $\underline{\Sigma}_A$ is the spectral presheaf, equipped with the internal topology generated by the closed open subobjects as in Section 2.4. Motivated by the locale maps of the previous subsection, we show that a unital $*$-homomorphism $f : A \to B$ induces a pair

$$(F, \underline{\Sigma}(\underline{f})) : ([\mathcal{C}_A^{op}, \mathbf{Set}], \underline{\Sigma}_A) \to ([\mathcal{C}_B^{op}, \mathbf{Set}], \underline{\Sigma}_B),$$

consisting of a geometric morphism $F : [\mathcal{C}_A^{op}, \mathbf{Set}] \to [\mathcal{C}_B^{op}, \mathbf{Set}]$ and a continuous map $\underline{\Sigma}(\underline{f}) : F^{*}\underline{\Sigma}_B \to \underline{\Sigma}_A$ in the topos $[\mathcal{C}_A^{op}, \mathbf{Set}]$. The first question which we need to address is how the object $F^{*}\underline{\Sigma}_B$ is an internal topological space.

As an object, $F^*\underline{\Sigma}_B$ is described as follows. The functor $\underline{\Sigma}_B$ can be described as an étale bundle $\pi_B : \Sigma_B^{\downarrow} \to \mathcal{C}_B^{\downarrow}$. As a set, $\Sigma_B^{\downarrow}$ is equal to $\coprod_{D \in \mathcal{C}_B} \Sigma_D$, and $U \subseteq \Sigma_B^{\downarrow}$ is open iff

$$\text{If } \ (D, \lambda) \in U \ \text{ and } \ D' \subseteq D, \ \text{ then } \ (D', \lambda|_{D'}) \in U. \tag{5.20}$$

As an étale bundle, $F^*\underline{\Sigma}_B$ is the pullback of the étale bundle π_B along $\hat{f} : \mathcal{C}_A^{\downarrow} \to \mathcal{C}_B^{\downarrow}$, $\hat{f}(C) = f[C]$. The bundle $\hat{f}^*\pi_B : \hat{f}^*\Sigma_B^{\downarrow} \to \mathcal{C}_A^{\downarrow}$ obtained in this way can be described as follows. As a set, the total space $\hat{f}^*\Sigma_B^{\downarrow}$ is equal to $\coprod_{C \in \mathcal{C}_A} \Sigma_{f[C]}$. A subset $U \subseteq \hat{f}^*\Sigma_B^{\downarrow}$ is open iff it satisfies the following condition: if, for $C \in \mathcal{C}_A$, $\lambda \in \Sigma_{f[C]}$, $(C, \lambda) \in U$, and $C' \subseteq C$ in $\mathcal{C}_A$, then $(C', \lambda|_{C'}) \in U$. The map $\hat{f}^*\pi_B$ is simply $(C, \lambda) \mapsto C$.
The internal topology on $\underline{\Sigma}_B$ corresponds to a topology on $\Sigma_B^{\downarrow}$, which is coarser than the étale topology, but with respect to which π_B is still continuous. With respect to this topology, $U \in \mathcal{O}\Sigma_B^{\downarrow}$ iff it is étale open in $\Sigma_B^{\downarrow}$, and, in addition, for each $D \in \mathcal{C}_B$, the set $U_D := U \cap \Sigma_D$ is open in Σ_D. We can take the pullback of π_B along $\hat{f}$, with this new topology on $\Sigma_B^{\downarrow}$, and obtain a coarser topology on $\hat{f}^*\Sigma_B^{\downarrow}$ than the étale topology. In fact $U \in \mathcal{O}\Sigma_B^{\uparrow}$ iff it is étale open and, for each $C \in \mathcal{C}_A$, $U_C = U \cap \Sigma_{f[C]}$ is open in $\Sigma_{f[C]}$. The bundle $\hat{f}^*\pi_B : \hat{f}^*\Sigma_B^{\downarrow} \to \mathcal{C}_A^{\downarrow}$ is continuous with respect to this new topology. We have thus defined an internal topology on $F^*\underline{\Sigma}_B$. It is the topology generated by the objects $F^*\underline{U}$, where $\underline{U}$ is a closed open subobject of $\underline{\Sigma}_B$. Whenever we consider $F^*\underline{\Sigma}_B$ as a topological space, it is with respect to this topology.
Now that we have identified $F^*\underline{\Sigma}_B$ as an internal topological space, we can define the function $\underline{\Sigma}(\underline{f})$ and check whether it is continuous.

Proposition 5.2.1. *The natural transformation $\underline{\Sigma}(\underline{f}) : F^*\underline{\Sigma}_B \to \underline{\Sigma}_A$, given by*

$$\underline{\Sigma}(\underline{f})_C : \Sigma_{f[C]} \to \Sigma_C, \quad \lambda \mapsto \lambda \circ f|_C \tag{5.21}$$

is a continuous map of topological spaces in $[\mathcal{C}_A^{op}, \mathbf{Set}]$.

Proof. We leave the verification that $\underline{\Sigma}(\underline{f})$ is indeed a natural transformation to the reader. At the level of étale bundles, $\underline{\Sigma}(\underline{f})$ corresponds to the commuting triangle

$$\begin{tikzcd}
\hat{f}^*\Sigma_B^{\downarrow} \arrow[rr, "\Sigma(\underline{f})"] \arrow[dr, "\hat{f}^*\pi_B"'] & & \Sigma_B^{\downarrow} \arrow[dl, "\pi_A"] \\
& \mathcal{C}_A^{\downarrow} &
\end{tikzcd}$$

of continuous maps, where the total spaces of the bundles are equipped with the étale topologies. Note that naturality of $\underline{\Sigma}(\underline{f})$ amounts to continuity of $\Sigma(\underline{f})$ with respect to the étale topologies. Also note that the function $\Sigma(\underline{f})$ is the same function as (5.16) from the covariant version. The only difference between the approaches is in the topologies.
The function $\underline{\Sigma}(\underline{f})$ is internally continuous iff $\Sigma(\underline{f})$ is also continuous with respect to the coarser topologies on the total spaces (corresponding to the internal topologies). Let $\Sigma(f|_C) : \Sigma_{f[C]} \to \Sigma_C$ be the Gelfand dual of the $*$-homomorphism $f|_C : C \to f[C]$. A straightforward check reveals that for any $U \in \mathcal{O}\Sigma_A^{\downarrow}$, and $C \in \mathcal{C}_A$, one has

$$\Sigma(\underline{f})^{-1}(U)_C = \Sigma(f|_C)^{-1}(U_C) \in \mathcal{O}\Sigma_{f[C]}. \tag{5.22}$$

Combined with étale continuity, this observation proves that $\Sigma(\underline{f})$ is continuous with respect to the desired topologies. Note that étale continuity can be deduced from (5.22), as for $\lambda \in \Sigma_{f[C]}$, and $C' \subseteq C$, clearly $(\lambda \circ f|_C)|_{C'} = \lambda|_{C'} \circ f|_{C'}$. $\square$

Proposition 5.2.1 is the contravariant counterpart to Proposition 5.1.9. A $*$-automorphism $h : A \to A$ induces a homeomorphism $\underline{\Sigma}(\underline{h}) : H^*\underline{\Sigma}_A \to \underline{\Sigma}_A$. In the following section we consider how elementary propositions $\underline{[a \in \Delta]}$ transform under the frame isomorphism $\underline{\Sigma}(\underline{h})^{-1}$.

If we ignore the étale topology of $\underline{\Sigma}_B$ and consider it to be a locale rather than an internal space, then the bundle map $\Sigma(\underline{f})$, from the previous proof, can be seen as an internal locale map $\underline{\Sigma}(\underline{f}) : F^\sharp\underline{\Sigma}_B \to \underline{\Sigma}_A$, as in the covariant case.

As in the previous section, if $f : A \to B$ reflects commutativity, we can define a continuous map of spaces $\underline{\Sigma}(\underline{g}) : \underline{\Sigma}_B \to G^*\underline{\Sigma}_A$ in the topos $[\mathcal{C}_B^{op}, \mathbf{Set}]$, or see it as a locale map $\underline{\Sigma}(\underline{g}) : \underline{\Sigma}_B \to G^\sharp\underline{\Sigma}_A$ in the same topos.

5.3 Automorphisms and Daseinisation

We show how the key objects of both topos approaches, in particular the truth values, transform under the action of a $*$-automorphism.

Let A be a von Neumann algebra and $h : A \to A$ a $*$-automorphism. In this subsection we investigate how daseinised self-adjoint operators transform under h. We will be working with the contravariant version. However, if we switch from internal spaces to locales, switch inner and outer daseinisation, replace $\underline{\Sigma}_A$ by $\underline{\Sigma}_{\underline{A}}$, switch order-reversing and order-preserving, and replace $\downarrow$ by $\uparrow$ whenever it occurs as a superscript, then this subsection is about the covariant version.

For $a \in A_{sa}$, outer daseinisation defines a continuous map $\underline{\delta^o(a)} : \underline{\Sigma}_A \to \underline{\mathbb{R}}_l$, and inner daseinisation defines a continuous map $\underline{\delta^i(a)} : \underline{\Sigma}_A \to \underline{\mathbb{R}}_u$, where $\underline{\mathbb{R}}_l$ and $\underline{\mathbb{R}}_u$ are the spaces of lower and upper reals respectively. From the previous subsection we know that h induces a continuous map $\underline{\Sigma}(\underline{h}) : H^*\underline{\Sigma}_A \to \underline{\Sigma}_A$. We can compose these maps to obtain continuous maps

$$\underline{\delta^o(a)}_h : H^*\underline{\Sigma}_A \to \underline{\mathbb{R}}_l, \quad \underline{\delta^i(a)}_h : H^*\underline{\Sigma}_A \to \underline{\mathbb{R}}_u.$$

If we look at [37, Section 10], we may suspect that there is a relation between $\underline{\delta^o(h(a))}$ and $\underline{\delta^o(a)}_h$ and also between their inner counterparts. This is indeed the case, and we proceed to describe this relation.

Lemma 5.3.1. *Let A be a von Neumann algebra, $a \in A_{sa}$, Δ a Borel subset of $\sigma(a)$ (the spectrum of a), and $h : A \to A$ a $*$-automorphism. Then*

$$h(\chi_\Delta(a)) = \chi_\Delta(h(a)).$$

Proof. Let $A \subseteq \mathcal{B}(\mathcal{H})$. First note that $\sigma(h(a)) = \sigma(a)$. If $p(a)$ denotes a polynomial in a with complex coefficients, then $h(p(a)) = p(h(a))$. By norm-continuity of h and the Stone-Weierstrass Theorem, h restricts to an isomorphism of unital C*-algebras

$$\tilde{h} : C^*(a, 1) \to C^*(h(a), 1).$$

Let $W^*(a) = C^*(a, 1)''$ denote the weak as well as the σ-weak closure of $C^*(a, 1)$ in $\mathcal{B}(\mathcal{H})$. Any $*$-automorphism is σ-weakly continuous, implying that $\tilde{h}$ extends to an isomorphism of abelian von Neumann algebras

$$\tilde{h} : W^*(a) \to W^*(h(a))$$

As shown in [26, I.7.2] there exists an isomorphism of von Neumann algebras $i : L^\infty(\sigma(a), \mu) \to W^*(a)$, where μ is any scalar-valued spectral measure on $\sigma(a)$. For $W^*(h(a))$ we can construct an isomorphism $j : L^\infty(\sigma(a), \mu) \to W^*(h(a))$, using the same μ, since the spectra of a and $h(a)$ coincide. With these identifications we obtain an automorphism $\hat{h} = j^{-1} \circ \tilde{h} \circ i$ of the abelian von Neumann algebra $L^\infty(\sigma(a), \mu)$. By construction, for each polynomial expression $p(x) : \sigma(a) \to \mathbb{C}$, we deduce $\hat{h}([p(x)]) = [p(x)]$. By σ-weak continuity, $\hat{h}$ is the identity map. The desired claim follows from

$$h(\chi_\Delta(a)) = \tilde{h}(\chi_\Delta(a)) = \tilde{h}(i([\chi_\Delta])) = j([\chi_\Delta]) = \chi_\Delta(h(a)).$$

$\square$

Lemma 5.3.2. *If $a \leq_s b$ in A_{sa}, then $h(a) \leq_s h(b)$ in A_{sa}.*

Proof. Let $a \leq_s b$ in A_{sa}, let $(e^a_x)_{x\in\mathbb{R}}$ be the spectral resolution of a, and let $(e^b_x)_{x\in\mathbb{R}}$ be the spectral resolution of b. By assumption, for each $x \in \mathbb{R}$, $e^b_x \leq e^a_x$. The family of projections $e^{h(a)}_x := h(e^a_x)$ defines a spectral resolution for $h(a)$. This can be verified as h, restricted to the projections of A, yields an isomorphism of complete lattices. Alternatively, it follows straight from the previous lemma. Likewise, $e^{h(b)}_x := h(e^b_x)$ is a spectral resolution for $h(b)$. Any $*$-homomorphism h is a positive map, so, from the assumption, we deduce that for each $x \in \mathbb{R}$, $e^{h(b)}_x \leq e^{h(a)}_x$. We conclude that $h(a) \leq_s h(b)$. $\square$

Corollary 5.3.3. *If $a \in A_{sa}$, and $C \in \mathcal{C}_A$, then*

$$h(\delta^o(a)_C) = \delta^o(h(a))_{h[C]}, \quad h(\delta^i(a)_C) = \delta^i(h(a))_{h[C]}. \tag{5.23}$$

Proof. By the previous lemma the bijection $h|_{A_{sa}} : A_{sa} \to A_{sa}$ is monotone with respect to the spectral order. It has an order-preserving inverse, making it an order-isomorphism. As a consequence, $h|_{A_{sa}}$ is an isomorphism of boundedly complete lattices.

$$\begin{aligned} h(\delta^o(a)_C) &= h\left(\bigwedge\{b \in C_{sa} \mid b \geq_s a\}\right) \\ &= \bigwedge\{h(b) \in h[C_{sa}] \mid b \geq_s a\} \\ &= \bigwedge\{h(b) \in h[C]_{sa} \mid h(b) \geq_s h(a)\} \\ &= \bigwedge\{c \in h[C]_{sa} \mid c \geq_s h(a)\} \\ &= \delta^o(h(a))_{h[C]}. \end{aligned}$$

Inner daseinisation can be treated in the same way. $\square$

The continuous map $\underline{\delta^o(a)}_h : H^*\underline{\Sigma}_A \to \underline{\mathbb{R}}_l$ is externally described by the triangle of continuous maps

$$\begin{tikzcd} \hat{h}^*\Sigma_A^{\downarrow} \arrow[rr, "\delta^o(a)_h"] \arrow[dr, "\hat{h}^*\pi_A"'] & & \mathcal{R}_l \arrow[dl, "\pi_l"] \\ & \mathcal{C}_A^{\downarrow} & \end{tikzcd}$$

where the elements of $\mathcal{R}_l$ are pairs (C, s), with $C \in \mathcal{C}_A$ and $s : (\downarrow C) \to \mathbb{R}$ is an order-reversing function. For $\lambda \in \Sigma_{h[C]}$,

$$\delta^o(a)_h(C, \lambda) :\downarrow C \to \mathbb{R}, \quad D \mapsto \langle \delta^o(a)_D, \lambda \circ h|_C \rangle.$$

Note that

$$\begin{aligned} \langle \delta^o(a)_D, \lambda \circ h|_C \rangle &= \langle \delta^o(a)_D, \Sigma(h|_C)(\lambda) \rangle \\ &= \langle (h|_C)(\delta^o(a)_D), \lambda \rangle \\ &= \langle \delta^o(h(a))_{h[D]}, \lambda \rangle, \end{aligned}$$

where, in the last step, we used Corollary 5.3.3.
We need one more definition before we can state the relations we are looking for. Define the continuous map of spaces

$$\underline{\mathbb{R}}_l(\underline{h}) : H^*\underline{\mathbb{R}}_l \to \underline{\mathbb{R}}_l,$$

$$\underline{\mathbb{R}}_l(\underline{h})_C : \mathrm{OR}(\downarrow h[C], \mathbb{R}) \to \mathrm{OR}(\downarrow C, \mathbb{R}), \quad s \mapsto s \circ \hat{h}|_{\downarrow C}.$$

Proposition 5.3.4. *Let $h : A \to A$ be a $*$-automorphism, and take $a \in A_{sa}$. Then the following square of continuous maps of spaces is commutative:*

$$\begin{tikzcd} H^*\underline{\Sigma}_A \arrow[r, "H^*(\underline{\delta^o(h(a))})"] \arrow[d, "\underline{\Sigma}(\underline{h})"'] & H^*\underline{\mathbb{R}}_l \arrow[d, "\underline{\mathbb{R}}_l(\underline{h})"] \\ \underline{\Sigma}_A \arrow[r, "\underline{\delta^o(a)}"'] & \underline{\mathbb{R}}_l \end{tikzcd}$$

The same holds for inner daseinisation if we replace $\underline{\mathbb{R}}_l$ by $\underline{\mathbb{R}}_u$.

Next, we look at the action of the automorphism on the elementary propositions. The map $\underline{\Sigma}(\underline{h}) : H^*\underline{\Sigma} \to \underline{\Sigma}$ is continuous, providing us with an

inverse image map $\underline{\Sigma}(\underline{h})^{-1} : \mathcal{O}\underline{\Sigma} \to \mathcal{O}H^*\underline{\Sigma}$. Let $\underline{[a \in \Delta]}$ be the elementary proposition obtained by outer daseinisation of the spectral projection $\chi_\Delta(a)$. To describe the open $\underline{\Sigma}(\underline{h})^{-1}(\underline{[a \in \Delta]})$ of $\mathcal{O}H^*\underline{\Sigma}$ it is convenient to take the external descriptions.
For a $\lambda \in \Sigma_{h[C]}$, by definition $(C, \lambda) \in \Sigma(\underline{h})^{-1}([a \in \Delta])$ iff $\lambda \circ h|_C \in [a \in \Delta]_C$, and this happens iff

$$1 = \langle \delta^o(\chi_\Delta(a))_C, \lambda \circ h|_C \rangle = \langle h(\delta^o(\chi_\Delta(a))_C), \lambda \rangle.$$

This can be simplified further using Lemma 5.3.1:

$$h(\delta^o(\chi_\Delta(a))_C) = \delta^o(h(\chi_\Delta(a)))_{h[C]} = \delta^o(\chi_\Delta(h(a)))_{h[C]}.$$

We conclude that

$$\Sigma(\underline{h})^{-1}([a \in \Delta]) = \{(C, \lambda) \in \hat{h}^*\Sigma_\downarrow \mid \langle \delta^o(\chi_\Delta(h(a)))_{h[C]}, \lambda \rangle = 1\}.$$

Note that if $\underline{[a \in \Delta]} = \underline{\delta(a)}^{-1}(\underline{\Delta})$, and

$$\underline{\delta(a)}_h = \langle \underline{\delta^i(a)}_h, \underline{\delta^o(a)}_h \rangle : H^*\underline{\Sigma} \to \underline{\mathbb{R}}_u \times \underline{\mathbb{R}}_l,$$

then

$$\underline{\Sigma}(\underline{h})^{-1}(\underline{[a \in \Delta]}) = \underline{\delta(a)}_h^{-1}(\underline{\Delta}).$$

In order to obtain a truth value, we would like to combine this open with a state, seen as a probability valuation. There is one problem however, as the open lies in $\mathcal{O}\hat{h}^*\Sigma_\downarrow$ and not in $\mathcal{O}\Sigma_\downarrow$. Recall that a state ψ defines a probability valuation by combining the probability valuations $\mu^C : \mathcal{O}\Sigma_C \to [0,1]$, corresponding to the local states $\psi|_C : C \to \mathbb{C}$. Viewed externally, the valuation $\underline{\mu}$ is given by

$$\mu : \mathcal{O}\Sigma_\downarrow \to \mathrm{OR}(\mathcal{C}, [0,1]) \quad \mu(U)(C) = \mu^C(U_C).$$

In very much the same way, a state ψ defines a function

$$\mu_h : \mathcal{O}\hat{h}^*\Sigma_\downarrow \to \mathrm{OR}(\mathcal{C}, [0,1]) \quad \mu_h(U)(C) = \mu^{h[C]}(U_C).$$

Note that for $U \in \mathcal{O}\hat{h}^*\Sigma_\downarrow$, $U_C \in \mathcal{O}\Sigma_{h[C]}$. The reader is invited to check that μ_h satisfies all conditions required to turn the corresponding internal function $\underline{\mu}_h : \mathcal{O}H^*\underline{\Sigma} \to \underline{[0,1]}_l$ into a probability valuation.
Using $\underline{\mu}_h$, we can once again obtain truth values. Since daseinisation and automorphisms interact in a straightforward way, we obtain the following result.

Theorem 5.3.5. *The following two forcing conditions are equivalent:*

1. $C \Vdash \underline{\mu}_h(\underline{\Sigma}(\underline{h})^{-1}(\underline{[a \in \Delta]})) = \underline{1}$,
2. $h[C] \Vdash \underline{\mu}(\underline{[h(a) \in \Delta]}) = \underline{1}$.

