Lecture Notes on Operator Algebras

Draft: December 14, 2011

N.P. Landsman

Institute for Mathematics, Astrophysics, and Particle Physics
Radboud University Nijmegen
Heyendaalseweg 135
6525 AJ NIJMEGEN
THE NETHERLANDS

email: landsman@math.ru.nl
website: http://www.math.ru.nl/∼landsman/OA2011.html
tel.: 024-3652874
office: HG03.740

1 Historical notes and overview

The theory of operator algebras tends to be rather technical. Some textbooks start straight away with the definition of at least three (and at most eleven) different topologies on the space of all bounded operators on a Hilbert space. Instead, we begin with an informal survey, which is partly historical in nature. One of our aims is to emphasize the origins of operator algebras in (quantum) physics.

In fact, the theory of operator algebras has two quite different sources in the 1930s and 1940s, respectively, with two associated great mathematicians:

- *Hilbert spaces* (John von Neumann),[1] leading to the theory of *von Neumann algebras* (originally called *rings of operators* by von Neumann himself).
- *Commutative Banach algebras* (Israel Gelfand),[2] giving rise to C^**-algebras*.

So, roughly speaking, the theory of operator algebras is the same as the theory of von Neumann algebras and C^*-algebras.[3] Let us elaborate on each of these in turn.

1.1 John von Neumann and quantum mechanics

John von Neumann (1903–1957) was a Hungarian prodigy; he wrote his first mathematical paper at the age of seventeen. Except for this first paper, his early work was in set theory and the foundations of mathematics. In the Fall of 1926, he moved to Göttingen to work with Hilbert, the most prominent mathematician of his time. Around 1920, Hilbert had initiated his *Beweistheory*, an approach to the axiomatization of mathematics that was doomed to fail in view of Gödel's later work. However, at the time that von Neumann arrived, Hilbert was also interested in quantum mechanics.

From 1900 onwards, physicists had begun to recognize that the classical physics of Newton and Maxwell could not describe all of Nature. The fascinating era that was thus initiated by Planck, to be continued mainly by Einstein and Bohr, ended in 1925 with the discovery of quantum mechanics. This theory replaced classical mechanics, and was initially discovered in two guises. Schrödinger was led to a formulation called 'wave mechanics,' in which the famous symbol Ψ, denoting a 'wave function,' played an important role. Heisenberg discovered a form of quantum mechanics that at the time was called 'matrix mechanics.' The relationship and possible equivalence between these alternative formulations of quantum mechanics, which at first sight looked completely different, was much discussed at the time.

Heisenberg's paper initiating matrix mechanics was followed by the 'Dreimännerarbeit' of Born, Heisenberg, and Jordan (1926); all three were in Göttingen at the time. Born was one of the few physicists of his day to be familiar with the

[1] Some of von Neumann's papers on operator algebras were coauthored by F.J. Murray.

[2] Some of Gelfand's papers on operator algebras were coauthored by M. Naimark.

[3] More recently, operator algebras that are not closed under involution have been studied, in close connection to so-called *operator spaces*. See, for example, [12].

concept of a matrix; in previous research he had even used infinite matrices.[4] Born turned to his former teacher Hilbert for mathematical advice. Hilbert had been interested in the mathematical structure of physical theories for a long time; his Sixth Problem (1900) called for the mathematical axiomatization of physics. Aided by his assistants Nordheim and von Neumann, Hilbert thus ran a seminar on the mathematical structure of quantum mechanics, and the three wrote a joint paper on the subject (which is now obsolete).

It was von Neumann alone who, at the age of 23, recognized the mathematical structure of quantum mechanics. In this process, he defined the abstract concept of a Hilbert space, which previously had only appeared in some examples. These examples went back to the work of Hilbert and his pupils in Göttingen on integral equations, spectral theory, and infinite-dimensional quadratic forms. Hilbert's famous memoirs on integral equations had appeared between 1904 and 1906. In 1908, his student E. Schmidt had defined the space ℓ^2 in the modern sense, and F. Riesz had studied the space of all continuous linear maps on ℓ^2 in 1912. Various examples of L^2-spaces had emerged around the same time. However, the abstract notion of a Hilbert space was missing until von Neumann provided it.

Von Neumann saw that Schrödinger's wave functions were unit vectors in a Hilbert space of L^2 type, and that Heisenberg's observables were linear operators on a different Hilbert space, of ℓ^2 type. A unitary transformation between these spaces provided the the mathematical equivalence between wave mechanics and matrix mechanics. (Similar, mathematically incomplete insights had been reached by Pauli and Dirac.) In a series of papers that appeared between 1927–1929, von Neumann defined Hilbert space, formulated quantum mechanics in this language, and developed the spectral theory of bounded as well as unbounded normal operators on a Hilbert space. This work culminated in his book *Mathematische Grundlagen der Quantenmechanik* (1932), which to this day remains the definitive account of the mathematical structure of elementary quantum mechanics.[5]

More precisely, von Neumann proposed the following mathematical formulation of quantum mechanics (cf. Ch. 2 below for minimal background on Hilbert spaces).

1. The **observables** of a given physical system are the self-adjoint linear operators a on a Hilbert space H.

2. The **states** of the system are the so-called **density operators** $\hat{\rho}$ on H, that is, the positive trace-class operators on H with unit trace.

3. The **expectation value** $< a >_{\hat{\rho}}$ of an observable a in a state $\hat{\rho}$ is given by

$$< a >_{\hat{\rho}} = \mathrm{Tr}\,(\hat{\rho} a). \tag{1.1}$$

[4]Heisenberg's fundamental equations of quantum mechanics, viz. $pq - qp = -\hbar i$, which initially were quite mysterious, could only be satisfied by infinite-dimensional matrices.

[5]Von Neumann's book was preceded by Dirac's *The Principles of Quantum Mechanics* (1930), which contains another brilliant, but this time mathematically questionable account of quantum mechanics in terms of linear spaces and operators. See e.g. [29, 34] for modern accounts.

As a special case, take a unit vector Ψ in H and form the associated projection

$$p_\Psi = |\Psi\rangle\langle\Psi|, \tag{1.2}$$

where we use the following notation (due to Dirac): for any two vectors Ψ, Φ in H, the operator $|\Psi\rangle\langle\Phi|$ is defined by[6]

$$|\Psi\rangle\langle\Phi|\Omega = (\Phi, \Omega)\Psi. \tag{1.3}$$

In particular, if $\Phi = \Psi$ is a unit vector, then (1.2) is the projection onto the one-dimensional subspace $\mathbb{C} \cdot \Psi$ of H spanned by Ψ (see Exercise 1 below for a *rappèl* on projections). In that case, it is easily shown (see Exercise 2) that the density operator $\hat{\rho} = p_\Psi$ leads to

$$< a >_{p_\Psi} \equiv < a >_\Psi = \mathrm{Tr}\,(p_\Psi a) = (\Psi, a\Psi). \tag{1.4}$$

Special states like p_Ψ (often confused with Ψ itself, which contains additional phase information) are called **pure states**, whereas all other states are said to be **mixed**.

Let $B(H)$ be the space of all bounded operators on H, with unit operator simply denoted by 1. A functional $\omega : B(H) \to \mathbb{C}$ (which is *linear* by definition) is called:

- **positive** when $\omega(a^*a) \geq 0$ for all $a \in B(H)$;
- **normalized** when $\omega(1) = 1$.

We will see that positivity implies continuity (see Exercise 3). For reasons to become clear shortly, a normalized positive functional on $B(H)$ is called a **state** on $B(H)$; our earlier use of this word should accordingly be revised a little. Indeed, it is trivial to show that for any density operator $\hat{\rho}$, the functional $\rho : B(H) \to \mathbb{C}$ defined by

$$\rho(a) = \mathrm{Tr}\,(\hat{\rho}a) \tag{1.5}$$

is a state on $B(H)$. Conversely, are all states on $B(H)$ of this kind?

Von Neumann implicitly assumed a certain continuity condition on states, which in modern terminology is called **σ-weak** (or **ultraweak**) continuity, which implies that the answer is yes; states à la (1.5) (in other words, σ-weakly continuous states) are called **normal** states on $B(H)$. However, without this continuity condition the set of states on $B(H)$ turns out not to be exhausted by density operators on H (unless H is finite-dimensional), although it is hard to give explicit examples.

The set of all states on $B(H)$ is obviously convex (within $B(H)^*$), as is its subset of all normal states on $B(H)$. The **extreme boundary** of a convex set K is the set of all $\omega \in K$ that are indecomposable, in the sense that if $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ for some $0 < \lambda < 1$ and $\omega_1, \omega_2 \in K$, then $\omega_1 = \omega_2 = \omega$.[7] Von Neumann saw that states (1.4) precisely correspond to the points in the extreme boundary of the convex set $S_n(B(H))$ of normal states on $B(H)$: on other words, a density operator $\hat{\rho}$ yields an extreme point ρ in $S_n(B(H))$ iff it is of the form (1.4). See also Exercise 4.

[6] Dirac wrote $|\Omega\rangle$ for Ω, etc., which is superfluous, but in this special case it leads to the neater expression $|\Psi\rangle\langle\Phi|\Omega\rangle = \langle\Phi|\Omega\rangle|\Psi\rangle$, where the inner product is written as $\langle\Phi|\Omega\rangle \equiv (\Phi, \Omega)$.

[7] In some examples of compact convex sets in $\mathbb{R}^n$, the extreme boundary of K coincides with its geometric boundary; cf. the closed unit ball. However, the extreme boundary of an equilateral triangle consists only of its corners. If K fails to be compact, its extreme boundary may even be empty, as illustrated by the open unit ball in any dimension.

1.2 Von Neumann algebras

In one of his papers on Hilbert space theory (1929), von Neumann defined a **ring of operators** M (nowadays called a **von Neumann algebra**) as a *-subalgebra of the algebra $B(H)$ of all bounded operators on a Hilbert space H that contains the unit 1 and is closed in the weak operator topology. This means that:

- M is a subalgebra of $B(H)$ with unit under operator multiplication;
- M is closed under taking the adjoint (or Hermitian conjugate) $a \mapsto a^*$;
- If $(v, (a_\lambda - a)w) \to 0$ for all $v, w \in H$ for some net (a_λ) in M, then $a \in M$.

For example, $B(H)$ is itself a von Neumann algebra. When H is finite-dimensional, any direct sum of matrix algebras containing 1 is a von Neumann algebra.

In the same paper, von Neumann proved what is still the first and most basic theorem of the subject, called the **Double Commutant Theorem**:[8]

> *Let M be a unital *-subalgebra of $B(H)$. Then the following condtions are equivalent (and hence each defines M to be a von Neumann algebra):*
>
> - $M'' = M$;
> - *M is closed in the weak operator topology;*
> - *M is closed in the strong operator topology;*
> - *M is closed in the σ-weak operator topology.*

Here, for any Hilbert space H, let $S \subset B(H)$ be some subset. The **commutant** of S is defined by

$$S' := \{b \in B(H) \mid ab = ba \, \forall a \in S\}. \tag{1.6}$$

Note that S' is a subalgebra of $B(H)$. Similarly, one defines the **bicommmutant** $S'' = (S')'$ of S (it makes no sense to go on, since $S''' = S'$, see Exercise 5).

The **strong** operator topology on $B(H)$ may be defined in saying that a net a_λ converges to a iff $a_\lambda v \to av$ for all $v \in H$. It should be mentioned that, though easily defined, neither the weak topology on a von Neumann algebra M nor the strong one is a natural one; the natural topology on a von Neumann algebra $M \subseteq B(H)$ turns out to be the **σ-weak** or **ultraweak** one. This topology is provided by the seminorms $\|a\|_{\hat{\rho}} = |\mathrm{Tr}\,(\hat{\rho}a)|$, where $\hat{\rho}$ is an element of the trace-class $B_1(H)$, cf. §2.10. Hence $a_\lambda \to a$ σ-weakly when $\mathrm{Tr}\,(\hat{\rho}(a_\lambda - a)) \to 0$ for all $\hat{\rho} \in B_1(H)$. Here it turns out that one could equally well restrict $\hat{\rho}$ to be a density operator on H, so that a physicist would be justified in saying that the σ-weak topology is the topology of pointwise convergence of quantum-mechanical expectation values.

Von Neumann's motivation in studying rings of operators was plurifold; beyond quantum theory, we mention probability theory, entropy, ergodic theory, discrete groups, representation theory, projective geometry, and lattice theory.

[8] This theorem is remarkable, in relating a topological condition to an algebraic one; one is reminded of the much simpler fact that a linear subspace K of H is closed iff $K^{\perp\perp} = K$, where $K^\perp$ is the orthogonal complement of K in H.

1.3 C*-algebras

Following the pioneering work of von Neumann, an important second step in the theory of operator algebras was the initiation of the theory of *C^*-algebras* by Gelfand and Naimark in 1943. It turns out that von Neumann's "rings of operators" are special cases of C^*-algebras, but von Neumann algebras also continue to be studied on their own. A fruitful mathematical analogy is that C^*-algebras provide a noncommutative generalization of *topology*, whereas von Neumann algebras comprise noncommutative *measure theory*.[9] To understand this, we state the most important fact about C^*-algebras, namely that there are two totally different way of approaching them:

- As norm-closed *-subalgebras $A \subset B(H)$;
- As noncommutative generalizations of the space $C(X) \equiv C(X, \mathbb{C})$ of complex-valued continuous functions on a compact space X.

We start with the second. In 1943, Gelfand and Naimark noted that the space $C(X)$ has the following additional structure beyond just being a commutative algebra over $\mathbb{C}$ (see Exercises). Firstly, it has a norm, given by ($a \in C(X)$)

$$\|a\|_\infty := \sup\{|a(x)|,\ x \in X\},$$

in which it is a Banach space. This Banach space structure of $C(X)$ is compatible with its structure as an algebra by the property

$$\|ab\| \le \|a\|\|b\|, \tag{1.7}$$

where we have written $\|\cdot\|$ for $\|\cdot\|_\infty$, and $a, b \in C(X)$. But more structure is needed!

An **involution** on an algebra A is a real-linear map $A \to A^*$ such that $a^{**} = a$, $(ab)^* = b^* a^*$, and $(\lambda a)^* = \overline{\lambda} a^*$ for all $a, b \in A$ and $\lambda \in \mathbb{C}$. An algebra with involution is also called a ***-algebra**. Secondly, then, $C(X)$ has an involution $a \mapsto a^*$, given by $a^*(x) = \overline{a(x)}$. This involution is related to the norm as well as to the algebraic structure by the property

$$\|a^* a\| = \|a\|^2. \tag{1.8}$$

We summarize these properties by saying that $C(X)$ is a **commutative C^*-algebra** with unit, in the following sense:

> *A (commutative) C^*-algebra is a Banach space that at the same time is a (commutative) algebra with involution, such that the compatibility conditions (1.7) and (1.8) hold.*

The first theorem of Gelfand and Naimark then reads as follows:

> *Every commutative C^*-algebra A with unit is isomorphic to $C(X)$ for some compact Hausdorff space X, unique up to homeomorphism.*

[9]These analogies form the basis of *noncommutative geometry* as developed by Connes [5].

The isomorphism is constructed as follows. The space X is often denoted by $\Sigma(A)$ and is called the **Gelfand spectrum** of A. It may be realized as the set of **multiplicative functionals** or **characters** on A, that is, nonzero linear maps $\omega : A \to \mathbb{C}$ that satisfy $\omega(ab) = \omega(a)\omega(b)$.[10] Thus one takes $X := \Delta(A)$, which turns out to be a compact Hausdorff space in the topology of pointwise convergence, and the map $A \to C(X)$ is the so-called **Gelfand transform** $a \mapsto \hat{a}$, where $\hat{a}(\omega) := \omega(a)$.

Similar to their characterization of commutative C^*-algebras above, Gelfand and Naimark also quite brilliantly clarified the nature of general C^*-algebras. Let us first note that $B(H)$ is a C^*-algebra (See Exercise 6). Moreover, any norm-closed *-subalgebra of $B(H)$ is a C^*-algebra (which is trivial, given the previous result). Perhaps surprisingly, this is also the most general kind of C^*-algebra, for the second theorem of Gelfand and Naimark (contained in the same paper as their first) reads:

> *Every C^*-algebra A is isomorphic to a norm-closed *-subalgebra of $B(H)$, for some Hilbert space H.*

Note that, in contrast to the previous theorem, H is by no means unique in any sense! The proof of this theorem is based on the so-called GNS**-construction** (after Gelfand, Naimark, and the American mathematician I.E. Segal), which basically explains why C^*-algebras are naturally related to Hilbert spaces, and which pervades the subject in every conceivable way. This construction starts with the concept of a **state** on a C^*-algebra A, which we have already encountered for $A = B(H)$:

> *A state on a C^*-algebra A with unit is a functional $\omega : A \to \mathbb{C}$ that satisfies $\omega(a^*a) \geq 0\, \forall a \in A$, (positivity) and $\omega(1) = 1$ (normalization).*[11]

The characters of a commutative C^*-algebra are examples of states. Let us first suppose that $\omega(a^*a) > 0$ for all a, and that A has a unit. In that case, A is a pre-Hilbert space in the inner product $(a, b)_\omega := \omega(a^*b)$, which may be completed into a Hilbert space H_ω. Then A acts on H_ω by means of $\pi_\omega : A \to B(H_\omega)$, given by $\pi_\omega(a)b := ab$.[12] It is easy to see that π_ω is injective: if $\pi_\omega(a) = 0$ then, taking $b = 1$, one infers that $a = 0$ as an element of H_ω, but then $(a, a)_\omega = \omega(a^*a) = 0$, contradicting the assumption that $\omega(a^*a) > 0$. Moreover, one checks that π_ω is a homomorphism of C^*-algebras, for example,

$$\pi_\omega(a)\pi_\omega(b)c = \pi_\omega(a)bc = abc = \pi_\omega(ab)c$$

for all c, which implies $\pi_\omega(a)\pi_\omega(b) = \pi_\omega(ab)$. Thus A is isomorphic to $\pi_\omega(A) \subset B(H_\omega)$.[13] In general, A may not possess such strictly positive functionals, but it always has sufficiently many states. For an arbitrary state ω, the Hilbert space H_ω is constructed by first dividing A by the kernel of $(\cdot,\cdot)_\omega$, and proceeding in the same way. The representation π_ω may then fail to be injective, but by taking the direct sum of enough such representations one always arrives at an injective one.

[10]These functionals lie in the dual space A^* of A. The topology in which the theorem holds is the (relative) weak* topology on $\Sigma(A) \subset A^*$, also called the **Gelfand topology**.

[11]Positive functionals on a C^*-algebra are continuous, with norm $\|\omega\| = \omega(1)$. A state on a C^*-algebra A without unit is defined as a positive functional $\omega : A \to \mathbb{C}$ that satisfies $\|\omega\| = 1$.

[12]Initially defined on the dense subspace $A \subset H_\omega$, and subsequently extended by continuity.

[13]A more technical argument shows that π_ω is isometric, so that its image is closed.

Exercises for Lecture 1

1. A (bounded) operator $p : H \to H$ is called a **projection** when $p^2 = p^* = p$. Prove that there is a bijective correspondence between projections in $B(H)$ and closed subspaces of H (which by definition are *linear*).

   Hint: the closed subspace K_p corresponding to p is $K_p = pH \equiv \text{Range}(p)$. Find the projection p_K corresponding to some given closed subspace $K \subset H$ and show that $p_{K_p} = p$ and $K_{p_K} = K$.

2. Prove (1.4).

3. Prove that a state ω on a C^*-algebra A (with unit) satisfies $|\omega(a)| \leq \|a\|$ (and hence is continuous with norm $\|\omega\| = \omega(1)$).

4. The convex structure of the state space is nicely displayed by $H = \mathbb{C}^2$, so that $B(H) = M_2(\mathbb{C})$, the C^*-algebra of 2×2 complex matrices. Put

$$\rho(x, y, z) = \tfrac{1}{2} \begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}. \tag{1.9}$$

   (a) Show that $\rho(x, y, z)$ is a density operator on $\mathbb{C}^2$ iff $(x, y, z) \in \mathbb{R}^3$ with

$$x^2 + y^2 + z^2 \leq 1.$$

   (b) Show that every state ω on $M_2(\mathbb{C})$ is of the form ω_ρ, with

$$\omega_\rho(a) = \text{Tr}\,(\rho a). \tag{1.10}$$

   (c) Conclude that the state space $S(M_2(\mathbb{C}))$ of $M_2(\mathbb{C})$ is isomorphic (as a convex set) to the three-ball $B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$.

