The cohomological descent method

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of Bram Buijs

under supervision of
prof. dr. N.P. Landsman

University of Amsterdam, Faculty of Science
Korteweg-de Vries Institute for Mathematics
Plantage Muidergracht 24, 1019 TV Amsterdam
The Netherlands

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Abstract. In this thesis we provide an algebraic approach to the cohomological descent method, which is used in gauge field theories to investigate anomalies. In pursuit of this, an algebraization of the principal bundle setting is put forward. Several concepts known from principal bundle theory are generalized to Lie algebra operations, and in particular we prove that the Weil homomorphism can be generalized. Finally, we introduce the Weil-B.R.S. algebra, and prove that the cohomological descent method is surjective and injective under certain circumstances, which was indicated by Dubois-Violette in [6].

Key-words: Anomalies, Cohomological descent, Lie algebra operations, B.R.S. algebras.

Contents

Preface

In this thesis, we consider the mathematical background to the so-called *descent equations*, which are used in physics (specifically in gauge field theories) to investigate possible anomalies. Gauge field theories are used in theoretical physics to describe interactions between particles and fields (e.g. the electromagnetical field), and to describe a certain physical situation one can apply either classical gauge field theory or quantum gauge field theory, as appropriate. When a certain symmetry or invariance is lost when passing from classical gauge field theory to quantum gauge field theory, one speaks of an *anomaly*.

In one particular case the anomalies can be interpreted as elements of a certain cohomology class and there exists an algorithm, known as the *cohomological descent method* which supplies elements of this cohomology class. The descent method can be considered from a purely mathematical point of view and the first part of our thesis introduces all the mathematical concepts needed for the framework of the descent method. Not only do we provide an introduction to mathematical concepts like principal bundles, connections, curvature, characteristic classes, the group of gauge transformations and its Lie algebra, but we also generalize these concepts to a more abstract theory of Lie algebra operations. This turns out to be useful when calculating some specific cohomology groups, which can be described using two universal objects: the *Weil algebra* and the *Weil-B.R.S. algebra*. We closely follow Dubois-Violette [6] in this, but hope to enhance his article by including (almost) all proofs of theorems used, and in being more thorough in our explanation and motivation. Furthermore, we hope to bridge the gap between the concrete example of a principal bundle and the constructions made in Dubois-Violette, by indicating clearly what motivates the specific generalizations.

Our main result will be the proof of a statement made in Dubois-Violette: the descent equations, which yield certain cohomology classes, provide *all* the classes of the cohomology considered, and the method is in this respect “surjective”. However, the restrictions and assumptions made by Dubois-Violette also have their consequences for the validity of this statement. These consequences are the subject of our research.

As we indicated, this thesis may be divided in two parts; the first part consists of three chapters which deal with the generalization of several concepts known from the theory of principal bundles and an additional chapter on Lie algebra cohomology. The second part is formed by the last two chapters, and concerns the complexes which accommodate the cohomological descent method. At the end of the last chapter we have included our conclusion, in which we summarize the results achieved in this thesis and discuss some questions left open by

Dubois-Violette's article. On various occasions we have also included references to recent articles which make use of the concepts we introduced or which provide further generalizations.

Finally we would like to inform the reader that we have included a page summarizing our notation, preceding the bibliography, for his or her convenience.

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Chapter 1

A brief introduction to principal bundles

1.1 The principal bundle

Though we will assume the principle bundle construction known, we will start with a concise treatment in order to generalize later on. Let $P(G, M)$ be a principal bundle

$$G \hookrightarrow P \xrightarrow{\pi} M \tag{1.1}$$

where P and M are smooth manifolds. G is a Lie group, which acts on P by a smooth free right-action ($R_g : p \mapsto pg$) and M has a covering $\{U_\alpha\}$ such that each inverse image $\pi^{-1}(U_\alpha) \subset P$ is diffeomorphic to $U_\alpha \times G$. The diffeomorphism should satisfy certain *local trivialisation* conditions (§1.1.4). Also, the action of G on P preserves fibers, so we have $\pi(p) = \pi(pg)$, and each fiber is diffeomorphic to G.

In the following we will call G the *structure group*, P the *total space* and M the *base manifold*.

1.1.1 Fundamental and vertical vector fields

Since G is a Lie group, it has a Lie algebra $Lie(G)$. With every element X in the Lie algebra $Lie(G)$ we can associate a vector field $X^\#$ on P, which we call the associated *fundamental vector field*. If we take a particular $p \in P$ and consider the mapping $\sigma_p : G \to P$ given by $\sigma_p(g) = R_g p = pg$, we notice that this mapping is smooth and has a derivative $(\sigma_p)_e^T$ at the identity $e \in G$. We define $X_p^\#$ in the tangent space T_pP at $p \in P$ as

$$X_p^\# = (\sigma_p)_e^T = \frac{d}{dt}\big(p \cdot \exp(tX)\big)\Big|_{t=0}, \tag{1.2}$$

with $\exp: Lie(G) \to G$ the exponential mapping of the Lie group G.

Furthermore, we define the *vertical subspace* $Vert_p(P) \subset T_p(P)$ as the kernel of π^T, with $\pi^T : T_pP \to T_mM$ the tangent mapping of the projection $\pi : P \to$

M. A *vertical vector field* Y is then a vector field on P such that $Y_p \in Vert_p(P)$ for all $p \in P$. So,

$$Y \text{ is vertical} \Leftrightarrow \pi^T(Y) \equiv 0 \text{ on } M. \tag{1.3}$$

The mapping $X \to X_p^{\#}$ supplies an isomorphism of $Lie(G)$ onto $Vert_p(P)$, and we also have $[X,Y]^{\#} = [X^{\#}, Y^{\#}]$. Proofs of these statements are not too hard, but can be found in Naber [15] (Th. 4.7.8, Cor. 4.7.9). We show the following:

Lemma 1.1.1 A fundamental vector field $X^{\#}$ is vertical.

Proof: Since G preserves fibers we have $\pi(p \cdot \exp(tX)) = \pi(p)$ for all t. So

$$\pi^T(X_p^{\#}) = \frac{d}{dt}\pi(p \cdot \exp(tX))\Big|_{t=0} = \frac{d}{dt}\pi(p)\Big|_{t=0} = 0 \tag{1.4}$$

and this is the definition of a vertical vector field. Because G acts along the fibers (in "vertical" direction) the fundamental vector field is vertical.

1.1.2 The connection form and horizontal subspaces

On a principal bundle one can define a *connection form* as a Lie(G)-valued one-form on P, i.e. $\omega \in Lie(G) \otimes \Omega^1(P)$, with the following properties:

1. $\omega(X^{\#}) = X$, by which we mean $\omega_p(X_p^{\#}) = X \quad \forall p \in P$, for all fundamental vector fields $X^{\#}$ associated with $X \in Lie(G)$.
2. $(R_g)^*\omega = Ad_{g^{-1}} \circ \omega$ for all $g \in G$.

Here Ad_g denotes the adjoint action $Ad_g : Lie(G) \to Lie(G)$ of the Lie group G on the Lie algebra $Lie(G)$, that is defined for an element $g \in G$ as[1]

$$Ad_g(X) = \frac{d}{dt}\big(g \cdot exp(tX) \cdot g^{-1}\big)\Big|_{t=0}. \tag{1.5}$$

One can also view the connection as an assignment of *horizontal subspaces* $Hor_p(P) \subset T_pP$ in every tangent space T_pP such that $T_pP = Vert_p(P) \oplus Hor_p(P)$.[2] Given a *connection form* as described above, we can define

$$Hor_p(P) := \{ v \in T_pP \mid \omega_p(v) = 0 \}. \tag{1.6}$$

When we use the decomposition of a tangent vector in horizontal and vertical parts, we will write for $v \in T_pP$

$$v = v^H + v^V \qquad \text{with } v^H \in Hor_p \text{ and } v^V \in Vert_p. \tag{1.7}$$

Horizontal vector fields are defined in the same way as vertical ones: a vector field $X \in \mathfrak{X}(P)$ is called horizontal iff. $X_p \in Hor_p \quad \forall p \in P$.

In later chapters we will consider the *space of connections* on a principal bundle $P(G,M)$, and we will denote this space with $\mathscr{C}(P)$. As a consequence of condition (1.) above, $\mathscr{C}(P)$ cannot be a vector space, e.g. $2 \cdot \omega(X^{\#}) = 2 \cdot X \neq X$. It is an *affine* space, however, so for any two connections $\omega_1, \omega_2 \in \mathscr{C}(P)$ we have $(1-t)\omega_1 + t\omega_2 \in \mathscr{C}(P)$ for all $t \in \mathbb{R}$.

[1] See the appendix §A.3 for a brief treatment of Lie groups, Lie algebras and adjoint actions.

[2] With the assignment of these subspaces there is a smoothness condition involved: for every $p \in P$ there should be a neighbourhood $U \subset P$ and smooth vector fields $\{X_i\}_{i \in I}$ such that the vector fields span Hor_p for every $p \in U$. When this condition is met, the assignment $p \mapsto Hor_p$ is called a *smooth distribution on P*.

1.1.3 Curvature

Once one has a connection on a principal bundle, it is possible to define the *curvature* of the chosen connection.

Let $\omega \in Lie(G) \otimes \Omega^1(P)$ be a connection form on the principal bundle $P(G, M)$. **The curvature $\mathbf{\Omega} \in$Lie(G) $\otimes$ $\mathbf{\Omega^2(P)}$ of the connection** ω is defined by

$$\Omega_p(v, w) \stackrel{\text{def}}{=} d\omega_p(v^H, w^H) \qquad \forall v, w \in T_pP. \tag{1.8}$$

It is the exterior derivative of the connection ω, working on the horizontal parts (v^H, w^H) of the vectors. For those unfamiliar with differential forms taking values in some kind of vector space and the differential on such forms, we refer to the appendix. In particular, if $\Omega \equiv 0$ the connection ω is called **flat**.

The curvature (or *curvature form*) Ω satisfies a couple of properties we would like to note. To begin with, it is clear from the definition that $\Omega(v, w) = 0$ if v or w is vertical (since $v = v^V$ implies $v^H = 0$).

Secondly, the right action $R_g : p \mapsto pg$ of the structure group G on the total space P induces an action on the differential forms on P by means of the pull-back $(R_g)^*$. The curvature satisfies the following transformation property under the pull-back $(R_g)^*$:

$$(R_g)^*\Omega = Ad(g^{-1}) \circ \Omega. \tag{1.9}$$

Notice the connection form ω has exactly the same transformation property (see 1.1.2). In section §1.1.5 we will see this is called *Ad-equivariance*.

Theorem 1.1.1 Let ω be a connection, and Ω its curvature. Then the **Cartan structural equation** holds, which asserts

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega]. \tag{1.10}$$

Proof: This is Theorem 2.1.3, [2]. See §A.2.2 for definition of $d\omega$ and $[\omega, \omega]$.

We will see that this formulation of the curvature will be used in subsequent chapters to provide the generalization to algebras.

Equipped with a curvature form Ω on a principal bundle $P(G, M)$ it is possible to define the *Chern class* of the bundle. The Chern class is a cohomology class in the de Rham cohomology $H_{DR}(M)$ of the base manifold M. It is called a *characteristic class* because it turns out that the cohomology class that is obtained, is *independent* of the connection chosen on the bundle (and its curvature). Thus it is truly characteristic of the principle bundle itself.

We will come back to this issue later in chapter 3.

1.1.4 Local sections and gauge potentials

A local *section* or *cross-section* of a principal bundle is a smooth map $s : U \to P$ with $U \subset M$ an open set in the base manifold, such that

$$\pi \circ s \equiv \text{id}_U \qquad \text{on } U \subset M.$$

A section associates with every $m \in M$ an element $s(m) \in \pi^{-1}(m)$ in the fiber above m in the total space P, and that in a smooth way. Since we have a covering $\{U_\alpha\}$ of the base manifold M, such that the inverse images $\pi^{-1}(U_\alpha) \subset P$ are diffeomorphic to $U_\alpha \times G$, we have on each of these U_α a local section: consider such a $U_\alpha \subset M$, and let

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$$

be the given diffeomorphism, also called a **local trivialisation**. This map should satisfy two conditions (the **local trivialisation conditions**):

1. $\pi \circ \Phi_\alpha^{-1}(u, g) = u$ for $u \in U_\alpha$.
2. $\Phi_\alpha^{-1}(u, gh) = R_h \circ \Phi_\alpha^{-1}(u, g)$ for $u \in U_\alpha$ and $g, h \in G$.

Now we can define a section $s_\alpha : U_\alpha \to P$ as

$$s_\alpha(u) = \Phi_\alpha^{-1}(u, e), \qquad u \in U_\alpha\,,$$

and the first condition (1.) on Φ_α makes sure this is truly a section.

Transition functions

From the trivializing cover $\{U_\alpha\}$ of a principal bundle, one can extract the so-called *transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$.

These are defined in the following way: let $s_\alpha : U_\alpha \to P$ and $s_\beta : U_\beta \to P$ be local sections subordinate to the trivializing cover. For any $x \in U_\alpha \cap U_\beta$ the elements $s_\alpha(x), s_\beta(x) \in P$ will both in the same fiber $\pi^{-1}(x)$. Since G acts transitively on the fibers in P, there exists an element $g \in G$ such that $s_\alpha(x) = R_g s_\beta(x) = s_\beta(x)g$. Defining this element for every $x \in U_\alpha \cap U_\beta$ provides us a smooth map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to G$, such that we have

$$s_\beta(x) = s_\alpha(x) g_{\alpha\beta}(x) \quad \text{for } x \in U_\alpha \cap U_\beta.$$

The functions $\{g_{\alpha\beta}\}$ are called the **transition functions** of the bundle.[3]

Gauge potentials

Suppose one has a connection form $\omega \in \mathit{Lie}\,(G) \otimes \Omega^1(P)$ on the principal bundle $P(G, M)$. Using the local sections, one can obtain $\mathit{Lie}\,(G)$-valued 1-forms $\{a_\alpha\}$ on the open sets $\{U_\alpha\}$ by defining[4]

$$a_\alpha = s_\alpha^*(\omega) \quad \in \mathit{Lie}\,(G) \otimes \Omega^1(U_\alpha),$$

with $s_\alpha : U_\alpha \to P$ a local section.

For principal bundles figuring in Yang-Mills gauge theories, these forms have a physical interpretation and are called **(local) gauge potentials**. The gauge potential a_α depends on the chosen section s_α (hence the subscript α) and a_α is called a gauge potential *in gauge* s_α.[5]

[3] One can verify that the transition functions satisfy a so-called *cocycle condition*: for $x \in U_\alpha \cap U_\beta \cap U_\gamma$ one has $g_{\gamma\beta}(x) g_{\beta\alpha}(x) = g_{\gamma\alpha}(x)$. Furthermore $g_{\alpha\alpha}(x) = e$ and $g_{\alpha\beta}(x)^{-1} = g_{\beta\alpha}(x)$. See §1.3 [2].

[4] Usual notation for the gauge potentials is $A_\alpha = s_\alpha^*(\omega)$, but we follow the notation in [6].

[5] In physics literature a local section $s : U_\alpha \to P$ is often called a **local gauge**.

Given a gauge potential a_α on U_α one can define the **(local) field strength** $f_\alpha \in Lie(G) \otimes \Omega^2(U_\alpha)$ (in gauge s_α) as

$$f_\alpha = d(a_\alpha) + \tfrac{1}{2}\,[\,a_\alpha, a_\alpha\,],$$

or equivalently

$$f_\alpha = s_\alpha^*(\Omega),$$

with $\Omega = d\omega + \frac{1}{2}\,[\omega, \omega\,]$ the curvature associated with the chosen connection form ω. In physics literature, the curvature form Ω is called a **gauge field**.[6]

Both the (local) gauge potentials as the (local) field strength depend on the chosen section s. The dependency on the chosen section can be made explicit by certain compatibility conditions, which we will first introduce for the gauge potentials. If we consider a neighborhood on M for which we have different sections, such as $U_\alpha \cap U_\beta$ with sections s_α and s_β, the gauge potentials $a_\alpha = s_\alpha^*(\omega)$ and $a_\beta = s_\beta^*(\omega)$ will generally not coincide (and hence do not provide a global $Lie(G)$-valued 1-form on M). Instead they satisfy the following compatibility condition:

$$a_{\beta,x}(v) = Ad_{g_{\alpha\beta}^{-1}(x)}\, a_{\alpha,x}(v) + \Theta_{\alpha\beta,\,x}(v), \tag{1.11}$$

for $x \in U_\alpha \cap U_\beta$ and $v \in T_xM$.

Remark: $g_{\alpha\beta}$ is the transition function related to the sections s_α, s_β on $U_\alpha \cap U_\beta$; Ad the adjoint action of the group G on its Lie algebra $Lie(G)$ according to eq. (1.5) (also §A.3.1); $\Theta_{\alpha\beta}$ is a $Lie(G)$-valued 1-form on $U_\alpha \cap U_\beta$ associated to the transition function $g_{\alpha\beta}$ defined as

$$\Theta_{\alpha\beta,\,x}(v) = (L^{-1}_{g_{\alpha\beta}(x)})^T\, d(g_{\alpha\beta})_x(v),$$

for $x \in U_\alpha \cap U_\beta$ and $v \in T_xM$. Here $d(g_{\alpha\beta}) : T_xM \to T_{g_{\alpha\beta}(x)}G$ is the differential of $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$; using the tangent mapping of $L^{-1}_{g_{\alpha\beta}(x)}$ gives us an element of $T_eG = Lie(G)$ (see section §A.3 for details). For a proof of the compatibility condition (1.11) we refer to Theorem 2.1.1 in de Azcárraga and Izquierdo [2].

In sloppy (but common) notation the compatibility condition of equation (1.11) is referred to as

$$a_\beta = g_{\alpha\beta}^{-1}\, a_\alpha\, g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}.$$

This is actually correct if G is a matrix Lie group.

We just state here without proof that for the (local) field strength we have

$$f_{\beta,x} = Ad_{g_{\alpha\beta}^{-1}(x)} f_{\alpha,x}, \tag{1.12}$$

for $x \in U_\alpha \cap U_\beta$; with again $g_{\alpha\beta}$ the associated transition function and Ad the group representation of G on $Lie(G)$. Again the local field strengths f_α defined on $U_\alpha \subset M$ do not coincide on intersections $U_\alpha \cap U_\beta$, and thus do not piece together a global 2-form on M. However, from equation (1.12) it follows that *if G is abelian* they do, since in this case $Ad_g = id_{Lie(G)}\ \forall g \in G$. This is a peculiarity of *abelian* gauge field theories.

The above discussion of gauge potentials and the local field strength is very brief, but it is not our intention to treat the subject in depth here. We just need above definitions in later chapters, and this knowledge will suffice for our goals. We refer to Naber [15](both volumes) and Chapter 2 of de Azcárraga and Izquierdo [2] for more background information.

[6] Cf. [15] Vol. II, Ch. 1.

1.1.5 Some terminology

Before going on, we will formalize some properties which we encountered so far. For instance, the curvature Ω of a connection ω was defined as the exterior derivative working only on the horizontal parts of the arguments. This can be defined for arbitrary forms, and is called *(exterior) covariant differentation.*

Definition 1.1.1 Let $P(G,M)$ be a principal bundle, with a connection ω. The **(exterior) covariant derivative** $\mathcal{D}\alpha$ of a differential form $\alpha \in \Omega^n(P)$ is defined by

$$\mathcal{D}\alpha(v_1,\ldots,v_{n+1}) = d\alpha(v_1^H,\ldots,v_{n+1}^H). \tag{1.13}$$

Since the definition of the "horizontal parts" depends on the connection ω (i.e. on Hor_p), the derivative $\mathcal{D}$ is in fact dependent on the chosen connection on the principal bundle. When, for clarity, we wish to stress this dependence we will write $\mathcal{D}^\omega$ instead of $\mathcal{D}$.

Corollary 1.1.1 The curvature is the *exterior covariant derivative* of the connection form, $\Omega = \mathcal{D}^\omega\omega$.

Also some terminology has been invented to describe the properties of the connection and curvature form.

Definition 1.1.2 Let $P(G,M)$ be a principle bundle, and let $D : G \to GL(\mathcal{V})$ be a representation of G on a vector space $\mathcal{V}$. If α is a $\mathcal{V}$-valued form on P, i.e. $\alpha \in \mathcal{V} \otimes \Omega(P)$, then it is called **(D)-equivariant** if we have[7]

$$(R_g)^*\alpha = D(g^{-1}) \circ \alpha \qquad \forall g \in G. \tag{1.14}$$

Such a form is also called **pseudotensorial** of type $(D,\mathcal{V})$.

We call $\alpha \in \Omega(P)$ (or in $\mathcal{V}\otimes\Omega^n(P)$) **invariant** if the pull-back $(R_g)^*$ doesn't affect it:

$$(R_g)^*\alpha = \alpha \qquad \forall g \in G.$$

Furthermore, a differential form $\alpha \in \Omega^n(P)$ (or $\alpha \in \mathcal{V}\otimes\Omega^n(P)$) is called **horizontal** if it is zero when one of the arguments is a *vertical* tangent vector. So, if for all $p \in P$

$$X_i(p) \in Vert_p \text{ for some } i \Rightarrow \alpha_p(X_1(p),\ldots,X_n(p)) = 0.$$

Notice this equivalent with saying

$$\alpha_p(v_1,\ldots,v_n) = \alpha_p(v_1^H,\ldots,v_n^H) \qquad \forall p \in P, v_i \in T_pP.$$

Finally, a differential form that is both pseudotensorial of type $(D,\mathcal{V})$ and horizontal is called **tensorial** of type $(D,\mathcal{V})$.

We will concern ourself with the case $\mathcal{V} = Lie(G)$, the Lie algebra of the structure group G, on which we have the adjoint representation $Ad : G \to GL(Lie(G))$ given by (1.5). We note the following.

Corollary 1.1.2 The *connection form* ω is pseudotensorial of type (Ad, Lie(G)), i.e. Ad-equivariant, by definition (see (1.1.2), property 2).

[7] For a $\mathcal{V}$-valued form ω one defines $(R_g)^* = id_\mathcal{V} \otimes (R_g)^* : \mathcal{V}\otimes\Omega(P) \to \mathcal{V}\otimes\Omega(P)$.

Corollary 1.1.3 (Claim) The *curvature form* Ω is tensorial of type (Ad, Lie(G)), i.e. it is Ad-equivariant and horizontal. Horizontality was implied by the definition, but we haven't shown Ad-equivariance. We will prove both properties in a more general setting in Lemma 1.2.4.

Now we are also interested in a special class of differential forms on P, namely, the forms which are the pull-back of a differential form on the base manifold M.

Definition 1.1.3 Let $P(G, M)$ be a principal bundle, and $\pi : P \to M$ the projection map. A differential form $\alpha \in \Omega(P)$ is called **basic** (or **projectable**) if

$$\alpha = \pi^*(\bar{\alpha}) \tag{1.15}$$

for some $\bar{\alpha} \in \Omega(M)$. We will use the bar $\bar{\alpha}$ to denote the differential forms on M which are projections of basic forms α on P.

In the following lemma necessary and sufficient conditions are given for a form to be projectable.

Lemma 1.1.2 A form $\alpha \in \Omega^n(P)$ is basic (projectable) iff. it is invariant and horizontal.

Proof: ($\Leftarrow$) Let $m \in M$ and $p \in \pi^{-1}(m)$ an arbitrary element in the fiber above m. Let $X_1(m), \ldots, X_n(m)$ be tangent vectors in $T_m M$. Since $\pi : P \to M$ is a submersion, we know there are vectors $Y_1(p), \ldots, Y_n(p) \in T_p P$ such that $\pi^T(Y_i(p)) = X_i(m)$ (the vectors Y_i project on the X_i). Now we define $\bar{\alpha} \in \Omega^n(M)$ as

$$\bar{\alpha}_m(X_1(m), \ldots, X_n(m)) = \alpha_p(Y_1(p), \ldots, Y_n(p)). \tag{1.16}$$

Of course we need to check this is independent of the choices we made ($p \in \pi^{-1}(m)$ and the $Y_i(p)$). Suppose $\tilde{p} \in \pi^{-1}(m)$ and $\tilde{p} \neq p$, and $\tilde{Y}_i(\tilde{p}) \in T_{\tilde{p}}P$ project also on $X_i(m)$. Since p and $\tilde{p}$ are both in the same fiber, we have $\tilde{p} = pg = R_g p$ for some $g \in G$. By invariance of α we have

$$\begin{aligned}
\alpha_p(Y_1(p), \ldots, Y_n(p)) &= (R_g)^* \alpha_p(Y_1(p), \ldots, Y_n(p)) \\
&= \alpha_{pg}(R_g^T Y_1(p), \ldots, R_g^T Y_n(p)) \\
&= \alpha_{\tilde{p}}(R_g^T Y_1(p), \ldots, R_g^T Y_n(p)) \\
&= \alpha_{\tilde{p}}(\tilde{Y}_1(\tilde{p}), \ldots, \tilde{Y}_n(\tilde{p})),
\end{aligned}$$

which proves $\bar{\alpha}$ is well-defined. The last equality holds because the difference $R_g^T Y_i(p) - \tilde{Y}_i(\tilde{p})$ between $R_g^T Y_i(p)$ and $\tilde{Y}_i(\tilde{p})$ (which are both vectors in $T_{\tilde{p}}P$) is a *vertical* vector, and α is a horizontal form that is zero on vertical vectors by definition. We show $R_g^T Y_i(p) - \tilde{Y}_i(\tilde{p})$ is vertical, i.e. it is projected to zero by π^T:

$$\begin{aligned}
\pi^T(R_g^T Y_i(p) - \tilde{Y}_i(\tilde{p})) &= \pi^T(R_g^T Y_i(p)) - \pi^T(\tilde{Y}_i(\tilde{p})) \\
&= (\pi \circ R_g)^T(Y_i(p)) - \pi^T(\tilde{Y}_i(\tilde{p})) \\
&= \pi^T(Y_i(p)) - \pi^T(\tilde{Y}_i(\tilde{p})) \\
&= X_i(m) - X_i(m) \\
&= 0,
\end{aligned}$$

where we used $\pi \circ R_g = \pi$ for the projection $\pi : P \to M$.

(⇒) We know α is basic, say $\alpha = \pi^*(\bar{\alpha})$ with $\bar{\alpha} \in \Omega^n(M)$. Then α is invariant since we have

$$(R_g)^*\alpha = (R_g)^*\pi^*\bar{\alpha} = (\pi \circ R_g)^*\bar{\alpha} = (\pi)^*\bar{\alpha} = \alpha.$$

Secondly, α is horizontal since

$$\alpha_p(X_1(p), \ldots, X_n(p)) = \bar{\alpha}_{\pi(p)}(\pi^T X_1(p), \ldots, \pi^T X_n(p)),$$

and if any $X_i(p) \in T_pP$ is vertical, it means by definition $\pi^T X_i(p) = 0$ and hence $\bar{\alpha}$ and α will be zero.

1.2 Generalizations

1.2.1 Lie algebra operations

A first step in generalizing the constructions made so far, is considering them from a purely algebraic point of view. We notice that the differential forms on the total space P, denoted by $\Omega(P)$, form a *graded-commutative differential algebra* (abbreviated as GCDA; definitions can be found in §A.1 in the appendix).

On any GCDA one can introduce the notion of a *Lie algebra action* or *Lie algebra operation.* For this one needs a finite-dimensional Lie algebra $\mathfrak{g}$ which maps linearly into graded derivations of the algebra by means of two maps i and L. The graded derivations should satisfy special commutation properties with the differential. To be precise:

Definition 1.2.1 Let ($\mathscr{A}$,d) be a graded-commutative differential algebra (with differential d) and $\mathfrak{g}$ a finite-dimensional Lie algebra. An **action of $\mathfrak{g}$ on $\mathscr{A}$** is a pair (i, L) of linear mappings from $\mathfrak{g}$ to the graded derivations $\mathrm{Der}^{(*)}(\mathscr{A})$ on the algebra $\mathscr{A}$

$$i : \mathfrak{g} \to \mathrm{Der}^{(-1)}(\mathscr{A}), \qquad i : X \mapsto i_X,$$
$$L : \mathfrak{g} \to \mathrm{Der}^{(0)}(\mathscr{A}), \qquad L : X \mapsto L_X,$$

such that

$$L_X = di_X + i_X d, \tag{1.17}$$
$$L_{[X,Y]} = [L_X, L_Y] = L_X L_Y - L_Y L_X, \tag{1.18}$$
$$i_{[X,Y]} = L_X i_Y - i_Y L_X, \tag{1.19}$$
$$(i_X)^2 = 0, \tag{1.20}$$

for all $X, Y \in \mathfrak{g}$.

So, for all $X \in \mathfrak{g}$ we have an anti-derivation i_X of degree -1, and a derivation L_X of degree zero on $\mathscr{A}$. If these conditions are met, the pair $(\mathscr{A}, i, L)$ (or simply $\mathscr{A}$) is called a **$\mathfrak{g}$-operation**. One also says $\mathfrak{g}$ **operates on** $\mathscr{A}$.

Remark: This notion is due to H. Cartan [5]. Unfortunately, there is no agreement upon the terminology as yet. Instead of $\mathfrak{g}$-*operation*, one can also encounter a *Cartan operation*[6] or $\mathfrak{g}$-*differential algebra* [1].

Remark II: This definition is taken from Kastler&Stora [11]. Since the derivation L_X is expressed by (1.17) as a combination of the differential d and anti-derivation i_X one could also define a $\mathfrak{g}$-operation by a mapping $i : \mathfrak{g} \to \mathrm{Der}^{(-1)}(\mathscr{A})$ for which $di_X + i_X d$ results in a derivation of degree zero satisfying equations (1.18) and (1.19). This equivalent definition is used in Dubois-Violette [6].

On the algebra $\Omega(P)$ of differential forms on P several graded derivations are known. We have the interior product i_X of a vector field $X \in \mathfrak{X}(P)$ with a form $\alpha \in \Omega^n(P)$ that is defined as $i_X\alpha(X_2,\ldots,X_n) = \alpha(X, X_2,\ldots,X_n)$. It is a anti-derivation of degree -1 on $\Omega(P)$. And we have the Lie derivative L_X of forms by vector fields, which can be defined as $L_X = di_X + i_X d$. That supplies us with an derivation of degree zero on $\Omega(P)$. Now, as we have seen, there is a way to extend elements of $\mathit{Lie}\,(G)$ to vector fields on P, by means of the fundamental vector field. This is all we need to make $\Omega(P)$ into a $\mathit{Lie}\,(G)$ -operation, which we state as the following corollary.

Corollary 1.2.1 Let $P(G, M)$ be a principal bundle. The GCDA $\Omega(P)$ is a $\mathit{Lie}\,(G)$ -operation, with the action

$$\begin{aligned} i : X \in \mathit{Lie}\,(G) &\mapsto i_{X^\#} \in \mathrm{Der}^{(-1)}(\Omega(P)), \\ L : X \in \mathit{Lie}\,(G) &\mapsto L_{X^\#} \in \mathrm{Der}^{(0)}(\Omega(P)), \end{aligned}$$

with $X^\# \in \mathfrak{X}(P)$ the fundamental vector field associated with $X \in \mathit{Lie}\,(G)$, $i_{X^\#}$ the interior product of differential forms with vector fields, and $L_{X^\#}$ the Lie derivative of forms by vector fields.

Proof: For the definition of the fundamental vector field, see (1.2) in §1.1.1. The interior product i_X and Lie derivative L_X are defined in §A.2.1 in the appendix. We know $i_V : \Omega^n(P) \to \Omega^{n-1}(P)$ is an anti-derivation on $\Omega(P)$ for any vector field $V \in \mathfrak{X}(P)$, so this is certainly true for the fundamental vector fields $X^\#$. Linearity of i follows from the definitions of the fundamental vector field (1.2) and properties of the interior product (§A.2.1):

$$c \cdot X \mapsto i_{(c\cdot X)^\#} = i_{c\cdot X^\#} = c \cdot i_{X^\#} \qquad (c \in \mathbb{R})$$

and

$$X + Y \mapsto i_{(X+Y)^\#} = i_{X^\# + Y^\#} = i_{X^\#} + i_{Y^\#}.$$

Furthermore we know the Lie derivative L_V is expressed as $L_V = di_V + i_V$ for $V \in \mathfrak{X}(P)$ (eq. (A.35) appendix) and that this is a derivation on $\Omega(P)$. So $L : X \mapsto L_{X^\#}$ is a linear mapping of $\mathit{Lie}\,(G)$ to $\mathrm{Der}^{(0)}(\Omega(P))$ by linearity of i and d.

Conditions (1.18)-(1.20) correspond to the relations described in §A.2.1, and this proves $\mathit{Lie}\,(G)$ operates on $\Omega(P)$.

1.2.2 Algebraic formulation of equivariance, invariance and horizontality

Now, having shown that the algebra $\Omega(P)$ of differential forms on the total space P of a principal bundle $P(G, M)$ is a $\mathit{Lie}\,(G)$ -operation, we can translate several properties defined on differential forms to elements of general $\mathfrak{g}$-operations, in particular *equivariance*,*invariance* and *horizontality*. Since on an arbitrary $\mathfrak{g}$-operation we just have the graded derivations i_X and L_X we will need to express equivariance, invariance and horizontality in terms of these operations. This is the content of the following three lemmas.

Lemma 1.2.1 Let $P(G, M)$ a principal bundle with G connected, $\mathfrak{g} = \mathit{Lie}\,(G)$, and $\alpha \in \mathfrak{g}\otimes\Omega(P)$ a $\mathfrak{g}$-valued form on P. For $X \in \mathfrak{g}$, let $X^\#$ denote the associated fundamental vector field. Then the following are equivalent:

1. α is Ad-equivariant, i.e. $(R_g)^*\alpha = Ad_{g^{-1}} \circ \alpha$.

2. $L_{X^\#}\alpha = [\alpha, X], \qquad \forall X \in \mathfrak{g}$;

   by which we mean $(L_{X^\#}\alpha)_p(v_p) = [\alpha_p(v_p), X]$ $\forall p \in P, v_p \in T_pP$ and the bracket is the ordinary Lie bracket in $\mathfrak{g}$.

Remark: ($1 \Rightarrow 2$) still holds if G is not connected, but for the converse we need connectivity.

Proof: ($1 \Rightarrow 2$) For this we recall the definition of the Lie derivative $L_{X^\#}$ by the vector field $X^\# \in \mathfrak{X}(P)$, described in the appendix §A.2.1 by (A.24)

$$L_{X^\#}\alpha = \lim_{t\to 0}\frac{\phi_t^*\alpha - \alpha}{t} = \frac{d}{dt}\,\phi_t^*\alpha\,\Big|_{t=0}.$$

Since the vector field $X^\#$ is defined in $p \in P$ as

$$X_p^\# = \frac{d}{dt}\,p\cdot\exp(tX)\,\Big|_{t=0} = \frac{d}{dt}\,R_{\exp(tX)}\,p\,\Big|_{t=0},$$

it follows the flow of $X^\#$ is given by $\phi_t = R_{\exp(tX)}$, and we get for $p \in P$ and $v_p \in T_pP$,

$$\begin{aligned}
(L_{X^\#}\alpha)_p(v_p) &= \frac{d}{dt}\,\phi_t^*\alpha_p(v_p)\,\Big|_{t=0}\\
&= \frac{d}{dt}\,(R_{\exp(tX)})^*\alpha_p(v_p)\,\Big|_{t=0}\\
&= \frac{d}{dt}\,Ad(\exp(tX)^{-1})\,\alpha_p(v_p))\,\Big|_{t=0}\\
&= \frac{d}{dt}\,Ad(\exp(-tX))\,\alpha_p(v_p)\,\Big|_{t=0}\\
&= ad(-X)\,\alpha_p(v_p)\\
&= [\,-X, \alpha_p(v_p)\,]\\
&= [\,\alpha_p(v_p), X\,].
\end{aligned}$$

We used (1) the Ad-equivariance of α which states $(R_{exp(tX)})^*\alpha = Ad(\exp(tX)^{-1})\circ\alpha$ and (2) the differential in e of the adjoint mapping $Ad : G \to GL(\mathfrak{g})$ is the adjoint action $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ of $\mathfrak{g}$ on itself given by $ad(X)(Y) = [X, Y]$ (see §A.3.1). It follows that

$$\frac{d}{dt}Ad(\,\exp(-tX)\,)\,\Big|_{t=0} = ad(-X).$$

A proof of these facts can be found in Duistermaat&Kolk [8], §1.1. We have now shown that $L_{X^\#}\alpha = [\alpha, X]$ for any Ad-equivariant differential form $\alpha \in \mathfrak{g}\otimes\Omega(P)$.

($1 \Leftarrow 2$) See Kastler and Stora [11].

Lemma 1.2.2 Let $P(G, M)$ a principal bundle with G connected, $\mathfrak{g} = Lie(G)$, and $\alpha \in \mathfrak{g}\otimes\Omega(P)$ a $\mathfrak{g}$-valued form on P. For $X \in \mathfrak{g}$ let $X^\#$ denote the associated fundamental vector field. Then the following are equivalent:

1. α is invariant, i.e. $(R_g)^*\alpha = \alpha$.
2. $L_{X^\#}\alpha = 0, \qquad \forall X \in \mathfrak{g}$.

Remark: Again, ($1 \Rightarrow 2$) still holds if G is not connected, but for the converse we need connectivity.

Proof: ($1 \Rightarrow 2$) The proof is similar to Lemma 1.2.1 above. This time we get

$$L_{X^\#}\alpha = \frac{d}{dt}\,\phi_t^*\alpha\,\Big|_{t=0} = \frac{d}{dt}\,(R_{\exp(tX)})^*\alpha\,\Big|_{t=0} = \frac{d}{dt}\,\alpha\,\Big|_{t=0} = 0.$$

(1 ⇐ 2) Following the equations the other way around shows us

$$\frac{d}{dt}\left(R_{\exp(tX)}\right)^*\alpha\Big|_{t=0} = 0,$$

for all $X \in \mathit{Lie}(G)$. Now this holds not only at $t = 0$, but at $t = t_0$ for any $t_0 \in \mathbb{R}$:

$$\begin{aligned}
0 &= (R_{\exp(t_0X)})^*(0)\\
&= (R_{\exp(t_0X)})^* \left.\frac{d}{dt}(R_{exp(tX)})^*\alpha\right|_{t=0}\\
&= \left.\frac{d}{dt}(R_{\exp(t_0X)})^*(R_{exp(tX)})^*\alpha\right|_{t=0}\\
&= \left.\frac{d}{dt}(R_{\exp((t+t_0)X)})^*\alpha\right|_{t=0}\\
&= \left.\frac{d}{dt}(R_{\exp(tX)})^*\alpha\right|_{t=t_0}.
\end{aligned}$$

Since $(R_{\exp(0)})^*\alpha = (R_e)^*\alpha = \alpha$ this implies $(R_{\exp(tX)})^*\alpha = \alpha$ for all $t \in \mathbb{R}$ and $X \in \mathit{Lie}(G)$. Since we assumed G to be connected it is generated by the image of exp ([8], Th. 1.9.1), and hence $(R_g)^*\alpha = \alpha$ for all $g \in G$.

Lemma 1.2.3 Let $P(G, M)$ a principal bundle with G connected, $\mathfrak{g} = \mathit{Lie}(G)$, and $\alpha \in \mathfrak{g} \otimes \Omega^n(P)$ a $\mathfrak{g}$-valued form on P. For $X \in \mathfrak{g}$ let $X^\#$ denote the associated fundamental vector field. Then the following are equivalent:

1. α is horizontal, i.e.
   $V_i(p) \in \mathit{Vert}_p$ for some $i \Rightarrow \alpha_p(V_1(p), \ldots, V_n(p)) = 0 \qquad V_i \in \mathfrak{X}(P)$.
2. $i_{X^\#}\alpha = 0, \qquad \forall X \in \mathfrak{g}$.

Proof: (1 ⇒ 2) This follows from the fact that fundamental vector fields are in particular *vertical* vector fields (Lemma 1.1.1) and horizontal forms are zero on vertical vector fields by definition.

(1 ⇐ 2) Suppose $V_i(p) \in Vert(p)$. In section 1.1.1 we claimed that the map $X \mapsto X^\#(p)$ is in fact an isomorphism between $\mathfrak{g}$ and $Vert(p)$ ([15], Corollary 4.7.9). So for every $V_i(p) \in Vert(p)$ there is a $X \in \mathfrak{g}$ such that $X^\#(p) = V_i(p)$. Then $i_{X^\#}\alpha = 0$ implies

$$\begin{aligned}
\alpha_p(V_1(p), \ldots, V_i(p), \ldots, V_n(p)) &= \alpha_p(V_1(p), \ldots, X^\#(p), \ldots, V_n(p))\\
&= \pm\, \alpha_p(X^\#(p), V_1(p), \ldots, V_n(p))\\
&= \pm\, (i_{X^\#}\alpha)_p(V_1(p), \ldots, V_n(p))\\
&= 0,
\end{aligned}$$

which proves the horizontality of α.

These three lemmas lead us to the generalization of equivariance, invariance and horizontality to arbitrary $\mathfrak{g}$-operations. Since Ad-equivariance is defined on $\mathit{Lie}(G)$ -valued differential forms (i.e. elements of $\mathit{Lie}(G) \otimes \Omega(P)$) we generalize this notion for elements of $\mathfrak{g} \otimes \mathscr{A}$, with $\mathscr{A}$ a $\mathfrak{g}$-operation. From Lemma A.1.1 in the appendix we know that if $\mathscr{A}$ is a GCDA and $\mathfrak{g}$ is a Lie algebra, $\mathfrak{g} \otimes \mathscr{A}$ is a differential graded Lie algebra (DGLA). Thus there is a graded Lie bracket on $\mathfrak{g} \otimes \mathscr{A}$ defined by $[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes (\alpha \cdot \beta)$ for $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathscr{A}$, and in particular there is a differential d defined on $\mathfrak{g} \otimes \mathscr{A}$ by $d(X \otimes \alpha) = X \otimes d\alpha$

for $X \in \mathfrak{g}, \alpha \in \mathscr{A}$. Now if $\mathscr{A}$ is a $\mathfrak{g}$ -operation we can make $\mathfrak{g} \otimes \mathscr{A}$ into a $\mathfrak{g}$ -operation as well by defining in the same way

$$L_Y(X \otimes \alpha) = X \otimes L_Y \alpha, \qquad X, Y \in \mathfrak{g}, \alpha \in \mathscr{A}$$

and

$$i_Y(X \otimes \alpha) = X \otimes i_Y \alpha, \qquad X, Y \in \mathfrak{g}, \alpha \in \mathscr{A}.$$

With this taken care of we are ready to define equivariance on arbitrary $\mathfrak{g}$ -operations. Invariance and horizontality cause less trouble since these notions are defined on elements of $\mathscr{A}$ itself.

Definition 1.2.2 Let $(\mathscr{A}, i, L)$ be a $\mathfrak{g}$-operation, then an element $A \in \mathfrak{g} \otimes \mathscr{A}$ is **equivariant** if we have

$$L_X A = [A, X] \qquad \forall X \in \mathfrak{g}. \tag{1.21}$$

Furthermore we define the set $\mathscr{I}(\mathscr{A})$ of **invariant** elements of $\mathscr{A}$, the set $\mathscr{H}(\mathscr{A})$ of **horizontal** elements of $\mathscr{A}$, and the set $\mathscr{B}(\mathscr{A})$ of **basic** elements of $\mathscr{A}$ by

$$\begin{aligned}
\mathscr{I}(\mathscr{A}) &= \{ \alpha \in \mathscr{A} \mid L_X \alpha = 0, \quad \forall X \in \mathfrak{g} \}, \\
\mathscr{H}(\mathscr{A}) &= \{ \alpha \in \mathscr{A} \mid i_X \alpha = 0, \quad \forall X \in \mathfrak{g} \}, \\
\mathscr{B}(\mathscr{A}) &= \{ \alpha \in \mathscr{A} \mid L_X \alpha = 0 \text{ and } i_X \alpha = 0, \quad \forall X \in \mathfrak{g} \}.
\end{aligned}$$

Remember we defined the *basic forms* on $\Omega(P)$ as the subset $\pi^*(\Omega(M)) \subset \Omega(P)$ of differential forms on P which were the pull-back of forms on M. We proved in Lemma 1.1.2 that an element $\alpha \in \Omega(P)$ was basic iff. it was invariant and horizontal. This motivates the definition of $\mathscr{B}(\mathscr{A})$.

In Lemma B.2.2 we prove $\mathscr{I}(\mathscr{A})$ is a graded differential subalgebra of $\mathscr{A}$; $\mathscr{H}(\mathscr{A})$ is graded subalgebra of $\mathscr{A}$, that is stable by L_X and furthermore $\mathscr{B}(\mathscr{A})$ is a differential subalgebra of $\mathscr{I}(\mathscr{A})$ as well as $\mathscr{A}$.

1.2.3 Algebraic connections and covariant derivatives

Now we wish to introduce corresponding notions of connections and curvature to arbitrary $\mathfrak{g}$-operations. With the properties of the connection form ω and curvature Ω expressed in the operations i_X and L_X on $\Omega(P)$, we can generalize this directly to arbitrary $\mathfrak{g}$ -operations.

First consider a connection form $\omega \in \mathfrak{g} \otimes \Omega^1(P)$ as defined in section 1.1.2. The first property $\omega(X^{\#}) = X$ for $X \in Lie(G)$ can be translated directly to

$$i_{X^{\#}} \omega = X, \qquad X \in Lie(G).$$

The second property $(R_g)^* \omega = Ad_{g^{-1}} \circ \omega$ states the equivariance of ω, which can be expressed as

$$L_{X^{\#}} \omega = [\omega, X], \qquad X \in Lie(G),$$

by Lemma 1.2.1. So we can define for arbitrary $\mathfrak{g}$ -operations the notion of an *algebraic connection.*

Definition 1.2.3 Let $(\mathscr{A}, i, L)$ be a $\mathfrak{g}$-operation, then an element $A \in \mathfrak{g} \otimes \mathscr{A}^1$ is called an **(algebraic) connection on** $\mathscr{A}$ if we have for all $X \in \mathfrak{g}$

$$i_X A = X \quad \text{and} \quad L_X A = [\,A, X\,].$$

Now for the corresponding notion of curvature. For the generalization we use the Cartan Structure Equation (1.10) which states $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$. We define the **curvature of an algebraic connection** $A \in \mathfrak{g} \otimes \mathscr{A}^1$ to be the element $F \in \mathfrak{g} \otimes \mathscr{A}^2$ given by

$$F = dA + \tfrac{1}{2}[\,A, A\,]$$

We claimed in Corollary 1.1.3 that the curvature form $\Omega \in \mathfrak{g} \otimes \Omega^2(P)$ is horizontal as well as equivariant, which means

$$i_{X^\#}\Omega = 0 \quad \text{and} \quad L_{X^\#}\Omega = [\Omega, X], \qquad X \in \mathit{Lie}\,(G),$$

by Lemma 1.2.3 and Lemma 1.2.1. Now we will prove these claims for the generalized curvature F on an arbitrary $\mathfrak{g}$ -operation, from which it follows that it holds for the curvature form Ω on a principal bundle as well since this is a $\mathit{Lie}\,(G)$ -operation with $\mathscr{A} = \Omega(P)$.

Lemma 1.2.4 Let $A \in \mathfrak{g} \otimes \mathscr{A}^1$ be an algebraic connection on the $\mathfrak{g}$ -operation $\mathscr{A}$ and let $F = dA + \frac{1}{2}[A, A] \in \mathfrak{g} \otimes \mathscr{A}^2$ be its curvature. Then F satisfies

$$i_X F = 0 \quad \text{and} \quad L_X F = [F, X] \qquad \forall\, X \in \mathfrak{g}. \tag{1.22}$$

Proof: This proof relies heavily on the fact that $\mathfrak{g} \otimes \mathscr{A}$ is a DGLA. In the appendix (§A.1.1) we have listed the basic properties of a DGLA, and proven several lemmas we will use now. For instance, by Lemma A.1.2 we know that i_X defined on $\mathfrak{g} \otimes \mathscr{A}$ as above is also an anti-derivation of degree -1 on $\mathfrak{g} \otimes \mathscr{A}$. Similarly, L_X is a derivation of degree zero. We use this in the following, where we have e.g. $i_X\,[\,A, A\,] = [\,i_X A, A\,] - [\,A, i_X A\,]$ since A has degree 1. Furthermore we use (i) $[\,X, A\,] = -[\,A, X\,]$ by the commutativity of the graded Lie bracket (A.11) (ii) $d(X) = 0$ for $X = X \otimes 1 \in \mathfrak{g} \otimes \mathscr{A}^0$ by Cor. A.1.2 (iii) $[\,[A, X], A\,] + [\,A, [A, X]\,] = [\,[\,A, A\,], X\,]$ by the graded Jacobi identity (A.12).

We have $i_X A = X$ and $L_X A = [A, X]$ for A, so

$$\begin{aligned}
i_X F &= i_X \;\; dA + \tfrac{1}{2}[\,A, A\,] \\
&= (L_X - d i_X)A + \tfrac{1}{2} \;\; i_X\,[\,A, A\,] \\
&= L_X A - d(X) + \tfrac{1}{2} \;\; [\,i_X A, A\,] - [\,A, i_X A\,] \\
&= [A, X] - 0 + \tfrac{1}{2} \;\; [\,i_X A, A\,] - [\,A, i_X A\,] \\
&= [A, X] + \tfrac{1}{2} \;\; [\,X, A\,] - [\,A, X\,] \\
&= [A, X] + \tfrac{1}{2} \;\; -[\,A, X\,] - [\,A, X\,] \\
&= [A, X] - [A, X] \\
&= 0,
\end{aligned}$$

which proves horizontality, and

$$\begin{aligned} L_X F &= L_X(dA + \tfrac{1}{2}[A,A]) \\ &= L_X(dA) + \tfrac{1}{2} L_X [A,A]) \\ &= d(L_X A) + \tfrac{1}{2} \quad [L_X A, A] + [A, L_X A] \\ &= d([A,X]) + \tfrac{1}{2} \quad [\,[A,X],A\,] + [\,A,[A,X]\,] \\ &= [\,dA, X\,] + [A, d(X)] + \tfrac{1}{2}\,[\,[\,A,A\,],X\,] \\ &= [\,dA, X\,] + [A, 0] + \tfrac{1}{2}\,[\,[\,A,A\,],X\,] \\ &= [\,dA, X\,] + 0 + \tfrac{1}{2}\,[\,[\,A,A\,],X\,] \\ &= [\,dA + \tfrac{1}{2}\,[\,A,A\,],X\,] \\ &= [\,F, X\,], \end{aligned}$$

which proves equivariance.

Continuing our generalizations, we follow Kastler&Stora [11] by defining the notion of covariant derivatives for an arbitrary $\mathfrak{g}$-operation.[8] If $\mathscr{A}$ is a $\mathfrak{g}$-operation, then in general any element of $\mathfrak{g} \otimes \mathscr{A}^1$ defines a covariant derivative.

Definition 1.2.4 Let $\mathcal{A} \in \mathfrak{g} \otimes \mathscr{A}^1$. We define the **covariant derivative** by $\mathcal{A}$ of an element $\omega \in \mathfrak{g} \otimes \mathscr{A}$, by

$$\mathcal{D}^{\mathcal{A}}(\omega) = d\omega + [\,\mathcal{A}, \omega\,],$$

and thus defined $\mathcal{D}^{\mathcal{A}} : \mathfrak{g} \otimes \mathscr{A}^k \to \mathfrak{g} \otimes \mathscr{A}^{k+1}$ is an anti-derivation of degree $+1$.

Proof: it is clearly a homogeneous endomorphism of degree $+1$, and since d and $ad(\mathcal{A}) = [\,\mathcal{A}, \cdot\,]$ are anti-derivations of degree $+1$ (see Lemma A.1.4) so is $\mathcal{D}^{\mathcal{A}}$.

Lemma 1.2.5 Let $\mathcal{A} \in \mathfrak{g} \otimes \mathscr{A}^1$, and let $\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\,\mathcal{A}, \mathcal{A}\,]$. Then

$$\mathcal{D}^{\mathcal{A}}\mathcal{F} = 0$$

and this is known as the (generalized) **Bianchi identity**.

Proof: We have

$$\begin{aligned} \mathcal{D}^{\mathcal{A}}\mathcal{F} &= d\mathcal{F} + [\,\mathcal{A}, \mathcal{F}\,] \\ &= d \;\; d\mathcal{A} + \tfrac{1}{2}[\,\mathcal{A},\mathcal{A}\,] \;\; + [\,\mathcal{A}, d\mathcal{A} + \tfrac{1}{2}[\,\mathcal{A},\mathcal{A}\,]\,] \\ &= \tfrac{1}{2}[\,d\mathcal{A},\mathcal{A}\,] - \tfrac{1}{2}[\,\mathcal{A}, d\mathcal{A}\,] + [\,\mathcal{A}, d\mathcal{A}\,] + \tfrac{1}{2}[\,\mathcal{A},[\,\mathcal{A},\mathcal{A}\,]\,] \\ &= -\tfrac{1}{2}[\,\mathcal{A}, d\mathcal{A}\,] - \tfrac{1}{2}[\,\mathcal{A}, d\mathcal{A}\,] + [\,\mathcal{A}, d\mathcal{A}\,] \\ &= 0, \end{aligned}$$

since $[\,d\mathcal{A},\mathcal{A}\,] = -[\,\mathcal{A}, d\mathcal{A}\,]$ by (A.11) and $[\,\mathcal{A},[\,\mathcal{A},\mathcal{A}\,]\,] = 0$ by the graded Jacobi identity (A.12).

Notice that there is a slight difference with the definitions as given in §1.1.5. In that section we defined the covariant derivative of a differential form α by $\mathcal{D}^{(\omega)}\alpha(X_1, .., X_k) = d\alpha(X_1^H, .., X_k^H)$. We saw the curvature form $\Omega \in Lie(G) \otimes \Omega^2(P)$ was equal to the covariant derivative $\mathcal{D}^{\omega}\omega$ of the connection form ω. In the generalized case above, for a connection form A, we have $\mathcal{D}^A A = dA + [\,A, A\,]$

[8] For this definition we do not need the derivations i_X and L_X, so in fact the following definition and lemma are valid for any differential graded Lie algebra (DGLA).

which is almost, but not quite, equal to $dA + \frac{1}{2}[A, A]$, the curvature of A. So in the general (algebraic) case, we have $F \neq \mathcal{D}^A A$.[9]

The finishes the first part of generalizing concepts taken from the principal bundle setting to Lie algebra operations on graded-commutative differential algebras. One might wonder if this is particularly useful. The use will become apparent when we will turn our attention to Lie algebra operations which are not of the form $\Omega(P)$ for some manifold P. We will show that in the category of $\mathfrak{g}$-operations with a connection on it, there is a universal object known as the *Weil algebra*. We will introduce the Weil algebra in chapter 3, but first we will turn to principal bundle homomorphisms and gauge transformations and their generalizations in the next chapter.

1.3 Notes

The entire construction of principal bundles is thoroughly treated in Naber [15] (Vol. I) and de Azcárraga and Izquierdo [2]. Asides from stressing the topological background, Naber also goes into the problems encountered in physics which motivated these mathematical constructions. There is much more to be said about this then we have done in this chapter, and we just sketch a few interesting theorems. Some terminology (e.g. trivial bundles and equivalent bundles) which we use in these notes will be introduced in Chapter 2.

From the trivializing cover $\{U_\alpha\}$ of a principal bundle, one can extract the so-called *transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ as we have seen. It turns out that these transition function contain all the "essential" information about the bundle. To be precise: given a manifold M, a covering $\{U_\alpha\}$, and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ one can construct a principal bundle $G \hookrightarrow P \to M$. If one took the transition functions belonging to a certain principle bundle, this *Reconstruction Theorem* (Th. 3.3.4 in Naber [15] Vol. I) will turn out an equivalent bundle.[10]

After having introduced the concept of a principal bundle, one might wonder how one obtains a principal bundle. A very important theorem in this context is the following: let P be a smooth manifold and G a Lie group acting on it. If the action is effective and proper the orbit space $M = P/G$ will be a smooth manifold, and so $G \hookrightarrow P \to M$, or in shorthand $P(G, M)$, will be a principal bundle (Theorem 1.11.4 in Duistermaat and Kolk [8]).

Examples are given by the Hopf bundles. For the complex Hopf bundle we have $P = SU(2), G = U(1), M = S^2$, so

$$U(1) \hookrightarrow SU(2) \to S^2.$$

Identifying $U(1) \cong S^1, SU(2) \cong S^3$ and $S^2 \cong CP^1$ (the complex projective space), one can also describe this as $S^1 \hookrightarrow S^3 \to CP^1$. Replacing complex numbers by the quaternions gives the quaternionic Hopf bundle

$$S^3 \hookrightarrow S^7 \to S^4 = HP^1.$$

Both bundles are described in §1.3 [2] and [15].

[9] This curious discordance is present in de Azcárraga[2] (compare eq. (2.1.17) with (2.1.11) and Def. 2.1.4) as well as Kastler&Stora [11](see eq. (B.1) and (B.2)).

[10] The *equivalence* of bundles is a notion which will be introduced in Chapter 2.

Another important remark to make is that fiber bundles (and in particular principal bundles) over a contractible base manifold are trivial.[11] Bundles with a sphere $M = S^n$ as base manifold (called *sphere bundles*, like e.g. the Hopf bundles) can thus be seen as the simplest non-trivial bundles. For sphere bundles there is a powerful classification theorem: the equivalence classes of bundles with S^n ($n \geq 2$) as base manifold and G (a pathwise connected topological group) as structure group are in bijective correspondence to the homotopy group $\pi_{n-1}(G)$ (Theorem 3.4.3 in Naber [15] Vol. I).

More generally, for principal bundles with a matrix Lie group as structure group one has a classification theorem based on the classification of complex vector bundles.[12] The theorem uses the construction of a so-called *universal bundle* and a *classifying map* to this universal bundle. Equivalence classes of bundles are then in bijective correspondence to homotopy classes of classifying maps. We refer to Chapter 2 §7 in Walschap [18] for further details.[13]

[11]We will introduce the formal notion of a *trivial principal bundle* in the next chaper.