Proof. Spelling out the first condition gives

$$\mu^{h[C]}(\{\lambda \in \Sigma_{h[C]} \mid \langle \delta^o(\chi_\Delta(h(a)))_{h[C]}, \lambda\rangle = 1\}) = 1$$

or, equivalently

$$\psi(\delta^o(\chi_\Delta(h(a)))_{h[C]}) = 1,$$

which is just the second forcing relation of the proposition. □

As daseinisation of self-adjoint operators commutes with $*$-automorphisms, the elementary propositions of both the covariant and contravariant approach transform in a simple way under the action of $\underline{\Sigma}(\underline{h})^{-1}$. As a consequence, the theorem given above states that if S is the cosieve or sieve of the proposition $\underline{\Sigma}(\underline{h})^{-1}(\underline{[a \in \Delta]})$ relative to some state ψ, then $h[S]$ is the cosieve or sieve of $\underline{[h(a) \in \Delta]}$ relative to that same state ψ.

6

Independence Conditions and Sheaves

In algebraic quantum field theory we work with a functor mapping regions of spacetime to C*-algebras, rather than with a single C*-algebra. Following Nuiten [68], extending the covariant approach, such a functor will be reformulated as a contravariant functor mapping regions of spacetime to pairs consisting of a topos and a commutative C*-algebra internal to that topos. Taking a physically motivated 'covering relation' on the spacetime regions, we can subsequently ask whether this functor is a sheaf. Although the full sheaf condition cannot be expected to hold for physically relevant quantum field theories, a slightly weaker condition turns out to be related to various kinematical independence conditions of algebraic quantum field theory.

After an introduction, in Section 6.2 we derive the sheaf condition in the original setting used by Nuiten. This leads to a new independence condition, called *strong locality*, which is discussed in Section 6.3. Finally, in Section 6.4 we derive another sheaf condition; the difference is that the original sheaf condition used a category of topoi with internal commutative rings, rather than topoi with internal commutative C*-algebras.

Theorem 6.4.6 is the central result of this chapter, as it relates the C*-algebraic version of the sheaf condition to C*-independence. Since the full sheaf condition turns out to be a strong condition, we do not succeed in finding a non-trivial topos model for algebraic quantum field theories in the sense that we do not find a single Grothendieck topos such that from within the internal perspective of the topos, quantum physics formally resembles classical physics. However, if we abandon this restricted view for a moment, we may hope that the relation between independence conditions and sheaf conditions found in this chapter turn out to be of use in future developments of quantum toposophy.

6.1 Introduction and Motivation

In the work of Nuiten a net of operator algebras corresponds to a contravariant functor mapping regions of spacetime to ringed topoi. Independence conditions on the net relate to sheaf conditions on the functor. We briefly sketch this relation and consider why it is of interest.

6.1.1 Nets of Operator Algebras as Functors

Let $\mathcal{V}(X)$ denote some poset of causally complete opens of a spacetime manifold X, partially ordered by inclusion. We assume that $\mathcal{V}(X)$, when ordered by inclusion, is a lattice[1]. Assume that we have a mapping $O \mapsto A(O)$, associating to each $O \in \mathcal{V}(X)$ a unital C*-algebra $A(O)$. Further, assume that the map $O \mapsto A(O)$ is *isotonic*, in the sense that if $O_1 \subseteq O_2$, then $A(O_1) \subseteq A(O_2)$. Such a mapping $O \mapsto A(O)$ is called a *net of operator algebras*.
For each $O \in \mathcal{V}(X)$, the covariant approach to quantum theory of [48] provides us with a pair $(\mathcal{T}_O, \underline{A(O)})$ consisting of a topos $\mathcal{T}_O = [\mathcal{C}_{A(O)}, \mathbf{Set}]$ and a unital commutative C*-algebra $\underline{A(O)}$ in this topos.
As we have seen in Section 5, and will briefly review in Section 6.2, if $O_1 \subseteq O_2$, we can associate to this inclusion a pair

$$(I, \underline{i}) : (\mathcal{T}_{O_2}, \underline{A(O_2)}) \to (\mathcal{T}_{O_1}, \underline{A(O_1)}), \tag{6.1}$$

where $I : \mathcal{T}_{O_2} \to \mathcal{T}_{O_1}$ is a geometric morphism, and $\underline{i} : I^* \underline{A(O_1)} \to \underline{A(O_2)}$ is a $*$-homomorphism , internal to the topos $\mathcal{T}_{O_2}$.
Associating to each O the pair $(\mathcal{T}_O, \underline{A(O)})$, and to each inclusion $O_1 \subseteq O_2$ a pair of arrows (6.1), a net $O \mapsto \underline{A(O)}$ defines a contravariant functor from $\mathcal{V}(X)$ to the category **cCTopos** of topoi with internal unital commutative C*-algebras. However, for technical reasons a different target category **RingSp** is used. This category differs from **cCTopos** in two ways. Instead of using all (Grothendieck) topoi, only topoi of the form $Sh(X)$, the topos of sheaves on a topological space X, will be considered.

[1] That is, any pair $O_1, O_2 \in \mathcal{V}(X)$ has a greatest lower bound $O_1 \wedge O_2$, and a least upper bound $O_1 \vee O_2$.

In addition, instead of using internal commutative C*-algebras, the more general class of internal commutative rings will be used. In Section 6.4 we will drop this last condition and consider a fully C*-algebraic version. Viewing the net $O \mapsto A(O)$ as a functor $\mathcal{V}(X)^{op} \to \mathbf{RingSp}$, and noting that **RingSp** is complete as a category, we can ask if this functor is a sheaf. Of course, we need to specify something like a covering relation before we can even ask this question. Let O_1 and O_2 be two spacelike separated regions of spacetime. Then we say that $O := O_1 \vee O_2$ is covered by O_1 and O_2.

The investigation of this sheaf condition was first performed by Nuiten [68]. As it turns out, for physically reasonable nets the functor $\mathcal{V}(X)^{op} \to \mathbf{RingSp}$ is not a sheaf. However, it does come close to a sheaf. Technically, what is meant by this is that under a mild condition the descent morphism is a local geometric surjection. This condition will be called *strong locality*. Strong locality implies microcausality, and is implied by C*-independence in the product sense (see Definition 6.3.1, taking $A = A(O_1)$ and $B = A(O_2)$, letting O_1 and O_2 be spacelike separated).

For the C*-algebraic version $\mathcal{V}(X)^{op} \to \mathbf{ucCSp}$, defined in Section 6.4, the sheaf condition is shown to be equivalent to C*-independence of the net, together with the condition

$$\forall C \in \mathcal{C}_{A(O_1 \vee O_2)} \quad (C \cap A(O_1)) \vee (C \cap A(O_2)) = C,$$

for all pairs (O_1, O_2) of spacelike separated regions.

6.1.2 Motivation

Why do we consider these constructions to be of interest? For a moment, suppose that we are sceptical about the specific topos models for quantum physics constructed so far [16, 37, 48]. Even so, the discussion in [16] may suggest that using the language of (pre)sheaves over posets of contexts is a natural step in studying the foundations of quantum theory. Indeed, the work [1, 3] studies non-locality and contextuality using (pre)sheaves, without invoking any topos theory. Furthermore, studying the relation between the poset $\mathcal{C}_A$ and the algebra A may be of interest in itself [45, 46]. The research described below fits nicely within such programs. As an example, consider the following simple lemma by Nuiten [68]. Let (A, B) be a pair of C*-algebras associated to spacelike separated regions by some net. This net is said to satisfy microcausality iff such algebras commute, i.e. $[A, B] = \{0\}$. This condition of microcausality, then, is equivalent to the claim that the poset morphism

$$\mathcal{C}_{A \vee B} \to \mathcal{C}_A \times \mathcal{C}_B, \quad C \mapsto (C \cap A, C \cap B)$$

has a left adjoint. Hence, at the level of contexts, microcausality can be captured categorically!
Next, assume that we are curious about the ideas of Isham and Döring to the effect that physical theories should be formulated in possibly non-trivial (i.e. non-**Set**) topoi. Apart from the copresheaf model $[\mathcal{C}_A, \mathbf{Set}]$, insofar as these count, the only nontrivial example is the motivating one $[\mathcal{C}_A^{op}, \mathbf{Set}]$. The discussion given below may be of help in finding new topos models, as a central theme is to encode independence conditions on nets of algebras as sheaf conditions. At this point, the reader may object that the sheaves discussed below are not sheaves on a site [63, Chapter III], but generalisations thereof. Nevertheless, as suggested by Subsection 6.3.2, there are relations between Nuiten's sheaves and sheaves on sites.
Finally, we can take the stance that we are interested in topos theory, but not so much in topos models for quantum theory. In this case, Section 6.4 may be of interest when seen as a discussion of particular C*-algebras in topoi.

The text below is divided into three parts. Section 6.2 discusses the sheaf condition of [68]. One difference from the original treatment is that we do not assume the net of operator algebras to be additive. The sheaf condition leads to a new independence condition, which we call *strong locality*. In Section 6.3 strong locality is compared to other locality conditions, such as microcausality and C*-independence. In relation to this, we describe these locality conditions at the level of commutative subalgebras $\mathcal{C}$. In Section 6.4 we revisit the sheaf condition in a slightly different setting. We wish to work with C*-algebras, rather than all unital commutative rings. This leads to certain technical complications. The sheaf condition is subsequently related to C*-independence of the net.

6.2 Nuiten's sheaves

This section reviews the sheaf condition as introduced by Joost Nuiten in [68]. This is a stepping stone to the C-algebraic version treated in Section 6.4, which uses the category* **ucCSp** *instead of* **RingSp***, whilst* *also motivating the notion of strong locality, the central concept of Section 6.3.Note that in [68], additivity of the net of operator algebras is assumed at certain places; our treatment below shows that we do not need to assume this.*

In the covariant topos model, to a unital C*-algebra A in **Set** we associate a topos $\mathcal{T}_A = [\mathcal{C}_A, \mathbf{Set}]$, as well as a unital commutative C*-algebra $\underline{A}$ in this topos. As in Section 5, any $*$-homomorphism $f : A \to B$ induces an order-preserving function

$$\hat{f} : \mathcal{C}_A \to \mathcal{C}_B, \quad \hat{f}(C) = f[C].$$

In turn, this function induces an essential geometric morphism $F : \mathcal{T}_A \to \mathcal{T}_B$, (and an internal $*$-homomorphism $\underline{A} \to F^*\underline{B}$). In this way, we obtain a covariant functor from the category of C*-algebras and $*$-homomorphisms in **Set** to the category of Grothendieck topoi and geometric morphisms. Alternatively, if the $*$-homomorphism reflects commutativity in the sense that

$$\forall a, b \in A, \quad [f(a), f(b)] = 0 \Rightarrow [a, b] = 0,$$

the inverse image map defines an order-preserving function

$$f^{-1} : \mathcal{C}_B \to \mathcal{C}_A, \quad C \mapsto f^{-1}(C),$$

which induces an essential geometric morphism $G : \mathcal{T}_B \to \mathcal{T}_A$, and defines a contravariant functor from the category of C*-algebras and $*$-homomorphisms that reflect commutativity to the category of Grothendieck topoi and geometric morphisms. It is this contravariant option with respect to which we formulate the sheaf condition. Before considering this sheaf condition, we first reflect on how restrictive the notion of reflecting commutativity is (no pun intended).

Lemma 6.2.1. *Let $\psi : A \to B$ is a unit-preserving $*$-homomorphism.*

1. *If $ker(\psi) = \{0\}$, then ψ reflects commutativity.*
2. *If A is simple, then ψ reflects commutativity.*
3. *If $A = \mathcal{B}(\mathcal{H})$, for a separable Hilbert space $\mathcal{H}$, then ψ reflects commutativity iff $ker(\psi) = \{0\}$.*

Proof. The first claim follows from

$$0 = [\psi(a), \psi(b)] = \psi([a, b]) \quad \Rightarrow \quad [a, b] \in ker(\psi) = \{0\}.$$

For a simple C*-algebra A, the kernel of a $*$-homomorphism $\psi : A \to B$ is either A or $\{0\}$. As ψ preserves the unit, it follows that $ker(\psi) = \{0\}$, and ψ reflects commutativity.
Next, let $A = \mathcal{B}(\mathcal{H})$. Assume that ψ reflects commutativity, and let $a \in ker(\psi)$. As $\psi(a) = 0$, $\psi(a)$ commutes with $\psi(b)$, for each $b \in \mathcal{B}(\mathcal{H})$. By assumption, a commutes with all elements of $\mathcal{B}(\mathcal{H})$. As the centre of $\mathcal{B}(\mathcal{H})$ is $\mathbb{C}1$, we deduce $a = \lambda 1$ for some $\lambda \in \mathbb{C}$. By assumption, $0 = \psi(\lambda 1) = \lambda\psi(1) = \lambda$. Consequently, $a = 0$, and hence $ker(\psi) = \{0\}$. □

Corollary 6.2.2. *For a a pure state ρ of $\mathcal{B}(\mathcal{H})$, the corresponding GNS representation $\pi_\rho : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}_\rho)$ reflects commutativity iff ρ is normal.*

Proof. If ρ is normal, then π_ρ is faithful, and therefore reflects commutativity. If ρ is not normal, then by [60, Theorem 10.4.6], $\pi_\rho(\mathcal{K}) = \{0\}$, where $\mathcal{K}$ denotes the ideal of compact operators. Clearly, π_ρ does not reflect commutativity in this case. □

As another example of a $*$-homomorphism that does not reflect commutativity, consider a continuous field of C*-algebras $(A, \{A_x, \psi_x\}_{x\in X})$ (see e.g. [25, Chapter 10]). Here A is a C*-algebra, X is a locally compact Hausdorff space, and for each $x \in X$ we are given a surjective $*$-homomorphism $\psi_x : A \to A_x$. If for $a \in A$ we define $a(x) := \psi_x(a)$, then a can be identified with the family $\{a(x)\}_{x\in X}$. Note that

$$[a, b]_A = 0 \quad \Leftrightarrow \quad \forall x \in X \; [a(x), b(x)]_{A_x} = 0.$$

If there exists an $y \in X$ such that A_y is commutative, whereas at least one A_x is non-commutative, then the $*$-homomorphism $\psi_y : A \to A_y$ does not reflect commutativity.
After these remarks on reflection of commutativity, we return to formulating the sheaf condition. In Section 5 we saw that the category of C*-algebras and commutativity-reflecting $*$-homomorphisms was mapped contravariantly to the category $\mathbf{cCTopos}_N$. We drop the subscript N as the covariant counterpart $\mathbf{cCTopos}_D$ does not appear in the rest of this chapter. For technical reasons[2], we replace **cCTopos** by the larger category **RingTopos**.

[2]This assumption helps in calculating limits, as rings behave in a simpler way under the action of inverse image functors.

Definition 6.2.3. *The category* **RingTopos** *of ringed topoi is given by:*

- *Objects are pairs* $(\mathcal{E}, \underline{R})$*, with* $\mathcal{E}$ *a topos, and* $\underline{R}$ *a commutative ring with unit, internal to* $\mathcal{E}$*.*
- *Arrows* $(F, \underline{f}) : (\mathcal{E}, \underline{R}) \to (\mathcal{F}, \underline{S})$ *are given by geometric morphisms* $F : \mathcal{E} \to \mathcal{F}$*, and ring homomorphisms* $\underline{f} : F^*\underline{S} \to \underline{R}$ *in* $\mathcal{E}$*.*
- *Composition is defined by* $(G, \underline{g}) \circ (F, \underline{f}) = (G \circ F, \underline{f} \circ F^*\underline{g})$*.*

If $(G, \underline{g})$ is induced by the commutativity-reflecting $*$-homomorphism $f : A \to B$, the inverse image functor $G^* : \mathcal{T}_A \to \mathcal{T}_B$ is given by: for $\underline{F} \in \mathcal{T}_A$ and $D \in \mathcal{C}_B$, then $G^*(\underline{F})(D) = \underline{F}(f^{-1}(D))$. The direct image can easily be described using $G_* = F^*$. Here $F : \mathcal{T}_A \to \mathcal{T}_B$ is the essential geometric morphism induced by $\hat{f} : \mathcal{C}_A \to \mathcal{C}_B$. So if $\underline{F} \in \mathcal{T}_B$ and $C \in \mathcal{C}_A$, then $G_*(\underline{F})(C) = \underline{F}(f[C])$. The ring part of the morphism of ringed topoi, $\underline{g} : G^*\underline{A} \to \underline{B}$, is the natural transformation $\underline{g}_D : f^{-1}(D) \to D$ defined as the restriction of f to $f^{-1}(D)$, i.e.,

$$\underline{g}_D = f|_{f^{-1}(D)}.$$

Instead of using the category **RingTopos**, we now restrict to the subcategory **RingSp** for this makes it easier to calculate limits later on.

Definition 6.2.4. *The category* **RingSp** *of ringed spaces is the following subcategory of* **RingTopos***:*

- *Objects are pairs* $(X, \underline{R})$*, with* X *a topological space, and* $\underline{R}$ *a commutative ring with unit internal to* $Sh(X)$*.*
- *Arrows* $(f, \underline{f}) : (X, \underline{R}) \to (Y, \underline{S})$ *are given by continuous maps* $f : X \to Y$*, and ring homomorphisms* $\underline{f} : F^*\underline{S} \to \underline{R}$ *in* $Sh(X)$*, where* $F : Sh(X) \to Sh(Y)$ *is the geometric morphism induced by* f*.*

With slight abuse of notation, we will also write $(Sh(X), \underline{R})$ *for the object* $(X, \underline{R})$*, as well as writing* $(F, \underline{f})$ *for an arrow* $(f, \underline{f})$*, emphasising that* **RingSp** *is indeed a subcategory of* **RingTopos***.*

As we have seen several times: if P is a poset, then P can be seen as a topological space $P_\uparrow$ by equipping it with the Alexandroff (upper set) topology, defined as

$$U \in \mathcal{O}P \quad \leftrightarrow \quad \forall p \in P \ \ (p \in U) \wedge (p \leq q) \to (q \in U).$$

If we identify the elements $p \in P$ with the Alexandroff opens $(\uparrow p) \in \mathcal{O}P_{\uparrow}$, the topos $Sh(P_{\uparrow})$ is isomorphic to the topos $[P, \mathbf{Set}]$. This implies that for any C*-algebra A, the pair $(\mathcal{T}_A, \underline{A})$ lies in **RingSp**. Any order-preserving map $P \to Q$ of posets is an Alexandroff continuous map. A straightforward check then reveals that the geometric morphism $G : Sh(\mathcal{C}_B) \to Sh(\mathcal{C}_A)$ induced by the continuous map $f^{-1} : \mathcal{C}_B \to \mathcal{C}_A$, is, under the identification $Sh(\mathcal{C}) \cong [\mathcal{C}, \mathbf{Set}]$, the same geometric morphism as the one induced by f^{-1}, seen as a functor on poset categories. The morphisms of ringed topoi induced by $*$-homomorphisms are present in **RingSp** as well.

Now that we have defined our category of interest, we move to *algebraic quantum field theory* (or AQFT for short) and derive Nuiten's sheaf condition. Consider the following situation: we are given a net $\mathcal{O} \mapsto A(\mathcal{O})$ of operator algebras. Throughout this chapter we assume that for each region of spacetime $\mathcal{O}$ under consideration, the C*-algebra $A(\mathcal{O})$ is unital. For the moment, the only other assumption on the net is isotony, i.e., if $O_1 \leq O_2$, then $A(O_1) \subseteq A(O_2)$.

Let $\mathcal{V}(X)$ denote a poset of certain causally complete opens of a spacetime manifold X, partially ordered by inclusion. As before, the details of $\mathcal{V}(X)$ are unimportant, but we will assume that the poset has binary joins and meets. Let $A : \mathcal{V}(X) \to \mathbf{CStar_{rc}}$ be a net of C*-algebras, where the subscript *rc* means that we restrict ourselves to morphisms that reflect commutativity. By isotony, the maps $A(O_1) \to A(O_2)$, corresponding to inclusions $O_1 \subseteq O_2$, are inclusion maps, clearly satisfying the constraint of reflecting commutativity. We therefore obtain a contravariant functor

$$\underline{A} : \mathcal{V}(X)^{\mathrm{op}} \to \mathbf{RingSp}, \quad \underline{A}(O) = (\mathcal{T}_{A(O)}, \underline{A(O)}),$$

where the inclusion $O_1 \subseteq O_2$ is mapped to $\underline{A}(O_1 \subseteq O_2) = (I, \underline{i})$, with

$$I^* : \mathcal{T}_{A(O_1)} \to \mathcal{T}_{A(O_2)} \quad I^*(\underline{F})(C) = \underline{F}(C \cap A(O_1)),$$

$$\underline{i} : I^*(\underline{A(O_1)}) \to \underline{A(O_2)}, \quad \underline{i}_C : C \cap A(O_1) \hookrightarrow C,$$

and where the ring morphisms are inclusion maps.