   (d) Under this isomorphism, show that the extreme boundary of $S(M_2(\mathbb{C}))$ corresponds to the two-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

   (e) Verify that the states in the extreme boundary of $S(M_2(\mathbb{C}))$ are exactly those of the form (1.4), where $\Psi \in \mathbb{C}^2$ is a unit vector.

5. Prove that $S''' = S'$, for any subset $S \subset B(H)$.

6. Prove that $B(H)$ is a C^*-algebra.

   Hint: the only reasonably difficult part is the proof of $\|a^* a\| = \|a\|^2$.

2 Review of Hilbert spaces

2.1 Inner product, norm, and metric

The following definitions are basic to all of functional analysis. Note that the concept of a metric applies to any set (i.e., not necessarily to a vector space).

Definition 2.1 *Let V be a vector space over $\mathbb{C}$.*

1. *An* ***inner product*** *on V is a map $V \times V \to \mathbb{C}$, written as $\langle f, g \rangle \mapsto (f, g)$, satisfying, for all $f, g, h \in V$, $t \in \mathbb{C}$:*
   (a) $(f, f) \in \mathbb{R}^+ := [0, \infty)$ *(positivity);*
   (b) $(g, f) = \overline{(f, g)}$ *(symmetry);*
   (c) $(f, tg) = t(f, g)$ *(linearity 1);*
   (d) $(f, g + h) = (f, g) + (f, h)$ *(linearity 2);*
   (e) $(f, f) = 0 \Rightarrow f = 0$ *(positive definiteness).*
2. *A* ***norm*** *on V is a function $\| \cdot \| : V \to \mathbb{R}^+$ such that for all $f, g, h \in V$, $t \in \mathbb{C}$:*
   (a) $\|f + g\| \leq \|f\| + \|g\|$ *(triangle inequality);*
   (b) $\|tf\| = |t|\|f\|$ *(homogeneity);*
   (c) $\|f\| = 0 \Rightarrow f = 0$ *(positive definiteness).*
3. *A* ***metric*** *on V is a function $d : V \times V \to \mathbb{R}^+$ satisfying, for all $f, g, h \in V$:*
   (a) $d(f, g) \leq d(f, h) + d(h, g)$ *(triangle inequality);*
   (b) $d(f, g) = d(g, f)$ *for all $f, g \in V$ (symmetry);*
   (c) $d(f, g) = 0 \Leftrightarrow f = g$ *(definiteness).*

These structures are related in the following way:

Proposition 2.2 1. *An inner product on V defines a norm on V by*

$$\|f\| = \sqrt{(f, f)}. \tag{2.11}$$

2. *This norm satisfies the* ***Cauchy–Schwarz inequality***

$$|(f, g)| \leq \|f\|\|g\|. \tag{2.12}$$

3. *A norm $\| \cdot \|$ on a complex vector space comes from an inner product iff*

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2), \tag{2.13}$$

in which case

$$(f, g) = \tfrac{1}{4}(\|f + g\|^2 - \|f - g\|^2 + i\|f - ig\|^2 - i\|f + ig\|^2). \tag{2.14}$$

4. *A norm on V defines a metric on V through $d(f, g) := \|f - g\|$.*

2.2 Completeness

Many concepts of importance for Hilbert spaces are associated with the metric rather than with the underlying inner product or norm. The main example is *convergence*:

Definition 2.3 *1. Let* $(x_n) := \{x_n\}_{n\in\mathbb{N}}$ *be a sequence in a metric space* (V, d)*. We say that* $x_n \to x$ *for some* $x \in V$ *when* $\lim_{n\to\infty} d(x_n, x) = 0$*, or, more precisely: for any* $\varepsilon > 0$ *there is* $N \in \mathbb{N}$ *such that* $d(x_n, x) < \varepsilon$ *for all* $n > N$*.*

In a normed space, hence in particular in a space with inner product, this therefore means that $x_n \to x$ *if* $\lim_{n\to\infty} \|x_n - x\| = 0$*.*

2. A sequence (x_n) *in* (V, d) *is called a* ***Cauchy sequence*** *when* $d(x_n, x_m) \to 0$ *when* $n, m \to \infty$*; more precisely: for any* $\varepsilon > 0$ *there is* $N \in \mathbb{N}$ *such that* $d(x_n, x_m) < \varepsilon$ *for all* $n, m > N$*.*

In a normed space, this means that (x_n) *is Cauchy when* $\|x_n - x_m\| \to 0$ *for* $n, m \to \infty$*, in other words, if* $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$*.*

Clearly, a convergent sequence is Cauchy: from the triangle inequality and symmetry one has $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x)$, so for given $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon/2$, et cetera. However, the converse statement does not hold in general, as is clear from the example of the metric space $(0, 1)$ with metric $d(x, y) = |x - y|$: the sequence $x_n = 1/n$ does not converge in $(0, 1)$. In this case one can simply extend the given space to $[0, 1]$, in which every Cauchy sequence does converge.

Definition 2.4 *A metric space* (V, d) *is called* ***complete*** *when every Cauchy sequence in* V *converges (i.e., to an element of* V*).*

- *A vector space with norm that is complete in the associated metric is called a* ***Banach space****. In other words: a vector space* B *with norm* $\|\cdot\|$ *is a Banach space when every sequence* (x_n) *such that* $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$ *has a limit* $x \in B$ *in the sense that* $\lim_{n\to\infty} \|x_n - x\| = 0$*.*
- *A vector space with inner product that is complete in the associated metric is called a* ***Hilbert space****. In other words: a vector space* H *with inner product* $(\ ,\ )$ *is a Hilbert space when it is a Banach space in the norm* $\|x\| = \sqrt{(x, x)}$*.*

A subspace of a Hilbert space may or may not be closed. A **closed subspace** $K \subset H$ of a Hilbert space H is by definition complete in the given norm on H (i.e. any Cauchy sequence in K converges to an element of K).[14] This implies that a closed subspace K of a Hilbert space H is itself a Hilbert space if one restricts the inner product from H to K. If K is not closed already, we define its **closure** $\overline{K}$ as the smallest closed subspace of H containing K; once again, this is a Hilbert space.

[14] Since H is a Hilbert space we know that the sequence has a limit in H, but this may not lie in K even when all elements of the sequence do. This is possible precisely when K fails to be closed.

2.3 Geometry of Hilbert space

The vector spaces $\mathbb{C}^n$ from linear algebra are Hilbert spaces in the usual inner product $(z, w) = \sum_{k=1}^{n} \overline{z_k} w_k$. Indeed, a finite-dimensional vector space is automatically complete in any possible norm. More generally, Hilbert spaces are the vector spaces whose geometry is closest to that of $\mathbb{C}^n$, because the inner product yields a notion of orthogonality: we say that two vectors $f, g \in H$ are **orthogonal**, written $f \perp g$, when $(f, g) = 0$.[15] Similary, two subspaces[16] $K \subset H$ and $L \subset H$ are said to be orthogonal ($K \perp L$) when $(f, g) = 0$ for all $f \in K$ and all $g \in L$. A vector f is called orthogonal to a subspace K, written $f \perp K$, when $(f, g) = 0$ for all $g \in K$, etc. We define the **orthogonal complement** $K^\perp$ of a subspace $K \subset H$ as

$$K^\perp := \{f \in H \mid f \perp K\}. \tag{2.15}$$

This set is linear, so that the map $K \mapsto K^\perp$, called **orthocomplementation**, is an operation from subspaces of H to subspaces of H. Clearly, $H^\perp = 0$ and $0^\perp = H$.

Closure is an analytic concept, related to convergence of sequences. Orthogonality is a geometric concept. However, both are derived from the inner product. Hence one may expect connections relating analysis and geometry on Hilbert space.

Proposition 2.5 *Let $K \subset H$ be a subspace of a Hilbert space.*

1. *The subspace $K^\perp$ is closed, with*

$$K^\perp = \overline{K}^\perp = \overline{K^\perp}. \tag{2.16}$$

2. *One has*

$$K^{\perp\perp} := (K^\perp)^\perp = \overline{K}. \tag{2.17}$$

3. *Hence for closed subspaces K one has $K^{\perp\perp} = K$.*

Definition 2.6 *An* ***orthonormal basis*** *(o.n.b.) in a Hilbert space is a set (e_k) of vectors satisfying $(e_k, e_l) = \delta_{kl}$ and being such that any $v \in H$ can be written as $v = \sum_k v_k e_k$ for some $v_k \in \mathbb{C}$, in that $\lim_{N\to\infty} \|v - \sum_{k=1}^{N} v_k e_k\| = 0$.*

If $v = \sum_k v_k e_k$, then, as in linear algebra, $v_k = (e_k, v)$, and $\sum_k |v_k|^2 = \|v\|^2$. This is called **Parseval's equality**; it is a generalization of Pythagoras's Theorem.

Once more like in linear algebra, all o.n.b. have the same cardinality, which defines the **dimension** of H. We call an infinite-dimensional Hilbert space **separable** when it has a *countable* o.n.b. Dimension is a very strong invariant: running ahead of the appropriate definition of isomorphism of Hilbert spaces in §2.4, we have

Theorem 2.7 *Two Hilbert spaces are isomorphic iff they have the same dimension.*

---

[15] By definition of the norm, if $f \perp g$ one has Pythagoras' theorem $\|f + g\|^2 = \|f\|^2 + \|g\|^2$.

[16] A subspace of a vector space is by definition a *linear* subspace.

2.4 The Hilbert spaces ℓ^2

We say that H_1 and H_2 are **isomorphic** as Hilbert space when there exists an invertible linear map $u : H_1 \to H_2$ that preserves the inner product, in that $(uf, ug)_{H_2} = (f, g)_{H_1}$ for all $f, g \in H_1$; this clearly implies that also the inverse of u preserves the inner product. Such a map is called **unitary**.

To prove Theorem 2.7, we first introduce a Hilbert spaces $\ell^2(S)$ for any set S (in the proof, S will be a set labeling some o.n.b., like $S = \mathbb{N}$ in the countable case).

- If S is finite, then $\ell^2(S) = \{f : S \to \mathbb{C}\}$ with inner product

$$(f, g) = \sum_{s\in S} \overline{f(s)}g(s). \tag{2.18}$$

The functions $(\delta_s)_{s\in S}$, defined by $\delta_s(t) = \delta_{st}$, $t \in S$, clearly form an o.n.b. of $\ell^2(S)$.

Now let H be an n-dimensional Hilbert space; a case in point is $H = \mathbb{C}^n$. By definition, H has an o.n.b. $(e_i)_{i=1}^n$. Take $S = \underline{n} = \{1, 2, \ldots, n\}$. The map $u : H \to \ell^2(\underline{n})$, given by linear extension of $ue_i = \delta_i$ is unitary and provides an isomorphism $H \cong \ell^2(\underline{n})$. Hence *all* n-dimensional Hilbert space are isomorphic.

- If S is countable, then $\ell^2(S) = \{f : S \to \mathbb{C} \mid \|f\|_2 < \infty\}$, with

$$\|f\|_2 := \left(\sum_{s\in S} |f(s)|^2\right)^{1/2}, \tag{2.19}$$

with inner product given by (2.18); this is finite for $f, g \in \ell^2(S)$ by the Cauchy–Schwarz inequality. Once again, the functions $(\delta_s)_{s\in S}$ form an o.n.b. of $\ell^2(S)$, and the same argument shows that all separable Hilbert space are isomorphic to $\ell^2(\mathbb{N})$ and hence to each other. A typical example is $\ell^2(\mathbb{Z})$.

- If S is uncountable, then $\ell^2(S)$ is defined as in the countable case, where the sum in (2.19) is now defined as the supremum of the same expression evaluated on each finite subset of S. Similarly, the sum in (2.18) is defined by first decomposing $f = f_1 - f_2 + i(f_3 - f_4)$ with $f_i \geq 0$, and g likewise; this decomposes (f, g) as a linear combination of 16 non-negative terms (f_i, g_j), each of which is defined as the supremum over finite subsets of S, as for $\|f\|_2$.

The previous construction of an o.n.b. of $\ell^2(S)$ still applies *verbatim*, as does the proof that any Hilbert space of given cardinality is isomorphic to $\ell^2(S)$ for some S of the same cardinality. In sum, we have proved (von Neumann's) Theorem 2.7.

Let us note that for infinite sets S we may regard $\ell^2(S)$ as the closure in the norm (2.19) of the (incomplete) space $\ell_c(S)$ of functions that are nonzero at finitely many $s \in S$; this means that for any $f \in \ell^2(S)$ there is a sequence (f_n) in $\ell_c(S)$ such that $\lim_{n\to\infty} \|f_n - f\|_2 = 0$. In what follows, we also encounter the Banach space

$$\ell^\infty(S) = \{f : S \to \mathbb{C} \mid \|f\|_\infty < \infty\}; \tag{2.20}$$

$$\|f\|_\infty := \sup_{s\in S}\{|f(s)|\}, \tag{2.21}$$

which is evidently the closure of $\ell_c(S)$ in the **supremum-norm** $\|\cdot\|_\infty$, in that for any $f \in \ell^\infty(S)$ there is a sequence (f_n) in $\ell_c(S)$ such that $\lim_{n\to\infty} \|f_n - f\|_\infty = 0$.

2.5 The Hilbert spaces L^2

A more complicated example of a Hilbert space is $L^2(\mathbb{R}^n)$, familiar from quantum mechanics. which can be defined either directly through measure theory (see §2.6), or indirectly, as a completion of $C_c(\mathbb{R}^n)$, the vector space of complex-valued continuous functions on $\mathbb{R}^n$ with compact support.[17] Two natural norms on $C_c(\mathbb{R}^n)$ are:

$$\|f\|_\infty := \sup\{|f(x)|, x \in \mathbb{R}^n\}, \tag{2.22}$$

$$\|f\|_2 := \left(\int_{\mathbb{R}^n} d^n x\, |f(x)|^2\right)^{1/2}. \tag{2.23}$$

The first norm is called the **supremum-norm** or **sup-norm**; see §2.7. The second norm is called the $\mathbf{L^2}$**-norm**. It is, of course, derived from the inner product

$$(f, g) := \int_{\mathbb{R}^n} d^n x\, \overline{f(x)} g(x). \tag{2.24}$$

Now, $C_c(\mathbb{R}^n)$ fails to be complete in either norm $\|\cdot\|_\infty$ or $\|\cdot\|_2$.

- The completion of $C_c(\mathbb{R}^n)$ in the norm $\|\cdot\|_\infty$ turns out to be $C_0(\mathbb{R}^n)$.[18]
- The completion of $C_c(\mathbb{R}^n)$ in the norm $\|\cdot\|_2$ is $L^2(\mathbb{R}^n)$, defined in two steps.

Definition 2.8 *The space $\mathcal{L}^2(\mathbb{R}^n)$ consists of all functions $f : \mathbb{R}^n \to \mathbb{C}$ for which there exists a Cauchy sequence (f_n) in $C_c(\mathbb{R}^n)$ with respect to $\|\cdot\|_2$ such that $f_n(x) \to f(x)$ for all $x \in \mathbb{R}^n \backslash N$, where $N \subset \mathbb{R}^n$ is a set of (Lebesgue) measure zero.*[19]

We can extend the inner product on $C_c(\mathbb{R}^n)$ to $\mathcal{L}^2(\mathbb{R}^n)$ by $(f, g) = \lim_{n\to\infty}(f_n, g_n)$, where (f_n) and (g_n) are Cauchy sequences in $\mathcal{L}^2(\mathbb{R}^n)$ w.r.t. the L^2-norm. However, this sesquilinear form fails to be positive definite (take a function f on $\mathbb{R}^n$ that is nonzero in finitely—or even countably—many points). To resolve this, introduce

$$L^2(\mathbb{R}^n) := \mathcal{L}^2(\mathbb{R}^n)/\mathcal{N}, \tag{2.25}$$

where

$$\mathcal{N} := \{f \in \mathcal{L}^2(\mathbb{R}^n) \mid \|f\|_2 = 0\}. \tag{2.26}$$

Using measure theory, it can be shown that $f \in \mathcal{N}$ iff $f(x) = 0$ for all $x \in \mathbb{R}^n \backslash N$, where $N \subset \mathbb{R}^n$ is some set of measure zero. If f is continuous, this implies that $f(x) = 0$ for all $x \in \mathbb{R}^n$. It is clear that $\|\cdot\|_2$ descends to a norm on $L^2(\mathbb{R}^n)$ by

$$\|[f]\|_2 := \|f\|_2, \tag{2.27}$$

where $[f]$ is the equivalence class of $f \in \mathcal{L}^2(\mathbb{R}^n)$ in the quotient space. However, we normally work with $\mathcal{L}^2(\mathbb{R}^n)$ and regard elements of $L^2(\mathbb{R}^n)$ as functions instead of equivalence classes thereof. So in what follows we should often write $[f] \in L^2(\mathbb{R}^n)$ instead of $f \in L^2(\mathbb{R}^n)$, which really means $f \in \mathcal{L}^2(\mathbb{R}^n)$, but who cares ...

---

[17] The **support** of a function is defined as the smallest closed set outside which it vanishes.

[18] This is the space of all continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ that *vanish at infinity* in the sense that for each $\epsilon > 0$ there is a compact subset $K \subset \mathbb{R}^n$ such that $|f(x)| < \epsilon$ for all x outside K.

[19] A subset $N \subset \mathbb{R}^n$ has **measure zero** if for any $\epsilon > 0$ there exists a covering of N by an at most countable set (I_n) of intervals for which $\sum_n |I_n| < \epsilon$, where $\sum_n |I_n|$ is the sum of the volumes of the I_n. (Here an interval in $\mathbb{R}^n$ is a set of the form $\prod_{k=1}^n [a_k, b_k]$). For example, any countable subset of $\mathbb{R}^n$ has measure zero, but there are many, many others.

2.6 Measure theory and Hilbert space

The construction of $L^2(\mathbb{R}^n)$ may be generalized to Hilbert spaces $L^2(X, \mu)$ defined for arbitrary *locally compact Hausdorff* spaces X; the concept of a measure μ underlying this generalization is very important also for (commutative) C^*-algebras.

Let $P(X)$ be the power set of X, i.e., the set of all subsets of X, and denote the topology of X (i.e., the set of open subsets of X) by $O(X)$. A **σ-algebra** on X is a subset Σ of $P(X)$ such that $\cup_n A_n \in \Sigma$ and $\cap_n A_n \in \Sigma$ whenever $A_n \in \Sigma$, $n \in \mathbb{N}$. Note that $\mathcal{O}(X)$ is generally *not* a σ-algebra on X; it is closed under taking *arbitrary* unions, but *finite* intersections only. Let $B(X)$ be the smallest σ-algebra on X containing $O(X)$; elements of $B(X)$ are called **Borel sets** in X.