[12]From a principal bundle one can construct associated vector bundles and vice versa. See §1.3 in de Azcárraga and Izquierdo [2] and (even better) Chapter 2 of Walschap [18].

[13]For the similar classification of complex vector bundles, also see Chapter IV §23 in Bott&Tu [4].

Chapter 2

The group of gauge transformations and its generalization

This chapter will start off with briefly defining the concept of principal bundle mappings in general, and then quickly move on to the notions of *equivalence maps* and bundle automorphisms. In particular the group of *vertical bundle automorphisms* is of interest to us. This group is also known as the *group of gauge transformations*. The group is of great importance for physicists, and its Lie algebra will play a central role in the cohomological descent method. We will introduce the group of gauge transformations and its most important properties in §2.3, and define its generalization with respect to $\mathfrak{g}$-operations.

When dealing with descent equations the considered principal bundle is usually assumed to be *trivial*. Therefore we will also pay some attention to the trivial bundle and its group of gauge transformations.

2.1 Principal bundle mappings

With the definition of a principal bundle comes also an appropriate definition of a *principal bundle mapping* which is as follows:

Definition 2.1.1 Let $P(G, M)$ and $P'(G', M')$ be two principal bundles, with right actions R, R' and projections π, π'. A **principle bundle mapping** (or **principal bundle homomorphism**) f is a triple $f = (f_{gr}, f_{ts}, f_b)$ with

$$f_{gr} : G \to G', \qquad f_{ts} : P \to P', \qquad f_b : M \to M',$$

such that

1. f_{gr} is a group homomorphism.
2. f_{ts} commutes with the right action: $f_{ts}(R_g p) = R'_{g'}(\ f_{ts}(p)\ )$ with $g' = f_{gr}(g)$. This implies that f_{ts} maps fibers to fibers.
3. f_b is the map induced by f_{ts} and the projections: $f_b(\pi(p)) = \pi'(f_{ts}(p))$.

In particular, we can consider mappings between principal bundles with the same structure group and base manifold (i.e. $G = G'$ and $M = M'$). This leads to the following notion.

Definition 2.1.2 An **equivalence** or **isomorphism** between two principal bundles $P(G, M)$ and $P'(G, M)$ with the same structure group G and base manifold M, is a principal bundle mapping $f = (f_{gr}, f_{ts}, f_b)$ for which

1. $f_{gr} \equiv \mathrm{id}_G : G \to G$, i.e. f_{gr} is the identity map on G.[1]
2. $f_{ts} : P \to P'$ is a diffeomorphism.
3. $f_b \equiv \mathrm{id}_M : M \to M$, i.e. f_b is the identity map on M.

As usual, this defines an equivalence relation on all principal bundles; two bundles are called *equivalent* if there exists an equivalence map between them. An important equivalence class is given by the *trivial* bundles, i.e. the principal bundles $P(G, M)$ equivalent to the trivial bundle described in the following section.

Now we would like to consider equivalence maps of a principal bundle $P(G, M)$ onto itself. One would suppose they are called *automorphisms* of the bundle following usual group terminology. They are, however, known as the ***vertical*** **bundle automorphisms** of the bundle, and denoted $\mathrm{Aut}_v(P)$[2]. They are determined by (and often identified with) the diffeomorphism $f_{ts} : P \to P$, so we have

$$\mathrm{Aut}_v(P) = \{\, f \in \mathit{Diff}(P) \mid f \circ R_g = R_g \circ f \text{ and } \pi \circ f = \pi \,\}.$$

The **bundle automorphisms**, denoted $\mathrm{Aut}(P)$, are defined as the diffeomorphisms on P commuting with the right action.

$$\mathrm{Aut}(P) = \{\, f \in \mathit{Diff}(P) \mid f \circ R_g = R_g \circ f \,\}.$$

Any element $f \in \mathrm{Aut}(P)$ maps fibers to fibers, and thus induces a diffeomorphism $f_M \in \mathit{Diff}(M)$. If $f \in \mathrm{Aut}_v(P)$, each fiber is mapped onto itself, so $f_M \equiv \mathrm{id}_M \in \mathit{Diff}(M)$. We can summarize this in the following short exact sequence

$$1 \to \mathrm{Aut}_v(P) \to \mathrm{Aut}(P) \to \mathit{Diff}(M) \to 1.$$

Later in this chapter we will change the notation and denote $\mathrm{Aut}_v(P)$ with $\mathscr{G}$, but first we will briefly discuss trivial bundles.

2.2 The trivial principal bundle

In this section we give the definition of a trivial principal bundle, and record some special properties of a trivial bundle.

Definition 2.2.1 The **trivial bundle** is the principal bundle $P(G, M)$ where the total space P is the Cartesian product of the base manifold and the structure group, $P = M \times G$.

[1] In some references (notably [2]) f_{gr} may be any group isomorphism. We will follow the majority and assume $f_{gr} = \mathrm{id}_G$ however.

[2] Sometimes denoted with $\mathrm{Aut}_{\mathrm{M}}(P)$.

Remark: The projection $\pi : P = M \times G \to M$ is then given by the ordinary Cartesian projection $\pi : (m, g) \mapsto m$, and the right action R_g of G on $P = M \times G$ by $R_g : (m, h) \mapsto (m, hg)$.

It is quite easy to check that this bundle satisfies all conditions of a principal bundle: π commutes with the right action R_g and P is globally diffeomorphic to $M \times G$, so certainly locally trivial.

A principal bundle $P'(G, M)$ is called **trivial** if it is equivalent to the trivial bundle $P(G, M)$ with $P = M \times G$. Concerning the trivial bundle, we would like to prove some essential propositions.

Proposition 2.2.1 A bundle is trivial iff. it admits a global section $s : M \to P$.

Proof: ([2], Th. 1.3.1)
($\Rightarrow$) On the trivial bundle define $s : M \to M \times G$ to be $s(m) = (m, e)$. On a bundle $P(M, G)$ equivalent with the trivial bundle, compose this map with the inverse of the diffeomorphism $f_{ts} : P \to M \times G$.
($\Leftarrow$) Let $P(G, M)$ a principal bundle, and let $s : M \to P$ be a global section. Because of the local trivialisations, each fiber is diffeomorphic to G, and G acts transitively and free on each fiber. If $p \in P$, and $\pi(p) = m \in M$, then p and $s(m)$ are in the same fiber. Hence there is a unique $g \in G$ such that $p = s(m)g$. So we have a map $\phi : P \to G$ such that $p = s(m)\phi(g)$, and it is smooth since the right-action R_g is smooth. Now we define

$$\Phi : P \to M \times G \quad \text{as} \quad \Phi(p) = (\pi(p), \phi(p)),$$

and this supplies us with the diffeomorphism $P \cong M \times G$.

Notice this proposition only holds in the case of *principal* bundles. For vector bundles for instance, where the fiber is a vector space, there is always a global section (namely the *zero section*) but this does not imply the triviality of the bundle.

The following proposition shows the Maurer-Cartan form Θ_{MC} on G (see appendix §A.3) supplies a connection form for the trivial bundle.

Proposition 2.2.2 On a trivial bundle there is a canonical flat connection.

Proof: Consider the trivial bundle $P(M, G)$ with $P = M \times G$. Let Θ_{MC} be the Maurer-Cartan form on G, and let $\pi_G : M \times G \to G$ be the ordinary Cartesian projection on G, i.e. $\pi_G : (m, g) \mapsto g$. Then the pull-back $\omega = (\pi_G)^*\Theta_{MC}$ is a connection form on $M \times G$.

Let $p = (m, g) \in P = M \times G$; a careful look at the definition of the fundamental vector field (1.2) and the right-action defined on the trivial bundle in Def. 2.2.1 shows that $(\pi_G)^T X_p^\# = X_g^L$, so for all $X \in Lie(G)$

$$\omega_p(X_p^\#) = (\pi_G^*\Theta_{MC})_p(X_p^\#) = \Theta_{MC}(g)(X_g^L) = X.$$

By Lemma A.3.2 we have $(R_g)^*\Theta_{MC} = Ad_{g^{-1}} \circ \Theta_{MC}$, from which it follows that for $g \in G$ we have

$$\begin{aligned}
(R_g)^*\omega &= (R_g)^*\pi_G^*\Theta_{MC} \\
&= (\pi_G \circ R_g)^*\Theta_{MC} \\
&= (\pi_G)^*(Ad_{g^{-1}} \circ \Theta_{MC}) \\
&= Ad_{g^{-1}} \circ (\pi_G)^*\Theta_{MC} \\
&= Ad_{g^{-1}} \circ \omega,
\end{aligned}$$

and so $\omega = (\pi_G)^*\Theta_{MC}$ is connection on $M \times G$.

By Lemma A.3.1 we know $\dot{\Theta}_{MC} + \frac{1}{2}[\Theta_{MC}, \Theta_{MC}] = 0$, and hence for the curvature of ω we have

$$\begin{aligned} d\omega + \tfrac{1}{2}[\omega,\omega] &= d(\pi_G)^*\Theta_{MC} + \tfrac{1}{2}[(\pi_G)^*\Theta_{MC}, (\pi_G)^*\Theta_{MC}] \\ &= (\pi_G)^* \;\; d\Theta_{MC} + \tfrac{1}{2}[\Theta_{MC}, \Theta_{MC}] \\ &= 0, \end{aligned}$$

since the pull-back $(\pi_G)^*$ is natural with the exterior derivative d and the bracket of forms. Thus the curvature $\Omega = d\omega + \frac{1}{2}[\omega,\omega] = 0$ and ω is flat.

Finally, we wish to consider gauge potentials on a trivial bundle. By Proposition 2.2.1 we know that for a trivial bundle $P(G,M)$ there exist global sections. Hence each global section $s : M \to P$ defines a global gauge potential

$$a = s^*(\omega) \quad \in \mathit{Lie}(G) \otimes \Omega^1(M),$$

with ω a connection form on P. Naturally a different connection form will lead to a different gauge potential, and we define the space of gauge potentials (for a trivial bundle) by

$$\mathfrak{a}_{pot}(M) = \{\, a \in \mathit{Lie}(G) \otimes \Omega^1(M) \mid a = s^*(\omega) \text{ for some } \omega \in \mathscr{C}(M \times G)\,\}. \quad (2.1)$$

It turns out that $\mathfrak{a}_{pot}(M) \cong \mathit{Lie}(G) \otimes \Omega^1(M) \cong \mathscr{C}(M \times G)$, which we will formalize in the following proposition.

Proposition 2.2.3 For the trivial bundle $P(M,G)$ with $P = M \times G$ one has

$$\mathfrak{a}_{pot}(M) \cong \mathit{Lie}(G) \otimes \Omega^1(M) \cong \mathscr{C}(M \times G).$$

Proof: We prove that every element $a \in \mathit{Lie}(G) \otimes \Omega^1(M)$ defines a connection form ω on $P = M \times G$, and that $a = s^*(\omega)$ where $s : M \to P$ is the canonical global section. This proves the proposition.

Let $a \in \mathit{Lie}(G) \otimes \Omega^1(M)$. Since $P = M \times G$ we have $T_pP = T_mM \oplus T_gG$ at $p = (m,g)$ and (using the splitting of horizontal and vertical subspaces) for $v \in T_{(m,g)}$ we have $v = v^H + v^V$ with $v^H \in T_mM$, $v^V \in T_gG$. We define

$$\omega_{(m,\,g)}(v^H + v^V) = Ad_{g^{-1}}\, a_m(v^H) + (\Theta_{MC})_g(v^V),$$

with Θ_{MC} the Maurer-Cartan form on G (§A.3). From the definition it follows that ω is a connection form and also we have $a = s^*(\omega)$.

2.3 The group of gauge transformations

We have defined a *vertical automorphism* of a principal bundle $P(G,M)$ as a diffeomorphism of P commuting with the right-action and inducing the identity on M. The vertical automorphisms were denoted with $\mathrm{Aut}_v(P)$, but we now abbreviate this to $\mathscr{G} = \mathrm{Aut}_v(P)$, i.e.

$$\mathscr{G} = \{\, f \in \mathrm{Diff}(P) \mid f \circ R_g = R_g \circ f \text{ and } \pi \circ f = \pi \,\}.$$

Naturally, $\mathscr{G}$ is a group with the composition and inverse of mappings, and id_P as identity. We call $\mathscr{G}$ the **group of gauge transformations**.

We now record some special properties of this group.

2.3.1 Correspondence with Ad-equivariant maps

First of all, notice that $f \in \mathscr{G}$ preserves fibers (i.e. $\pi \circ f = \pi$), and since the right-action of G on P is transitive and free on the fibers, we can express the map $f : p \mapsto f(p)$ for any element $p \in P$ by

$$f(p) = (R_{\phi(p)})p = p \cdot \phi(p),$$

where $\phi : P \to G$ is a smooth map; smooth because f and the right-action are smooth. This map $\phi : P \to G$ has the following property:

$$\phi(R_g p) = Ad_{g^{-1}} \circ \phi(p),$$

where $\phi(R_g p) = \phi(pg)$ and $Ad_{g^{-1}} \circ \phi(p) = g^{-1}\phi(p)g$.

Proof: Since f commutes with R_g, i.e. $f(pg) = f(p)g$, we have

$$p \cdot g\,\phi(pg) = pg \cdot \phi(pg) = f(pg) = f(p)g = p \cdot \phi(p)g,$$

so $g\,\phi(pg) = \phi(p)g$ and hence $\phi(pg) = g^{-1}\phi(p)g = Ad_{g^{-1}} \circ \phi(p)$.

This property is called **Ad-equivariance** in analogy with the definition of Ad-equivariance for differential forms (Def. 1.1.2), with the difference that here Ad means the action of the Lie group G on itself by conjugation:

$$Ad : G \to Aut(G) \quad \text{given by} \quad Ad_g(h) = ghg^{-1} \text{ for } g \in G.$$

Following the argument the other way around, one can verify that any smooth Ad-equivariant map defines a gauge transformation. The correspondence between automorphisms $f : P \to P$ with $f \in \mathscr{G}$ and Ad-equivariant maps $\phi : P \to G$ is in fact a bijection ([2], Th. 10.1.1).

Since the set $Map_{\mathrm{Ad}}(P, G)$ of Ad-equivariant maps is also a group with pointwise multiplication (i.e. for $\phi, \psi : P \to G$ we have $(\phi \cdot \psi)(p) = \phi(p) \cdot \psi(p)$) we can check this bijection is a group isomorphism.

Proof: Let $f, g \in \mathscr{G}$ and let $\phi, \psi : P \to G$ be the associated Ad-equivariant maps such that we have the identifications $f \leftrightarrow \phi$ and $g \leftrightarrow \psi$. Thus $f(p) = p \cdot \phi(p)$ and $g(p) = p \cdot \psi(p)$. Then for $f \cdot g = f \circ g \in \mathscr{G}$ we have

$$\begin{aligned} f \circ g(p) &= f(g(p)) \\ &= g(p) \cdot \phi(g(p)) \\ &= (p \cdot \psi(p)) \cdot \phi(p\psi(p)) \\ &= (p \cdot \psi(p)) \cdot \psi(p)^{-1}\phi(p)\psi(p) \\ &= p \cdot \phi(p)\psi(p) \\ &= p \cdot (\phi \cdot \psi)(p), \end{aligned}$$

so $f \circ g \leftrightarrow \phi \cdot \psi$ and hence the bijection is a group isomorphism.

2.3.2 Action on differential forms and connections

Since elements $f \in \mathscr{G}$ are diffeomorphisms $f : P \to P$ there is a natural action of $\mathscr{G}$ on the differential forms $\Omega(P)$ on P by means of the pull-back. Now if we want this action to be a representation, we should define a group homomorphism $R : \mathscr{G} \to \mathrm{Aut}(\Omega(P))$.[3]

[3] $\mathrm{Aut}(\Omega(P))$ denotes the group of automorphisms of the graded-commutative differential algebra $\Omega(P)$. Its automorphisms are linear mappings $\Omega(P) \to \Omega(P)$ which commute with the differential, and are homogeneous of degree zero; i.e. they map $\Omega^n(P)$ on $\Omega^n(P)$. See Def. A.1.7 in the appendix.

This can be done if we define $R(f) \in \mathrm{Aut}(\Omega(P))$ as

$$R(f) : \omega \mapsto (f^{-1})^* \omega, \qquad \omega \in \Omega(P).$$

We need to use the inverse $(f^{-1})^*$ instead of f^* because the pullback is contravariant, and otherwise R would not be a group homomorphism.[4] Since the pull-back commutes with the differential, and maps n-forms onto n-forms, $R(f)$ is indeed an algebra automorphism of $\Omega(P)$.

We now wish to concentrate on special kinds of differential forms on P, namely the *basic* forms defined in §1.1.5 and the connection form (§1.1.2), and consider the action of $\mathscr{G}$ on these forms. The results are stated in the following two propositions.

Proposition 2.3.1 The action of $\mathscr{G}$ on $\Omega(P)$ leaves basic forms invariant; i.e. for a basic form $\omega = \pi^*(\bar{\omega})$ with $\bar{\omega} \in \Omega(M)$ and $f \in \mathscr{G}$ we have

$$f^* \omega = \omega.$$

Proof: Since $f \in \mathscr{G}$ we have $\pi \circ f = \pi$, or what is the same, the map $f : P \to P$ which maps fibers into fibers, induces the identity map on M. Hence we have

$$f^* \omega = f^*(\pi^* \bar{\omega}) = (\pi \circ f)^* \bar{\omega} = \pi^* \bar{\omega} = \omega,$$

which shows ω is invariant under $f \in \mathscr{G}$.

The action of $\mathscr{G}$ on $\Omega(P)$ by pull-back extends naturally to Lie algebra valued forms in $\mathfrak{g} \otimes \Omega(P)$, see the appendix §A.2.2. Since connection forms are elements of $\mathit{Lie}(G) \otimes \Omega^1(P)$, the group of gauge transformations $\mathscr{G}$ acts on them too. We have the following proposition.

Proposition 2.3.2 The action of $\mathscr{G}$ on $\mathfrak{g} \otimes \Omega(P)$ maps connections onto connections; i.e. for a connection form $\omega \in \mathit{Lie}(G) \otimes \Omega^1(P)$ and $f \in \mathscr{G}$ the form $f^* \omega \in \mathit{Lie}(G) \otimes \Omega^1(P)$ will again be a connection.

Proof: Let $\omega \in \mathit{Lie}(G) \otimes \Omega^1(P)$ be a connection form, so that we have $\omega(X^{\#}) = X$ for $X \in \mathit{Lie}(G)$ and $(R_g)^* \omega = Ad_{g^{-1}} \circ \omega$. Now we need to prove these two properties for $f^* \omega$.
(I) We need to prove $f^* \omega(X^{\#}) = X$. Let $p \in P$ and $X_p^{\#} \in T_p P$, and remember $X_p^{\#}$ was defined as (cf. (1.2))

$$X_p^{\#} = \frac{d}{dt} \left. p \cdot \exp(tX) \right|_{t=0} = \frac{d}{dt} \left. R_{\exp(tX)}\, p \right|_{t=0}.$$

If we consider the image of this vector under the tangent map $f^T : T_p P \to T_{f(p)} P$ we get

$$f^T(X_p^{\#}) = \frac{d}{dt} f \left. \left(R_{\exp(tX)}\, p \right) \right|_{t=0} = \frac{d}{dt} R_{\exp(tX)} \left. \left(f(p) \right) \right|_{t=0} = X_{f(p)}^{\#},$$

by commutativity of f with the right-action R_g. It follows that

$$(f^* \omega)_p(X_p^{\#}) = \omega_{f(p)}(\, f^T(X_p^{\#})\,) = \omega_{f(p)}(\, X_{f(p)}^{\#}\,) = X \quad \text{ for } X \in \mathit{Lie}(G).$$

[4] Now we have $R(f \circ g) = ((f \circ g)^{-1})^* = (g^{-1} \circ f^{-1})^* = (f^{-1})^* \circ (g^{-1})^* = R(f) \circ R(g)$.

(II) We need to prove $(R_g)^*(f^*\omega) = Ad_{g^{-1}} \circ (f^*\omega)$. This follows easily, since

$$\begin{aligned}(R_g)^*(f^*\omega) &= (f \circ R_g)^*\omega \\ &= (R_g \circ f)^*\omega \\ &= f^*((R_g)^*\omega) \\ &= f^*(Ad_{g^{-1}} \circ \omega) \\ &= Ad_{g^{-1}} \circ f^*\omega,\end{aligned}$$

which proves the equivariance of $f^*\omega$.

This proposition leads to an important notion: two connections ω_1, ω_2 on a certain principal bundle $P(G, M)$ are called **gauge equivalent** if there is a gauge transformation $f \in \mathscr{G}$ such that $\omega_2 = f^*\omega_1$. As the name suggests, gauge equivalence defines an equivalence relation on connections.

In the so-called *Yang-Mills (gauge) theories* in physics, connections on a bundle are used to obtain the field strength of certain fields; for instance the electro-magnetic field. Gauge equivalent connections prescribe the same field, so physicists are more interested in the equivalence classes of connections. We already introduced the space of connections $\mathscr{C}(P)$ on a bundle in §1.1.2, and remarked this was an affine space. Above proposition gives us an action of $\mathscr{G}$ on $\mathscr{C}(P)$, and the space of equivalence classes $\mathscr{C}(P)/\mathscr{G}$ is called the *orbit space* or *moduli space of connections*. It is possible to define on this space a topology, and to construct a bundle $\mathscr{G} \hookrightarrow \mathscr{C}(P) \to \mathscr{C}(P)/\mathscr{G}$; all this in order to study the properties of the moduli space.

This is of course just a sketch of the situation to motivate the concepts we are discussing. We recommend the two books by Naber [15] for a (mathmatical) introduction to gauge theories, and the use of principal bundles, connections and group of gauge transformations in these theories. There is quite a lot of current research on moduli spaces; theorems in this field are published by famous mathematicians as Donaldson, Atiyah and Singer.

For an introduction to moduli spaces we refer to the first volume by Naber [15] (chapter 5 in particular) and §10.1 in [2].

2.3.3 Gauge transformations on a trivial bundle

The cohomological descent method, which will be treated in later chapters, is usually applied to a principal bundle that is assumed to be *trivial.* An important reason for this is that, aside from being the simplest example of a principal bundle, the trivial bundle also has a convenient group of gauge transformations. It can be shown that the group of gauge transformations $\mathscr{G}$ on a trivial bundle is in fact $Map\,(M, G)$.

Following this proposition we will show how the group of gauge transformations $\mathscr{G}$ acts on gauge potentials. This explains the origin of the terms *gauge equivalent* and *gauge transformation.* In particular we will discuss the action on the space of gauge potentials $a_{pot}(M)$ in the trivial bundle case.

Proposition 2.3.3 On a trivial bundle, the group of gauge transformations $\mathscr{G}$ identifies with $Map\,(M, G)$, i.e. the group of smooth mappings from M to G:

$$\mathscr{G} \cong Map\,(M, G) = \{\, F : M \to G \mid F \text{ smooth } \}.$$

Proof: Let $P(G,M)$ be a principal bundle. In §2.3.1 we saw that we could identify the group of gauge transformations $\mathscr{G} = \{ f \in \mathrm{Diff}(P) \,|\, f \circ R_g = R_g \circ f \text{ and } \pi \circ f = \pi \}$ with the smooth Ad-equivariant maps from P to G

$$Map_{Ad}(P,G) = \{ \phi : P \to G \,|\, \phi(R_g p) = Ad_{g^{-1}} \circ \phi(p) \}.$$

Now assume $P(G,M)$ is *trivial.* By Proposition 2.2.1 there exists a global section $s : M \to P$. Let $f \in \mathscr{G}$ be a gauge transformation, and let $\phi : P \to G$ be the Ad-equivariant map equivalent with it. We define the map $F : M \to G$ as

$$F = \phi \circ s,$$

and it will be smooth since s and ϕ are smooth. Hence $F \in Map(M,G)$.

Now suppose we have a map $F \in Map(M,G)$. We would like to define an Ad-equivariant smooth map $\phi : P \to G$, and we can do this in the following way. Take $p \in P$, and let $\pi(p) = m \in M$. Then $s(m)$ and p will be in the same fiber, hence there is a unique $g \in G$ such that $p = R_g s(m)$. Now define

$$\phi(p) \stackrel{\text{def}}{=} Ad_{g^{-1}} \circ F(m) = g^{-1} F(m) g.$$

One easily checks that ϕ is Ad-equivariant by definition.

Geometrically we can see it like this: the global section supplies us with an image $s[M] \subset P$, which we can identify with M. We can define the value of ϕ on this image, since we can use the map $F : M \to G$. Now we use Ad-equivalence to extend this map from $s[M] \to G$ to $P \to G$. An equivalent definition would thus be

$$\phi(s(m)g) \stackrel{\text{def}}{=} Ad_{g^{-1}} \circ F(m).$$

Since P is trivial every $p \in P$ can be written as $s(m) \cdot g$ for some $m \in M$ and $g \in G$, and hence $\phi : P \to M$ is a well-defined Ad-equivariant map.

It is smooth because s, F and the right-action are smooth. So ϕ defines a gauge transformation. Since the ϕ we constructed satisfies $\phi \circ s = F$, we have established a bijection between $\mathscr{G}$ and $Map(M,G)$. Just like the Ad-equivariant maps $Map_{Ad}(P,G)$, $Map(M,G)$ is a group with pointwise multiplication. And again the bijection is a group isomorphism: let $f, g \in \mathscr{G}$ with associated Ad-equivariant maps $\phi, \psi : P \to G$, and $F_1, F_2 : M \to G$ the associated elements of $Map(M,G)$. Then $(F_1 \cdot F_2) = (\phi \circ s) \cdot (\psi \circ s) = (\phi \cdot \psi) \circ s$. So we have the correspondences $f \circ g \leftrightarrow \phi \cdot \psi \leftrightarrow F_1 \cdot F_2$, and hence the bijection is a group homomorphism; hence a group isomorphism.

Action of $\mathscr{G}$ on gauge potentials

In section §2.4.1 we introduced the action of the group of gauge transformations on the space of connections $\mathscr{C}(P)$: for $\omega \in \mathscr{C}(P)$ and $f \in \mathscr{G}$ we had

$$R(f) : \omega \mapsto (f^{-1})^* \omega.$$

In general, this will give an action of $\mathscr{G}$ on the gauge potentials. Let a_α be a gauge potential defined on a neighborhood $U_\alpha \subset M$ through a section (gauge) $s_\alpha : U_\alpha \to M$. We can set

$$R(f) : a_\alpha = s_\alpha^*(\omega) \mapsto a'_\alpha = s_\alpha^*((f^{-1})^*\omega) = (f^{-1} \circ s_\alpha)^* \omega.$$

Notice the following: instead of interpreting $a'_\alpha = s_\alpha^*((f^{-1})^*\omega)$ as a gauge potential obtained by applying the section $s_\alpha : U_\alpha \to M$ to the newly defined connection form $(f^{-1})^*\omega$, we can also interpret a'_α as a gauge potential obtained from the same connection form ω as before, but using another section, namely $f^{-1} \circ s_\alpha : U_\alpha \to P$. This is an important observation, since for physical theories the specific choice of a section (gauge) should not be relevant (the *gauge-invariance principle*). It explains why connection forms in the same orbit under $\mathscr{G}$ are called *gauge equivalent* and why $\mathscr{G}$ is called the group of *gauge transformations.*

Let us now consider the trivial bundle $P = M \times G$. In this case gauge potentials are globally defined on M and we could identify $a_{pot}(M) \cong Lie\,(G) \otimes \Omega^1(M)$ as we saw in §2.3.3. For $a \in a_{pot}(M)$ we still can define (s the canonical global section)

$$R(f) : a = s^*(\omega) \mapsto a' = s^*((f^{-1})^*\omega).$$

However, we can use Proposition 2.3.3 which showed $\mathscr{G} \cong Map\,(M, G)$ to give an alternative description of this action. If $f \in \mathscr{G}$ corresponds to a map $g : M \to G$ such that $f(p) = p \cdot g(\pi(p))$, the section $s'(x) = (f^{-1} \circ s)(x)$ satisfies $s(x) = s'(x)g(x)$. In section §1.1.4 we discussed the compatibility conditions that a change of section induces, and as an extension of equation (1.11) we have

$$a'(x) = Ad_{g(x)^{-1}}\, a + (L^{-1}_{g(x)})^T (dg)_x \,, \tag{2.2}$$

which is often sloppily denoted as $a' = g^{-1}ag + g^{-1}dg$. Taking $g \in Map\,(M, G)$ we obtain the representation

$$\begin{aligned} &R : \mathscr{G} = Map\,(M, G) \to \mathrm{Aut}(a_{pot}(M)), \qquad R : g \mapsto R(g), \\ &\text{with } R(g) : a \mapsto a' = Ad_{g^{-1}}(a) + (L_g^{-1})^T(dg). \end{aligned} \tag{2.3}$$

In following sections we will derive representations of the Lie algebra $Lie\,(\mathscr{G})$ from these group representations, but first we will briefly discuss Lie theory for infinite-dimensional Lie groups such as $\mathscr{G}$.

2.3.4 Interlude: infinite-dimensional Lie groups

Before we can go on examining the group of gauge transformations associated with a principal bundle, we must first spend a few words on the theory of infinite-dimensional Lie groups. The reason is that the gauge group is such an infinite-dimensional Lie group, with a corresponding infinite-dimensional Lie algebra which will be very important in the following. The cohomological descent method which we will discuss involves this Lie algebra in an essential way.

For a thorough introduction to infinite-dimensional Lie groups (and algebras) we refer to the lecture of Milnor in *Relativity, Groups and Topology II* [14].[5] Also Chapter 9 of de Azcárraga and Izquierdo [2] provides a concise introduction to this subject.

A finite-dimensional Lie group is a group G, which is also a smooth manifold such that the group multiplication $g \cdot h \mapsto gh$ is a smooth map $(\cdot) : G \times G \to G$.

To generalize this to infinite-dimensional groups, we first need to define the infinite-dimensional analogue of an ordinary finite-dimensional manifold. Now an n-dimensional manifold M is locally homeomorphic to an open subset of $\mathbb{R}^n$ by means of coordinate charts, usually designated with $\{U_\alpha, \psi_\alpha\}$. The transition

[5] The same volume contains the lecture of Bruno Zumino on chiral anomalies and differential geometry, one of the first articles which presents and discusses the descent equations and the cohomological descent method for obtaining anomalies.

functions $\psi_\beta \circ \psi_\alpha^{-1}$ on $\mathbb{R}^n$ are required to be smooth if M is a smooth manifold. Furthermore, the tangent space at each point in M can be identified with $\mathbb{R}^n$ as well, since the tangent space of any point in $\mathbb{R}^n$ is $\mathbb{R}^n$ itself. For this reason one says that finite-dimensional manifolds are modelled on $\mathbb{R}^n$.

When generalizing to spaces of infinite-dimension, one substitutes an infinite-dimensional topological vector space E for $\mathbb{R}^n$, such that an infinite-dimensional manifold $\tilde{M}$ is locally homeomorphic to E by means of coordinate charts. One can define the tangent space in an analogue way, and for each point in $\tilde{M}$ it will be isomorphic to E. However, there are some difficulties: (I) what should be the choice for the infinite-dimensional topological vector space E? (II) what should be the analogue for *smooth* manifolds? How should one define *smoothness* for the transition maps in E?

In answer to (I): there are several possibilities, and the definition of an infinite-dimensional manifold should specify which choice for the model space E is made. The most common choices for E are Hilbert spaces, Banach spaces or locally convex spaces.[6] The former two have the advantage of a norm, and therefore some properties of finite-dimensional manifolds (and Lie groups) are easily transferred to their infinite-dimensional counterparts, e.g. the *Inverse Function Theorem* and *Existence and Uniqueness Theorems* for ordinary differential equations. Unfortunately, many of the infinite-dimensional groups one would like to consider, e.g. the group of gauge transformations, are not modelled on Banach spaces.

There is an answer to question (II). On locally convex topological vector spaces one can define a notion of smooth mappings, and introduce differential calculus. This is done for instance in §2,3 of Milnor [14].

Once one has a definition of infinite-dimensional manifolds, the notion of an infinite-dimensional Lie group is clear; it is an infinite-dimensional manifold with group structure, such that the group multiplication is smooth. As a special kind, we have the *Banach Lie groups* which are infinite-dimensional Lie groups modelled on a Banach space.

Now let us summarize some of the most important differences between Lie groups of finite and infinite dimension.

1. For infinite-dimensional Lie groups that are *not* Banach Lie groups, the exponential map is in general no longer locally a diffeomorphism. Hence there can be elements arbitrary close to the identity, which are not in any one-parameter subgroup.

2. Lie's Third Fundamental Theorem does not hold for infinite-dimensional Lie algebras. This theorem states that for any finite-dimensional real Lie algebra $\mathfrak{g}$ there is always a simply connected Lie group G with the given Lie algebra as its Lie algebra, i.e. $\mathfrak{g} = \mathit{Lie}(G)$. Even when $\mathfrak{g}$ is a infinite-dimensional *Banach* Lie algebra, this need not hold.

3. There are no classification theorems for infinite-dimensional Lie algebras, like the ones known for finite-dimensional Lie algebras.

[6] A Banach space is a complete normed vector space. A topological vector space is locally convex if every point has a convex neighboorhoud U, i.e. for all $x, y \in U$ and $t \in [0,1]$: $(1-t)x + ty \in U$. A Hilbert space is a Banach space with an inner product, from which the norm is derived.

Now let us consider a couple of examples which are of interest to us.

Diffeomorphism groups

Let $\mathit{Diff}(M)$ be the group of diffeomorphisms on a smooth compact finite-dimensional manifold M, without boundary. Remember it has a natural group structure, since we can compose mappings, take the inverse and have the identity map on M as identity in $\mathit{Diff}(M)$. This group can be made into an infinite-dimensional Lie group modelled on the locally convex topological vector space $\mathfrak{X}(M)$ of vector fields on M. This is done in §6 of Milnor [14], by constructing a coordinate chart around $\mathrm{id} \in \mathit{Diff}(M)$, and using the group structure of $\mathit{Diff}(M)$ to extend this to coordinate charts on all of $\mathit{Diff}(M)$.

A consequence of this is that the Lie algebra $\mathit{diff}(M)$ identifies with $\mathfrak{X}(M)$.[7] We would like to make this identification explicit.

The Lie algebra of $\mathit{Diff}(M)$ is identified with the tangent space $T_{\mathrm{id}}(\mathit{Diff}(M))$, which in turn can be identified with all the velocity vectors of smooth curves through $\mathrm{id} \in \mathit{Diff}(M)$. The smooth curves having the same velocity vectors on chart are declared equivalent, and for any vector in $T_{\mathrm{id}}(\mathit{Diff}(M))$ one can always find a curve representing it. (This construction is the same for finite-dimensional and infinite-dimensional Lie groups.) Such a curve is given by a map $\phi : \mathbb{R} \to \mathit{Diff}(M)$, with $\phi : t \mapsto \phi_t$ and $\phi_0 = \mathrm{id}_M$. Since ϕ also depends smoothly on $t \in \mathbb{R}$, one can also see this as a smooth map $\phi : \mathbb{R} \times M \to M$. Now define the vector field $X^\phi \in \mathfrak{X}(M)$ at $m \in M$ as

$$X^\phi_m = \frac{\partial}{\partial t}\, \phi(t, m)\Big|_{t=0} . \tag{2.4}$$

Thus $X^\phi \in \mathfrak{X}(M)$ is the vector field associated with an element of $\phi \in \mathit{diff}(M)$.

On $\mathit{Diff}(M)$ we can define *one-parameter subgroups* as usual. They are the curves $\phi : \mathbb{R} \to \mathit{Diff}(M)$ satisfying the condition $\phi_{t+s} = \phi_t \circ \phi_s$, and it is again possible to define an exponential mapping $\exp : \mathfrak{X}(M) \to \mathit{Diff}(M)$ which associates a one-parameter subgroup with an element of $\mathfrak{X}(M) = \mathit{diff}(M)$. This is a bijective correspondence, just like in the finite-dimensional case. However, $\mathfrak{X}(M)$ is not a Banach space, and hence $\mathit{Diff}(M)$ is not a Banach Lie group. In particular the *Inverse Function Theorem* does not hold for $\exp : \mathfrak{X}(M) \to \mathit{Diff}(M)$ at the identity, with the consequence that the exponential mapping is not longer locally a diffeomorphism. It implies that there are elements in $\mathit{Diff}(M)$ arbitrary close to the identity $\mathrm{id} \in \mathit{Diff}(M)$ which are not in the image of exp, or what is the same, which are not contained in any one-parameter subgroup. (For details see Milnor [14].)

Why is this important for us? As we have seen, the group of gauge transformations $\mathscr{G}$ is a subgroup of the diffeomorphisms on P, so $\mathscr{G} \subset \mathit{Diff}(P)$. Hence $\mathscr{G}$ can be made into an infinite-dimensional Lie group modelled over $\mathit{Lie}(\mathscr{G})$, where $\mathit{Lie}(\mathscr{G})$ is a subalgebra of $\mathfrak{X}(P)$.

[7] As usual with this notation, $\mathit{diff}(M) = \mathit{Lie}(\mathit{Diff}(M))$.

The group *Map* (M, G)

Now we consider the group *Map* (M, G) of smooth mappings from a manifold M to a Lie group G.[8] Since G is a group, *Map* (M, G) is a group by pointwise multiplication. With suitable topology and coordinate charts one can turn it into a infinite-dimensional Lie group modelled on *Map* $(M, \mathit{Lie}(G))$; with *Map* $(M, \mathit{Lie}(G))$ the smooth mappings from M to the Lie algebra *Lie* (G) of G. *Map* $(M, \mathit{Lie}(G))$ is a locally convex complete infinite-dimensional vector space, which inherits its vector space structure from *Lie* (G).

Now the identity in *Map* (M, G) is given by $\mathbb{1} : M \to G$ with $\mathbb{1}(m) \equiv e$ $\forall\, m \in M$, with $e \in G$ the identity element in G. Again we can explicitly identify $T_{\mathbb{1}}(\mathit{Map}(M, G))$ with the Lie algebra *Map* $(M, \mathit{Lie}(G))$.

Let $\phi : \mathbb{R} \to \mathit{Map}(M, G)$, with $\phi(0) = \mathbb{1}$, be a curve representing a tangent vector in $T_{\mathbb{1}}(\mathit{Map}(M, G))$. ϕ depends smoothly on $t \in \mathbb{R}$, so we can also interpret ϕ as a smooth map $\phi : \mathbb{R} \times M \to G$. Now we associate with ϕ an element $\xi \in \mathit{Map}(M, \mathit{Lie}(G))$ by defining for all $m \in M$

$$\xi(m) = \frac{\partial}{\partial t}\, \phi(t, m)\,\Big|_{t=0}\,. \tag{2.5}$$

Notice that since $\phi(0, m) = e$ for all $m \in M$ the map $t \mapsto \phi(t, m) : \mathbb{R} \to G$ is in fact a curve in G through the identity e. Hence the right-hand side of the equation supplies an element of $T_e(G) = \mathit{Lie}(G)$.

Examples of infinite-dimensional Lie groups of the form *Map* (M, G) which are of studied in mathematical physics are the *loop group* $LG = \mathit{Map}(S^1, G)$ of a Lie group G, and its higher dimensional analogue, the *sphere groups* $L^n G = \mathit{Map}(S^n, G)$. For us, when studying a principal bundle $P(G, M)$, with base manifold M and (finite) structure group G the group *Map* (M, G) will be interesting. The reason is given by Proposition 2.3.3 in the previous section: when $P(G, M)$ is trivial, the group of gauge transformations $\mathscr{G}$ identifies with *Map* (M, G). We have now shown that the Lie algebra *Lie* $(\mathscr{G})$ identifies with *Map* $(M, \mathit{Lie}(G))$ in that case.

2.4 Infinitesimal gauge transformations

As we remarked in the previous section, the Lie algebra *Lie* $(\mathscr{G})$ of the group of gauge transformations $\mathscr{G}$ will be very important to us. This Lie algebra is also known as the algebra of *infinitesimal gauge transformations*, since an element of *Lie* $(\mathscr{G})$ can be considered as the derivative of a gauge transformation, i.e. an infinitesimal gauge transformation.

We saw that *Lie* $(\mathscr{G})$ is an infinite-dimensional Lie algebra, since $\mathscr{G}$ is an infinite-dimensional Lie group. The identifications made in the previous sections give us the freedom to look at the Lie algebra of the group of gauge transformations in different ways.

1. First of all, we defined the group of gauge transformations $\mathscr{G}$ as a subgroup of *Diff* (P), the diffeomorphisms on P. In §2.3.4 we saw *diff* $(P) = \mathfrak{X}(P)$ so apparently we can identify *Lie* $(\mathscr{G})$ with a subalgebra of $\mathfrak{X}(P)$.[9]

[8] If we make no reference to the dimension of a manifold or Lie group in the following, then they are always meant to be *finite-dimensional*.

[9] To be precise, one can show $\mathit{Lie}(\mathscr{G}) = \{\, X \in \mathfrak{X}(P) \,|\, (R_g)^T X = X \text{ and } \pi^T X = 0 \,\} \subset \mathfrak{X}(P)$.

2. In §2.3.1 we proved that for an arbitrary bundle $\mathscr{G}$ could be identified with the Ad-equivariant mappings $P \to G$, and hence the Lie algebra is a subalgebra of $\mathit{Lie}\,(\mathit{Map}\,(P,G)) = \mathit{Map}\,(P, \mathit{Lie}\,(G))$.

3. For a trivial bundle $P(G,M)$ we had $\mathscr{G} \cong \mathit{Map}\,(M,G)$ so consequently $\mathit{Lie}\,(\mathscr{G}) \cong \mathit{Map}\,(M, \mathit{Lie}\,(G))$.

The second point of view will be the most useful to us. The following lemma identifies the subalgebra of $\mathit{Lie}\,(\mathit{Map}\,(P,G)) = \mathit{Map}\,(P, \mathit{Lie}\,(G))$ which corresponds with the Lie algebra of the group $\mathit{Map}_{Ad}\,(P,G)$ of Ad-equivariant maps.

Lemma 2.4.1 For the infinite-dimensional Lie group $\mathit{Map}_{Ad}\,(P,G)$ we have $\mathit{Lie}\,(\mathit{Map}_{Ad}\,(P,G)) = \mathit{Map}_{Ad}\,(P, \mathit{Lie}\,(G))$, with

$$\mathit{Map}_{Ad}\,(P, \mathit{Lie}\,(G)) = \{\ \xi : P \to \mathit{Lie}\,(G) \mid \xi(R_g p) = Ad_{g^{-1}}\xi(p)\ \}.$$

Here $Ad_{g^{-1}}$ is denoting the adjoint action of the Lie group G on its Lie algebra, cf. (1.5).

Proof: The elements of $\mathit{Lie}\,(\mathit{Map}_{Ad}\,(P,G))$ are in one-one correspondence with the one-parameter groups in $\mathit{Map}_{Ad}\,(P,G)$. Now let $\phi_t = \exp(t \cdot \xi) \in \mathit{Map}_{Ad}\,(P,G)$ be a one-parameter group in $\mathit{Map}_{Ad}\,(P,G)$. This is of course also a one-parameter group in $\mathit{Map}\,(P,G)$, and therefore corresponds to a Lie algebra element $\xi \in \mathit{Map}\,(P, \mathit{Lie}\,(G)) = \mathit{Lie}\,(\mathit{Map}\,(P,G))$, cf. (2.5)

$$\xi(p) = \frac{d}{dt}\,\phi_t(p)\Big|_{t=0}.$$

We will show ξ is in fact an element of $\mathit{Map}_{Ad}\,(P, \mathit{Lie}\,(G))$.

The $\{\phi_t\}$ are Ad-equivariant for each $t \in \mathbb{R}$, which means $\phi_t(R_g p) = g^{-1}\phi_t(p)g$ for all $g \in G$. Also, by definition, one has $\phi_t(p) = \exp(t \cdot \xi(p))$. It follows that

$$\begin{aligned}
\xi(R_g p) &= \frac{d}{dt}\,\phi_t(R_g p)\Big|_{t=0} \\
&= \frac{d}{dt}\,g^{-1}\phi_t(p)g\Big|_{t=0} \\
&= \frac{d}{dt}\,g^{-1}\exp(t \cdot \xi(p))g\Big|_{t=0} \\
&= Ad_{g^{-1}}(\,\xi(p)\,),
\end{aligned}$$

and hence ξ is Ad-equivariant with Ad here denoting the adjoint representation of G on $\mathit{Lie}\,(G)$, cf. (1.5). We denoted these Ad-equivariant maps with $\mathit{Map}_{Ad}\,(P, \mathit{Lie}\,(G))$ and thus established $\mathit{Lie}\,(\mathit{Map}_{Ad}\,(P,G)) = \mathit{Map}_{Ad}\,(P, \mathit{Lie}\,(G))$.

Corollary 2.4.1 Since we could identify $\mathscr{G}$ and $\mathit{Map}_{Ad}\,(P,G)$ (see §2.3.1), we have $\mathit{Lie}\,(\mathscr{G}) \cong \mathit{Map}_{Ad}\,(P, \mathit{Lie}\,(G))$.

The bracket on the Lie algebras $\mathit{Map}\,(P, \mathit{Lie}\,(G))$ (and $\mathit{Map}_{Ad}\,(P, \mathit{Lie}\,(G))$) is given by the pointwise Lie bracket of $\mathit{Lie}\,(G)$. Thus for $\xi_1, \xi_2 \in \mathit{Map}\,(P,G)$ we have

$$[\xi_1, \xi_2](p) = [\xi_1(p), \xi_2(p)].$$

(This stems from the fact that the multiplication in $\mathit{Map}\,(P,G)$ and $\mathit{Map}_{Ad}\,(P,G)$ was given by pointwise multiplication in G.)

We now continue the observations made in §2.3.2, and consider the action of $\mathit{Lie}\,(\mathscr{G})$ on connections and gauge potentials.

2.4.1 Representations of $\mathit{Lie}(\mathscr{G})$

In this section, we will describe two representations of the Lie algebra $\mathit{Lie}(\mathscr{G})$ of infinitesimal gauge transformations. These representations can be obtained from the group representations of $\mathscr{G}$ which we described in the previous sections. In general a linear representation $R : G \to GL(\mathcal{V})$ of a Lie group G leads to an associated representation $\rho = dR : \mathit{Lie}(G) \to \mathfrak{gl}(\mathcal{V}) = \mathit{Lie}(GL(\mathcal{V}))$ of its Lie algebra $\mathit{Lie}(G)$, called the *derived* or *induced representation*, defined as

$$\rho(X) \stackrel{\text{def}}{=} \frac{d}{dt} R(\exp(tX)) \Big|_{t=0}, \qquad X \in \mathit{Lie}(G),$$

with $\exp : \mathit{Lie}(G) \to G$ the exponential mapping of G.[10]

We start off with the action on the space of connections $\mathscr{C}(P)$, where

$$\mathscr{C}(P) = \{ \, \omega \in \mathit{Lie}(G) \otimes \Omega^1(P) \mid i_{X^\#}\omega = X \text{ and } L_{X^\#}\omega = [\omega, X] \, \}.$$

Now we remark the following. Any smooth map $P \to \mathit{Lie}(G)$ can be seen as an element of $\mathit{Lie}(G) \otimes \Omega^0(P)$, since the elements of $\Omega^0(P)$ are just the real-valued functions on P. Hence $\mathit{Lie}(G) \otimes \Omega^0(P)$ consists of smooth $\mathit{Lie}(G)$-valued functions on P, i.e. smooth maps $P \to \mathit{Lie}(G)$.

Since we have identified $\mathit{Lie}(\mathscr{G}) \cong \mathit{Map}_{Ad}(P, \mathit{Lie}(G))$ we can thus interpret any element $\xi \in \mathit{Lie}(\mathscr{G})$ as an element of $\mathit{Lie}(G) \otimes \Omega^0(P)$. The advantage is that we can apply the exterior differential to ξ and obtain an element $d\xi \in \mathit{Lie}(G) \otimes \Omega^1(P)$. Remembering that connection forms are also elements of $\mathit{Lie}(G) \otimes \Omega^1(P)$ we have the following theorem.

Theorem 2.4.1 The representation of $\mathscr{G}$ on $\mathscr{C}(P)$ induces a representation of $\mathit{Lie}(\mathscr{G})$ on $\mathscr{C}(P)$ given by

$$\rho(\xi) : \omega \mapsto -d\xi - [\omega, \xi], \tag{2.6}$$

or more explicitly,

$$(\rho(\xi)\omega)_p(v_p) = -(d\xi)_p(v_p) - [\omega_p(v_p), \xi(p)] \qquad p \in P, v_p \in T_pP,$$

for $\xi \in \mathit{Lie}(\mathscr{G})$ and $\omega \in \mathscr{C}(P)$.

Proof: A deduction of this formula is given in Kastler&Stora [11]; see equations (1.19),(1.20),(2.4),(2.5) and (2.12).

A similar formula describes the action of $\mathit{Lie}(\mathscr{G})$ on the space of gauge potentials on a trivial bundle. In the case of a trivial bundle we had

$$\mathscr{G} \cong \mathit{Map}(M, G), \quad \mathit{Lie}(\mathscr{G}) \cong \mathit{Map}(M, \mathit{Lie}(G)), \text{ and } a_{pot}(M) \cong \mathscr{C}(M \times G).$$

(cf. Proposition 2.3.3 and Proposition 2.2.3)

[10]We already have encountered an (infinite-dimensional) example of this. Let $P(G, M)$ be a principal bundle; the right-action of the structure group G on the total space P by $R_g : p \mapsto pg$ induces a fundamental vector field by

$$X^\#_p = \frac{d}{dt} R_{\exp(tX)}\, p \,\Big|_{t=0}.$$

The group homomorphism $R : G \to \mathit{Diff}(P)$, with $R : g \mapsto R_g$, thus results in a Lie algebra homomorphism $X \mapsto X^\# : \mathit{Lie}(G) \to \mathfrak{X}(P) = \mathit{Lie}(\mathit{Diff}(P))$.

The representation $R : \mathscr{G} \to \mathrm{Aut}(a_{pot}(M))$ was given by (2.3), i.e.

$$R : \mathscr{G} = Map\,(M, G) \to \mathrm{Aut}(a_{pot}(M)), \qquad R : g \mapsto R(g),$$
$$\text{with } R(g) : a \mapsto a' = Ad_{g^{-1}}(a) + (L_g^{-1})^T(dg),$$

and from this we obtain the following Lie algebra representation.

Proposition 2.4.1 For a trivial principal bundle $P(G, M)$ with $P = M \times G$ the representation of *Lie* $(\mathscr{G})$ on $a_{pot}(M)$ induced by the action of $\mathscr{G}$ on $a_{pot}(M)$ described by (2.3) is given explicitly by

$$\rho(\xi) : a \mapsto -d\xi - [\,a, \xi\,], \tag{2.7}$$

for $\xi \in Lie\,(\mathscr{G})$ (identified with $Lie\,(G) \otimes \Omega^0(M) = Map\,(M, Lie\,(G))$) and with $a \in a_{pot}(M)$ a gauge potential on M.

Proof: Notice the similarity between (2.7) and (2.6) is no coincidence, in the view of Proposition 2.2.3 which identified $\mathscr{C}(M \times G)$ and $a_{pot}(M)$. The above formula is derived as Proposition 10.3.1 in de Azcárraga and Izquierdo [2].

2.5 The generalized group of gauge transformations

In this section we will continue the generalizations we began in §1.2 of Chapter 1. There we remarked that for a principal bundle $P(G, M)$ the graded-commutative differential algebra $\Omega(P)$ can be considered as a *Lie* (G)-operation. We defined an arbitrary $\mathfrak{g}$-operation as a graded differential algebra with a Lie algebra $\mathfrak{g}$ operating on it. Futhermore many notions from the principal bundle setting could be generalized: horizontality, equivariance, basic elements, connection forms and curvature. In this section we will add two notions to this list, namely the *(generalized) group of gauge transformations* of a $\mathfrak{g}$-operation, and its Lie algebra: the *Lie algebra of infinitesimal gauge transformations.*

We would like to remark that the generalizations put forward in this section are not used in the rest of the thesis; they are just included to round off our general approach of generalizing concepts from the principal bundle setting. However, as Dubois-Violette remarks in his article, the generalized Lie algebra of infinitesimal gauge transformations can be used to construct a more general complex accommodating the cohomological descent method (which we will describe in Chapters 5 and 6). This generalization lacks a physical interpretation however, and we will not pursue it here.[11]

2.5.1 Mappings of $\mathfrak{g}$-operations

First of all we need to define a **homomorphism of $\mathfrak{g}$-operations**. This is simply a homomorphism of differential algebras which is natural with respect to the action of $\mathfrak{g}$. Thus, for two $\mathfrak{g}$-operations $(\mathscr{A}_1, i^1, L^1)$, $(\mathscr{A}_2, i^2, L^2)$ and a homomorphism of graded differential algebras $\Psi : \mathscr{A}_1 \to \mathscr{A}_2$ we should have (for $\alpha \in \mathscr{A}_1$)

$$\Psi(i^1_X \alpha) = i^2_X \Psi(\alpha) \quad \text{and} \quad \Psi(L^1_X \alpha) = L^2_X \Psi(\alpha).$$

[11] See [6] §2.2, pp. 547-551, for a brief sketch of these generalizations.

Isomorphisms and automorphisms are now defined in the usual way: an isomorphism of $\mathfrak{g}$-operations is a bijective homomorphism of $\mathfrak{g}$-operations. An automorphism of a $\mathfrak{g}$-operation $\mathscr{A}$ is an isomorphism onto itself. The group of automorphisms for a $\mathfrak{g}$-operation $\mathscr{A}$ is denoted with $\mathrm{Aut}(\mathscr{A})$.

If we have a homomorphism of $\mathfrak{g}$-operations $\Psi : \mathscr{A}_1 \to \mathscr{A}_2$ this can be naturally extended to a linear map from $\mathfrak{g}\otimes\mathscr{A}_1 \to \mathfrak{g}\otimes\mathscr{A}_2$, denoted also by Ψ, by declaring

$$\Psi(X \otimes \alpha) = X \otimes \Psi(\alpha).$$

Suppose now $A_1 \in \mathfrak{g} \otimes \mathscr{A}_1^1$ is a connection on $\mathscr{A}_1$, and A_2 a connection on $\mathscr{A}_2$. If $\Psi(A_1) = A_2$ then Ψ is called a **homomorphism of $\mathfrak{g}$-operations with connections**, with respect to the chosen connections of course. Note that in general the image $\Psi(A_1)$ of a connection A_1 on $\mathscr{A}_1$ will be a connection on $\mathscr{A}_2$.

2.5.2 Gauge transformations of a $\mathfrak{g}$-operation

After the work we did in the previous sections, the generalizations do not cost much effort. We recall the result stated in §2.3.2 by Proposition 2.3.1: the group of gauge transformations $\mathscr{G}$ of a principal bundle $P(G, M)$ leaves basic forms in $\Omega(P)$ invariant. Following Dubois-Violette [6], we will take this as a defining property.

Definition 2.5.1 Let $(\mathscr{A}, i, L)$ be a $\mathfrak{g}$-operation, with $\mathscr{B}(\mathscr{A})$ denoting the basic forms. An automorphism $\Psi : \mathscr{A} \to \mathscr{A}$ which leaves invariant every element of $\mathscr{B}(\mathscr{A})$ is called a *gauge transformation of the $\mathfrak{g}$-operation $\mathscr{A}$*. The **group of gauge transformations** of the $\mathfrak{g}$-operation $\mathscr{A}$ is denoted by

$$\mathrm{Aut}_{\mathcal{B}}(\mathscr{A}) = \{\, \Psi \in \mathrm{Aut}(\mathscr{A}) \mid \Psi(\alpha) = \alpha \;\forall \alpha \in \mathscr{B}(\mathscr{A}) \,\}.$$

Remark: This definition leaves an important question unanswered however. Does $\mathrm{Aut}_{\mathcal{B}}(\mathscr{A})$ coincide with the usual group of gauge transformations $\mathscr{G}$ if we take $\mathscr{A} = \Omega(P)$? In [6] p.549 Dubois-Violette remarks that "$\mathrm{Aut}_M(P)$ identifies with $\mathrm{Aut}_{\mathcal{B}}(\Omega(P))$", where $\mathrm{Aut}_M(P)$ is another notation for the group of gauge transformations $\mathscr{G}$. This is however not clear at all. The gauge transformations $f \in \mathscr{G}$ are automorphisms $f : P \to P$, whereas the elements of $\Psi \in \mathrm{Aut}_{\mathcal{B}}(\Omega(P))$ are differential algebra automorphisms $\Psi : \Omega(P) \to \Omega(P)$. Of course an automorphism $f : P \to P$ induces a GDA automorphism by its pull-back $f^* : \Omega(P) \to \Omega(P)$, but does this hold the other way around? Is every element $\Psi \in \mathrm{Aut}_{\mathcal{B}}(\Omega(P)) : \Omega(P) \to \Omega(P)$ the pull-back of some gauge transformation $f \in \mathscr{G}$? These questions are not easily answered, and should certainly require a proof.

For the Lie algebra of infinitesimal gauge transformations, we take the following generalization (cf. [6]):

Definition 2.5.2 The Lie algebra $\mathrm{aut}_{\mathcal{B}}(\mathscr{A})$ of **infinitesimal gauge transformations** of the $\mathfrak{g}$-operation $\mathscr{A}$ is defined as

$$\mathrm{aut}_{\mathcal{B}}(\mathscr{A}) = \{\, \theta \in \mathrm{Der}^{(0)}(\mathscr{A}) \mid i_X\theta = \theta i_X \;\forall X \in \mathfrak{g} \text{ and } \theta(\alpha) = 0 \;\forall \alpha \in \mathscr{B}(\mathscr{A}) \,\}.$$

Thus $\mathrm{aut}_{\mathcal{B}}(\mathscr{A})$ is a Lie subalgebra in $\mathrm{Der}^{(0)}(\mathscr{A})$ consisting of the derivations of degree zero which are zero on basic elements and which commute with the "contractions" i_X.