Let $O_1, O_2 \in \mathcal{V}(X)$. It will be convenient to introduce the following notation:

$$A_i := A(O_i), \quad A_{1\wedge 2} := A(O_1 \wedge O_2), \quad A_{1\vee 2} := A(O_1 \vee O_2),$$

$$(\mathcal{T}_i, \underline{A}_i) := (\mathcal{T}_{A_i}, \underline{A_i}),$$

$$(\mathcal{T}_{1\wedge 2}, \underline{A}_{1\wedge 2}) := (\mathcal{T}_{A_{1\wedge 2}}, \underline{A_{1\wedge 2}}),$$
$$(\mathcal{T}_{1\vee 2}, \underline{A}_{1\vee 2}) := (\mathcal{T}_{A_{1\vee 2}}, \underline{A_{1\vee 2}}).$$

Consider the following diagram in **RingSp**, where the morphisms $(I_i, \underline{i}_i)$ are induced by the inclusions $O_1 \wedge O_2 \subseteq O_i$ and the morphisms $(J_i, \underline{j}_i)$ are induced by the inclusions $O_i \subseteq O_1 \vee O_2$:

$$\begin{tikzcd}
(\mathcal{T}_{1\vee 2}, \underline{A}_{1\vee 2}) \arrow[dr, dotted, "{(H,\underline{h})}"] \arrow[drr, bend left, "{(J_1,\underline{j}_1)}"] \arrow[ddr, bend right, "{(J_2,\underline{j}_2)}"'] & & \\
& (\mathcal{F}, \underline{R}) \arrow[r, "{(P_1,\underline{p}_1)}"'] \arrow[d, "{(P_2,\underline{p}_2)}"] & (\mathcal{T}_1, \underline{A}_1) \arrow[d, "{(I_1,\underline{i}_1)}"'] \\
& (\mathcal{T}_2, \underline{A}_2) \arrow[r, "{(I_2,\underline{i}_2)}"] & (\mathcal{T}_{1\wedge 2}, \underline{A}_{1\wedge 2}).
\end{tikzcd}$$

The bottom square of the diagram is a pullback. As the category **RingSp** is complete, this pullback exists, and we will compute it below. We think of the pullback object $(\mathcal{F}, \underline{R})$ as the ringed topos of matching families for the cover $\{O_1, O_2\}$ of $O_1 \vee O_2$. We are now ready to formulate Nuiten's sheaf condition.

Definition 6.2.5. *The functor $A : \mathcal{V}(X)^{op} \to$ **RingSp** is said to be a sheaf iff for each pair $O_1, O_2 \in \mathcal{V}(X)$ of spacelike separated opens, the descent morphism*

$$(H, \underline{h}) : (\mathcal{T}_{1\vee 2}, \underline{A}_{1\vee 2}) \to (\mathcal{F}, \underline{R}).$$

is an isomorphism of ringed spaces.

Let us briefly compare this sheaf condition with the sheaf condition used for topoi $Sh(X)$, where X is a topological space. Let $F : \mathcal{O}X^{op} \to$ **Set** be a presheaf, and $U \in \mathcal{O}X$ an open subset covered by smaller open subsets $\{U_i\}_{i\in I}$, in the sense of $U = \bigcup_{i\in I} U_i$. Consider the equalizer

$$\begin{tikzcd}
E \arrow[r, hook] & \prod_{i\in I} F(U_i) \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{i\neq j} F(U_i \wedge U_j)
\end{tikzcd},$$

where

$$p((f_k)_{k\in I})_{ij} := f_i|_{U_i \wedge U_j}, \quad q((f_k)_{k\in I})_{ij} := f_j|_{U_i \wedge U_j}.$$

The presheaf F is a sheaf iff for each such U and $\{U_i\}_{i\in I}$, the descent morphism

$$F(U) \to E, \quad f \mapsto (f|_{U_i})_{i\in I},$$

is an isomorphism. Note that we can replace **Set** by any complete category, such as **RingSp**, leading to the sheaf condition of the previous definition.

The next step is to make the descent morphism $(H, \underline{h})$ explicit in order to understand the sheaf condition at the level of the net $A : \mathcal{V}(X) \to \mathbf{CStar}$, and to investigate if this mathematically sensible condition is plausible on physical grounds as well. We start by finding the space X of the topos $\mathcal{F} = Sh(X)$. The geometric morphisms I_i and J_i are induced by order-preserving functions

$$y_i : \mathcal{C}_{1\vee 2} \to \mathcal{C}_i, \quad y_i(C) = C \cap A_i,$$

$$x_i : \mathcal{C}_i \to \mathcal{C}_{1\wedge 2}, \quad x_i(C) = C \cap A_{1\wedge 2},$$

where we used the notation $\mathcal{C}_i := \mathcal{C}(A(\mathcal{O}_i))$, etc. Define the poset

$$\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2 = \{(C_1, C_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \mid C_1 \cap A_{1\wedge 2} = C_2 \cap A_{1\wedge 2}\},$$

with partial order $(D_1, D_2) \leq (C_1, C_2)$ iff $D_1 \subseteq C_1$ and $D_2 \subseteq C_2$, and (order-preserving) projection maps $\pi_i : \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2 \to \mathcal{C}_i$. In the category **Poset** we obtain the pullback square

$$\begin{tikzcd}
\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2 \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & \mathcal{C}_1 \arrow[d, "x_1"'] \\
\mathcal{C}_2 \arrow[r, "x_2"] & \mathcal{C}_{1\wedge 2}
\end{tikzcd} .$$

Taking the Alexandroff upper topology of a poset defines a functor Al : **Poset** $\to$ **Top**, where **Top** is the category of topological spaces and continuous maps. This functor preserves [3] limits [68]. With respect to the Alexandroff upper topologies, the previous square becomes a pullback in **Top**. It will turn out that $\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$, equipped with the Alexandroff upper topology, is the space we are looking for. Once this has been shown, we will conclude that $\mathcal{F} = [\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2, \mathbf{Set}]$. Let

$$P_i : [\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2, \mathbf{Set}] \to [\mathcal{C}_i, \mathbf{Set}]$$

[3] Note that if we replaced **Top** by the category of locales or topoi, then the functor Al would not preserve all limits.

denote the geometric morphisms corresponding to the projections π_i. The next step in describing the descent morphism is to compute the following pushout of rings in $\mathcal{F}$:

$$\begin{tikzcd} P_1^* I_1^* \underline{A}_{1\wedge 2} \arrow[r, "P_1^* \underline{i}_1"] \arrow[d, "P_2^* \underline{i}_2"'] & P_1^* \underline{A}_1 \arrow[d, "\underline{p}_1"] \\ P_2^* \underline{A}_2 \arrow[r, "\underline{p}_2"'] & \underline{R} \end{tikzcd} ,$$

where we used $P_1^* I_1^* = P_2^* I_2^*$.
In a functor category $[\mathcal{C}, \mathbf{Set}]$, an object $\underline{R}$ is an internal ring iff it is a functor $\underline{R} : \mathcal{C} \to \mathbf{Ring}$. This entails that we can compute the pushout $\underline{R}$ stage-wise. Taking $(C_1, C_2) \in \mathcal{C} \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$, we compute the pushout of rings in **Set**. Using

$$P_1^* I_1^* \underline{A}_{1\wedge 2}(C_1, C_2) = C_1 \cap A_{1\wedge 2} = C_2 \cap A_{1\wedge 2} = P_2^* I_2^* \underline{A}_{1\wedge 2}(C_1, C_2),$$

and $P_1^* \underline{A}_1(C_1, C_2) = C_1$ and $P_2^* \underline{A}_2(C_1, C_2) = C_2$, we obtain the pushout square

$$\begin{tikzcd} C_1 \cap A_{1\wedge 2} \arrow[r] \arrow[d] & C_1 \arrow[d, "-\otimes 1"] \\ C_2 \arrow[r, "1\otimes -"'] & C_1 \otimes_{C_1 \cap A_{1\wedge 2}} C_2 \end{tikzcd} ,$$

where the unlabelled arrows are inclusion maps, we used that $C_1 \cap A_{1\wedge 2} = C_2 \cap A_{1\wedge 2}$, and we used that for commutative rings, the pushout ring is given by the tensor product of C_1 and C_2, viewed as $C_1 \cap C_2 \cap A_{1\wedge 2}$-algebras.

Lemma 6.2.6. *Define $(\mathcal{F}, \underline{R}) \in$ **RingSp** as $\mathcal{F} = [\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2, \mathbf{Set}]$ and put*

$$\underline{R} : \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2 \to \mathbf{Set}, \quad \underline{R}(C_1, C_2) = C_1 \otimes_{C_{12}} C_2,$$

where we used the notation $C_{12} = C_1 \cap C_2 \cap A_{1\wedge 2}$. If $(D_1, D_2) \leq (C_1, C_2)$ in $\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$, the corresponding ring homomorphism is simply

$$\underline{R}(\leq) : D_1 \otimes_{D_{12}} D_2 \to C_1 \otimes_{C_{12}} C_2 \quad a \otimes b \mapsto a \otimes b.$$

Define $\underline{p}_i : P_i^* \underline{A}_i \to \underline{R}$ as

$$(\underline{p}_1)_{(C_1, C_2)} : C_1 \to C_1 \otimes_{C_{12}} C_2 \quad a \mapsto a \otimes 1,$$

$$(\underline{p}_2)_{(C_1,C_2)} : C_2 \to C_1 \otimes_{C_{12}} C_2 \quad b \mapsto 1 \otimes b.$$

Then the following diagram is a pullback in **RingSp***:*

$$\begin{tikzcd}
(\mathcal{F}, \underline{R}) \arrow[r, "(P_1,\underline{p}_1)"] \arrow[d, "(P_2,\underline{p}_2)"'] & (\mathcal{T}_1, \underline{A}_1) \arrow[d, "(I_1,\underline{i}_1)"] \\
(\mathcal{T}_2, \underline{A}_2) \arrow[r, "(I_2,\underline{i}_2)"'] & (\mathcal{T}_{1\wedge 2}, \underline{A}_{1\wedge 2})
\end{tikzcd} .$$

Proof. Suppose that we have the following commutative diagram in **RingSp**:

$$\begin{tikzcd}
(Sh(X), \underline{S}) \arrow[dr, dotted, "(H,\underline{h})"] \arrow[drr, bend left, "(F_1,\underline{f}_1)"] \arrow[ddr, bend right, "(F_2,\underline{f}_2)"'] & & \\
& (\mathcal{F}, \underline{R}) \arrow[r, "(P_1,\underline{p}_1)"'] \arrow[d, "(P_2,\underline{p}_2)"] & (\mathcal{T}_1, \underline{A}_1) \arrow[d, "(I_1,\underline{i}_1)"'] \\
& (\mathcal{T}_2, \underline{A}_2) \arrow[r, "(I_2,\underline{i}_2)"] & (\mathcal{T}_{1\wedge 2}, \underline{A}_{1\wedge 2})
\end{tikzcd} .$$

We need to show that there exists a unique $(H, \underline{h})$ completing the diagram. By definition of $\mathcal{F}$, there exists a unique continuous map $h : X \to \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$ such that

$$\begin{tikzcd}
X \arrow[dr, dotted, "h"] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "f_2"'] & & \\
& \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2 \arrow[r, "\pi_1"'] \arrow[d, "\pi_2"] & \mathcal{C}_1 \arrow[d, "i_1"'] \\
& \mathcal{C}_2 \arrow[r, "i_2"] & \mathcal{C}_{1\wedge 2}
\end{tikzcd}$$

is a commutative diagram. Let H be the geometric morphism corresponding to h. For the next step we consider the action of the inverse image functor H^* on pushout diagrams of rings in $\mathcal{F}$. If $F : \mathcal{E} \to \mathcal{F}$ is any geometric morphism, then F^* will map a pushout square of rings in $\mathcal{F}$ to a pushout square of rings in $\mathcal{E}$. This can be verified in a straightforward way using naturality of the adjunction $F^* \vdash F_*$, and the fact that F_* is left-exact. As a consequence, for a ring $\underline{R}$ in $\mathcal{E}$, the object $F_*\underline{R}$ is a ring in

$\mathcal{F}$. An arrow $F^*\underline{S} \to \underline{R}$ is a ring homomorphism in $\mathcal{E}$ iff the corresponding arrow $\underline{S} \to F_*\underline{R}$ is a ring homomorphism in $\mathcal{F}$ [4].
By the previous considerations, we know that the square below is a pushout of rings in $Sh(X)$.

$$\begin{tikzcd}
H^*P_1^*I_1^*\underline{A}_{1\wedge 2} \arrow[r, "H^*P_1^*\underline{i}_1"] \arrow[d, "H^*P_2^*\underline{i}_2"'] & H^*P_1^*\underline{A}_1 \arrow[d, "H^*\underline{p}_1"] \arrow[ddr, bend left, "\underline{f}_1"] & \\
H^*P_2^*\underline{A}_2 \arrow[r, "H^*\underline{p}_2"'] \arrow[drr, bend right, "\underline{f}_2"'] & H^*\underline{R} \arrow[dr, "\underline{h}"] & \\
 & & \underline{S}
\end{tikzcd} .$$

The pair $(H, \underline{h})$ exists and is unique. □

Using this lemma, we can write down an explicit expression for the descent morphism.

Lemma 6.2.7. *The descent morphism is given by*

$$(H, \underline{h}) : ([\mathcal{C}_{1\vee 2}, \mathbf{Set}], \underline{A}_{1\vee 2}) \to ([\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2, \mathbf{Set}], \underline{R}),$$

where H is the geometric morphism induced by the poset map

$$h : \mathcal{C}_{1\vee 2} \to \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2, \quad C \mapsto (C \cap A_1, C \cap A_2),$$

and the ring morphism $\underline{h} : H^*\underline{R} \to \underline{A}_{1\vee 2}$ in $[\mathcal{C}_{1\vee 2}, \mathbf{Set}]$ is given by the functions

$$\underline{h}_C : (C \cap A_1) \otimes_{C\cap A_{1\wedge 2}} (C \cap A_2) \to C, \quad a \otimes b \mapsto a \cdot b. \tag{6.2}$$

Note that (6.2) follows from

$$\underline{h}_C(a \otimes b) = \underline{h}_C(a \otimes 1 \cdot 1 \otimes b) = \underline{h}_C(a \otimes 1) \cdot \underline{h}_C(1 \otimes b) = a \cdot b.$$

In order to be a sheaf, by definition the morphism $(H, \underline{h})$ must be an isomorphism in **RingSp**. In particular, h has to be a homeomorphism and therefore a bijection. This seems like a strong requirement, as shown by the following example.

[4] Note that if we replace the algebraic theory of rings by a more general geometric theory, this need not be the case. One reason is that for a model $\underline{R}$ of a geometric theory, the object $F_*\underline{R}$ need not be a model of the same theory, as F_* need not preserve any colimits.

Example 6.2.8. *Let $A_{1\vee 2} = C([0,1]^2)$ be the C*-algebra of continuous complex-valued functions on the unit square. Let*

$$A_1 = \{f \in C([0,1]^2) \mid \exists g \in C([0,1]),\ \forall x, y,\ f(x,y) = g(x)\}, \tag{6.3}$$

$$A_2 = \{f \in C([0,1]^2) \mid \exists g \in C([0,1]),\ \forall x, y,\ f(x,y) = g(y)\}, \tag{6.4}$$

Note that $A_{1\vee 2} = A_1 \otimes A_2$ as C-algebras in this example. In particular, $A_1 \cap A_2 = \mathbb{C}$. Consider $C \in \mathcal{C}_{1\vee 2}$ given by*

$$C = \{f \in C([0,1]^2) \mid \forall x \in [0,1], \quad f(x,x) = f(0,0)\}. \tag{6.5}$$

Clearly, $C \cap A_1 = C \cap A_2 = \mathbb{C}$. Consequently, $h(C) = h(\mathbb{C})$ hence h is not injective, so that it does not define an isomorphism of posets or spaces. Note that in this example $\underline{h}_C : \mathbb{C} \to C$ is the inclusion map, which is a ring morphism that is not surjective.

Example 6.2.9. *We can simplify the previous example in order to demonstrate that the full sheaf condition can be expected to fail for physically reasonable nets. Let $\underline{2} = \{0,1\}$ be the two element discrete space. Define $A \vee B \cong A \otimes B \cong C(\underline{2} \times \underline{2})$.*

$$A = \{f : \underline{2} \times \underline{2} \to \mathbb{C} \mid f(0,0) = f(0,1),\ f(1,0) = f(1,1)\} \cong C(\underline{2}),$$

$$B = \{f : \underline{2} \times \underline{2} \to \mathbb{C} \mid f(0,0) = f(1,0),\ f(0,1) = f(1,1)\} \cong C(\underline{2}).$$

Let $e_{ij} = e_i \otimes e_j$ denote the characteristic function

$$e_{ij}(k,l) = \delta_{ik}\delta_{jl}, \quad i,j,k,l \in \{0,1\}.$$

Consider the unital subalgebra C of $A \otimes B$ generated by $e_{10} - e_{01}$. The C-algebra C consists of functions $f : \underline{2} \times \underline{2} \to \mathbb{C}$ of the form*

$$f = \alpha_0 1 + \alpha_1(e_{10} - e_{01}) + \alpha_2(e_{10} + e_{01}), \quad \alpha_0, \alpha_1, \alpha_2 \in \mathbb{C},$$

where 1 denotes the constant function. Note that $C \cap A = C \cap B = \mathbb{C}$. Consequently, $h(C) = h(\mathbb{C})$, and the sheaf condition does not hold.

If the full sheaf condition is too strong, we could consider weaker versions instead. Nuiten introduces what he calls strong locality as such an alternative. However, we first consider microcausality. Microcausality is the assumption that if O_1 and O_2 are spacelike separated, then $[A_1, A_2] = \{0\}$. This condition may be reformulated quite elegantly as

Proposition 6.2.10. *(Nuiten's Lemma [68]) Microcausality is equivalent to the property that the poset morphism $h : \mathcal{C}_{1\vee 2} \to \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$ has a left adjoint $\vee$.*

Proof. If we assume microcausality and $(C_1, C_2) \in \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$, then $C_1 \cup C_2$ is commutative in $A_{1\vee 2}$ and generates a context $\vee(C_1, C_2)$ in $\mathcal{C}_{1\vee 2}$, which we denote by $C_1 \vee C_2$. By construction,

$$C_1 \vee C_2 \subseteq C \;\text{ iff }\; (C_1 \subseteq C \cap A_1) \text{ and } (C_2 \subseteq C \cap A_2). \tag{6.6}$$

Conversely, assume that h has a left adjoint $\vee$. By setting $C = C_1 \vee C_2$ in (6.6) we find $C_1, C_2 \subseteq (C_1 \vee C_2)$. As $C_1 \vee C_2$ is commutative, $[C_1, C_2] = \{0\}$, for each $(C_1, C_2) \in \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$. As every normal operator appears in some context, and every operator is a linear combination of normal operators, we conclude that microcausality holds. $\square$

For a net A satisfying the sheaf condition, h needs to be an isomorphism of posets, implying that $\vee$ and h form an adjunction equivalence, which means that the inequalities of the unit and counit of this adjunction are equalities. To be more precise, the sheaf condition implies the equalities

$$C = (C \cap A_1) \vee (C \cap A_2); \tag{6.7}$$

$$(C_1 \vee C_2) \cap A_1 = C_1, \quad (C_1 \vee C_2) \cap A_2 = C_2, \tag{6.8}$$

for each $C \in \mathcal{C}_{1\vee 2}$ and each $(C_1, C_2) \in \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$. We already noted that (6.7) is too restrictive. However, the equality (6.8), introduced in [68] as strong locality, does not seem that restrictive at first glance.

Definition 6.2.11. *A net $A : \mathcal{V}(X) \to$ **Set** of operator algebras is called **strongly local** if it satisfies microcausality and if for any pair $O_1, O_2 \in \mathcal{V}(X)$ of spacelike separated opens, equality (6.8) holds.*

Strong locality states that h, seen as a functor of poset-categories, is a coreflector (i.e. it has a left adjoint which is a right inverse). We can describe strong locality as a condition on H, instead of h.

Definition 6.2.12. *A geometric morphism $F : \mathcal{E} \to \mathcal{F}$ is called a **local geometric morphism** is F_* is full and faithful.*

There are various equivalent ways of stating that a geometric morphism is local ([55, Theorem C3.6.1]). The important point is that for any pair

$\mathcal{C}$ and $\mathcal{D}$ of small categories[5], local geometric morphisms $F : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ correspond exactly to coreflectors $f : \mathcal{C} \to \mathcal{D}$.

Corollary 6.2.13. *A net $A : \mathcal{V}(X) \to \mathbf{Set}$ of operator algebras is strongly local iff for any pair $O_1, O_2 \in \mathcal{V}(X)$ of spacelike separated opens, the geometric morphism H of the descent morphism is local.*

It is attractive to think of strong locality as stating that although A may not be a sheaf, it is infinitesimally close to being one. To make this less sketchy, consider a geometric morphism $F : Sh(Y) \to Sh(X)$ coming from a continuous map $f : Y \to X$, of sober spaces, and assume that f is an infinitesimal thickening. By this we mean that f is a surjection with the property that for each fibre $f^{-1}(x)$ we can pick an element y_x such that the only neighbourhood of y_x in $f^{-1}(x)$ is $f^{-1}(x)$ itself, and the assignment $c : x \mapsto y_x$ defines a continuous section of f ([55, C3.6]). If this holds, F is a local geometric morphism.
This is relevant to strong locality. If we assume strong locality, and view $h : \mathcal{C}_{1\vee 2} \to \mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$ as an Alexandroff continuous map, then it is an infinitesimal thickening in the sense given above. The continuous section c is given by $\vee$. Thus we have found another way of looking at strong locality: the map h is an infinitesimal thickening.

6.3 Strong locality and independence conditions

Various independence conditions are discussed. Strong locality is shown to be implied by C-independence and to imply extended locality.*

6.3.1 Independence conditions

The previous section introduced strong locality as a weaker version of the sheaf condition. A net of observable algebras satisfying Einstein causality[6] is strongly local, and any strongly local net must satisfy microcausality. In this section we try to pinpoint strong locality among the various independence conditions used in AQFT. In what follows, we concentrate

[5] Where we assume that $\mathcal{C}$ is Cauchy-complete in the sense that each idempotent morphism splits. As we are concerned with poset categories, this condition holds trivially.

[6] Einstein causality is called C*-independence in the product sense in this chapter. It is defined in Definition 6.3.1.

on pairs (A, B) of unital C*-algebras, instead of whole nets of such algebras. Fo example, think of A and B as operator algebras associated to two spacelike separated regions of spacetime.