Definition 2.9 *A (Radon)* ***measure*** *on X is a map $\mu : B(X) \to [0, \infty]$ satisfying:*

1. *$\mu(\cup_n A_n) = \sum_n \mu(A_n)$ whenever $A_n \in B(X)$, $n \in \mathbb{N}$, $A_i \cap A_j = \emptyset$ for all $i \neq j$;*
2. *$\mu(K) < \infty$ for each compact subset K of X;*
3. *$\mu(A) = \sup\{\mu(K), K \subset A, K \text{compact}\}$ for each $A \in B(X)$.*

An ***integral*** *on $C_c(X)$ is a (complex) linear map $\int_X : C_c(X) \to \mathbb{C}$ such that $\int_X f$ is in $\mathbb{R}^+$ whenever $f(x) \in \mathbb{R}^+$ for all $x \in X$ (in which case we say $f \geq 0$).*

The **Riesz–Markov Theorem** states that these concepts are equivalent:

Theorem 2.10 *There is a bijective correspondence between integrals and measures:*

- *A measure μ on X defines an integral $\int_X d\mu$ on $C_c(X)$, given on $f \geq 0$ by*
$$\int_X d\mu\, f := \sup\left\{\int_X d\mu\, g \mid 0 \leq g \leq f, g \text{simple}\right\}, \tag{2.28}$$
where a ***simple*** *function is a finite linear combination of characteristic functions χ_K, $K \subset X$ compact, and if $g = \sum_i \lambda_i \chi_{K_i}$, then $\int_X d\mu\, g := \sum_i \lambda_i \mu(K_i)$.*
- *An integral $\int_X$ on $C_c(X)$ defines a measure μ on X, given on compact K by*
$$\mu(K) = \inf\left\{\int_X f \mid f \in C_c(X), \chi_K \leq f \leq 1\right\}. \tag{2.29}$$

For any $p > 0$, we define $\mathcal{L}^p(X, \mu)$ as the space of Borel functions[20] on X for which
$$\|f\|_p := \left(\int_X d\mu\, |f|^p\right)^{1/p} < \infty, \tag{2.30}$$
where the integral is defined à la (2.28). The map $\|\cdot\|_p : \mathcal{L}^p(X, \mu) \to \mathbb{R}^+$ has a p-independent null space $\mathcal{N}$, with associated Banach space $L^p(X, \mu) := \mathcal{L}^p(X, \mu)/\mathcal{N}$. For $p = 2$, the Banach space $L^2(X, \mu)$ is actually a Hilbert space with inner product
$$(f, g) := \int_X d\mu\, \overline{f} g \equiv \int_X d\mu(x)\, \overline{f(x)} g(x), \tag{2.31}$$
where similarly ambiguous notation has been used as for $L^2(\mathbb{R}^n)$ (cf. the end of §2.5).

[20] Here $f : X \to \mathbb{C}$ is **Borel** when $f_i^{-1}((s, t)) \in B(X)$ for each $0 \leq s < t$, $i = 1, 2, 3, 4$, where $f = f_1 - f_2 + i(f_3 - f_4)$ is the unique decomposition with $f_i \geq 0$ (e.g., $f_1(x) = \max\{\text{Re}(f(x)), 0\}$).

2.7 Operators on Hilbert space

An **operator** $a : H_1 \to H_2$ between two Hilbert space is simply a linear map (i.e., $a(\lambda v + \mu w) = \lambda a(v) + \mu a(w)$ for all $\lambda, \mu \in \mathbb{C}$ and $v, w \in H_1$). We write av for $a(v)$. Taking $H_1 = H_1 = H$, an operator $a : H \to H$ is just called an *operator on* H. Taking $H_1 = H$ and $H_2 = \mathbb{C}$, we obtain a *functional* on H. For example, any $f \in H$ yields a functional $\varphi : H \to \mathbb{C}$ by $\varphi(g) = (f, g)$. By Cauchy–Schwarz, $|\varphi(g)| \leq C\|g\|$ with $C = \|f\|$. Conversely, the **Riesz–Fischer Theorem** states that if some φ satisfies this bound, then it is of the above form, for a unique $f \in H$.

As in real analysis, where one deals with functions $f : \mathbb{R} \to \mathbb{R}$, it turns out to be useful to single out functions with good properties, notably continuity. So what does one mean by a 'continuous' operator $a : H_1 \to H_2$? One answer come from topology: the inner product on a Hilbert space defines a norm, the norm defines a metric, and finally the metric defines a topology, so one may use the usual definition of a continuous function $f : X \to Y$ between two topological spaces. We use an equivalent definition, in which continuity is replaced by *boundedness*:

Definition 2.11 *$a : H_1 \to H_2$ be an operator. Define $\|a\| \in \mathbb{R}^+ \cup \{\infty\}$ by*

$$\|a\| := \sup\{\|av\|_{H_2},\ v \in H_1,\ \|v\|_{H_1} = 1\}, \tag{2.32}$$

*where $\|v\|_{H_1} = \sqrt{(v, v)_{H_1}}$, etc. We say that a is **bounded** when $\|a\| < \infty$, in which case the number $\|a\|$ is called the **norm** of a.*

If a is bounded, then it is immediate that

$$\|av\|_{H_2} \leq \|a\|\ \|v\|_{H_1} \tag{2.33}$$

for all $v \in H_1$. This inequality is very important. For example, it implies that

$$\|ab\| \leq \|a\|\|b\|, \tag{2.34}$$

where $a : H \to H$ and $b : H \to H$ are any two bounded operators, and $ab := a \circ b$, so that $(ab)(v) := a(bv)$. Eq. (2.33) also implies the easy half of:

Proposition 2.12 *An operator on a Hilbert space H is bounded iff it is continuous in the sense that $f_n \to f$ implies $af_n \to af$ for all convergent sequences (f_n) in H.*

When H is finite-dimensional, any operator on H is bounded (and may be represented by a matrix). For an infinite-dimensional example, take $H = \ell^2(S)$ and $a \in \ell^\infty(S)$, for some set S. It is an exercise to show that if $f \in \ell^2(S)$, then $af \in \ell^2(S)$. Hence we may define a **multiplication operator** $\hat{a} : \ell^2(S) \to \ell^2(S)$ by

$$\hat{a}(f) := af, \tag{2.35}$$

that is, $(\hat{a}f)(x) = a(x)f(x)$. This operator is bounded, with

$$\|\hat{a}\| = \|a\|_\infty. \tag{2.36}$$

Similarly, take $H = L^2(\mathbb{R}^n)$ and $a \in C_0(\mathbb{R}^n)$. Once again, (2.35) defines a bounded multiplication operator $\hat{a} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, satisfying (2.36).

More generally, for locally compact X, a function $a \in C_0(X)$ defines a multiplication operator $\hat{a}$ on $H = L^2(X, \mu)$ satisfying $\|\hat{a}\| \leq \|a\|_\infty$, with equality iff the **support** of the measure μ is X (i.e., every open subset of X has positive measure).

2.8 The adjoint

Let $a : H \to H$ be a *bounded* operator. The inner product on H gives rise to a map $a \mapsto a^*$, which is familiar from linear algebra: if a is a matrix (a_{ij}) w.r.t. some o.n.b., then $a^* = (\overline{a_{ji}})$. In general, the adjoint a^* is uniquely defined by the property[21]

$$(a^* f, g) = (f, ag) \text{ for all } f, g \in H. \tag{2.37}$$

Note that $a \mapsto a^*$ is anti-linear: one has $(\lambda a)^* = \overline{\lambda} a$ for $\lambda \in \mathbb{C}$. Also, one has

$$\|a^*\| = \|a\|; \tag{2.38}$$
$$\|a^* a\| = \|a\|^2. \tag{2.39}$$

The adjoint allows one to define the following basic classes of bounded operators:

1. $n : H \to H$ is **normal** when $n^* n = n n^*$.
2. $a : H \to H$ is **self-adjoint** when $a^* = a$ (hence a is normal).
3. $a : H \to H$ is **positive**, written $a \geq 0$, when $(f, af) \geq 0$ for all $f \in H$.
4. $p : H \to H$ is a **projection** when $p^2 = p^* = p$ (hence p is positive).
5. $u : H \to H$ is **unitary** when $u^* u = u u^* = 1$ (hence u is normal).
6. $v : H \to H$ is an **isometry** when $v^* v = 1$, and a **partial isometry** when $v^* v$ is a projection (in which case $v v^*$ is automatically a projection, too).

Proposition 2.13

1. *An operator a is self-adjoint a iff $(f, af) \in \mathbb{R}$ for all $f \in H$ (and hence positive operators are automatically self-adjoint).*
2. *There is a bijective correspondence $p \leftrightarrow K$ between projections p on H and closed subspaces K of H: given p, put $K := pH$, and given $K \subset H$, define p on $f \in H$ by $pf = \sum_i (e_i, f) e_i$, where (e_i) is an arbitrary o.n.b. of K.*
3. *An operator u is unitary iff it is invertible (with $u^{-1} = u^*$) and preserves the inner product, i.e., $(uf, ug) = (f, g)$ for all $f, g \in H$.*
4. *An operator v is a partial isometry iff v is unitary from $(\ker v)^\perp$ to $\mathrm{ran}(v)$.*
5. *An operator v is an isometry iff $(vf, vg) = (f, g)$ for all $f, g \in H$.*

Similar definitions apply to (bounded) operators between different Hilbert spaces: e.g., the adjoint $a^* : H_2 \to H_1$ of $a : H_1 \to H_2$ satisfies $(a^* f, g)_{H_1} = (f, ag)_{H_2}$ for all $f \in H_2, g \in H_1$, and unitarity of $u : H_1 \to H_2$ means $u^* u = 1_{H_1}$ and $u u^* = 1_{H_2}$; equivalently, u is invertible and $(uf, ug)_{H_2} = (f, g)_{H_1}$ for all $f, g \in H_1$ (cf. §2.4).

[21] To prove existence of a^*, the Riesz–Fischer Theorem is needed. For fixed $a : H \to H$ and $f \in H$, one defines a functional $\varphi_f^a : H \to \mathbb{C}$ by $\varphi_f^a(g) := (f, ag)$. By Cauchy–Schwarz and (2.33), one has $|\varphi_f^a(g)| = |(f, ag)| \leq \|f\| \|ag\| \leq \|f\| \|a\| \|g\|$, so $\|\varphi_f^a\| \leq \|f\| \|a\|$. Hence there exists a unique $h \in H$ such that $\varphi_f^a(g) = (h, g)$ for all $g \in H$. Now, for given a the association $f \mapsto h$ is clearly linear, so that we may define $a^* : H \to H$ by $a^* f := h$; eq. (2.37) then trivially follows.

2.9 Spectral theory

The spectrum of an operator a generalizes the range of a (complex-valued) function, and is its only invariant under unitary transformations $a \mapsto u^*au$. To get started, we first restate the spectral theorem of linear algebra. In preparation, we call a family (p_i) of projections on a Hilbert space H **mutually orthogonal** if $p_iH \perp p_jH$ for $i \neq j$; this is the case iff $p_ip_j = \delta_{ij}p_i$. Such a family is called **complete** if $\sum_i p_if = f$ for all $f \in H$; of course, if $\dim(H) < \infty$, this simply means $\sum_i p_i = 1$.

Proposition 2.14 *Let $a : \mathbb{C}^n \to \mathbb{C}^n$ be a self-adjoint operator on $\mathbb{C}^n$ (i.e., an hermitian matrix). There exists a complete family (p_i) of mutually orthogonal projections so that $a = \sum_i \lambda_i p_i$, where λ_i are the eigenvalues of a. Consequently, p_i is the projection onto the eigenspace of a in H with eigenvalue λ_i, and the dimension of the subspace p_iH is equal to the multiplicity of the eigenvalue λ_i.*

This is no longer true for self-adjoint operators on infinite-dimensional Hilbert spaces. For example, if $a \in C_0(\mathbb{R}, \mathbb{R})$, then the associated multiplication operator $\hat{a}$ on $L^2(\mathbb{R})$ has no eigenvectors at all! However, is has *approximate* eigenvectors, in the following sense: for fixed $x_0 \in \mathbb{R}$, take $f_n(x) := (n/\pi)^{1/4}e^{-n(x-x_0)^2/2}$, so that $f_n \in L^2(\mathbb{R})$ with $\|f_n\| = 1$. The sequence f_n has no limit in $L^2(\mathbb{R})$.[22] Nonetheless, an elementary computation shows that $\lim_{n\to\infty} \|(\hat{a} - \lambda)f_n\| = 0$ for $\lambda = a(x_0)$, so that the f_n form approximate eigenvectors of $\hat{a}$ with 'eigenvalue' $a(x_0)$.

Definition 2.15 *Let $a : H \to H$ be a* normal *operator. The* ***spectrum*** *$\sigma(a)$ consists of all $\lambda \in \mathbb{C}$ for which there exists a sequence (f_n) in H with $\|f_n\| = 1$ and*

$$\lim_{n\to\infty} \|(a - \lambda)f_n\| = 0. \tag{2.40}$$

1. *If λ is an eigenvalue of a, in that $af = \lambda f$ for some $f \in H$ with $\|f\| = 1$, then we say that $\lambda \in \sigma(a)$ lies in the* ***discrete spectrum*** *$\sigma_d(a)$ of a.*
2. *If $\lambda \in \sigma(a)$ but $\lambda \notin \sigma_d(a)$, it lies in the* ***continuous spectrum*** *$\sigma_c(a)$ of a.*
3. *Thus $\sigma(a) = \sigma_d(a) \cup \sigma_c(a)$ is the union of the discrete and the continuous part.*

Indeed, in the first case (2.40) clearly holds for the constant sequence $f_n = f$ (for all n), whereas in the second case λ by definition has no associated eigenvector.

If a acts on a finite-dimensional Hilbert space, then $\sigma(a) = \sigma_d(a)$ consists of the eigenvalues of a. On the other hand, in the above example of a multiplication operator $\hat{a}$ on $L^2(\mathbb{R})$ we have $\sigma(\hat{a}) = \sigma_c(\hat{a})$. Our little computation shows that $\sigma_c(\hat{a})$ contains the range $\mathrm{ran}(a)$ of the function $a \in C_0(\mathbb{R})$, and it can be shown that $\sigma(\hat{a}) = \mathrm{ran}(a)^-$ (i.e., the topological closure of the range of $a : \mathbb{R} \to \mathbb{R}$ as a subset of $\mathbb{R}$). In general, the spectrum may have both a discrete and a continuous part.[23]

[22] It converges to Dirac's delta function $\delta(x - x_0)$ in a 'weak' sense, viz. $\lim_{n\to\infty}(f_n, g) = g(x_0)$ for each fixed $g \in C_c^\infty(\mathbb{R})$, but the δ 'function' is not an element of $L^2(\mathbb{R})$ (it is a distribution).

[23] If a is the Hamiltonian of a quantum-mechanical system, the eigenvectors corresponding to the discrete spectrum are *bound states*, whereas those related to the continuous spectrum form wavepackets defining *scattering states*. Just think of the hydrogen atom. It should be mentioned that such Hamiltonians are typically *unbounded* operators.

2.10 Compact operators

Even if H is infinite-dimensional, there is a class of operators whose spectrum is discrete. First, a **finite-rank operator** is an operator with finite-dimensional range. Using Dirac's notation, for $f, g \in H$ we write $|f\rangle\langle g|$ for the operator $h \mapsto (g, h)f$. An important special case is $g = f$ with $\|f\| = 1$, so that $|f\rangle\langle f|$ is the one-dimensional projection onto the subspace spanned by f. More generally, if (e_i) is an o.n.b. of some finite-dimensional subspace K, then $\sum_i |e_i\rangle\langle e_i|$ is the projection onto K. Any *finite* linear combination $\sum_i |f_i\rangle\langle g_i|$ is finite-rank, and *vice versa.*

Definition 2.16 *A bounded operator Hilbert space is called **compact** iff it is the norm-limit of a sequence of finite-rank operators.*

Note that multiplication operators of the type $\hat{a}$ on $L^2(\mathbb{R}^n)$ for $0 \neq a \in C_0(\mathbb{R}^n)$ are *never* compact. On the other hand, typical examples of compact operators on $L^2(\mathbb{R}^n)$ are integral operators of the kind $af(x) = \int d^n y\, K(x, y)f(y)$ with $K \in L^2(\mathbb{R}^{2n})$.

Theorem 2.17 *Let a be a self-adjoint compact operator on a Hilbert space H. Then the spectrum $\sigma(a)$ is discrete. All nonzero eigenvalues have finite multiplicity, so that only $\lambda = 0$ may have infinite multiplicity (if it occurs), and in addition 0 is the only possible accumulation point of $\sigma(a) = \sigma_d(a)$. If p_i is the projection onto the eigenspace corresponding to eigenvalue λ_i, then $a = \sum_i \lambda_i p_i$, where the sum converges strongly, i.e., in the sense that $af = \sum_i \lambda_i p_i f$ for each fixed $f \in H$.*

The compact operators are closed under multiplication and taking adjoints, so that, in particular, a^*a is compact whenever a is. Hence Theorem 2.17 applies to a^*a. Note that a^*a is self-adjoint and that its eigenvalues are automatically non-negative.

Definition 2.18 *We say that a compact operator $a : H \to H$ is **trace-class** if the **trace-norm** $\|a\|_1 := \sum_k \sqrt{\mu_k}$ is finite, where the μ_k are the eigenvalues of a^*a.*

Theorem 2.19 *Suppose a is trace-class. Then the **trace** of a, defined by*

$$\mathrm{Tr}\,(a) := \sum_i (e_i, ae_i), \tag{2.41}$$

is absolutely convergent and independent of the orthonormal basis (e_i). In particular, if $a = a^*$ with eigenvalues (λ_i), then $\mathrm{Tr}\, a = \sum_i \lambda_i$. Furthermore:

1. *If b is bounded and a is trace-class, then ab and ba are trace-class, with*

$$\mathrm{Tr}\,(ab) = \mathrm{Tr}\,(ba). \tag{2.42}$$

2. *If u is unitary and a is trace-class, then uau^{-1} is trace-class, with*

$$\mathrm{Tr}\,(uau^{-1}) = \mathrm{Tr}\,(a). \tag{2.43}$$

The following notion plays a fundamental role in von Neumann algebra theory:

Definition 2.20 *A trace-class operator $\rho : H \to H$ is called a **density operator** if ρ is* positive *and $\mathrm{Tr}\,(\rho) = 1$ (so that $\|\rho\|_1 = 1$). Equivalently, $\rho = \sum_i \lambda_i p_i$ (strongly) with $\dim(p_i) < \infty$ for all i, $0 < \lambda_i \leq 1$, and $\sum_i \lambda_i = 1$.*

3 C*-algebras

3.1 Basic definitions

If a and b are bounded operators on H, then so is their sum $a+b$, defined by $(a+b)(v) = av + bv$, and their product ab, given by $(ab)(v) = a(b(v))$. This follows from the triangle inequality for the norm and from (2.34), respectively. Also, homogeneity of the norm yields that ta is bounded for any $t \in \mathbb{C}$. Consequently, the set $B(H)$ of all bounded operators on a Hilbert space H forms an *algebra* over the complex numbers, having remarkable properties. To begin with (cf. (2.32)):

Proposition 3.1 *The space $B(H)$ of all bounded operators on a Hilbert space H is a Banach space in the operator norm*

$$\|a\| := \sup\{\|af\|_H,\ f \in H,\ \|f\|_H = 1\}. \tag{3.1}$$

This is a basic result from functional analysis; it even holds if H is a Banach space.

Definition 3.2 *A **Banach algebra** is a Banach space A that is simultaneously an algebra in which $\|ab\| \leq \|a\|\,\|b\|$ for all $a, b \in A$.*

According to (2.34), we see that $B(H)$ is not just a Banach *space* but even a Banach *algebra*. Also this would still be the case if H were merely a Banach space, but the fact that it is a Hilbert space gives a crucial further ingredient of the algebra $B(H)$.

Definition 3.3

1. *An **involution** on an algebra A is a real-linear map $A \to A^*$ such that $a^{**} = a$, $(ab)^* = b^*a^*$, and $(\lambda a)^* = \overline{\lambda} a^*$ for all $a, b \in A$ and $\lambda \in \mathbb{C}$. An algebra with involution is also called a ***-algebra**.*

2. *A C^***-algebra** is a Banach algebra A with involution in which for all $a \in A$,*

$$\|a^*a\| = \|a\|^2. \tag{3.2}$$

3. *A **homomorphism** between C^*-algebras A en B is a linear map $\varphi : A \to B$ that satisfies $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$, $b \in B$.*

4. *An **isomorphism** between two C^*-algebras is an invertible homomorphism.*[24]

In view of (2.39), we conclude that $B(H)$ *is a C^*-algebra* (with the identity operator as its unit) with respect to the involution defined by the operator adjoint (2.37).

Similarly, if $A \subset B(H)$ is a norm-closed subalgebra of $B(H)$ such that if $a \in A$, then $a^* \in A$ (so that A is an algebra with involution), then A is obviously a C^*-algebra (not necessarily with unit). A case in point is $A = K(H)$, the C^*-algebra of *compact operators* on H. If $\dim(H) = \infty$, there is a strict inclusion $K(H) \subset B(H)$; for one thing, the unit operator lies in $B(H)$ but not in $K(H)$, which has no unit. If $\dim(H) < \infty$, though, one has $K(H) = B(H) = M_n(\mathbb{C})$, the $n \times n$ matrices.