This generalization may not seem that straightforward. We identified $Lie(\mathscr{G})$ with $Map_{Ad}(P, Lie(G))$, which we can rephrase as

$$Lie(\mathscr{G}) = \{\, \alpha \in Lie(G) \otimes \Omega^0(P) \mid L_{X^\#}\alpha = [\,\alpha, X\,]\,\}.$$

Since all our generalizations originate from the case where $\mathscr{A} = \Omega(P)$ this generalizes easily to arbitrary $\mathfrak{g}$-operations $\mathscr{A}$:

$$\mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A}) = \{\, \xi \in \mathfrak{g} \otimes \mathscr{A}^0 \mid L_X\xi = [\,\xi, X\,]\,\}. \tag{2.8}$$

In Dubois-Violette [6] this is put forward as a kind of alternative definition of the Lie algebra of gauge transformations, and $\mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A})$ relates to $\mathrm{aut}_{\mathcal{B}}(\mathscr{A})$ in the following way.

Lemma 2.5.1 There exists a Lie algebra homomorphism $L : \xi \in \mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A}) \mapsto L_\xi \in \mathrm{aut}_{\mathcal{B}}(\mathscr{A})$ defined for $\xi = E_\alpha \otimes \xi^\alpha$ as

$$L_\xi(\omega) = \xi^\alpha L_{E_\alpha}(\omega) + (d\xi^\alpha) i_{E_\alpha}(\omega) \qquad \text{for } \omega \in \mathscr{A}. \tag{2.9}$$

Proof: First we check L_ξ is a derivation. Let $\omega, \eta \in \mathscr{A}$, then

$$\begin{aligned} L_\xi(\omega\eta) &= \xi^\alpha L_{E_\alpha}(\omega\eta) + (d\xi^\alpha) i_{E_\alpha}(\omega\eta) \\ &= \xi^\alpha L_{E_\alpha}(\omega)\eta + \xi^\alpha \omega L_{E_\alpha}(\eta) + (d\xi^\alpha) i_{E_\alpha}(\omega)\eta + (-1)^{\deg\,\omega}(d\xi^\alpha)\omega\, i_{E_\alpha}(\eta) \\ &= L_\xi(\omega)\eta + \omega\, \xi^\alpha L_{E_\alpha}(\eta) + \omega\,(d\xi^\alpha) i_{E_\alpha}(\eta) \\ &= L_\xi(\omega)\eta + \omega\, L_\xi(\eta). \end{aligned}$$

So $L_\xi \in \mathrm{Der}^{(0)}(\mathscr{A})$, but for $L_\xi \in \mathrm{aut}_{\mathcal{B}}(\mathscr{A})$ we also need to check (1) $L_\xi\, i_X = i_X\, L_\xi$ and (2) $L_\xi\omega = 0$ for $\omega \in \mathscr{B}(\mathscr{A})$. (2) follows immediately from the definition in eq. (2.9). For (1) we first prove (A) $\xi^\alpha i_{[E_\alpha, X]}\omega = i_X(d\xi^\alpha) i_{E_\alpha}\omega$.
Using $L_X = di_X + i_X d$ and $i_X(\xi^\alpha) = 0$ (since $\xi^\alpha \in \mathscr{A}^0$) we have

$$\begin{aligned} i_X(d\xi^\alpha) i_{E_\alpha}\omega &= (L_X - di_X)\xi^\alpha i_{E_\alpha}\omega \\ &= (L_X - di_X)\xi^\alpha i_{E_\alpha}\omega \\ &= L_X\xi^\alpha i_{E_\alpha}\omega \\ &= [\,\xi, X\,]^\alpha i_{E_\alpha}\omega \\ &= C^\alpha_{\beta\gamma}\,\xi^\beta X^\gamma i_{E_\alpha}\omega \\ &= \xi^\beta i_{(C^\alpha_{\beta\gamma}\, X^\gamma\, E_\alpha)}\omega \\ &= \xi^\beta i_{[E_\beta, X]}\omega \\ &= \xi^\alpha i_{[E_\alpha, X]}\omega, \end{aligned}$$

where we used $\xi \in \mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A}) \Rightarrow L_X\xi^\alpha = [\,\xi, X\,]^\alpha$. Since i is linear, and $X^\gamma, C^\alpha_{\beta\gamma} \in \mathbb{R}$ we also used $C^\alpha_{\beta\gamma}\, X^\gamma\, i_{E_\alpha} = i_{(C^\alpha_{\beta\gamma}\, X^\gamma\, E_\alpha)} = i_{[E_\beta, X]}$.

Now, using (A) we can prove property (1) $L_\xi\, i_X = i_X\, L_\xi$. From Def. 1.2.1 we also use $i_{[X,Y]} = L_X i_Y - i_Y L_X$, and $i_X i_Y = -i_Y i_X$ which follows from $(i_X)^2 = 0$.[12]

$$\begin{aligned} L_\xi(i_X\omega) &= \xi^\alpha L_{E_\alpha}(i_X\omega) + (d\xi^\alpha) i_{E_\alpha}(i_X\omega) \\ &= \xi^\alpha \;\; i_{[E_\alpha, X]}\omega + i_X L_{E_\alpha}\omega \;\; - (d\xi^\alpha) i_X i_{E_\alpha}(\omega) \\ &= i_X(d\xi^\alpha) i_{E_\alpha}\omega + \xi^\alpha i_X L_{E_\alpha}\omega - (d\xi^\alpha) i_X i_{E_\alpha}(\omega) \\ &= i_X \;\; \xi^\alpha L_{E_\alpha}\omega \;\; + i_X \;\; (d\xi^\alpha) i_{E_\alpha}(\omega) \\ &= i_X \;\; \xi^\alpha L_{E_\alpha}\omega + (d\xi^\alpha) i_{E_\alpha}(\omega) \\ &= i_X(L_\xi\omega). \end{aligned}$$

[12] By $0 = (i_{(X+Y)})(i_{(X+Y)}) = (i_X + i_Y)(i_X + i_Y) = i_X^2 + i_X i_Y + i_Y i_X + i_Y^2 = i_X i_Y + i_Y i_X$.

This concludes our proof of $L_\xi \in \mathrm{aut}_{\mathcal{B}}(\mathscr{A})$. We still have to prove that $L : \mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A}) \to \mathrm{aut}_{\mathcal{B}}(\mathscr{A})$ is a Lie algebra homomorphism, i.e. $L_{[\,\xi,\zeta\,]} = [\,L_\xi, L_\zeta\,]$. This is a tedious but straightforward calculation, which we have included in the appendix as Lemma B.4.1.

The action of $Lie\,(\mathscr{G})$ on $\mathscr{C}(P)$ which we described in the previous section generalizes to $\mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A})$ and the space of algebraic connections on $\mathscr{A}$, which we denote

$$\mathscr{C}(\mathscr{A}) = \{\, \mathcal{A} \in \mathfrak{g} \otimes \mathscr{A}^1 \mid i_X\mathcal{A} = X \text{ and } L_X\mathcal{A} = [\,\mathcal{A}, X\,]\,\}.$$

We thus have the following lemma.

Lemma 2.5.2 There is a representation of $\mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A})$ on $\mathscr{C}(\mathscr{A})$ given by $\rho : \xi \in \mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A}) \mapsto \rho(\xi) \in \mathrm{Aut}(\mathscr{C}(\mathscr{A}))$ with[13]

$$\rho(\xi) : \mathcal{A} \mapsto -d\xi - [\,\mathcal{A},\xi\,] \qquad \text{for } \mathcal{A} \in \mathscr{C}(\mathscr{A}).$$

Proof: We must prove $\rho(\xi)\mathcal{A} \in \mathscr{C}(\mathscr{A})$. Obviously $-d\xi - [\,\mathcal{A},\xi\,] \in \mathfrak{g}\otimes\mathscr{A}^1$. Now

$$\begin{aligned} i_X(-d\xi - [\,\mathcal{A},\xi\,]) &= -i_X\ \ d\xi - [\,\mathcal{A},\xi\,] \\ &= -(L_X - di_X)\xi - [\,i_X\mathcal{A},\xi\,] + [\,\mathcal{A}, i_X\xi\,] \\ &= -L_X\xi - [\,X,\xi\,] \\ &= -[\,\xi, X\,] + [\,\xi, X\,] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} L_X(-d\xi - [\,\mathcal{A},\xi\,]) &= L_X\ \ -d\xi - [\,\mathcal{A},\xi\,] \\ &= -d(L_X\xi) - [\,L_X\mathcal{A},\xi\,] - [\,\mathcal{A}, L_X\xi\,] \\ &= -d([\,\xi,X\,]) - [\,[\,\mathcal{A},X\,],\xi\,] - [\,\mathcal{A},[\,\xi,X\,]\,] \\ &= -[\,d\xi,X\,]) - [\,\xi,dX\,] - [\,[\,\mathcal{A},X\,],\xi\,] - [\,\mathcal{A},[\,\xi,X\,]\,] \\ &= -[\,d\xi,X\,]) - [\,[\,\mathcal{A},\xi\,],X\,] \\ &= [\,-d\xi - [\,\mathcal{A},\xi\,],X\,]), \end{aligned}$$

where we used the graded Jacobi identity in $[\,[\,\mathcal{A},X\,],\xi\,] + [\,\mathcal{A},[\,\xi,X\,]\,] = [\,[\,\mathcal{A},\xi\,],X\,]$. We checked the verticality and equivariance of $-d\xi - [\,\mathcal{A},\xi\,]$, which makes it an algebraic connection.

The definition of the generalized Lie algebra of infinitesimal gauge transformations $\mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A})$ for a $\mathfrak{g}$-operation $\mathscr{A}$, and its action on the algebraic connections $\mathscr{C}(\mathscr{A})$ on $\mathscr{A}$, can be used to construct for any $\mathfrak{g}$-operation $\mathscr{A}$ a bigraded complex which functions as the framework for the cohomological descent. This is indicated in [6].

We will turn our attention to the specific complex used in Dubois-Violette [6] to investigate possible anomalies in gauge field theories in Chapters 5 and 6. But first, in the next chapter, we will continue our generalization of the principal bundle setting by describing the classical Weil homomorphism for arbitrary $\mathfrak{g}$-operations.

[13] In Dubois-Violette [6] there is a sign difference. There $\rho(\xi) : \mathcal{A} \mapsto d\xi + [\,\mathcal{A},\xi\,]$ (on p.550). This defines ρ as a Lie algebra anti-homomorphism.

Chapter 3

The Weil algebra and Weil homomorphism

The theory of characteristic classes was developed in 1930-1950 and can be applied to both vector bundles and principal bundles.[1] This is not so surprising, since one can associate a principal bundle to a vector bundle, and vice versa. Connections are defined on vector bundles and principal bundles in different ways, but for associated bundles they are in correspondence. E.g. a connection on a vector bundle will define a connection form on the associated principal bundle.[2] We will pay no attention to vector bundles however, and describe the Weil homomorphism for principal bundles. On a principal bundle $P(G, M)$

$$G \hookrightarrow P \to M,$$

one chooses a connection form ω on P, with curvature Ω. If one takes an Ad-invariant[3] symmetric polynomial F on $\mathit{Lie}(G)$, i.e. $F : \mathit{Lie}(G) \times \ldots \times \mathit{Lie}(G) \to \mathbb{R}$ such that for $g \in G$ and $X_i \in \mathit{Lie}(G)$,

$$F(Ad_g X_1, \ldots, Ad_g X_k) = F(X_1, \ldots, X_k),$$

one can insert the curvature form in F, and thus obtain a (real-valued) differential form on P,

$$F(\Omega) = F(\Omega, \ldots, \Omega) \quad \in \Omega(P).$$

One can show this form is basic, so it corresponds to a differential form on M. It is also closed, so it defines an element of $H_{DR}(M)$, the de Rham cohomology of the base manifold M. Furthermore it turns out this cohomology class, known as the *Chern class* of the bundle, is independent of the chosen connection ω. Hence this construction yields a homomorphism, the *Weil homomorphism* from the algebra of symmetric invariant polynomials on $\mathit{Lie}(G)$, denoted $I(G)$, to the de Rham cohomology $H_{DR}(M)$.

This, in a nutshell, is the idea of the Weil homomorphism and of characteristic classes for principal bundles. Under certain circumstances the characteristic

[1] See the introduction in Bott&Tu [4]. It includes a sketch of the history of mathematical developments on the subject of differential forms and algebraic topology.

[2] This and more about characteristic classes is described in detail in Walschap [18].

[3] Where $Ad : G \to GL(\mathit{Lie}(G))$ is the adjoint action of the group G on its Lie algebra.

classes provide a powerful tool in classifying (principal) bundles. The procedure described above is treated in detail in §2.4-§2.6 of de Azcárraga&Izquierdo [2] and Ch. 6 of Naber [15](Vol. II). The books of Bott&Tu [4] (Ch. IV) and Walschap [18](Ch. 6) put the theory of characteristic classes in a broader (functorial) perspective.

In this chapter we generalize the Weil homomorphism (also known as the *Chern-Weil homomorphism*) to arbitrary $\mathfrak{g}$-operations. What we obtain is a homomorphism of the *invariant polynomials* $(S\mathfrak{g}^*)_{\text{inv}}$ in $S\mathfrak{g}^*$ to the basic cohomology $H_{\mathcal{B}}(\mathscr{A}) = H(\mathscr{B}(\mathscr{A}))$ of a $\mathfrak{g}$-operation $\mathscr{A}$. The original Weil homomorphism as described above follows from this generalization if we take $\mathscr{A} = \Omega(P)$. In this case the Ad-invariant polynomials on $\mathit{Lie}(G)$ identify as $(S\mathfrak{g}^*)_{\text{inv}}$ with $\mathfrak{g} = \mathit{Lie}(G)$, and the basic cohomology $H_{\mathcal{B}}(\Omega(P))$ is $H_{DR}(M)$, the de Rham cohomology of the base manifold M.

We just note here that these generalizations can be pursued even further. In our definition of a $\mathfrak{g}$-operation, we assumed the underlying differential algebra $\mathscr{A}$ to be graded-commutative. One can define the notion of a $\mathfrak{g}$-operation for *non-commutative* differential algebras as well. In that case one can construct a non-commutative analogue of the Weil algebra, and again obtain a unique Weil homomorphism. This is done in the article *"Lie theory and the Chern-Weil homomorphism"* of Alekseev and Meinrenken [1], which starts off almost precisely at our final point.

3.1 Preparations

(In what follows, let $\mathfrak{g}$ denote a finite-dimensional real Lie algebra, and let $\mathfrak{g}^*$ be its dual space, i.e. $\mathfrak{g}^* = \{\omega : \mathfrak{g} \to \mathbb{R} \,|\, \omega \text{ linear }\}$. If $\{E_\alpha\}$ is a basis of $\mathfrak{g}$, then $\{E^\alpha\}$ denotes the cobasis in $\mathfrak{g}^*$. The structure constants $C^\alpha_{\beta\gamma}$ are given by $[\,E_\beta, E_\gamma\,] = C^\alpha_{\beta\gamma}\,E_\alpha$.)

In the subsequent sections and chapters we will make extensive use of the symmetric algebra $S\mathfrak{g}^*$ and exterior algebra $\Lambda\mathfrak{g}^* = \Lambda(\mathfrak{g}^*)$ over $\mathfrak{g}^*$. Therefore we briefly recall some basic properties of these algebras.[4]

3.1.1 The symmetric algebra

The symmetric algebra $S\mathfrak{g}^*$ over $\mathfrak{g}^*$ can be interpreted as the graded algebra $S\mathfrak{g}^* = \oplus_{k\in\mathbb{N}} S^k\mathfrak{g}^*$, where $S^k\mathfrak{g}^*$ consists of the symmetric multilinear functions on $\mathfrak{g}^k$ (the k-fold Cartesian product of $\mathfrak{g}$ with itself)

$$S^k\mathfrak{g}^* = \{\,\omega : \underbrace{\mathfrak{g} \times \ldots \mathfrak{g}}_{k \text{ times}} \to \mathbb{R} \,|\, \omega \text{ symmetric and multilinear }\}.$$

The product is denoted by $\vee$ and defined as the symmetrizing product of two mappings, for $\omega \in S^k\mathfrak{g}^*$ and $\eta \in S^l\mathfrak{g}^*$ given by

$$\omega \vee \eta\,(X_1, .., X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma\in S_{k+l}} \omega(X_{\sigma(1)}, .., X_{\sigma(k)}) \cdot \eta(X_{\sigma(k+1)}, .., X_{\sigma(k+l)}). \tag{3.1}$$

Obviously this product is commutative, i.e. $\omega \vee \eta = \eta \vee \omega$.

[4] A reference for this section is Greub, Halperin and Vanstone [10], Vol. III, §0.4-§0.5. We also remark that the exterior algebra is sometimes called the Grassmann algebra.

As with any algebra which has a commutative product and a grading, we can make $S\mathfrak{g}^*$ into a graded-commutative algebra by applying an *even grading* to it. That is: we define

$$(S\mathfrak{g}^*)^{2k} = S^k\mathfrak{g}^* \quad (S\mathfrak{g}^*)^{2k+1} = \{0\} \qquad \text{for } k \in \mathbb{N}.$$

Hence all elements will have an *even* degree, and therefore the product will always satisfy graded-commutativity.

Furthermore notice that $S\mathfrak{g}^*$ is generated by the (co)basis elements $\{E^\alpha\}$ in degree one (degree two if one applies the even grading).

The elements of $S\mathfrak{g}^*$ as $\mathscr{A}$-valued polynomials on $\mathfrak{g} \otimes \mathscr{A}$

If we consider an arbitrary algebra $\mathscr{A}$, we can interpret the elements of $S\mathfrak{g}^*$ as multilinear maps which take $\mathfrak{g}$-valued algebra elements (i.e. elements of $\mathfrak{g} \otimes \mathscr{A}$) as arguments, and map these on an algebra element of $\mathscr{A}$. We do this by defining, for $P \in S^1\mathfrak{g}^* = \mathfrak{g}^*$ and $A = X \otimes \alpha$,

$$P(A) = P(X \otimes \alpha) = P(X)\alpha \quad (\in \mathscr{A}).$$

We can extend this to arbitrary $P \in S\mathfrak{g}^*$ by using formula (3.1) but reading the dot $(\cdot)$ there as the algebra product. To be precise: let $P \in S^k\mathfrak{g}^*$. We can interpret this as a multilinear map

$$P : \underbrace{(\mathfrak{g} \otimes \mathscr{A}) \times \ldots \times (\mathfrak{g} \otimes \mathscr{A})}_{k \text{ times}} \to \mathscr{A}$$

by defining

$$P\big(\, X_1 \otimes \alpha_1, .., X_k \otimes \alpha_k \,\big) = \frac{1}{k!} \sum_{\sigma \in S_k} P(X_{\sigma(1)}, .., X_{\sigma(k)})\, \alpha_{\sigma(1)} \cdots \alpha_{\sigma(k)}\,. \tag{3.2}$$

We used in this definition $X_1 \otimes \alpha_1$ as an element of $\mathfrak{g} \otimes \mathscr{A}$, but in general an element of $\mathfrak{g} \otimes \mathscr{A}$ will be a finite sum $\sum_i (X_1)_i \otimes \alpha_1^i$. However, since P is linear it is defined now for general elements of $\mathfrak{g} \otimes \mathscr{A}$. Notice that as a map $P : (\mathfrak{g} \otimes \mathscr{A}) \times .. \times (\mathfrak{g} \otimes \mathscr{A}) \to \mathscr{A}$ the polynomial P is again symmetric, and if the algebra elements $\{\alpha_1, .., \alpha_k\}$ commute then $P(X_1 \otimes \alpha_1, .., X_k \otimes \alpha_k) = P(X_1, .., X_k)\, \alpha_1 \cdots \alpha_k$.

3.1.2 The exterior algebra

The exterior algebra $\Lambda\mathfrak{g}^*$, by which we mean $\Lambda(\mathfrak{g}^*)$, has a similar interpretation as $S\mathfrak{g}^*$. It is the the graded algebra $\Lambda\mathfrak{g}^* = \oplus_{k\in\mathbb{N}} \Lambda^k\mathfrak{g}^*$, where $\Lambda^k\mathfrak{g}^*$ consists of antisymmetric multilinear functions on $\mathfrak{g}^k$:

$$\Lambda^k\mathfrak{g}^* = \{\, \omega : \underbrace{\mathfrak{g} \times \ldots \mathfrak{g}}_{k \text{ times}} \to \mathbb{R} \mid \omega \text{ antisymmetric and multilinear } \}.$$

The product is the well-known wedge product $\wedge$: the antisymmetrizing product of two mappings, for $\omega \in \Lambda^k\mathfrak{g}^*$ and $\eta \in \Lambda^l\mathfrak{g}^*$ given by

$$\omega\wedge\eta\, (X_1, .., X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \epsilon(\sigma)\, \omega(X_{\sigma(1)}, .., X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, .., X_{\sigma(k+l)}),$$

where $\epsilon(\sigma)$ denotes the sign of the permutation $\sigma \in S_{k+l}$. This product makes $\Lambda\mathfrak{g}^*$ into a graded-commutative algebra, generated by any (co)basis $\{E^\alpha\}$ of $\mathfrak{g}^*$.

3.1.3 The coadjoint action of $\mathfrak{g}$

On the finite-dimensional Lie algebra $\mathfrak{g}$ we have the adjoint representation given by the Lie algebra homomorphism $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, with

$$ad : X \mapsto ad(X), \quad ad(X) : Y \mapsto [X, Y], \qquad X, Y \in \mathfrak{g}.$$

This naturally leads to an representation of $\mathfrak{g}$ on its dual $\mathfrak{g}^*$, denoted by the Lie algebra homomorphism $ad^* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ with

$$ad^* : X \mapsto ad^*(X), \quad ad^*(X) : \omega \mapsto \omega \circ ad(-X), \qquad X \in \mathfrak{g}, \omega \in \mathfrak{g}^*,$$

which means $(ad^*(X)\omega)(Y) = \omega(-[X, Y]) = \omega([Y, X])$ for $Y \in \mathfrak{g}$.
This is called the *coadjoint action* or *coadjoint representation* of $\mathfrak{g}$ on $\mathfrak{g}^*$. We will adjust our notation here, and write L for ad^*, and L_X for $ad^*(X)$.

Since we have $S^1\mathfrak{g}^* = \Lambda^1\mathfrak{g}^* = \mathfrak{g}^*$ and both algebras are generated by the elements $\{E^\alpha\}$ in $\mathfrak{g}^*$, we can extend this representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ to a representation of $\mathfrak{g}$ on the algebras $S\mathfrak{g}^*$ and $\Lambda\mathfrak{g}^*$. We define this representation as above on the elements of $S^1\mathfrak{g}^*$ and $\Lambda^1\mathfrak{g}^*$, and extend it as an derivation of degree zero to homogeneous spaces of higher order.
If we consider the action of L_X on a monomial from $S\mathfrak{g}^*$, with $E^i \in \mathfrak{g}^*$, we see

$$\begin{aligned} L_X\big(\, E^1 \vee E^2 \vee .. \vee E^k \,\big) = &\, L_X(E^1) \vee E^2 \vee .. \vee E^k \, + \\ &\, E^1 \vee L_X(E^2) \vee .. \vee E^k \, + \\ &\, \dots \\ &+\; E^1 \vee E^2 \vee .. \vee L_X(E^k). \end{aligned}$$

Thus for $\omega \in S^k\mathfrak{g}^*$ interpreted as a mapping $\mathfrak{g} \times .. \times \mathfrak{g} \to \mathbb{R}$ we have

$$\begin{aligned} (L_X\omega)(Y_1, Y_2, ..., Y_k) = &\, \omega([Y_1, X], Y_2, .., Y_k) \, + \\ &\, \omega(Y_1, [Y_2, X], .., Y_k) \, + \\ &\, \dots \\ &+\; \omega(Y_1, Y_2, .., [Y_k, X]). \end{aligned}$$

Since $\omega \in S^k\mathfrak{g}^*$ is symmetric we can rewrite this to

$$(L_X\omega)(Y_1, Y_2, ..., Y_k) = \sum_{1 \le i \le k} \omega([Y_i, X], Y_2, .., \hat{Y_i}, .., Y_k). \tag{3.3}$$

For the exterior algebra, exactly the same story holds. The only difference is that $\eta \in \Lambda^k\mathfrak{g}^*$ interpreted as a mapping $\mathfrak{g} \times .. \times \mathfrak{g} \to \mathbb{R}$ is *anti*symmetric, so (3.3) becomes

$$(L_X\eta)(Y_1, Y_2, ..., Y_k) = \sum_{1 \le i \le k} (-1)^{i+1}\, \eta([Y_i, X], Y_2, .., \hat{Y_i}, .., Y_k). \tag{3.4}$$

We conclude this section by defining the subalgebras of *invariant elements* in $S\mathfrak{g}^*$ and $\Lambda\mathfrak{g}^*$ as

$$\begin{aligned} (S\mathfrak{g}^*)_{\text{inv}} &= \{\, \omega \in S\mathfrak{g}^* \mid L_X\omega = 0 \,\forall X \in \mathfrak{g} \,\}, \\ (\Lambda\mathfrak{g}^*)_{\text{inv}} &= \{\, \omega \in \Lambda\mathfrak{g}^* \mid L_X\omega = 0 \,\forall X \in \mathfrak{g} \,\}. \end{aligned}$$

These subalgebras will already play an important role in the next section, and we will see the notation L_X for the coadjoint action is not chosen by accident similar to the notation of a Lie action on an algebra.

3.1.4 $\Lambda\mathfrak{g}^*$ as a $\mathfrak{g}$-operation

The exterior algebra $\Lambda\mathfrak{g}^*$ with the wedge product $\wedge$ is a graded-commutative algebra. It is possible to make this into a graded-commutative *differential* algebra (GCDA) by defining the **Koszul differential**

$$d\eta(X,Y) = \eta([Y,X]), \qquad X,Y \in \mathfrak{g}, \tag{3.5}$$

for elements $\eta \in \Lambda^1\mathfrak{g}^* = \mathfrak{g}^*$. Since the Lie bracket is antisymmetric, $d\eta$ is antisymmetric, and we have $d\eta \in \Lambda^2\mathfrak{g}^*$. We extend d as an anti-derivation (of degree $+1$) to whole $\Lambda\mathfrak{g}^*$; we can do this since $\Lambda\mathfrak{g}^*$ is generated in degree 1 by any cobasis $\{E^\alpha\}$ of $\mathfrak{g}^*$. The only thing left to check is that $d^2 = 0$. It turns out this is equivalent to the Jacobi identity on $\mathfrak{g}$ (Appendix, Lemma B.3.3).

Furthermore, we already defined the coadjoint action L_X for $X \in \mathfrak{g}$ on $\Lambda\mathfrak{g}^*$ as a derivation of degree zero. Now we set

$$i_X\eta = X, \qquad X \in \mathfrak{g},$$

for elements $\eta \in \Lambda^1\mathfrak{g}^* = \mathfrak{g}^*$, and extend this as an anti-derivation of degree -1 on $\Lambda\mathfrak{g}^*$. One can verify $L_X = di_X + i_Xd$ this way, and hence $\Lambda\mathfrak{g}^*$ is a $\mathfrak{g}$-operation.

Remark 1: $\Lambda\mathfrak{g}^*$ has a canonical connection given by $\mathrm{id}_\mathfrak{g}$, i.e. $E_\alpha \otimes E^\alpha \in \mathfrak{g} \otimes \Lambda^1\mathfrak{g}^*$, since $i_X(\mathrm{id}_\mathfrak{g}) = \mathrm{id}_\mathfrak{g}(X) = X$ and $L_X(\mathrm{id}_\mathfrak{g})(Y) = \mathrm{id}_\mathfrak{g}(L_XY) = [\,Y, X\,] = [\,\mathrm{id}_\mathfrak{g}(Y), X\,]$.

Remark 2: Note that we can define i_X on $S\mathfrak{g}^*$ in a similar way. Still $S\mathfrak{g}^*$ is not a $\mathfrak{g}$-operation, since it cannot be made into a differential algebra.

Remarks on the geometrical interpretation of $\Lambda\mathfrak{g}^*$

In Dubois-Violette [6] $\Lambda\mathfrak{g}^*$ is put forward as an example of a $\mathfrak{g}$-operation not of the form $\Omega(P)$, with P the total space of a principal bundle.[5] This is true, but notetheless $\Lambda\mathfrak{g}^*$ has a geometric interpretation. As a consequence of Lie's Third Fundamental Theorem, there is a simply connected Lie group G with $\mathfrak{g} = \mathit{Lie}(G)$. We can identify $\Lambda\mathfrak{g}^*$ as the subalgebra of left-invariant forms $\Omega_{LI}(G)$ on G. The Koszul differential will correspond to the exterior derivative on these forms, as we will show in chapter 4 (this forms the basis of the Chevalley-Eilenberg approach to Lie algebra cohomology). If we identify elements $X \in \mathfrak{g}$ with the left-invariant vector fields $X^L \in \mathfrak{X}(G)$, then the anti-derivation i_X on $\Lambda\mathfrak{g}^*$ coincides with the contraction i_{X^L} on $\Omega_{LI}(G)$, and L_X with the Lie derivative L_{X^L}. Now the Maurer-Cartan form $\Theta_{MC} \in \mathfrak{g} \otimes \Omega_{LI}(G)$ supplies a canonical connection on $\Omega_{LI}(G)$, since it satisfies $i_{X^L}\Theta_{MC} = \Theta_{MC}(X^L) = X$ and $L_{X^L}\Theta_{MC} = [\,\Theta_{MC}, X\,]$ (see appendix §A.3.3). Θ_{MC} corresponds to the canonical connection $\mathrm{id}_\mathfrak{g}$ on $\Lambda\mathfrak{g}^*$.

3.2 The Weil algebra $\mathsf{W}(\mathfrak{g})$

In section 2.5.1 we defined a homomorphism of $\mathfrak{g}$-operations with connections, and hence we can consider the category of $\mathfrak{g}$-operations with a connection on it. In this category there is a universal object, known as the *Weil algebra* which we will now describe.

[5] On p.537: "Notice that,.., we already met a $\mathfrak{g}$-operation which is not of the type $\Omega(P)$, namely $\Lambda\mathfrak{g}^*$,.."

The idea behind the construction of the Weil algebra is the following: let $\mathscr{A}$ be any $\mathfrak{g}$-operation with a connection $\mathcal{A} \in \mathfrak{g} \otimes \mathscr{A}^1$. We can set $\mathcal{A} = E_\alpha \otimes \mathcal{A}^\alpha$, so the elements $\mathcal{A}^\alpha$ are the E_α-components of the algebraic connection $\mathcal{A}$. We can do this in same way for the curvature $\mathcal{F} = E_\alpha \otimes \mathcal{F}^\alpha$. Now we can look at the subalgebra in $\mathscr{A}$ generated by these elements $\mathcal{A}^\alpha$ and $\mathcal{F}^\alpha$. Every $\mathfrak{g}$-operation will have such a subalgebra, and the elements in it satisfy simple properties: the $\mathcal{A}^\alpha$'s anti-commute with each other since they are of degree one, and the $\mathcal{F}^\alpha$'s commute since they are of degree two. Furthermore we have the relation $\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]$. This simple observation leads us to the definition of the Weil algebra, which will be modelled upon this subalgebra. The unique homomorphism of the Weil algebra to any $\mathfrak{g}$-operation $\mathscr{A}$ will map it exactly onto the subalgebra we just described. This defines the Weil algebra as an universal initial object in the category of $\mathfrak{g}$-operations with connection.

Let us define the Weil algebra. As we saw in the previous section the exterior algebra $\Lambda\mathfrak{g}^*$ and the symmetric algebra $S\mathfrak{g}^*$ are both graded-commutative algebras (if we use the *even grading* for $S\mathfrak{g}^*$). Now the tensor product of two graded-commutative algebras (described in Def.A.1.5) will again be a graded-commutative algebra. We therefore define the Weil algebra $\mathsf{W}(\mathfrak{g})$ as a graded commutative algebra by

$$\mathsf{W}(\mathfrak{g}) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*.$$

Example: the first few homogeneous spaces of $\mathsf{W}(\mathfrak{g})$ are given by

$$\begin{aligned}
\mathsf{W}^0(\mathfrak{g}) &= \Lambda^0\mathfrak{g}^* \otimes (S\mathfrak{g}^*)^0 = \mathbb{K} \otimes \mathbb{K} = \mathbb{K},\\
\mathsf{W}^1(\mathfrak{g}) &= \Lambda^1\mathfrak{g}^* \otimes (S\mathfrak{g}^*)^0 \oplus \Lambda^0\mathfrak{g}^* \otimes (S\mathfrak{g}^*)^1\\
&= \Lambda^1\mathfrak{g}^* \otimes \mathbb{K} \oplus \mathbb{K} \otimes \{0\}\\
&= \Lambda^1\mathfrak{g}^* \otimes \mathbb{K}, \quad (\cong \Lambda^1\mathfrak{g}^*)\\
\mathsf{W}^2(\mathfrak{g}) &= \Lambda^2\mathfrak{g}^* \otimes (S\mathfrak{g}^*)^0 \oplus \Lambda^1\mathfrak{g}^* \otimes (S\mathfrak{g}^*)^1 \oplus \Lambda^0\mathfrak{g}^* \otimes (S\mathfrak{g}^*)^2\\
&= \Lambda^2\mathfrak{g}^* \otimes \mathbb{K} \oplus \Lambda^1\mathfrak{g}^* \otimes \{0\} \oplus \mathbb{K} \otimes S^1\mathfrak{g}^*\\
&= \Lambda^2\mathfrak{g}^* \otimes \mathbb{K} \oplus \mathbb{K} \otimes S^1\mathfrak{g}^*, \quad (\cong \Lambda^2\mathfrak{g}^* \oplus S^1\mathfrak{g}^*)\\
&\ldots (etc.) \ldots
\end{aligned}$$

Since $\Lambda\mathfrak{g}^*$ and $S\mathfrak{g}^*$ are both generated by cobasis elements $\{E^\alpha\}$ of $\mathfrak{g}^*$, $\mathsf{W}(\mathfrak{g})$ is generated by the elements $E^\alpha \otimes 1$ and $1 \otimes E^\alpha$. Therefore we introduce the elements $A^\alpha \in \mathsf{W}^1(\mathfrak{g})$ and $F^\alpha \in \mathsf{W}^2(\mathfrak{g})$ as

$$A^\alpha = E^\alpha \otimes 1 \quad \text{and} \quad F^\alpha = 1 \otimes E^\alpha.$$

Thus $\mathsf{W}(\mathfrak{g})$ is the free, connected, graded-commutative algebra generated by the $\{A^\alpha\}$ in degree one, and the $\{F^\alpha\}$ in degree two.

We wish to define a differential d on $\mathsf{W}(\mathfrak{g})$ which makes it a graded-commutative differential algebra (GCDA). Since $\mathsf{W}(\mathfrak{g})$ is generated by the A^α and F^α it suffices to define $d(A^\alpha)$ and $d(F^\alpha)$. We do this by introducing two elements in $\mathfrak{g} \otimes \mathsf{W}(\mathfrak{g})$ and using the bracket that is defined on elements of $\mathfrak{g} \otimes \mathsf{W}(\mathfrak{g})$ by

$$[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes (\alpha \cdot \beta),$$

where $X \otimes \alpha, Y \otimes \beta \in \mathfrak{g} \otimes \mathsf{W}(\mathfrak{g})$ with $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathsf{W}(\mathfrak{g})$.

Consider the elements $A \in \mathfrak{g} \otimes \mathsf{W}^1(\mathfrak{g})$ and $F \in \mathfrak{g} \otimes \mathsf{W}^2(\mathfrak{g})$ defined by[6]

$$A = \sum_\alpha E_\alpha \otimes A^\alpha \quad \text{and} \quad F = \sum_\alpha E_\alpha \otimes F^\alpha$$

We define $d(A^\alpha)$ as the E_α-component of dA, i.e. $d(A^\alpha) = (dA)^\alpha$ where $dA = E_\alpha \otimes (dA)^\alpha$. We now need to define dA of course, so let $dA \in \mathfrak{g} \otimes \mathsf{W}^2(\mathfrak{g})$ be given by

$$dA = -\tfrac{1}{2}[A, A] + F. \tag{3.6}$$

In the same way we define $d(F^\alpha)$ as the E_α-component of dF, i.e. $d(F^\alpha) = (dF)^\alpha$ with $dF = E_\alpha \otimes (dF)^\alpha$, where $dF \in \mathfrak{g} \otimes \mathsf{W}^3(\mathfrak{g})$ is defined as

$$dF = -[A, F]. \tag{3.7}$$

We have now defined $d(A^\alpha)$ and $d(F^\alpha)$ and extend this as an anti-derivation to the whole algebra $\mathsf{W}(\mathfrak{g})$. It is clear that d is a homogeneous linear mapping of degree $+1$, and hence it is a differential on $\mathsf{W}(\mathfrak{g})$, if we check $d^2 = 0$. This is done in the following lemma.

Lemma 3.2.1 Thus defined, $d^2 = 0$ on the Weil algebra $\mathsf{W}(\mathfrak{g})$.

Proof: By Lemma B.1.1 in the appendix we only need to check $d^2(A^\alpha) = 0$ and $d^2(F^\alpha) = 0$, since they generate $\mathsf{W}(\mathfrak{g})$. Since $d^2(A^\alpha) = (d^2(A))^\alpha$ we check

$$\begin{aligned}
d^2(A) &= d(-\tfrac{1}{2}[\,A, A\,] + F) \\
&= -\tfrac{1}{2}\,[\,dA, A\,] + \tfrac{1}{2}\,[\,A, dA\,] + dF \\
&= -\tfrac{1}{2}\,[\,dA, A\,] - \tfrac{1}{2}\,[\,dA, A\,] - [\,A, F\,] \\
&= -[\,dA, A\,] + [\,F, A\,] \\
&= -[\,-\tfrac{1}{2}\,[\,A, A\,] + F, A\,] + [\,F, A\,] \\
&= \tfrac{1}{2}\,[\,[\,A, A\,], A\,] - [\,F, A\,] + [\,F, A\,] \\
&= 0,
\end{aligned}$$

where we used a couple of properties of the bracket on $\mathfrak{g} \otimes \mathsf{W}(\mathfrak{g})$ which are described in §A.1.1 in the appendix. E.g. the commutation rule $[\,B, C\,] = (-1)^{(\deg B \cdot \deg C)+1}[\,C, B\,]$, which implies $[F, A] = -[A, F]$, and the graded Jacobi identity (A.12), which implies $[\,[\,A, A\,], A\,] = 0$. We also used Lemma A.1.2, which assures us $d([B, C]) = [dB, C] + (-1)^{\deg B}[B, dC]$. Now for F^α/F.

$$\begin{aligned}
d^2(F) &= d(-[\,A, F\,]) \\
&= -[\,dA, F\,] + [\,A, dF\,] \\
&= -[\,-\tfrac{1}{2}\,[\,A, A\,] + F, F\,] + [\,A, -[\,A, F\,]\,] \\
&= \tfrac{1}{2}\,[\,[\,A, A\,], F\,] - [\,F, F\,] - [\,A, [\,A, F\,]\,] \\
&= \tfrac{1}{2}\,[\,[\,A, A\,], F\,] - \tfrac{1}{2}\,[\,A, [\,A, F\,]\,] - \tfrac{1}{2}\,[\,A, [\,A, F\,]\,] \\
&= -\tfrac{1}{2}\,[\,F, [\,A, A\,]\,] - \tfrac{1}{2}\,[\,A, [\,A, F\,]\,] - \tfrac{1}{2}\,[\,A, [\,F, A\,]\,] \\
&= 0,
\end{aligned}$$

where we again used the graded Jacobi identity (A.12) for the last conclusion.

[6] We will omit the sum $\sum_\alpha$ from now on.

Thus $\mathsf{W}(\mathfrak{g})$ is a graded-commutative differential algebra (GCDA), which is free and connected. It is now straightforward to make $\mathsf{W}(\mathfrak{g})$ into a $\mathfrak{g}$-operation. For $X = E_\alpha \cdot X^\alpha \in \mathfrak{g}$ we define i_X on the $\{A^\alpha\}$ and $\{F^\alpha\}$ as

$$i_X(A^\alpha) = X^\alpha \quad \text{and} \quad i_X(F^\alpha) = 0, \tag{3.8}$$

and extend this as an anti-derivation on $\mathsf{W}(\mathfrak{g})$. We define L_X as $L_X = di_X + i_X d$ and this will be a derivation of degree zero on $\mathsf{W}(\mathfrak{g})$ for all $X \in \mathfrak{g}$. Furthermore $(i_X)^2 = 0$ on $\mathsf{W}(\mathfrak{g})$ by Lemma B.1.1. Hence $\mathsf{W}(\mathfrak{g})$ is a $\mathfrak{g}$-operation.

The definitions made so far can seem confusing, but the true reason for them appears at this moment. The differential d on $\mathsf{W}(\mathfrak{g})$ and the above operations i_X and L_X were defined in such a way that we can now remark that A is an (algebraic) connection on $\mathsf{W}(\mathfrak{g})$ with curvature F.

Lemma 3.2.2 $\mathsf{W}(\mathfrak{g})$ is a $\mathfrak{g}$-operation with connection A and curvature F.

Proof: The only thing perhaps not straightforward to check is that $A = E_\alpha \otimes A^\alpha \in \mathfrak{g} \otimes \mathsf{W}^1(\mathfrak{g})$ satisfies the connection properties, especially equivariance. From $i_X(A^\alpha) = X^\alpha$ it follows that $i_X(A) = i_X(E_\alpha \otimes A^\alpha) = E_\alpha \otimes i_X(A^\alpha) = E_\alpha \otimes X^\alpha = X$. Now we check the equivariance of A:

$$\begin{aligned} L_X A &= (di_X + i_X d)\, A \\ &= di_X(A) + i_X d(A) \\ &= d(X) + i_X(-\tfrac{1}{2}[\,A, A\,] + F) \\ &= 0 - \tfrac{1}{2}[\,i_X A, A\,] + \tfrac{1}{2}[\,A, i_X A\,] + i_X F \\ &= -\tfrac{1}{2}[\,X, A\,] + \tfrac{1}{2}[\,A, X\,] + 0 \\ &= \tfrac{1}{2}[\,A, X\,] + \tfrac{1}{2}[\,A, X\,] \\ &= [\,A, X\,]. \end{aligned}$$

Hence A satisfies the properties of an algebraic connection, as given in §1.2.3. In the same section the curvature associated with A was defined as $dA + \frac{1}{2}[\,A, A\,]$. So for $\mathsf{W}(\mathfrak{g})$ we indeed have $F = dA + \frac{1}{2}[\,A, A\,]$ as the curvature of A, again by definition. Finally we proved in that same section (in Lemma 1.2.4) that the curvature F satisfied $i_X F = 0$. This is in concordance with our definition $i_X(F^\alpha) = 0$.

After this conclusion and our earlier motivational remarks, the following theorem should not come as a surprise.

Theorem 3.2.1 The Weil algebra $\mathsf{W}(\mathfrak{g})$ is a universal object in the category of $\mathfrak{g}$-operations with connections. Thus for any $\mathfrak{g}$-operation $\mathscr{A}$ with algebraic connection $\mathcal{A} \in \mathfrak{g} \otimes \mathscr{A}^1$, there is a unique homomorphism of $\mathfrak{g}$-operations with connections $\Psi_{\mathsf{W}} : \mathsf{W}(\mathfrak{g}) \to \mathscr{A}$.

Proof: The requirement for such a homomorphism is that it maps A (the connection of $\mathsf{W}(\mathfrak{g})$) onto $\mathcal{A}$, the connection of $\mathscr{A}$. Thus $\Psi_{\mathsf{W}}(A) = \mathcal{A}$, and if $\mathcal{A}$ is given by $E_\alpha \otimes \mathcal{A}^\alpha$ it follows that $\Psi_{\mathsf{W}}(A^\alpha) = \mathcal{A}^\alpha$. Furthermore the homomorphism Ψ_{W} should be natural with respect to the differential and bracket. Thus $\Psi_{\mathsf{W}}(F) = \Psi_{\mathsf{W}}(dA + \frac{1}{2}[\,A, A\,]) = d(\Psi_{\mathsf{W}}(A) + \frac{1}{2}[\,\Psi_{\mathsf{W}}(A), \Psi_{\mathsf{W}}(A)\,] = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}\,] = \mathcal{F}$: the curvature F of $\mathsf{W}(\mathfrak{g})$ is mapped onto the curvature $\mathcal{F}$ of $\mathscr{A}$, and $\Psi_{\mathsf{W}}(F^\alpha) = \mathcal{F}^\alpha$. Since $\mathsf{W}(\mathfrak{g})$ was generated by the elements $\{A^\alpha\}$ and $\{F^\alpha\}$ the homomorphism Ψ_{W} is now defined on any element. The homomorphism is obviously unique, since we had no choice in defining Ψ_{W}.

The above homomorphism $\Psi_{\mathsf{W}} : \mathsf{W}(\mathfrak{g}) \to \mathscr{A}$ is called the **canonical homomorphism of $\mathsf{W}(\mathfrak{g})$ in $\mathscr{A}$**.

3.3 Subalgebras of the Weil algebra

Having introduced the Weil algebra, we will now study it in some more detail. The most important subalgebras of the Weil algebra are identified, and we define the *basic cohomology* which will be the key to the generalized Weil homomorphism.

Since $\mathsf{W}(\mathfrak{g}) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$, the symmetric algebra $S\mathfrak{g}^*$ is naturally embedded in $\mathsf{W}(\mathfrak{g})$ als $1 \otimes S\mathfrak{g}^*$. Similarly $\Lambda\mathfrak{g}^*$ is embedded as $\Lambda\mathfrak{g}^* \otimes 1 \subset \mathsf{W}(\mathfrak{g})$. The generating cobasis element $E^\alpha \in \Lambda^1\mathfrak{g}^*$ corresponds to $A^\alpha = E^\alpha \otimes 1$ in $\mathsf{W}(\mathfrak{g})$, and $E^\alpha \in S^1\mathfrak{g}^*$ to $F^\alpha = 1 \otimes E^\alpha$. We now argue these subalgebras are stable under the derivation L_X defined on $\mathsf{W}(\mathfrak{g})$ and that in fact L_X coincides with the coadjoint action on $\Lambda\mathfrak{g}^*$ and $S\mathfrak{g}^*$, as defined in §3.1.3.

Lemma 3.3.1 The action of L_X in $\mathsf{W}(\mathfrak{g})$ on $\Lambda\mathfrak{g}^* \otimes 1$ and $1 \otimes S\mathfrak{g}^*$ coincides with the coadjoint action L_X/ad^* on $\Lambda\mathfrak{g}^*$ and $S\mathfrak{g}^*$.

Proof: For clarity we use the notation ad^* for the coadjoint action on $\Lambda\mathfrak{g}^*$ and $S\mathfrak{g}^*$ in this proof. We defined the coadjoint ad^* on $\Lambda\mathfrak{g}^*$ for $E^\alpha \in \Lambda^1\mathfrak{g}^*$ as $ad^*(E^\alpha)(Y) = E^\alpha([Y,X])$, cf. equation (3.3). Now consider L_X on $A^\alpha = E^\alpha \otimes 1 \in \mathsf{W}(\mathfrak{g})$, that is defined as $L_X(A^\alpha) = [A,X]^\alpha$. But $A = E_\alpha \otimes A^\alpha$ interpreted as an element of $\mathfrak{g} \otimes \Lambda^1\mathfrak{g}^* = \mathfrak{g} \otimes \mathfrak{g}^*$ is $E_\alpha \otimes E^\alpha$: the identity $\mathrm{id}_\mathfrak{g}$ considered as a $\mathfrak{g}$-valued map on $\mathfrak{g}$. So we have $L_X(A^\alpha)(Y) = [A,X]^\alpha(Y) = [A(Y),X]^\alpha = [Y,X]^\alpha = E^\alpha([Y,X])$. Thus $L_X(A^\alpha) = L_X(E^\alpha \otimes 1) = (ad^*(E^\alpha)) \otimes 1$ like we needed to prove.

For $S\mathfrak{g}^*$ the proof is identical, since also $ad^*(E^\alpha)(Y) = E^\alpha([Y,X])$ for $E^\alpha \in S^1\mathfrak{g}^*$, and we have $L_X(F^\alpha) = [F,X]^\alpha$.

Corollary 3.3.1 The subalgebras $\Lambda\mathfrak{g}^* \otimes 1$ and $1 \otimes S\mathfrak{g}^*$ of $\mathsf{W}(\mathfrak{g})$ are stable by the derivation L_X (for all $X \in \mathfrak{g}$).

In section §1.2.2 we defined the subalgebras of horizontal elements $\mathscr{H}(\mathscr{A})$, invariant elements $\mathscr{I}(\mathscr{A})$ and basic elements $\mathscr{B}(\mathscr{A})$ of a $\mathfrak{g}$-operation $\mathscr{A}$. We now identify these subalgebras for the Weil algebra.

Notation: we will use the abbreviations $\mathscr{I}_\mathsf{W} \overset{\text{def}}{=} \mathscr{I}(\mathsf{W}(\mathfrak{g})), \mathscr{H}_\mathsf{W} \overset{\text{def}}{=} \mathscr{H}(\mathsf{W}(\mathfrak{g}))$ and $\mathscr{B}_\mathsf{W} \overset{\text{def}}{=} \mathscr{B}(\mathsf{W}(\mathfrak{g}))$.

Recall we defined $i_X(A^\alpha) = X^\alpha$ and $i_X(F^\alpha) = 0$, or equivalenty: $i_X(A) = X$ and $i_X(F) = 0$. It follows that the subalgebra of horizontal elements $\mathscr{H}_\mathsf{W}$ is

$$\mathscr{H}_\mathsf{W} = \{\, \alpha \in \mathsf{W}(\mathfrak{g}) \mid i_X\alpha = 0 \,\forall X \in \mathfrak{g} \,\} = 1 \otimes S\mathfrak{g}^* \subset \mathsf{W}(\mathfrak{g}).$$

The invariant elements $\mathscr{I}_\mathsf{W}$ were defined by

$$\mathscr{I}_\mathsf{W} = \{\, \alpha \in \mathsf{W}(\mathfrak{g}) \mid L_X\alpha = 0 \,\forall X \in \mathfrak{g} \,\}.$$

From Lemma 3.3.1 it follows that the invariant elements $(\Lambda\mathfrak{g}^*)_{\text{inv}}$ and $(S\mathfrak{g}^*)_{\text{inv}}$ of $\Lambda\mathfrak{g}^*$ and $S\mathfrak{g}^*$ embedded in $\mathsf{W}(\mathfrak{g})$ are certainly in $\mathscr{I}_\mathsf{W}$, i.e.

$$\begin{aligned}(\Lambda\mathfrak{g}^*)_{\text{inv}} \otimes 1 &= (\Lambda\mathfrak{g}^* \otimes 1) \cap \mathscr{I}_\mathsf{W},\\ 1 \otimes (S\mathfrak{g}^*)_{\text{inv}} &= (1 \otimes S\mathfrak{g}^*) \cap \mathscr{I}_\mathsf{W},\end{aligned}$$

and hence $(\Lambda\mathfrak{g}^*)_{\text{inv}} \otimes (S\mathfrak{g}^*)_{\text{inv}} \subset \mathscr{I}_\mathsf{W}$, but this inclusion is strict. However, since we defined the basic elements as $\mathscr{B}(\mathscr{A}) = \mathscr{H}(\mathscr{A}) \cap \mathscr{I}(\mathscr{A})$ we have

$$\begin{aligned}\mathscr{B}_\mathsf{W} &= \mathscr{H}_\mathsf{W} \cap \mathscr{I}_\mathsf{W}\\ &= (1 \otimes S\mathfrak{g}^*) \cap \mathscr{I}_\mathsf{W}\\ &= 1 \otimes (S\mathfrak{g}^*)_{\text{inv}}\,.\end{aligned}$$

We need one more lemma, before we can summarize these results in Corollary 3.3.2.

Lemma 3.3.2 For any $\omega = 1 \otimes P \in \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^* = \mathsf{W}(\mathfrak{g})$ we have

$$d\omega = A^\alpha \cdot L_{E_\alpha}\omega.$$

Thus $d\omega = E^\alpha \otimes L_{E\alpha}P$ since $L_{E_\alpha}\omega = 1 \otimes L_{E_\alpha}P$, by Lemma 3.3.1.

Proof: by induction. If $\omega = F^\alpha$, then $P = E^\alpha \in S^1\mathfrak{g}^*$. We have

$$d(F^\alpha) = -[A,F]^\alpha = -C^\alpha_{\beta\gamma}\, A^\beta F^\gamma = A^\beta\, [F, E_\beta]^\alpha = A^\beta\, L_{E_\beta}(F^\alpha).$$

Now we suppose the lemma is true for $\omega = (1 \otimes P)$ and $\eta = (1 \otimes Q)$, with $P, Q \in S\mathfrak{g}^*$. Noticing (1) that d and L_X both act as a derivation, (2) the degree of all elements in $1 \otimes S\mathfrak{g}^*$ is even and (3) for this reason A^α commutes with ω, we have

$$\begin{aligned} d(\omega \cdot \eta) &= d(\omega) \cdot \eta + \omega \cdot d(\eta) \\ &= A^\alpha \cdot L_{E_\alpha}(\omega) \cdot \eta + \omega \cdot \quad A^\alpha \cdot L_{E_\alpha}(\eta) \\ &= A^\alpha \cdot \quad L_{E_\alpha}(\omega) \cdot \eta + \omega \cdot L_{E_\alpha}(\eta) \\ &= A^\alpha \cdot L_{E_\alpha}(\omega \cdot \eta). \end{aligned}$$

So the lemma holds for all $\omega = 1 \otimes P \in \mathsf{W}(\mathfrak{g})$, since the F^α generate $1 \otimes S\mathfrak{g}^*$.

Corollary 3.3.2 For $\alpha \in \mathsf{W}(\mathfrak{g})$ the following are equivalent[7]

1. α is basic.
2. α is of the form $1 \otimes P$ with $P \in (S\mathfrak{g}^*)_{\text{inv}}$.
3. α is of the form $1 \otimes P$ and $d\alpha = 0$.

3.3.1 Basic cohomology

For any $\mathfrak{g}$-operation $\mathscr{A}$, the graded subalgebra of basic elements $\mathscr{B}_{\mathsf{W}}$ is a *differential* subalgebra (Lemma B.2.2), and therefore we can consider the cohomology of it, termed the **basic cohomology**, denoted with $\mathbf{H}_{\mathcal{B}}(\mathscr{A})$.

For the Weil algebra $\mathsf{W}(\mathfrak{g})$ we have $\mathscr{B}_{\mathsf{W}} = 1 \otimes (S\mathfrak{g}^*)_{\text{inv}}$, but since we applied even grading to $S\mathfrak{g}^*$ we have for $k \in \mathbb{N}$:

$$\begin{aligned} \mathscr{B}^{2k}_{\mathsf{W}} &= 1 \otimes (S\mathfrak{g}^*)^{2k}_{\text{inv}} = 1 \otimes (S^k\mathfrak{g}^*)_{\text{inv}} \,. \\ \mathscr{B}^{2k+1}_{\mathsf{W}} &= 1 \otimes (S\mathfrak{g}^*)^{2k+1}_{\text{inv}} = \{0\}. \end{aligned}$$

From Corollary 3.3.2 it follows that all elements of $\mathscr{B}_{\mathsf{W}}$ are cocycles, thus for the basic cohomology $H_{\mathcal{B}}(\mathsf{W}(\mathfrak{g}))$ we have

$$\begin{aligned} H^{2k}_{\mathcal{B}}(\mathsf{W}(\mathfrak{g})) &= 1 \otimes (S^k\mathfrak{g}^*)_{\text{inv}} \cong (S^k\mathfrak{g}^*)_{\text{inv}} \,, \\ H^{2k+1}_{\mathcal{B}}(\mathsf{W}(\mathfrak{g})) &= \{0\}. \end{aligned}$$

[7] This is Theorem 3 in [6].

3.4 The generalized Weil homomorphism

Let $\mathscr{A}$ be a $\mathfrak{g}$-operation and let $\mathcal{A}_1$ be any connection on $\mathscr{A}$. By the universal property of the Weil algebra we have a canonical homomorphism $\Psi_1 : \mathsf{W}(\mathfrak{g}) \to \mathscr{A}$, with $\Psi_1(A) = \mathcal{A}_1$. Since Ψ_1 is in particular a homomorphism of differential algebras it induces a homomorphism[8] from the cohomology of $\mathsf{W}(\mathfrak{g})$ to the cohomology of $\mathscr{A}$. Now Ψ_1 is also a homomorphism of $\mathfrak{g}$-operations, and thus commutes with the graded derivations i_X and L_X on $\mathsf{W}(\mathfrak{g})$ and $\mathscr{A}$. This implies Ψ_1 maps basic elements onto basic elements, and hence induces a homomorphism from the basic cohomology of $\mathsf{W}(\mathfrak{g})$ (which is isomorphic to $(S\mathfrak{g}^*)_{\text{inv}}$) to the basic cohomology of $\mathscr{A}$. This is the Weil homomorphism w. The map $\Psi_1 : \mathsf{W}(\mathfrak{g}) \to \mathscr{A}$ clearly depends on the chosen connection, but it can be proven that the induced homomorphism in basic cohomology does not.

The theorem stating this result is presented as Theorem 4 in the article by Dubois-Violette [6]. However, the proof is only sketched in a few lines and, as is apparent from this rather long section with preliminary results, it leaves quite a lot of problematic details to be solved by the reader. By including these results and treating the theorem in such detail we hope to satisfy the reader who is left puzzled by the sketch-of-proof presented in Dubois-Violette's article.