Definition 6.3.1. *([74]) Let A and B be two (not necessarily commutative) unital C*-subalgebras of some larger C*-algebra $\mathfrak{A}$. Then the pair (A, B) satisfies:*

1. ***microcausality** if the elements of A commute with those of B, i.e. $[A, B] = \{0\}$;*
2. ***extended locality** if it satisfies microcausality and $A \cap B = \mathbb{C}$;*
3. ***C*-independence** if it satisfies microcausality and if for each $a \in A$ and $b \in B$, $ab = 0$ implies $a = 0$ or $b = 0$. This last condition is called the **Schlieder property**;*
4. ***C*-independence in the product sense** if it satisfies microcausality and $A \vee B \cong A \otimes B$.*

These locality conditions are sorted in increasing strength. From their definitions we see that C*-independence in the product sense implies C*-independence, and that extended locality implies microcausality. It is not obvious at first sight that C*-independence implies extended locality.

Lemma 6.3.2. *C*-independence implies extended locality.*

Proof. By microcausality, $A \cap B$ is a commutative unital C*-algebra. Hence $A \cap B$ is isomorphic to $C(\Sigma)$, where Σ is the associated Gelfand spectrum. Under the assumption of microcausality, extended locality is equivalent to the compact Hausdorff space Σ being a singleton. We give a contrapositive proof of the lemma. Assume that $x, y \in \Sigma$ are two distinct points. By the Hausdorff property there exist open neighbourhoods U_x of x and U_y of y, such that $U_x \cap U_y = \emptyset$. A compact Hausdorff space is completely regular, therefore there exist nonzero continuous real-valued functions f and g on Σ, such that the support of f lies in U_x and the support of g lies in U_y. These $f, g \in (A \cap B)_{sa}$ satisfy $f \neq 0$, $g \neq 0$ and $f \cdot g = 0$, which implies that the Schlieder property fails for the pair (A, B). $\square$

The following examples show that none of the conditions of Definition 6.3.1 are equivalent.

Example 6.3.3. *Take $\mathfrak{A} = A_1 \oplus B_1$, with A_1 and B_1 C*-algebras. Defining $A = A_1 \oplus \mathbb{C}$, and $B = \mathbb{C} \oplus B_1$, the pair (A, B) satisfies microcausality, but not extended locality, since $A \cap B = \mathbb{C} \oplus \mathbb{C}$.*

Example 6.3.4. *Consider $\mathfrak{A} = C([0,1], \mathbb{C})$, the continuous complex-valued functions on the closed interval $[0,1]$. Define*

$$A = \{f \in \mathfrak{A} \mid f|_{[0,1/2]} \text{ is constant } \};$$

$$B = \{f \in \mathfrak{A} \mid f|_{[1/2,1]} \text{ is constant } \}.$$

Then the pair (A, B) satisfies extended locality, but the Schlieder property fails.

Example 6.3.5. *Let A be a von Neumann factor, and $B = A'$ its commutant. Then the pair (A, B) is C*-independent. But, as shown in [38, Corollary 4.6], it is C*-independent in the product sense iff A is semidiscrete.*

For commutative C*-algebras, C*-independence and C*-independence in the product sense are equivalent, as shown by the following lemma. The original proof of the lemma is included for the sake of completeness.

Lemma 6.3.6. *([44] Theorem 11.1.1) Let C and D be commutative unital C*-subalgebras of some larger C*-algebra. If (C, D) is C*-independent, then it is C*-independent in the product sense.*

Proof. Define the $*$-homomorphism

$$\Phi : C \otimes D \to C \vee D, \quad f \otimes g \mapsto f \cdot g.$$

We will show that Φ is a $*$-isomorphism. Assume that there is an element $h \in Ker\Phi$ that is nonzero and nonnegative. Let X and Y be the Gelfand spectra of C and D respectively. We will use the isomorphism

$$C \otimes D \to C(X \times Y), \quad f \otimes g \mapsto u, \quad u : X \times Y \to \mathbb{C}, \quad u(x,y) = f(x)g(y).$$

Under this isomorphism h can be seen as a nonzero, nonnegative function $h : X \times Y \to \mathbb{R}$. Let $(x, y) \in X \times Y$ be a point such that $h(x, y) > 0$. Consider compact neighbourhoods U_x of x in X, and U_y of y in Y, such that h restricted to $U_x \times U_y$ is strictly positive. By compactness of $U_x \times U_y$, there exists a constant $c > 0$, such that $h > c$ on $U_x \times U_y$. There exists a nonzero nonnegative continuous function f on X that vanishes outside of U_x. Likewise, there exists a continuous nonzero nonnegative function g that vanishes outside of U_y. Rescale these two functions such that $0 \leq f, g \leq \sqrt{c}$. Define $\bar{h} = f \otimes g$. By construction, $\bar{h} \leq h$, therefore $\Phi(\bar{h}) = 0$. This means that $f \cdot g = 0$, implying that (C, D) is not C*-independent. Contrapositively, if (C, D) is C*-independent, then $\ker \Phi = \{0\}$, implying $C \otimes D \cong C \vee D$. □

We know from the Proposition 6.2.10 that microcausality of (A, B) can be described at the level of contexts $\mathcal{C}_A$, and $\mathcal{C}_B$. Microcausality is equivalent to the claim that the poset-morphism

$$r : \mathcal{C}_{A \vee B} \to \mathcal{C}_A \times_{\mathcal{C}_{A \cap B}} \mathcal{C}_B \quad r(C) = (C \cap A, C \cap B)$$

has a left adjoint. Extended locality is now also easily described at the level of contexts, as it amounts to microcausality combined with the statement that $\mathcal{C}_{A \cap B}$ is a singleton set. Equivalently, the pair (A, B) satisfies extended locality iff it satisfies microcausality and the pullback square

$$\begin{tikzcd}
\mathcal{C}_A \times_{\mathcal{C}_{A \cap B}} \mathcal{C}_B \arrow[r] \arrow[d] & \mathcal{C}_A \arrow[d, "(-)\cap B"] \\
\mathcal{C}_B \arrow[r, "(-)\cap A"'] & \mathcal{C}_{A \cap B}
\end{tikzcd}$$

is equal to the product $\mathcal{C}_A \times \mathcal{C}_B = \mathcal{C}_A \times_{\mathcal{C}_{A \cap B}} \mathcal{C}_B$.
We proceed to describe C*-independence at the level of contexts.

Proposition 6.3.7. *The pair (A, B) is C*-independent iff*

$$\forall C \in \mathcal{C}_A \quad \forall D \in \mathcal{C}_B \quad C \vee D \cong C \otimes D.$$

Proof. Assume that (A, B) is C*-independent. If $C \in \mathcal{C}_A$ and $D \in \mathcal{C}_B$, then (C, D) satisfies the Schlieder property, because (A, B) does. The pair (C, D) is C*-independent, which implies C*-independence in the product sense, as we are working with commutative algebras. We conclude that $C \vee D \cong C \otimes D$.
Conversely, assume that $C \vee D \cong C \otimes D$. Then (C, D) is C*-independent, and satisfies the Schlieder property. All normal operators of A and B occur in contexts, so the Schlieder property holds when we restrict a and b to normal operators. Let $a \in A$ and $b \in B$ be arbitrary. Assume that $a \cdot b = 0$. Then

$$(a^* a) \cdot (b b^*) = a^* \cdot (ab) \cdot b^* = 0.$$

By the Schlieder property for normal operators, $a^*a = 0$ or $bb^* = 0$. This implies that $a = 0$ or $b = 0$, proving the Schlieder property for the pair (A, B). We conclude that (A, B) is C*-independent. □

6.3.2 C*-independence and the spectral presheaf

As we will see below, using an equivalent description of C*-independence, this condition resembles a sheaf condition on the topos $[\mathcal{C}^{op}, \mathbf{Set}]$. Strictly speaking, it is not really a sheaf condition, because the 'covering relation' in question fails to be a basis for a Grothendieck topology.
By a state on a C*-algebra we mean a normalised positive linear functional on the algebra. As argued in [74], a pair (A, B) is C*-independent iff, for any state ϕ_1 of A, and any state ϕ_2 on B, there exists a unique state ϕ on $A \vee B$, such that

$$\forall a \in A \;\; \forall b \in B \;\; \phi(a \cdot b) = \phi_1(a) \cdot \phi_2(b).$$

From the previous proposition it follows that (A, B) is C*-independent iff, for any $C \in \mathcal{C}_A$, any state ϕ_1 on C, any $D \in \mathcal{C}_B$, and any state ϕ_2 on D, there exists a unique state ϕ on $C \vee D$ such that $\phi(ab) = \phi_1(a)\phi_2(b)$.

Let Σ_C denote the Gelfand spectrum of the context C, and let $\mathrm{PV}(\Sigma_C)$ denote the set of probability valuations on Σ_C. Probability valuations $\mu : \mathcal{O}\Sigma_C \to [0,1]_l$ where discussed in Chapter 4. If $C_1 \subseteq C_2$, let $\rho_{C_2C_1} : \Sigma_{C_2} \to \Sigma_{C_1}$ denote the Gelfand dual of the inclusion map $C_1 \hookrightarrow C_2$. Define the function

$$\mathrm{PV}(i_{C_1C_2}) : \mathrm{PV}(\Sigma_{C_2}) \to \mathrm{PV}(\Sigma_{C_1}) \;\; \mu \mapsto \mu \circ \rho^{-1}_{C_2C_1}.$$

For any unital C*-algebra A, this assignment defines a presheaf

$$\mathrm{PV}(\underline{\Sigma}) : \mathcal{C}_A^{op} \to \underline{Set}.$$

For an element $\lambda \in \Sigma_{C_2}$, let $\delta_\lambda \in \mathrm{PV}(\Sigma_{C_2})$ denote the point valuation satisfying $\delta_\lambda(U) = 1$ iff $\lambda \in U$ and $\delta_\lambda(U) = 0$ otherwise. By definition,

$$\mathrm{PV}(i_{C_2C_1})(\delta_\lambda) = \delta_\lambda \circ \rho^{-1}_{C_2C_1} = \delta_{\rho_{C_2C_1}(\lambda)}.$$

This allows us to see the spectral presheaf $\underline{\Sigma}$ as a subobject of the presheaf $\mathrm{PV}(\underline{\Sigma})$. Recall that the spectral presheaf, assigns to each context C, its Gelfand spectrum Σ_C, and assigns to the inclusion $C_1 \subseteq C_2$, the (continuous) restriction map $\rho_{C_2C_1} : \Sigma_{C_2} \to \Sigma_{C_1}$.

Remark 6.3.8. *Given a locale $\underline{L}$ in the presheaf topos $[\mathcal{C}^{op}, \mathbf{Set}]$, we can assign to it the locale $\underline{PV}(\underline{L})$ of internal probability valuations on it. This assignment is part of an endofunctor*

$$\underline{PV} : \underline{Loc}_{[\mathcal{C}^{op},\mathbf{Set}]} \to \underline{Loc}_{[\mathcal{C}^{op},\mathbf{Set}]},$$

which is, in turn, part of a monad [79]. The inclusion $\underline{\Sigma} \subseteq PV(\underline{\Sigma})$ resembles of the unit $\eta : I \to \underline{PV}$ of this monad. Clearly, the inclusion cannot be completely identified as $\eta_{\underline{\Sigma}}$, as we do not view $\underline{\Sigma}$ as an internal locale here, and neither do we consider ***internal*** *probability valuations in the definition of $PV(\underline{\Sigma})$.*

Remark 6.3.9. *For finite-dimensional C*-algebras A, the presheaf $PV(\underline{\Sigma})$ (or rather the restriction of it to a finite subset of contexts) is easily identified with the quantum-mechanical realisation of the presheaf of R-distributions on the event sheaf $D_R(\mathcal{E})$, used in [1]. Here R is taken to be the non-negative real numbers.*

Using the Riesz–Markov theorem, C*-independence of the pair (A, B) can be translated to a condition on $\mathrm{PV}(\underline{\Sigma}) : \mathcal{C}_{A\vee B}^{op} \to \mathbf{Set}$, which resembles a sheaf condition. For any $C \in \mathcal{C}_A \subseteq \mathcal{C}_{A\vee B}$, any $D \in \mathcal{C}_B \subseteq \mathcal{C}_{A\vee B}$, for each pair of probability valuations $\mu_1 \in \mathrm{PV}(\Sigma_C)$, and $\mu_2 \in \mathrm{PV}(\Sigma_D)$, there exists a unique $\mu \in \mathrm{PV}(\Sigma_{C\vee D})$ such that

$$\mu|_C := \mathrm{PV}(i_{C,C\vee D})(\mu) = \mu_1, \quad \mu|_D := \mathrm{PV}(i_{D,C\vee D})(\mu) = \mu_2,$$

and, in addition, μ can be identified as the product probability valuation $\mu = \mu_1 \times \mu_2$ (see [79] for a discussion of product valuations).
If we think of the pair (C, D) as covering $C \vee D$, this resembles a sheaf condition on $\mathrm{PV}(\underline{\Sigma})$. Let us make this precise. Define a 'covering relation' $\lhd$, where for $E \in \mathcal{C}_{A\vee B}$ and $U \subseteq \mathcal{C}_{A\vee B}$, $E \lhd U$ means that E is covered by U. For each $E \in \mathcal{C}_{A\vee B}$ define $E \lhd \{E\}$, and, if $E = C \vee D$ with $C \in \mathcal{C}_A$, and $D \in \mathcal{C}_B$, then $E \lhd \{C, D\}$ as well. However, this relation $\lhd$ does not satisfy the necessary conditions for a basis for a Grothendieck topology on $\mathcal{C}_{A\vee B}$, in the sense of [63, III.2, Def. 2]. The obstruction is the stability axiom, which in our setting requires that for any $E \in \mathcal{C}_{A\vee B}$ such that $E \subseteq C \vee D$ for some $C \in \mathcal{C}_A$ and $D \in \mathcal{C}_B$, one has

$$E = (E \wedge C) \vee (E \wedge D).$$

Looking at Example 6.2.9 this condition does not hold. Indeed, the relation $\lhd$ does not define a basis for a Grothendieck topology for the same reason that the full sheaf condition of Definition 6.2.5 does not hold.

The conjunction of the claims that $\lhd$ defines a basis for a Grothendieck topology, and that $\mathrm{PV}(\Sigma)$ is a sheaf with respect to this topology, is equivalent to the sheaf condition that we find in Section 6.4, as we can see from Theorem 6.4.6.

6.3.3 Pinpointing strong locality

Next, we try to relate the previous locality conditions to the notion of strong locality. Recall:

Definition 6.3.10. *Let A and B be two (not necessarily commutative) unital C*-subalgebras of some larger C*-algebra $\mathfrak{A}$. Then the pair (A, B) is called **strongly local** if it satisfies microcausality, and*

$$\forall C \in \mathcal{C}_A \ \ \forall D \in \mathcal{C}_B \ \ (C \vee D) \cap A = C \ \text{and}\ (C \vee D) \cap B = D. \tag{6.9}$$

We wish to see where strong locality stands in the list of Definition 6.3.1. First, according to the following lemma it implies extended locality.

Lemma 6.3.11. *If the pair (A, B) is strongly local, then it satisfies extended locality.*

Proof. The proof is this lemma is contrapositive. Assume that (A, B) satisfies microcausality, but not extended locality. Then $A \cap B \neq \mathbb{C}$. Take $C = \mathbb{C} \in \mathcal{C}_A$ and $D = A \cap B \in \mathcal{C}_B$. Then

$$(C \vee D) \cap A = (\mathbb{C} \vee (A \cap B)) \cap A = (A \cap B) \cap A = A \cap B = D \neq C.$$

Hence, the pair (A, B) is not strongly local. □

Note that if (A, B) are C*-independent in the product sense, then the pair satisfies strong locality. We use this observation to prove that C*-independence implies strong locality.

Lemma 6.3.12. *If the pair (A, B) is C*-independent, then (A, B) satisfies strong locality.*

Proof. Let $C \in \mathcal{C}_A$ and $D \in \mathcal{C}_B$. Define $E = (C \vee D) \cap A$. Then E is a commutative C*-algebra containing C and is contained in A. Likewise, $F = (C \vee D) \cap B$ is a commutative C*-algebra that contains D, and is contained in B. As the pair (A, B) is C*-independent, so is the pair

(E, F). Since E and F are commutative, the pair (E, F) is also C*-independent in the product sense. Hence, it is strongly local. Note that

$$C = (C \vee D) \cap E = (C \vee D) \cap (C \vee D) \cap A = (C \vee D) \cap A,$$

and similarly $D = (C \vee D) \cap B$. We conclude that the pair (A, B) is strongly local as well. □

C*-independence implies strong locality, which in turn implies extended locality. Example 6.3.4 can be used to show that strong locality and C*-independence are inequivalent.

Lemma 6.3.13. *The pair (A, B) from Example 6.3.4 satisfies strong locality.*

Proof. We use the following observations. Let E be a unital commutative C*-subalgebra of $C([0,1])$. This algebra E defines a partition of the interval $[0,1]$ into closed subsets by means of the equivalence relation

$$x \sim_E y \text{ iff } \forall f \in E \;\; f(x) = f(y).$$

The algebra E consists of those $f \in C([0,1])$ for which f is constant on each of the equivalence classes $[x]_E$. Conversely, if we have a partition of $[0,1]$ into closed subsets $[x]$, then the set of $f \in C([0,1])$ such that f is constant on each equivalence class $[x]$, defines a C*-subalgebra of $C([0,1])$.
Take $C \in \mathcal{C}_A$ and $D \in \mathcal{C}_D$. These correspond to partitions $[x]_C$ and $[x]_D$ of $[0,1]$. Note that by definition of A and B, $[1/2]_C$ contains the interval $[0,1/2]$, and $[1/2]_D$ contains $[1/2,1]$. Define the finer partition consisting of classes $[x] = [x]_D \cap [x]_C$. Note that if $x \notin [1/2]_C$, then $[x] = [x]_C$, and if $x \notin [1/2]_D$, then $[x] = [x]_D$.
Let E be the C*-algebra consisting of those $h \in C([0,1])$ that are constant on each class $[x]$. Define

$$f : [0,1] \to \mathbb{C}, \quad x \mapsto \begin{cases} h(1/2) & \text{if } x \in [1/2]_C \\ h(x) & \text{if } x \notin [1/2]_C \end{cases}.$$

Note that $f \in C$. In addition, define

$$g : [0,1] \to \mathbb{C}, \quad x \mapsto \begin{cases} h(x) & \text{if } x \notin [1/2]_D \\ h(1/2) & \text{if } x \in [1/2]_D \end{cases}.$$

It follows that $g \in D$. Note that $h = f + g - h(1/2)$. As a consequence, $E \subseteq C \vee D$. It is straightforward to verify the converse, $C \vee D \subseteq E$. We conclude that $C \vee D = E$. To demonstrate strong locality we need to show that $E \cap A \subseteq C$ and $E \cap B \subseteq D$. Let $h \in E \cap A$. then, for each $x \in [0,1]$, h is constant on $[x]_D \cap [x]_C$. Let $[x]_C \neq [1/2]_C$. Then $[x]_C \cap [x]_D = [x]_C$. We conclude that h is constant on each $[x]_C$, except possibly $[1/2]_C$. As $h \in A$, h is constant on $[0,1/2]$, as well as on $[1/2]_C \cap [1/2]_D$. Note that

$$
\begin{aligned}
[0,1/2] \cup ([1/2]_D \cap [1/2]_C) &= ([0,1/2] \cup [1/2]_C) \cap ([0,1/2] \cup [1/2]_D) \\
&= [1/2]_C \cap [0,1] = [1/2]_C.
\end{aligned}
$$

We conclude that $h \in C$. This shows that $(C \vee D) \cap A = C$. The equality $(C \vee D) \cap B = D$ can be proven in the same way. □

It is an open question whether strong locality is equivalent to extended locality.

6.4 C*-Algebraic version

We derive Nuiten's sheaf condition once again, with the difference that the net of operator algebras does not map to a category of spaces and internal rings, but to spaces and internal commutative C-algebras. The obtained sheaf condition is related to C*-independence.*

In Section 6.2 a sheaf condition was derived using the category **RingSp**. This meant reducing internal unital commutative C*-algebras to rings. The aim of this section is to formulate the sheaf condition in the category **ucCSp** of spaces X and unital commutative C*-algebras internal to $Sh(X)$. The sheaf condition obtained in this way is equivalent to the unit law (6.7) together with C*-independence. We start by considering the following C*-algebraic counterpart to **RingTopos**, which was introduced in [68], and which was previously considered in Section 5

Definition 6.4.1. *The category* **cCTopos** *consists of the following:*

- *Objects are pairs $(\mathcal{E}, \underline{A})$, with $\mathcal{E}$ a topos and $\underline{A}$ a unital commutative C*-algebra internal to the topos $\mathcal{E}$.*
- *Arrows $(G, \underline{g}) : (\mathcal{E}, \underline{A}) \to (\mathcal{F}, \underline{B})$ are given by geometric morphisms $G : \mathcal{E} \to \mathcal{F}$ and $*$-homomorphisms $\underline{g} : G^*\underline{B} \to \underline{A}$ in $\mathcal{F}$.*

- *Composition of arrows is defined by* $(G,\underline{g})\circ(F,\underline{f}) = (G\circ F, \underline{f}\circ F^*\underline{g})$.