On the other hand, the set $B_1(H)$ of trace-class operators satisfies (3.2) in the operator norm (3.1) but fails to be complete in that norm, whereas in the trace-norm $\|\cdot\|_1$ it is complete but (3.2) fails. Either way, $B_1(H)$ *fails to be a C^*-algebra.*

[24]We will shortly prove that an isomorphism is automatically isometric.

3.2 Commutative C^*-algebras

The C^*-algebras $K(H)$ and $B(H)$ are highly noncommutative. For the opposite case, let X be a locally compact Hausdorff space (physicists may keep $X = \mathbb{R}^n$ in mind). The space $C_0(X)$ of all continuous functions $f : X \to \mathbb{C}$ that vanish at infinity[25] is an algebra under pointwise operations.[26] It has a natural involution

$$f^*(x) = \overline{f(x)}, \tag{3.3}$$

and a natural **supremum-norm** or **sup-norm** given by (cf. 2.22)

$$\|f\|_\infty := \sup\{|f(x)|, x \in X\}. \tag{3.4}$$

Then $C_0(X)$ is a commutative C^*-algebra; the axioms are easily checked. Let us note that $C_0(X)$ has a unit (namely the function equal to 1 for any x) iff X is compact. The converse, due to Gelfand and Naimark (1943), is a fundamental result:

Theorem 3.4 *Every commutative C^*-algebra A is isomorphic to $C_0(X)$ for some locally compact Hausdorff space X, which is unique up to homeomorphism.*

This space X is often denoted by $\Sigma(A)$ and is called the **Gelfand spectrum** of A. It may be realized as the set of all nonzero linear maps $\omega : A \to \mathbb{C}$ that satisfy $\omega(ab) = \omega(a)\omega(b)$ (i.e., of nonzero homomorphisms $A \to \mathbb{C}$ as C^*-algebras).[27] The **Gelfand transform** maps each $a \in A$ to a complex-valued function $\hat{a}$ on $\Sigma(A)$ by

$$\hat{a}(\omega) := \omega(a) \quad (a \in A,\ \omega \in \Sigma(A)). \tag{3.5}$$

The **Gelfand topology** is the weakest topology on $\Sigma(A)$ making all functions $\hat{a}$ continuous (i.e., the topology generated by the sets $\hat{a}^{-1}(U)$, $U \in \mathbb{C}$ open, $a \in A$). In this topology, $\Sigma(A)$ is *compact* iff $1 \in A$ (exercise), and *locally compact* otherwise (later).[28] The isomorphism $A \to C_0(\Sigma(A))$ is just given by the Gelfand transform.

It is immediate from the definition of $\Sigma(A)$ that $a \mapsto \hat{a}$ is an algebra homomorphism; the proof that $\omega(a^*) = \overline{\omega(a)}$, and hence that $\widehat{a^*} = (\hat{a})^*$, is an exercise. It If $1 \in A$, injectivity of the Gelfand transform as a map from A to $C(\Sigma(A))$ results from the difficult fact (proved in §3.4 below) that it is *isometric*, i.e., $\|\hat{a}\|_\infty = \|a\|$. Surjectivity then easily follows from the Stone–Weierstrass Theorem (exercise).

The hard part of the proof of Theorem 3.4, i.e., the isometry of the Gelfand transform, may be approached in two rather different ways. One, going back to Gelfand himself, heavily relies on the theory of (maximal) ideals in Banach algebras. The other, pioneered by Kadison and Segal, uses the state space (and especially the *pure* state space) of A in a central way. Since the same technique also applies to the proof of the second great theorem about C*-algebras (and also because it is closer to quantum-mechanical thinking), in these notes we favour the state space approach.

---

[25] I.e., for each $\epsilon > 0$ there is a compact subset $K \subset X$ such that $|f(x)| < \epsilon$ for all x outside K.

[26] Addition is given by $(f + g)(x) = f(x) + g(x)$, multiplication is $(fg)(x) = f(x)g(x)$, etc.

[27] For example, if A is already given as $A = C(X)$, then each $x \in X$ defines a functional ω_x on A by $\omega_x(f) = f(x)$, which is multiplicative by the pointwise definition of multiplication in A.

[28] In §3.3 we show that $\Sigma(A) \subset A^*$, where A^* is the dual of A, but the Gelfand topology on $\Sigma(A)$ (in which Theorem 3.4 holds) is not the norm-topology but the (relative) weak* topology. From that perspective, continuity of $\hat{a}$ follows from basic functional analysis: $a \mapsto \hat{a}$ maps A into the double dual A^{**} of A, and $A \subset A^{**}$ precisely consists of all w^*-continuous functionals on on A^*.

3.3 States

The concept of a state originates with quantum physics, but also purely mathematically it came to play a dominant (and beautiful) role in operator algebra theory.

Definition 3.5 *A* ***state*** *on a unital C^*-algebra A is a linear map $\omega : A \to \mathbb{C}$ that is* ***positive****, in that $\omega(a^*a) \geq 0$ for all $a \in A$, and* ***normalized****, in that $\omega(1) = 1$.*

If we define the dual A^* of A as the space of linear maps $\varphi : A \to \mathbb{C}$ for which

$$\|\varphi\| = \sup\{|\varphi(a)|, a \in A, \|a\| = 1\} \tag{3.6}$$

is finite (cf. (2.32)), then it can be shown that any state ω on A lies in A^*, with $\|\omega\| = 1$. This leads to an extension of Definition 3.5 to general (i.e., not necessarily unital) C^*-algebras: *a state on a C^*-algebra A is a functional $\omega : A \to \mathbb{C}$ that is positive and normalized in the sense that $\|\omega\| = 1$.* This implies $\omega(1) = 1$ whenever A does have a unit, so that the two definitions are consistent when they overlap.

The state space $S(A)$ of A (i.e., the set of all states on A) is a *convex set*: if ω_1 and ω_2 are states, then so is $\lambda\omega_1 + (1-\lambda)\omega_2$ for any $\lambda \in [0,1]$. It follows that if $(\omega_1, \omega_2, \ldots, \omega_n)$ are states, and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are numbers in $[0,1]$ such that $\sum_i \lambda_i = 1$, then $\sum_i \lambda_i\omega_i$ is a state. This extends to infinite sums if we equip $S(A)$ with the weak* topology inherited from A^* (in which $\omega_n \to \omega$ if $\omega_n(a) \to \omega(a)$ for each $a \in A$). If A has a unit, then $S(A)$ is a *compact convex set* in this topology.[29]

Definition 3.6 *A state ω is* ***pure*** *if $\omega = \lambda\,\omega_1 + (1-\lambda)\omega_2$ for some $\lambda \in (0,1)$ and certain states ω_1 and ω_2 implies $\omega_1 = \omega_2$. The pure states on A comprise the* pure state space *of A, denoted by $P(A)$ or $\partial S(A)$. If a state is not pure, it is* ***mixed****.*[30]

The convex structure of the state space is nicely displayed in the noncommutative case by $A = M_2(\mathbb{C})$, the C^*-algebra of 2×2 complex matrices. Put

$$\rho = \tfrac{1}{2}\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}; \tag{3.7}$$

then ρ is a density matrix on $\mathbb{C}^2$ iff $(x,y,z) \in \mathbb{R}^3$ with $x^2+y^2+z^2 \leq 1$; this set is the three-ball B^3 in $\mathbb{R}^3$. It is easy to see that ρ defines a state ω_ρ on the $M_2(\mathbb{C})$ by

$$\omega_\rho(a) = \mathrm{Tr}\,(\rho a). \tag{3.8}$$

Conversely, every state on $M_2(\mathbb{C})$ is of this form, so that the state space $S(M_2(\mathbb{C}))$ is isomorphic (as a convex set) to B^3. The pure states ∂B^3 then correspond to the two-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^2 \mid x^2+y^2+z^2 = 1\}$ (see exercises).

In the commutative case, we have the key behind the proof of Theorem 3.4:

Lemma 3.7 *The pure state space $P(A) \subset S(A) \subset A^*$ of a commutative C^*-algebra A coincides with its Gelfand spectrum $\Sigma(A)$ (seen as a subspace of A^*).*

The proof is an exercise, but one can see the point from the example $A = C(X)$.

---

[29] This follows from the *Banach–Alaoglu Theorem* of functional analysis; see [8] or exercises.

[30] The *Krein–Milman Theorem* of functional analysis [8] guarantees the abundance of pure states in compact convex sets in that any state is a convex sumof pure states (or limit thereof).

3.4 Spectrum

To prove isometry of the Gelfand transform (and hence Theorem 3.4) from Lemma 3.7, we need a nice result with an ugly proof based on the Axiom of Choice (AC):

Lemma 3.8 *Let A be a C^*-algebra with unit. For any self-adjoint $a \in A$, there is a pure state $\omega_0 \in P(A)$ such that $|\omega_0(a)| = \|a\|$.*

This will be proved in a minute; for now, we just point out that for $A = C(X)$, X compact, this is immediate from Weierstrass' Theorem stating that a continuous function on a compact set assumes its maximum (and its minimum). Given Lemma 3.8, if $a^* = a$, then $\|a\| = |\omega_0(a)| = |\hat{a}(\omega_0)| \leq \|\hat{a}\|_\infty \leq \|a\|$, the last inequality arising because $\|\hat{a}\|_\infty = \sup\{|\hat{a}(\omega)|, \omega \in \Sigma(A)\}$, (3.5), and $|\omega(a)| \leq \|a\|$ (since ω is a state). Hence $\|\hat{a}\|_\infty = \|a\|$ for self-adjoint a, and therefore for any a (exercise).

The proof of Theorem 3.4 is now complete up to Lemma 3.8. To prove the latter, and for many other reasons, we introduce the following extremely important notion.

Definition 3.9 *Let A be a Banach algebra with unit. The **spectrum** $\sigma(a)$ of $a \in A$ is the set of all $z \in \mathbb{C}$ for which $a - z \equiv a - z \cdot 1$ has no (two-sided) inverse in A. The **spectral radius** $r(a)$ of $a \in A$ is defined as $r(a) := \sup\{|z|, z \in \sigma(a)\}$.*

We quote two basic results from functional analysis [8]:[31]

Proposition 3.10 *1. The spectrum $\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$.*

2. The spectral radius is given by $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

If A is a C^*-algebra, then $\|a^2\| = \|a\|^2$ for self-adjoint a, so the second property implies $\|a\| = r(a)$ whenever $a^* = a$. The first part ensures the existence of $\lambda_0 \in \sigma(a)$ with $|\lambda_0| = r(a)$, so jointly we have $\|a\| = |\lambda_0|$ for some $\lambda_0 \in \sigma(a)$. Furthermore:

Lemma 3.11 *Let A be a C^*-algebra with unit. For any $a \in A$ and $\lambda \in \sigma(a)$, there is a pure state $\omega \in P(A)$ such that $\omega(a) = \lambda$.*

Choosing $\lambda = \lambda_0$ as above immediately yields Lemma 3.8. The existence of a *general* state $\omega \in S(A)$ achieving $\omega(a) = \lambda$ is an exercise, based on the Hahn–Banach Theorem of functional analysis (which in turn relies on AC).[32] Furthermore, a clever use of the Krein–Milman Theorem (which once again relies on AC) shows that ω may be chosen in $P(A)$; see [16, Corollary 1.4.4, Theorem 4.3.8].

We have now completed the proof of Theorem 3.4, up to the uniqueness of X (up to homeomorphism).[33] This is easily seen to be equivalent to the property

$$\Sigma(C(X)) \cong X, \tag{3.9}$$

where the pertinent map $X \to \Sigma(C(X))$ is given by $x \mapsto \omega_x$, $\omega_x(f) = f(x)$. The proof is a highly nontrivial but dull exercise in topology; see [8, Theorem VII.8.7].

[31] For completeness' sake, the proofs are given in the appendix below.

[32] Gelfand and Naimark's proof of Theorem 3.4 also relies on AC through a Zorn's Lemma argument finding some maximal ideal. A constructive version of Theorem 3.4 exists; see [9].

[33] The nonunital case follows from the unital one, using a technique called *unitization*. See §3.6.

Exercises for Lecture 2

Unless stated otherwise, A is a commutative C^*-algebra with unit. To do nos. 4 and 7 in the simplest way, you may use a few facts about spectral theory and about positive operators we will prove later on. First, $\sigma(\mu \cdot 1 + \lambda a) = \mu + \lambda\sigma(a) \equiv \{\mu + \lambda z : z \in \sigma(a)\}$ for all $a \in A$ and $\mu, \lambda \in \mathbb{C}$. Second, $a \in A$ is positive (in that $a = b^*b$ for some $b \in A$) iff a is self-adjoint with $\sigma(a) \subset [0, \infty)$. In particular, a functional $\omega : A \to \mathbb{C}$ is positive iff $\omega(a) \geq 0$ for all self-adjoint a with $\sigma(a) \subset [0, \infty)$. We use the notation $a \geq 0$ for positive a and write $a \geq b$ if $a - b$ is positive. Similarly, we write $\omega \geq \tau$ for functionals ω and τ such that $\omega - \tau$ is positive.

1. Show that if $\omega \in \Sigma(A)$, then $\omega(a^*) = \overline{\omega(a)}$ for all $a \in A$.

2. Given that $\|\hat{a}\|_\infty = \|a\|$ for self-adjoint $a \in A$, prove the same equality for general $a \in A$.

3. For $a \in C(X)$, X compact Hausdorff, show that $\sigma(a) = \{a(x), x \in X\}$.

4. Show that $\Sigma(A) = P(A)$, as follows:

   (a) Prove that each $\omega \in \Sigma(A)$ is a state, and use the Cauchy-Schwarz inequality for an appropriate semi-inner product to show that ω is pure. *Hint*: Start by showing that $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ with $\lambda \neq 0, 1$ implies that $\omega_1(a) = \omega_2(a)$ for self-adjoint a.

   (b) Prove for all self adjoint $a \in A$ that there is a positive scalar λ such that $a + \lambda$ and $\lambda - a$ are positive in A.

   (c) Let ω be a pure state. Prove that if $\tau : A \to \mathbb{C}$ is a functional such that $0 \leq \tau \leq \omega$, then we can find a scalar β such that $\tau = \beta\omega$.

   (d) For $1 \geq b \geq 0$ use $\omega_0(a) := \omega(ab)$ to show that
   $$\omega(ab) = \omega(a)\omega(b). \tag{3.10}$$

   (e) Finally, prove that (3.10) holds for general $b \in A$.

5. Using isometry, prove surjectivity of the Gelfand transform from the Stone-Weierstrass theorem [8]: Let X be a compact Hausdorff space. Let $\hat{A}$ be a subalgebra of $C(X)$ (regarded as a commutative C^*-algebra) that:

   (a) separates points on X (i.e., if $x \neq y$ there is $f \in \hat{A}$ such that $f(x) \neq f(y)$);

   (b) is closed under complex conjugation (i.e., if $f \in \hat{A}$ then $\overline{f} \in \hat{A}$);

   (c) contains the unit function 1_X (where $1_X(x) = 1$ for all $x \in X$).

   Then $\hat{A}$ is dense in $C(X)$ in the sup-norm.

6. Use the Banach–Alaoglu Theorem [8] to prove that $\Sigma(A)$ is compact in the Gelfand topology. You may assume that $\Sigma(A) \subset A^*$, so it only remains to be proved that $\Sigma(A)$ is a closed subset of the unit ball in A^*.

7. For an arbitrary C^*-algebra with unit, and self-adjoint $a \in A$, show that for each $\lambda \in \sigma(a)$ there is a state $\omega \in S(A)$ for which $\omega(a) = \lambda$. You may proceed as follows:

   (a) Define ω on an appropriate two-dimensional subspace of A and use the Hahn-Banach Theorem to extend it to all of A.

   (b) Let $b \in A$ be positive and write $\omega(b) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. Use the spectral radius to show that $\alpha \geq 0$.

   (c) Consider the sequence $a_n = a - \alpha + in\beta$ to show that $\beta = 0$.

3.5 Continuous functional calculus

Let A be a C^*-algebra with unit, and let $a \in A$ be *normal* (i.e., $aa^* = a^*a$). In that case, the "C^*-algebra $C^*(a,1)$ generated (within A) by a and the unit" is well defined as the smallest C^*-subalgebra of A containing a and 1: for this is simply given by the norm-closure of all polynomials in a and a^*. By normality of a, all such polynomials commute, and by the Banach algebra axiom $\|ab\| \leq \|a\|\|b\|$ this commutativity is preserved by norm-limits of sequences (or even nets) of polynomials. Hence $C^*(a,1)$ is commutative, and according to Theorem 3.4 we have $C^*(a,1) \cong C(X)$ for some compact Hausdorff space X. What is X explicitly?

Theorem 3.12 *Let $a \in A$ be normal. Then $\Sigma(C^*(a,1))$ is homeomorphic with $\sigma(a)$, so that $C^*(a,1) \cong C(\sigma(a))$. This isomorphism may be chosen such that (and is uniquely defined if) $a \in C^*(a,1)$ maps to the function $\mathrm{id}_{\sigma(a)} : t \to t$ in $C(\sigma(a))$.*

We prove this in case that $a^* = a$; the general case involves some extra notational complications only. Define a map $f \mapsto f(a)$ from $C(\sigma(a))$ to $C^*(a,1)$ as follows:

1. Polynomials p are mapped into the corresponding polynomials $p(a) \in C^*(a,1)$;[34]

2. Arbitrary functions $f \in C(\sigma(a))$ are first approximated in the sup-norm by polynomials p_n, i.e., $\lim_n p_n = f$ uniformly on $\sigma(a)$, upon which $f(a)$ is defined as the norm-limit of $p_n(a)$ in A.[35] By construction, $f(\mathrm{id}_{\sigma(a)}) = a$.

To show that $f \mapsto f(a)$ is an isomorphism, we first note that $f(a) = \mathrm{GT}^{-1} \circ \hat{a}^*(f)$ (exercise),[36] where both the Gelfand transform $\mathrm{GT} : A \to C(\Sigma(A))$ and the map $\hat{a} : \Sigma(A) \to \sigma(a)$ are given by (3.5), in this case with $A = C^*(a,1)$. Note that $\hat{a}$ is initially defined as a function from $\Sigma(A)$ to $\mathbb{C}$, but since $\omega(a) \in \sigma(a)$ for all $\omega \in \Sigma(A)$ (exercise), it actually takes values in $\sigma(a) \subset \mathbb{C}$. Since GT is an isomorphism, it remains to be shown that $\hat{a}^* : C(\sigma(a)) \to C(\Sigma(C^*(a,1)))$ is an isomorphism, which in turn will be the case iff $\hat{a}$ is a homeomorphism. This is indeed the case: the inverse of $\hat{a} : \Sigma(A) \to \sigma(a)$, $\omega \mapsto \omega(a)$, is $\lambda \mapsto \omega_\lambda$, where $\lambda \in \sigma(a)$ and $\omega_\lambda \in \Sigma(C^*(a,1))$ is defined by $\omega_\lambda(f(a)) = f(\lambda)$. Finally, continuity of $\hat{a}$ and its inverse is an easy consequence of the definition of the Gelfand topology. ■

It immediately follows from Theorem 3.12 that

$$\|f(a)\| = \|f\|_\infty; \tag{3.11}$$
$$\sigma(f(a)) = f(\sigma(a)). \tag{3.12}$$

For $f = \mathrm{id}_{\sigma(a)}$ this recovers a result we already knew, viz. $r(a) = \|a\|$ if $a^* = a$, and

$$\|a\| = \sqrt{r(a^*a)} \tag{3.13}$$

in general. This shows that the norm is determined by the spectrum, so that:

Corollary 3.13 *The norm in a C^*-algebra is unique. That is, given a C^*-algebra A (and especially its norm), there is no other norm in which A is a C^*-algebra.*

---

[34] That is, if $p(t) = \sum_n c_n t^n$, then $p(a) = \sum_n c_n a^n$.

[35] This procedure is validated by Weierstrass' Theorem (recall that $\sigma(a)$ is compact) and the fact that if also $q_n \to f$, then $\lim_n q_n(a) = \lim_n p_n(a)$ (exercise, based on $\|f(a)\| \leq \|f\|_\infty$).

[36] Here $g^* : C(Y) \to C(X)$ is the pullback of a continuous map $g : X \to Y$, i.e., $g^*\varphi = \varphi \circ g$.