Preliminary results

Before proving the generalized theorem concerning the Weil homomorphism, we need some preliminary results. We turn to them now. Suppose $\mathscr{A}$ is a $\mathfrak{g}$-operation, and $\mathcal{A}_1$ and $\mathcal{A}_2$ are two algebraic connections on $\mathscr{A}$. Since the space of connections is affine, $\mathcal{A}_t = (1-t)\mathcal{A}_0 + t\mathcal{A}_1$ will also be a connection on $\mathscr{A}$ for $t \in [0,1]$. We denote the corresponding curvature $\mathcal{F}_t = d\mathcal{A}_t + \frac{1}{2}[\,\mathcal{A}_t, \mathcal{A}_t\,]$, and we introduce the element $\eta \overset{\text{def}}{=} \mathcal{A}_1 - \mathcal{A}_0$, such that $\mathcal{A}_t = \mathcal{A}_0 + t\eta$.

Lemma 3.4.1 Defining the covariant derivatives by the elements $\mathcal{A}_t$ as $\mathcal{D}_t \overset{\text{def}}{=} \mathcal{D}^{\mathcal{A}_t}$ where $\mathcal{D}^{\mathcal{A}_t}\omega = d\omega + [\,\mathcal{A}_t, \omega\,]$ (Def. 1.2.4); we have the following

1. $\mathcal{F}_t = \mathcal{F}_0 + t\mathcal{D}_0\eta + \frac{t^2}{2}[\,\eta,\eta\,]$.
2. $\frac{d}{dt}\,\mathcal{F}_t = \mathcal{D}_t\eta$.

Proof: (1.) We have

$$\begin{aligned}
\mathcal{F}_t &= d\mathcal{A}_t + \tfrac{1}{2}[\,\mathcal{A}_t, \mathcal{A}_t\,] \\
&= d(\mathcal{A}_0 + t\eta) + \tfrac{1}{2}[\,\mathcal{A}_0 + t\eta, \mathcal{A}_0 + t\eta\,] \\
&= d\mathcal{A}_0 + td\eta + \tfrac{1}{2}[\,\mathcal{A}_0, \mathcal{A}_0\,] + t[\,\mathcal{A}_0, \eta\,] + \tfrac{1}{2}\cdot t^2[\,\eta,\eta\,] \\
&= \mathcal{F}_0 + t\mathcal{D}_0\eta + \frac{t^2}{2}[\,\eta,\eta\,].
\end{aligned}$$

(2.) Using (1.) we have

$$\begin{aligned}
\frac{d}{dt}\,\mathcal{F}_t &= \mathcal{D}_0\eta + t[\,\eta,\eta\,] = d\eta + [\,\mathcal{A}_0, \eta\,] + t[\,\eta,\eta\,] \\
&= d\eta + [\,\mathcal{A}_0 + t\eta, \eta\,] = d\eta + [\,\mathcal{A}_t, \eta\,] = \mathcal{D}_t\eta,
\end{aligned}$$

which completes the lemma.

[8] See §A.1.7.

By Definition 1.2.4 the covariant derivatives $\mathcal{D}_t$ are anti-derivations of degree $+1$ on $\mathfrak{g}\otimes\mathscr{A}$, but they are not defined on $\mathscr{A}$ itself. We can define them on a certain subalgebra of $\mathscr{A}$ however. Fix a basis $\{E_\alpha\}$ of $\mathfrak{g}$ and consider the components of the elements $\mathcal{A}_t$ and $\mathcal{F}_t$ in $\mathfrak{g}\otimes\mathscr{A}$ relative to this basis, i.e. $\mathcal{A}_t = E_\alpha \otimes \mathcal{A}_t^\alpha$ and $\mathcal{F}_t = E_\alpha \otimes \mathcal{F}_t^\alpha$. Let us denote the set of these components with

$$\mathscr{A}_{\text{comp}} \stackrel{\text{def}}{=} \{\, \mathcal{A}_{t_1}^\alpha, \mathcal{F}_{t_2}^\beta \mid t_1, t_2 \in [0,1] \text{ and } \alpha, \beta = 1, .., \dim \mathfrak{g} \,\}.$$

Now we can define the covariant derivatives $\mathcal{D}_t$ on the subalgebra $\mathscr{A}_{\text{sub}} \subset \mathscr{A}$ generated by the set of components $\mathscr{A}_{\text{comp}}$, i.e.

$$\mathscr{A}_{\text{sub}} \stackrel{\text{def}}{=} \langle \mathscr{A}_{\text{comp}} \rangle = \langle\, \mathcal{A}_{t_1}^\alpha, \mathcal{F}_{t_2}^\beta \mid t_1, t_2 \in [0,1] \text{ and } \alpha, \beta = 1, .., \dim \mathfrak{g} \,\rangle.$$

We do this by defining

$$\mathcal{D}_t(\mathcal{A}_s^\alpha) = (\mathcal{D}_t \mathcal{A}_s)^\alpha \quad \text{and} \quad \mathcal{D}_t(\mathcal{F}_s^\alpha) = (\mathcal{D}_t \mathcal{F}_s)^\alpha,$$

and extending this as an anti-derivation of degree $+1$ on $\mathscr{A}_{\text{sub}}$.

An important consequence is that by definition we have $P(\mathcal{D}_t\omega) = \mathcal{D}_t P(\omega)$ for $\omega = \mathcal{A}_s$ or $\omega = \mathcal{F}_s$ with $P = E^\alpha \in S^1\mathfrak{g}^*$, which generalizes to arbitrary $P \in S\mathfrak{g}^*$ as demonstrated in the following lemma.

Lemma 3.4.2 Let $P \in S^k\mathfrak{g}^*$, interpreted as a polynomial on $\mathfrak{g}\otimes\mathscr{A}$ as described in §3.1.1. Then for elements $\omega_1, .., \omega_k \in \mathfrak{g}\otimes\mathscr{A}$ with $\omega_i = E_\alpha \otimes \omega_i^\alpha$ such that $\mathcal{D}_t\omega_i = E_\alpha \otimes \mathcal{D}_t\omega_i^\alpha$ we have

$$\mathcal{D}_t P(\omega_1, .., \omega_k) = \sum_{1\le i\le k} P(\omega_1, .., \mathcal{D}_t\omega_i, .., \omega_k).$$

Proof: We first consider the case that $\omega_1 = X_1 \otimes \alpha_1, \ldots, \omega_k = X_k \otimes \alpha_k$. Using the definition of P as polynomial on $\mathfrak{g}\otimes\mathscr{A}$ cf. (3.2), we have

$$\begin{aligned}
\mathcal{D}_t P\big(X_1\otimes\alpha_1, .., X_k\otimes\alpha_k\big) &= \frac{1}{k!}\sum_{\sigma\in S_k} P(X_{\sigma(1)}, .., X_{\sigma(k)})\, \mathcal{D}_t\big(\alpha_{\sigma(1)}\cdots\alpha_{\sigma(k)}\big) \\
&= \frac{1}{k!}\sum_{\sigma\in S_k}\sum_{1\le i\le k} P(X_{\sigma(1)}, .., X_{\sigma(k)})\, \alpha_{\sigma(1)}\cdots\mathcal{D}_t\alpha_{\sigma(i)}\cdots\alpha_{\sigma(k)} \\
&= \sum_{1\le i\le k} P\big(X_1\otimes\alpha_1, .., X_i\otimes\mathcal{D}_t\alpha_i, .., X_k\otimes\alpha_k\big).
\end{aligned}$$

Now by linearity of P and $\mathcal{D}_t$ we have for general $\omega_i = \sum_{\alpha_i=1}^{\dim\mathfrak{g}} E_{\alpha_i}\otimes\omega_i^{\alpha_i}$ with $\mathcal{D}_t\omega_i = E_{\alpha_i}\otimes\mathcal{D}_t\omega_i^{\alpha_i}$ the following:

$$\begin{aligned}
\mathcal{D}_t P\big(\omega_1, \ldots, \omega_k\big) &= \mathcal{D}_t P\Big(\sum_{\alpha_1} E_{\alpha_1}\otimes\omega_1^{\alpha_1}, .., \sum_{\alpha_k} E_{\alpha_k}\otimes\omega_k^{\alpha_k}\Big) \\
&= \sum_{\alpha_1,..,\alpha_k} \mathcal{D}_t P\big(E_{\alpha_1}\otimes\omega_1^{\alpha_1}, .., E_{\alpha_k}\otimes\omega_k^{\alpha_k}\big) \\
&= \sum_{\alpha_1,..,\alpha_k}\sum_{1\le i\le k} P\big(E_{\alpha_1}\otimes\omega_1^{\alpha_1}, .., E_{\alpha_i}\otimes\mathcal{D}_t\omega_i^{\alpha_i}, .., E_{\alpha_k}\otimes\omega_k^{\alpha_k}\big) \\
&= \sum_{1\le i\le k} P\Big(\sum_{\alpha_1} E_{\alpha_1}\otimes\omega_1^{\alpha_1}, .., \sum_{\alpha_i} E_{\alpha_i}\otimes\mathcal{D}_t\omega_i^{\alpha_i}, .., \sum_{\alpha_k} E_{\alpha_k}\otimes\omega_k^{\alpha_k}\Big) \\
&= \sum_{1\le i\le k} P\big(\omega_1, .., \mathcal{D}_t\omega_i, .., \omega_k\big),
\end{aligned}$$

this proves the lemma.

If we again interpret $S\mathfrak{g}^*$ as the symmetric multilinear mappings from $\mathfrak{g} \otimes \mathscr{A}$ to $\mathscr{A}$, we can view this last lemma as saying that we could have equally well defined $\mathcal{D}_t$ on $S\mathfrak{g}^*$ by $\mathcal{D}_t : E^\alpha \mapsto E^\alpha \circ \mathcal{D}_t$ and extending it to $S\mathfrak{g}^*$ as a derivation of degree zero. We could have taken this as a definition; however, in that case we would have to prove that $\mathcal{D}_t$ is a derivation on $\mathscr{A}_{\mathrm{sub}}$.

Things done as we did, we now turn to the following lemma, which gives an explicit form of $\mathcal{D}_t$ defined on $\mathscr{A}_{\mathrm{sub}}$.

Lemma 3.4.3 For $\zeta \in \mathscr{A}_{\mathrm{sub}}$ we have

$$\mathcal{D}_t\zeta = d\zeta + \mathcal{A}_t^\beta \cdot L_{E_\beta}\,\zeta.$$

Proof: We first prove this for the generators $\mathscr{A}_{\mathrm{comp}}$ of $\mathscr{A}_{\mathrm{sub}}$. Since connections and curvatures are equivariant, we have $L_X\mathcal{A}_s = [\,\mathcal{A}_s, X\,]$ and $L_X\mathcal{F}_s = [\,\mathcal{F}_s, X\,]$ for all $s \in [0,1]$. Since $L_X(\mathcal{A}_s^\alpha) = L_X(\mathcal{A}_s)^\alpha$, we have (using structure constants) $L_X(\mathcal{F}_s^\alpha) = C^\alpha_{\beta\gamma}\,\mathcal{F}_s^\beta X^\gamma$, e.g. for $X = E_\beta$ this results in $L_{E_\beta}(\mathcal{F}_s^\alpha) = C^\alpha_{\beta\gamma}\,\mathcal{F}_s^\beta$; and the same for $\mathcal{A}_s$ of course. Taking $\mathcal{F}_s^\alpha$ as example, we have

$$\begin{aligned} \mathcal{D}_t\mathcal{F}_s^\alpha &= \ \ \mathcal{D}_t\mathcal{F}_s \ \ {}^\alpha \\ &= \ \ d\mathcal{F}_s + [\,\mathcal{A}_t, \mathcal{F}_s\,] \ \ {}^\alpha \\ &= d(\mathcal{F}_s^\alpha) + C^\alpha_{\beta\gamma}\,\mathcal{A}_t^\beta \cdot \mathcal{F}_s^\gamma \\ &= d(\mathcal{F}_s^\alpha) + \mathcal{A}_t^\beta \cdot L_{E_\beta}\mathcal{F}_s^\alpha\,. \end{aligned}$$

The same proof holds for $\mathcal{A}_s^\alpha$ of course. We have proven $\mathcal{D}_t\zeta = d\zeta + \mathcal{A}_t^\beta \cdot L_{E_\beta}\,\zeta$ for $\zeta \in \mathscr{A}_{\mathrm{comp}}$, so if we prove $\zeta \mapsto d\zeta + \mathcal{A}_t^\beta \cdot L_{E_\beta}\,\zeta$ is an anti-derivation (of degree +1) like $\mathcal{D}_t$ we are finished. Let $\zeta, \xi \in \mathscr{A}_{\mathrm{sub}}$, then

$$\begin{aligned} d(\zeta\xi) + \mathcal{A}_t^\beta L_{E_\beta}(\zeta\xi) &= (d\zeta)\xi + (-1)^{\deg\zeta}\zeta(d\xi) + \mathcal{A}_t^\beta \ \ L_{E_\beta}(\zeta)\xi + \zeta L_{E_\beta}(\xi) \\ &= (d\zeta)\xi + (-1)^{\deg\zeta}\zeta(d\xi) + \mathcal{A}_t^\beta L_{E_\beta}(\zeta)\xi + (-1)^{\deg\zeta}\zeta\mathcal{A}_t^\beta L_{E_\beta}(\xi) \\ &= \ \ d\zeta + \mathcal{A}_t^\beta L_{E_\beta}\zeta \ \ \xi + (-1)^{\deg\zeta}\zeta \ \ d\xi + \mathcal{A}_t^\beta L_{E_\beta}(\xi) \ \ . \end{aligned}$$

Since $\mathscr{A}_{\mathrm{sub}}$ is generated by $\mathscr{A}_{\mathrm{comp}}$, $\mathcal{D}_t\zeta$ and $d\zeta + \mathcal{A}_t^\beta \cdot L_{E_\beta}\,\zeta$ coincide on $\mathscr{A}_{\mathrm{sub}}$.

There are two more lemmas to go. The following lemma proves $L_X(P(\omega_1, .., \omega_k))$ is equal to $(L_XP)(\omega_1, .., \omega_k)$: something which seems to be obvious and therefore could be easily overlooked. Yet the L_X on the lefthand side is the L_X defined on $\mathscr{A}$, and in the r.h.s. L_X is defined on $S\mathfrak{g}^*$ (as the coadjoint action). The second lemma proves $\frac{d}{dt}$ acts as a derivation on $P(\mathcal{F}_t, .., \mathcal{F}_t)$.

Lemma 3.4.4 Let $P \in S^k\mathfrak{g}^*$, interpreted as a polynomial on $\mathfrak{g} \otimes \mathscr{A}$. Then for elements $\omega_1, .., \omega_k \in \mathfrak{g} \otimes \mathscr{A}$ with $\omega_i = E_\alpha \otimes \omega_i^\alpha$ such that $L_X\omega_i = [\,\omega_i, X\,]$ and $L_X\omega_i = E_\alpha \otimes L_X\omega_i^\alpha$ we have

$$L_X\big(\,P(\omega_1, .., \omega_k)\,\big) = (L_XP)(\omega_1, .., \omega_k).$$

Proof: By following exactly the same proof as in Lemma 3.4.2, we have

$$L_X \ \ P(\omega_1, .., \omega_k) \ \ = \sum_{1 \le i \le k} P(\omega_1, .., L_X\omega_i, .., \omega_k).$$

Continuing from this, and using $L_X\omega_i = [\,\omega_i, X\,]$ we obtain

$$\begin{aligned}
L_X \;\; P(\omega_1,..,\omega_k) &= \sum_{1\le i\le k} P(\omega_1,..,L_X\omega_i,..,\omega_k) \\
&= \sum_{1\le i\le k} P(\omega_1,..,[\,\omega_i,X\,],..,\omega_k) \\
&= \sum_{\alpha_1,..,\alpha_k}\sum_{1\le i\le k} P\;\; E_{\alpha_1}\otimes\omega_1^{\alpha_1},..,[\,E_{\alpha_i},X\,]\otimes\omega_i^{\alpha_i},..,E_{\alpha_k}\otimes\omega_k^{\alpha_k} \\
&= \sum_{\alpha_1,..,\alpha_k}\sum_{1\le i\le k} P(\,E_{\alpha_1},..,[\,E_{\alpha_i},X\,],..,E_{\alpha_k}\,)\,\omega_1^{\alpha_1}\cdots\omega_k^{\alpha_k} \\
&= \sum_{\alpha_1,..,\alpha_k} (L_XP)(\,E_{\alpha_1},..,E_{\alpha_k}\,)\,\omega_1^{\alpha_1}\cdots\omega_k^{\alpha_k} \\
&= (L_XP)(\omega_1,..,\omega_k)
\end{aligned}.$$

This proves the lemma.

Lemma 3.4.5 For $P\in S^k\mathfrak{g}^*$ we have

$$\frac{d}{dt}\,P(\mathcal{F}_t,\mathcal{F}_t,..,\mathcal{F}_t) = k\cdot P(\frac{d}{dt}\mathcal{F}_t,\mathcal{F}_t,..,\mathcal{F}_t).$$

Proof: This is essentially due to the product rule. We prove the lemma by induction. For $P=E^\alpha\in S^1\mathfrak{g}^*$ it holds, since $\frac{d}{dt}\,F_t = E_\alpha\otimes(\frac{d}{dt}\,F_t^\alpha)$. Now suppose it holds for $Q\in S^{(k-1)}\mathfrak{g}^*$. We prove the lemma holds for $E^\alpha\vee Q$. It follows that the lemma holds for all monomials in $S\mathfrak{g}^*$, and since $\frac{d}{dt}$ is linear it thus holds for all $P\in S\mathfrak{g}^*$. In the first and last line we use the definition of the symmetric ($\vee$) product on $S\mathfrak{g}^*$ given by equation (3.1) in §3.1.1.

$$\begin{aligned}
&\frac{d}{dt}\,(E^\alpha\vee Q)(\mathcal{F}_t,..,\mathcal{F}_t) \\
&= \frac{d}{dt}\,E^\alpha(\mathcal{F}_t)\cdot Q(\mathcal{F}_t,..,\mathcal{F}_t) \\
&= \;\; \frac{d}{dt}\,E^\alpha(\mathcal{F}_t)\;\;\cdot Q(\mathcal{F}_t,..,\mathcal{F}_t)+E^\alpha(\mathcal{F}_t)\cdot\;\;\frac{d}{dt}\,Q(\mathcal{F}_t,..,\mathcal{F}_t) \\
&= \;\; E^\alpha(\frac{d}{dt}\,\mathcal{F}_t)\;\;\cdot Q(\mathcal{F}_t,..,\mathcal{F}_t)+E^\alpha(\mathcal{F}_t)\cdot\;\;(k-1)Q(\frac{d}{dt}\,\mathcal{F}_t,..,\mathcal{F}_t) \\
&= E^\alpha(\frac{d}{dt}\,\mathcal{F}_t)\cdot Q(\mathcal{F}_t,..,\mathcal{F}_t)+E^\alpha(\mathcal{F}_t)\cdot\;\;Q(\frac{d}{dt}\,\mathcal{F}_t,\mathcal{F}_t,..,\mathcal{F}_t)+ \\
&\qquad\qquad Q(\mathcal{F}_t,\frac{d}{dt}\,\mathcal{F}_t,..,\mathcal{F}_t)+\ldots+Q(\mathcal{F}_t,..,\frac{d}{dt}\,\mathcal{F}_t) \\
&= E^\alpha(\frac{d}{dt}\,\mathcal{F}_t)\cdot Q(\mathcal{F}_t,..,\mathcal{F}_t)+E^\alpha(\mathcal{F}_t)\cdot Q(\frac{d}{dt}\,\mathcal{F}_t,\mathcal{F}_t,..,\mathcal{F}_t)+ \\
&\qquad\qquad E^\alpha(\mathcal{F}_t)\cdot Q(\mathcal{F}_t,\frac{d}{dt}\,\mathcal{F}_t,..,\mathcal{F}_t)+\ldots+E^\alpha(\mathcal{F}_t)\cdot Q(\mathcal{F}_t,..,\frac{d}{dt}\,\mathcal{F}_t) \\
&= k\cdot(E^\alpha\vee Q)\,(\frac{d}{dt}\,\mathcal{F}_t,\mathcal{F}_t,..,\mathcal{F}_t).
\end{aligned}$$

Let us explain the appearance of the last factor k. The symmetrizing product sums over $k!$ permutations and includes a factor $1/k!$, but since in the fifth line $(k-1)$ arguments of $(E^\alpha\vee Q)$ are the same, $(k-1)!$ of these permutations turn out the same element. Thus we obtain a factor $1/(k-1)!\cdot k! = k$. Notice also that we used the fact that Q is symmetric in the fourth and fifth line. As we remarked at the end of §3.1.1, we can use the less complex definition of P because the arguments $\{\frac{d}{dt}\,\mathcal{F}_t,\mathcal{F}_t,..,\mathcal{F}_t\}$ all commute.

Theorem and proof

With these preliminary results behind us, we are finally in the position to state and prove the theorem which generalizes the Weil homomorphism to general $\mathfrak{g}$-operations.

Theorem 3.4.1 For a $\mathfrak{g}$-operation $\mathscr{A}$ that admits connections, one has an algebra homomorphism

$$\mathsf{w} : (S\mathfrak{g}^*)_{\text{inv}} \to H_B(\mathscr{A})$$

such that $\mathsf{w}[(S^k\mathfrak{g}^*)_{\text{inv}}] \subset H_B^{2k}(\mathscr{A})$. This homomorphism is called the **Weil homomorphism**.

Proof: The proof is an exact imitation of the specific case of the Weil homomorphism for a principal bundle. That the proof generalized to arbitrary $\mathfrak{g}$-operation seems to have been noticed already by H. Cartan [5]. We need to prove the following claim: let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two different connections on $\mathscr{A}$, and Ψ_1 and Ψ_2 the corresponding canonical homomorphisms of $\mathsf{W}(\mathfrak{g})$ into $\mathscr{A}$; such that $\Psi_1(A) = \mathcal{A}_1$ and $\Psi_2(A) = \mathcal{A}_2$. Then

(Claim) The induced maps $\Psi_1^\#, \Psi_2^\# : H_\mathcal{B}(\mathsf{W}(\mathfrak{g})) \to H_\mathcal{B}(\mathscr{A})$ are identical: $\Psi_1^\# \equiv \Psi_2^\#$.

Let us define $\mathcal{A}_t$, $\mathcal{F}_t$, η and $\mathcal{D}_t$ as we did in the preliminary paragraph. We denote $\Psi_t : \mathsf{W}(\mathfrak{g}) \to \mathscr{A}$ with $\Psi_t(A) = \mathcal{A}_t$ the canonical homomorphism corresponding to the connection $\mathcal{A}_t$. If we restrict these Ψ_t to $\mathscr{B}_\mathsf{W} = 1 \otimes (S\mathfrak{g}^*)_{\text{inv}} \subset \mathsf{W}(\mathfrak{g})$ we get homomorphisms $\widetilde{\Psi}_t : (S^k\mathfrak{g}^*)_{\text{inv}} \to \mathscr{B}^{2k}(\mathscr{A})$. The image of $\widetilde{\Psi}_t$ consists of cocycles in $\mathscr{A}$, since the elements of $1 \otimes (S\mathfrak{g}^*)_{\text{inv}}$ are cocycles in $\mathsf{W}(\mathfrak{g})$ and Ψ is a homomorphism of differential algebras. In order to prove the claim (and theorem), we first establish the following results.

(A) For $P \in (S^k\mathfrak{g}^*)_{\text{inv}}$ we have $\widetilde{\Psi}_t(P) = P(\mathcal{F}_t, \dots, \mathcal{F}_t)$.

A polynomial $P \in (S^k\mathfrak{g}^*)_{\text{inv}}$ is in general a sum of monomials, and the monomials are products of the cobasis elements $E^\alpha \in S^1\mathfrak{g}^*$. Now the homomorphism $\widetilde{\Psi}_t$ maps the cobasis elements E^α onto $\mathcal{F}_t^\alpha \in \mathscr{A}$, and hence maps P onto the element in $\mathscr{A}$ that is obtained from P by substituting every E^α in P by $\mathcal{F}_t^\alpha$. This is exactly $P(\mathcal{F}_t, .., \mathcal{F}_t)$ as defined in §3.1.1.

(B) $\frac{d}{dt}\widetilde{\Psi}_t(P) = k\, dP(\eta, \mathcal{F}_t, .., \mathcal{F}_t)$.

We prove this using the preliminary results:

$$\begin{aligned}
\frac{d}{dt}\widetilde{\Psi}_t(P) &= \frac{d}{dt} P(\mathcal{F}_t, \dots, \mathcal{F}_t) \qquad \text{(by (A))}\\
&= k\, P(\frac{d}{dt}\mathcal{F}_t, \mathcal{F}_t, \dots, \mathcal{F}_t) \qquad \text{(by Lemma 3.4.5)}\\
&= k\, P(\mathcal{D}_t\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) \qquad \text{(by Lemma 3.4.1)}\\
&= k\, \mathcal{D}_t P(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) \qquad \text{(by Lemma 3.4.2 and since } \mathcal{D}_t\mathcal{F}_t = 0\\
&\qquad\qquad \text{by the Bianchi identity, Lemma 1.2.5)}\\
&= k \quad d + \mathcal{A}_t^\alpha L_{E_\alpha} \quad P(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) \qquad \text{(by Lemma 3.4.3)}\\
&= k \quad dP(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) + \mathcal{A}_t^\alpha \cdot L_{E_\alpha} P(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t)\\
&= k \quad dP(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) + \mathcal{A}_t^\alpha \cdot (L_{E_\alpha} P)(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) \qquad \text{(by Lemma 3.4.4)}\\
&= k\, dP(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t). \qquad \text{(since } P \text{ was invariant)}
\end{aligned}$$

(C) $P(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t)$ is basic, i.e. element of $\mathscr{B}(\mathscr{A})$.

By Lemma 3.4.4 we have $L_X(P(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t)) = (L_X P)(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) = 0$ since P is invariant. For $\mathcal{F}_t$ and $\eta = \mathcal{A}_1 - \mathcal{A}_0$ we have $i_X\mathcal{F}_t = 0$ and $i_X\eta = i_X\mathcal{A}_1 - i_X\mathcal{A}_0 = X - X = 0$, thus i_X will be zero on all the components of η and $\mathcal{F}_t$ as well. Since $P(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) \in \mathscr{A}$ consists of these components $i_X(P(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t))$ will be zero. Hence $P(\eta, \mathcal{F}_t, \dots, \mathcal{F}_t) \in \mathscr{B}(\mathscr{A})$.

(Final proof) We already noticed that since $\widetilde{\Psi}_t(P)$ with $P \in (S\mathfrak{g}^*)_{\text{inv}}$ will be a basic element of $\mathscr{A}$, since $1 \otimes P$ is basic in $\mathsf{W}(\mathfrak{g})$ and Ψ_t is a homomorphism of $\mathfrak{g}$-operations which maps basic elements on basic elements. The elements $\widetilde{\Psi}_t(P)$ are also cocycles, since $1 \otimes P$ is a cocycle in $\mathsf{W}(\mathfrak{g})$. We now show that $\widetilde{\Psi}_1(P)$ and $\widetilde{\Psi}_0(P)$ are in the same basic cohomology class. Using (B) and (C) we have $\widetilde{\Psi}_1(P) - \widetilde{\Psi}_0(P) = \int_0^1 \frac{d}{dt}\widetilde{\Psi}_t(P) = \int_0^1 k\, dP(\eta, \mathcal{F}_t, \ldots, \mathcal{F}_t) = d\int_0^1 k\, P(\eta, \mathcal{F}_t, \ldots, \mathcal{F}_t)$ and thus $\widetilde{\Psi}_1(P)$ and $\widetilde{\Psi}_0(P)$ differ a coboundary in $\mathscr{B}(\mathscr{A})$. We conclude the induced maps in basic cohomology coincide: $\Psi_1^{\#} \equiv \Psi_2^{\#}$. Hence we proved the claim and theorem.

As we already noted in the introduction to this chapter, the classical theorem on the Weil homomorphism follows from this generalization, which gives us the following corollary.

Corollary 3.4.1 For a principal bundle $P(G, M)$ there exists an algebra homomorphism from $I(G)$, the invariant symmetric polynomials on *Lie* (G), to the de Rham cohomology $H_{DR}(M)$ of the base manifold M. This homomorphism is called the *Weil homomorphism* or *Chern-Weil homomorphism*.[9]

3.5 Notes

Though Theorem 3.4.1 is not proved in the article by Dubois-Violette [6], we later found a proof in Greub, Halperin and Vanstone [10](Vol. III) where the theorem is stated in Chapter VIII, §4 (8.20 Theorem V). The proof of the theorem there builds on a lot of work done in previous chapters and makes use of spectral sequences.

On the particular case that one consider a complex vector bundle (or associated principal bundle) and one has the classical Weil homomorphism from $I(G)$ to $H_{DR}(M)$, we would like to make a few remarks.

There are various choices for a basis of the symmetric invariant polynomial $I(G)$. On can define the *total Chern form* of the bundle by

$$c(\Omega) \overset{\text{def}}{=} \det\big(1 + \frac{i}{2\pi}\Omega\big) = \sum_{l \in \mathbb{N}} P_l(\Omega),$$

where Ω is a curvature form on the bundle. The $2l$-forms $P_l(\Omega)$ are projectable; and one defines the *Chern forms* $c_l(\bar{\Omega})$ as the forms for which one has $\pi^* c_l(\bar{\Omega}) = P_l(\Omega)$. The Chern forms are closed forms on M and thus define the *Chern classes* in $H_{DR}(M)$ given by $[c_l(\bar{\Omega})]$.

It turns out that the Chern forms $c_l(\bar{\Omega})$ are of integer class: i.e. their integral over any $2l$-cycle on M with integer coefficients is an integer that does not depend on $\bar{\Omega}$. Now from the algebra generated by the Chern forms, one can pick the forms of degree $\dim M$ and integrate them over the base manifold M: the resultant integers are topological invariants and are called the *Chern numbers* of the bundle.

Under certain circumstances, these characteristic classes provide a powerful tool in classifying principal bundles. As we remarked in the *Notes* at Chapter1, for sphere bundles (i.e. bundles with base manifold $M = S^n$) we have an important classifying theorem. It states that the equivalence classes of bundles

[9] See e.g. Theorem 2.4.1. in de Azcárraga and Izquierdo [2].

with $M = S^n$ and structure group G are in bijective correspondence with the homotopy group $\pi_{n-1}(G)$. For the particular example $G = U(1)$ and $M = S^2$ the equivalent bundles are indexed by $\pi_1(U(1)) = \pi_1(S^1) = \mathbb{Z}$. The classifying integer is then supplied provided by the Chern number of the bundle (there is only one such number since $\dim M = \dim S^2 = 2$).

There is another definition that is important to us, since it is used in the cohomological descent method. This is the definition of the *Chern character forms*, which follow from taking another choice of invariant symmetric polynomials in $I(G)$. For this, one starts with the element

$$ch(\Omega) = \mathrm{Tr}\big(\exp(\frac{i}{2\pi}\Omega)\big) = \sum_{l\in\mathbb{N}} ch_l(\Omega),$$

which is an inhomogeneous element of $\Omega(P)$. Again the forms $ch_l(\Omega)$ project to closed forms on M, denoted with $ch_l(\bar{\Omega})$. They are called the *Chern character forms* $ch_l(\bar{\Omega})$, given explicitly by

$$ch_l(\bar{\Omega}) = \frac{1}{l\,!}\big(\frac{i}{2\pi}\big)^l\,\mathrm{Tr}\big(\underbrace{\bar{\Omega}\wedge\ldots\wedge\bar{\Omega}}_{l \text{ times}}\big).$$

As a final remark we wish to quote a theorem from Bott and Tu, which concerns the fundamental role of the Chern classes in the theory of characteristic classes. For the proof and implications of this theorem we refer to Bott&Tu [4] (Ch. IV) and Walschap [18](Ch. 6), who both put the theory of characteristic classes in a much broader (functorial) perspective.

Proposition 3.5.1 (Prop. 23.11 Bott&Tu [4]) *Every natural transformation from the isomorphism classes of complex vector bundles over a manifold of finite type to the de Rham cohomology can be given as a polynomial in the Chern classes.*

Chapter 4

Cohomology of Lie algebras

The cohomology of Lie algebras will occupy a prominent place in the subsequent chapters, so therefore we will take some effort to treat different approaches to this subject, and show how they generally coincide.

The standard treatment of Lie algebra cohomology starts with spaces of cochains $C^*(\mathfrak{g})$ and the coboundary operator s defined on them, which acts like a differential since $s^2 = 0$. The advantage of this definition is that it can be easily extended to allow cohomology with values in some vector space $\mathcal{V}$. In this case $\mathcal{V}$ should be a representation space of the Lie algebra.

A second, simple route to Lie algebra cohomology for finite-dimensional Lie algebras $\mathfrak{g}$ is described by Dubois-Violette in [6]. Instead of considering cochains we consider the cohomology of $\Lambda(\mathfrak{g}^*)$ as a differential algebra. Essentially, this is of course the same, since a cochain $\omega_n \in C^n(\mathfrak{g})$ can also be interpreted as an element of $\Lambda^n(\mathfrak{g}^*)$ when $\dim \mathfrak{g} < \infty$.

Finally, we also take what is called the *Chevally-Eilenberg* approach to Lie algebra cohomology, which relates it to the more common *de Rham cohomology* on a Lie group G. It relies on the fact that $\mathfrak{g}$ can be identified with the left-invariant vector fields on G and $\Lambda^n(\mathfrak{g}^*)$ can be identified with the left-invariant forms on G. It shows the definition of Lie algebra cohomology is quite natural when related thus with $H_{DR}(G)$, and is in fact isomorphic with the de Rham cohomology $H_{DR}(G)$ if G is a compact connected Lie group.

4.1 Standard approach to cohomology of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, not necessarily finite-dimensional. We will occupy ourself mainly with real algebras ($\mathbb{K} = \mathbb{R}$) since *Lie*(G) is real for a Lie group G. However the definitions are valid for $\mathbb{K} = \mathbb{C}$ as well, so we will take a more general approach.

We define the vector space $C^n(\mathfrak{g})$ of **n-cochains** as

$$C^n(\mathfrak{g}) = \{\, \alpha : \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{n \text{ times}} \to \mathbb{K} \mid \alpha \text{ multilinear and antisymmetric} \,\},$$

and call $\omega_n \in C^n(\mathfrak{g})$ a **n-cochain on** $\mathfrak{g}$. By convention one has $C^0(\mathfrak{g}) = \mathbb{K}$.

We can now make the complex $C^*(\mathfrak{g}) = \oplus_{n\in\mathbb{N}} C^n(\mathfrak{g})$ into a graded-commutative algebra by defining the product $(\omega_r \cdot \eta_s)$ for $\omega_r \in C^r(\mathfrak{g}), \eta_s \in C^s(\mathfrak{g})$ by

$$(\omega_r \cdot \eta_s)(X_1, .., X_{r+s}) = \frac{1}{(r+s)!} \sum_{\sigma\in S_{r+s}} \epsilon(\sigma)\, \omega_r(X_{\sigma(1)}, .., X_{\sigma(r)}) \;\cdot\; \eta_r(X_{\sigma(r+1)}, .., X_{\sigma(r+s)}), \tag{4.1}$$

where S^{r+s} is the permutation group on $r+s$ elements, and $\epsilon(\sigma) = \pm 1$ the sign of the permutation σ. The second dot $(\cdot)$ is the multiplication in $\mathbb{K}$.

On this complex we define the **coboundary operator** $s : C^n(\mathfrak{g}) \to C^{n+1}(\mathfrak{g})$ by

$$s\omega_n(X_0, .., X_n) = \sum_{0\le i<j\le n} (-1)^{i+j} \omega_n([\,X_i, X_j\,], X_0, .., \hat{X}_i, .., \hat{X}_j, .., X_n). \tag{4.2}$$

On $C^0(\mathfrak{g})$ one defines $s \equiv 0$. One can show $s^2 = 0$ by direct computation, but this is quite cumbersome. One can verify that s is in fact an anti-derivation of degree $+1$ on the graded-commutative algebra $C^*(\mathfrak{g})$, which makes it a differential.

$C^*(\mathfrak{g})$ equipped with the coboundary operator s is now a GCDA, and we can define its cohomology as usual (see Def. A.1.4).

The vector space $Z^n(\mathfrak{g})$ of **cocycles** on $\mathfrak{g}$ are defined as

$$Z^n(\mathfrak{g}) = \{\, \omega_n \in C^n(\mathfrak{g}) \mid s\omega_n = 0 \,\}.$$

The **coboundary** spaces $B^n(\mathfrak{g})$ are defined similarly

$$B^n(\mathfrak{g}) = s[C^{n-1}(\mathfrak{g})],$$

and we have the **n-th cohomology space** $H^n(\mathfrak{g})$ defined as

$$H^n(\mathfrak{g}) = Z^n(\mathfrak{g})/B^n(\mathfrak{g}).$$

The complex $H^*(\mathfrak{g}) = \oplus_{n\in\mathbb{N}} H^n(\mathfrak{g})$ is a graded-commutative algebra, since $C(\mathfrak{g})$ was a GCDA. $H^*(\mathfrak{g})$ is the **cohomology** of $\mathfrak{g}$.

4.1.1 Cohomology with values in a vector space

We can generalize the definitions in the preceeding section, and consider cohomology with values in a representation space of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, and $\mathcal{V}$ a vector space over the same field $\mathbb{K}$. A representation of $\mathfrak{g}$ on $\mathcal{V}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathcal{V})$, i.e. a linear map that satisfies

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X), \qquad \forall X, Y \in \mathfrak{g}.$$

We define the vector space $C^n(\mathfrak{g}, \mathcal{V})$ of n-dimensional $\mathcal{V}$-cochains by

$$C^n(\mathfrak{g}, \mathcal{V}) = \{\, \alpha : \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{n \text{ times}} \to \mathcal{V} \mid \alpha \text{ multilinear and antisymmetric} \,\}.$$

Again, we have a **coboundary operator s** on $C^n(\mathfrak{g}, \mathcal{V})$ which now includes a new term using the representation ρ. For $\omega_n \in C^n(\mathfrak{g}, \mathcal{V})$ we have

$$\begin{aligned} s\omega_n(X_0, .., X_n) = &\sum_{0 \le i \le n} (-1)^i \rho(X_i)\omega_n(X_0, .., \hat{X}_i, .., X_n) \\ &+ \sum_{0 \le i < j \le n} (-1)^{i+j} \omega_n([\,X_i, X_j\,], X_0, .., \hat{X}_i, .., \hat{X}_j, .., X_n). \end{aligned} \tag{4.3}$$

For $\omega_0 \in C^0(\mathfrak{g}, \mathcal{V}) = \mathcal{V}$ one sets $s\omega_0(X_0) = \rho(X_0)\omega_0$. Again, one has $s^2 = 0$, and one defines cocycles $Z^n(\mathfrak{g}, \mathcal{V})$ and coboundaries $B^n(\mathfrak{g}, \mathcal{V})$ in the same way as before. We cannot define a product on $C^*(\mathfrak{g}, \mathcal{V})$ however, since $\mathcal{V}$ is not necessarily an algebra. So the cohomology space $H^*(\mathfrak{g}, \mathcal{V})$ will in general be a graded vector space (i.e. $H^n(\mathfrak{g}, \mathcal{V})$ is a vector space for each $n \in \mathbb{N}$). In case that $\mathcal{V}$ is an algebra and $\rho(X)$ is a derivation on $\mathcal{V}$ for every $X \in \mathfrak{g}$,[1] we can define a product on $C^*(\mathfrak{g}, \mathcal{V})$ as in eq. (4.1), and $H^*(\mathfrak{g}, \mathcal{V})$ will be a graded algebra, graded-commutative if $\mathcal{V}$ is graded-commutative.

Notice we retrieve the original definitions from §4.1 if we take $\mathcal{V} = \mathbb{K}$ and ρ the trivial representation on $\mathbb{K}$, i.e. $\rho(X) = 0$ for all $X \in \mathfrak{g}$.

4.2 An algebraic approach

(Let $\mathfrak{g}$ be a finite-dimensional Lie algebra in what follows.)

An elegant and simple way to define the cohomology on a finite-dimensional Lie algebra $\mathfrak{g}$ is the following. Consider the dual space $\mathfrak{g}^*$ and take the exterior algebra $\Lambda(\mathfrak{g}^*)$ over it. Then $\Lambda(\mathfrak{g}^*)$ is a graded algebra, and we can define a differential on it by defining the Koszul differential

$$d\eta(X, Y) = \eta([Y, X]), \qquad X, Y \in \mathfrak{g}, \tag{4.4}$$

for elements $\eta \in \Lambda^1(\mathfrak{g}^*) = \mathfrak{g}^*$. Since $d\eta$ is antisymmetric, we have $d\eta \in \Lambda^2(\mathfrak{g}^*)$. We extend d as an anti-derivation (of degree +1) to whole $\Lambda(\mathfrak{g}^*)$; we can do this since $\Lambda(\mathfrak{g}^*)$ is generated in degree 1 by any cobasis $\{E^\alpha\}$ of $\mathfrak{g}^*$. The only thing left to check is that $d^2 = 0$. It turns out this is equivalent to the Jacobi identity on $\mathfrak{g}$ (appendix, Lemma B.3.3).

Now $(\Lambda(\mathfrak{g}^*), d)$ forms a graded differential algebra, or more generally a *differential complex*, and we can consider the cohomology of this algebra as described in the appendix, §A.1 (in Definition A.1.4). For a finite-dimensional Lie algebra $\mathfrak{g}$ we define its cohomology $H^*(\mathfrak{g})$ as the cohomology $H^*(\Lambda(\mathfrak{g}^*), d)$ of $\Lambda(\mathfrak{g}^*)$.

Notice that for finite dimensional Lie algebras $\mathfrak{g}$ the complexes $(\Lambda(\mathfrak{g}^*), d)$ and $(C(\mathfrak{g}), s)$ are isomorphic. An element $\eta^1 \wedge \cdots \wedge \eta^n$ with $\eta^i \in \mathfrak{g}^*$ corresponds to the n-cochain

$$(X_1, \ldots, X_n) \mapsto \eta^1(X_1) \cdots \eta^n(X_n) \in C^n(\mathfrak{g}).$$

[1] If $\rho(X)$ is a derivation $\forall X \in \mathfrak{g}$, then $\rho : \mathfrak{g} \to \mathrm{Der}^{(ev)}(\mathcal{V})$ is in fact a Lie algebra homomorphism, since the derivations on any algebra form a Lie algebra. We write $\mathrm{Der}^{(ev)}$ for the set of derivations, by which we mean $\mathrm{Der}^{(ev)} = \oplus_{k \in \mathbb{Z}} \mathrm{Der}^{(2k)}$, since $\mathrm{Der}^{(k)}$ are anti-derivations if k is odd.

Vice versa, a cochain $\alpha \in C^n(\mathfrak{g})$ defines an element of $\eta \in \Lambda^n(\mathfrak{g}^*)$ by

$$\eta = \sum_{I=(i_1..i_n)} C_{i_1..i_n} E^{i_1} \wedge \ldots \wedge E^{i_n} ,$$

(the sum taken over all tuples $I = (i_1..i_n)$ of length n with $i_1 < i_2 < .. < i_n$, $i_j \in \{1, .., \dim \mathfrak{g}\}$ and $\{E^\alpha\}$ denoting a cobasis of $\mathfrak{g}$) with $C_{i_1..i_n} = \alpha(E_{i_1}, .., E_{i_n})$. One can compare equations (4.4) and (4.2) to verify the differentials coincide.

Obstructions for infinite-dimensional Lie algebras

When we try to define the cohomology of an infinite-dimensional Lie algebra $\mathfrak{g}$ in this algebraic manner, we run into trouble when defining the differential $d : \Lambda^1(\mathfrak{g}^*) \to \Lambda^2(\mathfrak{g}^*)$ as

$$d\eta(X, Y) = \eta([Y, X]),$$

since in general d maps into $\Lambda(\mathfrak{g})^*$ instead of $\Lambda(\mathfrak{g}^*)$, which do not correspond when $\mathfrak{g}$ is infinite-dimensional[2]. When $\mathfrak{g}$ is finite-dimensional we have $\Lambda(\mathfrak{g})^* \cong \Lambda(\mathfrak{g}^*)$, but if $\mathfrak{g}$ if infinite-dimensional we just have the inclusion $\Lambda(\mathfrak{g}^*) \subset \Lambda(\mathfrak{g})^*$. We will illustrate this with an example.

Example 4.2.1 Suppose the infinite-dimensional Lie algebra $\mathfrak{g}$ has a countable basis $\{E_\alpha\}$, with cobasis $\{E^\alpha\}$. Then the element $\eta : \Lambda^2(\mathfrak{g}) \to \mathbb{K}$ defined by

$$\eta = \sum_{i=2}^{\infty} E^1 \wedge E^i ,$$

is an element of $\Lambda^2(\mathfrak{g})^*$ but not of $\Lambda^2(\mathfrak{g}^*)$ since the exterior algebra over any space consists of *finite* sums of wedge products.

Suppose for this countable basis, $\mathfrak{g}$ has structure constants $C^\alpha_{\beta\gamma}$. Now it is also clear that the differential d does not necessarily map into $\Lambda^2(\mathfrak{g}^*)$, since in general we have by a natural extension of Lemma B.3.2

$$d(E^\alpha) = \sum_{\beta,\gamma=1}^{\infty} C^\alpha_{\beta\gamma} E^\gamma \wedge E^\beta ,$$

and this will generally not be a finite sum (it depends on the structure constants $C^\alpha_{\beta\gamma}$ of the Lie algebra).

4.3 Chevalley-Eilenberg cohomology

Let G be a connected Lie group, and let $\mathfrak{g} = \mathit{Lie}(G)$ denote its Lie algebra. Standard Lie theory shows $\mathfrak{g} = T_eG$, the tangent space at the identity $e \in G$, and we can identify the elements $X \in \mathfrak{g}$ with left-invariant vector fields (LIVF's) X^L on G by defining $X^L(g) = L_g^T X$. Here L_g denotes the left multiplication on G. In the same way we can interpret an element $\omega \in C^n(\mathfrak{g})$ as an antisymmetric multilineair mapping

$$\omega : \underbrace{T_eG \times \ldots T_eG}_{n \text{ times}} \to \mathbb{R},$$

[2]By $\Lambda(\mathfrak{g})^*$ we mean $(\Lambda(\mathfrak{g}))^* = \{\, \eta : \Lambda(\mathfrak{g}) \to \mathbb{K} \,|\, \eta \text{ linear } \}$, the dual space of $\Lambda(\mathfrak{g})$

and denote it ω_e. We can make this a left-invariant form on G by defining

$$\omega_g(v_1, \ldots, v_n) = \omega_e(L^T_{g^{-1}} v_1, \ldots, L^T_{g^{-1}} v_n), \qquad v_i \in T_g G,$$

for all $g \in G$. This is the same as saying $\omega_g = (L_g)^* \omega_e$, and so $\omega \in \Omega^n(G)$ is left-invariant by definition.

Now on differential forms we have the exterior derivative d at our disposal, which is given (cf. A.19) by

$$d\omega(X_0, \ldots, X_n) = \sum_{i=0}^{n} (-1)^i X_i \cdot \omega(X_0, \ldots, \hat{X}_i, \ldots, X_n) + \sum_{0 \leq i<j \leq p} (-1)^{i+j} \omega(\,[X_i, X_j]\,, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n) \tag{4.5}$$

for $X_i \in \mathfrak{X}(G)$. If we now take n left-invariant vector fields $X_i^L \in \mathfrak{X}(G)$, the expression

$$\omega_g(X_1^L(g), \ldots, X_n^L(g))$$

will be constant on G, since ω was left-invariant itself. Hence the first part of the exterior derivative

$$\sum_{i=0}^{n} (-1)^i X_i^L \cdot \omega(X_0^L, \ldots, \hat{X}_i^L, \ldots, X_n^L)$$

will vanish if $X_0^L, .., X_n^L$ are LIVF's. In that case we are left with

$$d\omega(X_0^L, .., X_n^L) = \sum_{0 \leq i<j \leq p} (-1)^{i+j} \omega(\,[X_i^L, X_j^L]\,, X_0^L, .., \hat{X}_i^L, .., \hat{X}_j^L, .., X_n^L). \tag{4.6}$$

So $d\omega$ is an element of $\Omega^{n+1}(G)$, and by restricting it to the tangent space $T_e G = \mathfrak{g}$ we have again a multilinear antisymmetric map

$$(d\omega)_e : \underbrace{T_e G \times \ldots T_e G}_{n+1 \text{ times}} \to \mathbb{R},$$

thus $(d\omega)_e \in C^{n+1}(\mathfrak{g})$. One can verify the exterior derivative $d : \Omega^n(G) \to \Omega^{n+1}(G)$ induces a differential $d : C^n(\mathfrak{g}) \to C^{n+1}(\mathfrak{g})$ this way. Moreover, by comparing equation (4.2) and (4.6) one sees that this differential is exactly the one defined on $C(\mathfrak{g})$ in section 4.1.

Relation to de Rham cohomology

In the above section we considered the Lie algebra cohomology of a Lie algebra $\mathfrak{g}$ belonging to a Lie group G: that is, $\mathfrak{g} = \mathit{Lie}(G)$. We identified n-cochains $\omega_n \in C^n(\mathfrak{g})$ with left-invariant differential forms $\omega \in \Omega^n_{LI}(G)$ on the group G. Since the differentials on $C^n(\mathfrak{g})$ and $\Omega_{LI}(G)$ coincide under this identification, we have an isomorphism

$$H(\mathit{Lie}(G)) \cong H(\Omega_{LI}(G), d).$$

Here $H(\Omega_{LI}(G), d)$ means the cohomology of the left-invariant forms on G with the exterior derivative as differential.

Since the left-invariant forms $\Omega_{LI}(G)$ form a subset of the ordinary differential forms $\Omega(G)$, the inclusion map $i : \Omega_{LI}(G) \to \Omega(G)$ induces a map between the cohomology spaces:

$$i^\sharp : H(\Omega_{LI}(G), d) \to H(\Omega(G), d) = H_{DR}(G).$$

In general this map will neither be surjective nor injective, but for compact connected Lie groups G one can prove this map to be an isomorphism. A way to prove this is to show that for such a Lie group every de Rham cohomology class contains a bi-invariant differential form (this is Theorem 6.7.2 in de Azcárraga and Izquierdo [2]). We just state the result and refer to a proof in Greub, Halperin and Vanstone [10].

Theorem 4.3.1 For a compact connected real Lie group G, the de Rham cohomology of G and the Lie algebra cohomology of $\mathfrak{g} = \mathit{Lie}(G)$ are isomorphic. That is

$$H_{DR}(G) \cong H(\mathfrak{g}), \qquad \text{with } \mathfrak{g} = \mathit{Lie}(G).$$

Proof: This is Theorem III of section IV (§4.10) in Volume II of Greub, Halperin and Vanstone [10]. Another approach is included in §6.7 in de Azcárraga and Izquierdo [2] as indicated.

4.4 Notes

We just note here that the cohomology classes of Lie algebras have various interpretations. As an example, the cohomology classes of $H^2(\mathfrak{g}, \mathfrak{a})$ correspond to equivalence classes of extensions of $\mathfrak{g}$ with the abelian algebra $\mathfrak{a}$. Many more of such results can be found in Fuks [9]. Lie algebra cohomology can also be linked to the cohomology of Lie groups. We refer to Fuks [9] and Chapter 6 in de Azcárraga and Izquierdo [2] for a more thorough treatment of the cohomology of Lie algebras. The book by Fuks [9] pays attention in particular to the cohomology of infinite-dimensional Lie algebras (as the title indicates).

Chapter 5

B.R.S. algebras

In this chapter we will introduce the abstract notion of a B.R.S. algebra, named after Becchi, Rouet and Stora.[1] Just like the generalization of the algebra of differential forms $\Omega(P)$ on a principal bundle P to a $\mathfrak{g}$-operation, this abstract notion of a B.R.S. algebra is inspired by a complex, denoted $\mathcal{B}^{*,*}$ which we will introduce in section 5.2. This complex, whose cohomology is linked to the appearance of anomalies in gauge field theories, can be introduced in several different ways which we will discuss in this section.

Following the introduction of the cohomology complex, we will explain the argument applied by Dubois-Violette to motivate the constructions he makes in his article[6], and also point out some weak points in his line of reasoning. After this rather long section explaining the main strategy we will plunge into a more detailed treatment of all constructions and theorems concerned.

Recapitulation

Let us return to the concrete example of the principal bundle $P(G, M)$ as described in chapter 1. We had a Lie group G acting on a total space P by a right-action $R_g : p \mapsto pg$,

$$G \hookrightarrow P \xrightarrow{\pi} M,$$

such that the quotient M is locally, in a neighbourhood $U \subset M$, diffeomorphic to the product $U \times G$. In the following we will assume the base manifold **M is a compact manifold without boundary**.[2]

A connection on the principal bundle was denoted with ω, and the associated curvature form with $\Omega = d\omega + \frac{1}{2}[\,\omega, \omega\,]$. In chapter 2 we defined the group of gauge transformations $\mathscr{G}$ as

$$\mathscr{G} = \{\, f \in \mathrm{Diff}(P) \mid f \circ R_g = R_g \circ f \text{ and } \pi \circ f = \pi \,\}.$$

[1] Who also, including Tyutin, gave their name to the related B.R.S.T. transformations, discussed in §6.8 [2].

[2] Cf. [11] (p. 473) and [2] (p. 398). The assumption that M is compact is necessary to ensure that local functionals yield finite values. Moreover, the condition that M has no boundary is essential for the cohomological descent method, which centers on δ-cocycles *modulo d*. The 'modulo d'-parts of these cocycles vanish because of Stokes theorem and $\partial M = 0$.

As an infinite-dimensional Lie group (c.f. §2.3.4) it came with an infinite-dimensional Lie algebra $\mathit{Lie}(\mathscr{G})$, which had multiple interpretations. The definition of $\mathit{Lie}(\mathscr{G})$ which we generalized to arbitrary $\mathfrak{g}$-operations was the following:

$$\mathit{Lie}(\mathscr{G}) = \{ \alpha \in \mathit{Lie}(G) \otimes \Omega^0(P) \mid L_{X^\#}\alpha = [\alpha, X] \; \forall X \in \mathit{Lie}(G) \}.$$

So far, so good. We now move on to define the cohomology complex which is of interest to us, but the motivation behind these definitions we can only sketch, for it lies in the realm of physics. In Chapter 10 of [2] there is some attention paid to the physics background, but it is hardly understandable for a mathematician and does not really serve as an introduction. We more or less quote the following section from the article by Dubois-Violette [6].

The cohomology space $H^1(\mathit{Lie}(\mathscr{G}), \mathcal{P}_{\text{loc}})$

In quantum gauge field theories one works with a so-called *quantum action functional*, denoted $\Gamma(a, \psi)$ in [6], which depends on a gauge potential a on the quotient space M of a principal bundle $P(G, M)$ and on *field* variables, which are denoted by ψ. Now one of the main principles on which modern gauge field theories are based, is the principle of gauge invariance. We encountered this notion in section 2.3.2. In the theories of classical mechanics and special relativity one principal notion is that all physics and laws of nature should be independent of the chosen frame of reference; hence the theories should be invariant under Galilei or Lorentz transformations. Gauge invariance can be considered as a "gauge field theory" equivalent of this. We refer to the books of Naber [15] for an introduction and historical background.

In some gauge field theories, the functional $\Gamma(a, \psi)$ is *not* gauge invariant. If the equivalent functional in classical field theory *did* possess gauge invariance, this is called an *anomaly*: a particular kind of symmetry or invariance is lost in the quantization process. Citing [6] (p. 526):

> The lack of gauge invariance of $\Gamma(a, \psi)$ manifests itself by the non-vanishing of the variation $\Delta = \delta\Gamma(a, \psi; \xi)$ of Γ under infinitesimal gauge transformations (ξ are in the Lie algebra of the group of gauge transformations). It turns out that $\delta\Gamma = \Delta$, which is a linear functional in ξ, only depends on a, (and ξ of course), and is local in the sense that one has $\Delta(a; \xi) = \int Q(a; \xi)$, where the integral is taken over the n-dimensional space-time M and where $Q(a; \xi)$ is a n-form on M (which is a functional of a and ξ) such that its value at $x \in M$ only depends on the values at x of a, ξ and a finite number of their derivatives; i.e. $(a, \xi) \mapsto Q(a, \xi)$ is a differential operator which is linear in ξ. By a finite renormalization, Δ is modified by the addition of a term $\delta\Gamma^{\text{loc}}$, where $\Gamma^{\text{loc}}(a) = \int L(a)$ is a local functional of a. It follows that the obstructions to invariance is only Δ modulo such $\delta\Gamma^{\text{loc}}$.

It is now possible to place these observations in the mathematical framework which we have already developed. We start with the following definition.

Definition 5.0.1 We define $\mathcal{P}$ to be the vector space of polynomial functionals on $a_{pot}(M)$, the space of gauge potentials on the principal bundle $P(G, M)$. I.e.

$$\mathcal{P} = \{ F : a_{pot}(M) \to \mathbb{C} \mid F \text{ is linear and polynomial in } a \}.$$

We now define a *local functional* as a polynomial functional that is defined in terms of an integrated differential form; to be precise: an element $F_{\rm loc} \in \mathcal{P}$ given by

$$F_{\rm loc}(a) = \int_M \hat{F}_{\rm loc}(a)\,,$$

(an element of $\mathbb{C}$ for every $a \in a_{pot}(M)$, since M is assumed to be compact)

where $\hat{F}_{\rm loc}(a) \in \Omega^{(\dim\,{\rm M})}(M)$ is a differential form of top degree on M, and $\hat{F}_{\rm loc} : a_{pot}(M) \to \Omega^{(\dim {\rm M})}(M)$ is a linear mapping (differential operator) that is dependent of a, *local* in the sense that the value of $\hat{F}_{\rm loc}$ in $x \in M$ only depends on the value in $x \in M$ of a and derivatives of a up to finite order.[3] The local functionals of $\mathcal{P}$ we denote with $\mathcal{P}_{\rm loc}$.