As C*-algebras cannot be completely captured by a geometric theory, we cannot guarantee that $G^*\underline{B}$ is a C*-algebra internal to $\mathcal{F}$. We do know that it is a semi-normed $*$-algebra over the complex rationals, where the semi-norm has the C*-property (See Section 2.1 and Appendix A.3). If by a $*$-homomorphism we mean a function that preserves all algebraic structure, then this definition makes sense. But we can question if it is the right definition.
For the semi-normed $*$-algebra $G^*\underline{B}$ we can consider the Cauchy-completion and obtain a C*-algebra. Just as in **Set**, the algebra $G^*\underline{B}$ is everywhere norm-dense in its completion. In order to extend a $*$-homomorphism (in the algebraic sense given above) $\underline{g} : G^*\underline{B} \to \underline{A}$ to the completion of $G^*\underline{B}$ in a continuous way, we require $\underline{g}$ to satisfy

$$\forall b \in G^*\underline{B},\ \forall q \in \mathbb{Q}^+ \quad (b,q) \in \underline{N}_{G^*\underline{B}} \to (\underline{g}(b), q) \in \underline{N}_{\underline{A}}, \tag{6.10}$$

which simply states that $\|\underline{g}(b)\| \leq \|b\|$. In the topos **Set**, this is a necessary condition for the extension to exist as every $*$-homomorphism between C*-algebras is norm decreasing (i.e. contractive). We demand that each $*$-homomorphism in our category is continuous in this sense. Note that (6.10) is an expression within geometric logic. If $\underline{f}$ is a continuous $*$-homomorphism in a topos $\mathcal{E}$ and $F : \mathcal{F} \to \mathcal{E}$ is a geometric morphism, then $F^*\underline{f}$ is a continuous $*$-homomorphism. This is important, as it entails that continuous $*$-homomorphisms are closed under the composition of arrows in **cCTopos**.

Remark 6.4.2. *Let* $\underline{A}$ *and* $\underline{B}$ *be semi-normed* $\underline{\mathbb{Q}[i]}$*-algebras. If we see* $\underline{A}$ *and* $\underline{B}$ *as models of the geometric theory of such algebras, then the structure homomorphisms of these models are precisely the continuous* $*$*-homomorphisms.*

In formulating the C*-algebraic version of Nuiten's sheaf condition we use the following subcategory of **cCTopos**.

Definition 6.4.3. *The category* **ucCSp** *consists of the following:*

- *Objects are pairs* $(X,\underline{A})$*, with* X *a topological space and* $\underline{A}$ *a unital commutative C*-algebra internal to the topos* $Sh(X)$*.*

- *Arrows* $(g,\underline{g}) : (X,\underline{A}) \to (Y,\underline{B})$ *are given by continuous maps* $g : X \to Y$ *and continuous, unit-preserving* $*$*-homomorphisms*

$\underline{g} : G^*\underline{B} \to \underline{A}$ in $Sh(Y)$, where G is the geometric morphism induced by g.

- *Composition of arrows is defined by $(G, \underline{g}) \circ (F, \underline{f}) = (G \circ F, \underline{f} \circ F^*\underline{g})$.*

Let $\mathcal{O} \to A(\mathcal{O})$ be an isotonic net of operator algebras, assumed to be unital C*-algebras. Such a net defines a contravariant functor $A : \mathcal{V}(X)^{op} \to$ **ucCSp**. Let O_1, and O_2 be two spacelike separated opens. If we think of $O_1 \vee O_2$ as being covered by O_1 and O_2, we want to know under what conditions A is a sheaf, just like in Section 6.2. Recall that the sheaf condition was formulated using the descent morphism. This descent morphism used the ringed topos of matching families, which was defined as a pullback. In the C*-algebraic version, does the corresponding pullback even exist? Proving that such pullbacks exist will be our first step in describing the sheaf condition.

Assume for convenience that the net satisfies extended locality. Using the notation of Section 6.2, consider the poset $\mathcal{C}_1 \times_{\mathcal{C}_{1\wedge 2}} \mathcal{C}_2$, which, under the assumption of extended locality, simplifies to $\mathcal{C}_1 \times \mathcal{C}_2$. As before, we use the topos $\mathcal{F} = [\mathcal{C}_1 \times \mathcal{C}_2, \mathbf{Set}]$. In this topos, define the internal unital commutative C*-algebra

$$\underline{A}_1 \otimes \underline{A}_2 : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathbf{Set} \quad (\underline{A}_1 \otimes \underline{A}_2)(C_1, C_2) = C_1 \otimes C_2, \tag{6.11}$$

where $C_1 \otimes C_2$ is the C*-algebraic tensor product[7]. If $(D_1, D_2) \leq (C_1, C_2)$, the corresponding $*$-homomorphism is the inclusion $D_1 \otimes D_2 \hookrightarrow C_1 \otimes C_2$. The pair $(\mathcal{F}, \underline{A}_1 \otimes \underline{A}_2)$ comes equipped with projection morphisms

$$(P_i, \underline{p}_i) : ([\mathcal{C}_1 \times \mathcal{C}_2, \mathbf{Set}], \underline{A}_1 \otimes \underline{A}_2) \to ([\mathcal{C}_i, \mathbf{Set}], \underline{A}_i),$$

where the geometric morphism P_i is induced by the projection $\pi_i : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_i$, and

$$\underline{p}_i : P_i^* \underline{A}_i \to \underline{A}_1 \otimes \underline{A}_2,$$
$$(\underline{p}_1)_{(C_1, C_2)} : C_1 \to C_1 \otimes C_2 \quad a \mapsto a \otimes 1,$$
$$(\underline{p}_2)_{(C_1, C_2)} : C_2 \to C_1 \otimes C_2 \quad b \mapsto 1 \otimes b.$$

The following theorem is needed to describe the descent morphism, and is consequently needed for the sheaf condition in the C*-algebraic setting. The theorem shows that the pushout of rings, used in Nuiten's

[7]Since commutative C*-algebras are nuclear, there is a unique C*-algebraic tensor product completing the algebraic tensor product $C_1 \odot C_2$.

original sheaf condition, are replaced by C*-algebraic tensor products of the contexts. As a consequence, the sheaf condition can be related to C*-independence.

Theorem 6.4.4. *The diagram*

$$(\mathcal{C}_1, \underline{A}_1) \xleftarrow{(P_1, \underline{p}_1)} (\mathcal{C}_1 \times \mathcal{C}_2, \underline{A}_1 \otimes \underline{A}_2) \xrightarrow{(P_2, \underline{p}_2)} (\mathcal{C}_2, \underline{A}_2),$$

is a product in **ucCSp**.

Before we start with the proof, note the similarity with the pullback in Section 6.2. By extended locality, $(\mathcal{T}_{1\wedge 2}, \underline{A}_{1\wedge 2})$ can be identified with the pair $(\mathbf{Set}, \mathbb{C})$, which is the terminal object of **ucCSp**. It is by extended locality that the pullback coincides with the product.

Proof. The proof is presented in six steps.

Step 1: We start by showing that

$$P_1^* \underline{A}_1 \xleftarrow{\underline{p}_2} \underline{A}_1 \otimes \underline{A}_2 \xrightarrow{\underline{p}_1} P_2^* \underline{A}_2,$$

is a coproduct of unital commutative C*-algebras in $[\mathcal{C}_1 \times \mathcal{C}_2, \mathbf{Set}]$. By Proposition 2.1.2, $\underline{B}$ is a unital commutative C*-algebra in $\mathcal{F}$ iff each $\underline{B}(C_1, C_2)$ is a unital commutative C*-algebra in **Set**, and each restriction map is a unit-preserving $*$-homomorphism. Note that for each $(C_1, C_2) \in \mathcal{C}_1 \times \mathcal{C}_2$, the diagram above gives

$$C_1 \xleftarrow{-\otimes 1} C_1 \otimes C_2 \xrightarrow{1 \otimes -} C_1 \ ,$$

which is a coproduct of unital commutative C*-algebras in **Set**. Suppose we have internal $*$-homomorphisms (which in this case are automatically norm-continuous) $\underline{f}_i : P_i^* \underline{A}_i \to \underline{B}$. This provides us with $*$-homomorphisms

$$\forall (C_1, C_2) \in \mathcal{C}_1 \times \mathcal{C}_2 \quad (\underline{f}_i)_{(C_1, C_2)} : C_i \to \underline{B}(C_1, C_2).$$

Using the universal property of the stagewise tensor products, we obtain unique maps

$$\underline{h}_{(C_1, C_2)} : C_1 \otimes C_2 \to \underline{B}(C_1, C_2),$$
$$\underline{h}_{(C_1, C_2)}(a \otimes b) = (\underline{f}_1)_{(C_1, C_2)}(a) (\underline{f}_2)_{(C_1, C_2)}(b). \tag{6.12}$$

We need to check whether these local $*$-homomorphisms $\underline{h}_{(C_1,C_2)}$ piece together to a single natural transformation. Let

$$\rho : \underline{B}(D_1, D_2) \to \underline{B}(C_1, C_2), \quad (D_1, D_2) \leq (C_1, C_2)$$

denote the arrow part of the functor $\underline{B}$, corresponding to the given inequality of $\mathcal{C}_1 \times \mathcal{C}_2$. Using naturality of $\underline{f}_1$, $\underline{f}_2$, and (6.12), we find that $\rho \circ \underline{h}_{(D_1,D_2)}$ and $\underline{h}_{(C_1,C_2)}$ agree on the algebraic tensor product $D_1 \odot D_2$. By continuity, they agree on $D_1 \otimes D_2$, showing naturality. Hence, we found an internal $*$-homomorphism $\underline{h} : \underline{A}_1 \otimes \underline{A}_2 \to \underline{B}$ with the desired properties. Note that $\underline{h}$ is unique, as each component $\underline{h}_{(C_1,C_2)}$ is unique.

Step 2: In this step we concentrate on the algebraic part of $\underline{A}_1 \otimes \underline{A}_2$. Since we have established that $\underline{A}_1 \otimes \underline{A}_2$ is a coproduct of unital commutative C*-algebras, we ask how this coproduct behaves under the action of an inverse image functor coming from a geometric morphism. In the setting of rings this was straightforward, as we were dealing with an essentially algebraic theory. The theory of C*-algebras is not geometric, let alone essentially algebraic. Instead of working with the whole full-fledged C*-algebra, forget about the norm for a moment. We consider $*$-algebras over $\underline{\mathbb{Q}[i]}$. Looking at $P_1^*\underline{A}_1$ and $P_2^*\underline{A}_2$ as such algebras, we can find their coproduct $\underline{A}_1 \odot \underline{A}_2$ in $[\mathcal{C}_1 \times \mathcal{C}_2, \mathbf{Set}]$. As the notation suggests, this coproduct is also computed context-wise, i.e. $(\underline{A}_1 \odot \underline{A}_2)(C_1, C_2) = C_1 \odot C_2$. Furthermore, if $H : \mathcal{E} \to [\mathcal{C}_1 \times \mathcal{C}_2, \mathbf{Set}]$ is a geometric morphism, then we can identify $H^*(\underline{A}_1 \odot \underline{A}_2)$ with the coproduct of $H^*P_1^*\underline{A}_1$ and $H^*P_2^*\underline{A}_2$. The functor H^* preserves coproducts of $\underline{\mathbb{Q}[i]}$-algebras for the same reasons it preserves coproducts of rings. If $F : \mathcal{E} \to \mathcal{F}$ is any geometric morphism, and $\underline{A}$ is a $\underline{R}$-algebra in $\mathcal{F}$ for some commutative ring, then $F^*\underline{A}$ is an $F^*\underline{R}$-algebra in $\mathcal{E}$. If $\underline{B}$ is a $F^*\underline{R}$-algebra in $\mathcal{E}$ with action $\mu_{\underline{B}} : F^*\underline{R} \times \underline{B} \to \underline{B}$, then $F_*\underline{B}$ is a $\underline{R}$-algebra in $\mathcal{F}$ with action

$$\mu_{F_*\underline{B}} : \underline{R} \times F_*\underline{B} \to F_*\underline{B}, \quad \mu_{F_*\underline{B}} = F_*(\mu_{\underline{B}}) \circ (\eta_{\underline{R}} \times F_*\underline{B}),$$

where $\eta_{\underline{R}} : \underline{R} \to F_*F^*\underline{R}$ is the unit of the adjunction $F^* \dashv F_*$. In this way, the $F^*\underline{R}$-algebra homomorphisms $F^*\underline{A} \to \underline{B}$, in $\mathcal{E}$ correspond, by the adjunction, to $\underline{R}$-algebra homomorphisms $\underline{A} \to F_*\underline{B}$ in $\mathcal{F}$.
Returning to case at hand; if we are given $*$-preserving algebra morphisms $\underline{f}_i : H^*P_i^*\underline{A}_i \to \underline{B}$, then there exists a unique $*$-preserving algebra morphism $\underline{h} : H^*(\underline{A}_1 \odot \underline{A}_2) \to \underline{B}$ such that $\underline{h} \circ H^*\underline{p}_i = \underline{f}_i$.

Step 3: The previous step showed that $H^*(\underline{A}_1 \odot \underline{A}_2)$ was the coproduct of $H^*\underline{A}_1$ and $H^*\underline{A}_2$ as $\underline{\mathbb{Q}[i]}$-algebras in $\mathcal{E}$. As such, for $*$-preserving $\underline{\mathbb{Q}[i]}$-algebra homomorphisms $\underline{f}_i : H^*P_i^*\underline{A}_i \to \underline{B}$, there exists a corresponding unique $*$-preserving $\underline{\mathbb{Q}[i]}$-algebra homomorphisms $\underline{h} : H^*(\underline{A}_1 \odot \underline{A}_2) \to \underline{B}$. From this point onwards we include the norms back into the discussion. We consider $\underline{A}_1 \odot \underline{A}_2$ normed, using the restriction of the norm $\underline{N}$ on $\underline{A}_1 \otimes \underline{A}_2$ to the subobject $\underline{A}_1 \odot \underline{A}_2$. Equipped with this norm, the last three steps are devoted to showing that $\underline{h}$ is norm-continuous whenever both $\underline{f}_i$ are norm-continuous.
Note that with respect to the norm $\underline{N}$ on $\underline{A}_1 \otimes \underline{A}_2$, the subset $\underline{A}_1 \odot \underline{A}_2$ is everywhere dense in the sense that

$$\forall x \in \underline{A}_1 \otimes \underline{A}_2 \; \forall n \in \mathbb{N} \;\; (\exists y \in \underline{A}_1 \odot \underline{A}_2 \;\; (x - y, 1/n) \in \underline{N}).$$

This is a suitable axiom for a geometric theory, and this remark is relevant in two ways. First, the axiom holds internally for $\underline{A}_1 \otimes \underline{A}_2$, because it holds at each stage (C_1, C_2) (Lemma A.3.2). Second, it also holds for $H^*(\underline{A}_1 \otimes \underline{A}_2)$ and $H^*(\underline{A}_1 \odot \underline{A}_2)$, relative to the semi-norm $H^*\underline{N}$. This implies that the elements of $H^*(\underline{A}_1 \otimes \underline{A}_2)$ can be seen as Cauchy-approximations of $H^*(\underline{A}_1 \odot \underline{A}_2)$, relative to $H^*\underline{N}$. This, in turn, implies that we can extend any norm-continuous $*$-homomorphism $\underline{h} : H^*(\underline{A}_1 \odot \underline{A}_2) \to \underline{B}$ to $H^*(\underline{A}_1 \otimes \underline{A}_2)$.

Step 4: For any pair $\underline{f}_i : H^*P_i^*\underline{A}_i \to \underline{B}$ of continuous $*$-homomorphisms, we show that the $*$-preserving algebra morphism $\underline{h} : H^*(\underline{A}_1 \odot \underline{A}_2) \to \underline{B}$ satisfying $\underline{h} \circ H^*\underline{p}_i = \underline{f}_i$ is norm-decreasing for elements of the form $H^*\underline{p}_1(a) =: a \otimes 1$ and $H^*\underline{p}_2(b) =: 1 \otimes b$.
Note that the norm on $\underline{A}_1 \odot \underline{A}_2$ satisfies the following (geometric) condition, expressing the properties $\|a \otimes 1\| = \|a\|$ and $\|1 \otimes b\| = \|b\|$:

$$\forall a \in P_i^*\underline{A}_i \;\; \forall q \in \mathbb{Q}^+ \;\; (a, q) \in \underline{N}_{P_i^*\underline{A}_i} \leftrightarrow (\underline{p}_i(a), q) \in \underline{N}_{\underline{A}_1 \odot \underline{A}_2}.$$

Fitting within geometric logic, this identity is preserved by H^*. Consequently,

$$(H^*\underline{p}_i(a), q) \in \underline{N}_{H^*(\underline{A}_1 \odot \underline{A}_2)} \to (a, q) \in \underline{N}_{H^*P_i^*\underline{A}_i} \to (\underline{f}_i(a), q) \in \underline{N}_{\underline{B}},$$

where we used the fact that by definition $\underline{N}_{H^*P_i^*\underline{A}_i} = H^*\underline{N}_{P_i^*\underline{A}_i}$. This argument shows that $\underline{h}$ is (semi-)norm-continuous on elements of the form $\underline{p}_1(a) = a \otimes 1$ and $\underline{p}_2(b) = 1 \otimes b$.

Step 5: Next, we show that $\underline{h}$ is norm-decreasing for simple tensors $a \otimes b = H^*\underline{p}_1(a) \cdot H^*\underline{p}_2(b)$. We pursue the same strategy as in the previous step. Take a property of the tensor product in **Set**. Describe it geometrically. Then it holds internally to $\mathcal{F}$, and is preserved by H^*. In **Set** the norm on the tensor product is a cross-norm, which means that it satisfies $\|a \otimes b\| = \|a\| \cdot \|b\|$. We can reformulate this as as

$$\forall a \in P_1^*\underline{A}_1 \;\; \forall b \in P_2^*\underline{A}_2 \;\; \forall q \in \mathbb{Q}^+ \; (\underline{p}_1(a) \cdot \underline{p}_2(b), q) \in \underline{N}_{\underline{A}_1 \odot \underline{A}_2} \leftrightarrow$$
$$(\exists p_1, p_2 \in \mathbb{Q}^+((a, p_1) \in \underline{N}_{P_1^*\underline{A}_1}) \wedge ((b, p_2) \in \underline{N}_{P_2^*\underline{A}_2}) \wedge (p_1 \cdot p_2 < q)).$$

This condition states that for any positive rational $q > 0$, one has $\|a \otimes 1 \cdot 1 \otimes b\| < q$ iff there exist rational numbers $p_1, p_2 > 0$, satisfying

$$\|a \otimes 1\| \cdot \|1 \otimes b\| < p_1 \cdot p_2 < q.$$

Note that the condition is preserved by H^*, as the axiom can be expressed using geometric logic.
We are now ready to prove norm-continuity for simple tensors. Let $q > \|H^*\underline{p}_1(a) \cdot H^*\underline{p}_2(b)\|$. There exist p_1, p_2 as above. If $p_1 > \|a\|$, then $p_1 > \|\underline{f}_1(a)\|$. Likewise, $p_2 > \|\underline{f}_2(b)\|$. Using submultiplicativity of the norm, we conclude that

$$q > p_1 \cdot p_2 > \|\underline{f}_1(a)\| \cdot \|\underline{f}_2(b)\| \geq \|\underline{f}_1(a) \cdot \underline{f}_2(b)\| = \|\underline{h}(H^*\underline{p}_1(a) \cdot H^*\underline{p}_2(b))\|.$$

This proves norm-continuity of $\underline{h}$ on the simple tensors.

Step 6: For the last step in demonstrating norm-continuity, we consider linear combinations of simple tensors, i.e. arbitrary elements of the algebraic tensor product. The proof relies on the fact that the norm which we defined on $\underline{A}_1 \odot \underline{A}_2$ is the projective cross-norm. In the topos **Set**, for two unital commutative C*-algebras A and B, a $q \in \mathbb{Q}^+$, and an element $x \in A \odot B$, we have $q > \|x\|$ iff

$$\exists n \in \mathbb{N} \; \exists a_1, ..., a_n \in A \;\; \exists b_1, ..., b_n \in B$$
$$\left(x = \sum_{i=1}^{n} a_i \otimes b_i\right) \wedge \left(q > \sum_{i=1}^{n} \|a_i\| \cdot \|b_i\|\right).$$

Noting that finite subsets or finite lists falls within the domain of geometric logic, this property can be described geometrically. Therefore, it can be applied to $H^*(\underline{A}_1 \odot \underline{A}_2)$. Using both the triangle inequality and

the submultiplicativity of the norm, one proves continuity of $\underline{h}$ for all elements of $H^*(\underline{A}_1 \odot \underline{A}_2)$.
This proves that $\underline{h}$ can be extended to a continuous $*$-homomorphism $\underline{h}' : H^*(\underline{A}_1 \otimes \underline{A}_2) \to \underline{B}$. Note that $\underline{h}'$ is unique, as $\underline{h}$ is unique by construction.
To complete the proof of the theorem, consider morphisms in **ucCSp**, viz.

$$(f_i, \underline{f}_i) : (X, \underline{B}) \to (\mathcal{C}_i, \underline{A}_i).$$

There exists a unique $h : X \to \mathcal{C}_1 \times \mathcal{C}_2$, such that $f_i = \pi_i \circ h$. In particular, $F_i^* = H^* P_i^*$. We are given continuous $*$-homomorphisms $\underline{f}_i : H^* P_i^* \underline{A}_i \to \underline{B}$. By the previous reasoning, there exists a unique $\underline{h} : H^*(\underline{A}_1 \otimes \underline{A}_2) \to \underline{B}$ such that $\underline{h} \circ H^* \underline{p}_i = \underline{f}_i$. The pair $(h, \underline{h})$ is the unique arrow such that $(f_i, \underline{f}_i) = (\pi_i, \underline{p}_i) \circ (h, \underline{h})$. □

Using the product of the previous theorem, we can write down the descent morphism.