3.6 C^*-algebras without unit

We still need to prove Theorem 3.4 for the nonunital case, and will use this opportunity to introduce a general technique for handling C^*-algebras without a unit. When a Banach algebra A does not contain a unit, we can always add one, as follows. Form the vector space $A \oplus \mathbb{C}$, and turn this into an algebra by means of

$$(a + \lambda 1)(b + \mu 1) := ab + \lambda b + \mu a + \lambda\mu 1, \tag{3.14}$$

where we have prophetically written $a + \lambda 1$ for (a, λ), et cetera. In other words, the number 1 in $\mathbb{C}$ is identified with the unit 1. Now, the norm on $A \oplus \mathbb{C}$ defined by $\|a + \lambda 1\| := \|a\| + |\lambda|$ makes $A \oplus \mathbb{C}$ a Banach algebra with unit, but that isn't the right one for C^*-algebras (exercise). Indeed, when A is a C^*-algebra we equip $A \oplus \mathbb{C}$ with the natural and obvious involution

$$(a + \lambda 1)^* := a^* + \overline{\lambda} 1, \tag{3.15}$$

and the correct C^*-norm is as follows (by Corollary 3.13, this is the only possibility).

Theorem 3.14 *Let A be a C^*-algebra without unit.*[37]

1. *The map $\rho : A \to L(A)$ given by $\rho(a)b := ab$ yields an isomorphism (of Banach algebras) between A and $\rho(A) \subset L(A)$. (N.B. This is true also for unital A.)*

2. *Define a norm on $A \oplus \mathbb{C}$ by $\|a + \lambda 1\| := \|\rho(a) + \lambda 1\|_{L(A)}$, where 1 on the right-hand side is the unit operator in $L(A)$. With the natural algebraic operations (given above), this norm turns $A \oplus \mathbb{C}$ into a C^*-algebra with unit, called $\dot{A}$.*

3. *There is an isometric (hence injective) morphism $A \to \dot{A}$, such that $\dot{A}/A \cong \mathbb{C}$ as C^*-algebras, and $\dot{A}$ is the unique unital C^*-algebra with this property.*

The proof of the first claim is an exercise. It is clear from (3.14) and (3.15) that the map $a + \lambda 1 \to \rho(a) + \lambda 1$ (where the symbol 1 on the left-hand side is defined below (3.14), and the 1 on the right-hand side is the unit in $L(A)$) is a morphism. Hence the norm defined in claim 2 satisfies (2.39), because the latter is satisfied in the Banach algebra $L(A)$. Moreover, in order to prove that the norm on $\dot{A}$ satisfies (3.2), by Lemma 3.19 (see exercises) it suffices to prove that for all $a \in A$ and $\lambda \in \mathbb{C}$,

$$\|\rho(a) + \lambda 1\|^2 \leq \|(\rho(a) + \lambda 1)^*(\rho(a) + \lambda 1)\|. \tag{3.16}$$

Indeed, for given $c \in L(A)$ and $\epsilon > 0$ there exists a $b \in A$, with $\|b\| = 1$, such that $\|c\|^2 - \epsilon \leq \|c(b)\|^2$. Applying this with $c = \rho(a) + \lambda 1$, we infer that for every $\epsilon > 0$,

$$\|\rho(a) + \lambda 1\|^2 - \epsilon \leq \|(\rho(a) + \lambda 1)b\|^2 = \|ab + \lambda b\|^2 = \|(ab + \lambda b)^*(ab + \lambda b)\|.$$

Here we used (3.2) in A. The right-hand side may be rearranged as

$$\|\rho(b^*)\rho(a^* + \overline{\lambda}1)\rho(a + \lambda 1)b\| \leq \|\rho(b^*)\| \, \|(\rho(a) + \lambda 1)^*(\rho(a) + \lambda 1)\| \, \|b\|.$$

Since $\|\rho(b^*)\| = \|b^*\| = \|b\| = 1$ by claim 1 and (3.22), and $\|b\| = 1$ also in the last term, the inequality (3.16) follows by letting $\epsilon \to 0$. ∎

[37]For any Banach space A, the Banach algebra of all bounded linear maps $A \to A$ is called $L(A)$.

3.7 Commutative C^*-algebras without unit

In the commutative case, the unitization procedure has a simple topological meaning, which illustrates the general principle that the use of commutative C^*-algebras often allows one to trade topological properties for algebraic ones. Recall that the **one-point compactification** $\dot{X}$ of a non-compact topological space X is the set $\dot{X} = X \cup \infty$, whose open sets are the open sets in X plus those subsets of $X \cup \infty$ whose complement is compact in X. The injection $i : X \hookrightarrow \dot{X}$ is continuous, and any continuous function $f \in C_0(X)$ extends uniquely to a function $f \in C(\dot{X})$ satisfying $f(\infty) = 0$. The space $\dot{X}$ is the solution (unique up to homeomorphism) of a so-called universal problem by Alexandroff's theorem: If $\varphi : X \to Y$ is a map between locally compact Hausdorff spaces such that $Y \backslash f(X)$ is a point and f is a homeomorphism onto its image, then there is a unique homeomorphism $\psi : \dot{X} \to Y$ such that $\varphi = \psi \circ i$. The proof of the following lemmas is an easy exercise.

Lemma 3.15 *Let $A = C_0(X)$ for some noncompact locally compact Hausdorff space X. Then $\dot{A} \cong C(\dot{X})$, where $1 \in \dot{A}$ is identified with the constant function $1_{\dot{X}}$ in $C(\dot{X})$. Conversely, removing $\mathbb{C}1_{\dot{X}}$ from $C(\dot{X})$ corresponds to removing $\mathbb{C}$ from $\dot{A} = A \oplus \mathbb{C}$ (as a vector space), leaving one with $C_0(X)$.*

Hence the unitization of $C_0(X)$ corresponds to the one-point compactification of X.

Lemma 3.16 *Let A be a commutative C^*-algebra without unit.*[38]

1. *Each $\omega \in \Sigma(A)$ extends to a character $\dot{\omega}$ on $\dot{A}$ by*
$$\dot{\omega}(a + \lambda 1) := \omega(a) + \lambda. \tag{3.17}$$

2. *The functional ω_∞ on $\dot{A}$, defined by*
$$\omega_\infty(a + \lambda 1) := \lambda, \tag{3.18}$$
is a character of $\dot{A}$.

3. *There are no other characters on $\dot{A}$.*

4. *$\Sigma(\dot{A})$ is homeomorphic to the one-point compactification of $\Sigma(A)$.*

We may now prove Theorem 3.4 also in the nonunital case. Applying the unital case of Theorem 3.4 to $\dot{A}$ and using Lemma 3.16, one finds $\dot{A} \cong C(\dot{X})$ with $X := \Sigma(A)$. Formally, we now use a little lemma stating that if A and B are C^*-algebras without unit, then $\dot{A} \cong \dot{B}$ iff $A \cong B$. Informally: removing $\mathbb{C}$ from $\dot{A} = A \oplus \mathbb{C}$ precisely leaves one with $C_0(X)$ by Lemma 3.15, so that finally $A \cong C_0(X)$. ■

Note that the Gelfand transform on a commutative C^*-algebra without unit indeed takes values in $C_0(\Sigma(A))$, since by (3.18) one has $\hat{a}(\omega_\infty) = \omega_\infty(a + 0) = 0$ for the (unique continuous) extension of $\hat{a}$ from $\Sigma(A)$ to its one-point compactification.

[38] In fact, this lemma is true for any commutative Banach algebra, with respect to any unitization.

3.8 Commutative harmonic analysis

One of the most beautiful applications of Theorem 3.4 is to commutative harmonic analysis. Let G be a locally compact *abelian* group (think of $G = \mathbb{R}$, $G = \mathbb{Z}$, or $G = \mathbb{T}$ if you like), with Haar measure dx (think of Lebesgue measure on $\mathbb{R}$, the counting measure on $\mathbb{Z}$, so that $\int_{\mathbb{Z}} dx\, f(x) = \sum_{n\in\mathbb{Z}} f(n)$, whilst $\int_{\mathbb{T}} dx\, f(x) = \int_0^{2\pi} \frac{d\theta}{2\pi} f(e^{i\theta})$). For $f, g \in C_c(G)$, the *convolution product* $f * g$ is defined by ($x - y \equiv xy^{-1}$)

$$f * g(x) := \int_G dy\, f(x-y)g(y). \tag{3.19}$$

Using the invariance of Haar measure, it is trivial to verify that this product is commutative if G is abelian. We also define an involution on $C_c(G)$ by $f^*(x) = \overline{f(-x)}$, where $-x \equiv x^{-1}$.[39] Of course, we would now like to turn $C_c(G)$ into a commutative C^*-algebra, but the most obvious norms like the L^p-ones do not accomplish this.

Instead, for $f \in C_c(G)$ we define an operator $\pi(f)$ on the Hilbert space $L^2(G)$ by $\pi(f)\psi = f * \psi$; here we initially pick $\psi \in C_c(G)$ and show that

$$\|\pi(f)\| \le \|f\|_1 := \int_G dx\, |f(x)|, \tag{3.20}$$

so that $\pi(f)$ is bounded and may be extended to all of $L^2(G)$ by continuity. Associativity of convolution then implies $\pi(f * g) = \pi(f)\pi(g)$, and also one has $\pi(f^*) = \pi(f)^*$. The map $f \mapsto \pi(f)$ from $C_c(G)$ to $B(L^2(G))$ is injective (exercise), so that $\|f\| = \|\pi(f)\|$ defines a norm on $C_c(G)$. One immediately sees that the axioms (1.7) and (1.8) are satisfied, so that the completion of $C_c(G)$ in this norm, called $C^*(G)$, is a commutative C^*-algebra.[40] What is its Gelfand spectrum?

Recall that, for any locally compact abelian group G, the *dual group* or *character group* $\hat{G}$ is defined as $\hat{G} = \mathrm{Hom}(G, \mathbb{T})$, i.e., the continuous group homomorphisms from G to $\mathbb{T}$, equipped with the compact-open topology.[41] For example, for $G = \mathbb{R}$ we have $\hat{G} \cong \mathbb{R}$, where $p \in \mathbb{R}$ defines a character $\chi_p \in \hat{\mathbb{R}}$ by $\chi_p(x) = \exp(ipx)$. On the other hand, for $G = \mathbb{T}$ one finds $\hat{G} \cong \mathbb{Z}$, where $n \in \mathbb{Z}$ defines $\chi_n(z) = z^n$, $z \in \mathbb{T}$.

Theorem 3.17 *Let G be a locally compact abelian group. The Gelfand spectrum $\Sigma(C^*(G))$ is homeomorphic to $\hat{G}$, so that $C^*(G) \cong C_0(\hat{G})$, and the Gelfand transform $f \mapsto \hat{f}$ implementing this isomorphism coincides with the Fourier transform*

$$\hat{f}(\chi) = \int_G dx\, \overline{\chi(x)} f(x). \tag{3.21}$$

The homeomorphism in question maps $\chi \in \hat{G}$ to $\omega_\chi \in \Sigma(C^*(G))$, given by $\omega_\chi(f) = \hat{f}(\chi)$, as in (3.21). We defer a proof of this beautiful theorem to the exercises.

[39] This choice, rather than the more natural $f^*(x) = \overline{f(x)}$, is made in order to satisfy the axioms for an involution *with respect to the convolution product* (as opposed to the pointwise one).

[40] Because of (1.7), commutativity is preserved by the completion procedure.

[41] The compact-open topology on $\hat{G}$ is the restriction to $\mathrm{Hom}(G, \mathbb{C})$ of the topology on $C(G, \mathbb{C})$ generated by the open sets $O(K, U) = \{\varphi \in C(G, \mathbb{C}) \mid \varphi(K) \subset U, K \subset G \text{ compact}, U \subset \mathbb{C} \text{ open}\}$. In general, G is compact iff $\hat{G}$ is discrete, as exemplified by $G = \mathbb{T}$ and $\hat{G} = \mathbb{Z}$. The space $\hat{G}$ is itself a locally compact abelian group under pointwise multiplication, and the famous Pontryagin Duality Theorem states that $\hat{\hat{G}} \cong G$. We will not need this group structure, though.

Exercises for Lecture 3

1. Prove the following theorem.

   Theorem 3.18 *Let a be a normal element of a unital C^*-algebra A. Then the spectrum of a in A coincides with the spectrum of a in $C^*(a,1)$, so that we may unambiguously speak of* the *spectrum $\sigma(a)$.*

2. Let A be a Banach algebra without unit. Show that $A \oplus \mathbb{C}$ is a Banach algebra with unit in the norm $\|a + \lambda 1\| := \|a\| + |\lambda|$. Give an example where A is a C^*-algebra showing that $A \oplus \mathbb{C}$ is not a C^*-algebra in this norm.

3. Prove Part 1 of Theorem 3.14.

4. Prove:

   Lemma 3.19 *(a) If an involution $a \mapsto a^*$ on a Banach algebra A satisfies the inequality $\|a\|^2 \leq \|a^*a\|$, then $\|a\|^2 = \|a^*a\|$ and hence A is a C^*-algebra.*

   (b) For any element a of a C^*-algebra one has

   $$\|a^*\| = \|a\|. \tag{3.22}$$

5. Prove Lemma 3.15.

6. Prove Lemma 3.16.

7. Prove Theorem 3.17 by showing that:

   (a) $\Sigma(C^*(G)) = \Sigma(L^1(G))$, where the Banach algebra $L^1(G)$ is the completion of $C_c(G)$ in the L^1-norm $\|\cdot\|_1$.

   (b) $\Sigma(L^1(G)) \cong \hat{G}$, first by proving that each ω_χ is a character of $L^1(G)$, $\chi \in \hat{G}$, and secondly that there are no others.

3.9 The structure of C^*-algebras

Having understood the structure of commutative C^*-algebras, we now turn to the general case. We already know that the algebra $B(H)$ of all bounded operators on some Hilbert space H is a C^*-algebra in the obvious way (i.e., the algebraic operations are the natural ones, the involution is the operator adjoint $a \mapsto a^*$, and the norm is the operator norm of Banach space theory). Moreover, each (operator) norm-closed *-algebra in $B(H)$ is a C^*-algebra. Our goal is to prove the converse:

Theorem 3.20 *Each C^*-algebra A is isomorphic to a norm-closed *-algebra in $B(H)$, for some Hilbert space H. In other words, for any C^*-algebra A there exist a Hilbert space H and an injective homomorphism $\pi : A \to B(H)$.*

A homomorphism $\pi : A \to B(H)$ is often called a **representation** of A on H. The equivalence between the two statements in the theorem is a consequence of:

Proposition 3.21 *Let $\varphi : A \to B$ be a nonzero morphism between C^*-algebras.*

1. *φ is continuous, with norm ≤ 1 (i.e., $\|\varphi(a)\|_B \leq \|a\|_A$ for all $a \in A$);*
2. *If φ is injective, then it is isometric (i.e., $\|\varphi(a)\|_B = \|a\|_A$ for all $a \in A$).*

The proof is an exercise. Let us note that Theorems 3.4 and 3.20 are compatible, in that any measure μ on X satisfying $\mu(U) > 0$ for each open $U \subset X$ leads to an injective representation of $C_0(X)$ on $L^2(X, \mu)$ by multiplication operators, that is, $\pi(f)\psi(x) = f(x)\psi(x)$, where $f \in C_0(X)$ and $\psi \in L^2(X, \mu)$. See [8, §VIII.5].

The proof of Theorem 3.20 uses the beautiful GNS**-construction**,[42] which is important in its own right. We assume that A is unital (and return to the non-unital case at the end). First, we call a representation π **cyclic** if its carrier space H contains a **cyclic vector** Ω for π; this means that the closure of $\pi(A)\Omega$ coincides with H. Such representation are the building blocks of any representation.[43]

Theorem 3.22 *Let ω be a state on a C^*-algebra A. There exists a cyclic representation π_ω of A on a Hilbert space H_ω with cyclic unit vector Ω_ω such that*

$$\omega(a) = (\Omega_\omega, \pi_\omega(a)\Omega_\omega) \quad \forall a \in A. \tag{3.23}$$

We first give the idea of the proof in the special case that $\omega(a^*a) > 0$ for any $a \neq 0$. Define a sesquilinear form $(-,-)$ on A by $(a, b) := \omega(a^*b)$. This form is positive definite by assumption, so that we may complete A to a Hilbert space called H_ω. For each $a \in A$ we then define a map $\pi_\omega(a) : A \to A$ by $\pi_\omega(a)b = ab$. Regarding A as a dense subspace of H_ω, this defines an operator $\pi_\omega(a)$ on a dense domain in H_ω. This operator turns out to be bounded, so that it may be extended from A to H_ω by continuity and we obtain a map $\pi_\omega : A \to B(H_\omega)$. Trivial computations show that π_ω is a representation. The special vector Ω_ω is simply $1 \in A$, seen as an element of H_ω. Indeed, $\|\Omega_\omega\|^2 = (\Omega_\omega, \Omega_\omega) = \omega(1^*1) = \omega(1) = 1$ and $(\Omega_\omega, \pi_\omega(a)\Omega_\omega) = \omega(1^*a1) = \omega(a)$. Hence the only difficulty of the proof lies in the boundedness of $\pi_\omega(a)$ and in the removal of the assumption $\omega(a^*a) > 0$.

[42] For *Gelfand-Naimark-Segal.* This construction is very important also in mathematical physics.

[43] Any non-degenerate representation π is a direct sum of cyclic representations. Here one says that a representation $\pi(A)$ is **non-degenerate** if $\pi(a)v = 0$ for all $a \in A$ implies $v = 0$.

3.10 Proof of Theorems 3.22 and 3.20

The inequality establishing boundedness of $\pi_\omega(a)$, or specifically $\|\pi_\omega(a)\| \leq \|a\|$, is

$$\omega(b^*a^*ab) \leq \|a\|^2\omega(b^*b), \tag{3.24}$$

which is a transcription of the inequality $\|\pi_\omega(a)b\|^2_{\mathcal{H}_\omega} \leq \|a\|^2_A\|b\|^2_{\mathcal{H}_\omega}$. We defer the proof of this to subsection 3.12. Under our standing assumption, i.e., $\omega(a^*a) > 0$ iff $a \neq 0$, this not only proves Theorems 3.22, but also Theorem 3.20: for $\pi_\omega(a) = 0$ implies $\|\pi_\omega(a)\Omega_\omega\|^2 = 0$, whose left-hand side is precisely $(\Omega_\omega, \pi_\omega(a^*a)\Omega_\omega) = \omega(a^*a)$.

In general, a C^*-algebra may lack such states, and we must adapt the proof of both theorems. The GNS-construction is easy: for an arbitrary state ω, we introduce $N_\omega = \{a \in A \,|\, \omega(a^*a) = 0\}$, which is a left-ideal in A (i.e. a C^*-subalgebra of A such that $ab \in N_\omega$ whenever $a \in A$ and $b \in N_\omega$). If $p_\omega a$ is the image of $a \in A$ in A/N_ω, we may define an inner product on the latter by $(p_\omega a, p_\omega b) = \omega(a^*b)$; this is well defined and positive definite, and we define the Hilbert space H_ω as the completion of A/N_ω in this inner product. Furthermore, we define $\pi_\omega(a) : A/N_\omega \to H_\omega$ by $\pi_\omega(a)p_\omega b := p_\omega ab$; this is indeed well defined because N_ω is a left ideal in A. Finally, we define $\Omega_\omega := p_\omega 1$. The proof that everything works is then an exercise.