Having defined the (local) functionals we can now use the constructions introduced in Chapter 4 (on Lie algebra cohomology) to consider the Lie algebra cohomology of $\mathit{Lie}(\mathscr{G})$. We are interested in the cohomology of $\mathit{Lie}(\mathscr{G})$ with values in $\mathcal{P}_{\rm loc}$ however. Remembering the definition of the coboundary operator s in eq. (4.3) we first need to have a representation of $\mathit{Lie}(\mathscr{G})$ in $\mathcal{P}_{\rm loc}$ in order to define this cohomology.

In §2.4.1 we described the action of the Lie algebra of gauge transformations $\mathit{Lie}(\mathscr{G})$ on the space of gauge potentials $a_{pot}(M)$, by $\rho(\xi) : a \mapsto -d\xi - [\,a, \xi\,]$ for $\xi \in \mathit{Lie}(\mathscr{G})$ and $a \in a_{pot}(M)$. This naturally leads to a representation of $\mathit{Lie}(\mathscr{G})$ on $\mathcal{P}$ (and $\mathcal{P}_{\rm loc}$) by

$$\rho(\xi) : F \mapsto F \circ \rho(\xi)^{-1} \qquad \text{for } F \in \mathcal{P}.$$

With this representation in hand we can define the cohomology of $\mathit{Lie}(\mathscr{G})$ with values in $\mathcal{P}$, as we defined in §4.1.1. One considers the space of s-cochains of $\mathit{Lie}(\mathscr{G})$ with values in $\mathcal{P}$, denoted $C^s(\mathit{Lie}(\mathscr{G}), \mathcal{P})$, that was defined as

$$C^s(\mathit{Lie}(\mathscr{G}), \mathcal{P}) = \{\, \alpha : \underbrace{\mathit{Lie}(\mathscr{G}) \times \ldots \times \mathit{Lie}(\mathscr{G})}_{s \text{ times}} \to \mathcal{P} \mid \alpha \text{ multilinear and antisymmetric} \,\}.$$

The complex $C^*(\mathit{Lie}(\mathscr{G}), \mathcal{P}) = \oplus_{s\in\mathbb{N}}\, C^s(\mathit{Lie}(\mathscr{G}), \mathcal{P})$ forms a graded-commutative differential algebra, with the differential given by the coboundary operator s (see §4.1.1). The cohomology with values in $\mathcal{P}_{\rm loc}$ is now included as the following subcomplex:

The subset $C^s(\mathit{Lie}(\mathscr{G}), \mathcal{P}_{\rm loc}) \subset C^s(\mathit{Lie}(\mathscr{G}), \mathcal{P})$ can be defined as those $\alpha \in C^s(\mathit{Lie}(\mathscr{G}), \mathcal{P})$ that are defined as[4]

$$\alpha(a, \xi_1, .., \xi_s) = \int_M Q(a, \xi_1, .., \xi_s)\,,$$

such that $Q : (a, \xi_1, .., \xi_s) \mapsto Q(a, \xi_1, .., \xi_s)$ is a differential operator with values in $\Omega^{(\dim\,{\rm M})}(M)$. We will see in the next section that this defines Q as an element of a complex denoted with $\widetilde{\mathcal{B}}^{\dim M, s}$.

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[3] The local functionals that are obtained in the cohomological descent method are constructed as integrals of (wedge-product) polynomials in a (and da). With derivatives of a up to finite order, one specifically means higher order derivatives of the functions $a_\mu(x)$, where $a(x) = a_\mu(x)dx^\mu$ in local coordinates.

[4] For some reason Dubois-Violette takes a slightly different approach (see [6] p. 526), and defines a sub-complex $C^s_{\rm loc}(\mathit{Lie}(\mathscr{G}), \mathcal{P})$, which is in fact the same as $C^s(\mathit{Lie}(\mathscr{G}), \mathcal{P}_{\rm loc})$.

From the definition of the coboundary operator s it follows that $C^s(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$ is stable under s, and therefore determines a cohomology complex, that we denote $H^*(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$.

Returning to the part we literally quoted from Dubois-Violette, we notice that the variation $\Delta(a;\xi)$ of the quantum action functional Γ we just described is in fact an element of $C^1(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$. The δ appearing in the quote coincides with our coboundary operator s, and since $\Delta(a;\xi) = \delta\Gamma(a,\psi;\xi)$ (in the quote) we have $s\Delta = 0$ which is known as the *Wess-Zumino consistency condition.*[5] It asserts the variation $\Delta(a;\xi)$ is a cocycle, and hence defines an element of $H^1(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$. The elements that can be added through finite renormalization were defined as $\delta\Gamma^{\text{loc}}$ with $\Gamma^{\text{loc}} \in C^0(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$: the elements $\delta\Gamma^{\text{loc}}$ are precisely the coboundaries in $C^1(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$. Thus, if the variation $\Delta(a;\xi)$ persists and does not vanish by finite renormalization it defines a non-trivial cohomology class in $H^1(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$. This explains the interest in $H^1(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$: it can be interpreted as the space of possible (candidate) anomalies.[6]

The $\widetilde{\mathscr{B}}$ complex

The approach employed by Dubois-Violette to investigate $H^1(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$ makes use of the following complex:

1. The complex $\widetilde{\mathscr{B}}^{*,*}$. This complex is a bigraded algebra, whose homogeneous space $\widetilde{\mathscr{B}}^{r,s}$ of bidegree (r,s) is given by the space of differential operators of $a_{pot}(M)\times(Lie(\mathscr{G}))^s$ in $\Omega^r(M)$ that are polynomial in $a_{pot}(M)$ (i.e. as a function of $a \in a_{pot}(M)$ only depending on finitely many derivatives of a) and s-linear antisymmetric in $(Lie(\mathscr{G}))^s$. Thus

$$\widetilde{\mathscr{B}}^{r,s}(P) = \{\, \omega : a_{pot}(M) \times (Lie(\mathscr{G}))^s \to \Omega^r(M) \mid \text{with } \omega \text{ polynomial in } a_{pot}(M) \text{ and multilinear \& antisymmetric in } (Lie(\mathscr{G}))^s \,\}.$$

The link between this complex and the cohomology spaces discussed previously is the observation that an element $Q \in \widetilde{\mathscr{B}}^{r,s}$ with $r = \dim M$ supplies an cochain $\alpha \in C^s(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$ by setting

$$\alpha(\xi_1, .., \xi_s)(a) = \int_M Q(a;\, \xi_1, ..\xi_s).$$

Vice versa, every cochain $\alpha \in C^s(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$ defines an element of $\widetilde{\mathscr{B}}^{r,s}$, as we already remarked: recall that $\alpha(\xi_1, .., \xi_s)$ is an element of $\mathcal{P}_{\text{loc}}$ for every set of Lie algebra variables $(\xi_1, .., \xi_s)$, so that one has

$$\alpha(\xi_1, .., \xi_s)(a) = \int_M Q(a;\, \xi_1, .., \xi_s),$$

with $Q(a;\, \xi_1, .., \xi_s)$ a differential form on M of top degree. The linear map $Q : (a;\, \xi_1, .., \xi_s) \mapsto \hat{\alpha}_{\text{loc}}(a)$ is then an element of $\widetilde{\mathscr{B}}^{r,s}$.

[5] Cf. [2] Ch. 10; Lemma 10.7.1 in particular. Also see [11](§6).

[6] We just note here that the second cohomology space $H^2(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$ has a similar physical interpretation. According to Dubois-Violette "obstructions to the elemination of anomalous Schwinger terms in the equal-time commutation relations of currents are elements of $H^2(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$." ([6], p. 526-527). This remark is Lemma 10.7.3 in de Azcárraga[2].

We will treat the $\widetilde{\mathcal{B}}$ complex in more detail later, but here we give a general outline of its structure and relate this to our discussion of the cohomology $H^1(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$ of interest for anomalies.

On the $\widetilde{\mathcal{B}}$ complex one defines two differentials (and a product) which make it a bigraded algebra. The first differential $d : \widetilde{\mathcal{B}}^{r,s} \to \widetilde{\mathcal{B}}^{r+1,s}$ is defined as an extension of the exterior derivative on $\Omega(M)$:

$$(d\omega)(a; \xi_1, .., \xi_s) = d\big(\omega(a; \xi_1, .., \xi_s)\big).$$

The second differential $\delta : \widetilde{\mathcal{B}}^{r,s} \to \widetilde{\mathcal{B}}^{r,s+1}$ extends the coboundary operator s we encountered in Chapter 4 (ρ is a representation of $Lie(\mathscr{G})$ which we will define later on):

$$(\delta\omega)(a; \xi_0, .., \xi_s) = (-1)^r \Big(\sum_{0 \le i \le s} (-1)^i \rho(\xi_i)\omega(\, a\,; \xi_1, .., \hat{\xi}_i, .., \xi_s) + \sum_{0 \le i < j \le s} (-1)^{i+j} \omega(a; [\,\xi_i, \xi_j\,], .., \hat{\xi}_i, .., \hat{\xi}_j, .., \xi_s) \Big).$$

The definition of these two differentials has important consequences. First of all, if one considers an element $\alpha \in C^s(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$ with corresponding element $Q \in \widetilde{\mathcal{B}}^{r,\,s}$, one has

$$s\alpha(\xi_1, .., \xi_{s+1})(a) = \int_M \delta Q(a;\, \xi_1, .., \xi_{s+1}),$$

with $\delta Q \in \widetilde{\mathcal{B}}^{r,\,s+1}$. In fact δ is defined by this equation.

Now consider a cocycle $\alpha \in C^s(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$ for which one has $s\alpha = 0$. For the corresponding element $Q \in \widetilde{\mathcal{B}}^{r,\,s}$ this means

$$\int_M \delta Q = 0.$$

Recall that we assumed M to be a compact manifold without boundary, so Stokes theorem assures us that exact forms added to Q will vanish under the integral. The above equation translates into the following condition on $Q \in \widetilde{\mathcal{B}}^{r,\,s}$: there is an element $Q' \in \widetilde{\mathcal{B}}^{r-1,\,s+1}$ such that

$$\delta Q + dQ' = 0.$$

An element satisfying this condition is called a *δ-cocycle modulo d*. It defines a cohomology class in the cohomology space $H(\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}, \delta)$.[7] Suppose now that the cocycle $\alpha \in C^s(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$ was a coboundary, thus there exists an element $\beta \in C^{s-1}(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$ such that $s\beta = \alpha$. Let L denote the element in $\widetilde{\mathcal{B}}^{r,\,s-1}$ corresponding to β, such that

$$\beta(a; \xi_1, .., \xi_{s-1}) = \int_M L(a; \xi_1, .., \xi_{s-1}). \tag{5.1}$$

Notice that (again) one can modify L by adding an element dL' with $L' \in \widetilde{\mathcal{B}}^{r-1,\,s-1}$ and equation (5.1) will still hold. (We see that any element $\alpha, \beta \in$

[7] The modulo space $\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}$ has a well-defined cohomology space since it is stable under δ: this is due to the anti-commuting of the (nilpotent) differentials d and δ ($d\delta = -\delta d$).

$C^s(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$ in fact defines an element of the modulo space $\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}$.) Now, from $s\beta = \alpha$ it follows that there exist $L \in \widetilde{\mathcal{B}}^{r,s-1}$ and $L' \in \widetilde{\mathcal{B}}^{r-1,s-1}$ such that

$$\delta L + dL' = Q,$$

which defines Q as a *δ-coboundary modulo d*.

The conclusion we derive from these observations is that one can equivalently consider the cohomology space $H(\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}, \delta)$ of the $\widetilde{\mathcal{B}}$ complex we introduced above, instead of the cohomology space $H^1(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$, since they are isomorphic.

The Weil-B.R.S. algebra

After describing the isomorphism between $H^1(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$ and $H(\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}, \delta)$, Dubois-Violette investigates the structure of $\widetilde{\mathcal{B}}$. It turns out that $\widetilde{\mathcal{B}}$ is a particular kind of bigraded algebra: a so-called *B.R.S. algebra*. For such algebras one has an universal initial object, which is called the *Weil B.R.S. algebra* $\mathsf{A}^{*,*}(\mathfrak{g})$.

It is at this point that, in my opinion, Dubois-Violette takes a questionable approach. He limits his considerations to a subalgebra $\mathcal{B}^{*,*} \subset \widetilde{\mathcal{B}}^{*,*}$ and for this subalgebra Dubois-Violette proves that the canonical homomorphism from $\mathsf{A}^{*,*}(\mathfrak{g})$ to $\mathcal{B}^{*,*}$ is in fact an isomorphism for the bihomogeneous spaces we wish to consider. As a consequence we have

$$H(\mathcal{B}/d\mathcal{B}, \delta) \cong H(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}), \delta).$$

The last part of his article [6] deals with the $H(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}), \delta)$ cohomology space. This bigraded space is also suited to accommodate a double cohomological chain, generated by the so-called *descent equations*.

Using this descent method one starts off with an element $P(F) \in \mathsf{A}^{2k+2,0}(\mathfrak{g})$ that is a $(d+\delta)$-cocycle in $\mathsf{A}(\mathfrak{g})$. Using cohomological properties of $\mathsf{A}(\mathfrak{g})$ one arrives at an element of $H^{2k,1}(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}), \delta)$. Dubois-Violette shows that in this particular case, the descent method yields all the cohomology classes of $H^{2k,1}(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}), \delta)$.

The implications of this result are not so clear, however. Since $\mathcal{B}$ is just a subalgebra of $\widetilde{\mathcal{B}}$, the cohomology spaces $H(\mathcal{B}/d\mathcal{B}, \delta)$ and $H(\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}, \delta)$ are not necessarily isomorphic. The inclusion $i : \mathcal{B} \to \widetilde{\mathcal{B}}$ yields a map

$$i^\sharp : H(\mathcal{B}/d\mathcal{B}, \delta) \to H(\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}, \delta),$$

which need neither be injective nor surjective. Thus, though $H^{2k,1}(\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}, \delta)$ identifies with $H^1(Lie(\mathscr{G}), \mathcal{P}_{\rm loc})$ (the cohomology space of candidate anomalies), the result of Dubois-Violette obtained for $H(\mathcal{B}/d\mathcal{B}, \delta)$ has no direct implications for the surjectivity of the descent method in general.

Some criticism

The most important criticism we already put forward in the previous section, where we noted that Dubois-Violette did not prove surjectivity of the descent method in general. However, there is another point at which the article by Dubois-Violette lacks a convincing proof. It concerns the generalization of all constructions and theorems to the case that the underlying principal bundle P

is not trivial. In almost all articles on the descent method the ambient principal bundle P is assumed to be trivial (i.e. $P = M \times G$), cf. [2][3][6][7][11]. In his introduction Dubois-Violette claims he will include the non-triviality case in his considerations, but the article lacks any argument concerning the case that P is not trivial. In fact, an important theorem extends a result obtained for the trivial bundle case to the non-trivial bundle case with the following justification as proof:[8]

> Since all this comes from local considerations (in fact jets of finite orders) and since a principal G-bundle is locally trivializable one has the following theorem. [9]

According to my point of view, the theorem thus introduced is by no means obvious. An exact indication of the use of 'local considerations' which can be readily generalized to the non-trivial case is not easy to provide, and thus the theorem remains unconvincing. The argument we cited above is referred to at another point, which justifies another extension:[10]

> Again, by the same argument as the one leading from lemma 1 to theorem 7, one has the following extension of theorem 7.[11]

The lack of an explicit proof makes this statement rather unsatisfactory. In general however, if one looks at other references, authors all agree that the descent method can be extended to the non-trivial case, although a thorough treatment is usually not included. We will return to this issue in section §6.2.1.

5.1 B.R.S. $\mathfrak{g}$-operations

Having discussed the main line of proof employed by Dubois-Violette, we will now follow and exhibit his constructions, starting off with the definition of bigraded $\mathfrak{g}$-operations, B.R.S. algebras and B.R.S. $\mathfrak{g}$-operations. We do this in slightly more detail than is done in [6].

Bigraded $\mathfrak{g}$-operations

Let $\mathscr{A}$ be a bigraded commutative differential algebra, with differentials d and δ, and let $\mathfrak{g}$ be a finite-dimensional Lie algebra.[12] $\mathscr{A}$ is also a ordinary graded-commutative differential algebra with differential $d + \delta$. Suppose now $\mathscr{A}$ is a $\mathfrak{g}$-operation $(\mathscr{A}, i, L)$ considered as a singular graded GCDA. In principle, this means that we have two linear maps

$$\begin{cases} i : \mathfrak{g} \to \mathrm{Der}^{(-1)}(\mathscr{A}) \\ L : \mathfrak{g} \to \mathrm{Der}^{(0)}(\mathscr{A}) \end{cases}$$

In the case that i_X is bihomogeneous of degree $(-1, 0)$ and L_X is bihomogeneous of degree $(0, 0)$, then $\mathscr{A}$ called a *bigraded* $\mathfrak{g}$*-operation*. From the relation $L_X =$

[8] This is the theorem in which the isomorphism between $\mathsf{W}^r(Lie(G))$ and $\mathcal{B}^{r,0}(M \times G)$ is extended to the case of $\mathcal{B}^{r,0}(P)$: Theorem 7 in [6].

[9] [6], p. 544.

[10] Of an isomorphism $\mathsf{A}(\mathfrak{g}) \cong \mathcal{B}$ to $\mathsf{A}(\mathfrak{g}) \cong \mathcal{B}(P)$: Theorem 11 in [6].

[11] [6], p. 554.

[12] For the definition of a bigraded algebra see definition A.1.8 in section §A.1 of the appendix.

$(d+\delta)i_X + i_X(d+\delta)$ it follows that

$$\begin{cases} L_X &= di_X + i_X d, \\ 0 &= \delta i_X + i_X \delta. \end{cases}$$

We record this in the following definition.

Definition 5.1.1 Let $\mathfrak{g}$ be a finite-dimensional algebra, and $\mathscr{A}$ a bigraded commutative differential algebra with differentials d and δ. $\mathscr{A}$ is called a **bigraded $\mathfrak{g}$-operation** if there exists a pair (i, L) of linear mappings from $\mathfrak{g}$ to the graded derivations $\mathrm{Der}^{(*,*)}(\mathscr{A})$ on the algebra $\mathscr{A}$

$$\begin{aligned} i &: \mathfrak{g} \to \mathrm{Der}^{(-1,0)}(\mathscr{A}), \qquad & i &: X \mapsto i_X, \\ L &: \mathfrak{g} \to \mathrm{Der}^{(0,0)}(\mathscr{A}), \qquad & L &: X \mapsto L_X, \end{aligned}$$

such that

$$L_X = di_X + i_X d, \tag{5.2}$$

$$0 = \delta i_X + i_X \delta, \tag{5.3}$$

$$L_{[X,Y]} = [L_X, L_Y] = L_X L_Y - L_Y L_X, \tag{5.4}$$

$$i_{[X,Y]} = L_X i_Y - i_Y L_X, \tag{5.5}$$

$$(i_X)^2 = 0, \tag{5.6}$$

for all $X, Y \in \mathfrak{g}$.

Notice that for a bigraded commutative differential algebra $(\mathscr{A}, d+\delta)$, the subalgebra $(\mathscr{A}^{*,0}, d)$ is a GCDA. If $\mathscr{A}$ is a bigraded $\mathfrak{g}$-operation, then $(\mathscr{A}^{*,0}, d)$ together with the mapping (i, L) forms an "ordinary" $\mathfrak{g}$-operation, since i_X will be a graded derivation of degree -1 on $\mathscr{A}^{*,0}$.

B.R.S. algebras

Using the previous concepts we can now define the general notion of a B.R.S. algebra. This definition formalizes the special structure that appears in the complexes used in the descent method. In these complexes one can identify two elements A and χ (some authors use other symbols), that satisfy special relations, dubbed the *B.R.S. relations.*[13]

Definition 5.1.2 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. A **B.R.S. algebra over $\mathfrak{g}$** is a bigraded commutative differential algebra $\mathscr{A}$ together with an element $\omega \in \mathfrak{g} \otimes \mathscr{A}^1$, that decomposes as $\omega = A + \chi$ with $A \in \mathfrak{g} \otimes \mathscr{A}^{1,0}$ and $\chi \in \mathfrak{g} \otimes \mathscr{A}^{0,1}$ such that

$$(d+\delta)\omega + \tfrac{1}{2}[\,\omega, \omega\,] \in \mathfrak{g} \otimes \mathscr{A}^{2,0}, \tag{5.7}$$

where d and δ are defined on $\mathfrak{g} \otimes \mathscr{A}$ as usual by $d(X \otimes \alpha) = X \otimes (d\alpha)$ and $\delta(X \otimes \alpha) = X \otimes (\delta\alpha)$; plus $[\,X \otimes \alpha, Y \otimes \beta\,] = [\,X, Y\,] \otimes (\alpha \cdot \beta)$. We can use the decomposition $\omega = A + \chi$ and write this out in components to obtain

$$(d+\delta)(A+\chi) + \tfrac{1}{2}[\,A+\chi, A+\chi\,] = dA + \tfrac{1}{2}[\,A, A\,], \tag{5.8}$$

[13]In the $\mathcal{B}$ complex we will discuss later on A will be identified as the identity operator on the space of gauge potentials, while χ can be seen as the Maurer-Cartan form of the Lie algebra $\mathit{Lie}(\mathscr{G})$. The element χ is also called a *ghost (field)* in many references.

which is known in some literature as *the Russian formula.*[14]

By grouping the terms which belong to the same homogeneous space of $\mathscr{A}$ one sees that (5.8) is equivalent with the following two equations, called the *B.R.S. relations*

$$\begin{cases} \delta A &= -d\chi - [\,A,\chi\,], \\ \delta\chi &= -\frac{1}{2}[\,\chi,\chi\,]. \end{cases} \tag{5.9}$$

The element $\omega = A+\chi$ is called the **(algebraic) connection on $\mathscr{A}$**. The correspondence to the notion of algebraic connections on $\mathfrak{g}$-operations will become clear in the following subsection when we introduce *B.R.S. $\mathfrak{g}$-operations*.

B.R.S. $\mathfrak{g}$-operations

Definition 5.1.3 A **B.R.S. $\mathfrak{g}$-operation** is a B.R.S. algebra $(\mathscr{A},\omega)$ for which $\mathscr{A}$ is a bigraded $\mathfrak{g}$-operation, such that $\omega = A+\chi \in \mathfrak{g}\otimes\mathscr{A}^1$ is an algebraic connection on $\mathscr{A}$ considered as a $\mathfrak{g}$-operation.

In the following two small lemmas we establish the explicit form of the actions i_X and L_X on a B.R.S. $\mathfrak{g}$-operation $\mathscr{A}$.

Lemma 5.1.1 For a B.R.S. $\mathfrak{g}$-operation $(\mathscr{A},\omega)$ (with $\omega = A+\chi$) the following relations hold for the anti-derivation i_X

$$i_Y(A) = Y, \tag{5.10}$$
$$i_Y(\chi) = 0, \tag{5.11}$$
$$i_Y(dA) = [\,A,Y\,], \tag{5.12}$$
$$i_Y(d\chi) = [\,\chi,Y\,]. \tag{5.13}$$

Proof: (i) Since i_Y has bidegree $(-1,0)$ it maps $\mathscr{A}^{0,1}$ on $\mathscr{A}^{-1,1}=\{0\}$, thus $i_Y(\chi)=0$ since $\chi\in\mathfrak{g}\otimes\mathscr{A}^{0,1}$. (ii) Because $\omega = A+\chi$ is a connection on $\mathscr{A}$, we have $Y = i_Y(\omega) = i_Y(A)+i_Y(\chi) = i_Y(A)$. (iii) Invoking Lemma 1.2.4, and using equations (5.7) and (5.8) we have $0 = i_Y((d+\delta)\omega + \frac{1}{2}[\,\omega,\omega\,]) = i_Y(dA+\frac{1}{2}[\,A,A\,])$ from which it follows that

$$\begin{aligned} i_Y(dA) &= -i_Y(\tfrac{1}{2}[\,A,A\,]) \\ &= -\tfrac{1}{2}[\,i_YA,A\,] + \tfrac{1}{2}[\,A,i_YA\,] \\ &= -\tfrac{1}{2}[\,Y,A\,] + \tfrac{1}{2}[\,A,Y\,] \\ &= [\,A,Y\,]. \end{aligned}$$

(iv) Now using the first equation of (5.9) it follows that

$$\begin{aligned} i_Y(d\chi) &= i_Y(-\delta A) - i_X(\,[\,A,\chi\,]\,) \\ &= \delta i_Y(A) - [\,i_YA,\chi\,] + [\,A,i_Y\chi\,] \\ &= \delta(Y) - [\,Y,\chi\,] + [\,A,0\,] \\ &= [\,\chi,Y\,], \end{aligned}$$

which finishes our lemma.

[14] Cf. Mañes, Stora and Zumino [13]: eq. (9) p. 159.

Lemma 5.1.2 For a B.R.S. $\mathfrak{g}$-operation $(\mathscr{A}, \omega)$ (with $\omega = A + \chi$) the following relations hold for the anti-derivation L_X

$$\begin{aligned} L_Y(A) &= [\,A, Y\,], & (5.14)\\ L_Y(\chi) &= [\,\chi, Y\,], & (5.15)\\ L_Y(dA) &= [\,dA, Y\,], & (5.16)\\ L_Y(d\chi) &= [\,d\chi, Y\,]. & (5.17) \end{aligned}$$

Proof: (i-ii) ω is an algebraic connection on $\mathscr{A}$, thus $L_Y(A+\chi) = L_Y(\omega) = [\,\omega, Y\,] = [\,A + \chi, Y\,]$. Splitting this in bihomogeneous spaces gives us $L_Y(A) = [\,A, Y\,]$ and $L_Y(\chi) = [\,\chi, Y\,]$. For (iii) we use $L_Y(dA) = (di_Y + i_Y d)(dA) = di_Y(dA)$ and

$$di_Y(dA) = d([\,A, Y\,]) = [\,dA, Y\,] - [\,A, d(Y)\,] = [\,dA, Y\,],$$

since $d(Y) = 0$ by Lemma A.1.3. The same reasoning gives (iv) $L_Y(d\chi) = [\,d\chi, Y\,]$.

The curvature F of a B.R.S. $\mathfrak{g}$-operation $\mathscr{A}$

Following earlier definitions, we define the element $F \in \mathfrak{g} \otimes \mathscr{A}^{2,\,0}$ as

$$F = dA + \tfrac{1}{2}[\,A, A\,].$$

Notice that by formula (5.8) we have $F = (d + \delta)\omega + \frac{1}{2}[\,\omega, \omega\,]$ as well. Thus F fulfills a double role. The above observation means that F is the curvature associated with ω interpreted as the algebraic connection on the $\mathfrak{g}$-operation $\mathscr{A}$ (using the total grading). But moreover, F is also the curvature associated with A, whereby A is interpreted as the algebraic connection on the $\mathfrak{g}$-operation $\mathscr{A}^{*,\,0}$ (which is a $\mathfrak{g}$-operation itself, as noted in §5.1).

5.2 The $\widetilde{\mathscr{B}}$ complex

We will now investigate the $\widetilde{\mathscr{B}}^{*,*}$ complex which was introduced at the start of this chapter in some more detail. We first recall the various parts which were put together in the complex.

We start off with a principal bundle $P(G, M)$, **which we will assume to be trivial**. In this case we can identify the space of gauge potentials as

$$a_{pot}(M) = Lie\,(G) \otimes \Omega^1(M) \cong \mathscr{C}(M \times G).$$

(This is Proposition 2.2.3.) Second, we had a representation of $\mathscr{G}$ (identified as $Map\,(M, G)$) on the space of gauge potentials $a_{pot}(M)$, given by (2.3):

$$\begin{aligned} &R : \mathscr{G} = Map\,(M, G) \to \mathrm{Aut}(a_{pot}(M)), \qquad R : g \mapsto R(g), \\ &R(g) : a \mapsto a' = Ad_{g^{-1}}(a) + (L_g^{-1})^T(dg), \end{aligned} ,$$

with $g : M \to G$ and $a \in a_{pot}(M)$.

The complex $\widetilde{\mathscr{B}}$ as bigraded algebra

The homogeneous space of bidegree (r, s) of $\widetilde{\mathscr{B}}$, i.e. $\widetilde{\mathscr{B}}^{r,s}$, we define as the space of differential operators of $a_{pot}(M) \times (Lie\,(\mathscr{G}))^s$ in $\Omega^r(M)$ which are polynomial in $a_{pot}(M)$ (i.e. as a function of $a \in a_{pot}(M)$ only depending on finitely many derivatives of a) and s-linear antisymmetric in $(Lie(\mathscr{G}))^s$. We thus define the complex $\widetilde{\mathcal{B}}^{*,*}$ as

$$\widetilde{\mathcal{B}}^{*,*} = \bigoplus_{r,s \in \mathbb{N}} \widetilde{\mathcal{B}}^{r,s},$$

with

$$\widetilde{\mathcal{B}}^{r,s} = \{\, \omega : a_{pot}(M) \times (Lie(\mathscr{G}))^s \to \Omega^r(M) \mid \text{with } \omega \text{ a differential operator, polynomial in } a_{pot}(M) \text{ and multilinear \& antisymmetric in } (Lie(\mathscr{G}))^s \,\}.$$

We can turn this bigraded complex into a bigraded commutative algebra by defining the following product of $\omega \in \widetilde{\mathcal{B}}^{r,s}$ and $\omega' \in \widetilde{\mathcal{B}}^{r',s'}$:

$$\omega \cdot \omega'(a;\xi_1,\ldots,\xi_{s+s'}) = \frac{(-1)^{r's}}{(s+s')!} \sum_{\sigma \in S_{s+s'}} \epsilon(\sigma)\, \omega(a;\xi_{\sigma(1)},..,\xi_{\sigma(s)}) \wedge \omega'(a;\xi_{\sigma(s+1)},..,\xi_{\sigma(s+s')}), \tag{5.18}$$

with, as before, $S_{s+s'}$ the permutation group of $s+s'$ elements, and $\epsilon(\sigma) = \pm 1$ the sign of the permutation. One can check this product indeed satisfies graded-commutativity, i.e. $\omega \cdot \omega' = (-1)^{(r+s)(r'+s')}\, \omega' \cdot \omega$.

The complex $\widetilde{\mathcal{B}}$ as differential algebra

We will now turn $\widetilde{\mathcal{B}}^{*,*}$ in a bigraded commutative *differential* algebra by defining two anti-commuting differentials on it. First we define

$$d : \widetilde{\mathcal{B}}^{r,s} \to \widetilde{\mathcal{B}}^{r+1,s}$$

for $\omega \in \widetilde{\mathcal{B}}^{r,s}$ as the natural extension of the exterior derivative on $\Omega(M)$ to $\widetilde{\mathcal{B}}$, by

$$(d\omega)(a;\xi_1,..,\xi_s) = d\big(\,\omega(a;\xi_1,..,\xi_s)\,\big). \tag{5.19}$$

Thus in the r.h.s. d indicates the exterior derivative of the r-form on M given by $\omega(a;\xi_1,..,\xi_s)$. For the second differential, denoted δ, we extend the coboundary operator s defined in §4.1.1. For this, we need a representation of $Lie(\mathscr{G})$ on the space of operators $\widetilde{\mathcal{B}}^{*,*}$. We obtain this representation by extending the representation of $\mathscr{G}$ on $a_{pot}(M)$ to a representation of $Lie(\mathscr{G})$ in the following way; for $\xi \in Lie(\mathscr{G})$ we define $\rho(\xi) : \omega \in \widetilde{\mathcal{B}}^{r,s} \mapsto \rho(\xi)\omega \in \widetilde{\mathcal{B}}^{r,s}$ by

$$(\rho(\xi)\omega)\,(a;\xi_1,..,\xi_s) = \frac{d}{dt}\,\omega(\,R(\exp(t\xi))a\,;\,\xi_1,..,\xi_s)\,\Big|_{t=0},$$

where $\exp : Lie(\mathscr{G}) \to \mathscr{G}$ is the exponential mapping of the infinite-dimensional group of gauge transformations $\mathscr{G}$, and $R : \mathscr{G} \to \mathrm{Aut}(a_{pot}(M))$ is the representation of $\mathscr{G}$ defined in §2.3.3. We obtain

$$\delta : \widetilde{\mathcal{B}}^{r,s} \to \widetilde{\mathcal{B}}^{r,\,s+1}$$

defined on $\omega \in \widetilde{\mathcal{B}}^{r,\,s}$ by

$$(\delta\omega)(a;\xi_0,..,\xi_s) = \sum_{0 \le i \le s} (-1)^{i+r}\, \rho(\xi_i)\omega(\,a\,;\,\xi_1,..,\hat{\xi}_i,..,\xi_s) + \sum_{0 \le i<j \le s} (-1)^{i+j+r}\, \omega(a;[\,\xi_i,\xi_j\,],..,\hat{\xi}_i,..,\hat{\xi}_j,..,\xi_s).$$

Or, putting the common factor $(-1)^r$ in front,

$$(\delta\omega)(a;\xi_0,..,\xi_s) = (-1)^r \Big(\sum_{0\leq\, i\leq s} (-1)^i\, \rho(\xi_i)\omega(\, a\,;\, \xi_1,..,\hat{\xi}_i,..,\xi_s) + \sum_{0\leq\, i<j\,\leq s} (-1)^{i+j}\, \omega(a;[\,\xi_i,\xi_j\,],..,\hat{\xi}_i,..,\hat{\xi}_j,..,\xi_s)\,\Big). \tag{5.20}$$

For $\omega \in \widetilde{\mathscr{B}}^{r,\,0}$ the second part of δ with the Lie bracket is omitted, and one defines $\delta\omega(a;\xi_0) = (-1)^r\rho(\xi_0)\omega(a)$.
The two differential satisfy $d^2 = 0$ and $\delta^2 = 0$ and anti-commute, i.e. $d\delta = -\delta d$, due to the factor $(-1)^r$ in the definition of $\delta\omega$. Together they define a differential $d+\delta$ on $\widetilde{\mathscr{B}}$ considered as singular graded algebra using the total grading.

The complex $\widetilde{\mathscr{B}}$ as B.R.S. algebra

Since the definition of a general B.R.S. algebra by Dubois-Violette was modelled on the special properties of the $\widetilde{\mathscr{B}}$ complex, it is no surprise that $\widetilde{\mathscr{B}}$ fulfills the requirements of Definition 5.1.2, and can be considered as a B.R.S. algebra over *Lie* (G). In order to prove this we need to designate an element $\omega \in Lie\,(G)\otimes\widetilde{\mathscr{B}}^{\,1}$ that decomposes as

$$\omega = A + \chi,$$

with $A \in Lie\,(G) \otimes \widetilde{\mathscr{B}}^{\,1,0}$ and $\chi \in Lie\,(G) \otimes \widetilde{\mathscr{B}}^{\,0,1}$, such that A and χ satisfy the B.R.S. relations (5.9)

$$\begin{cases} \delta A &= -d\chi - [\,A,\chi\,], \\ \delta\chi &= -\frac{1}{2}[\,\chi,\chi\,]. \end{cases}$$

Decomposing $A = E_\alpha \otimes A^\alpha$ and $\chi = E_\alpha \otimes \chi^\alpha$ we have components A^α belonging to $\widetilde{\mathscr{B}}^{\,1,0}(P)$, and elements χ^α of $\widetilde{\mathscr{B}}^{\,0,1}(P)$.[15] Let us consider $\widetilde{\mathscr{B}}^{\,1,0}$ and $\widetilde{\mathscr{B}}^{\,0,1}$. Elements of $\widetilde{\mathscr{B}}^{\,1,0}$ are linear mappings $\eta : a_{pot}(M) \to \Omega^1(M)$, such that $\eta(a) \in \Omega^1(M)$ depends on only finitely many derivatives of $a \in a_{pot}(M)$. Elements of $\widetilde{\mathscr{B}}^{\,0,1}$ are linear mappings $\eta : a_{pot}(M) \times Lie\,(\mathscr{G}) \to \Omega^0(M)$, also depending polynomially on $a_{pot}(M)$.

We first define the A^α's. The gauge potentials $a \in a_{pot}(M)$ are *Lie* (G)-valued forms on M, i.e. $a = E_\alpha \otimes a^\alpha$ with $a^\alpha \in \Omega^1(M)$. Now set

$$A^\alpha : a \to \Omega^1(P) \text{ with } A^\alpha(a) = a^\alpha. \tag{5.21}$$

Notice we can identify $Lie\,(G) \otimes \widetilde{\mathscr{B}}^{\,1,0}$ with the linear mappings $\eta : a_{pot}(M) \to Lie\,(G) \otimes \Omega^1(M)$: $E_\alpha \otimes \eta^\alpha$ (with $\eta^\alpha \in \widetilde{\mathscr{B}}^{\,1,0}$) corresponds to $\eta : a \mapsto E_\alpha \otimes \eta^\alpha(a)$ and vice versa. Interpreted this way, $A = E_\alpha \otimes A^\alpha \in Lie\,(G) \otimes \widetilde{\mathscr{B}}^{\,1,0}$ is simply the identity map:

$$A(a) = a \qquad \text{with } A : a_{pot}(M) \to Lie\,(G) \otimes \Omega^1(M).$$

It maps $a \in a_{pot}(M)$ to itself, considered to be an element of $Lie\,(G) \otimes \Omega^1(M)$.
Similarly, we define the $\chi^\alpha \in \widetilde{\mathscr{B}}^{\,0,1}$. Recall that we identified *Lie* $(\mathscr{G})$ as a the Ad-equivariant *Lie* (G)-valued functions on M (see §2.4), that is, a subspace

[15] From now on $\{E_\alpha\}$ will designate a fixed chosen basis of *Lie* (G). All decompositions will be with respect to this basis.

of $Lie(G) \otimes \Omega^0(M)$. An element $\xi \in Lie(\mathscr{G})$ can therefore be decomposed as $E_\alpha \otimes \xi^\alpha$. We set

$$\chi^\alpha : a_{pot}(M) \times Lie(\mathscr{G}) \to \Omega^0(M) \text{ with } \chi^\alpha(a;\xi) = \xi^\alpha\,. \tag{5.22}$$

Similar to the interpretation of A described above, we can identify $\chi = E_\alpha \otimes \chi^\alpha \in Lie(G) \otimes \widetilde{\mathcal{B}}^{0,1}$ as a map from $a_{pot}(M) \times Lie(\mathscr{G})$ to $Lie(G) \otimes \Omega^0(M)$. Since we can interpret an element $\xi \in Lie(\mathscr{G})$ as an (Ad-equivariant) element of $Lie(G) \otimes \Omega^0(M)$, we have

$$\chi(a;\xi) = \xi \qquad \text{with } \chi : a_{pot}(M) \times Lie(\mathscr{G}) \to Lie(G) \otimes \Omega^0(M).$$

Having defined $A \in Lie(G) \otimes \widetilde{\mathcal{B}}^{1,0}$ and $\chi \in Lie(G) \otimes \widetilde{\mathcal{B}}^{0,1}$, we can set $\omega \in Lie(G) \otimes \widetilde{\mathcal{B}}^1$ as $\omega = A + \chi$. We need to check that ω satisfies the B.R.S. relations. Instead of checking the Russian formula (5.7) directly, we check the equivalent relations (i) $\delta A = -d\chi - [A,\chi]$ and (ii) $\delta\chi = -\frac{1}{2}[\chi,\chi]$, c.f. (5.9).

Lemma 5.2.1 The complex $\widetilde{\mathcal{B}}^{*,*}$ is a B.R.S. algebra. That is, the elements $A = E_\alpha \otimes A^\alpha \in Lie(G) \otimes \widetilde{\mathcal{B}}^{1,0}$, $\chi = E_\alpha \otimes \chi^\alpha \in Lie(G) \otimes \widetilde{\mathcal{B}}^{0,1}$ and $\omega = A + \chi$ defined above, satisfy the B.R.S. relations

$$(i)\ \delta A = -d\chi - [A,\chi], \qquad (ii)\ \delta\chi = -\tfrac{1}{2}[\chi,\chi].$$

Proof: Consider $\delta A = \delta(E_\alpha \otimes A^\alpha) = E_\alpha \otimes \delta A^\alpha$, where δA^α are elements of $\widetilde{\mathcal{B}}^{1,1}$ defined by (5.20) as

$$\begin{aligned}\delta A^\alpha(a;\xi) &= -\rho(\xi)A^\alpha(a)\\ &= -\frac{d}{dt}\, A^\alpha(R(\exp(t\xi))a)\Big|_{t=0}\\ &= -\frac{d}{dt}\, R(\exp(t\xi))a^\alpha\Big|_{t=0}\\ &= \rho(\xi)\, a^\alpha\\ &= -(d\xi)^\alpha - [a,\xi]^\alpha\,.\end{aligned}$$

This implies (i):

$$\delta A = E_\alpha \otimes (-(d\xi)^\alpha - [a,\xi]^\alpha) = -d\xi - [a,\xi] = \big(-d\chi - [A,\chi]\big)(a,\xi).$$

(ii) Again applying the definition of δ on $\widetilde{\mathcal{B}}^{*,*}$ we obtain

$$\begin{aligned}\delta\chi(a;\xi_0,\xi_1) &= \rho(\xi_0)\chi(a;\xi_1) - \rho(\xi_1)\chi(a;\xi_0) - \chi(a;[\xi_0,\xi_1])\\ &= \frac{d}{dt}\,\chi(R(\exp(t\xi_0))a\,;\xi_1)\Big|_{t=0} - \frac{d}{dt}\,\chi(R(\exp(t\xi_1))a;\xi_0)\Big|_{t=0} - \chi(a;[\xi_0,\xi_1])\\ &= \frac{d}{dt}\,\xi_1\Big|_{t=0} - \frac{d}{dt}\,\xi_0\Big|_{t=0} - [\xi_0,\xi_1]\\ &= -[\xi_0,\xi_1].\end{aligned}$$

Now consider $-\frac{1}{2}[\chi,\chi]$ $(\in Lie(G) \otimes \widetilde{\mathcal{B}}^{0,2})$. The computation

$$\begin{aligned}-\tfrac{1}{2}[\chi,\chi]\,(a;\xi_0,\xi_1) &= -\tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, E_\alpha \otimes \big(\chi^\beta \cdot \chi^\gamma(a;\xi_0,\xi_1)\big)\\ &= -\tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, E_\alpha \otimes \big(\chi^\beta \cdot \chi^\gamma(a;\xi_0,\xi_1)\big)\\ &= -\tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, E_\alpha \otimes \big(\tfrac{1}{2}\,\chi^\beta(a;\xi_0)\chi^\gamma(a;\xi_1) - \tfrac{1}{2}\,\chi^\beta(a;\xi_1)\chi^\gamma(a;\xi_0)\big)\\ &= -\tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, E_\alpha \otimes \big(\tfrac{1}{2}\,\xi_0^\beta\xi_1^\gamma - \tfrac{1}{2}\,\xi_1^\beta\xi_0^\gamma\big)\\ &= -\tfrac{1}{2}\big(\tfrac{1}{2}[\xi_0,\xi_1] - \tfrac{1}{2}[\xi_1,\xi_0]\big) = -\tfrac{1}{2}[\xi_0,\xi_1],\end{aligned}$$

proves the lemma.

Remark: Notice that $\chi \in Lie(G) \otimes \widetilde{\mathcal{B}}^{0,1}$, interpreted as a map $a_{pot}(M) \times Lie(\mathscr{G}) \to Lie(G) \otimes \Omega^0(M)$, was in fact the identity map on $Lie(\mathscr{G})$. The B.R.S. relation $\delta\chi = -\frac{1}{2}[\chi, \chi]$ can be considered an analogue to equation (A.43) (appendix §A.3.3) i.e. the structure equation for the Maurer-Cartan form on a Lie group. Since the Maurer-Cartan form Θ_{MC} was defined as $id_{\mathfrak{g}} \in \mathfrak{g} \otimes \Omega^1(G)$, the element χ can be regarded as the infinite-dimensional analogue of this: the Maurer-Cartan form on $\mathscr{G}$.

With this last lemma, we have established that $\widetilde{\mathcal{B}}^{*,*}$ is a B.R.S. algebra over $Lie(G)$. We now turn to a sub-complex of $\widetilde{\mathcal{B}}$, denoted by $\mathcal{B}$, which is also a B.R.S. $Lie(G)$-operation. In the article by Dubois-Violette [6], the author restricts himself to this sub-complex from the start. The reason for this is that several isomorphism theorems are not valid for the bigger complex $\widetilde{\mathcal{B}}$, and in fact the sub-complex $\mathcal{B}$ is exactly chosen in such a way that the surjectivity of several isomorphisms is trivial. This has its implications however for the validity of the final results achieved by Dubois-Violette. We will come to speak of these implications later on.

The sub-complex $\mathcal{B}$ of $\widetilde{\mathcal{B}}$

Following Dubois-Violette, we define $\mathcal{B}^{*,*}$ as the smallest bigraded differential subalgebra (with unit) of $\widetilde{\mathcal{B}}^{*,*}$ containing the components of $\omega = A + \chi$. That is, $\mathcal{B}^{*,*}$ is the algebra generated by $A^\alpha, \chi^\alpha, dA^\alpha$ and $d\chi^\alpha$:

$$\mathcal{B}^{*,*} = \langle\, 1, A^\alpha, \chi^\alpha, dA^\alpha, d\chi^\alpha \,\rangle.$$

Because of the B.R.S. relations (5.9), i.e. (i) $\delta A = -d\chi - [A, \chi]$ and (ii) $\delta\chi = -\frac{1}{2}[\chi, \chi]$, this subalgebra is closed under δ as well (δA^α and $\delta\chi^\alpha$ are expressible in the other generating elements). Obviously $A \in Lie(G) \otimes \mathcal{B}^{1,0}$ and $\chi \in Lie(G) \otimes \mathcal{B}^{0,1}$, and hence $\omega = A + \chi \in Lie(G) \otimes \mathcal{B}^1$, and it still satisfies the B.R.S. relations. In other words, $\mathcal{B}^{*,*}$ is also a B.R.S. algebra over $Lie(G)$, and is in fact a sub-complex of $\widetilde{\mathcal{B}}^{*,*}$.

5.3 The Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$

In this section we will construct the Weil B.R.S. algebra, denoted $\mathsf{A}(\mathfrak{g})$, over a given finite-dimensional Lie algebra $\mathfrak{g}$, which is a universal object in the category of B.R.S. algebras as well as the category of B.R.S. $\mathfrak{g}$-operations. The construction is similar to the Weil algebra $\mathsf{W}(\mathfrak{g})$ which we described in chapter 3. We start with defining a homomorphism of B.R.S. algebras and of B.R.S. $\mathfrak{g}$-operations.

Definition 5.3.1 A **homomorphism of B.R.S. algebras** over $\mathfrak{g}$ is a homomorphism of the underlying bigraded differential algebras, that maps the connection on the connection. A **homomorphism of B.R.S. $\mathfrak{g}$-operations** is a homomorphism of B.R.S. algebras which is also a homomorphism of $\mathfrak{g}$-operations.

Similar to the situation with the Weil algebra, described in section §3.2, a B.R.S. algebra $\mathscr{A}$ has a certain subalgebra on which the Weil-B.R.S. algebra is modelled. For this one considers the designated connection $\omega = A + \chi \in \mathfrak{g} \otimes \mathscr{A}^1$, with $A \in \mathfrak{g} \otimes \mathscr{A}^{1,0}$ and $\chi \in \mathfrak{g} \otimes \mathscr{A}^{0,1}$, and the elements $F = dA + [A, A] \in \mathfrak{g} \otimes \mathscr{A}^{2,0}$

and $d\chi \in \mathfrak{g} \otimes \mathscr{A}^{1,1}$. Now, one fixes a basis $\{E_\alpha\}$ of $\mathfrak{g}$, and decomposes these elements as

$$A = E_\alpha \otimes A^\alpha, \quad F = E_\alpha \otimes F^\alpha, \quad \chi = E_\alpha \otimes \chi^\alpha, \quad d\chi = E_\alpha \otimes (d\chi)^\alpha.$$

The characteristic subalgebra of $\mathscr{A}$ is the one generated by the elements $\{ A^\alpha, F^\alpha, \chi^\alpha, (d\chi)^\alpha \}$. Again these elements satisfy certain relations to each other, and on this subalgebra the Weil-B.R.S. algebra is modelled.

The Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ as GDA and B.R.S. $\mathfrak{g}$-operation

As a *graded algebra*, we define $\mathsf{A}(\mathfrak{g})$ as the tensor product

$$\mathsf{A}(\mathfrak{g}) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*, \tag{5.23}$$

where we apply the even grading to $S\mathfrak{g}^*$ (see Def. A.1.5 in the appendix for the definition of graded algebra tensor products.) Since $\Lambda\mathfrak{g}^*$ and $S\mathfrak{g}^*$ are both generated by any cobasis $\{E^\alpha\}$ of $\mathfrak{g}^*$, we can easily identify a set of generators of $\mathsf{A}(\mathfrak{g})$ and denote these as

$$\begin{aligned} A^\alpha &= E^\alpha \otimes 1 \otimes 1 \otimes 1, \\ F^\alpha &= 1 \otimes E^\alpha \otimes 1 \otimes 1, \\ \chi^\alpha &= 1 \otimes 1 \otimes E^\alpha \otimes 1, \\ \psi^\alpha &= 1 \otimes 1 \otimes 1 \otimes E^\alpha. \end{aligned}$$

$\mathsf{A}(\mathfrak{g})$ is thus the free connected graded-commutative algebra generated by the A^α's and the χ^α's in degree one, and the F^α's and ψ^α's in degree two. Again, we interpret these elements as the components of four elements in $\mathfrak{g} \otimes \mathsf{A}(\mathfrak{g})$, to wit

$$A = E_\alpha \otimes A^\alpha, \quad F = E_\alpha \otimes F^\alpha, \quad \chi = E_\alpha \otimes \chi^\alpha, \quad \psi = E_\alpha \otimes \psi^\alpha.$$

To define differentials d and δ on $\mathsf{A}(\mathfrak{g})$ it suffices to define d and δ on the generating elements of $\mathsf{A}(\mathfrak{g})$. We do this by defining differentials d and δ on the elements A, F, χ and ψ in $\mathfrak{g} \otimes \mathsf{A}(\mathfrak{g})$, and declaring $d(A^\alpha) = (dA)^\alpha, \ldots, d(\psi^\alpha) = (d\psi)^\alpha$. The definitions of $dA, dF, d\chi, d\psi$ and $\delta A, \delta F, \delta\chi, \delta\psi$ reflect the B.R.S. relations we wish them to satisfy: we set

$$dA = -\tfrac{1}{2}[A, A] + F, \tag{5.24}$$
$$dF = -[A, F], \tag{5.25}$$
$$d\chi = \psi, \tag{5.26}$$
$$d\psi = 0, \tag{5.27}$$
$$\delta A = -\psi - [A, \chi], \tag{5.28}$$
$$\delta F = [F, \chi], \tag{5.29}$$
$$\delta\chi = -\tfrac{1}{2}[\chi, \chi], \tag{5.30}$$
$$\delta\psi = [\psi, \chi]. \tag{5.31}$$

From equations (5.24) and (5.25) one can infer that A and F will play the role of algebraic connection and curvature in the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$, just like they did in the Weil algebra $\mathsf{W}(\mathfrak{g})$. One also notices that, by definition, the B.R.S. relations are satisfied by A and χ, cf. equations (5.26), (5.28) and (5.30).

Being the tensor product of graded algebras, the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ is already equipped with a total grading. We define an underlying bigrading, such that $\mathsf{A}(\mathfrak{g})$ turns into a graded-commutative *bi*graded algebra. We do this by assigning bidegree $(1,0)$ to the A^α's, bidegree $(2,0)$ to the F^α's, bidegree $(0,1)$ to the χ^α's and finally, bidegree $(1,1)$ to the ψ^α's. One can easily check this is compatible with the total grading, and with our definitions of the differentials d and δ. It follows that $\mathsf{A}(\mathfrak{g})$ is a bigraded commutative differential algebra, with elements $A \in \mathfrak{g} \otimes \mathsf{A}^{1,0}(\mathfrak{g}), F \in \mathfrak{g} \otimes \mathsf{A}^{2,0}(\mathfrak{g})$ and $\chi \in \mathfrak{g} \otimes \mathsf{A}^{0,1}(\mathfrak{g})$ satisfying the B.R.S. relations

$$F = dA + \tfrac{1}{2}\,[\,A, A\,] = (d+\delta)(A+\chi) + \tfrac{1}{2}\,[\,A+\chi, A+\chi\,].$$

We conclude $\mathsf{A}(\mathfrak{g})$ is a B.R.S. algebra over $\mathfrak{g}$ with connection $A + \chi$. We now turn this into a B.R.S $\mathfrak{g}$-operation by defining i_X and L_X on the generating elements: we set

$$i_X(A) = X, \quad i_X(F) = 0, \quad i_X(\chi) = 0, \quad i_X(\psi) = [\,\psi, X\,],$$

and apply our usual convention $i_X(A^\alpha) = i_X(A)^\alpha, \ldots, i_X(\psi^\alpha) = i_X(\psi)^\alpha$. Since L_X is defined as $L_X = di_X + i_X d$ it is not really necessary to define L_X explicitly, but one can verify L_X is given by

$$L_X(A) = [\,A, X\,], \quad L_X(F) = [\,F, X\,], \quad L_X(\chi) = [\,\chi, X\,], \quad L_X(\psi) = [\,\psi, X\,].$$

Notice that these definitions are in accordance with Lemma 5.1.1 and Lemma 5.1.2, which gave the explicit form of i_X and L_X on the components of the algebraic connection $A + \chi$ of any B.R.S. $\mathfrak{g}$-operation.

The universal property of the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$

As stated in the beginning of this section, the Weil-B.R.S. algebra is a universal object in the category of B.R.S. algebras as well as the category of B.R.S. $\mathfrak{g}$-operations. Just as the construction of the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ was analogue to the construction of the Weil algebra $\mathsf{W}(\mathfrak{g})$, the following proof of the universal property is entirely similar to the proof of Theorem 3.2.1 concerning the Weil algebra.

Theorem 5.3.1 The Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ over a finite-dimensional Lie algebra $\mathfrak{g}$ is a universal object in the categories of B.R.S. algebras and B.R.S. $\mathfrak{g}$-operations. That is, for any B.R.S. algebra $\mathscr{A}$ over $\mathfrak{g}$, there is a unique homomorphism of B.R.S. algebras $\Phi_{\mathsf{A}} : \mathsf{A}(\mathfrak{g}) \to \mathscr{A}$. If $\mathcal{A}$ is B.R.S. $\mathfrak{g}$-operation, then Φ_{A} is a homomorphism of B.R.S. $\mathfrak{g}$-operations.

Proof: (This is Theorem 8 in Dubois-Violette [6].)
Since we will again use the symbols A, F and χ for the arbitrary B.R.S. algebra $\mathscr{A}$, we will denote elements of $\mathsf{A}(\mathfrak{g})$ with a subscript A for clarity, e.g. $A_{\mathsf{A}}, F_{\mathsf{A}}, \chi_{\mathsf{A}}$ and ψ_{A}.

The condition for Φ_{A} to be a homomorphism of B.R.S. algebras is that it maps the connection on the connection, besides being a homomorphism of bigraded differential algebras. Let $A + \chi$ be the connection on $\mathscr{A}$. Then we have $f(A^\alpha_{\mathsf{A}}) = A^\alpha$ and $f(\chi^\alpha_{\mathsf{A}}) = \chi^\alpha$. Since $F_{\mathsf{A}} = dA_{\mathsf{A}} + \frac{1}{2}\,[\,A_{\mathsf{A}}, A_{\mathsf{A}}\,]$ by (5.24) and $\psi = d\chi$ by (5.26), and Φ_{A} commutes with d and the Lie bracket, we have $\Phi_{\mathsf{A}}(F^\alpha_{\mathsf{A}}) = (dA + \frac{1}{2}\,[\,A, A\,])^\alpha$ and

$\Phi_{\mathsf{A}}(\psi_{\mathsf{A}}^{\alpha}) = (d\chi)^{\alpha}$. Being defined on the generating elements of $\mathsf{A}(\mathfrak{g})$ the homomorphism Φ_{A} is clearly unique; it is determined by the simple requirement that is should be a homomorphism of B.R.S. algebras.

Now in the case that $\mathscr{A}$ is a B.R.S. $\mathfrak{g}$-operation, $\mathfrak{g}$ operates on $\mathscr{A}$ by i_X and L_X. By Lemma 5.1.1 and Lemma 5.1.2 we know how i_X and L_X operate on the elements A and χ of $\mathscr{A}$ and the differentials dA and $d\chi$. This is identical to the action of $\mathfrak{g}$ on the elements $A_{\mathsf{A}}, \chi_{\mathsf{A}}, dA_{\mathsf{A}}$ and $d\chi_{\mathsf{A}}$ of $\mathsf{A}(\mathfrak{g})$, thus Φ_{A} will commute with respect to i_X and L_X. This makes Φ_{A} a homomorphism of B.R.S. $\mathfrak{g}$-operations.

The close relationship between the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ and the Weil algebra $\mathsf{W}(\mathfrak{g})$ is also evident in the following remark by Dubois-Violette: as graded differential algebras we have $\mathsf{A}^{*,0}(\mathfrak{g}) \cong \mathsf{W}(\mathfrak{g})\otimes 1\otimes 1$, and hence $\mathsf{A}^{*,0}(\mathfrak{g})$ is isomorphic to $\mathsf{W}(\mathfrak{g})$ as $\mathfrak{g}$-operation. For any B.R.S. $\mathfrak{g}$-operation $\mathscr{A}$ the sub-complex $\mathscr{A}^{*,0}$ is an ordinary $\mathfrak{g}$-operation[16] and hence $\Phi_{\mathsf{A}} : \mathsf{A}(\mathfrak{g}) \to \mathscr{A}$ restricted to $\mathsf{A}^{*,0}(\mathfrak{g})$ will yield the canonical homomorphism $\Psi_{\mathsf{W}} : \mathsf{W}(\mathfrak{g}) \to \mathscr{A}^{*,0}$ described in §3.2.