Proposition 6.4.5. *In* **ucCSp***, the descent morphism is given by*

$$(h, \underline{h}) : ([\mathcal{C}_{1\vee 2}, \mathbf{Set}], \underline{A}_{1\vee 2}) \to ([\mathcal{C}_1 \times \mathcal{C}_2, \mathbf{Set}], \underline{A}_1 \otimes \underline{A}_2),$$

$$h : \mathcal{C}_{1\vee 2} \to \mathcal{C}_1 \times \mathcal{C}_2 \quad C \mapsto (C \cap A_1, C \cap A_2),$$

$$\underline{h} : H^*(\underline{A}_1 \otimes \underline{A}_2) \to \underline{A}_{1\vee 2},$$

$$\underline{h}_C : (C \cap A_1) \otimes (C \cap A_2) \to C; \quad a \otimes b \mapsto a \cdot b.$$

Using this description of the descent morphism, we now come to our main and final result, which relates the C*-algebraic version of Nuiten's sheaf condition to C*-independence.

Theorem 6.4.6. *Let (A, B) satisfy extended locality. Then the pair (A, B) satisfies the sheaf condition in* **ucCSp** *iff the pair is C*-independent, and satisfies*

$$\forall C \in \mathcal{C}_{1\vee 2} \quad (C \cap A_1) \vee (C \cap A_2) = C. \tag{6.13}$$

Proof. Let (A, B) satisfy the conditions of Theorem 6.4.6. We will define an inverse $(j, \underline{j})$ for the descent morphism $(h, \underline{h})$. C*-independence implies strong locality. Combined with (6.13), we deduce that the poset morphism h from Proposition 6.4.5 is an isomorphism. The inverse for h is given by $\vee$, so we simply define $j = \vee$. The poset map j

defines an essential geometric morphism J, with inverse image functor $J^* : [\mathcal{C}_{1\vee 2}, \mathbf{Set}] \to [\mathcal{C}_1 \times \mathcal{C}_2, \mathbf{Set}]$. In order to define

$$\underline{j} : J^* \underline{A}_{1\vee 2} \to \underline{A}_1 \otimes \underline{A}_2,$$

for each $(C, D) \in \mathcal{C}_1 \times \mathcal{C}_2$ define

$$\underline{j}_{(C,D)} : C \vee D \to C \otimes D$$

to be the inverse of the isomorphism of commutative C*-algebras $\underline{h}_{C\vee D}$. Here we used C*-independence and (6.13) to deduce that $\underline{h}_{C\vee D}$ is indeed an isomorphism $C \otimes D \to C \vee D$. We conclude that the pair (A, B) satisfies the sheaf condition.
The converse claim, to the effect that the sheaf condition implies the conditions of the theorem is straightforward to prove. □

Because of (6.13), the full sheaf condition cannot be assumed for physically reasonable nets $O \mapsto A(O)$. The decent morphism $(H, \underline{h})$ does satisfy the weaker conditions that H is a local geometric morphism, and for each context of the form $C \vee D$, the $*$-homomorphism $\underline{h}_{C\vee D}$ is an isomorphism of C*-algebras.
The full sheaf condition seems to strong a requirement on physical grounds. If this was not the case, then by Subsection 6.3.2 we would have found a nontrivial Grothendieck topos which we could use as a topos model for algebraic quantum field theories. It is still an open question of how we can define nontrivial topos models for quantum field theories.

7

Epilogue

A central theme in the previous material is that, from the internal perspective of the topoi at hand, the topos models for quantum theory resemble classical physics. However, working internal to a nontrivial topos differs from working with sets (as we would in classical physics) in ways other than having a multivalued logic. In this section we consider the relevance of these other differences, in particular the absence of the axiom of choice and the law of excluded middle.

A topos is a mathematical universe of discourse. But the axiom of choice and the law of excluded middle may very well lead to contradictions in such a universe. This gives possibilities which are not allowed in the topos **Set**. As an example there is a topos ($Sh(\mathbf{T})$, where $\mathbf{T}$ is a site of topological spaces with the open cover topology) such that all functions $\underline{\mathbb{R}}_d \to \underline{\mathbb{R}}_d$ are continuous [63, Section VI.9].
But are any of these new possibilities relevant to physics? And how impractical is it to lose the axiom of choice? In Analysis, a branch of mathematics used extensively throughout physics, this axiom plays an important role. Consider the following quotation, taken from the well-known text [70] by G.K. Pedersen on functional analysis.

> This means that our acceptance of the axiom of choice determines what sort of mathematics we want to create, and it may in the end affect our mathematical description of physical realities. The same is true (albeit on a smaller scale) with the parallel axiom in euclidean geometry. But as the advocates of the axiom of choice, among them Hilbert and von Neumann,

> point out, several key results in modern mathematical analysis (e.g. the Tychonoff theorem, the Hahn-Banach theorem, the Krein-Milman theorem and Gelfand theory) depend crucially on the axiom of choice.

Even so, we have used a topos-valid version of Gelfand duality theory for the covariant model. Indeed, this version of Gelfand theory [9] does not rely on the axiom of choice; it works because locales take the place of topological spaces. In the same way, the other theorems mentioned by Pedersen have a localic counterpart which do not require the axiom of choice. At the risk of presenting things somewhat simpler than they really are, we might say that it is the emphasis on points in the notion of a topological space, which makes us need an axiom that generates enough points. Without emphasis on points, the axiom of choice becomes less powerful, possibly even superfluous.
Removing emphasis on points is relevant when we consider Isham's motivation for using topos theory in physics. In particular, consider the potential problem in quantum gravity of using real numbers as values for physical quantities, and, associated to that, the use of smooth manifolds for space-time. In [53] we read:

> Indeed, it is not hard to convince oneself that, from a physical perspective, the important ingredient in a space-time model is not the 'points' in that space, but rather the 'regions' in which physical entities can reside. In the context of a topological space, such regions are best modelled by open sets: the closed sets may be too 'thin' to contain a physical entity, and the only physically meaningful closed sets are those with a non-empty interior. These reflections lead naturally to the subject of 'pointless topology' and the theory of locales-a natural step along the road to topos theory.

The arguments thus far only claimed that dropping the axiom of choice might not be as big a problem as one would expect at first. On the other hand, thus far no arguments have shown that there is an actual advantage in dropping the axiom. In all honesty, the author does not know of any physically motivated arguments, but there are some mathematical ones. The extent to which the axiom of choice holds (in the internal language) depends on the topos [39]. In the topos **Set** we can assume the full axiom of choice. For a category of presheaves $[\mathcal{C}^{op}, \mathbf{Set}]$ (or copresheaves) the

weaker version called the axiom of dependent choice can be assumed (by assuming the full axiom of choice for **Set**), but for many categories $\mathcal{C}$, assuming the full axiom of choice leads to contradictions. If the topos is of the form $Sh(X)$, with X a topological space, even the axiom of dependent choice may lead to contradictions.
Assuming that the axiom of choice holds internally for a topos has large consequences for the topos. In particular, the topos is boolean, which means that the internal Heyting algebra Ω is an internal boolean algebra [10]. The contravariant model was founded on considerations of coarse-graining. In particular, the idea that truth values should correspond to down-closed subsets of $\mathcal{C}$ is a key ingredient. This leads to a topos which is not boolean, and therefore does not satisfy the (internal) axiom of choice. For the covariant model, using up-closed subsets of $\mathcal{C}$ as truth values does not seem a motivational point, but rather a consequence of the choice of topos. In [72], Spitters considers the dense topology on the copresheaf topos $[\mathcal{C}, \mathbf{Set}]$. The resulting topos of sheaves is a boolean topos, which satisfies the axiom of choice. Although it would be interesting to investigate this topos, we postpone further discussion.

One of the more striking possibilities granted by the absence of the law of excluded middle is synthetic differential geometry. Certain topoi [66] can act as models for differential geometry, using (nilpotent) infinitesimals. In the presence of the law of excluded middle all such infinitesimals would be forced to be equal to zero. As argued in detail in [11] (following Lawvere), these infinitesimals could allow us to deal with the continuum in a mathematically more natural way. At this point we might frown and say: thinking of quantum gravity we would like to get rid of the continuum rather than giving it a face-lift! Maybe so, but the problem of continuous versus discrete in quantum gravity is deep and subtle. Having mathematical universes that capture the subtleties of the continuum may assist in understanding this problem better. In any case, the current topos models $[\mathcal{C}^{op}, \mathbf{Set}]$ and $[\mathcal{C}, \mathbf{Set}]$ are not models for synthetic differential geometry. Speculating for a moment, it may be interesting to see if the categories used for studying locally covariant quantum field theories [15], can be extended to models of synthetic differential geometry, in such a way that the quantum field theories can be studied internally.

Appendix A

Topos Theory

There are various ways of looking at topoi, and it is the interplay between these different points of view that makes topos theory interesting [55, 63]. For our goals, thinking of a topos as a generalised universe of sets, or a universe of mathematical discourse, is the most important perspective. In the first section however, we think of topoi as generalisations of topological spaces. This helps in introducing geometric morphisms and locales, two important notions. In Section A.2 we view a topos as a generalised universe of sets, and in Section A.3 we focus on the part of topos internal mathemtaics which is expressible using geometric logic.

A.1 Topoi as Generalised Spaces

We review some basic theory of geometric morphisms and locales.

If X is a topological space, then we can associate to it the topos $Sh(X)$, of sheaves on that topological space [63, Chapter II]. From the topos $Sh(X)$ we can recover the topology $\mathcal{O}X$ of X by considering the subobject classifier Ω of the topos. If the space X is Hausdorff (and hence sober), we can subsequently retrieve X as the set of points of $\mathcal{O}X$.
Given $f : X \to Y$, a continuous map of topological spaces, we can associate to f a geometric morphism $F : Sh(X) \to Sh(Y)$. For topoi $\mathcal{E}$, and $\mathcal{F}$, a **geometric morphism** $F : \mathcal{E} \to \mathcal{F}$ is an adjoint pair of functors where the right adjoint $F_* : \mathcal{E} \to \mathcal{F}$ is called the **direct image functor**, and the left adjoint $F^* : \mathcal{F} \to \mathcal{F}$ is called the **inverse image functor** and where F^* is assumed to be left-exact [63, Chapter VII]. Conversely,

if $F : Sh(X) \to Sh(Y)$ is a geometric morphism and the spaces are Hausdorff, then F comes from a unique continuous map $f : X \to Y$.

The topos $Sh(X)$ depends on the topology $\mathcal{O}X$, rather than on the underlying set of points of the space. For example, if X has the trivial topology $\{\emptyset, X\}$, then $Sh(X)$ can be identified with **Set**, regardless of the set X. In this sense we should not see topoi as generalisations of topological spaces, but of locales. In order to define locales, we first need to consider frames. A **frame** F is a complete lattice, satisfying the following distributivity law

$$\forall U \in F, \ \ \forall S \subseteq F, \ \ U \wedge \left(\bigvee S\right) = \bigvee\{U \wedge V \mid V \in S\}.$$

The motivating example of a frame is a topology $\mathcal{O}X$, where $\wedge$ is simply the intersection $\cap$, and $\bigvee$ corresponds to the union $\bigcup$. However, not every frame comes from a topology, as we will see below. If F and G are frames, then a morphism of frames, or a **frame homomorphism** $f : F \to G$, is a function that preserves finite meets and all joins. If $f : X \to Y$ is a continuous map of topological spaces, then the inverse image map $f^{-1} : \mathcal{O}Y \to \mathcal{O}X$ is a frame homomorphism.

The category of locales **Loc** is defined to be the dual category of the category of frames **Frm**. So a **locale** L corresponds to a unique frame, which we will denote as $\mathcal{O}L$, and a **locale map**, $f : K \to L$, also called a continuous map, is the same as a frame map $f^* : \mathcal{O}L \to \mathcal{O}K$. In particular, a topological space X defines a locale $L(X)$ through the topology $\mathcal{O}X$, and a continuous map of spaces $f : X \to Y$ induces, through the inverse image map, a locale map $L(f) : L(X) \to L(Y)$.

Points of a space X correspond to continuous maps $x : 1 \to X$, where 1 is the one-point topological space. The inverse image of $x : 1 \to X$ is a frame morphism $x^{-1} : \mathcal{O}X \to \underline{2}$, where $\underline{2}$ is the frame of two elements. If we are given the topology $\mathcal{O}X$, we consider the set $pt(X)$ consisting of frame homomorphisms $\mathcal{O}X \to \underline{2}$. The frame $\mathcal{O}X$ defines a topology on $pt(X)$: if $U \in \mathcal{O}X$, let $pt(U)$ consist of the $p \in pt(X)$ such that $p(U)$ is the top element of $\underline{2}$. The space X is called **sober** iff it is homeomorphic to $pt(X)$. In other words, we can reconstruct the points from the topology. In particular, any Hausdorff space is sober. For any sober space X, we can retrieve X from the locale $L(X)$, or from the topos $Sh(X)$.

As mentioned before, a frame, and therefore a locale, need not come from a topological space. For example, let F consist of the subsets U of the real line $\mathbb{R}$, satisfying the condition that taking the interior of the closure of U is equal to U. This set, counting all open intervals (r, s)

amongst its elements, and partially ordered by inclusion, defines a frame. However, contrary to topologies, there are no points in the sense of frame homomorphisms $F \to 2$. The only topology that has no points is the unique topology on $\emptyset$, and obviously F is not isomorphic to this topology. The theory of frames and their morphisms can be interpreted in the internal language of any topos $\mathcal{E}$. As a consequence we can work with frames and locales internal to a topos. We proceed to discussing the internal language of topoi.

A.2 Topoi as Generalised Universes of Sets

We briefly discuss the internal language of topoi, and presheaf semantics, a simple form of Kripke-Joyal semantics.

Whenever we talk about taking an **internal** perspective or internal picture of topoi such as $[\mathcal{C}^{op}, \mathbf{Set}]$ and $[\mathcal{C}, \mathbf{Set}]$, we are working with the objects and arrows of that topos without referring to the topos **Set**. This could mean that we are considering these objects and arrows in terms of abstract category theory. However, we use it to indicate that we are using the **internal language** of the topos. Any topos has an associated internal or Mitchell-Bénabou language [14, 10, 63]. With respect to this language, working with objects and arrows of the topos resembles set theory, but the category **Set** itself is not used. Whenever we use **Set** in our descriptions, we take on an **external** perspective. For example, if we think of the spectral presheaf $\underline{\Sigma}$ as a functor mapping into the category **Set**, we are using an external perspective. If we consider $\underline{\Sigma}$ as a 'set' (or topological space) with respect to the internal language, we are dealing with an internal perspective.

We briefly describe the internal language associated to a topos $\mathcal{E}$, following [14]. The internal language is typed, and the objects of the topos are the **types** or **sorts** of the language. To each object X we associate the following formal symbols:

- A countable set of variables $x, x', x'', \ldots$ of type X.
- For each arrow $1 \to X$, a constant of type X.

Using this data, and the arrows of the topos, we inductively generate **terms** by the rules below. Note that by construction each term has both a type and a finite set of free variables associated to it.

- Any constant of type X is a term of type X that has no free variables.
- Any variable x of type X is a term of type X with free variable x.
- If σ is a term of type X with free variables $x_1, \ldots, x_n$, and $f : X \to Y$ is an arrow of $\mathcal{E}$, then there is a term $f \circ \sigma$ of type X with the same free variables $x_1, \ldots, x_n$.
- If σ is a term of type X with free variables $x_1, \ldots, x_n$ and τ is a term of type Y with the same free variables, then there is a term $\langle \sigma, \tau \rangle$ of type $X \times Y$ with the same free variables $x_1, \ldots, x_n$.
- If σ is a term of type X with free variables $x_1, \ldots, x_n$, and $x_1, \ldots x_m$ is a larger set of free variables, then there is a term $\sigma_{(x_1, \ldots, x_m)}$ of type X with free variables $x_1, \ldots, x_m$.
- Let τ be a term of type X, with free variables $x_1, \ldots, x_n$ of types $X_1, \ldots X_n$, and $\sigma_1, \ldots, \sigma_n$ be terms of types $X_1, \ldots, X_n$.[1] Then there is a term $\tau(\sigma_1, \ldots, \sigma_n)$ of type X and the free variables are the free variables of the terms $\sigma_1 \ldots \sigma_n$.
- Let ϕ be a term of type Ω with free variables[2] $x_1, \ldots, x_n, x_{n+1}, \ldots, x_m$ of types $X_1, \ldots X_m$, then there is a term $\{(x_1, \ldots, x_n \mid \phi\}$ of type $\mathcal{P}(X_1 \times \ldots \times X_n)$ with free variables $x_{n+1}, \ldots, x_m$.

Note that in the last rule special types occur. For any type X there is a power type $\mathcal{P}X$ corresponding to the associated power object. In paticular, if $X = 1$, the associated power type is Ω, the object which is part of the subobject classifier. Terms of type Ω play a special role. A term of type Ω is called a **formula**. The internal language of a topos has additional rules for formulae. We present these rules after a brief discussion of the subobject classifier of a topos.

[1] We should require that no bound variables of τ occur as free variables of the $\sigma_1, \ldots, \sigma_n$. A free variable x in a formula ϕ can be bound by quantifiers $\forall x$ and $\exists x$, or as $\{x \mid \phi\}$.

[2] We require the first n variables to be distinct from the last $m - n$ variables.

For a topos $\mathcal{E}$, the **subobject classifier** is an object Ω, together with an arrow $\mathbf{true} : 1 \to \Omega$, such that for any subobject $U \hookrightarrow X$, there is a unique arrow $X \to \Omega$ turning the following square into a pullback.

$$\begin{tikzcd} U \arrow[r, "!"] \arrow[d, hook] & 1 \arrow[d, "\mathbf{true}"] \\ X \arrow[r] & \Omega \end{tikzcd}$$

For the topos **Set**, Ω is the two-element set $\underline{2}$, and the function corresponding to an inclusion $U \hookrightarrow X$ is just the characteristic function $\chi_U : X \to \underline{2}$. Returning to an arbitrary topos, the arrows $1 \to \Omega$ are called **truth values**. Like the topos **Set** there are always truth values **true** and **false**, but unlike **Set**, there may be many others. For the topos $[\mathcal{C}^{op}, \mathbf{Set}]$, the truth values correspond bijectively to the sieves on $\mathcal{C}$, and for $[\mathcal{C}, \mathbf{Set}]$ the truth values can be identified with the cosieves on $\mathcal{C}$.
We return to the internal language of the topos $\mathcal{E}$. The inductive rules of term formation are supplemented by the following rules for formulae:

- There are formulae **true** and **false** that have no free variables.
- If σ and τ are terms of the same type, having the same free variables, then there is a formula $\sigma = \tau$ with the same free variables.
- If σ is a term of type X and τ a term of type $\mathcal{P}X$, having the same set of free variables, then there is a formula $\sigma \in \tau$ with the same free variables.
- If ϕ and ψ are formulae with the same free variables, then there are formulae $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$ and $\phi \to \psi$ with the same free variables.
- If ϕ is a formula with (distinct) free variables $x, x_1, \ldots, x_n$, then there are formulae $\exists x\ \phi$ and $\forall x\ \phi$ with free variables $x_1, \ldots, x_n$.

The typed language formed by the terms and formulae constructed above, can be interpreted in the topos $\mathcal{E}$. With respect to this interpretation, any term σ of type X, with free variables of types $X_1, \ldots, X_n$ is represented by an arrow

$$\ulcorner \tau \urcorner : X_1 \times \ldots \times X_n \to X.$$

In particular, a formula ϕ with free variables $x_1, \ldots, x_n$ is respresented as an arrow

$$\ulcorner \phi \urcorner : X_1 \times \ldots \times X_n \to \Omega,$$

or, equivalently, as a subobject

$$\{(x_1, \ldots, x_n) \mid \phi(x_1, \ldots, x_n)\} \subseteq X_1 \times \ldots \times X_n.$$

If the formula ϕ has no free variables, it is interpreted as a truth value $1 \to \Omega$.

The interpretation of the internal language can be defined inductively as treated in [14, (Definition 6.3.3)]. Starting from the internal language with interpretation, the next step logical step is to introduce Kripke–Joyal semantics ([63, (VI.6)], [14, (6.6)]). It is with respect to the Kripke–Joyal semantics that the internal language turns the topos into a universe of mathematical discourse, resembling set theory. However, there are some important differences. In the topos **Set**, an element of a set X corresponds to a function $x : 1 \to X$, where 1 is a singleton set, the terminal object of **Set**. For an arbitrary topos $\mathcal{E}$ this is not a fruitful notion of element. Consider, for example, the spectral presheaf. Whenever the Kochen–Specker theorem applies, this object $\underline{\Sigma}$ has no elements in the sense of arrows $1 \to \underline{\Sigma}$. We need a more generalised notion of element that allows us to describe arbitrary sets in topoi. Define a **generalised element** of a set X in a topos $\mathcal{E}$ to be any arrow $Y \to X$.

Note that in the topos **Set** these generalised elements of a set X are *all* functions $f : Y \to X$ that have X as a codomain. We know from set theory that we need not consider $Y \neq 1$. This follows from the observation that the object 1 generates the category **Set**, which means that for any pair of functions $f, g : X \to Y$, the functions are equal $f = g$, iff $f \circ x = g \circ x$ for each $x : 1 \to X$.