When A has no unit, use the GNS-construction for the unitization $\dot{A}$ and simply restrict $\pi_{\dot{\omega}}(\dot{A})$ to A to define $\pi_\omega(A)$. This completes the proof of Theorem 3.22. ■

We now take up the proof of Theorem 3.20. To solve the problem of the possible lack of injectivity of π_ω, we replace H_ω by the *c*razy Hilbert space $H_c = \bigoplus_{\omega\in P(A)} H_\omega$, where $P(A)$ is the pure state space of A. The elements of this space are sequences $\Psi \equiv (\Psi_\omega)_{\omega\in P(A)}$, such that: *(i)* $\Psi_\omega \in H_\omega$; *(ii)* only countably many vectors Ψ_ω are nonzero; and *(iiii)* $\sum_{\omega\in P(A)} \|\Psi_\omega\|^2_{H_\omega} < \infty$ (note that the sum makes sense!). Addition and scalar multiplication are defined pointwise, and the inner product is

$$(\Psi, \Phi)_{H_c} = \sum_{\omega\in P(A)} (\Psi_\omega, \Phi_\omega)_{H_\omega}. \tag{3.25}$$

The crazy space H_c carries a representation $\pi(A)$ defined by $(\pi(a)\Psi)_\omega = \pi_\omega(a)\Psi_\omega$, and the point of all this is that π is injective: if $\pi(a) = 0$, then $(\pi(a)\Psi)_\omega = 0$ for each $\Psi \in H_c$ and each $\omega \in P(A)$, hence also for the pure state ω_0 of Lemma 3.8, and for the vector Ψ given by $\Psi_{\omega_0} = \Omega_{\omega_0}$ and $\Psi_\omega = 0$ for all $\omega \neq \omega_0$. But this implies $\pi_{\omega_0}(a)\Omega_{\omega_0} = 0$, so that $|(\Omega_{\omega_0}, \pi_{\omega_0}(a)\Omega_{\omega_0})| = |\omega_0(a)| = 0$. If $a^* = a$ this implies $\|a\| = 0$ by Lemma 3.8, hence $a = 0$. For general a, we replace a in the above calculation by a^*a, which is self-adjoint. This yields the inference $\pi(a) = 0 \Rightarrow \pi(a^*a) = 0 \Rightarrow \|a^*a\| = 0$. The C*-axiom equates $\|a^*a\|$ to $\|a\|^2$, so that once again $\|a\| = 0$. It follows that π is injective, and Theorem 3.20 is proved. ■

It should be noted that this proof relies on incredible overkill, in that H_c is far larger than necessary (indeed, in all but the most trivial cases, H is non-separable). For example, already for $A = M_2(\mathbb{C})$ we have $P(A) \cong S^2$, so that $H_c = \bigoplus_{\omega\in S^2} \mathbb{C}^2$; this Hilbert space is non-separable, whereas A has an injective representation on $\mathbb{C}^2$. More generally, $K(H)$ or $B(H)$ has an injective representation on H by definition, whereas H_c is non-separable. In the commutative case, $A = C_0(X)$ yields the non-separable $H_c = \bigoplus_{x\in X} \mathbb{C}$, although A has an injective representation on $L^2(X, \mu)$.

3.11 Easy examples of the GNS-construction

In the following examples we say that two representations $\pi_1(A)$ and $\pi_2(A)$ of a C^*-algebra A on Hilbert spaces H_1 and H_2 are **(unitarily) equivalent** ($\pi_1 \cong \pi_2$), when there is a *unitary* map $u : H_1 \to H_2$ such that $\pi_2(a) = u\pi_1(a)u^*$ for all $a \in A$. The GNS-representation is often equivalent to some 'familiar' representation:

• Let $A = C_0(X)$. The Riesz–Markov Theorem 2.10 implies that the state space of A is isomorphic to the convex set of (Radon) probability measures μ on X; we write $\omega_\mu(f) = \int_X d\mu\, f$ for this correspondence, cf. (2.28). The pure states (which correspond to point or Dirac measures) take the form $\omega_x(f) = f(x)$ for some $x \in X$, and clearly $N_x \equiv N_{\omega_x} = \{f \in C_0(X) \mid f(x) = 0\}$. It follows that $A/N_x \cong \mathbb{C}$ under the unitary map $p_{\omega_x} f \mapsto f(x)$, and the corresponding GNS-representation is equivalent to $\pi_x(f) = f(x)$ on $H_x = \mathbb{C}$ (exercise). In the 'opposite' case where $\mu(U) > 0$ for any open $U \subset X$, we have $N_{\omega_\mu} = 0$ and the Hilbert space $H_{\omega_\mu} \equiv H_\mu$ is given ('on the nose', i.e. literally rather than up to equivalence) by the completion $L^2(X, \mu)$ of $C_0(X)$ with respect to the inner product $(f, g) = \int_X d\mu\, \overline{f} g$. The corresponding GNS-representation π_μ is obviously given by multiplication operators, and the cyclic vector Ω_μ is simply the function identically equal to 1. We verify (3.23) by computing $(\Omega_\mu, \pi_\mu(f)\Omega_\mu) = \int_X d\mu\, f = \omega_\mu(f)$; for $f = 1_X$ this yields $\|\Omega_\mu\|^2 = 1$.

• For a noncommutative example, take $A = M_n(\mathbb{C})$, with a state necessarily of the form $\rho(a) = \mathrm{Tr}\,(\hat{\rho}a)$, for some density matrix $\hat{\rho}$. So $N_\rho = \{a \in A \mid \mathrm{Tr}\,(\hat{\rho}a^*a) = 0\}$. If we expand $\hat{\rho} = \sum_i \lambda_i p_i$ (cf. §2.10), and for simplicity assume that $p_i = |e_i\rangle\langle e_i|$ with respect to the standard basis (e_i) of $\mathbb{C}^n$, then two cases of special interest arise:

1. If $\hat{\rho} = |e_j\rangle\langle e_j|$ is pure, the state is just $\rho(a) = (e_j, ae_j)$, having null space $N_\rho = \{a \in A \mid ae_j = 0\}$. Hence $a \in N_\rho$ iff the j'th column $C_j(a)$ of a vanishes, so that $a - b \in N_\rho$ iff $C_j(a) = C_j(b)$. Thus the equivalence class $p_\rho a \in A/N_\rho$ may be identified with $C_j(a)$, so that $H_\rho \equiv A/N_\rho \cong \mathbb{C}^n$ (with the standard inner product) under the unitary $u : p_\rho a \mapsto C_j(a)$ from A/N_ρ to $\mathbb{C}^n$, with inverse $u^{-1} : z \mapsto p_\rho a$, where $a_{ij} = z_i$ and $a_{ik} = 0$ for all i and $k \neq j$ (that is, a has $C_j(a) = z$ and zeros elsewhere). We likewise write $u^{-1}w = p_\rho b$, with $b_{ij} = w_i$ and $b_{ik} = 0$ for all i and $k \neq j$. Hence $u(p_\rho a) = z$ and $u(p_\rho b) = w$, and unitarity follows by computing

$$(p_\rho a, p_\rho b)_{H_\rho} = \rho(a^*b) = \sum_i \overline{a_{ij}} b_{ij} = \sum_i \overline{z}_i w_i = (z, w)_{\mathbb{C}^n} = (u(p_\rho a), u(p_\rho b))_{\mathbb{C}^n}.$$

(Physicists beware: no sum over j!) The GNS-representation, originally given on H_ρ by $\pi_\rho(a)p_\rho b = p_\rho(ab)$, is transformed to $u\pi_\rho(a)u^{-1} \equiv \hat{\pi}_\rho$ on $\mathbb{C}^n$, which is given by

$$\hat{\pi}_\rho(a)w = u\pi_\rho(a)p_\rho b = up_\rho(ab) = C_j(ab) = aw.$$

The cyclic vector $u\Omega_\rho = \hat{\Omega}_\rho$ in $\mathbb{C}^n$ is just the basis vector e_j from which we started. More generally, for a pure state $\psi(a) = (\Psi, a\Psi)$ the GNS-representation $\pi_\psi(M_n(\mathbb{C}))$ induced by ψ is equivalent to the defining representation on $\mathbb{C}^n$, with $\hat{\Omega}_\psi = \Psi$.

2. The 'opposite' case where $\rho(a^*a) > 0$ for all $a \neq 0$ occurs when $\lambda_i > 0$ for all i. In that case, $H_\rho \cong M_n(\mathbb{C})$ with inner product $(a, b) = \mathrm{Tr}\,(a^*b)$, and $\pi_\rho \cong \hat{\pi}_\rho$ with $\hat{\pi}_\rho(a)b = ab$. The cyclic vector in $M_n(\mathbb{C})$ then becomes $\hat{\Omega}_\rho = \rho^{1/2}$ (see exercises).

3.12 Positivity in C^*-algebras

To prove the inequality (3.24), as well as some deeper results not contained in this course, we need to develop the theory of positivity in C^*-algebras. This theory will unify the following special cases: a bounded operator a on a Hilbert space H is called positive when $(v, av) \geq 0$ for all $v \in H$, whilst a function f on some space X is called positive when it is pointwise positive, that is, when $f(x) \geq 0$ for all $x \in X$.

Definition 3.23 *An element a of a C^*-algebra A is* ***positive*** *when $a = b^*b$ for some $b \in A$. We then write $a \geq 0$, and define $a \geq b$ for self-adjoint a, b if $a - b \geq 0$.*

If A is commutative, this clearly reproduces the second definition above, and if $A \subset B(H)$, the equivalence with the first notion is a consequence of the following:

Proposition 3.24 *An element $a \in A$ is positive iff $a^* = a$ and $\sigma(a) \subset \mathbb{R}^+$.*

The proof is an exercice. Hilbert space theory then equates the characterization of $a \geq 0$ given in this proposition to the condition that $(v, av) \geq 0$ for all $v \in H$.

We are now going to prove the inequality (3.24). Here is the first step.

Lemma 3.25 *1. If C is a C^*-subalgebra of A and $a \leq b$ in C, then $a \leq b$ in A.*

2. If $\varphi : A \to B$ is an isomorhpism, then $a \geq b$ in A iff $\varphi(a) \geq \varphi(b)$ in B.

3. If $a \geq d$, then $b^*ab \geq b^*db$ for any $b \in A$.

4. If $a^* = a$ and A is unital, then $-\|a\| \cdot 1 \leq a \leq \|a\| \cdot 1$.

The first three claims are trivial. The fourth is first proved in $C(\sigma(a))$, where it reads $-r(a) \cdot 1_{\sigma(a)} \leq \mathrm{id}_{\sigma(a)} \leq r(a) \cdot 1_{\sigma(a)}$ (here $1_{\sigma(a)}$ is the function $t \to 1$ whilst $\mathrm{id}_{\sigma(a)} : t \to t$, and we have used $\|\mathrm{id}_{\sigma(a)}\|_\infty = r(a)$). This obviously holds pointwise, and hence also in the sense of Definition 3.23. Under the inverse Gelfand transform $C(\sigma(a)) \to C^*(a, 1) \subset A$, i.e., by the continuous functional calculus (see §3.5), the function $1_{\sigma(a)}$ is mapped to $1 \in A$, whereas $\mathrm{id}_{\sigma(a)}$ is mapped to $a \in A$. In combination with $r(a) = \|a\|$ and the second part of the lemma, this gives the fourth part. ∎

For any a, the third and the fourth part give $b^*a^*ab \leq \|a\|^2 \cdot b^*b$. Since states by definition preserve inequalities as in Definition 3.23, we finally obtain (3.24). ∎

Finally, an interesting perspective on positivity is given by the concept of a **convex cone** in a real vector space V, which is a subspace V^+ such that:
(i): if $v \in V^+$ and $t \in \mathbb{R}^+$ then $tv \in V^+$; *(ii):* if $v, w \in V^+$ then $v + w \in V^+$;
(iii): $V^+ \cap -V^+ = 0$.
Such a convex cone is equivalent to a **linear partial ordering** in V, which is a partial ordering $\leq$ in which $v \leq w$ implies $v + f \leq w + f$ for all $f \in V$ and $tv \leq tw$ for all $t \in \mathbb{R}^+$. Indeed, given $V^+ \subset V$ one defines $\leq$ by putting $v \leq w$ if $w - v \in V^+$, and given $\leq$ one defines $V^+ = \{v \in V \mid 0 \leq v\}$. Without proof, we state:

Proposition 3.26 *The set A^+ of all positive elements of a C^*-algebra A is a convex cone in the real vector space $A_{\mathrm{sa}} := \{a \in A \mid a^* = a\}$.*

Exercises for Lecture 4

1. Prove Proposition 3.21 (you may assume that A and B have units).
   Hint for part 1: show that $\sigma(\varphi(a)) \subseteq \sigma(a)$. Hint for part 2: Assume there is an $b \in A$ for which $\|\varphi(b)\| \neq \|b\|$. Show that this implies $\|\varphi(b^*b)\| \neq \|b^*b\|$. Put $a := b^*b$ and show that $\sigma(\varphi(a)) \subset \sigma(a)$. By Urysohn's lemma there is a nonzero $f \in C(\sigma(a))$ that vanishes on $\sigma(\varphi(a))$, so that $f(\varphi(a)) = 0$. Prove that $\varphi(f(a)) = 0$, which contradicts injectivity of φ.

2. Fill in the details of the GNS-construction for general states ω:

   (a) Show that the inner product $(p_\omega a, p_\omega b) = \omega(a^*b)$ on A/N_ω is well defined;

   (b) Show that the representation $\pi_\omega(a)p_\omega b := p_\omega ab$ is well defined on A/N_ω;

   (c) Show that $\pi_\omega(a)$ is bounded.

3. Show in detail that the GNS-representation of $C_0(X)$ induced by a pure state $\omega_x(f) = f(x)$ for some $x \in X$ is equivalent to $\pi_x(f) = f(x)$ on $H = \mathbb{C}$.

4. Show in detail that the GNS-representation of $M_n(C)$ induced by a mixed state $\rho(a) = \mathrm{Tr}\,(\hat{\rho}a)$, where $\hat{\rho} = \sum_i \lambda_i p_i$ with $\lambda_i > 0$ for all i, is equivalent to the representation $\pi(a)b = ab$ on $H = M_n(\mathbb{C})$ with inner product $(a, b) = \mathrm{Tr}\,(a^*b)$.

5. Prove Proposition 3.24.

6. A representation π of a C^*-algebra A on a Hilbert space H is called **irreducible** if a closed subspace K of H that is stable under $\pi(A)$ (in the sense that if $\psi \in K$, then $\pi(a)\psi \in K$ for all $a \in A$) is either H or 0.

   (a) Prove that each of the following conditions is equivalent to irreducibility:
   - $\pi(A)' = \mathbb{C} \cdot 1$ (where S' is the commutant of $S \subset B(H)$);
   - $\pi(A)'' = B(H)$;
   - Every vector in H is cyclic for $\pi(A)$.

   Hint: from the theory of von Neumann algebras, use the fact that if $\pi(A)' \neq \mathbb{C} \cdot 1$, then $\pi(A)'$ contains a nontrivial projection (i.e. $0 \neq p \neq 1$).

   (b) Prove that if ω is pure, then the GNS-representation π_ω is irreducible.

   (c) Prove that if π_ω is irreducible, then ω is pure (difficult!).

Appendix 1: proof of Proposition 3.10

The proof of part 1 uses two lemmas. We assume that A is unital.

Lemma 3.27 *When* $\|a\| < 1$ *the sum* $\sum_{k=0}^{n} a^k$ *converges to* $(1-a)^{-1}$. *Hence* $(a - z1)^{-1}$ *always exists when* $|z| > \|a\|$, *so that*

$$r(a) \leq \|a\|. \tag{3.26}$$

We first show that the sum is a Cauchy sequence. Indeed, for $n > m$ one has

$$\|\sum_{k=0}^{n} a^k - \sum_{k=0}^{m} a^k\| = \|\sum_{k=m+1}^{n} a^k\| \leq \sum_{k=m+1}^{n} \|a^k\| \leq \sum_{k=m+1}^{n} \|a\|^k.$$

For $n, m \to \infty$ this goes to 0 by the theory of the geometric series. Since A is complete, the Cauchy sequence $\sum_{k=0}^{n} a^k$ converges for $n \to \infty$. Now compute

$$\sum_{k=0}^{n} a^k(1-a) = \sum_{k=0}^{n} (a^k - a^{k+1}) = 1 - a^{n+1}.$$

Hence

$$\|1 - \sum_{k=0}^{n} a^k(1-a)\| = \|a^{n+1}\| \leq \|a\|^{n+1}.$$

which $\to 0$ for $n \to \infty$, as $\|a\| < 1$ by assumption. Thus $\lim_{n\to\infty} \sum_{k=0}^{n} a^k(1-a) = 1$. By a similar argument,

$$\lim_{n\to\infty} (1-a) \sum_{k=0}^{n} a^k = 1.$$

so that, by continuity of multiplication in a Banach algebra, one finally has

$$\lim_{n\to\infty} \sum_{k=0}^{n} a^k = (1-a)^{-1}.$$

The second claim of the lemma follows because $(a-z)^{-1} = -z^{-1}(1-a/z)^{-1}$, which exists because $\|a/z\| < 1$ when $|z| > \|a\|$. ■

To prove that $\sigma(a)$ is compact, it remains to be shown that it is closed.

Lemma 3.28 *The set* $G(A) := \{a \in A | \, a^{-1} \text{ exists}\}$ *of invertible elements in* A *is open in* A.

Given $a \in G(A)$, take a $b \in A$ for which $\|b\| < \|a^{-1}\|^{-1}$. This implies

$$\|a^{-1}b\| \leq \|a^{-1}\| \, \|b\| < 1. \tag{3.27}$$

Hence $a + b = a(1 + a^{-1}b)$ has an inverse, namely $(1 + a^{-1}b)^{-1}a^{-1}$, which exists by (3.27) and Lemma 3.27. It follows that all $c \in A$ for which $\|a - c\| < \epsilon$ lie in $G(A)$, for $\epsilon \leq \|a^{-1}\|^{-1}$. ■

To resume the proof of Proposition 3.10.1, given $a \in A$ we now define a function $f : \mathbb{C} \to A$ by $f(z) := z - a$. Since $\|f(z+\delta) - f(z)\| = \delta$, we see that f is continuous (take $\delta = \epsilon$ in the definition of continuity). Because $G(A)$ is open in A by Lemma 3.28, it follows from the topological definition of a continuous function that $f^{-1}(G(A))$ is open in $\mathbb{C}$. But $f^{-1}(G(A))$ is the set of all $z \in \mathbb{C}$ where $z - a$ has an inverse, so that $f^{-1}(G(A)) = \rho(a)$. This set being open, its complement $\sigma(a)$ is closed. Finally, define $g : \rho(a) \to A$ by $g(z) := (z-a)^{-1}$. For fixed $z_0 \in \rho(a)$, choose $z \in \mathbb{C}$ such that $|z - z_0| < \|(a - z_0)^{-1}\|^{-1}$. From the proof of Lemma 3.28, with a replaced by $a - z_0$ and c replaced by $a - z$, we see that $z \in \rho(a)$, as $\|a - z_0 - (a - z)\| = |z - z_0|$. Moreover, the power series

$$\frac{1}{z_0 - a} \sum_{k=0}^{n} \left(\frac{z_0 - z}{z_0 - a} \right)^k$$

converges for $n \to \infty$ by Lemma 3.27, because

$$\|(z_0 - z)(z_0 - a)^{-1}\| = |z_0 - z| \, \|(z_0 - a)^{-1}\| < 1.$$

By Lemma 3.27, the limit $n \to \infty$ of this power series is

$$\frac{1}{z_0 - a} \sum_{k=0}^{\infty} \left(\frac{z_0 - z}{z_0 - a} \right)^k = \frac{1}{z_0 - a} \left(1 - \left(\frac{z_0 - z}{z_0 - a} \right) \right)^{-1} = \frac{1}{z - a} = g(z).$$

Hence

$$g(z) = \sum_{k=0}^{\infty} (z_0 - z)^k (z_0 - a)^{-k-1} \tag{3.28}$$

is a norm-convergent power series in z. For $z \neq 0$ we write $\|g(z)\| = |z|^{-1} \|(1 - a/z)^{-1}\|$ and observe that $\lim_{z\to\infty} 1 - a/z = 1$, since $\lim_{z\to\infty} \|a/z\| = 0$. Hence $\lim_{z\to\infty}(1 - a/z)^{-1} = 1$, and

$$\lim_{z\to\infty} \|g(z)\| = 0. \tag{3.29}$$

Let $\rho \in A^*$ be a functional on A; since ρ is bounded, (3.28) implies that the function $g_\rho : z \to \rho(g(z))$ is given by a convergent power series, and (3.29) implies that

$$\lim_{z\to\infty} g_\rho(z) = 0. \tag{3.30}$$

Now suppose that $\sigma(a) = \emptyset$, so that $\rho(a) = \mathbb{C}$. The function g, and hence g_ρ, is then defined on $\mathbb{C}$, where it is analytic and vanishes at infinity. In particular, g_ρ is bounded, so that by Liouville's theorem it must be constant. By (3.30) this constant is zero, so that $g = 0$.[44] This is absurd, so that $\rho(a) \neq \mathbb{C}$ hence $\sigma(a) \neq \emptyset$. This finishes the proof of Proposition 3.10.1. ∎

[44] This follows by a basic result in Banach spaces B: if $v \in B$ is such that $\rho(v) = 0$ for all $\rho \in B^*$, then $v = 0$.