5.3.1 Cohomology theorems concerning the Weil-B.R.S. algebra

In this section we will deal with several cohomological properties of the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$. These cohomological properties of $\mathsf{A}(\mathfrak{g})$ will be related to the cohomology of the $\mathcal{B}$ complex by an isomorphism theorem which we introduce in the next section.

For a bigraded algebra such as the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ there are several cohomology spaces one can consider, for example:

1. $H(\mathsf{A}(\mathfrak{g}), d) = \mathsf{A}(\mathfrak{g})/d(\mathsf{A}(\mathfrak{g}))$,
2. $H(\mathsf{A}(\mathfrak{g}), \delta) = \mathsf{A}(\mathfrak{g})/\delta(\mathsf{A}(\mathfrak{g}))$,
3. $H(\mathsf{A}(\mathfrak{g}), d+\delta) = \mathsf{A}(\mathfrak{g})/(d+\delta)\mathsf{A}(\mathfrak{g})$.

These are all well-defined cohomology spaces. In fact, there is one more cohomology space of interest to us, as we indicated in the introduction to this chapter: the so-called δ-cohomology modulo d. For this one considers the modulo space $\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g})$. Since δ anti-commutes with d this modulo space is stable under δ, and we can consider the following cohomology, which we call the *δ-cohomology modulo d*:

$$H(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}), \delta).$$

This cohomology will be discussed in our last chapter, dealing with the descent equations. We now quote two theorems from [6] which concern the d, δ and $(d+\delta)$-cohomology of $\mathsf{A}(\mathfrak{g})$. We include quite detailed proofs: the reason for this is that the proofs presented in [6] are a bit unsatisfactory and only consist of a few short statements. Especially concerning Theorem 5.3.3, which will be a crucial theorem for all further developments, we felt a detailed proof was appropriate.

Theorem 5.3.2 The d-cohomology and $(d+\delta)$-cohomology of $\mathsf{A}(\mathfrak{g})$ are trivial.
Proof: (This is Theorem 9 in Dubois-Violette [6])

[16] See the remark following the definition of a bigraded $\mathfrak{g}$-operation in §5.1.

One notices that for any $t \in \mathbb{R}$ the elements $A^\alpha, (d+t\delta)A^\alpha, \chi^\alpha$ and $(d+t\delta)\chi^\alpha$ form a free system of homogeneous generators of $\mathsf{A}(\mathfrak{g})$: we can obtain our normal generators F^α and ψ^α by

$$dA^\alpha + \tfrac{1}{2}[A, A]^\alpha = F^\alpha$$

and

$$(d+t\delta)\chi^\alpha + \tfrac{1}{2}\, t\, [\chi, \chi]^\alpha = d\chi^\alpha + t\delta\chi^\alpha - t\delta\chi^\alpha = \psi^\alpha\,.$$

This shows $\mathsf{A}(\mathfrak{g})$ equipped with the differential $d + t\delta$ is isomorphic to the contractible algebra $\bigotimes_\alpha \mathscr{C}(A^\alpha, (d+t\delta)A^\alpha) \otimes \mathscr{C}(\chi^\alpha, (d+t\delta)\chi^\alpha)$ and hence has trivial cohomology (see Definition A.1.6 in the appendix). To explicitly show this, we construct a contracting homotopy $\hat{k}_t$. We first define its precursor k_t: it is the unique anti-derivation satisfying

$$\begin{aligned} k_t A^\alpha &= 0, & k_t(d+t\delta)A^\alpha &= A^\alpha, \\ k_t \chi^\alpha &= 0, & k_t(d+t\delta)\chi^\alpha &= \chi^\alpha. \end{aligned}$$

One can check k_t satisfies $\big(k_t(d+t\delta) + (d+t\delta)k_t\big)(x) = \mathrm{gendeg}(x)\, x$ for $x \in \mathsf{A}(\mathfrak{g})$ where $\mathrm{gendeg}(x)$ designates the degree of x in the generators $\{ A^\alpha, \chi^\alpha, (d+t\delta)A^\alpha, (d+t\delta)\chi^\alpha \}$.[17] Now if we set $\hat{k}_t(x) = \frac{1}{\mathrm{gendeg}(x)}\, k_t(x)$ then $\hat{k}_t(d+t\delta) + (d+t\delta)\hat{k}_t = \mathrm{id}_{\mathsf{A}}$, and hence $\hat{k}_t$ is a contracting homotopy for $\mathsf{A}(\mathfrak{g})$ with differential $d+t\delta$, and hence it has trivial cohomology with respect to this differential.[18] Inserting $t = 0$ and $t = 1$ gives us the theorem.

Theorem 5.3.3 As a bigraded algebra $\mathsf{A}(\mathfrak{g})$ is isomorphic to the tensor product of the contractible algebra $\oplus_\alpha \mathscr{C}(A^\alpha, \delta A^\alpha)$ and the algebra $\mathsf{A}_{\mathsf{sub}}$ generated by elements $1, F^\alpha, \chi^\alpha \in \mathsf{A}(\mathfrak{g})$, i.e.

$$\mathsf{A}(\mathfrak{g}) \cong \big(\oplus_\alpha \mathscr{C}(A^\alpha, \delta A^\alpha) \big) \otimes \langle 1, F^\alpha, \chi^\alpha \rangle, \tag{5.32}$$

and the subalgebra $\mathsf{A}_{\mathsf{sub}} = \langle 1, F^\alpha, \chi^\alpha \rangle$ identifies with the algebra $C(\mathfrak{g}, S\mathfrak{g}^*)$ of Lie algebra cochains with values in $S\mathfrak{g}^*$. For every $r \in \mathbb{N}$ we have

$$(\mathsf{A}^{r,*}(\mathfrak{g}), \delta) \cong C^*(\mathfrak{g}, (S\mathfrak{g}^*)^r). \tag{5.33}$$

As a consequence, the δ-cohomology of $\mathsf{A}(\mathfrak{g})$ identifies with the Lie algebra cohomology $C(\mathfrak{g}, S\mathfrak{g}^*)$ of the Lie algebra $\mathfrak{g}$ with values in $S\mathfrak{g}^*$. We have

$$H^{2k+1,s}(\mathsf{A}(\mathfrak{g}), \delta) = 0 \quad \text{and} \quad H^{2k,s}(\mathsf{A}(\mathfrak{g}), \delta) = H^s(\mathfrak{g}^*, S^k\mathfrak{g}^*), \tag{5.34}$$

for any $k, s \in \mathbb{N}$.

Proof: (This is Theorem 10 in Dubois-Violette [6].)
First notice that the $A^\alpha, \delta A^\alpha, \chi^\alpha$ and F^α generate $\mathsf{A}(\mathfrak{g})$: they form a free system of bihomogeneous generators of $\mathsf{A}(\mathfrak{g})$. To show this we construct the missing 'ordinary' generator ψ^α by $\psi^\alpha = -[A, \chi]^\alpha - \delta A^\alpha$ (using the B.R.S. relation (5.28)). That gives us equation (5.32).

Now consider the subalgebra $\mathsf{A}_{\mathsf{sub}} \subset \mathsf{A}(\mathfrak{g})$ (with unit) generated by the χ^α and F^α:

$$\mathsf{A}_{\mathsf{sub}} = \langle 1,\, \chi^\alpha,\, F^\alpha \rangle.$$

It is stable under δ since we have $\delta F = [F, \chi]$ and $\delta\chi = -\frac{1}{2}[\chi, \chi]$, cf. equations (5.29) and (5.30).

The Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ considered as a differential algebra with differential δ is therefore isomorphic to the tensor product $\bigotimes_\alpha \mathscr{C}(A^\alpha, \delta A^\alpha) \otimes (\mathsf{A}_{\mathsf{sub}}, \delta)$.

[17] For example: $\mathrm{gendeg}(A^{\alpha_1} \cdot \chi^{\alpha_2}) = 2$ and $\mathrm{gendeg}(\,(d+t\delta)\chi^{\alpha_1}\,) = 1$.
[18] See Definition A.1.6 in the appendix.

Recall that the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ was given as graded algebra by $\mathsf{A}(\mathfrak{g}) = \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$, and we had $F^\alpha = 1 \otimes E^\alpha \otimes 1 \otimes 1$ and $\chi^\alpha = 1 \otimes 1 \otimes E^\alpha \otimes 1$. We conclude

$$\mathsf{A}_{\mathsf{sub}} = 1 \otimes S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes 1 \subset \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^* \otimes \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^* = \mathsf{A}(\mathfrak{g}).$$

In chapter 4 we saw that $C(\mathfrak{g}, S\mathfrak{g}^*)$ identifies with $\Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$ as algebra. The usual grading $C^k(\mathfrak{g}, S\mathfrak{g}^*) = \Lambda^k\mathfrak{g}^* \otimes S\mathfrak{g}^*$, can be extended to a bigrading $C^{k,r}(\mathfrak{g}, S\mathfrak{g}^*) = \Lambda^k\mathfrak{g}^* \otimes S^r\mathfrak{g}^*$; notice that in that case s is a differential of bidegree $(1, 0)$.

We need to show however that (*Claim:*) the differential s defined on $C(\mathfrak{g}, S\mathfrak{g}^*) \cong \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$ coincides with the differential δ defined on $\mathsf{A}_{\mathsf{sub}} \cong \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$.

Proof of the claim.
We check this for our generating elements χ^α and F^α, which correspond to the elements $E^\alpha \otimes 1 \in \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$ and $1 \otimes E^\alpha \in \Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$ respectively. These in their turn correspond to (1) the element of $C^1(\mathfrak{g}, S^0\mathfrak{g}^*)$ given by the linear mapping $X \in \mathfrak{g} \mapsto X^\alpha \in \mathbb{R}$ (which we will designate with $\chi^\alpha \in C^1(\mathfrak{g}, S^0\mathfrak{g}^*)$), and (2) the constant map $E^\alpha \in C^0(\mathfrak{g}, S^1\mathfrak{g}^*)$ (which we will designate with $F^\alpha \in C^0(\mathfrak{g}, S^1\mathfrak{g}^*)$).

First consider χ^α. In $\mathsf{A}_{\mathsf{sub}}$ we have $\delta\chi^\alpha = -\frac{1}{2}[\chi, \chi]^\alpha = -\frac{1}{2} C^\alpha_{\beta\gamma} \chi^\beta \cdot \chi^\gamma$ which corresponds to $-\frac{1}{2} C^\alpha_{\beta\gamma} (E^\beta \wedge E^\gamma) \otimes 1 \in \Lambda^2\mathfrak{g}^* \otimes S^0\mathfrak{g}^*$. Considering this as an element of $C^2(\mathfrak{g}, S^0\mathfrak{g}^*)$, i.e. an antisymmetric multilinear map from $\mathfrak{g} \times \mathfrak{g}$ to $S^0\mathfrak{g}^*$, we have (inserting two basis vectors E_i and E_j)

$$-\tfrac{1}{2} C^\alpha_{\beta\gamma} (E^\beta \wedge E^\gamma)(E_i, E_j) \otimes 1 = -\tfrac{1}{2} C^\alpha_{ij} + \tfrac{1}{2} C^\alpha_{ji} = -C^\alpha_{ij}\,. \quad (*)$$

Now consider $s\chi^\alpha$ for $\chi^\alpha \in C^1(\mathfrak{g}, S^0\mathfrak{g}^*)$. Recalling the definition of the coboundary operator s cf. (4.2) and inserting two basis vectors E_i and E_j we have

$$s\chi^\alpha(E_i, E_j) = \rho(E_i)\chi^\alpha(E_j) - \rho(E_j)\chi^\alpha(E_i) - \chi^\alpha([E_i, E_j]).$$

The representation ρ of $\mathfrak{g}$ on $S\mathfrak{g}^*$ is given by the co-adjoint action described in §3.1.3, and since $\chi^\alpha(E_j)$ and $\chi^\alpha(E_i)$ are both in $S^0\mathfrak{g}^*$ the action of $\mathfrak{g}$ will be trivial and we have $\rho(E_i)\chi^\alpha(E_j) = \rho(E_j)\chi^\alpha(E_i) = 0$. We are left with

$$s\chi^\alpha(E_i, E_j) = -\chi^\alpha([E_i, E_j]) = -\chi^\alpha(C^k_{ij} E_k) = -C^\alpha_{ij}\,,$$

which equals $(*)$. So the differentials coincides for the element χ^α.

In the same way we tackle $F^\alpha = 1 \otimes E^\alpha \in \Lambda^0\mathfrak{g}^* \otimes S^1\mathfrak{g}^*$. We have $\delta F^\alpha = -[\chi, F]^\alpha = -C^\alpha_{\beta\gamma} \chi^\beta \cdot F^\gamma$ which corresponds to the element $-C^\alpha_{\beta\gamma} (E^\beta \otimes E^\gamma)$ of $\Lambda^1\mathfrak{g}^* \otimes S^1\mathfrak{g}^*$. Considering this as an element of $C^1(\mathfrak{g}, S^1\mathfrak{g}^*)$, i.e. as a linear map from $\mathfrak{g}$ to $S^1\mathfrak{g}^*$ we have (inserting a basis vector E_i)

$$-C^\alpha_{\beta\gamma} E^\beta(E_i) \otimes E^\gamma = -C^\alpha_{i\gamma} E^\gamma\,.$$

This is an element of $S^1\mathfrak{g}^*$. Inserting another basis vector $E_j \in \mathfrak{g}$ gives us

$$-C^\alpha_{i\gamma} E^\gamma(E_j) = -C^\alpha_{ij}\,. \quad (**)$$

Now for $F^\alpha \in C^0(\mathfrak{g}, S^1\mathfrak{g}^*)$. Following the definition of s we have $sF^\alpha \in C^1(\mathfrak{g}, S^1\mathfrak{g}^*)$ given by

$$sF^\alpha(E_i) = \rho(E_i)F^\alpha = \rho(E_i)E^\alpha\,.$$

This is an element of $S^1\mathfrak{g}^*$, but to make clear that it equals δF^α we insert the basisvector $E_j \in \mathfrak{g}$ and obtain

$$\rho(E_i)E^\alpha(E_j) = E^\alpha(-[E_i, E_j]) = E^\alpha(-C^k_{ij} E_k) = -C^\alpha_{ij}\,,$$

which equals $(**)$. The differentials therefore coincide on both χ^α and F^α. Since they generate $\Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$, they coincide on the whole algebra. *End of proof of the claim.*

We have shown s and δ coincide as differentials on $\Lambda\mathfrak{g}^*\otimes S\mathfrak{g}^*$, and hence $(\mathsf{A}_{\mathsf{sub}},\delta)$ and $(C(\mathfrak{g},S\mathfrak{g}^*),s)$ are isomorphic as graded differential algebras. For every $r\in\mathbb{N}$ we have an isomorphism $(\mathsf{A}^{r,*}_{\mathsf{sub}},\delta)\cong C^*(\mathfrak{g},(S\mathfrak{g}^*)^r)$, where $\mathsf{A}^{r,s}_{\mathsf{sub}}=\mathsf{A}^{r,s}(\mathfrak{g})\cap\mathsf{A}_{\mathsf{sub}}$. This proves (5.33). By the Künneth formula we obtain the desired consequence

$$H^{2k+1,s}(\mathsf{A}(\mathfrak{g}),\delta)=0 \quad\text{and}\quad H^{2k,s}(\mathsf{A}(\mathfrak{g}),\delta)=H^s(\mathfrak{g}^*,S^k\mathfrak{g}^*).$$

Notice $H^{2k+1,s}(\mathsf{A}(\mathfrak{g}),\delta)=0$ since $S^{2k+1}\mathfrak{g}^*=\{0\}$: this because we applied the even grading to $S\mathfrak{g}^*$.

Corollary 5.3.1 As a consequence of Theorem 5.3.3 we have, for $k\in\mathbb{N}$

$$H^{2k,0}(\mathsf{A}(\mathfrak{g}),\delta)=1\otimes(S^k\mathfrak{g}^*)_{\mathrm{inv}}\otimes 1\otimes 1,\qquad H^{2k+1,0}(\mathsf{A}(\mathfrak{g}),\delta)=\{0\}. \tag{5.35}$$

(This result will be used in Chapter 6, Theorem 6.1.3.)

Proof: Let $n\in\mathbb{N}$. It follows from Theorem 5.3.3 that if $X\in\mathsf{A}^{n,0}(\mathfrak{g})$ defines a non-trivial cohomology class in $[X]\in H^{n,0}(\mathsf{A}(\mathfrak{g}),\delta)$ then $X\in\mathsf{A}_{\mathsf{sub}}$. Now $\mathsf{A}^{n,0}(\mathfrak{g})\cap\mathsf{A}_{\mathsf{sub}}=1\otimes S\mathfrak{g}^*\otimes 1\otimes 1$. Thus X is of the form $1\otimes P\otimes 1\otimes 1$, with $P\in S\mathfrak{g}^*$. We conclude $n=2k$ if $P\in S^k\mathfrak{g}^*$; or $X=0$ otherwise. By the isomorphism $H^{2k,s}(\mathsf{A}(\mathfrak{g}),\delta)\cong H^s(\mathfrak{g},S^k\mathfrak{g}^*)$ we have $P\in H^0(\mathfrak{g},S^k\mathfrak{g}^*)$. But by definition of s on $C(\mathfrak{g},S\mathfrak{g}^*)$ we know $H^0(\mathfrak{g},S^k\mathfrak{g}^*)=(S^k\mathfrak{g}^*)_{\mathrm{inv}}$, thus $P\in(S^k\mathfrak{g}^*)_{\mathrm{inv}}$. Since $H^{2k,0}(\mathsf{A}(\mathfrak{g}),\delta)=Z(\mathsf{A}^{2k,0}(\mathfrak{g}),\delta)$ every cohomology class has an unique representative δ-cocycle, which proves (5.35).

We now finished our last cohomological theorem, and move on to the isomorphism theorem, which links the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ to the $\mathcal{B}$ complex.

5.3.2 Isomorphism theorem

As we discussed in the introduction to this chapter, the approach taken by Dubois-Violette is to concentrate on the Weil-B.R.S. algebra and link its cohomology to the δ-cohomology modulo d of the $\mathcal{B}$ complex by an isomorphism theorem. In this section we discuss the isomorphism theorem presented in Dubois-Violette [7]. This is one of the few cases in which Dubois-Violette provides a quite detailed proof, which we will quote here. There reason for including the proof (instead of referring to it) is that we need to indicate the steps in the proof that need to be generalized if one wishes to consider anomalies on a non-trivial bundle.

Theorem 5.3.4 The unique canonical homomorphism $\Phi_{\mathsf{A}}:\mathsf{A}(\mathit{Lie}(G))\to\mathcal{B}$ induces isomorphism of vector spaces

$$\mathsf{A}^{r,s}(\mathit{Lie}(G))\;\cong\;\mathcal{B}^{r,s}\,,$$

for any $r,s\in\mathbb{N}$ with $r\leq\dim(M)$.

Proof: (This is Lemma 2 in [6]; Proposition 8.5 in [7])
In [6] there is no proof presented, but fortunately in [7] there is. We will more or less quote it here.

The Weil-B.R.S. algebra $\mathsf{A}(\mathit{Lie}(G))$ is freely generated by the elements $\{A^\alpha,\chi^\alpha,\}$ plus their d and δ differentials, while $\mathcal{B}^{*,*}$ was defined as the subcomplex in $\widetilde{\mathcal{B}}^{*,*}$ generated by the elements $A^\alpha,\chi^\alpha\in\widetilde{\mathcal{B}}$ (and closed under d and δ). Hence the homomorphism from $\mathsf{A}(\mathit{Lie}(G))$ to $\mathcal{B}^{*,*}$ is surjective by the definition of $\mathcal{B}^{*,*}$ as subalgebra of $\widetilde{\mathcal{B}}^{*,*}$. (This is the main reason for introducing $\mathcal{B}^{*,*}$ in the first place.) We will now prove the homomorphism is injective by proving that a basis for any bihomogeneous space $\mathsf{A}^{r,s}(\mathfrak{g})$ $(r \leq \dim M)$ is mapped to a set of linear independent elements in $\mathcal{B}^{r,s}$.

Since we introduced elements A, F and χ both for the Weil-B.R.S. algebra $\mathsf{A}(Lie(G))$ and the complex $\mathcal{B}$, we will indicate all elements of $\mathsf{A}(Lie(G))$ with a subscript A for clarity. If we fix $r, s \in \mathbb{N}, r \leq \dim M$ a basis for $\mathsf{A}^{r,s}(\mathfrak{g})$ is given by all elements of the form

$$A_{\mathsf{A}}^{\alpha_1} \cdots A_{\mathsf{A}}^{\alpha_a} (F_{\mathsf{A}}^{\beta_1})^{m_1} \cdots (F_{\mathsf{A}}^{\beta_b})^{m_b} \chi_{\mathsf{A}}^{\gamma_1} \cdots \chi_{\mathsf{A}}^{\gamma_c} (d\chi_{\mathsf{A}}^{\delta_1})^{n_1} \cdots (d\chi_{\mathsf{A}}^{\delta_d})^{n_d} ,$$

where all exponents are such that this element is part of $\mathsf{A}^{r,s}(\mathfrak{g})$, thus

$$a + 2\sum_{i=1}^{i=b} m_i + \sum_{j=1}^{j=d} n_j = r \quad \text{and} \quad c + \sum_{j=1}^{j=d} n_j = s \,,$$

and $\alpha_1 < \alpha_2 < .. < \alpha_a$, $\beta_1 < \beta_2 < .. < \beta_b$, $\gamma_1 < \gamma_2 < .. < \gamma_c$, $\delta_1 < \delta_2 < .. < \delta_d$ with $\alpha_a, \beta_b, \gamma_c$ and δ_d smaller than $\dim(\mathfrak{g})$.

We now need to show that the functional of the gauge potential and Lie algebra elements (ghost fields) in $\mathcal{B}^{r,s}$ corresponding to this basis element is linearly independent from the other functionals. Remembering that an element of $\mathcal{B}^{r,s}$ is a linear map $a_{pot}(M) \times (Lie(\mathscr{G}))^s \to \Omega^r(M)$, we do this by choosing a point $x_0 \in M$, a gauge potential $a \in a_{pot}(M)$ and a ghost field $\xi \in Lie(\mathscr{G})$ such that at $x_0 \in M$ we have

$$\begin{aligned} a(x_0)^{\alpha_1} \cdots a(x_0)^{\alpha_a} (f(x_0)^{\beta_1})^{m_1} \cdots (f(x_0)^{\beta_b})^{m_b} \xi(x_0)^{\gamma_1} \cdots \xi(x_0)^{\gamma_c} \\ \cdot \, (d\xi(x_0)^{\delta_1})^{n_1} \cdots (d\xi(x_0)^{\delta_d})^{n_d} \neq 0, \end{aligned}$$

while all the other products vanish (i.e. all functionals in $\mathcal{B}^{r,s}$ corresponding to other basis elements of $\mathsf{A}^{r,s}(\mathfrak{g})$ vanish at $x_0 \in M$ when the same gauge potential $a(x)$ and ghost field $\xi(x)$ are inserted). We thus must construct a gauge potential a and ghost field ξ satisfying this condition. For this, we notice that $a_{pot}(M)$ is identifiable with $\mathfrak{g} \otimes \Omega^1(M)$ since we assumed the underlying principal bundle $P(G, M)$ was trivial. Now, given a $\mathfrak{g}$-valued 1-form a_0 at x_0 and a $\mathfrak{g}$-valued 2-form f_0 at x_0 there is a $\mathfrak{g}$-valued 1 form $a(x)$ on M such that $a(x_0) = a_0$ and $f(x_0) = da(x_0) + \frac{1}{2}[\, a(x_0), a(x_0)\,] = f_0$, and that similar consideration applies to $\chi(x)$ and $d\chi(x)$. Thus, there is a gauge potential $a(x) \in a_{pot}(M)$ (with field strength $f(x)$) such that (i) $a^{\alpha_1}(x_0) = dx^1, .., a^{\alpha_a}(x_0) = dx^a$ and the other components of $a(x_0)$ vanish; (ii) for the corresponding field strength $f(x_0) = da(x_0) + \frac{1}{2}[\, a(x_0), a(x_0)\,]$ we have

$$f^{\beta_1}(x_0) = \frac{1}{m_1!} \sum_{k=a+1}^{a+m_1} dx^k \wedge dx^{k+m_1}, .., f^{\beta_b}(x_0) = \frac{1}{m_b!} \sum_{k=a+\sum_{i=1}^{b-1} 2m_i+1}^{a+\sum_{i=1}^{b-1} 2m_i + m_b} dx^k \wedge dx^{k+m_b}$$

and the other components of $f(x_0)$ vanish; (iii) there is a Lie algebra element (ghost field) $\xi \in Lie(\mathscr{G})$ (identified with $Map(M, Lie(G))$) such that $\xi^{\gamma_1}(x_0) = \xi^{\gamma_1}, .., \xi^{\gamma_c}(x_0) = \xi^{\gamma_c}$ and the other components of $\xi(x_0)$ vanish, and (iv)

$$d\xi^{\delta_1}(x_0) = \frac{1}{n_1!} \sum_{k=a+2\sum_{i=1}^{b-1} m_i+1}^{a+2\sum_{i=1}^{b-1} 2m_i+n_1} dx^k \xi_k^{\delta_1}, \ldots,$$

$$d\xi^{\delta_d}(x_0) = \frac{1}{n_d!} \sum_{k=a+2\sum_{i=1}^{b-1} m_i+\sum_{j=1}^{d-1} n_j+1}^{a+2\sum_{i=1}^{b-1} 2m_i+\sum_{j=1}^{d} n_j} dx^k \xi_k^{\delta_d}$$

and the other components of $d\xi(x_0)$ vanish. Here the x^k are local coordinates around x_0 on M, and the ξ^{γ_λ} and $\xi_k^{\delta_\sigma}$ are linearly independent. Such a configuration satisfies the above conditions, and hence the functionals in $\mathcal{B}^{r,s}$ corresponding to the basis elements of $\mathsf{A}^{r,s}(\mathfrak{g})$ are linearly independent. We conclude the canonical homomorphism Φ_{A} : $\mathsf{A}^{r,s}(\mathfrak{g}) \to \mathcal{B}^{r,s}$ is an isomorphism for $r \leq \dim M$.

For those who wonder at the condition $r \leq \dim M$ we just remark that $\mathcal{B}^{r,s} = \{0\}$ for $r > \dim M$ since $\Omega^r(M)$ is trivial in that case. Since the canonical unique homomorphism $\Phi_{\mathsf{A}} : \mathsf{A}(\mathit{Lie}(G)) \to \mathcal{B}$ (described in Theorem 5.3.1) was a homomorphism of bigraded algebras, we have the following corollary.

Corollary 5.3.2 The canonical homomorphism $\Phi_{\mathsf{A}} : \mathsf{A}(\mathit{Lie}(G)) \to \mathcal{B}$ induces isomorphisms of their δ-cohomology and their δ-cohomology modulo d in bidegree (r, s) for $r \leq \dim M$. (This is Corollary 8.6 in [7].)

Chapter 6

Descent equations

The cohomological descent method is an algorithm used in gauge field theories to obtain elements of the cohomology $H^1(Lie(\mathscr{G}), \mathcal{P}_{\mathrm{loc}})$. The method is described in various articles and books; apart from Dubois-Violette [6][7] good references are Kastler and Stora [11], and de Azcárraga and Izquierdo [2]. In fact the construction of a double cohomological chain that is used in this process is possible in any bigraded algebra with two nilpotent anti-commuting differentials ($d^2 = 0, \delta^2 = 0$ and $d\delta = -\delta d$) which has trivial $(d+\delta)$-cohomology.[1]

The Weil-B.R.S. algebra will serve as an example. We are interested in elements of $H^{2k,1}(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$ with $2k = \dim M$. We start our cohomological chain with an element $P(F) \in \mathsf{A}^{2k+2,\,0}(\mathfrak{g})$, an element of $\mathsf{A}^{2k+2}(\mathfrak{g})$ with respect to the total grading, that is a $(d+\delta)$-cocycle: $(d+\delta)P(F) = 0$.

Since the $(d+\delta)$-cohomology is trivial, we know $P(F)$ is a coboundary, hence there exists an element $Q \in \mathsf{A}^{2k+1}(\mathfrak{g})$ such that $(d+\delta)Q = P(F)$. If we split this element in its bihomogeneous components,

$$Q = \bigoplus_{r+s=2k+1} Q^{r,s} \qquad \text{with } Q^{r,s} \in \mathsf{A}^{r,s}(\mathfrak{g}),$$

the condition $(d+\delta)Q = P(F)$ gives rise to a set of equations known as the *descent equations*:[2]

$$\begin{aligned} dQ^{2k+1,0} &= P(F), \\ \delta Q^{2k+1,0} + dQ^{2k,1} &= 0, \\ \delta Q^{2k,1} + dQ^{2k-1,2} &= 0, \\ &\vdots \\ \delta Q^{1,2k} + dQ^{0,2k+1} &= 0, \\ \delta Q^{0,2k+1} &= 0. \end{aligned}$$

A schematic figure (Figure 1) may illustrate the situation.

[1] A recent article by Langmann [12] describes a generalization of the cohomological descent method which can be applied in non-commutative geometry.

[2] Compare to [11] p. 473.

Figure 1 A schematic illustration of the descent equations. In this figure, the squares may be interpreted as the homogeneous spaces $\mathsf{A}^{r,s}(\mathfrak{g})$ of bidegree (r,s) of the Weil-B.R.S algebra $\mathsf{A}(\mathfrak{g})$. The elements are all put in their corresponding bihomogenous space. The element $Q \in \mathsf{A}^{2k+1}(\mathfrak{g})$ can be considered as the diagonal running from $Q^{0,2k+1}$ to $Q^{2k+1,0}$, just like $P(F) \in \mathsf{A}^{2k+2}(\mathfrak{g})$ (as an element in the singular graded algebra) can be seen as the diagonal consisting of all the zero's and the element $P(F) \in \mathsf{A}^{2k+2,0}(\mathfrak{g})$.

These equations define the elements $Q^{2k+1-p,p}$ as δ-cocycles modulo d for $0 \leq p \leq 2k+1$, i.e. $Q^{2k+1-p,p} \in H^{2k+1-p,p}(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$. The third relation is of particular interest, since it defines $Q^{2k,1}$ as an element of $H^{2k,1}(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$. Since $\mathsf{A}^{r,s}(\mathfrak{g})$ was isomorphic (as bigraded algebra) to $\mathcal{B}^{r,s}$ for $(r,s) \in \mathbb{N}^2$ and $r \leq \dim M$, this also provides an element $Q^{2k,1} \in H^{2k,1}(\mathcal{B}, \delta\text{-mod } d)$, the cohomology that is also relevant to candidate anomalies according to Dubois-Violette.

6.1 The cohomological descent in $\mathsf{A}(\mathfrak{g})$

The route which Dubois-Violette follows in order to prove the surjectivity of the cohomological descent procedure in the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ is to apply a fundamental theorem from homological algebra. The theorem is used to construct several long exact sequences which relate the $H(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$ cohomology and the $H(\mathsf{A}(\mathfrak{g}), \delta)$ cohomology. Since we explicitly computed the δ-cohomology of $\mathsf{A}(\mathfrak{g})$ in §5.3.1 we can also identify most of the bihomogeneous spaces of the $H(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$ cohomology complex.

In the first subsection we prove several preliminary results, which are then used in the second subsection where we deal with surjectivity theorem stated by Dubois-Violette. This theorem entails that there exists an isomorphism from $(S^{k+1}\mathfrak{g}^*)_{\mathrm{inv}}$, the invariant polynomials on $\mathfrak{g}$, to $H^{2k,1}(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$.

6.1.1 Preliminary results

We first briefly introduce the concepts from homological algebra which we will need, and then move on and apply them to the Weil-B.R.S. algebra.

Homological algebra

Though the following concepts have more general definitions, we concentrate on the case of interest to us and restrict ourselves to differential complexes; i.e. graded vector spaces equipped with a differential (see §A.1.3).

Let us consider an infinite sequence of differential complexes $\{ (\mathcal{V}_i, d_i) \}_{i\in\mathbb{Z}}$,

$$\ldots \mathcal{V}_{i-1} \xrightarrow{\phi_{i-1}} \mathcal{V}_i \xrightarrow{\phi_i} \mathcal{V}_{i+1} \xrightarrow{\phi_{i+1}} \ldots,$$

where the ϕ_i are all homomorphisms of differential spaces (also called *chain maps*). The ϕ_i are linear maps homogeneous of degree zero ($\phi_i(\mathcal{V}_i^n) \subset \mathcal{V}_{i+1}^n$) and we have $\phi_{i-1} \circ d_{i-1} = d_i \circ \phi_{i-1}$ for all $i \in Z$. Such an infinite sequence is called **exact at** $\mathcal{V}_i$ if $\mathrm{im}(\phi_{i-1}) = \ker(\phi_i)$. If it is exact for every $i \in \mathbb{Z}$ than the sequence is called a **long exact sequence**. In general a sequence (not necessarily infinite) is called **exact** if it is exact at every space occurring in the sequence.

In the long exact sequences we will encounter, we will try to identify exact subsequences of the form

$$(\ldots \longrightarrow)\, 0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow 0\, (\longrightarrow \ldots),$$

since this implies that A and B are isomorphic. Exactness at A states $\ker(\phi) = \{0\}$ (ϕ is injective); exactness at B states $\mathrm{im}(\phi) = B$ (ϕ is surjective), hence ϕ is an isomorphism.

A **short exact sequence** is a sequence of the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0,$$

that is exact everywhere. Notice that, similar to the sequence above, exactness at A implies the injectivity of $i : A \to B$ and exactness at C implies $p : B \to C$ must be surjective.[3] Exactness at B implies C is isomorphic to $B/i(A)$.

[3] The i stands for inclusion, and p for projection.

We assumed A, B and C were differential complexes: let d_A, d_B and d_C denote their respective differentials. Since both i and p are natural with respect to the differentials, they induce maps in cohomology (see §A.1.7)

$$i^\sharp : H^n(A) \to H^n(B), \qquad p^\sharp : H^n(B) \to H^n(C). \qquad (\text{for } n \in \mathbb{N})$$

One now has the following theorem which puts the cohomology spaces $H(A), H(B)$ and $H(C)$ together in a long exact sequence.

Theorem 6.1.1 Fundamental theorem of homological algebra. Given a short exact sequence of differential space

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0,$$

there exists a linear mapping $\partial : H^n(C) \to H^{n+1}(A)$, called the **connecting homomorphism**, such that the following long sequence is exact

$$\ldots \longrightarrow H^n(A) \xrightarrow{i^\sharp} H^n(B) \xrightarrow{p^\sharp} H^n(C) \xrightarrow{\partial} H^{n+1}(A) \xrightarrow{i^\sharp} \ldots.$$

Proof: (Dubois-Violette [6] Proposition 1; Bott and Tu [4] Ch. 1 §1.)
We just recall the definition of ∂. Let $[c]$ be a cohomology class in $H^n(C)$, and let $c \in C^n$ be a representative cocycle ($d_C(c) = 0$). Since $p : B \to C$ was surjective there is an element $b \in B^n$ such that $p(b) = c$. Since c was a cocycle and p commutes with the differential, we have $p(d_B b) = d_C p(b) = d_C c = 0$, thus $d_B b \in \ker(p)$. Exactness at B implies $\ker(p) = \mathrm{im}(i)$ thus $\exists a \in A^{n+1}$ such that $i(a) = d_B b$. The element a is a cocycle since $i(d_A(a)) = d_B(i(a)) = d_B^2 b = 0$ and i was injective. Hence a defines a cohomology class $[a] \in H^{n+1}(A)$. We now set $\partial([c]) = [a] \in H^{n+1}(A)$. One can verify by a similar argument that this definition of ∂ is independent of the chosen representative c of $[c] \in H^n(C)$.

Application to the Weil-B.R.S. algebra

In order to apply Theorem 6.1.1 to our situation, we first need to construct a short exact sequence. In this sequence we put the following spaces:

1. $\mathsf{A}(\mathfrak{g})$. Naturally, for every $r \in \mathbb{N}$, $\mathsf{A}^{r,*}(\mathfrak{g})$ forms a differential complex with differential δ.
2. $d\mathsf{A}(\mathfrak{g})$. For every $r \in \mathbb{N}$, $(d\mathsf{A}(\mathfrak{g}))^{r,*} = d\mathsf{A}(\mathfrak{g}) \cap \mathsf{A}^{r,*}(\mathfrak{g})$ is a graded vector space. It is closed under the nilpotent differential $\delta : (d\mathsf{A}(\mathfrak{g}))^{r,s} \to (d\mathsf{A}(\mathfrak{g}))^{r,s+1}$ since d and δ anti-commute: if $da \in d\mathsf{A}(\mathfrak{g})$, then $\delta(da) = d(-\delta a) \in d\mathsf{A}(\mathfrak{g})$. Hence $(d\mathsf{A}(\mathfrak{g}))^{r,*}$ is a differential complex for every $r \in \mathbb{N}$.
3. $\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g})$. Since $d\mathsf{A}(\mathfrak{g})$ was closed under δ, the modulo space $\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g})$ also is a differential space with differential δ. For every $r \in \mathbb{N}$ we set $(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))^{r,*} = \mathsf{A}^{r,*}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g})$ as a graded vector space. This makes $(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))^{r,*}$ a differential complex for every $r \in \mathbb{N}$.

The chain maps connecting these spaces are obvious. We define $i : d\mathsf{A}(\mathfrak{g}) \to \mathsf{A}(\mathfrak{g})$ as the inclusion $i(a) = a$, and $p : \mathsf{A}(\mathfrak{g}) \to (\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))$ is the projection $p(a) = [a] \in (\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))$. Hence we have the following short exact sequence

$$0 \longrightarrow d\mathsf{A}(\mathfrak{g}) \xrightarrow{i} \mathsf{A}(\mathfrak{g}) \xrightarrow{p} \mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}) \longrightarrow 0.$$

More precisely: for every $r \in \mathbb{N}$ we have

$$0 \longrightarrow (d\mathsf{A}(\mathfrak{g}))^{r,*} \xrightarrow{i} \mathsf{A}^{r,*}(\mathfrak{g}) \xrightarrow{p} (\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))^{r,*} \longrightarrow 0. \tag{6.1}$$

By Theorem 6.1.1 this short exact sequence induces a long exact sequences for every $r \in \mathbb{N}$. In order to reduce notational weight, we will use the following shorthands (cf. [6]) $H(\delta) = H(\mathsf{A}(\mathfrak{g}), \delta)$ and $H(\delta\text{-mod } d) = H(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$; and write

$$\begin{aligned} \ldots \longrightarrow H^{r,s}(d\mathsf{A}(\mathfrak{g}), \delta) \xrightarrow{i^\sharp} H^{r,s}(\delta) \xrightarrow{p^\sharp} H^{r,s}(\delta\text{-mod } d) \\ \xrightarrow{\partial} H^{r,s+1}(d\mathsf{A}(\mathfrak{g}), \delta) \xrightarrow{i^\sharp} \ldots. \end{aligned} \tag{6.2}$$

Now we use the following Lemma to identify certain parts of $H(\delta\text{-mod } d)$ and $H(d\mathsf{A}(\mathfrak{g}), \delta)$.

Lemma 6.1.1 There exists an isomorphism

$$(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))^{r,s} \cong (d\mathsf{A}(\mathfrak{g}))^{r+1,s},$$

for $r + s \geq 1$, induced by the map $Q^{r,s} \mapsto (-1)^r dQ^{r,s}$. The isomorphisms commute with the δ-differential so one has

$$H^{r,s}(\delta\text{-mod } d) \cong H^{r+1,s}(d\mathsf{A}(\mathfrak{g}), \delta) \quad \text{for } r, s \in \mathbb{N} : r + s \geq 1.$$

Proof: Take $r, s \in \mathbb{N}$ such that $r + s \geq 1$. For $d : \mathsf{A}^{r,s}(\mathfrak{g}) \to \mathsf{A}^{r+1,s}(\mathfrak{g})$ we have $\ker(d) = d(\mathsf{A}^{r-1,s}(\mathfrak{g})$ since the d-cohomology of $\mathsf{A}(\mathfrak{g})$ is trivial (we proved this in Theorem 5.3.2). From $\mathsf{A}^{r,s}(\mathfrak{g})/\ker(d) \cong \mathrm{im}(d)$ it follows that

$$\mathsf{A}^{r,s}(\mathfrak{g})/d\mathsf{A}^{r-1,s}(\mathfrak{g}) \cong d[\mathsf{A}^{r,s}(\mathfrak{g})] = (d\mathsf{A}(\mathfrak{g}))^{r+1,s}.$$

If we include a factor $(-1)^r$ to the differential and define $\phi : \mathsf{A}^{r,s}(\mathfrak{g}) \to \mathsf{A}^{r+1,s}(\mathfrak{g})$ as $\phi : Q^{r,s} \mapsto (-1)^r dQ^{r,s}$ the isomorphism still holds, while the anti-commutativity of d and δ turns into commutativity. Hence we have $\phi \circ \delta = \delta \circ \phi$ and for the cohomology spaces we have $H^{r,s}(\delta\text{-mod } d) \cong H^{r+1,s}(d\mathsf{A}(\mathfrak{g}), \delta)$.

We now insert this isomorphism in the long exact sequence given by (6.2) and obtain (for $r \in \mathbb{N}$)

$$\begin{aligned} \ldots \longrightarrow H^{r-1,s}(\delta\text{-mod } d) \xrightarrow{i^\sharp} H^{r,s}(\delta) \xrightarrow{p^\sharp} H^{r,s}(\delta\text{-mod } d) \\ \xrightarrow{\partial} H^{r-1,s+1}(\delta\text{-mod } d) \xrightarrow{i^\sharp} H^{r,s+1}(\delta) \xrightarrow{p^\sharp} \ldots. \end{aligned} \tag{6.3}$$

Since $\mathsf{A}^{r,s}(\mathfrak{g}) = \{0\}$ if $r < 0$ or $s < 0$, the cohomology spaces $H^{r,s}(\delta)$ and $H^{r,s}(\delta\text{-mod } d)$ will be trivial as well in that case. It follows from (6.3) that if we take $r = 0$ we have isomorphisms

$$H^{0,s}(\delta\text{-mod } d) \cong H^{0,s}(\delta),$$

for every $s \in \mathbb{N}$ (induced by $p^\sharp$). If we look at the starting point of these long sequences, we can verify they start off with either

$$0 \longrightarrow H^{1,0}(\delta) \xrightarrow{p^\sharp} H^{1,0}(\delta\text{-mod } d) \xrightarrow{\partial} \dots,$$

if $r = 1$; or

$$0 \longrightarrow H^{r-1,0}(\delta\text{-mod } d) \xrightarrow{i^\sharp} H^{r,0}(\delta) \xrightarrow{p^\sharp} \dots,$$

for $r \geq 2$.

As we indicated in the previous section we can use subsequences of the form $0 \to A \to B \to 0$ to establish isomorphisms. Except from the observation that

$$H^{r,s}(\delta) = H^{r,s}(\delta\text{-mod } d) = \{0\} \qquad \text{for } r \text{ or } s < 0,$$

we can use Theorem 5.3.3 on the δ-cohomology of $\mathsf{A}(\mathfrak{g})$ which stated

$$H^{2k+1,s}(\delta) = \{0\} \qquad H^{2k,s}(\delta) \cong H^s(\mathfrak{g}, S^k\mathfrak{g}^*) \qquad \text{for } k, s \in \mathbb{N}.$$

Using this knowledge we obtain the following isomorphisms from the long exact sequences given by (6.3). This theorem also concludes our preliminary work, and in the next subsection we turn to the descent equations in $\mathsf{A}(\mathfrak{g})$.

Theorem 6.1.2 The following isomorphisms exist:

1. $H^{0,s}(\delta\text{-mod } d) \cong H^s(\mathfrak{g})$, $\forall s \in \mathbb{N}$ (induced by $p^\sharp$).
2. $H^{2k+2,0}(\delta\text{-mod } d) = \{0\}$, $\forall k \in \mathbb{N}$ (induced by $i^\sharp$).
3. $H^{2k+1,0}(\delta\text{-mod } d) \cong H^0(\mathfrak{g}, S^{k+1}\mathfrak{g}^*) = (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$, $\forall k \in \mathbb{N}$. (induced by $i^\sharp$).
4. $H^{2k+1,s}(\delta\text{-mod } d) = H^{2k,s+1}(\delta\text{-mod } d)$, $\forall k, s \in \mathbb{N}$ (induced by ∂).

Proof: (This is Theorem 12 in [6], but no proof/verification is included.)
First of all recall the long exact sequence given by (*) (6.3)

$$\dots \longrightarrow H^{r-1,s}(\delta\text{-mod } d) \xrightarrow{i^\sharp} H^{r,s}(\delta) \xrightarrow{p^\sharp} H^{r,s}(\delta\text{-mod } d)$$
$$\xrightarrow{\partial} H^{r-1,s+1}(\delta\text{-mod } d) \xrightarrow{i^\sharp} H^{r,s+1}(\delta) \xrightarrow{p^\sharp} \dots .$$

(1.) Take $r = 0$, then $H^{r-1,s}(\delta\text{-mod } d) = \{0\}$ and $H^{r-1,s+1}(\delta\text{-mod } d) = \{0\}$, thus (*) gives us

$$\dots \longrightarrow \{0\} \xrightarrow{i^\sharp} H^{0,s}(\delta) \xrightarrow{p^\sharp} H^{0,s}(\delta\text{-mod } d) \xrightarrow{\partial} \{0\} \xrightarrow{i^\sharp} \dots$$

and hence $H^{0,s}(\delta\text{-mod } d) \cong H^{0,s}$. By Theorem 5.3.3 we have $H^{0,s} = H^s(\mathfrak{g}, S^0\mathfrak{g}^*) = H^s(\mathfrak{g})$ (since $S^0\mathfrak{g}^* = \mathbb{R}$) thus $H^{0,s}(\delta\text{-mod } d) \cong H^s(\mathfrak{g})$ for $s \in \mathbb{N}$.

(2.) Let $k \in \mathbb{N}$. Use (*) with $r = 2k+3$ and $s = -1$. Since $H^{2k+3,-1}(\delta\text{-mod } d) = \{0\}$ and $H^{2k+3,0}(\delta\text{-mod } d) = \{0\}$ (by Theorem 5.3.3) we have

$$\dots \xrightarrow{p^\sharp} H^{2k+3,-1}(\delta\text{-mod } d) = \{0\} \xrightarrow{\partial} H^{2k+2,0}(\delta\text{-mod } d) \xrightarrow{i^\sharp} H^{2k+3,0}(\delta) = \{0\} \xrightarrow{p^\sharp} \dots,$$

from which it follows that $H^{2k+2,0}(\delta\text{-mod } d) = \{0\}$.

(3.) Use (*) with $r = 2k+2$ and $s = -1$. Since $H^{2k+2,-1}(\delta\text{-mod } d) = \{0\}$ and $H^{2k+2,0}(\delta\text{-mod } d) = \{0\}$ (by 2.) we get

$$\dots \xrightarrow{p^\sharp} H^{2k+2,-1}(\delta\text{-mod } d) = \{0\} \xrightarrow{\partial} H^{2k+1,0}(\delta\text{-mod } d) \xrightarrow{i^\sharp} H^{2k+2,0}(\delta)$$
$$\xrightarrow{p^\sharp} H^{2k+2,0}(\delta\text{-mod } d) = \{0\} \dots,$$

from which we obtain $H^{2k+1,0}(\delta\text{-mod } d) \cong H^{2k+2,0}(\delta)$. By Theorem 5.3.3 we know $H^{2k+2,0}(\delta) = H^0(\mathfrak{g}, S^{k+1}\mathfrak{g}^*)$. But $H^0(\mathfrak{g}, S^{k+1}\mathfrak{g}^*)$ consists of all the elements in $S^{k+1}\mathfrak{g}^*$ that are mapped to zero by the coboundary operator s as defined in 4.1.1. Since this coboundary operator was defined using the action of $\mathfrak{g}$ on $S\mathfrak{g}^*$, these elements are precisely the invariant elements of $\{P \in S^{k+1}\mathfrak{g}^* | L_X P = 0 \, \forall X \in \mathfrak{g}\}$, i.e. $(S^{k+1}\mathfrak{g}^*)_{\text{inv}}$. Thus we have $H^{2k+1,0}(\delta\text{-mod } d) \cong (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$.

(4.) Use (*) with $r = 2k+1$. We know $H^{2k+1,s}(\delta) = \{0\}$ and $H^{2k+1,s+1}(\delta) = \{0\}$ by Theorem 5.3.3, thus (*) gives us

$$\dots \xrightarrow{i^\sharp} H^{2k+1,s}(\delta) = \{0\} \xrightarrow{p^\sharp} H^{2k+1,s}(\delta\text{-mod } d) \xrightarrow{\partial} H^{2k,s+1}(\delta\text{-mod } d)$$
$$\xrightarrow{i^\sharp} H^{2k+1,s+1}(\delta) = \{0\} \xrightarrow{p^\sharp} \dots,$$

from which it follows that $H^{2k+1,s}(\delta\text{-mod } d) \cong H^{2k,s+1}(\delta\text{-mod } d)$ for all $k, s \in \mathbb{N}$. This finishes our proof.

From this theorem, the fourth statement will be the most important to us: for every $k, s \in \mathbb{N}$ the map induced by ∂ is an isomorphism

$$H^{2k+1,s}(\delta\text{-mod } d) \cong H^{2k,s+1}(\delta\text{-mod } d).$$

In the following lemma we will identify the mapping $\partial : H^{r,s}(\delta\text{-mod } d) \to H^{r-1,s+1}(\delta\text{-mod } d)$ as being simply the map

$$[Q^{r,s}] \mapsto [Q^{r-1,s+1}],$$

where $Q^{r,s}\mathsf{A}^{r,s}(\mathfrak{g})$ is a representative of a cohomology class $[Q^{r,s}] \in H^{r,s}(\delta\text{-mod } d)$, and $[Q^{r-1,s+1}]$ is the cohomology class induced by the element $Q^{r-1,s+1} \in \mathsf{A}^{r-1,s+1}(\mathfrak{g})$, for which we have

$$\delta Q^{r,s} + dQ^{r-1,s+1} = 0.$$

Such an element exists since $Q^{r,s}$ is a δ-cocycle modulo d.

Lemma 6.1.2 Let $Q^{r,s} \in \mathsf{A}^{r,s}(\mathfrak{g})$ be the representative of a δ-modulo d cohomology class $[Q^{r,s}] \in H^{r,s}(\delta\text{-mod } d)$. For every element $Q^{r-1,s+1} \in \mathsf{A}^{r-1,s+1}(\mathfrak{g})$ such that

$$\delta Q^{r,s} + dQ^{r-1,s+1} = 0,$$

the element $Q^{r-1,s+1}$ defines itself a δ-modulo d cohomology class

$$[Q^{r-1,s+1}] \in H^{r-1,s+1}(\delta\text{-mod } d).$$

Moreover, if $Q^{r,s}$ is a δ-coboundary modulo d, i.e. $[Q^{r,s}] = [0] \in H^{r,s}(\delta\text{-mod } d)$, than $Q^{r-1,s+1}$ is a δ-coboundary modulo d as well.

This defines $[Q^{r,s}] \mapsto [Q^{r-1,s+1}]$ as a well defined mapping between $H^{r,s}(\delta\text{-mod } d)$ and $H^{r-1,s+1}(\delta\text{-mod } d)$, and it coincides with the connecting homomorphism

$$\partial : H^{r,s}(\delta\text{-mod } d) \to H^{r-1,s+1}(\delta\text{-mod } d)$$

which we encountered in the long exact sequence given by (6.3) in the previous subsection.

Proof: First we notice that since $Q^{r,s}$ is a δ-cocycle modulo d (by definition) there always will exists an element $Q^{r-1,s+1} \in \mathsf{A}^{r-1,s+1}(\mathfrak{g})$ such that

$$(*)\ \delta Q^{r,s} + dQ^{r-1,s+1} = 0.$$

We need to prove $Q^{r-1,s+1}$ is a δ-cocycle modulo d. If we apply δ to $(*)$ we obtain

$$\delta(dQ^{r-1,s+1}) = d(-\delta Q^{r-1,s+1}) = 0.$$

Since the d-cohomology of $\mathsf{A}(\mathfrak{g})$ is trivial (Th. 5.3.2) there exists an element $Q^{r-2,s+2} \in \mathsf{A}^{r-1,s+2}(\mathfrak{g})$ such that

$$dQ^{r-2,s+2} = -\delta Q^{r-1,s+1} \Rightarrow \delta Q^{r-1,s+1} + dQ^{r-2,s+2} = 0.$$

Thus $Q^{r-1,s+1}$ is a δ-cocycle modulo d and defines an element $[Q^{r-1,s+1}]$ in the cohomology space $H^{r-1,s+1}(\delta\text{-mod } d)$.

Now suppose $Q^{r,s}$ is a δ-coboundary modulo d, i.e. there exist elements $L^{r,s-1} \in \mathsf{A}^{r,s-1}(\mathfrak{g})$ and $L^{r-1,s} \in \mathsf{A}^{r-1,s}(\mathfrak{g})$ such that

$$Q^{r,s} = \delta L^{r,s-1} + dL^{r-1,s}.$$

If we apply δ we obtain $(**)\ \delta Q^{r,s} = \delta dL^{r-1,s} = -d\delta L^{r-1,s}$. If we have an element $Q^{r-1,s+1} \in \mathsf{A}^{r-1,s+1}(\mathfrak{g})$ satisfying $(*)$, then combining $(*)$ and $(**)$ leads to

$$0 = dQ^{r-1,s+1} - d\delta L^{r-1,s+1} = d\left(Q^{r-1,s+1} - \delta L^{r-1,s+1}\right).$$

Hence, again by Th. 5.3.2, there exists an element $L^{r-2,s+1} \in \mathsf{A}^{r-2,s+1}(\mathfrak{g})$ such that $dL^{r-2,s+1} = Q^{r-1,s+1} - \delta L^{r-1,s+1}$. That is,

$$Q^{r-1,s+1} = \delta L^{r-1,s+1} + dL^{r-2,s+1},$$

which defines $Q^{r-1,s+1}$ as a δ-coboundary modulo d. Hence $[Q^{r-1,s+1}] = [0] \in H^{r-1,s+1}(\delta\text{-mod } d)$.

That the map $Q^{r,s} \mapsto Q^{r-1,s+1}$ induces a well-defined map in δ-modulo d cohomology is now clear: let $Q^{r,s}\mathsf{A}^{r,s}(\mathfrak{g})$ be a representative of $[Q^{r,s}] \in H^{r,s}(\delta\text{-mod } d)$, and let $K^{r,s}$ be another representative. Then $K^{r,s} = Q^{r,s} + L^{r,s}$ with $L^{r,s}$ a δ-coboundary modulo d. We have

$$\begin{aligned}[K^{r-1,s+1}] =& [Q^{r-1,s+1} + L^{r-1,s+1}] = [Q^{r-1,s+1}] + [L^{r-1,s+1}] \\ &= [Q^{r-1,s+1}] + [0] = [Q^{r-1,s+1}] \in H^{r-1,s+1}(\delta\text{-mod } d)\end{aligned}$$

and thus $K^{r,s}$ and $Q^{r,s}$ define the same cohomology class in $H^{r-1,s+1}(\delta\text{-mod } d)$.

That the induced map is exactly the connecting homomorphism ∂ follows from checking the definition of ∂, as given in Theorem 6.1.1. We will identify $\partial([Q^{r,s}])$ with $[Q^{r,s}]$ a cohomology class in $H^{r,s}(\delta\text{-mod } d)$. Since $H^{r,s}(\delta\text{-mod } d)$ is the δ-cohomology of the modulo space $\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g})$ a representative is technically a class $\langle Q^{r,s}\rangle \in (\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))^{r,s}$. We usually pick an element $Q^{r,s} \in \mathsf{A}^{r,s}(\mathfrak{g})$ to represent this class (this is possible because of the obvious surjectivity of p). For this element we have $\delta Q^{r,s} + dQ^{r-1,s+1} = 0$ for some element $dQ^{r-1,s+1} \in (d\mathsf{A}(\mathfrak{g}))^{r,s+1}$, that is a δ-cocycle in $d\mathsf{A}(\mathfrak{g})$. Thus $[dQ^{r-1,s+1}] \in H^{r,s+1}(d\mathsf{A}(\mathfrak{g}), \delta)$. By the isomorphism $H^{r,s}(\delta\text{-mod } d) \cong H^{r+1,s}(d\mathsf{A}(\mathfrak{g}), \delta)$ described in Lemma 6.1.1, this also gives an cohomology class in $H^{r-1,s+1}(\delta\text{-mod } d)$ defined by $[Q^{r-1,s+1}]$.

Thus $\partial([Q^{r,s}]) = [Q^{r-1,s+1}] \in H^{r-1,s+1}(\delta\text{-mod } d)$.

6.1.2 A description of the descent method in $\mathsf{A}(\mathfrak{g})$

We have now obtained all the results we need to describe the descent method in $\mathsf{A}(\mathfrak{g})$ and prove the surjectivity theorem.

As described in the introduction to this chapter, a cohomological chain is started from an element $P(F) \in \mathsf{A}^{2k+2,\,0}(\mathfrak{g})$ that is a $(d+\delta)$-cocycle: $(d+\delta)P(F) = 0$.

One obtains this element by choosing an invariant polynomial $P \in (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$. Similar to the construction of the Weil homomorphism described in Chapter 3, we insert the curvature $F \in \mathfrak{g} \otimes \mathsf{A}^{2,0}(\mathfrak{g})$ in this polynomial which gives us the element

$$P(F) = P(\underbrace{F, \ldots, F}_{k+1 \text{ times}}) = 1 \otimes P \otimes 1 \otimes 1 \in \mathsf{A}^{2k+2,0}(\mathfrak{g}).$$

(Conform the interpretation of P as a map $P : (\mathfrak{g} \otimes \mathsf{A}(\mathfrak{g}))^{k+1} \to \mathsf{A}(\mathfrak{g}))$ as described in §3.1.)