In fact, for any Grothendieck topos there is a set of objects that generate the topos, and can therefore be used to reduce the number of generalised elements that we need to consider. If $y : \mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$ denotes the Yoneda embedding, then the objects $y(C)$ generate the presheaf category $[\mathcal{C}^{op}, \mathbf{Set}]$, meaning that for any pair of natural transformations $f, g : X \to Y$, we have $f = g$ iff for each element of the form $x : y(C) \to X$, $f \circ x = g \circ x$. This means that in doing internal mathematics, we can restrict to generalised elements of the form $y(C) \to X$. Exploiting this observation leads to **presheaf semantics** which we use several times throughout the text when proving internal claims.

By the Yoneda Lemma, a generalised element $\alpha : y(C) \to X$ corresponds to an element $\alpha \in X(C)$. The following conditions are equivalent:

1. The generalised element α factors through the inclusion $\{x \mid \phi(x)\} \subseteq X$;

2. $\alpha \in \{x \mid \phi(x)\}(C)$.

We denote these two equivalent conditions by the **forcing relation** $C \Vdash \phi(\alpha)$. Note that we can think of $C \Vdash \phi(\alpha)$, either internally (no reference to **Set**) using condition (1) to express the fact that the generalised element α of the 'set' X is an element of the 'set' $\{x \in X \mid \phi(x)\}$, or, externally, using condition (2). The internal and external view are connected by the Yoneda Lemma.

For the copresheaf topos $[\mathcal{C}, \mathbf{Set}]$, we can also use a version of presheaf semantics. This amounts to replacing the contravariant Hom-functors $y(C)$ of the Yoneda embedding by the covariant Hom-functors $k(C)$, and considering the generating set of objects $k(C)$. Presheaf semantics, its rules, and the more general sheaf semantics, are explained in [63, Section VI.7].

A.3 Geometric Logic

We describe geometric logic, geometric theories formed from this logic, their models, and the way these models transform under the action of inverse image functors.

The discussion is based on [55, (Section D1)] where a much more detailed presentation is given. Section 3 of [78] treats this material as well (with more emphasis on geometric logic). Another good source is [63, Chapter X], but note that what is called *geometric* there is called *coherent* in [55] and [78] (the difference is in allowing infinite disjunctions).
The language of geometric logic is that of an infinitary, first-order, many-sorted predicate logic with equality. We start with a first-order **signature** Σ (where Σ has nothing to do with Gelfand spectra or the spectral presheaf). The signature consists of a set S of **sorts**, a set F of **function symbols** and a set R of **relations**. Each function symbol $f \in F$ has a type, which is a non-empty finite list of sorts $A_1, ..., A_n, B$. We write this as $f : A_1 \times ... \times A_n \to B$, in anticipation of the categorical interpretation of the function symbols. The number n is called the **arity** of the function symbol. If the arity of a function symbol f is 0, i.e., when its type is a single sort B (we will write $f : 1 \to B$), f is called a **constant**. Each relation $P \in R$ also has a type $A_1, ..., A_n$. For relations we allow the empty set as a type. We write $P \subseteq A_1 \times ... \times A_n$, in anticipation of the

categorical representation. A relation with arity equal to 0, written as $P \subseteq 1$, is called a **propositional symbol**.
Unlike in the previous section, the sorts need not correspond to objects of a topos, and neither need a function symbol correspond to an arrow; we are not working with the internal language of some topos. However, when we have defined geometric theories, their interpretation in topoi works in the same way as the interpretation of the internal language of that topos.
As with the internal language of a topos, we supplement the signature Σ with variables $x, y, z, ...$ associated with each sort, taking as many variables as we need, and inductively define terms and formulae over Σ.
Terms, which all have a sort and a finite set of free variables assigned to them, are inductively defined as follows:

- Each variable x of sort A is a term of sort A with unique free variable x.

- If $f : A_1 \times ... \times A_n \to B$ is a function symbol, and $t_1, ..., t_n$ are terms of sort $A_1, ..., A_n$, then $f(t_1, ..., t_n)$ is a term of sort B. In particular, every constant is a term. The free variables of $f(t_1, ..., t_n)$ are the free variables of $t_1, \ldots, t_n$.

We proceed to define geometric formulae, and, associated to each geometric formula, a finite set of free variables. The **geometric formulae** over Σ form the smallest class that is closed under the rules:

- If $R \subseteq A_1 \times ... \times A_n$ is a relation, and $t_1, ..., t_n$ are terms of sort $A_1, ..., A_n$, then $R(t_1, ...t_n)$ is a formula. The free variables of $R(t_1, ...t_n)$ are all the variables of the t_i. In particular every propositional symbol is a formula without free variables.

- If s and t are terms of the same sort, then $s = t$ is a formula. The free variables of $s = t$ are all the variables in s and t.

- Truth $\top$ is a formula without free variables. If ϕ and ψ are formulae, then so is the conjunction $\phi \wedge \psi$. The free variables of $\phi \wedge \psi$ are those of ϕ and ψ.

- Let I be any (index) set and let for every $i \in I$, ϕ_i be a formula such that the total number of free variables in these formulae is finite. Then $\bigvee_{i \in I} \phi_i$ is a formula, such that the free variables are all the free variables of the ϕ_i.

- If ϕ is a formula with (distinct) free variables $x, x_1, \ldots, x_n$, then $(\exists x)\phi$ is a formula with free variables $x_1, \ldots, x_n$.

Note that implication, negation and the universal quantifier are not allowed in the construction of geometric formulae.

We do not consider formulae and terms by themselves, but always in context. A **context** (which should not be confused with the non-logical notion of context used in the topos approaches to quantum theory) is a finite vector $\vec{x} = (x_1, ..., x_n)$ where each x_i is a variable of some sort and all the variables are distinct. A context $\vec{x}$ is called *suitable* for a formula ϕ (or term t) if all free variables of ϕ (or all variables in t) occur in $\vec{x}$. We denote a formula-in-context (or term-in-context) by $\vec{x}.\phi$ (or $\vec{x}.t$), where $\vec{x}$ is a suitable context.

We will also consider sequents $\phi \vdash_{\vec{x}} \psi$, where ϕ and ψ are geometric formulae and $\vec{x}$ is a suitable context for both formulae. We will think of the sequent as expressing that ψ is a logical consequence of ϕ in context $\vec{x}$. A **geometric theory** $\mathbb{T}$ over Σ is simply a set of such sequents $\phi \vdash_{\vec{x}} \psi$.

The next step is to consider interpretations of geometric theories in toposes. Our discussion will be very brief, but the details can be found in the references stated at the beginning of this subsection. Let $\mathcal{E}$ be a topos (actually any category that has finite products would suffice at this point, and also assuming pullbacks makes the definition of a homomorphism of Σ-structures nicer). Given a signature Σ, a **Σ-structure** M in $\mathcal{E}$ is defined as follows. For every sort A in Σ there is an associated object MA in $\mathcal{E}$. For every function symbol $f : A_1 \times ...A_n \to B$ there is an arrow $Mf : MA_1 \times ... \times MA_n \to MB$ in $\mathcal{E}$. A constant $c : 1 \to B$ is interpreted as an arrow $Mc : 1 \to MB$, where 1 denotes the terminal object of $\mathcal{E}$. A relation R of type $A_1 \times ... \times A_n$ is interpreted as a monic arrow $MR \rightarrowtail MA_1 \times ... \times MA_n$.

If M and N are Σ-structures in $\mathcal{E}$, then a **homomorphism of Σ-structures** h is defined as follows. For each sort A in Σ there is an arrow $h_A : MA \to NA$. For each function symbol $f : A_1 \times ... \times A_n \to B$, we demand $h_B \circ Mf = Nf \circ (h_{A_1} \times ... \times h_{A_n})$. If $R \subseteq A_1 \times ... \times A_n$ is a relation, then we demand that $MR \subseteq (h_{A_1} \times ... \times h_{A_n})^*(NR)$ holds as subobjects of $MA_1 \times ... \times MA_n$, where the right hand side means pulling the monic arrow $NR \rightarrowtail NA_1 \times ... \times NA_n$ back along $h_{A_1} \times ... \times h_{A_n}$.

The Σ-structures in a topos $\mathcal{E}$ and their homomorphisms define a category $\mathbf{Str}_{\mathbf{\Sigma}}(\mathcal{E})$. Let $F : \mathcal{E} \to \mathcal{F}$ be a functor between toposes. F need not come from a geometric morphism, but we do assume it to be left exact.

It is straightforward to check that any such functor induces a functor $\mathbf{Str}_{\mathbf{\Sigma}}(\mathcal{E}) \to \mathbf{Str}_{\mathbf{\Sigma}}(\mathcal{F})$.

Next, we introduce models of a geometric theory $\mathbb{T}$ over Σ. In order to do this we need to interpret terms and formulae-in-context for a Σ-structure in $\mathcal{E}$. This can be done inductively, in much the same way as in using the internal language of the topos. Details can be found in [55, Subsection D1.2]. For a given Σ-structure M, the end result is that a formula-in-context $\vec{x}.\phi$, where $\vec{x} = (x_1, ..., x_n)$ are variables with associated sorts $A_1, ..., A_n$ is interpreted as a subobject $M(\vec{x}.\phi)$ of $MA_1 \times ... \times MA_n$. A Σ-structure M in a topos $\mathcal{E}$ is called a **model** for a geometric theory $\mathbb{T}$ if for every sequent $\phi \vdash_{\vec{x}} \psi$ in $\mathbb{T}$ we have $M(\vec{x}.\phi) \subseteq M(\vec{x}.\psi)$, where we view the interpretation of the formulae as subobjects of $MA_1 \times ... \times MA_n$.
We relate this definition of a model to the internal language of the topos.

Proposition A.3.1. *A Σ-structure in a topos $\mathcal{E}$ is a model for the geometric theory $\mathbb{T}$ iff for each sequent $\phi \vdash_{\vec{x}} \psi$ in $\mathbb{T}$ the proposition:*

$$\forall x_1 \in MX_1 \;, \dots, \forall x_n \in MX_n \;(\phi \to \psi),$$

of the internal language of $\mathcal{E}$ is interpreted as the truth value **true***.*

Note that in the proposition, the geometric formulae ϕ and ψ are seen as formulae with free variables $\vec{x} = (x_1, \dots, x_n)$ in the internal language[3] of $\mathcal{E}$.

Geometric theories and their models are of interest because of their relation to geometric morphisms. Although any left-exact functor $F : \mathcal{E} \to \mathcal{F}$ induces a functor between the associated categories of Σ-structures, this functor does not restrict to the full subcategories $\mathbf{Mod}_{\mathcal{E}}(\mathbb{T})$ of models of a geometric theory $\mathbb{T}$ over Σ. However, if $F = f^*$, i.e., the inverse image of a geometric morphism, then the functor $\mathbf{Str}_{\mathbf{\Sigma}}(\mathcal{E}) \to \mathbf{Str}_{\mathbf{\Sigma}}(\mathcal{F})$ does restrict to a functor map $\mathbf{Mod}_{\mathcal{E}}(\mathbb{T}) \to \mathbf{Mod}_{\mathcal{F}}(\mathbb{T})$. A nice proof of this claim is given in [63, Section X.3]. Although only finite joins are considered in this proof, the proof is interesting in particular because it also treats (non-geometric) formulae that use implication and the universal quantifier, and shows that models of a theory using this additional

[3]The attentive reader may have noticed that the geometric formulae may contain infinite disjunctions, whereas the internal language only considered finite disjunctions. However, we can extend the internal language to deal with infinite disjunctions by using the interpretation given in [55, (D1.2)].

structure are only preserved by the inverse image functor of a geometric morphism when the geometric morphism is open.
Models of geometric theories are respected by inverse image functors, as the categorical interpretation of geometric logic relies on finite limits and arbitrary colimits, which are all preserved by inverse image functors. In particular, an inverse image functor F^* preserves certain objects, such as the terminal object $F^*(\underline{1}_{\mathcal{F}}) \cong \underline{1}_{\mathcal{E}}$, the natural numbers object $F^*(\underline{\mathbb{N}}_{\mathcal{F}}) \cong \underline{\mathbb{N}}_{\mathcal{E}}$, and the objects $\underline{\mathbb{Q}}^+$, and $\underline{\mathbb{Q}[i]}$.

For functor categories, the fact that geometric constructions are preserved under the inverse image functor of any geometric morphism entails the following lemma.

Lemma A.3.2. *([55, Corollary D1.2.14(i)]) Let $\mathbb{T}$ be a geometric theory over a signature Σ and let $\mathcal{C}$ be any small category. A Σ-structure M in the topos $[\mathcal{C}, \mathbf{Set}]$ is a $\mathbb{T}$-model iff for every object $C \in \mathcal{C}_0$ the Σ-structure $ev_C(M)$ is a $\mathbb{T}$-model in $\mathbf{Set}$. Here $ev_C : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ denotes the functor that evaluates at the object C. There is an isomorphism*

$$\mathbf{Mod}_{\mathbb{T}}([\mathcal{C}, \mathbf{Set}]) \cong [\mathcal{C}, \mathbf{Mod}_{\mathbb{T}}(\mathbf{Set})].$$

In this lemma the "only if" part follows from the fact that ev_C is the inverse image part of a geometric morphism. The observation that we have an isomorphism of categories of models uses the fact that a homomorphism of Σ-structures in $[\mathcal{C}, \mathbf{Set}]$ can be identified with a natural transformation between the Σ-structures, viewed as functors $\mathcal{C} \to \mathbf{Str}_{\mathbf{\Sigma}}(\mathbf{Set})$.

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Index

Kwantum-Toposofie in een Notendop

Als we de titel van het proefschrift naar het Nederlands vertalen, dan staat er *Kwantum-Toposofie.* De term *toposofie* verwijst naar het gebruik van de tak van de wiskunde die *topos-theorie* heet. Topos-theorie speelt een centrale rol in dit boekje. Het voorvoegsel *kwantum* is een verwijzing naar kwantum-mechanica, of algemener, naar kwantum-theorie, een tak van de natuurkunde. Kwantum-theorie is berucht omdat de theorie geen bevredigende interpretatie heeft. Met een interpretatie bedoel ik een beschrijving die ons uitlegt hoe wij, volgens kwantum-theorie, kijken naar de natuur. Dit boekje draait om een herformulering van de wiskundige beschrijving van kwantum-theorie met behulp van topos-theorie in de hoop dat dit ons iets dichter brengt bij een bevredigende interpretatie.

Deze samenvatting is als volgt opgebouwd. Het eerste deel gaat over de zoektocht naar een kwantum-theorie van ruimte en tijd. Dit stuk dient als motivatie voor de zoektocht naar een interpretatie van kwantum-theorie. Daarna bespreek ik enkele problemen bij het vinden van een interpretatie van kwantum-theorie. Vervolgens draait het om de relatie tussen wis- en natuurkunde. Immers, als wiskunde en natuurkunde over heel verschillende zaken gaan, waarom zouden we dan verwachten dat een wiskundige herformulering van kwantum-theorie ons nieuwe natuurkundige inzichten oplevert? Daarna bespreek ik topos-theorie en de topos theoretische modellen voor kwantum-theorie die belangrijk zijn voor dit boekje. Tot slot worden de hoofdresultaten uit dit boekje doorgenomen en sta ik even stil bij de vraag in hoeverre we daadwerkelijk zijn opgeschoten met het vinden van een interpretatie.

Als je dit leest is er een redelijke kans dat je geen achtergrond hebt in de hogere wiskunde of natuurkunde. In de hoop dat dit niet al te pedant overkomt, wil ik deze lezer een leestip meegeven. Mochten bepaalde stukken tekst onduidelijk of te moeilijk zijn, sla ze dan over en ga verderop lezen. Wiskundige artikelen en boeken zijn vaak moeilijk en worden dan ook meestal niet van begin tot eind gelezen. De theoretisch natuurkundige Chen Ning Yang[4] schijnt gezegd te hebben dat er maar twee soorten (moderne) wiskunde boeken zijn: boeken waarbij je na één bladzijde al vastloopt en boeken waarbij je na één zin al vastloopt. Misschien niet de meest geruststellende woorden, maar helaas heeft professor Yang hiermee wel een punt.

Kwantum-Theorie van Ruimte en Tijd

De theoretisch natuurkundige Chris Isham is de spin in het web van de kwantum-toposofie. Veel van de kernideeën uit dit boekje zijn afkomstig van, of geïnspireerd door Isham's werk. De zoektocht naar een kwantum-theorie van ruimte en tijd vormt een belangrijke motivatie voor zowel Isham als mijzelf om topos-theorie toe te passen op kwantum-theorie.
De huidige theoretische natuurkunde rust op twee steunpilaren. De ene is de *Algemene Relativiteitstheorie* die zwaartekracht beschrijft. De andere heet het *Standaard Model*[5], een kwantum-theorie die de overige bekende natuurkrachten beschrijft, van elektromagnetisme tot de krachten die atomen bij elkaar houden. In plaats van deze twee theorieën, zouden we liever over één enkele theorie beschikken die alle bekende natuurkrachten beschrijft. Zo'n theorie noem ik hier een *kwantum-theorie van ruimte en tijd*[6]. De *kwantum* in de naam komt van het Standaard Model, en de *ruimte en tijd* van de Algemene Relativiteitstheorie waarin ruimte en tijd een sleutelrol spelen.
Al meer dan zestig jaar spannen een groot aantal van de knapste koppen die onze wereld te bieden heeft zich in om dit probleem te tackelen, maar een alom geaccepteerde theorie lijkt nog steeds ver weg. Het is zelfs al lastig om na te gaan wat een *goede* kwantum-theorie van ruimte en tijd inhoudt. Zo is het niet van te voren duidelijk dat zo'n theorie voorspellingen doet die we via experimenten kunnen testen. Er zijn tal van redenen waarom het vinden van een kwantum-theorie van ruimte en tijd een lastige onderneming is, maar voor deze samenvatting zal ik er maar één noemen; *het probleem van de tijd.* De kern van dit probleem zit in

[4]Hij won de Nobelprijs in 1957 voor zijn werk aan pariteitsschending.

[5]Dit is geen grapje, deze naam wordt echt gebruikt.

[6]Deze naam is misschien wat voorbarig. Mogelijk is de uiteindelijke theorie helemaal geen kwantum-theorie.

het feit dat tijd in de Algemene Relativiteitstheorie een heel andere rol speelt dan in het Standaard Model.
We verwachten dat een kwantum-theorie van ruimte en tijd ons helpt om de conceptuele problemen van kwantum-theorie en de Algemene Relativiteitstheorie (deels) op te lossen. Het vermoeden van Isham is echter dat we *eerst* een deel van de conceptuele problemen al opgelost moeten hebben voordat we een kwantum-theorie van ruimte en tijd kunnen vinden. Hierbij gaat het met name om de problemen van kwantum-theorie. Het is gebruikelijk om een *instrumentalistische* kijk te nemen op kwantum-theorie. Dit betekent dat we ons niet druk maken over wat de theorie zegt over hoe de natuur in elkaar steekt, maar dat we de theorie reduceren tot het voorspellen van uitkomsten van metingen[7]. kwantum-theorie wordt dus gezien als een theorie van metingen. In dit plaatje is tijd niet veel meer dan wat de klok aangeeft in het laboratorium waarin de meting wordt uitgevoerd. Tijd, in deze zin, is erg lastig te vergelijken met het begrip tijd zoals deze in de relativiteitstheorie wordt gebruikt. Het vermoeden van Isham is dan ook dat elke beschrijving van kwantum-theorie die leunt op metingen, als een dronkelap die leunt tegen een lantarenpaal, een groot obstakel vormt voor het vinden van een kwantum-theorie van ruimte en tijd. Dit is een sentiment dat ik met hem deel.

De Kopenhaagse Interpretatie

De lezer die bekend is met de geschiedenis van kwantum-mechanica weet dat het vinden van een bevredigende interpretatie een zware (misschien wel onmogelijke) opgave is. Beroemd is dan ook het debat van Einstein en Bohr over dit probleem. De *Kopenhaagse Interpretatie* van de kwantum-mechanica, voorgesteld door Bohr en anderen, is bedoeld om de diepe beerput van conceptuele problemen te omzeilen. Eigenlijk is de Kopenhaagse Interpretatie geen interpretatie in de zin dat het ons uitlegt hoe wij via de theorie kijken naar de natuur. Het is eerder een stel nauwkeurig geplaatste verbodsborden die aangeven welke vragen je vooral niet moet stellen. Het is de Kopenhaagse Interpretatie die van kwantum-mechanica een theorie van enkel metingen maakt.
Stel dat we kwantum-mechanica gebruiken om een systeem van subatomaire deeltjes, bijvoorbeeld elektronen, te bestuderen. Volgens de Kopenhaagse Interpretatie moeten we vooral niet vragen wat de positie

[7]Nauwkeuriger is: het voorspellen van kansen op uitkomsten bij metingen

van een elektron is op een bepaald tijdstip. Als we dit zouden doen, komen we in de knel wanneer we, even naïef, ook vragen gaan stellen over bijvoorbeeld de snelheid van het elektron. Het is alleen zinvol om te vragen met welke waarschijnlijkheid we het elektron ergens aantreffen wanneer we een positiemeting[8] uitvoeren.