The proof of Proposition 3.10.2 is as follows. By Lemma 3.27, for $|z| > \|a\|$ the function g in the proof of Lemma 3.28 has the norm-convergent power series

$$g(z) = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k. \tag{3.31}$$

On the other hand, we have seen that for any $z \in \rho(a)$ one may find a $z_0 \in \rho(a)$ such that the power series (3.28) converges. If $|z| > r(a)$ then $z \in \rho(a)$, so (3.28) converges for $|z| > r(a)$. At this point the proof relies on the theory of analytic functions with values in a Banach space, which says that, accordingly, (3.31) is norm-convergent for $|z| > r(a)$, uniformly in z. Comparing with (3.26), this sharpens what we know from Lemma 3.27. The same theory says that (3.31) cannot norm-converge uniformly in z unless $\|a^n\|/|z|^n < 1$ for large enough n. This is true for all z for which $|z| > r(a)$, so that

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a). \tag{3.32}$$

To derive a second inequality we use the following **polynomial spectral mapping property**.

Lemma 3.29 *For a polynomial p on $\mathbb{C}$, define $p(\sigma(a))$ as $\{p(z) | z \in \sigma(a)\}$. Then*

$$p(\sigma(a)) = \sigma(p(a)). \tag{3.33}$$

To prove this equality, choose $z, \alpha \in \mathbb{C}$ and compare the factorizations

$$\begin{aligned} p(z) - \alpha &= c\prod_{i=1}^{n}(z - \beta_i(\alpha)); \\ p(a) - \alpha 1 &= c\prod_{i=1}^{n}(a - \beta_i(\alpha)1). \end{aligned} \tag{3.34}$$

Here the coefficients c and $\beta_i(\alpha)$ are determined by p and α. When $\alpha \in \rho(p(a))$ then $p(a) - \alpha 1$ is invertible, which implies that all $a - \beta_i(\alpha)1$ must be invertible. Hence $\alpha \in \sigma(p(a))$ implies that at least one of the $a - \beta_i(\alpha)1$ is not invertible, so that $\beta_i(\alpha) \in \sigma(a)$ for at least one i. Hence $\alpha \in p(\beta_i(\alpha)) - \alpha = 0$, i.e., $\alpha \in p(\sigma(a))$. This proves the inclusion $\sigma(p(a)) \subseteq p(\sigma(a))$.

Conversely, when $\alpha \in p(\sigma(a))$ then $\alpha = p(z)$ for some $z \in \sigma(a)$, so that for some i one must have $\beta_i(\alpha) = z$ for this particular z. Hence $\beta_i(\alpha) \in \sigma(a)$, so that $a - \beta_i(\alpha)$ is not invertible, implying that $p(a) - \alpha 1$ is not invertible, so that $\alpha \in \sigma(p(a))$. This shows that $p(\sigma(a)) \subseteq \sigma(p(a))$, and (3.33) follows. ■

To conclude the proof of Proposition 3.10.2, we note that since $\sigma(a)$ is closed there is an $\alpha \in \sigma(a)$ for which $|\alpha| = r(a)$. Since $\alpha^n \in \sigma(a^n)$ by Lemma 3.29, one has $|\alpha^n| \le \|a^n\|$ by (3.26). Hence $\|a^n\|^{1/n} \ge |\alpha| = r(a)$. Combining this with (3.32) yields

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a) \le \|a^n\|^{1/n}.$$

Hence the limit must exist, and

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \inf_n \|a^n\|^{1/n} = r(a).$$ ■

4 von Neumann algebras

See §1.2 for notation and preliminary remarks on von Neumann algebras. Apart from von Neumann's own motivation, later developments brought contact with quantum field theory [13], quantum statistical mechanics [2], number theory [5, 6], knot theory [15], and algebraic geometry [6].[45] In 1982 the Fields Medal was awarded to Alain Connes for his contributions to the classification of von Neumann algebras [5], whereas Vaughan Jones received the same prize in 1990 for his new invariant in knot theory, obtained through a study of von Neumann algebras of type II_1.

4.1 The Double Commutant Theorem

Operator algebra theory started with the following theorem of von Neumann:[46]

Theorem 4.1 *Let M be a unital *-subalgebra of $B(H)$. Then the following conditions are equivalent (and hence each defines M to be a von Neumann algebra):*

(1) $M'' = M$;

(2) M is closed in the weak operator topology;

(3) M is closed in the strong operator topology;

(4) M is closed in the σ-weak operator topology.

The essence of the proof is already contained in the finite-dimensional case $H = \mathbb{C}^n$, where the nontrivial claim in Theorem 4.1 is: *if M is a unital *-subalgebra of $M_n(\mathbb{C})$, then $M'' = M$.* In fact, all we need to prove is $M'' \subseteq M$, since the converse inclusion is trivial. The idea is to take n arbitrary vectors $v_1, \ldots, v_n$ in H, and, given $a \in M''$, find some $b \in M$ such that $av_i = bv_i$ for all $i = 1, \ldots, n$. Hence $a = b$, so $a \in M$.

For fixed $v \in H$, form the linear subspace $Mv = \{mv \mid m \in M\}$ of H, with associated projection p (i.e. $pw = w$ if $w \in Mv$ and $pw = 0$ if $w \in (Mv)^{\perp}$). Then $p \in M'$ (exercise). Hence $a \in M''$ commutes with p. Since $1 \in M$, we have $v \in Mv$, so $v = pv$, and $av = apv = pav \in Mv$. Hence $av = bv$ for some $b \in M$.

Now run the same argument with the substitutions $H \rightsquigarrow H^n = H \oplus \cdots \oplus H$ (with n terms), $M \rightsquigarrow M_n = \{\text{diag}(m, \ldots, m) \mid m \in M\}$, and $v \rightsquigarrow \mathsf{v} = \oplus_i v_i \equiv (v_1, \ldots, v_n)$. We then have $(M_n)'' = (M'')_n$ (exercise), so for any matrix $\mathsf{a} = \text{diag}(a, \ldots, a)$ in $(M'')_n$ the previous argument yields a matrix $\mathsf{b} = \text{diag}(b, \ldots, b) \in M_n$ such that $\mathsf{a}v = \mathsf{b}v$. But this is $av_i = bv_i$ for all $i = 1, \ldots, n$, so that $a = b$ and hence $M'' \subseteq M$.

The implication (3) $\Rightarrow$ (1), then, is easily shown by adapting the above proof to infinite-dimensional H (exercise). Furthermore, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is quite trivial, whereas (1) $\Leftrightarrow$ (4) may be proved either as in exercise 6 or through the following functional-analytic result (cf. [33, Vol. I, Thm. II.2.6] or [28, Thm. 4.6.7]): *A linear subspace of $B(H)$ is strongly closed iff it is weakly closed iff it is σ-weakly closed.* ■

[45] Jakob Lurie (hailed by some as a successor to the algebraic geometer and category theorist Alexandre Grothendieck) is now teaching a course on von Neumann algebras at Harvard [19].

[46] See [23], and compare Proposition 2.5, reformulated as: *A subspace $K \subset H$ is closed iff $K^{\perp\perp} = K$.* In either case, a topological condition is equivalent to an algebraic one.

4.2 From spectral theory to von Neumann algebras

If we combine the continuous functional calculus (i.e. Theorem 3.12) with the Riesz–Markov Theorem 2.10 from measure theory (see also §3.11), we obtain:

Proposition 4.2 *Let $a^* = a \in B(H)$. For each density operator $\hat{\rho}$ on H (with associated state ρ) there exists a unique probability measure μ_ρ on $\sigma(a)$ such that*

$$\rho(f(a)) \equiv \mathrm{Tr}\,(\hat{\rho}f(a)) = \int_{\sigma(a)} d\mu_\rho\, f \quad \text{for all } f \in C(\sigma(a)). \tag{4.35}$$

Since the (Lebesgue) integral appearing here is defined for the far larger class $\mathcal{B}(\sigma(a))$ of bounded Borel functions on $\sigma(a)$, we may try to extend the continuous functional calculus $f \mapsto f(a)$ to a "Borel functional calculus." To do so, we regard $\mathcal{B}(\sigma(a))$ as a commutative C^*-algebra under pointwise operations and the sup-norm, and introduce the commutative von Neumann (and hence C^*) algebra $W^*(a) = C^*(a,1)''$.

Theorem 4.3 *Let $a^* = a \in B(H)$. The isomorphism $C(\sigma(a)) \to C^*(a,1)$ of Theorem 3.18 has a unique extension to a homomorphism $\mathcal{B}(\sigma(a)) \to W^*(a)$ that satisfies $\|f(a)\| \leq \|f\|_\infty$ for each $f \in \mathcal{B}(\sigma(a))$, and (4.35) remains valid for such f.*

The proof [28, Thm. 4.5.4] is based on the following fact [28, Lemma 3.2.2].

Lemma 4.4 *There is a bijective correspondence between bounded sesquilinear forms $Q : H \times H \to \mathbb{C}$ on a Hilbert space H and bounded operators on H, as follows:*

- *Any $b \in B(H)$ defines a bounded sesquilinear form Q_b by $Q_b(v,w) = (v, bw)$, which satisfies the bound $|Q_b(v,w)| \leq C\|v\|\|w\|$ (indeed, take $C = \|b\|$).*
- *Conversely, for any sequilinear form Q that satisfies $|Q(v,w)| \leq C\|v\|\|w\|$ there is unique $b \in B(H)$ such that $Q = Q_b$, with $\|b\| \leq C$.*

It suffices to define Q on the diagonal in $H \times H$, that is, $Q(v,w)$ is determined by $Q(v,v) \equiv Q(v)$ through the polarization identity $Q(v,w) = \frac{1}{4}\sum_{k=0}^{3} i^k Q(w + i^k v)$.

This correspondence applies to (4.35) by taking the special case $\hat{\rho} = |\Psi\rangle\langle\Psi|$, so that $\rho(b) \equiv \psi(b) = (\Psi, b\Psi)$: if $Q(\Psi) = \int_{\sigma(a)} d\mu_\psi\, f$, then $Q = Q_{f(a)}$, at least for $f \in C(\sigma(a))$. The point now is that for $f \in \mathcal{B}(\sigma(a))$ the (Lebesgue) integral on the right-hand side of (4.35) remains well defined, yielding a sesquilinear form bounded by $C = \|f\|_\infty$. This, then, *defines* the operator $f(a)$ through $Q = Q_{f(a)}$.

To prove that $f(a) \in W^*(a)$, we use a much deeper lemma [28, Prop. 6.2.9]:[47]

Lemma 4.5 $\mathcal{B}(\sigma(a), \mathbb{R})$ *is the bounded monotone sequential completion of* $C(\sigma(a), \mathbb{R})$.

Now define $\mathcal{B}_! = \{f \in \mathcal{B}(\sigma(a), \mathbb{R}) \mid f(a) \in W^*(a)\}$, so $C(\sigma(a), \mathbb{R}) \subseteq \mathcal{B}_! \subseteq \mathcal{B}(\sigma(a), \mathbb{R})$. If (f_n) is some bounded monotone sequence in $\mathcal{B}_!$ with pointwise limit f, then—since the map $f \mapsto f(a)$ is a homomorphism and hence preserves positivity—$(f_n(a))$ is a bounded monotone sequence in $B(H)$, which *strongly* converges to $f(a)$ [28, Prop. 4.5.2]. Since $W^*(a)$ is closed under strong limits, it follows that $\mathcal{B}_!$ is monotone sequentially complete, so that by Lemma 4.5, $\mathcal{B}_! = \mathcal{B}(\sigma(a), \mathbb{R})$. Complexifying, we obtain $f(a) \in W^*(a)$ for all $f \in \mathcal{B}(\sigma(a))$. The rest of the proof is an exercise. ■

[47] I.e., $\mathcal{B}(\sigma(a), \mathbb{R})$ is the smallest space of bounded real functions on $\sigma(a)$ that contains $C(\sigma(a), \mathbb{R})$ and is closed under bounded pointwise limits of monotone (increasing or decreasing) sequences.

4.3 Projections in von Neumann algebras

By Theorem 4.3, the *spectral projection* $P(\Delta) \equiv \chi_\Delta(a)$ is defined for any Borel subset $\Delta \subset \sigma(a)$ (or even $\Delta \subset \mathbb{R}$, in which case $P(\Delta)$ is defined as $\chi_{\Delta\cap\sigma(a)}$). For any $f \in \mathcal{B}(\sigma(a))$, we are going to define an operator $\int dP\, f$, in such a way that

$$f(a) = \int dP\, f \equiv \int_{\sigma(a)} dP(\lambda)\, f(\lambda). \tag{4.36}$$

The special case $f = 1_{\sigma(a)}$ gives the **resolution of the identity** $1_H = \int_{\sigma(a)} dP(\lambda)$, whilst $f = \mathrm{id}_{\sigma(a)}$ yields the **spectral decomposition** of a, i.e., $a = \int_{\sigma(a)} dP(\lambda)\, \lambda$ (one could equally well integrate over $\mathbb{R}$ here). To define the integral, we need a result like Lemma 4.5, involving the set $\mathcal{E}(\sigma(a), \mathbb{R})$ of *elementary functions* on $\sigma(a)$, i.e., finite linear combinations $f = \sum_k c_k \chi_{B_k}$, with $c_k \in \mathbb{R}$ and $B_k \subset \sigma(a)$ Borel.

Lemma 4.6 *$\mathcal{B}(\sigma(a), \mathbb{R})$ is the bounded monotone sequential completion of $\mathcal{E}(\sigma(a), \mathbb{R})$. Moreover, any $f \in \mathcal{B}(\sigma(a), \mathbb{R}^+)$ is a pointwise limit of a bounded (from above) monotone increasing sequence f_n in $\mathcal{E}(\sigma(a), \mathbb{R}^+)$, denoted by $f_n \nearrow f$.*

See e.g. [11, Prop. 4.1.5]. So first assume $f \geq 0$, and find (f_n) as in this lemma. As in the proof of Lemma 4.5, the bounded monotonicity of the sequence (f_n) in $\mathcal{B}(\sigma(a), \mathbb{R})$ is inherited by the sequence $(f_n(a))$ in $B(H)$, which strongly converges to $f(a)$. Since the underlying principle will recur, we isolate it [28, Prop. 4.5.2]):

Lemma 4.7 *Any monotone increasing (decreasing) sequence (b_n) of self-adjoint operators on H that is bounded from above (below) converges strongly to a unique limit $b = b^* \in B(H)$. In that case we write $b_n \nearrow b$ ($b_n \searrow b$).*

So the applications of this Lemma we have had so far were all of the following nature: if $f_n \nearrow f$ in $\mathcal{B}(\sigma(a), \mathbb{R})$, then $f_n(a) \nearrow f(a)$ in $W^*(a) \subset B(H)$. We now use Lemma 4.7 once more to turn (4.36) into a *tautology* (!), as follows. Initially, assume $f \geq 0$.

1. Approximate f by a sequence $f_n \nearrow f$, as in Lemma 4.6.
2. For elementary functions $f_n = \sum_k c_k \chi_{B_k}$, define $\int dP\, f_n = \sum_k c_k P(B_k)$.
3. The ensuing sequence $b_n = \int dP\, f_n$ of self-adjoint operators is bounded monotone in $W^*(a)$ and hence has a strong limit b. By definition, $\int dP\, f = b$.

Finally, write a general (complex-valued) f as a sum of positive terms and define the integral by linearity. Thus (4.36) holds just by definition of the right-hand side.

Crucially, it follows that a von Neumann algebra is generated by its projections:[48]

Theorem 4.8 *Let $\mathcal{P}(M) = \{p \in M \mid p^2 = p^* = p\}$ be the* ***projection lattice*** *in a von Neumann algebra $M \subset B(H)$. Then $M = \mathcal{P}(M)''$.*

The inclusion $P(M)'' \subseteq M$ follows from $P(M) \subset M$ and $M'' = M$. Conversely, by Theorem 4.3, the spectral projections of $a^* = a \in M$ lie in $W^*(a) \subseteq M$, and hence in $\mathcal{P}(M)$. Let $\mathcal{E}(M)$ be the finite linear span of $\mathcal{P}(M)$; our proof of (4.36) yields a as a strong limit of some sequence in $\mathcal{E}(M)$, so that $a \in \mathcal{E}(M)'' = P(M)''$. ∎

[48] This is not true for general C^*-algebras! Just think of $A = C_0(\mathbb{R})$, with $\mathcal{P}(A) = \{0\}$.

4.4 von Neumann algebras as C^*-algebras

Since the strong and the weak topologies on $B(H)$ are weaker than the norm topology, it is clear from Theorem 4.1 that a von Neumann algebra is norm-closed and hence is a C^*-algebra. In the spirit of Definition 3.3, one may look for a characterization of von Neumann algebras as C^*-algebras with some extra property that is independent of the embedding $M \subset B(H)$. This problem was solved by Sakai [30].[49]

Theorem 4.9 *A C^*-algebra M is isomorphic to a von Neumann algebra iff it is the (Banach) dual of a Banach space M_* (called the **predual** of M).*

There is a canonical embedding $M_* \hookrightarrow M^*$, $\hat{\varphi} \mapsto \varphi$, with $\varphi(a) = a(\hat{\varphi})$. Elementary functional analysis [8, Thm. V.1.3] shows that (the image of) M_* consists precisely of the weak*-continuous functionals on M (where the weak*-topology on M is the topology of pointwise convergence, seeing M as the dual of M_*).

For example, in the commutative case $M = L^\infty(X, \mu)$, acting on $H = L^2(X, \mu)$ as multiplication operators, standard measure theory gives $M_* = L^1(X, \mu)$ under the pairing $\varphi(f) = \int_X d\mu\, f \hat{\varphi}$ [28, Thm. 6.5.11]. In the noncommutative case, Theorem 4.42 in Appendix 2 or [28, Thm. 3.4.13] yields $M_* = B_1(H)$ for $M = B(H)$ under the familiar identification $\rho(a) = \mathrm{Tr}\,(\hat{\rho}a)$, the trace-norm $\|\hat{\rho}\|_1$ on $B_1(H)$ coinciding with the norm $\|\rho\|$ in $B(H)^*$. This also shows that the σ-weak topology on $B(H)$ coincides with the weak*-topology, which is true for all von Neumann algebras:

Theorem 4.10 *Let $M \subset B(H)$ be a von Neumann algebra. The predual M_* of M (seen as a subspace of M^*) coincides with the space of σ-weakly continuous functionals on M (so that the σ-weak topology on M coincides with the weak*-topology).*

We just sketch the proof, leaving details to the exercises and Appendix 3. Let

$$\begin{aligned} M^\perp &:= \{\hat{\rho} \in B(H)_* \mid \rho(a) = 0\ \forall a \in M\}; \\ M^{\perp\perp} &:= \{a \in B(H) \mid \rho(a) = 0\ \forall \hat{\rho} \in M^\perp\}. \end{aligned}$$

Assuming the theorem for $M = B(H)$ (i.e. Theorem 4.42), the key is to show that

$$M^{\perp\perp} = M; \tag{4.37}$$

$$M_* \cong B(H)_*/M^\perp, \tag{4.38}$$

where (4.38) denotes an isometric isomorphism of normed spaces. Since the right-hand side of (4.38) is a Banach space, so is the left-hand side. This yields the first claim. Combining (4.38) with the duality $B(H) = B_1(H)^*$ and (4.37), we have

$$M_*^* \cong (B(H)_*/M^\perp)^* = M^{\perp\perp} = M.$$

This is the second claim. The first equality sign is true, because if Y is a closed subspace of a Banach space Y, then $(X/Y)^* = \{\varphi \in X^* \mid \varphi \restriction Y = 0\}$.