We will now show that $P(F)$ is a d-cocycle and δ-cocycle in $\mathsf{A}(\mathfrak{g})$, and hence a $(d+\delta)$-cocycle as well. To prove this we will rely on the work we did in Chapter 3 on the Weil algebra, which is imbedded in the Weil-B.R.S. algebra, as we remarked in §5.3.

Lemma 6.1.3 Let $P \in (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$, and $P(F) = 1 \otimes P \otimes 1 \otimes 1 \in \mathsf{A}^{2k+2,0}(\mathfrak{g})$. Then

$$dP(F) = 0 \text{ and } \delta P(F) = 0.$$

Proof: Since $(\mathsf{A}^{*,\,0}(\mathfrak{g}), d)$ is isomorphic to the Weil algebra $(\mathsf{W}^*(\mathfrak{g}), d)$, we can interpret $P(F)$ as the element $1 \otimes P \in \mathsf{W}(\mathfrak{g})$. Corollary 3.3.2 then assures us $dP(F) = 0$.

For $\delta P(F) = 0$ we go back to the definitions of δ and L_X on $\mathsf{A}(\mathfrak{g})$, as described in §5.3. We notice that $\delta(F) = [\,F, \chi\,]$ and $L_X F = [\,F, X\,]$ from which it follows that

$$\delta(F^\alpha) = \sum_\gamma \chi^\gamma \cdot L_{E_\gamma}(F^\alpha).$$

This holds for every element in the subalgebra $1 \otimes S\mathfrak{g}^* \otimes 1 \otimes 1 \subset \mathsf{A}(\mathfrak{g})$ generated by the F^α's, since δ and L_X both act as a derivation on this subalgebra (the elements of which all have even degree). Since P was an invariant polynomial, we have $L_X P(F) = 0$ by Lemma 3.4.4 and thus $\delta P(F) = \sum_\gamma \chi^\gamma \cdot L_{E_\gamma} P(F) = 0$.

We can now continue with the construction of the cohomological chain. Since the $(d+\delta)$-cohomology of $\mathsf{A}(\mathfrak{g})$ is trivial (by Theorem 5.3.2), we know $P(F)$ is a coboundary, hence there exists an element $Q \in \mathsf{A}^{2k+1}(\mathfrak{g})$ such that $(d+\delta)Q = P(F)$. If we split this element in its bihomogeneous components,

$$Q = \bigoplus_{r+s=2k+1} Q^{r,s} \qquad \text{with } Q^{r,s} \in \mathsf{A}^{r,s}(\mathfrak{g}),$$

the condition $(d+\delta)Q = P(F)$ gives rise to the descent equations:

$$\begin{aligned}
dQ^{2k+1,0} &= P(F), \\
\delta Q^{2k+1,0} + dQ^{2k,1} &= 0, \\
\delta Q^{2k,1} + dQ^{2k-1,2} &= 0, \\
&\;\;\vdots \\
\delta Q^{1,2k} + dQ^{0,2k+1} &= 0, \\
\delta Q^{0,2k+1} &= 0.
\end{aligned}$$

These equations define the elements $Q^{2k+1-p,p}$ as δ-cocycles modulo d for $0 \leq p \leq 2k+1$, i.e. $Q^{2k+1-p,p} \in H^{2k+1-p,p}(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$.

The achievement of the previous section was the observation that the cohomology classes thus defined are related by

$$[Q^{2k-p,p+1}] = \partial([Q^{2k+1-p,p}]) \quad \text{for } 0 \leq p \leq 2k,$$

with $\partial : H^{2k+1-p,p}(\delta\text{-mod } d) \to H^{2k-p,p+1}(\delta\text{-mod } d)$ the connecting homomorphism. We will formalize this result in the following lemma.

Lemma 6.1.4 The procedure described above provides a well-defined map $j^{\,k,p+1} : (S^{k+1}\mathfrak{g}^*)_{\text{inv}} \to H^{2k+1-p,p}(\delta\text{-mod } d)$ given by

$$j^{\,k,p+1} : P \mapsto [Q^{2k-p,p+1}],$$

for $0 \leq p \leq 2k+1$, and we have $j^{\,k,p+1} = \partial \circ j^{\,k,p}$.

Proof: (This is a lemma described at p. 561, [6])
That $j^{\,k,p+1}$ is well-defined follows from the following observation: let $Q, K \in \mathsf{A}^{2k+1}(\mathfrak{g})$ such that $(d+\delta)Q = (d+\delta)K = P(F)$. Then $(d+\delta)(Q-K) = 0$ so there exists an element $L \in \mathsf{A}^{2k}(\mathfrak{g})$ such that $(d+\delta)L = Q-K$ by Theorem 5.3.2. This implies $[Q^{2k+1-p,p}] = [K^{2k+1-p,p}] \in H^{2k+1-p,p}(\delta\text{-mod } d)$ for $0 \leq p \leq 2k+1$. That $j^{\,k,p+1} = \partial \circ j^{\,k,p}$ follows from Lemma 6.1.2.

Now let us look at the start of the cohomological chain. It is the cohomology class of $H^{2k,1}(\delta\text{-mod } d)$ that is of interest to us, since it was linked to the δ-modulo d cohomology class $H^{2k,1}(\mathcal{B}, \delta\text{-mod } d)$ by the isomorphism described in §5.3.2. Theorem 6.1.2 parts (3)-(4) tells us

$$H^{2k+1,0}(\delta\text{-mod } d) \cong (S^{k+1}\mathfrak{g}^*)_{\text{inv}} \quad \text{and} \quad H^{2k+1,0}(\delta\text{-mod } d) \cong H^{2k,1}(\delta\text{-mod } d).$$

We conclude $H^{2k,1}(\delta\text{-mod } d) \cong (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$, and hence it is no surprise that the map $j^{\,k,0} : (S^{k+1}\mathfrak{g}^*)_{\text{inv}} \to H^{2k,1}(\delta\text{-mod } d)$ is an isomorphism. It is the content of the following theorem.

Theorem 6.1.3 Surjectivity and injectivity theorem of the descent method in $\mathsf{A}(\mathfrak{g})$. The map

$$j^{\,k,1} : (S^{k+1}\mathfrak{g}^*)_{\text{inv}} \to H^{2k,1}(\delta\text{-mod } d)$$

given by

$$j^{\,k,1} : P \mapsto [Q^{2k,1}]$$

(with $[Q^{2k,1}]$ defined as above) is an isomorphism.

Proof: We know $H^{2k+1,0}(\delta\text{-mod } d) \cong H^{2k,1}(\delta\text{-mod } d)$ by Theorem 6.1.2 (4), thus we only need to prove the map $j^{\,k,0} : (S^{k+1}\mathfrak{g}^*)_{\text{inv}} \to H^{2k+1,0}(\delta\text{-mod } d)$ is an isomorphism.

Starting from an invariant polynomial $P \in (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$ we could always obtain a cohomology class in $H^{2k+1,0}(\delta\text{-mod } d)$; the question to answer is if, starting with a cohomology class $Q^{2k+1,0} \in H^{2k+1,0}(\delta\text{-mod } d)$, there exists a polynomial $P \in (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$ such that $dQ^{2k+1,0} = P(F)$. The answer to this question is positive.

As the zeroth homology space of $(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}), \delta)$, the space $H^{2k+1,0}(\delta\text{-mod } d)$ just consists of the δ-cocycles modulo d in $(\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))^{2k+1,0}$. Since there are no cohomologous elements, every cohomology class $[Q^{2k+1,0}] \in H^{2k+1,0}(\delta\text{-mod } d)$ has just one unique representative $\langle Q^{2k+1,0}\rangle \in (\mathsf{A}(\mathfrak{g})/d\mathsf{A}(\mathfrak{g}))^{2k+1,0}$. By the nilpotency of d ($d^2 = 0$) the d-differential of this class, $d(\langle Q^{2k+1,0}\rangle) = dQ^{2k+1,0}$, is a well-defined element of $\mathsf{A}^{2k+2,0}(\mathfrak{g})$. Now, $dQ^{2k+1,0} \in \mathsf{A}^{2k+2,0}(\mathfrak{g})$ is a d-cocycle (by $d^2 = 0$) as well as an δ-cocycle: $\delta dQ^{2k+1,0} = -d(\delta Q^{2k+1,0}) = -d(0) = 0$. We now refer to Corollary 5.3.1 to Theorem 5.3.3, which stated that

$$H^{2k+2,0}(\mathsf{A}(\mathfrak{g}), \delta) = Z(\mathsf{A}^{2k+2,0}(\mathfrak{g}), \delta) = \mathbb{1} \otimes (S^{k+1}\mathfrak{g}^*)_{\text{inv}} \otimes \mathbb{1} \otimes \mathbb{1}.$$

($Z(\mathsf{A}^{2k+2,0}(\mathfrak{g}), \delta)$ denoting the δ-cocycles in $\mathsf{A}^{2k+2,0}(\mathfrak{g})$.) We conclude $dQ^{2k+1,0} \in \mathsf{A}^{2k+2,0}(\mathfrak{g})$ is of the form $\mathbb{1} \otimes P \otimes \mathbb{1} \otimes \mathbb{1}$ for some $P \in (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$; i.e. $dQ^{2k+1,0} = P(F)$ for this $P \in (S^{k+1}\mathfrak{g}^*)_{\text{inv}}$. This finished our proof.

It is this theorem that states the surjectivity and injectivity of the descent method in $\mathsf{A}(\mathfrak{g})$, and we will discuss the implications of the theorem in the last two sections of this thesis.

6.2 Remarks

We have almost come to the end of this thesis, but before going on to the conclusion, we would like to discuss two topics that might otherwise remain unclear. First we look at a question left open up until now: we address the case of the possible non-triviality of the ambient principal bundle $P(G, M)$ and investigate what is left of the cohomological descent method in that case. We try to indicate what, in our opinion, is unsatisfactory in Dubois-Violette's treatment of this case and sketch his main line of argument. Second, we briefly comment on some results obtained by Dubois-Violette in the articles [6][7] which we have not included in our thesis.

6.2.1 Non-triviality of the bundle $P(G, M)$

For a non-trivial bundle $P(G, M)$ on can still set up a cohomological descent method, as is sketched at the end of §7 in Kastler and Stora [11] and §10.9 in de Azcárraga and Izquierdo [2]. However, the only article that truly goes into this case instead of just sketching how things could be done is the article by Mañes, Stora and Zumino [13]. Since gauge potentials are not globally defined anymore on a non-trivial bundle, one supposes all elements formerly depending on $a \in a_{pot}(M)$ now depend on a connection $\omega \in \mathscr{C}(P)$. To be able to use the cohomological descent method one needs to fix a background connection ω_0: the physical interpretation of this is that there exists a background field (given by ω_0). One also assumes that all objects depending formerly on $a \in a_{pot}(M)$ now also depend on ω_0: for instance, the quantum action functional $\Gamma(a, \psi; \xi)$ turns into $\Gamma(\omega, \omega_0, \psi; \xi)$. In this case the cohomology space of candidate anomalies is still $H^1(Lie(\mathscr{G}), \mathcal{P}_{\text{loc}})$, but these anomalies are anomalies *in the presence of a background field* ω_0.

In order to generalize his constructions to non-trivial principal bundles $P(G, M)$ Dubois-Violette introduces the following complexes:

1. The complex $\widetilde{\mathcal{B}}^{*,*}(P)$. This complex is very similar to the $\widetilde{\mathcal{B}}^{**}$ complex, only with the space of gauge potentials $a_{pot}(M)$ replaced with the space

$\mathscr{C}(P)$ of connections on P, and $\Omega(M)$ replaced by $\Omega(P)$. The complex $\widetilde{\mathcal{B}}^{*,*}(P)$ is also a bigraded algebra, whose homogeneous space $\widetilde{\mathcal{B}}^{r,s}(P)$ of bidegree (r,s) is defined as the space of differential operators of $\mathscr{C}(P) \times (\mathit{Lie}(\mathscr{G}))^s$ in $\Omega^r(P)$ which are polynomial in $\mathscr{C}(P)$ (i.e. as a function of $\mathcal{A} \in \mathscr{C}(P)$ only depending on finitely many derivatives of $\mathcal{A}$) and s-linear antisymmetric in $(\mathit{Lie}(\mathscr{G}))^s$. We thus define the complex $\widetilde{\mathcal{B}}^{*,*}(P)$ as

$$\widetilde{\mathcal{B}}^{r,s}(P) = \{\, \omega : \mathscr{C}(P) \times (\mathit{Lie}(\mathscr{G}))^s \to \Omega^r(P) \mid \text{with } \omega \text{ polynomial in } \mathscr{C}(P) \text{ and multilinear \& antisymmetric in } (\mathit{Lie}(\mathscr{G}))^s \,\}.$$

2. The complex $\widetilde{\mathcal{B}}^{*,*}(M{\times}G)$. This is just the complex $\widetilde{\mathcal{B}}(P)$ in the case that the principal bundle $P(G,M)$ is trivial ($P = M{\times}\, G$).

For these complexes one defines a product and differential d and δ in exactly the same way as for $\widetilde{\mathcal{B}}^{*,*}$ (only here one uses the representation of $\mathit{Lie}(\mathscr{G})$ on $\mathscr{C}(P)$ described in §2.4.1). Notice that for $\widetilde{\mathcal{B}}^{*,*}(P)$ one can define the elements A and χ satisfying the B.R.S. relations in almost the same way as we did for $\widetilde{\mathcal{B}}^{*,*}$. For $P(G,M)$ non-trivial we had

$$\mathscr{G} \cong \mathit{Map}_{\mathit{Ad}}(P,G) \quad \text{and} \quad \mathit{Lie}(\mathscr{G}) \cong \mathit{Map}_{\mathit{Ad}}(P,\mathit{Lie}(G)).$$

We can thus define $A \in \mathit{Lie}(G) \otimes \widetilde{\mathcal{B}}^{1,0}(P)$ interpreted as map $A : \mathscr{C}(P) \to \mathit{Lie}(G) \otimes \Omega^1(P)$ as

$$A : \omega \mapsto \omega, \qquad \omega \in \mathscr{C}(P).$$

(Since $\mathscr{C}(P) \subset \mathit{Lie}(G) \otimes \Omega^1(P)$).

Because we know $\mathit{Map}_{\mathit{Ad}}(P,\mathit{Lie}(G)) \subset \mathit{Lie}(G) \otimes \Omega^0(P)$ we can set $\chi \in \mathit{Lie}(G) \otimes \widetilde{\mathcal{B}}^{0,1}(P)$, interpreted as a map $\chi : \mathscr{C}(P) \times \mathit{Lie}(\mathscr{G}) \to \mathit{Lie}(G) \otimes \Omega^0(P)$, as

$$\chi : (\omega,\xi) \mapsto \xi, \qquad \xi \in \mathit{Lie}(\mathscr{G}).$$

The element A and χ thus defined will satisfy the B.R.S. relations and $\widetilde{\mathcal{B}}^{*,*}(P)$ (and $\widetilde{\mathcal{B}}^{*,*}(M{\times}G)$) form *B.R.S. algebras (over Lie(G))*.[4] Similar to the $\widetilde{\mathcal{B}}^{*,*}$ complex one can define sub-complexes $\mathcal{B}^{*,*}(P) \subset \widetilde{\mathcal{B}}^{*,*}(P)$ and $\mathcal{B}^{*,*}(M{\times}G) \subset \widetilde{\mathcal{B}}^{*,*}(M{\times}G)$ as the sub-complexes generated by the elements A and χ. The sub-complexes are also B.R.S. algebras.

Being a B.R.S. algebra, there is a canonical homomorphism $\Phi_{\mathsf{A}} : \mathsf{A}(\mathit{Lie}(G)) \to \mathcal{B}^{*,*}(P)$ by Theorem 5.3.1. Now the difficulty lies in the fact that Dubois-Violette claims that the isomorphism of $\mathsf{A}(\mathit{Lie}(G))$ to $\mathcal{B}$ can be extended to $\mathcal{B}(P)$ without proving this result. First, he remarks that the isomorphism $\mathsf{A}^{r,s}(\mathit{Lie}(G)) \cong \mathcal{B}^{r,s}$ for $r \leq \dim M$ extends to an isomorphism

$$\mathsf{A}^{r,s}(\mathit{Lie}(G)) \cong \mathcal{B}^{r,s}(M{\times}G), \qquad r \leq \dim M.$$

This can be proved reasonably easy and is not so remarkable in view of Proposition 2.2.3, which stated that for $P(G,M)$ trivial $a_{pot}(M) \cong \mathscr{C}(M \times G)$: hence we can substitute $a_{pot}(M)$ by $\mathscr{C}(M \times G)$ in the $\mathcal{B}^{*,*}$ complex and retain the isomorphism.

[4]In fact one can show that $\widetilde{\mathcal{B}}^{*,*}(P)$ forms a B.R.S. $\mathit{Lie}(G)$-operation cf. Definition 5.1.3.

However, the next step employed by Dubois-Violette is saying that the isomorphism $\mathsf{A}^{r,s}(Lie(G)) \cong \mathcal{B}^{r,s}(M\times G)$ can be extended to an isomorphism

$$\mathsf{A}^{r,s}(Lie(G)) \cong \mathcal{B}^{r,s}(P), \qquad r, s \in \mathbb{N}, r \leq \dim M, \tag{6.4}$$

(Th. 11 in [6]) with the following justification as proof:

> Since all this comes from local considerations (in fact jets of finite orders) and since a principal G-bundle is locally trivializable one has the following theorem. ([6], p. 544.)

As we noted earlier, the lack of an explicit proof makes this statement rather unsatisfactory. If one looks closely at the proof of Theorem 5.3.4, one observes that if one wishes to use the same structure of proof for (6.4) one needs to construct a global connection ω fulfilling a whole set of conditions; that such an $\omega \in \mathscr{C}(P)$ exists is by no means obvious.

Finally, we would like to remark that even if the isomorphism of (6.4) would hold, then the result of Dubois-Violette for the surjectivity and injectivity of the descent method would have the same limited validity as the trivial bundle case due to the restriction of Dubois-Violette to the subcomplex $\mathcal{B}^{*,*}(P) \subset \widetilde{\mathcal{B}}^{*,*}(P)$. We will discuss the implications of this restriction in our Conclusion (§ 6.3).

6.2.2 Further results by Dubois-Violette

While studying the proofs presented in this chapter, one could possibly wonder about the use of homological algebra. The results needed for Theorem 6.1.3 could probably be attained without resorting to homological algebra. However, Dubois-Violette puts the homological algebra methods and results of Theorem 6.1.2 to a much broader use. His strategy is as follows: using Theorem 6.1.2 he constructs what is called a *exact couple*; that is, an exact triangle of vector spaces with two different vector spaces (one appearing twice) and linear mappings between them. From such an exact couple one can construct a *derived exact couple* and, repeating the process, one obtains for each $r \in \mathbb{N}$ the r^{th} *derived exact couple*. From this chain of derived exact couples one can construct a *spectral sequence*: that is, a sequence of differential spaces $(E_r, d_r)_{r\in\mathbb{N}}$ such that $E_{r+1} = H(E_r, d_r)$. Using all this theory, Dubois-Violette shows that for the δ-modulo d cohomology of $\mathsf{A}(\mathfrak{g})$ one has[5]

$$H_{+}^{ev,*}(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d) \cong \bigoplus_{r\in\mathbb{N}} p_r(E_r)\,,$$

where the p_r are linear mappings and the E_r are the vector spaces of a certain spectral sequence.

In case of a reductive Lie algebra[6] it is possible to give a explicit form of the spaces $p_r(E_r)$ such that one can 'compute' the $H(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$ cohomology space.

[5] [6], p. 559.

[6] A *reductive* Lie algebra is the direct product of a semi-simple Lie algebra and an abelian Lie algebra.

6.3 Conclusion

In this thesis we have attempted to give a thorough treatment of the generalization of various concepts from the principal bundle setting to a more algebraic theory of Lie algebra operations. These generalization were put forward in the articles by the physicists Dubois-Violette [6], Kaster and Stora [11] without too many proofs or motivational remarks, and we hope to have clarified most of these definitions. In particular, we provided a proof of the generalized Weil homomorphism in the algebraic setting, which was missing in [6]. On these generalizations proposed by Dubois-Violette we only had one point of critique: it is unclear to the author if the generalization of the group of gauge transformations $\mathrm{Aut}_{\mathcal{B}}(\mathscr{A})$ for a $\mathfrak{g}$-operation $\mathscr{A}$ (Definition 2.5.1) supplies the usual group of gauge transformations $\mathscr{G}$ if one takes the key example from which the generalizations arise (that is, $\mathscr{A} = \Omega(P)$ for a principal bundle $P(G, M)$). To provide a meaningful generalization this should be the case, of course.

The second part of our thesis concerned the cohomological descent method. We followed the constructions proposed in Dubois-Violette [6] and proved an isomorphism theorem (Theorem 6.1.3) which showed the cohomological descent method is surjective (and injective) under certain conditions. The most important question we had concerned the validity of this result and we will address this question now.

At the beginning of Chapter 5 we gave a brief outline of the argument applied by Dubois-Violette, which we will recapitulate here. After identifying the cohomology space $H^1(Lie(\mathscr{G}), \mathcal{P}_{\mathrm{loc}})$ as the space of candidate anomalies, we showed that we could equivalently consider the cohomology space $H^{2k,1}(\widetilde{\mathcal{B}}, \delta\text{-mod } d)$ in the δ-modulo d cohomology of the $\widetilde{\mathcal{B}}^{*,*}$ complex introduced by Dubois-Violette. In this complex we identified a subcomplex $\mathcal{B}^{*,*} \subset \widetilde{\mathcal{B}}^{*,*}$ and showed that it was isomorphic to the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ (with $\mathfrak{g} = Lie(G)$). As a consequence the δ-modulo d cohomology $H(\mathcal{B}, \delta\text{-mod } d)$ of $\mathcal{B}^{*,*}$ was isomorphic to $H(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$, the δ-modulo d cohomology of the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$.

In this chapter (i.e. Chapter 6) we discussed the cohomological descent method: an algorithm used to obtain elements of the $H^{2k,1}(\delta\text{-mod } d)$ cohomology space, starting off with an invariant polynomial $P \in (S^{k+1}\mathfrak{g}^*)_{\mathrm{inv}}$. We saw that in the Weil-B.R.S. algebra $\mathsf{A}(\mathfrak{g})$ we could accommodate the cohomological descent and for $\mathsf{A}(\mathfrak{g})$ this method yielded all cohomology classes of $H^{2k,1}(\mathsf{A}(\mathfrak{g}), \delta\text{-mod } d)$ (Theorem 6.1.3). By the established isomorphism, this result also holds for the $\mathcal{B}^{*,*}$ complex. The question left open is how to interpret this result.

As remarked before, the inclusion $i : \mathcal{B} \to \widetilde{\mathcal{B}}$ yields a map

$$i^\sharp : H(\mathcal{B}/d\mathcal{B}, \delta) \to H(\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}, \delta),$$

which need not be injective nor surjective. Thus, at first sight, though the space $H^{2k,1}(\widetilde{\mathcal{B}}/d\widetilde{\mathcal{B}}, \delta)$ identifies with $H^1(Lie(\mathscr{G}), \mathcal{P}_{\mathrm{loc}})$ (the cohomology space of candidate anomalies), the result of Dubois-Violette obtained for $H(\mathcal{B}/d\mathcal{B}, \delta)$ has no direct implications for the surjectivity of the descent method in general. Although the larger complex $\widetilde{\mathcal{B}}$ is also a B.R.S. algebra, and hence there exists a canonical unique homomorphism of $\mathsf{A}(\mathfrak{g})$ into $\widetilde{\mathcal{B}}$ it is clear from the proof of Theorem 5.3.4 that this will never be an isomorphism; in fact the subcomplex $\mathcal{B}$ could just as well be defined as the image of $\mathsf{A}(\mathfrak{g})$ in $\widetilde{\mathcal{B}}$. The results obtained for $\mathcal{B}^{*,*}$ are therefore not easily transferred to the larger $\widetilde{\mathcal{B}}^{*,*}$ complex, which makes the application of Dubois-Violette's surjectivity theorem quite difficult.

However, much depends on the exact form of the anomalies: if one could show that the cocycles in $H^1(\mathit{Lie}\,(\mathscr{G}), \mathcal{P}_{\mathrm{loc}})$ representing anomalies correspond to elements of the $H^{2k,1}(\mathcal{B}\,, \delta\text{-mod } d)$ cohomology space (instead of the more general $H^{2k,1}(\widetilde{\mathcal{B}}\,, \delta\text{-mod } d)$ cohomology space), then Dubois-Violette's result *would* be applicable. Referring to our brief sketch at the start of Chapter 5, this would mean that the variation of the quantum action functional $\Delta = \delta\Gamma(a, \psi; \xi)$ and the elements $\delta\Gamma^{\mathrm{loc}}$ that could be added by finite renormalization, should all be local functionals of $a \in a_{pot}(M)$ and $\xi \in \mathit{Lie}\,(\mathscr{G})$ such that in the differential forms that make up the local functionals only a, ξ and their d and δ derivatives appear. This would imply that the elements corresponding to $\Delta(a, \psi; \xi)$ and $\delta\Gamma^{\mathrm{loc}}$ in $\widetilde{\mathcal{B}}^{*,*}$ are in fact in $\mathcal{B}^{*,*}$ and hence that it would be the cohomology $H^{2k,1}(\mathcal{B}\,, \delta\text{-mod } d)$ that is of interest for anomalies.

Dubois-Violette himself has the following to say about this (in [7]):

> We have computed all possible anomalous terms which are (exterior) products of gauge potential 1-forms, ghost field and their d and δ differentials. It would be desirable to extend these results to more general expressions containing arbitrary derivatives of the fields since, in principle, such expressions could occur in some models (although no non-trivial examples are known up to now). We shall apply our results to specific examples in a forthcoming publication. (Conclusion of [7], pp. 121-122)

And in [6], following the definition of the subcomplex $\mathcal{B}^{*,*}$ he explains (here $\tilde{H}(\delta, \mathrm{mod}\ (d)) = H(\widetilde{\mathcal{B}}\,, \delta\text{-mod } d)$)

> Of course the elements of $\mathcal{B}^{*,*}$ are very special types of differential operators, for instance they are first order at most, and it would be nice to compute $\tilde{H}(\delta, \mathrm{mod}\ (d))$; the $\tilde{H}^{k,0}(\delta \text{ mod } (d))$ contain more elements than the ones coming from $\mathcal{B}^{k,0}$, the Yang-Mills lagrangian for instance, but one may expect that it is essentially all what is lost by working with $\mathcal{B}^{*,*}$ instead of $\widetilde{\mathcal{B}}^{*,*}$. ([6], p. 529)

Our conclusion is that it depends on the physical model that is used and the exact form of the anomalies whether the result of Dubois-Violette (concerning the surjectivity and injectivity of the descent method) is applicable. The answer to this question lies beyond our (mathematically oriented) knowledge and hence we leave it to physicists to judge whether the results of Dubois-Violette on the cohomological descent method are conclusive.

Appendix A

Preliminaries

A.1 Algebras

In this section we define the different kinds of algebras we will encounter. As in Dubois-Violette [6], all definitions are standard except that all algebras are assumed to be *associative* and *unital*. So if we speak of an algebra we mean an *associative and unital* algebra.[1]

Definition A.1.1 Algebras.

An **algebra** $\mathscr{A}$ is a vector space over a field $\mathbb{K}$ (with usually $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), together with a bilinear multiplication $\cdot : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$. We will assume all our algebras to be **associative**, i.e. $x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \forall x, y, z \in \mathscr{A}$. We also assume they are **unital**, i.e. $\exists 1 \in \mathscr{A} : 1 \cdot x = x \cdot 1 = x \quad \forall x \in \mathscr{A}$.

On a **graded algebra** there exists a direct sum decomposition

$$\mathscr{A} = \bigoplus_{n \in \mathbb{N}} \mathscr{A}^n ,$$

and the multiplication satifies

$$\mathscr{A}^n \cdot \mathscr{A}^m \subset \mathscr{A}^{n+m}.$$

Elements of $\mathscr{A}^n$ are called **homogeneous elements of degree** n, and we will write $deg\, x = n$ for $x \in \mathscr{A}^n$.

A **graded-commutative algebra** is a graded algebra with a multiplication that satisfies

$$x \cdot y = (-1)^{nm} y \cdot x, \qquad x \in \mathscr{A}^n, y \in \mathscr{A}^m,$$

or, which is the same, $xy = (-1)^{(deg\, x)(deg\, y)} yx$.

Definition A.1.2 Graded algebra endomorphisms.

Consider a graded algebra $\mathscr{A} = \oplus \mathscr{A}^n$. A linear mapping $L : \mathscr{A} \to \mathscr{A}$ is called **homogeneous of degree** k ($k \in \mathbb{Z}$) if

$$L[\mathscr{A}^n] \subset \mathscr{A}^{n+k} \qquad \forall n \in \mathbb{N}.$$

[1]References for this section: [6], [10] (Vol. II, Ch. 0). For differential graded Lie algebras: see appendix [11].

A **derivation** of $\mathscr{A}$ is a homogeneous linear mapping of even degree of $\mathscr{A}$ into itself (an *endomorphism*) which satisfies the **Leibniz rule**; also known as the **derivation property**:

$$\theta(x \cdot y) = \theta(x) \cdot y + x \cdot \theta(y), \qquad \forall x, y \in \mathscr{A}. \tag{A.1}$$

An **anti-derivation**[2] is a homogeneous linear endomorphism δ of $\mathscr{A}$ of odd degree, satisfying

$$\delta(x \cdot y) = \delta(x) \cdot y + (-1)^n x \cdot \delta(y), \qquad \forall x \in \mathscr{A}^n, y \in \mathscr{A}. \tag{A.2}$$

Derivation and anti-derivations are also called graded derivations. We denote the set of graded derivations of degree k on $\mathscr{A}$ with $\mathrm{Der}^{(k)}(\mathscr{A})$; so if $\alpha \in \mathrm{Der}^{(k)}(\mathscr{A})$ then it is a *derivation* if k is even, and an *anti-derivation* if k is odd. We can merge equations (A.1) and (A.2) together in a general **graded derivation property** for a graded derivation $\alpha \in \mathrm{Der}^{(k)}(\mathscr{A})$ of degree k:

$$\alpha(x \cdot y) = \alpha(x) \cdot y + (-1)^{nk} x \cdot \alpha(y), \qquad \forall x \in \mathscr{A}^n, y \in \mathscr{A}. \tag{A.3}$$

Definition A.1.3 Differential spaces and (co)homology
Let $\mathcal{V}$ be a vector space, and $d : \mathcal{V} \to \mathcal{V}$ a linear mapping. If d satisfies $d^2 = 0$ then $\mathcal{V}$ is called a **differential (vector) space**. If $\mathcal{V} = \mathscr{A}$ is an algebra and d is a derivation, i.e. d satisfies (A.1), then $\mathscr{A}$ is called a **differential algebra**. In any case d is called the **differential**.

Let $(\mathcal{V}, d)$ be a differential space. We can define the **homology** $H(\mathcal{V}, d)$ of $\mathcal{V}$ by $H(\mathcal{V}, d) = (\mathrm{Ker}\ d)/(\mathrm{Im}\ d)$. In general this will be a vector space. If $\mathcal{V} = \mathscr{A}$ is a differential algebra, then $H(\mathscr{A}, d)$ will be an algebra, and is called the **cohomology** of $\mathscr{A}$. When $\mathscr{A}$ is a graded algebra the definitions of the differential and cohomology are slightly different, so we treat this in the next definition.

For a **graded vector space** $\mathcal{V} = \oplus_{\mathbb{N}} \mathcal{V}^n$, a differential is nilpotent linear map $d : \mathcal{V} \to \mathcal{V}$, which decomposes as a set of nilpotent linear maps $d_n : \mathcal{V}^n \to \mathcal{V}^{n+1}$. The pair $(\mathcal{V}, d)$ is also called a **differential complex**. The induced cohomology is also a graded vector space given by $H(\mathcal{V}, d) = \oplus_{\mathbb{N}} H^n(\mathcal{V}, d)$ with $H^n(\mathcal{V}, d) = (\mathrm{Ker}\ d_n)/(\mathrm{Im}\ d_{n-1})$.

Definition A.1.4 Graded differential algebras. Cohomology algebras.
Let $\mathscr{A}$ be a graded algebra. A **differential** on $\mathscr{A}$ is an anti-derivation of degree $+1$ which satisfies $d^2 = 0$. $\mathscr{A}$ together with a differential d is called a **graded differential algebra**, or GDA for short.

Now we can define the cohomology of $\mathscr{A}$ in the usual way. If $dx = 0$ ($x \in \mathrm{ker} d$) for $x \in \mathscr{A}$ we call x a *cocycle*. The set of cocycles $Z(\mathscr{A})$ is a graded subalgebra of $\mathscr{A}$, with $Z^n(\mathscr{A}) = Z(\mathscr{A}) \cap \mathscr{A}^n$. The set of *coboundaries* is $B(\mathscr{A}) = d\mathscr{A}$. It is a graded two-sided ideal in $\mathscr{A}$ with grading $B^n(\mathscr{A}) = B(\mathscr{A}) \cap \mathscr{A}^n$. The graded algebra

$$H(\mathscr{A}) = \bigoplus_{n \in \mathbb{N}} H^n(\mathscr{A}), \qquad H^n(\mathscr{A}) = Z^n(\mathscr{A})/B^n(\mathscr{A}),$$

is called the **cohomology algebra** of $\mathscr{A}$, with $H^n(\mathscr{A})$ the n-th *cohomology space*. If $\mathscr{A}$ is a *graded-commutative* differential algebra, then $H(\mathscr{A})$ is graded-commutative as well.

[2] Sometimes called a *skew-derivation*.

Definition A.1.5 The tensor product of graded algebras.
Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be graded algebras. We can equip the vector space[3]$\mathscr{A}_1 \otimes \mathscr{A}_2$ with the grading

$$(\mathscr{A}_1 \otimes \mathscr{A}_2)^n = \bigoplus_{m=0}^{n} \mathscr{A}_1^m \otimes \mathscr{A}_2^{n-m} \,, \tag{A.4}$$

and the product

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (-1)^{mn} x_1 y_1 \,, \otimes x_2 y_2 \tag{A.5}$$

for $x_1 \in \mathscr{A}_1, x_2 \in \mathscr{A}_1^m, y_1 \in \mathscr{A}^n, y_2 \in \mathscr{A}$. With this grading and product $\mathscr{A}_1 \otimes \mathscr{A}_2$ has become a graded algebra called **the tensor product of the graded algebras $\mathscr{A}_1$ and $\mathscr{A}_2$**. If $\mathscr{A}_1$ and $\mathscr{A}_2$ were graded-commutative, so is $\mathscr{A}_1 \otimes \mathscr{A}_2$.

If $\mathscr{A}_1$ and $\mathscr{A}_2$ are *differential* algebras with differentials d_1 and d_2 respectively, we can make $\mathscr{A}_1 \otimes \mathscr{A}_2$ a differential algebra by defining:

$$d(x_1 \otimes x_2) = d_1 x_1 \otimes x_2 + (-1)^n x_1 \otimes d_2 x_2 \,, \tag{A.6}$$

for $x_1 \in \mathscr{A}_1$ and $x_2 \in \mathscr{A}_2$. For cohomology the Künneth formula holds, which expresses the naturality of the cohomology functor w.r.t. the algebra tensor product:

$$H(\mathscr{A}_1 \otimes \mathscr{A}_2) = H(\mathscr{A}_1) \otimes H(\mathscr{A}_2). \tag{A.7}$$

In particular, one has $H^n(\mathscr{A}_1 \otimes \mathscr{A}_2) = \bigoplus_{m=0}^{n} H^m(\mathscr{A}_1) \otimes H^{n-m}(\mathscr{A}_2)$.

If one calls to mind the algebra $\Omega(M)$ of (real or complex valued) differential forms on a manifold M, one notices that this a graded-commutative differential algebra, with the wedge product $\wedge$ and exterior differential d. The cohomology $H(\Omega(M))$ is the *de Rham* cohomology, and often denoted $H^*_{DR}(M)$. One of the main aims of this thesis is to translate the machinery available for the algebra $\Omega(P)$ of differential forms on the total space P of a principal bundle $P(G, M)$ to arbitrary algebras $\mathscr{A}$.

Definition A.1.6 Connected, free, minimal and contractible algebras.
Let $\mathscr{A}$ be a graded-commutative algebra over $\mathbb{K}$. If $\mathscr{A}^0 = \mathbb{K}$ and $\mathscr{A} = \mathbb{K} \oplus \mathscr{A}^+$ with $\mathscr{A}^+ = \oplus_{n \geq 1} \mathscr{A}^n$ we call $\mathscr{A}$ **connected**. This terminology comes from the example above, when $\mathscr{A} = \Omega(M)$ and $H^0_{DR}(M) = \mathbb{K}^k$, with k the number of connected components of the manifold M.

A connected graded-commutative algebra $\mathscr{A}$ is called **free** if $\mathscr{A}$ is finitely generated by a set of homogeneous elements $\{e_\alpha\}$ of $\mathscr{A}^+$ which are free of algebraic relations except for graded-commutativity. If so, every element of $\mathscr{A}^+$ can be written as a linear combination of products of the e_α's.

Let $\mathscr{A}$ be a graded-commutative differential algebra which is connected and free. If

$$d\mathscr{A} \subset \mathscr{A}^+ \cdot \mathscr{A}^+ \tag{A.8}$$

we call $\mathscr{A}$ **minimal**.

Next we consider the graded-commutative differential algebra $\mathscr{C}(x, dx)$ generated by one single element x. It is obviously free. If x has odd degree $x^2 = 0$

[3] If $\mathscr{A}_1$ and $\mathscr{A}_2$ are algebras over the field $\mathbb{K}$, the tensor product $\mathscr{A}_1 \otimes_{\mathbb{K}} \mathscr{A}_2$ is understood. We will omit this notation, since confusion is unlikely.

because of graded-commutativity. A basis for $\mathscr{C}(x, dx)$ is then given by the elements

$$\{1, x, dx, xdx, dx^2, \ldots, (dx)^n, x(dx)^n, \ldots\}.$$

If the degree of x is even, dx has odd degree, so $(dx)^2 = 0$. In that case a basis is the set

$$\{1, x, dx, x^2, xdx, \ldots, x^{n+1}, x^n dx, \ldots\}.$$

These two cases have in common that all cocycles in the algebra $\mathscr{C}(x, dx)$ are coboundaries, whatever the degree of the generating element x. So the cohomology $H(\mathscr{C}(x, dx))$ satisfies

$$H^0(\mathscr{C}(x, dx)) = \mathbb{K}, \qquad H^n(\mathscr{C}(x, dx)) = \{0\}, \quad n \geq 1. \tag{A.9}$$

In [6] a **contractible** differential algebra $\mathscr{C}$ is defined as the tensor product of algebras of the type above:

$$\mathscr{C} = \mathscr{C}(x_1, dx_1) \otimes \ldots \otimes \mathscr{C}(x_p, dx_p). \tag{A.10}$$

For such algebra we also have (A.9), i.e. $H^0(\mathscr{C}) = \mathbb{K}$ and $H^+(\mathscr{C}) = \{0\}$ because of the Künneth formula (A.7). This terminology is also derived from the case $\mathscr{C} = \Omega(M)$ with M a contractible (homotopically trivial) manifold, in which case $H^+_{DR}(M) = \{0\}$.

To show a GDA $(\mathscr{A}, d)$ has trivial cohomology, one can construct a **contracting homotopy** $k : \mathscr{A}^n \to \mathscr{A}^{n-1}$. It is an anti-derivation of degree -1, for which one has $kd + dk = id_{\mathscr{A}}$. Equipped with a contracting homotopy one can prove every cocycle $\alpha \in \mathscr{A}$ $(d\alpha = 0)$ is a coboundary by showing $d(k(\alpha)) = \alpha$.

Next we define the natural notions of algebra homomorphisms and isomorphisms, and the corresponding notions for differential algebras and graded algebras.

Definition A.1.7 Algebra morphisms.

Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras over the same field $\mathbb{K}$. An **algebra homomorphism** $\Psi : \mathscr{A} \to \mathscr{B}$ is a $\mathbb{K}$-linear mapping that is natural with respect to the algebra multiplication:

$$\Psi(x \cdot y) = \Psi(x) \cdot \Psi(y), \qquad \forall x, y \in \mathscr{A}.$$

An algebra **isomorphism** is a bijective algebra homomorphism. An algebra **automorphism** is an isomorphism of an algebra to itself.

For differential algebras these notions are defined in the same way, only they should be commute with the differential d. Let $(\mathscr{A}, d_A)$ and $(\mathscr{B}, d_B)$ be differential algebras. For $\Psi : \mathscr{A} \to \mathscr{B}$ we should have

$$\Psi(d_A\, \alpha) = d_B\, \Psi(\alpha).$$

For graded (differential) algebras the additional condition is that the homomorphisms should be homogeneous of degree zero.

Homomorphisms of differential algebras induce linear mappings in cohomology. In our example we have $\Psi^\sharp : H(\mathscr{A}, d_A) \to H(\mathscr{B}, d_B)$, defined by

$$\Psi^\sharp([\alpha]) = [\Psi(\alpha)].$$

One can easily verify this is a well-defined mapping. If the cohomology spaces are algebras, then $\Psi^\sharp$ will be an algebra homomorphism. If they are graded algebras then $\Psi^\sharp$ will be homogeneous of degree zero.

We denote the set of automorphisms of an algebra $\mathscr{A}$ with $\mathrm{Aut}(\mathscr{A})$.

Definition A.1.8 Bigraded algebras.

A **bigraded algebra** is an associative algebra $\mathscr{A}$ which has a bigrading

$$\mathscr{A} = \bigoplus_{(r,s)\in\mathbb{N}^2} \mathscr{A}^{r,s} ,$$

and a product which respects this bigrading

$$\mathscr{A}^{r,s} \cdot \mathscr{A}^{r',s'} \subset \mathscr{A}^{r+r',s+s'} , \qquad \forall r, r', s, s' \in \mathbb{N}.$$

We could also extend the definition to $\mathbb{Z}^2$-graded algebras, by allowing $(r, s) \in \mathbb{Z}^2$ instead of $\mathbb{N}^2$. However, in this thesis we deal with bigraded algebras $\mathscr{A}$ for which $\mathscr{A}^{r,s} = \{0\}$ if $r < 0$ or $s < 0$. We will take this as a general definition.

The definitions in Def. (A.1.1) and Def. (A.1.2) can be easily adjusted to this situation. An element $x \in \mathscr{A}^{r,s}$ is called **bihomogeneous of bidegree (r,s)**. If a linear mapping $L : \mathscr{A} \to \mathscr{A}$ satisfies

$$L[\mathscr{A}^{r,s}] \subset \mathscr{A}^{r+k,s+l} , \qquad \forall (r, s) \in \mathbb{N}^2,$$

it is called **bihomogeneous of degree (k,l)**.

A bigraded algebra allows a natural **total grading** for which it is a "normal" graded algebra. Take

$$\mathscr{A}^n = \bigoplus_{r+s=n} \mathscr{A}^{r,s} .$$

The notions related to normal graded algebras therefore also apply to bigraded algebras. The elements of $\mathscr{A}^n$ will be called **homogeneous of total degree n**, and an element $x \in \mathscr{A}^{r,s}$ of bidegree (r, s) thus has total degree $r + s$. Furthermore a bigraded algebra will be called *(bigraded) commutative* iff. it is graded-commutative with respect to the total grading.

Now consider the case that there are two anti-commuting anti-derivations $d^{1,0}$ and $d^{0,1}$ on the bigraded algebra $\mathscr{A}$, with $d^{1,0} : \mathscr{A}^{r,s} \to \mathscr{A}^{r+1,s}$ of bidegree $(1, 0)$ and $d^{0,1} : \mathscr{A}^{r,s} \to \mathscr{A}^{r,s+1}$ of bidegree $(0, 1)$ satisfying $(d^{1,0})^2 = (d^{0,1})^2 = 0$. If we define $d = d^{1,0} + d^{0,1}$ then d will be a anti-derivation of degree +1 with respect to the total degree:

$$d : \mathscr{A}^n \to \mathscr{A}^{n+1},$$

and since $d^{1,0}$ and $d^{0,1}$ anti-commute, i.e. $d^{1,0}d^{0,1} = -d^{0,1}d^{1,0}$, we have

$$\begin{aligned} d^2 &= (d^{1,0} + d^{0,1})(d^{1,0} + d^{0,1}) \\ &= (d^{1,0})^2 + d^{1,0}d^{0,1} + d^{0,1}d^{1,0} + (d^{0,1})^2 \\ &= 0. \end{aligned}$$

So d is a differential on $\mathscr{A}$ with respect to the total grading. If such $d^{1,0}$ and $d^{0,1}$ (and hence d) exist, $\mathscr{A}$ is called a **bigraded differential algebra**.

Notice that for a bigraded algebra $\mathscr{A}$ the subspaces $\mathscr{A}^{*,0} = \oplus\mathscr{A}^{n,0}$ and $\mathscr{A}^{0,*} = \oplus\mathscr{A}^{0,n}$ are closed under multiplication and hence are graded subalgebras. If $\mathscr{A}$ is a bigraded differential algebra, then $(\mathscr{A}^{*,0}, d^{1,0})$ and $(\mathscr{A}^{0,*}, d^{0,1})$ are both graded differential algebras with differential $d^{1,0}$ and $d^{0,1}$ respectively.

A.1.1 Differential graded Lie algebras

Definition A.1.9 Differential graded Lie algebra.
A *differential graded Lie algebra* $\mathscr{L}$, or DGLA for short, is a graded differential algebra (GDA) $\mathscr{L}$ which has a product $[\cdot,\cdot] : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ that satisfies

$$[x, y] = (-1)^{nm+1}[y, x], \qquad x \in \mathscr{L}^n, y \in \mathscr{L}^m, \tag{A.11}$$

and

$$(-1)^{np}[x, [y, z]] + (-1)^{mn}[y, [z, x]] + (-1)^{pm}[z, [x, y]] = 0, \tag{A.12}$$

with $x \in \mathscr{L}^n, y \in \mathscr{L}^m, z \in \mathscr{L}^p$.

Such a product is called a **graded Lie bracket** and is therefore denoted by a bracket instead of a dot $(\cdot)$. Equation (A.11) generalizes the normal antisymmetry of a Lie bracket on a Lie algebra, and property (A.12) is known as the **graded Jacobi identity**.

The key example of a differential graded Lie algebra is the algebra $\mathfrak{g}\otimes\Omega(M)$ of $\mathfrak{g}$-valued differential forms on a manifold M, with $\mathfrak{g}$ a Lie algebra. It is a special case of the following lemma (with $\mathscr{A} = \Omega(M)$).

Lemma A.1.1 Let $\mathscr{A}$ be a graded-commutative differential algebra (GCDA), with product $(\cdot)$ and differential d, and $\mathfrak{g}$ a Lie algebra. Then $\mathfrak{g} \otimes \mathscr{A}$ is a differential graded Lie algebra (DGLA), with the following grading

$$(\mathfrak{g} \otimes \mathscr{A})^n := \mathfrak{g} \otimes \mathscr{A}^n \qquad \forall n \in \mathbb{N}, \tag{A.13}$$

the following product $[\cdot,\cdot] : (\mathfrak{g} \otimes \mathscr{A}) \times (\mathfrak{g} \otimes \mathscr{A}) \to (\mathfrak{g} \otimes \mathscr{A})$

$$[X \otimes \alpha, Y \otimes \beta] = [X, Y] \otimes (\alpha \cdot \beta), \qquad X, Y \in \mathfrak{g},\ \alpha, \beta \in \mathscr{A}, \tag{A.14}$$

(the dot in $(\alpha \cdot \beta)$ denoting the product in $\mathscr{A}$);

and differential $d : (\mathfrak{g} \otimes \mathscr{A})^n \to (\mathfrak{g} \otimes \mathscr{A})^{n+1}$ defined by

$$d(X \otimes \alpha) = X \otimes (d\alpha), \qquad X \in \mathfrak{g},\ \alpha \in \mathscr{A}^n\,. \tag{A.15}$$

Remark: Note that we just defined the product and differential on elements $X \otimes \alpha \in \mathfrak{g} \otimes \mathscr{A}$, but a general element $S \in \mathfrak{g} \otimes \mathscr{A}$ is a finite sum of these elements:

$$S = \sum_i X_i \otimes \alpha_i\,, \qquad X_i \in \mathfrak{g},\ \alpha_i \in \mathscr{A}.$$

The product and differential are defined on these general elements by linearity.

Proof: First we prove d is a differential on $\mathfrak{g} \otimes \mathscr{A}$. We still have $d^2 = 0$ because this is true for d on $\mathscr{A}$. It is also obviously a homogeneous linear mapping of degree $+1$. We only need to show it satisfies the anti-derivation property; so let $A = X \otimes \alpha$, $B = Y \otimes \beta \in \mathfrak{g} \otimes \mathscr{A}$ with $X, Y \in \mathfrak{g}$ and $\alpha, \beta \in \mathscr{A}$, then

$$\begin{aligned}
d\ \ [A, B] &= d\ \ [X \otimes \alpha, Y \otimes \beta] \\
&= d\ \ [X, Y] \otimes (\alpha \cdot \beta) \\
&= [X, Y] \otimes\ \ d(\alpha \cdot \beta) \\
&= [X, Y] \otimes\ \ d\alpha \cdot \beta + (-1)^n \alpha \cdot d\beta \\
&= [X, Y] \otimes\ \ d\alpha \cdot \beta\ \ + (-1)^n [X, Y] \otimes\ \ \alpha \cdot d\beta \\
&= [X \otimes (d\alpha), Y \otimes \beta] + (-1)^n [X \otimes \alpha, Y \otimes (d\beta)] \\
&= [d(X \otimes \alpha), Y \otimes \beta] + (-1)^n [X \otimes \alpha, d(Y \otimes \beta)] \\
&= [dA, B] + (-1)^n [A, dB],
\end{aligned}$$

which is exactly (A.2) for the graded Lie bracket as product $(\cdot)$.

Next we show the product satisfies (A.11). Take again $A = X\otimes\alpha \in (\mathfrak{g}\otimes\mathscr{A})^n$, $B = Y\otimes\beta \in (\mathfrak{g}\otimes\mathscr{A})^m$ with $X, Y \in \mathfrak{g}$ and $\alpha\in\mathscr{A}^n, \beta\in\mathscr{A}^m$, then

$$\begin{aligned}
[A,B] &= [X\otimes\alpha, Y\otimes\beta]\\
&= [X,Y]\otimes \quad \alpha\cdot\beta\\
&= -[Y,X]\otimes \quad \alpha\cdot\beta\\
&= -[Y,X]\otimes \quad (-1)^{mn}\beta\cdot\alpha\\
&= (-1)^{mn+1}\,[Y,X]\otimes \quad \beta\cdot\alpha\\
&= (-1)^{mn+1}\,[Y\otimes\beta, X\otimes\alpha]\\
&= (-1)^{mn+1}\,[B,A].
\end{aligned}$$

Now we are left with the graded Jacobi identity. Let A, B be as above, and let $C = Z\otimes\gamma\in(\mathfrak{g}\otimes\mathscr{A})^p$ with $Z\in\mathfrak{g}$ and $\gamma\in\mathscr{A}^p$. Now consider (A.12)

$$(-1)^{np}[A,[B,C]] + (-1)^{mn}[B,[C,A]] + (-1)^{pm}[C,[A,B]] = 0.$$

For computational convenience we will start out with an equivalent expression that is obtained from the above equation by multiplying the whole expression with $(-1)^{np}$, such that the first factor $(-1)^{np}$ disappears $((-1)^{np}(-1)^{np} = (-1)^{2np} = 1)$ and we get

$$\begin{aligned}
& (-1)^{np}[A,[B,C]] + (-1)^{mn}[B,[C,A]] + (-1)^{pm}[C,[A,B]] = 0\\
\Leftrightarrow\quad & [A,[B,C]] + (-1)^{np}(-1)^{mn}[B,[C,A]] + (-1)^{np}(-1)^{pm}[C,[A,B]] = 0\\
\Leftrightarrow\quad & [A,[B,C]] + (-1)^{n(p+m)}[B,[C,A]] + (-1)^{p(m+n)}[C,[A,B]] = 0.
\end{aligned}$$

Now using $\alpha\beta\gamma = (-1)^{n(p+m)}\beta\gamma\alpha$ and $\alpha\beta\gamma = (-1)^{p(m+n)}\gamma\alpha\beta$ (by graded-commutativity) we can prove

$$\begin{aligned}
& [A,[B,C]] + (-1)^{n(m+p)}[B,[C,A]] + (-1)^{p(m+n)}[C,[A,B]]\\
=\; & [X,[Y,Z]]\otimes\alpha\beta\gamma + (-1)^{n(m+p)}[Y,[Z,X]]\otimes\beta\gamma\alpha + (-1)^{p(m+n)}[Z,[X,Y]]\otimes\gamma\alpha\beta\\
=\; & [X,[Y,Z]]\otimes\alpha\beta\gamma + [Y,[Z,X]]\otimes\alpha\beta\gamma + [Z,[X,Y]]\otimes\alpha\beta\gamma\\
=\; & ([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]])\otimes\alpha\beta\gamma\\
=\; & 0\otimes\alpha\beta\gamma\\
=\; & 0,
\end{aligned}$$

in the end using the ordinary Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ of the Lie algebra $\mathfrak{g}$. This proves the algebra $\mathfrak{g}\otimes\mathscr{A}$ is a DGLA.

Corollary A.1.1 The algebra $\mathfrak{g}\otimes\Omega(M)$ of $\mathfrak{g}$-valued differential forms on a manifold M, with grading, product and differential defined as in Lemma (A.1.1) is a DGLA, and satisfies properties (A.11) and (A.12).

We also prove the following useful lemma, which allows us to extend graded derivations on $\mathscr{A}$ (a GDA) to $\mathfrak{g}\otimes\mathscr{A}$ (a DGLA).

Lemma A.1.2 Let $\mathscr{A}$ be a GDA, $\mathfrak{g}$ a Lie algebra.
Then if $\delta : \mathscr{A}^n \to \mathscr{A}^{n+k}$ is a graded derivation of degree k on $\mathscr{A}$ we have:
$\mathrm{id}\otimes\delta : (\mathfrak{g}\otimes\mathscr{A})^n \to (\mathfrak{g}\otimes\mathscr{A})^{n+k}$ is a graded derivation of degree k on $\mathfrak{g}\otimes\mathscr{A}$.

If $\mathfrak{g}$ is non-abelian the converse statement is also true.

Proof: We assume $\delta \in \mathrm{Der}^{(k)}(\mathscr{A})$. Let $A = X \otimes \alpha \in (\mathfrak{g} \otimes \mathscr{A})^n$ and $B = Y \otimes \beta \in (\mathfrak{g} \otimes \mathscr{A})$ with $X, Y \in \mathfrak{g}$ and $\alpha \in \mathscr{A}^n, \beta \in \mathscr{A}$. Then, using the graded derivation property (A.3) for δ we have

$$\begin{aligned}
(\mathrm{id} \otimes \delta)(\,[A, B]\,) &= (\mathrm{id} \otimes \delta)(\,[X \otimes \alpha, Y \otimes \beta]\,) \\
&= (\mathrm{id} \otimes \delta)(\,[X, Y] \otimes (\,\alpha \cdot \beta\,)) \\
&= [X, Y] \otimes \delta(\,\alpha \cdot \beta\,) \\
&= [X, Y] \otimes (\delta\alpha \cdot \beta + (-1)^{nk} \alpha \cdot \delta\beta) \\
&= [X, Y] \otimes (\delta\alpha \cdot \beta) + (-1)^{nk} [X, Y] \otimes (\alpha \cdot \delta\beta) \\
&= [X \otimes \delta\alpha, Y \otimes \beta] + (-1)^{nk} [X \otimes \alpha, Y \otimes \delta\beta] \\
&= [(\mathrm{id} \otimes \delta)A, B] + (-1)^{nk} [A, (\mathrm{id} \otimes \delta)B].
\end{aligned}$$

This proves the first part; now for the converse we can use the above equations. We know

$$(\mathrm{id} \otimes \delta)(\,[A, B]\,) = [(\mathrm{id} \otimes \delta)A, B] + (-1)^{nk} [A, (\mathrm{id} \otimes \delta)B].$$

So if we chose $X, Y \in \mathfrak{g}$ such that $[X, Y] \neq 0$ (here we use $\mathfrak{g}$ is non-abelian), and $\alpha \in \mathscr{A}^n, \beta \in \mathscr{A}$ arbitrary (and take again $A = X \otimes \alpha \in (\mathfrak{g} \otimes \mathscr{A})^n$ and $B = Y \otimes \beta \in (\mathfrak{g} \otimes \mathscr{A})$) it follows from above equations that

$$[X, Y] \otimes \delta(\,\alpha \cdot \beta\,) = [X, Y] \otimes (\delta\alpha \cdot \beta + (-1)^{nk} \alpha \cdot \delta\beta),$$

which implies $\delta(\alpha \cdot \beta) = \delta\alpha \cdot \beta + (-1)^{nk} \alpha \cdot \delta\beta$ for arbitrary $\alpha, \beta \in \mathscr{A}$.

Notice that the condition that $\mathfrak{g}$ should not be abelian necessary for the converse statement in Lemma A.1.2 is the same as saying that the DGLA $\mathfrak{g} \otimes \mathscr{A}$ should not be trivial: if $\mathfrak{g}$ were abelian (i.e. $[X, Y] = 0\ \forall X, Y \in \mathfrak{g}$) the product on $\mathfrak{g} \otimes \mathscr{A}$ (the graded Lie bracket) would be trivially zero, and so $\mathfrak{g} \otimes \mathscr{A}$ would be a trivial algebra on the vector space $\mathfrak{g} \otimes \mathscr{A}$.

Although the notation $(id \otimes \delta)$ for the graded derivation on $\mathfrak{g} \otimes \mathscr{A}$ is more precise, we will often abbreviate this to δ, and write

$$\delta(X \otimes \alpha) := X \otimes (\delta\alpha),$$

just as we did for the differential d in Lemma A.1.1.