In de klassieke natuurkunde zoals bijvoorbeeld mechanica beschreven door de wetten van Newton, is de situatie heel anders. Stel, je hebt een systeem dat je met klassieke mechanica wilt beschrijven zoals een slinger, of een blok op een helling. In de klassieke mechanica kan je aan zo'n systeem een *faseruimte* toekennen. De punten van de faseruimte zijn alle mogelijke (scherp bepaalde) toestanden waarin het systeem zich kan bevinden. Een fysische grootheid zoals bijvoorbeeld de positie of snelheid van de slinger of de potentiële energie van het blok op de helling, kan worden weergegeven als een functie van de faseruimte naar de verzameling van de reële getallen. Deze functie kent aan een toestand (dus een punt van de faseruimte) de waarde van de grootheid (bijvoorbeeld de positie of snelheid van de slinger) in die toestand toe. Bij kwantum-mechanica is de situatie heel anders. Immers, we mogen niet eens spreken over de positie van een object, tenzij we dat doen vanuit de context van een (klassiek beschreven) positiemeting.

In de herformuleringen van kwantum-theorie door middel van topostheorie die in dit boekje besproken worden fysische grootheden *wel* voorgesteld als afbeeldingen van een ruimte van toestanden naar een verzameling van waarden die deze grootheden aannemen. Voor Isham en andere kwantum toposofen is het belangrijk dat een fysische grootheid zoals positie of snelheid van een deeltje op deze manier kan worden beschreven. De hoop is dat deze beschrijving, die lijkt op klassieke natuurkunde, helpt om kwantum-theorie begrijpelijker te maken (zoals klassieke natuurkunde). Echter, de verzameling van waarden die een grootheid aanneemt hoeft niet de verzameling van reële getallen te zijn. We gebruiken reële getallen voor waarden van fysische grootheden onder andere omdat we dezelfde reële getallen gebruiken voor ruimte en tijd. In de queeste voor een kwantum-theorie van ruimte en tijd staat het gebruik van de reële getallen ter discussie en dus moeten we dus ook andere opties kunnen toelaten voor de waarden voor fysische grootheden. Deze extra vrijheid voor de mogelijke waarden van grootheden is natuurlijk alleen nuttig als we ook ideeën hebben over hoe we die vrijheid kunnen benutten.

[8]Deze meting wordt dan beschreven met behulp van klassieke natuurkunde

Wiskunde en Natuurkunde

In dit boekje hoop ik dus een bijdrage te leveren aan het krijgen van meer inzicht in de conceptuele problemen in de kwantum-theorie, door de wiskundige beschrijving van deze theorie te veranderen middels topos-theorie, wat dat ook moge inhouden. Maar waarom zouden we verwachten dat het bestuderen van het wiskundig raamwerk ons ook maar iets vertelt over kwantum-theorie? Is het niet zo dat wiskunde en natuurkunde over heel andere zaken gaan? Natuurkunde onderzoekt wat we over de natuur kunnen zeggen op een fundamenteel niveau. Wiskunde gaat niet over de natuur, maar over abstracties zoals bijvoorbeeld cirkels en algebras, en over hun onderlinge relaties. Het is eigenlijk al een klein wonder dat de taal van de wiskunde een rol speelt binnen de natuurkunde. Er wordt dan ook wel eens gesproken over de onredelijke effectiviteit van de wiskunde (Wigner). Als wiskunde in de natuurkunde geen grotere rol speelt dan van taal hoef je niet te verwachten dat een wiskundige herformulering iets dieps oplevert. Immers, als je een moeilijk probleem formuleert maakt het voor het vinden van een oplossing niet zo veel uit of je die vraag in het Nederlands of in het Engels stelt.

Uiteraard kan ik mij niet in het bovenstaande plaatje vinden. De rol van de wiskunde binnen de natuurkunde is diep en subtiel. Voor het grootste deel van de geschiedenis van deze twee disciplines was er dan ook geen strak onderscheid; dat is er pas hooguit sinds de twintigste eeuw. Maar ook in de moderne wis- en natuurkunde is het vaak lastig te zien waar de ene discipline begint en de andere ophoudt. Wat mij betreft gaat wiskunde niet over zuivere abstracties die op één of andere manier bestaan in een wereld buiten ons. Concepten uit de wiskunde worden niet ontdekt maar uitgevonden. Ze zijn het product van de wiskundige en daarmee gekleurd door hoe wij naar de wereld kijken. De bovenstaande metafoor van iemand die een moeilijke vraag formuleert in het Nederlands of het Engels lijkt me ook misplaats om een andere reden. Eigenlijk zijn we niet op zoek naar een antwoord op een vraag als we kwantum-theorie beter willen begrijpen. Het antwoord hebben we al: dat is kwantum-theorie. We zijn op zoek naar een begrijpelijke vraag die bij dit antwoord past.

Topos-Theorie

Het startpunt van het boekje is de gebruikelijke beschrijving van kwantumtheorie. Voor natuurkundigen ziet het er wellicht een beetje vreemd uit omdat het in de taal van abstracte operator-algebras is gegoten, maar de beschrijving staat dicht bij de vertrouwde beschrijving in termen van Hilbert ruimten. Uit deze beschrijving wordt vervolgens een topos gemaakt. Maar wat is een topos?

In dit boekje zie ik een topos vooral als een wiskundig universum. Een *topos* is een plek waar je wiskunde kunt doen. Een topos heeft echter ook hele andere gezichten die allemaal belangrijk zijn, maar het gaat te ver om deze in deze samenvatting te bespreken. De meeste wiskunde is gegoten in de taal van *verzamelingen* en *functies* tussen die verzamelingen. Ook hogere wiskunde zoals differentiaalmeetkunde of functionaalanalyse voldoet hier aan. Om wiskunde te doen hebben we natuurlijk naast de definities van allerlei (mogelijk zeer complexe) verzamelingen en functies ook redeneerregels die bijvoorbeeld gebruikt worden om relaties tussen de definities te bewijzen. Verzamelingen en functies vormen samen een voorbeeld van een topos. Behalve de topos van verzamelingen (en functies) zijn er echter tal van andere mogelijkheden. Een topos is een voorbeeld van een *categorie*, dat betekent dat het bestaat uit een aantal objecten (zoals bijvoorbeeld verzamelingen) en een aantal pijlen (zoals bijvoorbeeld functies).

Een topos is niet zo maar een categorie, maar een categorie met bijzonder veel structuur. Deze structuur wordt gebruikt om bijvoorbeeld nieuwe objecten en pijlen te maken uit oude. Deze structuur zorgt er ook voor dat we over de objecten kunnen nadenken alsof het verzamelingen zijn en over de pijlen kunnen nadenken alsof het functies zijn. Een topos is dus een categorie die sterk lijkt op de categorie van verzamelingen. De structuur van een topos maakt het mogelijk om wiskundige begrippen zoals bijvoorbeeld algebras en topologische ruimtes toe te passen op objecten en pijlen in een topos. We kunnen met deze objecten redeneren en eigenschappen bewijzen, op dezelfde manier als met verzamelingen. Echter, niet alle redeneerregels die we gewend zijn uit de theorie van de verzamelingen mogen zomaar gebruikt worden voor elke willekeurige topos. Het keuzeaxioma en de wet van de uitgesloten derde zijn regels die we in de theorie van verzamelingen kunnen aannemen zonder in de problemen te komen. Voor een willekeurige topos daarentegen leidt het aannemen van deze regels mogelijk tot tegenstrijdigheden en absurditeiten.

In dit boekje beginnen we in de topos van verzamelingen, waarin kwantum-theorie normaal beschreven wordt. In deze topos van verzamelingen maken we een nieuwe topos $\mathcal{T}$. De wiskundige ingrediënten van kwantum-theorie worden vervolgens vertaald naar objecten en pijlen in de topos $\mathcal{T}$. We kunnen op twee verschillende manieren kijken naar kwantum-theorie:

- Vanuit het *interne perspectief* van de topos $\mathcal{T}$. Dit betekent dat we over de objecten en pijlen in $\mathcal{T}$ nadenken alsof het verzamelingen en functies zijn (ook al is dit strikt genomen niet zo). We beschouwen $\mathcal{T}$ als een wiskundig universum.

- Vanuit het *externe perspectief* van de topos $\mathcal{T}$. De topos $\mathcal{T}$ is zelf gedefiniëerd in termen van 'gewone' verzamelingen en functies. We kunnen dus ook naar de objecten en pijltjes in de topos uitschrijven in termen van (echte) verzamelingen en functies.

Beide perspectieven op een topos spelen een belangrijke rol in dit boekje. Wanneer we het formalisme van kwantum-theorie vanuit het interne perspectief bekijken zien we sterke overeenkomsten met het formalisme van de klassieke natuurkunde (een tak van de natuurkunde met heel wat minder interpretatieproblemen). We bekijken diezelfde constructies ook vanuit het externe perspectief (dus in termen van de meer vertrouwde wiskunde van verzamelingen) om na te gaan of de topos theoretische herformulering van kwantum-theorie begrijpelijk is wanneer we deze vanuit de Kopenhaagse Interpretatie bekijken.

Topos Modellen

Om de inhoud van het boekje daadwerkelijk samen te vatten moet ik de bovenstaande discussie preciezer maken. Vanaf dit punt wordt de presentatie dan ook technischer. Het startpunt van het boekje is een unitale C*-algebra A, en de (zelf-geadjungeerde) elementen van dit algebra corresponderen met de observeerbare grootheden van een kwantum-theorie. De toestanden van de kwantum-theorie worden wiskundig beschreven als positieve genormaliseerde lineaire functionalen, $\phi : A \to \mathbb{C}$, op de algebra A. Uit de algebra A maken we twee verschillende, maar nauw samenhangende, topoi als volgt.

Zoals benadrukt in de Kopenhaagse Interpretatie van de kwantum-mechanica is het enkel zinvol om over concepten uit kwantum-theorie te spreken vanuit een context, beschreven aan de hand van een klassieke natuurkunde.

In dit boekje is een *context* wiskundig opgevat als een *commutatieve* unitale C*-subalgebra C van de algebra A. Een sleutelrol is weggelegd voor de verzameling $\mathcal{C}_A$ van alle contexten van A. We zien $\mathcal{C}_A$ als een poset waarbij de orde wordt gegeven door inclusie. De twee topoi die worden gebruikt zijn $[\mathcal{C}_A^{op}, \mathbf{Set}]$, de categorie van preschoven op $\mathcal{C}_A$, en $[\mathcal{C}_A, \mathbf{Set}]$, de categorie van copreschoven op $\mathcal{C}_A$. Een object van zo'n topos is niet een verzameling maar een collectie van verzamelingen, één voor elke context C van A. Het verschil tussen deze twee topoi zit in hoe deze context-geïndexeerde verzamelingen met elkaar samenhangen. Het gebruik van de topos $[\mathcal{C}_A^{op}, \mathbf{Set}]$ is in 1998 voorgesteld door Jeremy Butterfield en Chris Isham en vanaf 2007 verder uitgewerkt door Andreas Döring en Isham. Het gebruik van de topos $[\mathcal{C}_A, \mathbf{Set}]$ is in 2009 voorgesteld door Chris Heunen, Klaas Landsman en Bas Spitters.

Eindelijk: De Daadwerkelijke Samenvatting

In Hoofdstuk 2 zien we hoe in de twee topos modellen voor kwantumtheorie waarmee de vorige sectie eindigde, de algebra A wordt gebruikt om een *faseruimte* te definiëren in de topoi. Hierbij wordt intensief gebruik gemaakt van *Gelfand dualiteit*. Gelfand dualiteit, in de versie die werkt voor elke topos, kent aan een commutatieve unitale C*-algebra C een ruimte toe, een compact volledig regulier locale, Σ_C zodanig dat C (tot op isomorfisme) gelijk is aan de algebra van continue complexwaardige functies $\Sigma_C \to \mathbb{C}$. In het copreschoof topos model $[\mathcal{C}_A, \mathbf{Set}]$ wordt de algebra A vervangen door een object $\underline{A}$ van de topos, die aan elke context C de verzameling C toekent. Vanuit het interne perspectief van de topos is dit object een unitale commutatieve C*-algebra, en heeft dus een Gelfand spectrum $\Sigma_{\underline{A}}$. In Sectie 2.2 beschrijf ik de externe presentatie van dit locale. Dit Gelfand spectrum fungeert als de faseruimte voor het copreschoof topos model. In de preschoof topos speelt de *spectrale preschoof* een centrale rol. Deze preschoof wordt gemaakt uit de Gelfand spectra Σ_C van alle contexten C. In Sectie 2.4 wordt de spectrale preschoof bekeken vanuit het interne perspectief van de preschoof topos en beschreven als topologische ruimte. Deze beschrijving laat zien dat de preschoof en copreschoof modellen sterk aan elkaar gerelateerd zijn.

Hoofdstuk 3 draait om fysische grootheden. Via de techniek van *daseinisatie*, geleend uit het preschoof model, geeft elk zelf-geadjungeerd element $a \in A$ een continue functie van de faseruimte naar de eenzijdige

reële getallen (een functie naar de zogenaamde 'lower reals' en een functie naar de 'upper reals'). Dit wordt op dezelfde manier gedaan voor beide topos modellen. Voor het copreschoof model betekent dit dat we de oorspronkelijke daseinisatie techniek van dit model iets moeten aanpassen. Ik denk dat dit nodig is om uiteindelijk, vanuit het externe perspectief, natuurkundig zinvolle resultaten te krijgen. Voor het preschoof model is de observatie dat vanuit het interne perspectief daseinisatie continue reëelwaardige functies levert nieuw. Ook worden de elementaire proposities uit dit model in direct verband gebracht met deze continue functies. Om technische redenen wordt in dit hoofdstuk de klasse van C*-algebras beperkt tot de kleinere klasse van von Neumann algebras.
Hoofdstuk 4 gaat over het beschrijven van toestanden in de topos modellen. In beide aanpakken geven toestanden, in de zin van functionalen op A, vanuit het interne perspectief kansmaten op de faseruimtes. Voor het preschoof model is deze observatie nieuw, al is hier de open vraag of er ook kansmaten bestaan op de faseruimte die niet afkomstig zijn van (quasi-)toestanden op A. Dit hoofdstuk bespreekt ook de logica's voor kwantum-theorie die de twee topos modellen voorstellen als alternatief voor de kwantum logica van Birkhoff en von Neumann uit 1936. Het oordeel is helaas dat het voor beide topos modellen nog helemaal niet duidelijk is of we met deze alternatieven beter af zijn.
Hoofdstuk 5 is technisch van aard. Het gaat over hoe tijdsevolutie op een wiskundig natuurlijke manier beschreven kan worden in de topos modellen. De technische complicatie hierbij is het feit dat een automorfisme op de algebra A een niet-triviale actie heeft op de verzameling $\mathcal{C}_A$ van contexten. Net als in de eerdere hoofdstukken wordt dit probleem voor beide modellen op dezelfde manier aangepakt.
Ten slotte onderzoekt Hoofdstuk 6 mogelijke uitbreidingen van het copreschoof model naar het Haag-Kastler formalisme van algebraïsche kwantum velden theorie. Het hoofdresultaat brengt een kinematische onafhankelijksconditie, C*-onafhankelijkheid genaamd, in verband met een schoof-conditie in de setting van topos modellen.

Reflectie

In de beide topos modellen die aan bod kwamen lijkt de herformulering van kwantum-theorie, vanuit het interne perspectief van de topos bekeken, heel wat meer op het formalisme van klassieke natuurkunde dan in de gebruikelijke versie van deze theorie. Maar helpt dit ons ook daadwerkelijk om verder te komen met de conceptuele problemen van kwantum-theorie? Dat lijkt mij nog onduidelijk. In de toposmodellen zijn de fysische grootheden continue functies van een faseruimte naar een (intern beschreven) verzameling van reële getallen. Wanneer we deze grootheden (gepaard met toestanden) extern beschouwen dan zijn de kansen uit de Born-regel van kwantum-theorie terug te vinden. Het is fijn dat we de Born waarschijnlijkheden houden, maar toch intern een klassiek uitziende beschrijving hebben. De topos theoretische modellen bieden een fraaie beschrijving, maar er lijkt nog een groot gat te zitten tussen deze beschrijving en het aanpakken van de conceptuele problemen, zoals het probleem van de tijd dat in het begin van deze samenvatting werd geschetst. Het is de vraag hoe dit gat overbrugd kan worden.

Het is niet zo dat ik betwijfel of topos-theorie rijk genoeg is om te helpen met conceptuele problemen in de moderne natuurkunde. Integendeel, topos-theorie biedt toegang tot veel nieuwe wiskundige werelden en, omdat de interne wiskunde van een topos geen last heeft van het keuzeaxioma of de wet van de uitgesloten derde, biedt het een subtielere en minder vooringenomen kijk op noties zoals de reële getallen. Ik verwacht dat de moeilijkheid vooral zit in het scherp genoeg stellen van de conceptuele problemen van kwantum-theorie om deze wiskundige middelen goed in te zetten.

Misschien gaat het ons niet lukken om meer verfijnde topos-theoretische modellen te vinden die ons helpen bij de analyse van conceptuele problemen zoals het probleem van de tijd. Dan zijn de topos-theoretische modellen besproken in dit boekje nog steeds interessant. Niet vanuit het perspectief van een topos als wiskundig universum, maar omdat de modellen werken met functoren over de verzameling $\mathcal{C}_A$. Contextualiteit is belangrijk in de studie van de grondslagen van kwantum-theorie en de taal van preschoven lijkt een natuurlijk hulpmiddel in de studie van contextualiteit. Het lijkt mij zeer interessant om dieper inzicht te krijgen hoe de structuur van $\mathcal{C}_A$ als poset of topologische ruimte samenhangt met de mate problemen rondom contextualitiet en nonlocaliteit in kwantum-theorie. In feite past dit perspectief naadloos bij de kwantum toposofie van Butterfield en Isham uit 1998. Het past ook in de recentere studie naar contextualiteit en nonlocaliteit door Samson Abramsky, Adam Brandenburgur en anderen.

Acknowledgements

Vier jaar werk heeft een fraai uitzicht over het landschap van de mathematische fysica opgeleverd. Uiteraard heb ik alleen zo ver kunnen kijken omdat ik op de schouders van een reus stond. De reus in kwestie is natuurlijk promotor Klaas Landsman. Uit respect voor zijn aversie voor overdreven dankwoorden zal ik het kort houden. Klaas, bedankt voor het delen van jouw passie voor de wetenschap, voor de oprechte betrokkenheid en de wijsheid die je hebt gedeeld. Ik had me geen betere promotor kunnen wensen.

Het is gebruikelijk om het dankwoord af te sluiten met het bedanken van de partner (indien van toepassing). Omdat dit teveel aanvoelt als een verplicht nummertje wijk ik hier van af. Vooral omdat Miranda een cruciale rol heeft gespeeld in de totstandkoming van dit boekje. Miranda, ontzettend bedankt voor alle steun die je hebt gegeven in de afgelopen jaren. Het is uiterst waardevol om te weten dat er iemand is op wie je altijd kan rekenen.

Bas Spitters wil ik graag bedanken voor het altijd enthousiast delen van zijn vele ideeën. Verder wil ik Hans Halvorson bedanken voor de uiterst plezierige samenwerking tijdens en na mijn twee maanden lange bezoek aan Princeton University. Ook dank aan Hans voor het mogelijk maken van dit onvergetelijke bezoek. Thanks go out to Cecilia Flori for sharing her ideas, as well as inviting me twice to visit the Perimeter Institute in Waterloo, Ontario. These weeks were inspiritional and have contributed much to my resolve to work on foundational issues in physics. Special thanks to Steve Vickers and Bertfried Fauser for sharing their insights and making me feel welcome during the two week visit to the University of Birmingham.

Thanks go out to the thesis committee, and in particular to Urs Schreiber, who went beyond the call of duty as a member of the thesis committee, sharing his many valuable thoughts regarding the material in this thesis.

Paranimfen Ralph van Gelderen en Willem Maat, hartelijk dank voor de rugdekking.
Het kloppende hart van de wiskunde afdeling op de Radboud Universiteit is natuurlijk het secretariaat. Mijn dank gaat uit naar Astrid en Greta, in het bijzonder voor hun hulp bij het organiseren van de kwantum toposofie workshop. Natuurlijk gaat mijn dank ook uit naar alle collega's van deze afdeling. Immers, zij hebben geholpen om van de afgelopen vier jaar een mooie tijd te maken. In het bijzonder wil ik uit deze groep kwantum toposoof Bert Lindenhovius bedanken voor zijn betrokkenheid en Noud Aldenhoven omdat ik hem nog wel eens lastig viel met (domme) vragen.
Tenslotte wil ik mijn ouders bedanken voor hun steun. Uiteraard kan ik het dankwoord niet zo laten eindigen, zeker gezien de bovenstaande opmerking over verplichtte nummertjes. Daarom sluit af met het bedanken van mijzelf, voor mijn tomeloze inzet.

Curriculum Vitae

Sander Wolters was born on February 17, 1980 in Zevenaar. He attended the LATS (Lagere Agrarische Technische School) *de Agron* in the period 1993-1996, after which he studied chemistry at the *Middelbare Laboratorium Onderwijs Arnhem* (vocational education) in the period 1996-2000. In the period 2000-2003 he studied analytic chemistry at the *Hogeschool van Arnhem en Nijmegen* where he obtained a bachelor of applied science with honours. In the period 2003-2006 he studied physics and mathematics at the *University of Utrecht* where he obtained both bachelors of science with honours. In 2009 he completed a master in mathematics with honours. The subject of the masters thesis was the use of Tannaka-Krein duality in algebraic quantum field theory. On June 15, 2009 he started as a research assistant in mathematical physics, under the supervision of professor Nicolaas P. Landsman. The end result of this research can be found in the previous pages.