Furthermore we wish to point out that the Lie algebra $\mathfrak{g}$ is imbedded in $\mathfrak{g} \otimes \mathscr{A}$ since we assumed $\mathscr{A}$ to be unital. Hence we have

$$X \in \mathfrak{g} \hookrightarrow X \otimes 1 \in \mathfrak{g} \otimes \mathscr{A}, \qquad \forall X \in \mathfrak{g},$$

and we will often write X for $X \otimes 1$. More generally, for any (unital) GDA $\mathscr{A}$ over $\mathbb{K}$ we have the ground field $\mathbb{K}$ imbedded in $\mathscr{A}^0$ by $c \in \mathbb{K} \mapsto c \cdot 1 \in \mathscr{A}^0$. We prove the differential d is zero on these elements by the following small lemma.

Lemma A.1.3 Let $\mathscr{A}$ be a unital GDA over $\mathbb{K}$, and let $c := c \cdot 1 \in \mathscr{A}^0$ for $c \in \mathbb{K}$. Then

$$d(c) = d(c \cdot 1) = 0.$$

Proof: by definition the differential d is $\mathbb{K}$-linear on $\mathscr{A}$, and also an anti-derivation. So we have for all $x \in \mathscr{A}$ ($c \in \mathbb{K}$)

$$c \cdot d(x) = d(c \cdot x) = d(c) \cdot x + c \cdot d(x),$$

which implies $d(c) = 0$ for all $c \in \mathbb{K}$.

Corollary A.1.2 $d(X) = 0$ for elements $X = X \otimes 1 \in \mathfrak{g} \otimes \mathscr{A}$ with $\mathfrak{g} \otimes \mathscr{A}$ a DGLA. This follows from $d(X \otimes 1) = X \otimes d(1) = X \otimes 0$ by Lemma A.1.3 above.

Lemma A.1.4 [4] Let $\mathscr{L}$ be a DGLA, and $\alpha \in \mathscr{L}^n$. Then α defines a graded derivation of degree n on $\mathscr{L}$ by

$$ad(\alpha) : \lambda \mapsto [\,\alpha, \lambda\,].$$

Proof: It is clear that $ad(\alpha)$ is homogeneous of degree $+n$. We need to prove the graded derivation property (A.3). For $\lambda \in \mathscr{L}^p, \mu \in \mathscr{L}^q$ the graded Jacobi identity (A.12) states

$$(-1)^{nq}[\,\alpha, [\,\lambda, \mu\,]\,] + (-1)^{np}[\,\lambda, [\,\mu, \alpha\,]\,] + (-1)^{pq}[\,\mu, [\,\alpha, \lambda\,]\,] = 0,$$

or equivalently,

$$[\,\alpha, [\,\lambda, \mu\,]\,] = -(-1)^{n(p+q)}[\,\lambda, [\,\mu, \alpha\,]\,] - (-1)^{q(n+p)}[\,\mu, [\,\alpha, \lambda\,]\,].$$

Now, using $[\,\mu, \alpha\,] = -(-1)^{nq}[\,\alpha, \mu\,]$ and $[\,\mu, [\,\alpha, \lambda\,]\,] = -(-1)^{q(n+p)}[\,[\,\alpha, \lambda\,], \mu\,]$ we have

$$[\,\alpha, [\,\lambda, \mu\,]\,] = [\,[\,\alpha, \lambda\,], \mu\,] + (-1)^{np}[\,\lambda, [\,\alpha, \mu\,]\,],$$

which proves $ad(\alpha)$ is a graded derivation of degree n.

A.2 Differential forms

In the following (as in all of this document) we will often suppress the notion *smooth* when speaking of manifolds, differential forms and vector fields. All manifolds, forms and vector fields in this thesis are considered to be *smooth.*

We consider known the essential definitions of the calculus of differential forms. In particular, the notions of *push-forward* and *pull-back* should be understood. If M and N are manifolds, and $\phi : M \to N$ is a smooth diffeomorphism, we denote the push-forward $\phi_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$. For the *pull-back* we use $\phi^* : \Omega(N) \to \Omega(N)$. We reserve the notation $\phi^T : T_m(M) \to T_{\phi(m)}(N)$ for the *tangent mapping* in the point $m \in M$, and for a vector field $X \in \mathfrak{X}(M)$ we have

$$\big(\phi_*(X)\big)_n = \phi^T(X_{\phi^{-1}(n)}), \qquad (n \in N). \tag{A.16}$$

Note: The concepts defined below (such as the interior product and the Lie derivative) can be given a still more general definition, defining them on arbitrary tensor fields instead of differential forms (which are antisymmetric tensor fields). This is done for instance in de Azcárraga[2] Sec. 1.4. We do not need this generalization however, and therefore we omit it.

A.2.1 Derivations

Exterior derivative

Let M be a manifold, and $\Omega(M)$ the real vector space of all smooth differential forms on M. The *exterior derivative* is the unique linear map $d : \Omega(M) \to \Omega(M)$ satisfying ([18], Th. 11.1)

[4] This is part of the Proposition in Appendix B of Kastler&Stora [11]. We include the same proof.

1. $d : \Omega^p(M) \to \Omega^{p+1}(M)$.
2. $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge (d\beta), \quad$ for $\alpha \in \Omega^p(M), \beta \in \Omega^q(M)$.
3. $d^2 = 0$.
4. df is the *differential* of f for $f \in C^\infty(M) = \Omega^0(M)$.

Above properties qualify d as a *differential* or *anti-derivation of degree +1* on the graded algebra of differential forms $\Omega(M)$.

Let α a p-form ($p <$ dim M) expressed in local coordinates $x = (x_1, \ldots, x_n)$ on M

$$\alpha = \sum_I a_I(x) dx^I \,, \tag{A.17}$$

with the sum ranging over all p-tuples I, with $I = (i_1, \ldots, i_n)$, with $i_j \in \{1, \ldots,$ dim $M\}$. The exterior derivative d is usually defined as

$$d\alpha = \sum_I da_I(x) \wedge dx^I \,. \tag{A.18}$$

There is however another definition of the exterior derivative, that is more practical to us.

A p-form is completely known if we know its value

$$\alpha(X_1, \ldots, X_p) \in C^\infty(M)$$

on p arbitrary vector fields $X_1, \ldots, X_p \in \mathfrak{X}(M)$, and inserting these vector fields gives us a C^∞-function on M. Therefore we can view $\alpha \in \Omega^p(M)$ as a mapping $\alpha : \mathfrak{X}(M) \times \ldots \times \mathfrak{X}(M) \to C^\infty(M)$ that is $C^\infty(M)$-multilinear ($\mathfrak{X}(M)$ is a $C^\infty(M)$-module) and antisymmetric.

Also remember that a vector field $X \in \mathfrak{X}(M)$ acts as a derivation on smooth functions on M (i.e. $C^\infty(M)$), and $\mathfrak{X}(M)$ is a Lie algebra (over $C^\infty(M)$) with the commutator of vector fields $[X, Y] \in \mathfrak{X}(M)$ as Lie bracket. Now we use this in order to give another (equivalent) definition of the exterior derivative d that is as follows ([18] Th. 11.3, [11] §1.2)

$$d\alpha(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i X_i \cdot \alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p) + \\ \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha(\, [X_i, X_j]\,, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \tag{A.19}$$

Interior product

The *interior product* i_X of a p-form $\alpha \in \Omega^p(M)$ and a vector field $X \in \mathfrak{X}(M)$ is defined as

$$(i_X \alpha)(X_2, \ldots, X_p) := \alpha(X, X_2, \ldots, X_p). \tag{A.20}$$

As we can see, i_X lowers the degree of the differential form by one, so $i_X : \Omega^p(M) \to \Omega^{p-1}(M)$. i_X respects the vector space structure of $\Omega(M)$, since $i_X(\alpha + \beta) = i_X(\alpha) + i_X(\beta)$ and $i_X(c \cdot \alpha) = c \cdot i_X(\alpha)$. We also have, $\alpha \in \Omega^p(M), \beta \in \Omega^q(M)$

$$i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X \beta), \tag{A.21}$$

which qualifies i_X as an anti-derivation of degree -1 on the graded algebra of differential forms $\Omega(M)$. The *interior product* is also known as the *interior antiderivative* [6], the *contraction* [2], or the *interior multiplication* [18].

Lemma A.2.1 The interior product is natural with respect to diffeomorphisms. Let M, N be manifolds, and $\phi : M \to N$ a smooth diffeomorphism. Let $\alpha \in \Omega^q(N), X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N)$ be such that $\phi_*(X) = Y$ (Y is the push-forward of X). Then

$$\phi^*(i_Y\alpha) = i_X(\phi^*\alpha). \tag{A.22}$$

Proof: As in [2] 1.4.32. Let $m \in M$ and $n = \phi(m) \in N$ then

$$\begin{aligned} i_X(\phi^*\alpha)(m)(X_2(m), \ldots X_q(m)) &= (\phi^*\alpha)(m)(X(m), X_2(m), \ldots X_q(m)) \\ &= \alpha(n)(\phi^T(X(m)), \phi^T(X_2(m)), \ldots \phi^T(X_q(m))) \\ &= \alpha(n)(Y(n)), \phi^T(X_2(m)), \ldots \phi^T(X_q(m))) \\ &= (i_Y\alpha)(n)(\phi^T(X_2(m)), \ldots \phi^T(X_q(m))) \\ &= (\phi^*(i_Y\alpha))(m)(X_2(m), \ldots X_q(m)), \end{aligned}$$

which is equation (A.22).

Lie derivative

The Lie derivative has a definition which gives an intuitive feeling of why this is a derivative. Let us remind that for a vector field $X \in \mathfrak{X}(M)$, and a point $m \in M$ the *integral curve* of X through m is given by the unique curve $\gamma : I \to M$ which satisfies:

$$\frac{d}{dt}\gamma(t)\Big|_{t=t_0} = X_{\gamma(t_0)} \qquad \forall t_0 \in I. \tag{A.23}$$

With I we refer to an open interval around 0 in $\mathbb{R}$, and assume $\gamma(0) = m$. In the case of $M = G$ a Lie group, and a curve through the identity e, this interval is extendable to whole $\mathbb{R}$, and we get a one-parameter subgroup of G.

If, for $t \in I$, we define the diffeomorphism $\phi_t : M \to M$ by $m \mapsto \gamma(t)$ with γ the integral curve through m belonging by X, we get a one-parameter group $\{\phi_t\}$ of diffeomorphisms of M. We have $(\phi_t)^{-1} = \phi_{-t}$, and $\phi_0 = \mathrm{id}_M$. This group of diffeomorphisms is called the *flow* of X.

Now, given a vector field $X \in \mathfrak{X}(M)$ with flow ϕ_t, we can define the Lie derivative of it on a differential form $\alpha \in \Omega^p(M)$ by

$$L_X\alpha \;=\; \lim_{t\to 0}\frac{\phi_t^*\alpha - \alpha}{t} \;=\; \frac{d}{dt}\,\phi_t^*\alpha\,\Big|_{t=0}. \tag{A.24}$$

Notice that the limit is just an ordinary limit in $\mathbb{R}$ once we have inserted a point $m \in M$ and p tangent vectors from T_mM in $\phi_t^*\alpha$ and α. So $L_X\alpha$ is a p-form on M, and we have $L_X : \Omega^p(M) \to \Omega^p(M)$. Together with the property

$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta) \tag{A.25}$$

and the linearity $L_X(\alpha + \beta) = L_X\alpha + L_X\beta$, $i_X(c \cdot \alpha) = c \cdot i_X(\alpha)$ this makes L_X a *derivation of degree zero* on $\Omega(M)$.

Now we would like to give an algebraic description of the Lie derivative, like (A.19) for the exterior differential. It is given by ([2], 1.4.35)

$$(L_X\alpha)(X_1,\ldots,X_p) = Y\cdot\alpha(X_1,\ldots,X_p) - \sum_{i=1}^{q}\alpha(X_1,\ldots,[Y,X_i],\ldots,X_q), \tag{A.26}$$

with $\alpha\in\Omega^p(M), Y$ and $X_i\in\mathfrak{X}(M)$.

Although we will be primarily concerned with the Lie derivative on *forms*, we also want to note that the Lie derivative can be defined on *vector fields* $Y\in\mathfrak{X}(M)$ as well. In this case

$$L_XY = \lim_{t\to 0}\frac{\phi^T_{-t}Y_{\phi(t)} - Y}{t} \tag{A.27}$$

and $L_X:\mathfrak{X}(M)\to\mathfrak{X}(M)$. This limit can be shown to be equal to the commutator of the vector fields X and Y (see [2], §1.4)

$$L_XY = [X,Y]. \tag{A.28}$$

In the following L_X will mean the derivative operating on *forms* unless explicitly stated otherwise.

Relations between d, i_X and L_X

In the following we take $X,Y\in\mathfrak{X}(M)$. For both i_X and L_X we have

$$i_{X+Y} = i_X + i_Y, \tag{A.29}$$
$$L_{X+Y} = L_X + L_Y, \tag{A.30}$$

which follows from the definition and the linearity of the forms on which i_X and L_X act. On 0-forms $f\in\Omega^0(M)=C^\infty(M)$ we have

$$i_X(df) = df(X) = X\cdot f, \tag{A.31}$$
$$L_X(df) = X\cdot f. \tag{A.32}$$

For i_X the the antisymmetry of differential forms immediately implies

$$i_Xi_X = 0. \tag{A.33}$$

The Lie derivative L_X commutes with the exterior derivative d

$$[L_X,d] = 0, \tag{A.34}$$

which follows immediately from the following equation, which is known as the *Cartan decomposition*, and could be taken as a definition for L_X:

$$L_X = di_X + i_Xd. \tag{A.35}$$

This definition will be used to generalize the Lie derivative to arbitrary $\mathfrak{g}$-operations. From (A.35) it also follows

$$[L_X,i_Y] = i_{[X,Y]}, \tag{A.36}$$
$$[L_X,i_X] = 0, \tag{A.37}$$
$$[L_X,L_Y] = L_{[X,Y]}. \tag{A.38}$$

A.2.2 Vector-valued differential forms

In this thesis we will make extensive use of differential forms with values in a vector space (usually a Lie algebra). The connection form and curvature form on a principal bundle are examples of Lie(G)-valued forms (with G the structure group).

Let $\mathcal{V}$ be an arbitrary finite-dimensional real vector space. In principle, a $\mathcal{V}$-valued differential p-form ω on M (a manifold) means a multilinear mapping

$$\omega_m : \underbrace{T_m(M) \times \ldots \times T_m(M)}_{p \text{ times}} \to \mathcal{V},$$

for every $m \in M$. If we take any basis $\{E_\alpha\}_{\alpha=1}^{\dim \mathcal{V}}$ of $\mathcal{V}$ we can project the form ω on each E_α-axis to obtain a real valued differential form $\omega^\alpha = \pi_{E_\alpha} \circ \omega$ and write

$$\omega = E_\alpha \cdot \omega^\alpha,$$

were summation over α is understood (the so-called *Einstein summation* convention). Since both $\mathcal{V}$ and $\Omega(M)$ are linear (vector) spaces over $\mathbb{R}$ we can characterize ω as an element of $\mathcal{V} \otimes \Omega(M)$. Now for an arbitrary vector space $\mathcal{V}$ there will not be a wedge product for $\mathcal{V}$-valued forms, unless there is a bilinear map $\rho : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ (see [15](Vol II), §1.2). Therefore we will now concentrate on the case $\mathcal{V} = \mathfrak{g}$, with $\mathfrak{g}$ a finite-dimensional real Lie algebra.

Definition A.2.1 A $\mathfrak{g}$-valued differential form ω on M is an element of $\mathfrak{g} \otimes \Omega(M)$. If $\{E_\alpha\}$ is a basis for $\mathfrak{g}$ we usually write

$$\omega = E_\alpha \otimes \omega^\alpha,$$

with $\omega^\alpha \in \Omega(M)$ (Einstein summation used).

Since $\Omega(M)$ is a graded-commutative differential algebra, we can apply Lemma A.1.1 (this is in fact Corollary A.1.1), which says $\mathfrak{g} \otimes \Omega(M)$ is a differential graded Lie algebra (DGLA). So we have a bracket on these $\mathfrak{g}$-valued forms defined by

$$[\omega, \eta] = [E_\beta \otimes \omega^\beta, E_\gamma \otimes \eta^\gamma] = [E_\beta, E_\gamma] \otimes (\omega^\beta \wedge \eta^\gamma), \tag{A.39}$$

for $\omega = E_\beta \otimes \omega^\beta \in \mathfrak{g} \otimes \Omega^p(M)$ and $\eta = E_\gamma \otimes \eta^\gamma \in \mathfrak{g} \otimes \Omega^q(M)$.

This bracket is called the **Schouten product** ([11], §1.5), and can also be expressed as

$$[\omega, \eta](X_1, .., X_{p+q}) = \frac{1}{p\,!\,q!} \sum_{\sigma \in S_{p+q}} \epsilon(\sigma) [\, \omega(X_{\sigma(1)}, .., X_{\sigma(p)}), \eta(X_{\sigma(q+1)}, .., X_{\sigma(q+p)}) \,] \tag{A.40}$$

with $X_i \in \mathfrak{X}(M)$, S_{p+q} the permutation group of $p + q$ elements, and $\epsilon(\sigma)$ the sign of the permutation.

Most other operations defined on differential forms have a natural extension to vector-valued forms. For instance, the exterior differential d of a form $\omega \in \mathfrak{g} \otimes \Omega(M)$ is defined as

$$d\omega = d(E_\alpha \otimes \omega^\alpha) = E_\alpha \otimes (d\omega^\alpha),$$

in accordance with Lemma A.1.1.

We will also often use the pull-back of a vector-valued differential form. If we have a diffeomorphism $f : M \to N$ then the pull-back $f^* : \Omega(N) \to \Omega(M)$ has a natural extension to $\tilde{f}^* : \mathfrak{g} \otimes \Omega(N) \to \mathfrak{g} \otimes \Omega(M)$ by defining $\tilde{f}^* = \mathrm{id}_{\mathfrak{g}} \otimes f^*$, or equivalently

$$\tilde{f}^*\omega = \tilde{f}^*(E_\alpha \otimes \omega^\alpha) = E_\alpha \otimes (f^*\omega^\alpha),$$

for $\omega \in \mathfrak{g} \otimes \Omega(M)$. We will often write f^* for $\tilde{f}^*$, and thus use the notation f^* for both ordinary and vector-valued differential forms.

A.3 Lie group theory

(For background information on Lie group theory we recommend [8],[17] and [2].)

Let G be a (finite-dimensional) Lie group, i.e. G is both a group and a smooth manifold such that the group multiplication $(\cdot) : G \times G \to G$ is smooth. It follows that the left and right multiplication, denoted with $L_g : h \mapsto gh$ and $R_g : h \mapsto hg$, are also smooth.

Let T_eG be the tangent space to the identity $e \in G$. Any element $X \in T_eG$ can be extended to a left-invariant vector field $X^L \in \mathfrak{X}(G)$ by defining $X^L_g = (L_g)^T X$. The commutator of such left-invariant vector fields (LIVF's) is again left-invariant, and thus defines a Lie bracket on T_eG (with $[X,Y] = [X,Y]^L_e = [X^L, Y^L]_e$). This makes T_eG into Lie algebra, and we therefore denote it with $\mathit{Lie}(G)$.

For any element $X \in \mathit{Lie}(G)$ one can consider the integral curve $\gamma(t)$ of the vector field $X^L \in \mathfrak{X}(G)$ through $\gamma(0) = e \in G$. By defining $\exp(X) = \gamma(1)$ one can define an exponential mapping $\exp : \mathit{Lie}(G) \to G$, such that we can associate a one-parameter group in G with every element $X \in \mathit{Lie}(G)$.

A.3.1 Adjoint actions

Another way to derive the Lie bracket on $\mathit{Lie}(G) = T_eG$ is the following. We have the action of G on itself by conjugation, denoted by

$$\mathbf{Ad} : G \to Aut(G), \quad \mathbf{Ad} : g \mapsto \mathbf{Ad}_g, \quad \mathbf{Ad}_g : h \mapsto ghg^{-1}.$$

Take any $g \in G$ and consider the tangent mapping $\mathbf{Ad}_g^T$. Since $\mathbf{Ad}_g(e) = e$ for any $g \in G$, and $T_eG = \mathit{Lie}(G)$, we have a linear map

$$\mathbf{Ad}_g^T : \mathit{Lie}(G) \to \mathit{Lie}(G).$$

We denote this map also by $Ad_g : \mathit{Lie}(G) \to \mathit{Lie}(G)$. For $X \in \mathit{Lie}(G)$ it is explicitly given by

$$Ad_g(X) = \frac{d}{dt}\, g \cdot \exp(tX) \cdot g^{-1} \Big|_{t=0}. \tag{A.41}$$

On can show this is a group homomorphism $Ad : G \to GL(\mathit{Lie}(G))$ by applying the chain rule for tangent mappings (see §1.1 [8]), and Ad is called the adjoint action of the group G on its Lie algebra $\mathit{Lie}(G)$.

Now we can take the tangent mapping Ad^T of the map $Ad : G \to GL(\mathit{Lie}(G))$, and denote it with ad

$$ad : \mathit{Lie}(G) \to \mathfrak{gl}(\mathit{Lie}(G)),$$

where we substituted $T_eG = \mathit{Lie}(G)$ and $T_e(GL(\mathit{Lie}(G))) = \mathfrak{gl}(\mathit{Lie}(G))$. One can define a bracket on $\mathit{Lie}(G)$ by $[X, Y] = (adX)(Y)$. When a bracket is already known, one reads this equation the other way around and defines $ad : \mathit{Lie}(G) \to \mathfrak{gl}(\mathit{Lie}(G))$ by $(adX)(Y) = [X, Y]$. Either way ad is a Lie algebra homomorphism, and this is called the adjoint action of the Lie algebra $\mathit{Lie}(G)$ on itself:

$$ad : \mathit{Lie}(G) \to \mathfrak{gl}(\mathit{Lie}(G)), \quad ad : X \mapsto (adX), \quad (adX)(Y) = [X, Y].$$

A.3.2 Differential forms on G and the Maurer-Cartan form

For convenience let $\mathfrak{g} = \mathit{Lie}(G)$ in the following, and let $\mathfrak{g}^*$ denote the dual space of $\mathfrak{g}$, i.e.

$$\mathfrak{g}^* = \{ \, \omega : \mathfrak{g} \to \mathbb{R} \mid \text{linear} \, \}.$$

We saw we could identify $X \in \mathfrak{g}$ with left-invariant vector fields $X^L \in \mathfrak{X}(G)$. In the same way we can identify elements $\omega \in \mathfrak{g}^*$ with left-invariant differential forms ω^Lon G by defining

$$\omega^L_g(v_g) = \omega(L^T_{g^{-1}} v_g), \qquad v_g \in T_gG.$$

Now $\omega^L \in \Omega(G)$ is left-invariant by definition. From now on, let $\Omega^L(G)$ denote the LI forms on G, and $\mathfrak{X}^L(G)$ the LIVFs.

We now introduce the **Maurer-Carter form** on the Lie group G. It is the unique $\mathit{Lie}(G)$-valued 1-form $\Theta_{MC} \in \mathit{Lie}(G) \otimes \Omega^1(G)$ on G satisfying

$$\Theta_{MC}(X^L) = X \qquad \forall\, X \in \mathit{Lie}(G),$$

by which we mean: $\Theta_{MC}(g)(X^L_g) = X$ for all $g \in G$. Θ_{MC} is thus completely specified, since we have for arbitrary $v_g \in T_gG$,

$$\Theta_{MC}(g)(v_g) = L^T_{g^{-1}} v_g \in T_eG = \mathit{Lie}(G),$$

or in other words $\Theta_{MC}(e) = \mathrm{id}_{\mathfrak{g}}$ and $\Theta_{MC}(g) = (L_{g^{-1}})^* \Theta_{MC}(e)$. This shows Θ_{MC} is left-invariant. We call Θ_{MC} the *Maurer-Cartan form* on G.

A.3.3 Properties of the Maurer-Cartan form

Let $\{E_\alpha\}$ be a basis for $\mathfrak{g}$, and $\{E^\alpha\}$ the corresponding cobasis of $\mathfrak{g}^*$. Let $\{X^L_\alpha\}$ denote the LIVFs corresponding to the elements $\{E_\alpha\}$ in $\mathfrak{g}$. Since $L^T_g : T_eG \to T_gG$ supplies an isomorphism between any tangent space T_gG and $T_eG = \mathfrak{g}$, we know the set $\{X^L_\alpha(g)\}$ is a basis for T_gG for each $g \in G$.

Similarly, we can extend each element $E^\alpha \in \mathfrak{g}^*$ to a LI form $\theta^\alpha \in \Omega^L(G)$, and we have $\theta^\alpha(X^L_\beta) = \delta^\alpha_\beta$. With respect to this choice of basis we can write the Maurer-Cartan form $\Theta_{MC} \in \mathfrak{g} \otimes \Omega^L(G)$ as

$$\Theta_{MC} = E_\alpha \otimes \theta^\alpha.$$

We now consider the exterior derivative $d\theta^\alpha$ of the LI form θ^α. In the previous section d was defined in a coordinate-free way in (A.19), which for the 1-form θ^α states

$$d\theta^\alpha(X, Y) = X \cdot \theta^\alpha(Y) - Y \cdot \theta^\alpha(X) - \theta^\alpha([X, Y]), \quad X, Y \in \mathfrak{X}(G).$$

Since for all $g \in G$, the LIVFs $\{X^L_\alpha\}$ span the tangent space T_gG we may assume $X = X^L_\beta$ and $Y = X^L_\gamma$, and since LI forms like θ^α are constant on LIVFs the first two terms vanish. We are left with

$$d\theta^\alpha(X^L_\beta, X^L_\gamma) = -\theta^\alpha([X^L_\beta, X^L_\gamma]). \tag{A.42}$$

Now we introduce the *structure constants* of the Lie algebra $\mathfrak{g}$ with respect to the chosen basis:

$$[E_\beta, E_\gamma] = C^\alpha_{\beta\gamma}\, E_\alpha\,, \qquad C^\alpha_{\beta\gamma} \in \mathbb{R},$$

from which follows $[X^L_\beta, X^L_\gamma] = C^\alpha_{\beta\gamma}\, X^L_\alpha$. Inserting this in (A.42) results in

$$d\theta^\alpha(X^L_\beta, X^L_\gamma) = -\theta^\alpha(C^\alpha_{\beta\gamma}\, X^L_\alpha) = -C^\alpha_{\beta\gamma}\, X_\alpha$$

and hence

$$d\theta^\alpha = -\tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, \theta^\beta \wedge \theta^\gamma\,. \tag{A.43}$$

This is known as the **Maurer-Cartan structure equation**. The factor $\frac{1}{2}$ appears, because the summation is by convention taken over *all* β and γ in $\{1, .., \dim\, \mathfrak{g}\}$. We could also write

$$d\theta^\alpha = \sum_{1\leq\, \beta<\gamma\leq \dim\, \mathfrak{g}} -\, C^\alpha_{\beta\gamma}\, \theta^\beta \wedge \theta^\gamma\,,$$

since $C^\alpha_{\beta\gamma} = -C^\alpha_{\gamma\beta}$ and $\theta^\beta \wedge \theta^\gamma = -\theta^\gamma \wedge \theta^\beta$.

We prove two important properties of the Maurer-Cartan form Θ_{MC} in the following lemmas.

Lemma A.3.1 The Maurer-Cartan form Θ_{MC} satisfies

$$d\Theta_{MC} = -\tfrac{1}{2}[\,\Theta_{MC}, \Theta_{MC}\,]. \tag{A.44}$$

Proof: Using the Maurer-Cartan structure equation (A.43), and the definition (A.39) of the bracket on $\mathfrak{g}$-valued forms, we have

$$\begin{aligned} d\Theta_{MC} &= d(E_\alpha \otimes \theta^\alpha) \\ &= E_\alpha \otimes\, d\theta^\alpha \\ &= -\tfrac{1}{2}\, C^\alpha_{\beta\gamma}\, E_\alpha \otimes \;\; \theta^\beta \wedge \theta^\gamma \\ &= -\tfrac{1}{2}\, [\, E_\beta, E_\gamma\,] \otimes \;\; \theta^\beta \wedge \theta^\gamma \\ &= -\tfrac{1}{2}\, [\, E_\beta \otimes \theta^\beta, E_\gamma \otimes \theta^\gamma\,] \\ &= -\tfrac{1}{2}\, [\,\Theta_{MC}, \Theta_{MC}\,], \end{aligned}$$

QED.

Lemma A.3.2 The Maurer-Cartan form on G is Ad-equivariant under the right-action of G on itself, i.e.

$$(R_g)^*\Theta_{MC} = Ad_{g^{-1}} \circ \Theta_{MC}.$$

Proof: Let $h \in G$ and $v_h \in T_h G$. We have

$$\begin{aligned}((R_g)^* \Theta_{MC})_h(v_h) &= \Theta_{MC}(hg)((R_g)^T v_h) \\ &= (L_{(hg)^{-1}})^T (R_g)^T v_h) \\ &= (L_{g^{-1}})^T (L_{h^{-1}})^T (R_g)^T v_h \\ &= (L_{g^{-1}})^T (R_g)^T (L_{h^{-1}})^T v_h \\ &= Ad_{g^{-1}} \ \ (L_{h^{-1}})^T v_h \\ &= Ad_{g^{-1}} \ \ \Theta_{MC}(h)(v_h) \ \ .\end{aligned}$$

Where we used (I) $(L_{h^{-1}})^T (R_g)^T = (R_g)^T (L_{h^{-1}})^T$ since the right and left actions of G commute, and (II) $(L_{g^{-1}})^T (R_g)^T : \mathfrak{g} \to \mathfrak{g}$ corresponds to the adjoint action $Ad_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$ as defined above in equation (A.41).

We assumed G was a finite-dimensional Lie group, but in fact the constructions above can be applied to infinite-dimensional Lie groups (and algebras) as well, with some adjustments. When considering the infinite-dimensional Lie group $\mathscr{G}$ of gauge transformations on a principal bundle, its Lie algebra $Lie(\mathscr{G})$, and their role in the descent equations, we will see the so-called *ghost fields* that appear there can be considered as the Maurer-Cartan forms on the infinite-dimensional Lie algebra $Lie(\mathscr{G})$. See Chapter 5 §5.2.

Appendix B

Additional proofs

B.1 General algebra lemmas

Lemma B.1.1 Let $\mathscr{A}$ be a graded algebra, generated by the elements $\{E^\alpha\}$. Suppose δ an anti-derivation on $\mathscr{A}$ of degree k (so k is odd), and we know $\delta^2(E^\alpha) = 0$. Then

$$\delta^2(\eta) = 0,$$

for all $\eta \in \mathscr{A}$.

Proof: We will use induction on the length of a product. Suppose we know $\delta^2(\alpha) = 0$ and $\delta^2(\beta) = 0$, then we know for $\alpha \cdot \beta$

$$\begin{aligned}
\delta^2(\,\alpha \cdot \beta\,) &= \delta(\,\delta(\,\alpha \cdot \beta\,)\,) \\
&= \delta(\,\delta\alpha \cdot \beta + (-1)^{\deg\,\alpha}\alpha \cdot \delta\beta\,) \\
&= \delta(\,\delta\alpha \cdot \beta\,) + (-1)^{\deg\,\alpha}\delta(\,\alpha \cdot \delta\beta\,) \\
&= \delta^2\alpha \cdot \beta + (-1)^{\deg\,\alpha + k}\delta\alpha \cdot \delta\beta + (-1)^{\deg\,\alpha}\delta\alpha \cdot \delta\beta + \alpha \cdot \delta^2\beta \\
&= 0 - (-1)^{\deg\,\alpha}\delta\alpha \cdot \delta\beta + (-1)^{\deg\,\alpha}\delta\alpha \cdot \delta\beta + 0 \\
&= 0,
\end{aligned}$$

where we used that k is odd, and so $(-1)^{\deg\,\alpha + k} = -(-1)^{\deg\,\alpha}$.

Since $\mathscr{A}$ is generated by $\{E^\alpha\}$, we know every element η of $\mathscr{A}$ can be written as a linear combination of products of the $\{E^\alpha\}$. Since $\delta^2 = 0$ on any E^α it will be zero on any product of the E^α's, as we just showed. Combining this with the linearity of δ this proves $\delta^2 = 0$ on any arbitrary element η of $\mathscr{A}$.

B.2 Lie algebra operations

Let $(\mathscr{A}, i, L)$ be a $\mathfrak{g}$-operation, such that by Definition 1.2.1 we have linear mappings

$$\begin{aligned}
i &: \mathfrak{g} \to \mathrm{Der}^{(-1)}(\mathscr{A}), \quad i : X \mapsto i_X, \\
L &: \mathfrak{g} \to \mathrm{Der}^{(0)}(\mathscr{A}), \quad L : X \mapsto L_X,
\end{aligned}$$

which satisfy (cf. (1.17)-(1.20))

$$\begin{aligned}
L_X &= di_X + i_X d, \\
L_{[X,Y]} &= [L_X, L_Y] = L_X L_Y - L_Y L_X, \\
i_{[X,Y]} &= L_X i_Y - i_Y L_X, \\
(i_X)^2 &= 0.
\end{aligned}$$

We now have the following lemmas.

Lemma B.2.1 $[d, L_X] = 0$.
Proof: We have $L_X = di_X + i_X d$ by (1.17) so

$$\begin{aligned}
[d, L_X] &= dL_X - L_X d \\
&= d(di_X + i_X d) - (di_X + i_X d)d \\
&= d^2 i_X + di_X d - di_X d + i_X d^2 \\
&= 0 + di_X d - di_X d + 0 \\
&= 0,
\end{aligned}$$

since $d^2 = 0$.

The following lemma concerns the sets of **invariant**, **horizontal** and **basic** elements which were defined in Def. 1.2.2 by

$$\begin{aligned}
\mathscr{I}(\mathscr{A}) &= \{\, \alpha \in \mathscr{A} \mid L_X \alpha = 0, \quad \forall X \in \mathfrak{g} \,\}, \\
\mathscr{H}(\mathscr{A}) &= \{\, \alpha \in \mathscr{A} \mid i_X \alpha = 0, \quad \forall X \in \mathfrak{g} \,\}, \\
\mathscr{B}(\mathscr{A}) &= \{\, \alpha \in \mathscr{A} \mid L_X \alpha = 0 \text{ and } i_X \alpha = 0, \quad \forall X \in \mathfrak{g} \,\}.
\end{aligned}$$

Lemma B.2.2 For $\mathscr{I}(\mathscr{A}), \mathscr{H}(\mathscr{A})$ and $\mathscr{B}(\mathscr{A})$ we have the following

1. $\mathscr{I}(\mathscr{A})$ is a graded differential subalgebra of $\mathscr{A}$.
2. $\mathscr{H}(\mathscr{A})$ is graded subalgebra of $\mathscr{A}$ that is stable by L_X
3. $\mathscr{B}(\mathscr{A})$ is a differential subalgebra of $\mathscr{I}(\mathscr{A})$ as well as $\mathscr{A}$.

Proof: That $\mathscr{I}(\mathscr{A}), \mathscr{H}(\mathscr{A})$ and $\mathscr{B}(\mathscr{A})$ are graded subalgebras of $\mathscr{A}$ follows from the fact that $\mathscr{A}$ is graded and i_X and L_X are antiderivations, so $L_X(\alpha \cdot \beta) = L_X\alpha \cdot \beta + \alpha \cdot L_X \beta = 0$ if $\alpha, \beta \in \mathscr{I}(\mathscr{A})$ and similarly for i_X and $\mathscr{H}(\mathscr{A})$.

(1.) Since L_X commutes with the differential d by Lemma B.2.1 above, we have for $\alpha \in \mathscr{I}(\mathscr{A})$ that $L_X(d\alpha) = d(L_X\alpha) = d(0) = 0$, for all $X \in \mathfrak{g}$.

(2.) Cf. (1.19) we have for $\alpha \in \mathscr{H}(\mathscr{A})$ and $X, Y \in \mathfrak{g}$

$$\begin{aligned}
i_Y(L_X\alpha) &= (L_X i_Y - i_{[X,Y]})\alpha \\
&= L_X(i_Y\alpha) - i_{[X,Y]}\alpha \\
&= L_X(0) - 0 = 0,
\end{aligned}$$

which proves $L_X\alpha \in \mathscr{H}(\mathscr{A})\, \forall\, X \in \mathfrak{g}$.

(3.) Let $\alpha \in \mathscr{B}(\mathscr{A})$. We already now $L_X(d\alpha) = 0$ by (1.) and now we show $i_X(d\alpha) = 0$ as well (using (1.17))

$$\begin{aligned}
i_X(d\alpha) &= (L_X - di_X)\alpha \\
&= L_X\alpha) - d(i_X\alpha) \\
&= 0 - d(0) = 0
\end{aligned}$$

and this shows $d\alpha \in \mathscr{B}(\mathscr{A})$. Since $\mathscr{B}(\mathscr{A}) \subset \mathscr{I}(\mathscr{A})$ it is also a differential subalgebra of $\mathscr{I}(\mathscr{A})$.

B.3 Lie algebra cohomology

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, with dual space $\mathfrak{g}^*$. Let $\{E_\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$, with cobasis $\{E^\alpha\}$ of $\mathfrak{g}^*$ such that $E^\alpha(E_\beta) = \delta^\alpha_\beta$. Let $C^\alpha_{\gamma\beta}$ denote the structure constants of $\mathfrak{g}$ in this basis, i.e. $[E_i, E_j] = C^k_{ij}\, E_k$.

Let $\Lambda(\mathfrak{g}^*)$ be the exterior algebra over $\mathfrak{g}^*$. If d is defined on $\Lambda(\mathfrak{g}^*)$ by $d\eta(X, Y) = \eta([Y, X])$ for $\eta \in \Lambda^1(\mathfrak{g}^*)$ we have the following lemmas.

Lemma B.3.1 The Jacobi identity in structure constants, with respect to the basis $\{E_\alpha\}$ of $\mathfrak{g}$ is given by

$$C^l_{ka}C^k_{bc} + C^l_{kb}C^k_{ca} + C^l_{kc}C^k_{ab} = 0, \tag{B.1}$$

for all a, b, c and l in the set $\{1, \dots, \dim \mathfrak{g}\}$.

Proof:

$$\begin{aligned}
0 &= [E_a, [E_b, E_c]] + [E_b, [E_c, E_a]] + [E_c, [E_a, E_b]] \\
&= [E_a, C^k_{bc}\, E_k] + [E_b, C^k_{ca}\, E_k] + [E_c, C^k_{ab}\, E_k] \\
&= C^k_{bc}\, [E_a, E_k] + C^k_{ca}\, [E_b, E_k] + C^k_{ab}\, [E_c, E_k] \\
&= C^k_{bc}C^l_{ak}\, E_l + C^k_{ca}C^l_{bk}\, E_l + C^k_{ab}C^l_{ck}\, E_l \\
&= \left(C^k_{bc}C^l_{ak} + C^k_{ca}C^l_{bk} + C^k_{ab}C^l_{ck} \right) E_l
\end{aligned}$$

which implies for all $l = 1, \dots, \dim \mathfrak{g}$

$$\begin{aligned}
0 &= C^k_{bc}C^l_{ak} + C^k_{ca}C^l_{bk} + C^k_{ab}C^l_{ck} \\
&= C^l_{ak}C^k_{bc} + C^l_{bk}C^k_{ca} + C^l_{ck}C^k_{ab} \\
&= -C^l_{ka}C^k_{bc} - C^l_{kb}C^k_{ca} - C^l_{kc}C^k_{ab} \\
&= C^l_{ka}C^k_{bc} + C^l_{kb}C^k_{ca} + C^l_{kc}C^k_{ab}
\end{aligned}$$

which completes the proof.

Lemma B.3.2 $d(E^\alpha) = C^\alpha_{\gamma\beta}\, E^\beta \wedge E^\gamma$.

Proof: We know $\{E^\beta \wedge E^\gamma\}$ is a basis for $\Lambda^2(\mathfrak{g}^*)$, and for any element $\eta \in \Lambda^2(\mathfrak{g}^*)$ we have

$$\eta = \eta_{\beta\gamma}\, E^\beta \wedge E^\gamma\,,$$

with $\eta_{\beta\gamma} = \eta(E_\beta, E_\gamma)$ a constant. So we consider the case $\eta = E^\alpha$, and show $E^\alpha(E_\beta, E_\gamma) = C^\alpha_{\gamma\beta}$.

$$\begin{aligned}
(dE^\alpha)(E_\beta, E_\gamma) &= E^\alpha([E_\gamma, E_\beta]) \\
&= E^\alpha(C^k_{\gamma\beta}E_k) \\
&= C^\alpha_{\gamma\beta}\,.
\end{aligned}$$

This proves $d(E_\alpha) = C^\alpha_{\gamma\beta}\, E^\beta \wedge E^\gamma$.

Lemma B.3.3 $d^2 = 0$.

Proof: By Lemma B.1.1 we just need to prove $d^2 = 0$ on an arbitrary cobasis element E^α since $\Lambda(\mathfrak{g}^*)$ is generated by them, and d is an anti-derivation. The condition $d^2 = 0$

turns out to be equivalent with the Jacobi identity on $\mathfrak{g}$, but unfortunately it is a messy ugly proof.

$$\begin{aligned}
d(dE_\alpha) &= d(C^\alpha_{\gamma\beta}\, E^\beta \wedge E^\gamma) \\
&= C^\alpha_{\gamma\beta} \quad dE^\beta \wedge E^\gamma - E^\beta \wedge dE^\gamma \\
&= C^\alpha_{\gamma\beta} \quad (C^\beta_{\epsilon\delta}\, E^\delta \wedge E^\epsilon) \wedge E^\gamma - E^\beta \wedge (C^\gamma_{\epsilon\delta}\, E^\delta \wedge E^\epsilon) \\
&= C^\alpha_{\gamma\beta} C^\beta_{\epsilon\delta}\, E^\delta \wedge E^\epsilon \wedge E^\gamma - C^\alpha_{\gamma\beta} C^\gamma_{\epsilon\delta}\, E^\beta \wedge E^\delta \wedge E^\epsilon \\
&= C^\alpha_{\gamma\beta} C^\beta_{\epsilon\delta}\, E^\delta \wedge E^\epsilon \wedge E^\gamma + C^\alpha_{\gamma\beta} C^\gamma_{\epsilon\delta}\, E^\delta \wedge E^\beta \wedge E^\epsilon\,. \qquad \text{(B.2)}
\end{aligned}$$

Up to this point we have just inserted the result from Lemma B.3.2 for dE^β and dE^γ, making more or less natural choices for the index-symbols $(\alpha, \beta, \gamma, \delta, \epsilon)$. Unfortunately we will have to regroup these expressions quite a bit before we are able to show that (B.2) is in fact the Jacobi identity written out in the structure constants.

Consider the first term in the last equation (B.2), i.e. $C^\alpha_{\gamma\beta} C^\beta_{\epsilon\delta}\, E^\delta \wedge E^\epsilon \wedge E^\gamma$. By using $E^i \wedge E^j = -E^j \wedge E^i$ and $C^k_{ij} = -C^k_{ji}$ we shuffle this around and get

$$\begin{aligned}
&C^\alpha_{\gamma\beta} C^\beta_{\epsilon\delta}\, E^\delta \wedge E^\epsilon \wedge E^\gamma \\
=& - C^\alpha_{\beta\gamma} C^\beta_{\epsilon\delta}\, E^\delta \wedge E^\epsilon \wedge E^\gamma \\
=& - C^\alpha_{\beta\gamma} C^\beta_{\epsilon\delta}\, E^\gamma \wedge E^\delta \wedge E^\epsilon \\
=& C^\alpha_{\beta\gamma} C^\beta_{\epsilon\delta}\, E^\gamma \wedge E^\epsilon \wedge E^\delta\,.
\end{aligned}$$

Now we are going to make things worse by renaming all the indices. If we substitute $a = \gamma, b = \epsilon, c = \delta, k = \beta, l = \alpha$ we get

$$C^l_{ka} C^k_{bc}\, E^a \wedge E^b \wedge E^c\,.$$

Now for the second term in (B.2), i.e. $C^\alpha_{\gamma\beta} C^\gamma_{\epsilon\delta}\, E^\delta \wedge E^\beta \wedge E^\epsilon$. Again by rearranging we get

$$\begin{aligned}
&C^\alpha_{\gamma\beta} C^\gamma_{\epsilon\delta}\, E^\delta \wedge E^\beta \wedge E^\epsilon \\
=& - C^\alpha_{\gamma\beta} C^\gamma_{\epsilon\delta}\, E^\beta \wedge E^\delta \wedge E^\epsilon \\
=& C^\alpha_{\gamma\beta} C^\gamma_{\epsilon\delta}\, E^\beta \wedge E^\epsilon \wedge E^\delta\,.
\end{aligned}$$

Now substituting $a = \beta, b = \epsilon, c = \delta, k = \gamma, l = \alpha$ we get

$$C^l_{ka} C^k_{bc}\, E^a \wedge E^b \wedge E^c\,.$$

So after some work, we now see that the two terms are in fact the same. Now we continue from equation (B.2), adding the terms up, then splitting this new term in three pieces which will turn out to form the three terms in the Jacobi identity

$$\begin{aligned}
d(dE_\alpha) &= \ldots \\
&= 2 \cdot \quad C^l_{ka} C^k_{bc}\, E^a \wedge E^b \wedge E^c \\
&= \frac{2}{3} \quad C^l_{ka} C^k_{bc}\, E^a \wedge E^b \wedge E^c + C^l_{ka} C^k_{bc}\, E^a \wedge E^b \wedge E^c + C^l_{ka} C^k_{bc}\, E^a \wedge E^b \wedge E^c \\
&= \frac{2}{3} \quad C^l_{ka} C^k_{bc}\, E^a \wedge E^b \wedge E^c + C^l_{kb} C^k_{ca}\, E^b \wedge E^c \wedge E^a + C^l_{kc} C^k_{ab}\, E^c \wedge E^a \wedge E^b \\
&= \frac{2}{3} \quad C^l_{ka} C^k_{bc}\, E^a \wedge E^b \wedge E^c + C^l_{kb} C^k_{ca}\, E^a \wedge E^b \wedge E^c + C^l_{kc} C^k_{ab}\, E^a \wedge E^b \wedge E^c \\
&= \frac{2}{3} \quad C^l_{ka} C^k_{bc} + C^l_{kb} C^k_{ca} + C^l_{kc} C^k_{ab} \quad E^a \wedge E^b \wedge E^c \\
&= \frac{2}{3} \quad 0 \quad E^a \wedge E^b \wedge E^c \\
&= 0,
\end{aligned}$$

which finally completes our proof, since we retrieved the Jacobi identity as given in Lemma B.3.1. Note that we again renamed some variables in the fourth equation; for the second term $b = a, c = b, a = c$ and for the third term $c = a, a = b, b = c$.

B.4 Miscellaneous

Lemma B.4.1 The mapping $L : \xi \in \mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A}) \mapsto \mathrm{L}_\xi \in \mathrm{aut}_{\mathcal{B}}(\mathscr{A})$ defined in equation (2.9) as

$$L_\xi = \xi^\alpha L_{E_\alpha} + (d\xi^\alpha) i_{E_\alpha} \ ,$$

for $\xi = E_\alpha \otimes \xi^\alpha$, is a Lie algebra homomorphism.

Proof: Let $\xi = E_\alpha \otimes \xi^\alpha$ and $\zeta = E_\alpha \otimes \zeta^\alpha$ in $\mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A}) \subset \mathfrak{g} \otimes \mathscr{A}^0$. We prove $[\, L_\xi, L_\zeta \,] = L_{[\,\xi,\zeta\,]}$.

$$\begin{aligned}
[\, L_\xi, L_\zeta \,] &= L_\xi L_\zeta - L_\zeta L_\xi \\
&= \ \ \xi^\beta L_{E_\beta} + (d\xi^\beta) i_{E_\beta} \quad \zeta^\gamma L_{E_\gamma} + (d\zeta^\gamma) i_{E_\gamma} \\
&\quad - \ \ \zeta^\gamma L_{E_\gamma} + (d\zeta^\gamma) i_{E_\gamma} \quad \xi^\beta L_{E_\beta} + (d\xi^\beta) i_{E_\beta} \\
&= \xi^\beta \zeta^\gamma \ \ L_{E_\beta} L_{E_\gamma} - L_{E_\gamma} L_{E_\beta} \ \ + (d\xi^\beta)\zeta^\gamma \ \ i_{E_\beta} L_{E_\gamma} - L_{E_\gamma} i_{E_\beta} \\
&\quad + \xi^\beta (d\zeta^\gamma) \ \ L_{E_\beta} i_{E_\gamma} - i_{E_\gamma} L_{E_\beta} \ \ + (d\xi^\beta)(d\zeta^\gamma) i_{E_\beta} i_{E_\gamma} - (d\zeta^\gamma)(d\xi^\beta) i_{E_\gamma} i_{E_\beta} \,.
\end{aligned}$$

Now the last two terms cancel since $(d\zeta^\gamma)(d\xi^\beta) = -(d\xi^\beta)(d\zeta^\gamma)$ (both elements have degree 1) and $i_{E_\gamma} i_{E_\beta} = -i_{E_\beta} i_{E_\gamma}$. We continue, using $[\, L_X, L_Y \,] = L_{[\,X,Y\,]}$ and $L_X i_Y - i_Y L_X = i_{[\,X,Y\,]}$, and obtain

$$\begin{aligned}
\ldots &= \xi^\beta \zeta^\gamma \ \ L_{E_\beta} L_{E_\gamma} - L_{E_\gamma} L_{E_\beta} \ \ + (d\xi^\beta)\zeta^\gamma \ \ i_{E_\beta} L_{E_\gamma} - L_{E_\gamma} i_{E_\beta} \\
&\quad + \xi^\beta (d\zeta^\gamma) \ \ L_{E_\beta} i_{E_\gamma} - i_{E_\gamma} L_{E_\beta} \\
&= \xi^\beta \zeta^\gamma \ \ L_{[\,E_\beta, E_\gamma\,]} \ \ + (d\xi^\beta)\zeta^\gamma \ \ -i_{[\,E_\gamma, E_\beta\,]} \ \ + \xi^\beta (d\zeta^\gamma) \ \ i_{[\,E_\beta, E_\gamma\,]} \\
&= \xi^\beta \zeta^\gamma \ \ L_{(C^\alpha_{\beta\gamma}\, E_\alpha)} \ \ + \ \ (d\xi^\beta)\zeta^\gamma + \xi^\beta (d\zeta^\gamma) \ \ i_{(C^\alpha_{\beta\gamma}\, E_\alpha)} \\
&= C^\alpha_{\beta\gamma}\, \xi^\beta \zeta^\gamma L_{E_\alpha} + d(C^\alpha_{\beta\gamma}\, \xi^\beta \zeta^\gamma) i_{E_\alpha} \\
&= [\,\xi,\zeta\,]^\alpha L_{E_\alpha} + d(\,[\,\xi,\zeta\,]^\alpha\,) i_{E_\alpha} \\
&= L_{[\,\xi,\zeta\,]},
\end{aligned}$$

which finishes our proof.

Notation

Symbol	***Meaning***
$\mathsf{1}$	unit in an algebra
$\mathscr{A}$	algebra (possibly a GCDA, DGLA or $\mathfrak{g}$-operation)
a	gauge potential
$\mathcal{A}$	algebraic connection
$a_{pot}(M)$	space of gauge potentials on base manifold M for a trivial bundle
$\mathsf{A}(\mathfrak{g})$	(universal) Weil-B.R.S. algebra over $\mathfrak{g}$
$\mathrm{Aut}_v(P)$	vertical bundle automorphism on a principal bundle $P(G, M)$
$\mathrm{Aut}_{\mathcal{B}}(\mathscr{A})$	generalized group of gauge transformations for a $\mathfrak{g}$-operation $\mathscr{A}$
$\mathrm{aut}_{\mathcal{B}}(\mathscr{A})$	generalized Lie algebra of infinitesimal gauge transformations for a $\mathfrak{g}$-operation $\mathscr{A}$
$\mathrm{aut}^{(0)}_{\mathcal{B}}(\mathscr{A})$	generalized Lie algebra of infinitesimal gauge transformations for a $\mathfrak{g}$-operation $\mathscr{A}$ (alternative definition)
$\mathscr{B}(\mathscr{A})$	basic elements of a $\mathfrak{g}$-operation $\mathscr{A}$
$\mathscr{C}(\mathscr{A})$	space of algebraic connection forms on a $\mathfrak{g}$-operation $\mathscr{A}$
$\mathscr{C}(P)$	space of connection forms on principal bundle $P(G, M)$
$C^n(\mathfrak{g})$	space of n-cochains of a Lie algebra $\mathfrak{g}$
$C^n(\mathfrak{g}, \mathcal{V})$	space of $\mathcal{V}$-valued n-cochains of a Lie algebra $\mathfrak{g}$
$\mathrm{Der}^{(k)}(\mathscr{A})$	graded derivations of degree k on the algebra $\mathscr{A}$
E_α	element of a basis (usually of $\mathfrak{g}$)
E^α	element of a cobasis (usually of $\mathfrak{g}$)
$(...)^\alpha$	the α-component of some $\mathfrak{g}$-valued object, with respect to some fixed basis $\{E_\alpha\}$
$\mathcal{F}$	algebraic curvature
f	field strength
$\mathscr{G}$	group of gauge transformations
$\mathfrak{g}$	Lie algebra
$\mathfrak{g}^*$	dual of a Lie algebra $\mathfrak{g}$

Symbol	***Meaning***
$\mathscr{H}(\mathscr{A})$	horizontal elements of a $\mathfrak{g}$-operation $\mathscr{A}$
$\mathscr{I}(\mathscr{A})$	invariant elements of a $\mathfrak{g}$-operation $\mathscr{A}$
$Lie(G)$	Lie algebra associated to a Lie group G
$Lie(\mathscr{G})$	Lie algebra of infinitesimal gauge transformations
$\Lambda\mathfrak{g}^*$	exterior algebra over $\mathfrak{g}^*$
$R(...)$	group representation
$\rho(...)$	Lie algebra representation
$S\mathfrak{g}^*$	symmetric algebra over $\mathfrak{g}^*$
$(S\mathfrak{g}^*)_{\rm inv}$	invariant elements of $S\mathfrak{g}^*$ (under the coadjoint action of $\mathfrak{g}$)
Θ_{MC}	Maurer-Cartan form
$X, Y, ..$	Lie algebra elements (usually of $Lie(G)$) or vector fields
$\mathfrak{X}(P)$	space of vector fields on a manifold P
$\mathsf{W}(\mathfrak{g})$	(universal) Weil algebra over $\mathfrak{g}$
ω	(i) arbitrary algebra element (ii) connection form on a principal bundle
$\xi, \zeta, ..$	elements of the Lie algebra $Lie(\mathscr{G})$ of infinitesimal gauge transformations
Ω	curvature form associated to a connection form
$\mathcal{V}$	vector space

Bibliography

[1] A. Alekseev, E. Meinrenken. *Lie Theory and the Chern-Weil Homomorphism.* Arxiv:math.RT/0308135, 2003.

[2] J.A. de Azcárraga, J.M. Izquierdo. *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics.* Cambridge University Press 1995.

[3] L. Bonora, P. Cotta-Ramusino. *Some Remarks on B.R.S. Transformations, Anomalies and the Cohomology of the Lie Algebra of the Group of Gauge Transformations.* Commun. Math. Phys. 87, 1983.

[4] R. Bott, L.W. Tu. *Differential Forms in Algebraic Topology.* Springer-Verlag 1982.

[5] H. Cartan. *Notion d'algébre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie* and *La transgression dans un groupe de Lie et dans un espace fibré principal* in: Colloque de topologie, Bruxelles 1950.

[6] M. Dubois-Violette. *The Weil-B.R.S. algebra of a Lie algebra and the anomalous terms in gauge theory.* Journal of Geometry and Physics, Vol. 3, n. 4, 1986.

[7] M. Dubois-Violette, M. Talon, C.M. Viallet. *B.R.S. algebras. Analysis of the Consistency Equations in Gauge Theory.* Commun. Math. Phys. 102, 1985.

[8] J.J. Duistermaat, J.A.C. Kolk. *Lie Groups.* Springer-Verlag Heidelberg 2000.

[9] D.B. Fuks. *Cohomology of Infinite-Dimensional Lie Algebras.* Consultants Bureau (Plenum Publishing Corporation), New York 1986.

[10] W. Greub, S. Halperin, R. Vanstone. *Connections, Curvature, and Cohomology.* Vols. II, III. Academic Press, 1973.

[11] D. Kastler, R. Stora. *A differential geometric setting for BRS transformations and anomalies I (II).* Journal of Geometry and Physics, Vol. 3, n. 4, 1986.

[12] E. Langmann. *Descent equations of Yang-Mills anomalies in noncommutative geometry.* Arxiv:hep-th/9508003, 1995.

[13] J. Mañes, R. Stora, B. Zumino. *Algebraic Study of Chiral Anomalies.* Commun. Math. Phys. 102, 1985.

[14] J. Milnor. *Remarks on infinite-dimensional Lie groups.* In: Relativity, groups and topology II. *Les Houches 1983, Session XL.* B.S. DeWitt, R. Stora (eds). Amsterdam, North-Holland 1984.

[15] G.L. Naber. *Topology, Geometry, and Gauge Fields.* Volume I: *Foundations.* Springer-Verlag New York 1997. Volume II: *Interactions.* Springer-Verlag New York 2000.

[16] R. Schmid. *Local cohomology in gauge theories, BRST transformations and anomalies.* Diff. Geom. Appl. 4 (107), 1994.

[17] V.S. Varadarajan. *Lie Groups, Lie Algebras, and Their Representations.* Prentice-Hall 1974.

[18] G. Walschap. *Metric Structures in Differential Geometry.* Springer-Verlag New York 2004.