For the remainder of the theorem, note that $a_\lambda \to a$ σ-weakly in M whenever $\rho(a_\lambda - a) \to 0$ for all $\varphi \in B(H)_*$. By (4.38), this is equivalent to $a_\lambda \to a$ in the weak*-topology, since a possible component of φ in $M^\perp$ drops out. ∎

Corollary 4.11 *Each $\varphi \in M_*$ is of the form $\varphi(a) = \mathrm{Tr}\,(\hat{\rho}a)$, for some $\hat{\rho} \in B_1(H)$.*

[49] A proof of this theorem will be sketched Appendix 3.

Exercises for Lecture 5

1. (a) Let $S \subset B(H)$ be such that $a \in S$ iff $a^* \in S$. Show that S' is a von Neumann algebra.
   (b) Apply this to show that $U(G)'$ is a von Neumann algebra for any unitary group representation $U : G \to B(H)$ on a Hilbert space H.
   (c) Show that any von Neumann algebra arises in this way. In other words, given a von Neumann algebra $M \subset B(H)$, give a group G and a unitary representation $U : G \to B(H)$ such that $M = U(G)'$.

2. Derive the following useful reformulation of Theorem 4.1 from the latter:

   Let M be a unital *-subalgebra of $B(H)$. Then the closures of M in the strong, weak, and σ-weak topologies coincide with each other and with M''.

3. Let M be a unital *-algebra in $B(H)$, take a vector $v \in H$, and let p be the (orthogonal) projection onto the closure of $Mv \subset H$. Prove that $p \in M'$.

4. In the notation of the proof of Theorem 4.1, prove that $(M_n)'' = (M'')_n$.

5. In the proof of Theorem 4.1 we showed that if M is a unital *-subalgebra of $M_n(\mathbb{C})$, then $M'' = M$. Prove the following claim for infinite-dimensional H: *if M is a unital *-subalgebra of $B(H)$, then $M'' = M$ iff M is strongly closed.*

6. For possibly infinite-dimensional H, define a new Hilbert space H^∞ whose elements v are infinite sequences of vectors $(v_1, v_2, \ldots)$ in H with $\sum_i \|v_i\|^2 < \infty$; the inner product is given by $(\mathsf{v}, \mathsf{w})_{H^\infty} = \sum_i (v_i, w_i)_H$. (Note: $H^\infty = H \otimes \ell^2$.) There is an obvious (diagonal) embedding of $B(H)$ in $B(H^\infty)$, whose image is denoted by $B(H)_\infty$. Similarly, the image of $M \subset B(H)$ is denoted by M_∞.

   (a) Show that the σ-weak topology on $B(H)$ is the relative weak topology on $B(H)_\infty$ (i.e., the weak topology on $B(H^\infty)$ restricted to $B(H)_\infty$).
   (b) Define a new topology on $B(H)$, called the σ-**strong** topology, by restricting the strong topology on $B(H^\infty)$ to $B(H)_\infty$. Use the same trick as above (i.e., the passage of H to H^d) to prove the following version of the Double Commutant Theorem (and hence (1) $\Leftrightarrow$ (4) in Theorem 4.1):

   Theorem 4.12 *Let M be a unital *-algebra in $B(H)$. The following conditions are equivalent:*

   i. M is a von Neumann algebra;
   ii. M is closed in the σ-weak operator topology;
   iii. M is closed in the σ-strong operator topology.

7. Complete the proof of Theorem 4.3 by showing that the map $f \mapsto f(a)$, initially from $\mathcal{B}(\sigma(a))$ to $B(H)$, is a homomorphism of C^*-algebras.

8. Prove (4.37) and (4.38) (the answer is in Appendix 3).

4.5 Isomorphisms between von Neumann algebras

An isomorphism between C^*-algebras in the sense of *-algebras is automatically isometric; see Proposition 3.21. An even better result holds for von Neumann algebras:

Theorem 4.13 *An isomorphism $\varphi : M \to N$ between von Neumann algebras (as *-algebras) is an isomorphism of Banach spaces and a homeomorphism w.r.t. the σ-weak topologies on M and N (hence M_* and N_* are isomorphic as Banach spaces).*

The claim about the norm-topology follows from Proposition 3.21, for von Neumann algebras are C^*-algebras. Since φ is isometric, it induces a dual isomorphism (of Banach spaces) $\varphi^* : N^* \to M^*$, with the property that $M \cong (\varphi^*(N_*))^*$ under the map $a \mapsto (\varphi^*(\omega) \mapsto \omega(\varphi(a)))$, $a \in M$, $\omega \in N_*$. Uniqueness of the predual (cf. [33, Vol. I Cor. III.3.9]) then yields $\varphi^*(N_*) \cong M_*$, which in turn implies that φ preserves pointwise convergent nets: if $\omega'(a_\lambda) \to \omega'(a)$ for all $\omega' \in M_*$, then $\omega(\varphi(a_\lambda)) \to \omega(\varphi(a))$ for all $\omega \in N_*$. Hence φ is σ-weakly continuous. ■

A second instructive proof is based on the projection lattice; cf. Theorem 4.8.

Proposition 4.14 *The set $\mathcal{P}(M)$ of projections in a von Neumann algebra M is a complete lattice under the partial ordering $p \leqslant q$ iff $pq = qp = p$.*

Since $p \leqslant q$ in $M \subset B(H)$ iff $pH \subseteq qH$ (exercise 1), the supremum $p \vee q$ is the projection on $\overline{pH + qH}$, whilst the infimum $p \wedge q$ is the projection on $pH \cap qH$ (see exercise 2 for intrinsic expressions independent of H). For arbitrary families $(p_\lambda)_{\lambda\in\Lambda}$ of projections, $\vee_\lambda p_\lambda$ equals the projection on the closure of the linear span of all subspaces $H_\lambda \equiv p_\lambda H$, whereas $\wedge_\lambda p_\lambda \equiv p$ is the projection on their intersection. To show that the latter lies in M (provided all the p_λ do, of course), note that each unitary $u \in M'$ satisfies $uH_\lambda = H_\lambda$ for all λ, so that also $u(\cap_\lambda H_\lambda) = \cap_\lambda H_\lambda$. Hence $pu = up$ and so $p \in M'' = M$ (for each element of a von Neumann algebra is a linear combination of at most four unitaries in it). Finally, by de Morgan's Law we have $\vee_\lambda p_\lambda = (\wedge_\lambda p_\lambda^\perp)^\perp$, with $f^\perp = 1 - f$ for any $f \in \mathcal{P}(M)$. Hence also $\vee_\lambda p_\lambda \in M$. ■

Definition 4.15 *A map $\varphi : M \to N$ of von Neumann algebras is called* ***completely additive*** *if $\varphi(\vee_\lambda p_\lambda) = \vee_\lambda \varphi(p_\lambda)$ for any family (p_λ) in $\mathcal{P}(M)$.*

Theorem 4.13 then follows from two key properties of completely additive maps:

Proposition 4.16 *Let $\varphi : M \to N$ be a homomorphism of von Neumann algebras.*

1. *φ is σ-weakly continuous iff it is completely additive.*[50]
2. *If φ is an isomorphism, then it is completely additive.*

The proof of claim 2 is an exercise, as is the implication from σ-weak continuity to completely additivity in claim 1. The converse implication, however, is quite difficult; we refer to [10, §I.4.2], [33, Vol. I, Cor. III.3.11], [19, Lecture 11].

[50] Similarly, a functional $\rho : M \to \mathbb{C}$ is σ-weakly continuous iff it is completely additive.

4.6 Classification of abelian von Neumann algebras

Theorem 4.13 shows that the notion of isomorphism to be used in the classification of von Neumann algebras M is unambiguous. There are two totally different cases:[51]

- *Abelian* von Neumann algebras, which equal their center ($M \cap M' = M$);
- *Factors*, i.e., von Neumann algebras with trivial center ($M \cap M' = \mathbb{C} \cdot 1$).

Using the technique of *direct integrals*, the classification of general von Neumann algebras may be reduced to these cases [10, Part II], [33, Vol. I, Ch. IV, V].

As to the abelian case, we first sharpen Theorem 4.3 (cf. [8, §IX.8.10]).

Theorem 4.17 *Let $a^* = a \in B(H)$. There exists a measure μ on $\sigma(a)$ such that the isomorphism $C(\sigma(a)) \to C^*(a, 1)$ of C^*-algebras (cf. Theorem 3.18) has a unique extension to an isomorphism $L^\infty(\sigma(a), \mu) \to W^*(a)$ of von Neumann algebras.*[52]

Such a measure μ has the defining property that $\mu(\Delta) = 0$ iff $\mu_\psi(\Delta) = 0$ for all unit vectors $\Psi \in H$ (see §4.2 for notation), and hence is defined up to equivalence by its measure class (if H is separable, μ may even be taken to be a probability measure). We omit a complete proof,[53] but would like to clarify the key points (cf. [8, §IX.8]):

1. The Borel functional calculus $B_a : f \mapsto f(a)$, $\mathcal{B}(\sigma(a)) \to W^*(a)$, is surjective.
2. It is not injective; for real-valued f one has $f(a) = 0$ iff $(\Psi, f(a)\Psi) = 0$ for all unit vectors $\Psi \in H$, which by the last claim of Theorem 4.3 is the case iff $\int_{\sigma(a)} d\mu_\psi f = 0$ for all Ψ, which by construction is the case iff $\|f\|_\infty^{ess,\mu} = 0$.
3. Hence $W^*(a) \cong \mathcal{B}(\sigma(a))/\ker B_a \cong L^\infty(\sigma(a), \mu)$ as Banach spaces (via B_a).

Theorem 4.18 *Let $M \subset B(H)$ be an abelian von Neumann algebra (H separable). Then $M \cong L^\infty(X, \mu)$ for some compact space X and probability measure μ on X.*

This follows from the previous theorem by a result of von Neumann himself [23]:

Proposition 4.19 *Let $M \subset B(H)$ be an abelian von Neumann algebra (H separable). Then $M = W^*(a)$ for some $a = a^* \in B(H)$ (i.e., M is "singly generated").*

The proof is a (difficult) exercise. Finally, the most general result is as follows:[54]

Theorem 4.20 *Let $M \subset B(H)$ be an abelian von Neumann algebra. Then one has $M \cong L^\infty(X, \mu)$ for some locally compact space X and Borel measure μ on X.*

Conversely, $L^\infty(X, \mu)$ defines a von Neumann algebra of multiplication operators on $H = L^2(X, \mu)$, with operator norm equal to the norm $\| \cdot \|_\infty^{ess,\mu}$ [8, Thm. II.1.5], so that we have found a complete characterization of abelian von Neumann algebras!

---

[51] Of course, $M = \mathbb{C}$ falls in both classes, but is unique as such.

[52] Here $L^\infty(\sigma(a), \mu)$ denotes the space of (equivalence classes of) Borel functions $f : \sigma(a) \to \mathbb{C}$ for which f is bounded on $\sigma(a)\backslash\Delta_0$ for some Borel subset $\Delta_0 \subset \sigma(a)$ with $\mu(\Delta_0) = 0$. The canonical norm on $L^\infty(\sigma(a), \mu)$ is $\|f\|_\infty^{ess,\mu} = \inf\{\sup\{|f(x)|, x \in \sigma(a)\backslash\Delta\}\Delta \subset \sigma(a), | \mu(\Delta) = 0\}$.

[53] For the existence of μ see [8, §IX.8] for separable H, and [10, §I.7.2] in general.

[54] The proof is technical (cf. [10, Thm. I.7.3.1] or [33, Vol. I, §III.1]), but the idea is to find an abelian C^*-algebra A for which $M = A''$, upon which $X = \Sigma(A)$, and the measure μ is constructed such that $\mu(\Delta) = 0$ iff $\mu_\psi(\Delta) = 0$ for all unit vectors $\Psi \in H$, with μ_ψ defined similarly to (4.35). One cannot take $A = M$, since $\Sigma(M)$ may not support such measures, cf. [33, Vol. I, Thm. III.1.18].

4.7 Abelian von Neumann algebras and Boolean lattices

Theorem 4.20 is not as good as its counterpart Theorem 3.4 for C^*-algebras, since in the latter X is unique up to homeomorphism, whereas in the former the pair (X, μ) lacks intrinsic uniqueness properties.[55] Thus it also makes sense to apply Theorem 3.4 to abelian von Neumann algebras, so that $M \cong C(X)$. Since by Theorem 4.8, M has plenty of projections, which as elements of $C(X)$ are realized by characteristic functions χ_Δ ($\Delta \subset X$), the space X must have lots of clopen (i.e. closed and open) sets. It can be shown that X arises as the Gelfand spectrum of some abelian von Neumann algebra iff it is *hyperstonean*, which (besides compact Hausdorff) means:

- X is **Stone** if the only connected subsets of it are single points.[56]
- X is **Stonean** if it is Stone and the closure of every open set is (cl)open;
- X is **hyperstonean** if it is Stonean, and for any nonzero $f \in C(X, \mathbb{R}^+)$ there exists a completely additive positive measure μ such that $\mu(f) > 0$.

See [33, Vol. I, §III.1]. Now, the only other area of mathematics where such crazy spaces appear is *logic*. Indeed, recall that a *Boolean lattice* is an orthocomplemented distributive lattice,[57] and also recall *Stone's Theorem*, which states that a lattice L is Boolean iff it is isomorphic to the lattice of all clopen subsets of a Stone space X (where the partial ordering is given by set-theoretic inclusion, so that the lattice operations are $U \vee W = U \cup W$ and $U \wedge V = U \cap W$). The space $X \equiv \hat{\Sigma}(L)$ is called the **Stone spectrum** of L, and is determined by L up to homeomorphism [14].

Theorem 4.21 *The projection lattice $\mathcal{P}(M)$ of a von Neumann algebra M is Boolean iff M is abelian, in which case the Gelfand spectrum $\Sigma(M)$ of M (as a commutative C^*-algebra) is homeomorphic to the Stone spectrum $\hat{\Sigma}(\mathcal{P}(M))$ of $\mathcal{P}(M)$. Hence $M \cong C(\hat{\Sigma}(\mathcal{P}(M)))$, whilst $\mathcal{P}(M)$ is isomorphic to the lattice of clopens in $\Sigma(M)$.*

Towards a proof, recall that the Stone spectrum $\hat{\Sigma}(L)$ is the set of homomorphisms $\hat{\varphi} : L \to \{0, 1\}$ of Boolean lattices (where $\{0, 1\}$ is a lattice under $0 \leqslant 1$), topologized by saying that the basic opens in Σ_L are those of the form $U_x = \{\hat{\varphi} \in \hat{\Sigma}(L) \mid \hat{\varphi}(x) = 1\}$, for each $x \in L$. The homeomorphism $\Sigma(M) \cong \hat{\Sigma}(\mathcal{P}(M))$ then arises as follows:

- $\varphi \in \Sigma(M)$, $\varphi : M \to \mathbb{C}$, restricts to $\hat{\varphi} \in \hat{\Sigma}(\mathcal{P}(M))$, $\hat{\varphi} : \mathcal{P}(M) \to \{0, 1\}$.
- $\hat{\varphi} \in \hat{\Sigma}(\mathcal{P}(M))$ extends to a character $\varphi \in \Sigma(M)$ by the spectral theorem.

---

[55] Under special assumptions, at least some good models for (X, μ) obtain. To explain the two main examples, let us call a projection $p \in \mathcal{P}(M)$ *minimal* if $p \neq 0$ and there exists no $q \in \mathcal{P}(M)$ such that $0 < q < p$ (where $q < p$ iff $q \leqslant p$ and $q \neq p$). Then M is called *atomless* if it has no minimal projections, whereas it is said to be *atomic* if for any nonzero $p \in \mathcal{P}(M)$ there is a minimal projection $q \in \mathcal{P}(M)$ such that $q \leqslant p$. Under the assumption that H is separable, we then have $M \cong L^\infty(0, 1)$ (w.r.t. Lebesgue measure) iff M is atomless [33, Vol. I, Thm. III.1.22], whereas if M is atomic, it is isomorphic to either ℓ^∞ or to a finite direct sum of copies of $\mathbb{C}$ (exercise).

[56] Equivalently, a Stone space is compact, T_0, and has a basis of clopen sets.

[57] Recall that a lattice L is called *distributive* when $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and *orthocomplemented* when there exists a map $\perp : L \to L$ that satisfies $x^{\perp\perp} = x$, $y^\perp \leqslant x^\perp$ when $x \leqslant y$, $x \wedge x^\perp = 0$, and $x \vee x^\perp = 1$. For example, $\mathcal{P}(M)$ is orthocomplemented by $p^\perp = 1 - p$.

Exercises for Lecture 6

1. For any two projections p, q on H, show that $pq = qp = p$ iff $pH \subseteq qH$.
2. Show that for any two projections $p, q \in M \subset B(H)$ one has
$$p \wedge q = \lim_{n\to\infty} (pq)^n \overset{M\,abelian}{=} pq; \quad (4.39)$$
$$p \vee q = (p^\perp \wedge q^\perp)^\perp \overset{M\,abelian}{=} p + q - pq, \quad (4.40)$$
where the limit is taken in either the strong or the σ-weak topology on M, $p^\perp = 1 - p$, and the expressions on the right apply when M is abelian.
3. Prove that an isomorphism of von Neumann algebras is completely additive.
4. Prove that a σ-weakly continuous homomorhism of von Neumann algebras is completely additive. *Hint 1:* Use the following sharpening of Lemma 4.7:

   Lemma 4.22 *Any monotone increasing net* (a_μ) *of self-adjoint operators that is bounded from above strongly converges to its supremum* $\vee_\mu a_\mu$ *in* $B(H)$.

   See [16, Lemma 5.1.4]. *Hint 2:* Show that strong convergence in this lemma may be replaced by σ-weak convergence, as follows: show that strong convergence of a bounded monotone increasing net in $B(H)$ implies strong convergence of the corresponding net of diagonal operators in $B(H^\infty)$, which by definition establishes σ-strong *and hence σ-weak* convergence in $B(H)$.
5. Use Lemma 4.22 to give a new proof of Proposition 4.14, in which the existence of $\vee_\lambda p_\lambda$ is shown first (from which the existence of $\wedge_\lambda p_\lambda$ is derived). *Hint:* With $\lambda \in \Lambda$, use the net $\mathcal{P}_f(\Lambda)$ of finite subsets of Λ (ordered by inclusion).
6. Around Theorem 4.17, assume that H is separable with o.n.b. (e_n) and prove that the measure $\mu(\Delta) = \sum_{n=1}^\infty 2^{-n}(e_n, p(\Delta)e_n)$ is a probability measure on $\sigma(a)$, which satisfies $\mu(\Delta) = 0$ iff $\mu_\psi(\Delta) = 0$ for all unit vectors $\Psi \in H$.
7. Prove Proposition 4.19. *Hint:* give a self-contained version of von Neumann's own proof [23, Satz 10] (see `www.math.ru.nl/~landsman/Johnny1929.pdf`).[58]
8. Show that a Boolean lattice may equivalently be defined as a commutative ring in which $x^2 = x$ for all x (see [19, Lecture 14]).
9. Give an example illustrating that $\mathcal{P}(M)$ is not distributive (and hence not Boolean) whenever M is noncommutative.
10. Is the canonical example of a Stone space, viz. the Cantor set C,[59] Ston*ean*?

---

[58] This proof has been rephrased and streamlined by Takesaki [33, Vol. I, Prop. III.1.21]. See also Exercise f on p. 134 of [10], which we paraphrase as follows. Since H is separable, M is generated by a countable family (p_n) of projections. From this, construct a new family $(p')_{r\in\mathbb{Q}\cap[0,1]}$ of projections, such that if $r \leq r'$ then $p'_r \leqslant p'_{r'}$, $p'_0 = 0$, $p'_1 = 1$, and $s\text{-}\lim_{r\uparrow r'} p_r = p_{r'}$. Then construct a self-adjoint $a \in M$ such that for each $r \in \mathbb{Q}\cap[0,1]$, p'_r is the greatest spectral projection for a such that $p'_r a \leqslant r \cdot p'_r$. Finally, prove that a generates M, i.e., that $M = W^*(a)$.

[59] This is either $C = \prod_{\mathbb{N}} \underline{2}$ or, homeomorphically, $f(C) \subset [0,1]$ defined by $f(x) = \sum_{n=1}^\infty x_n 2^{-n}$.

4.8 The von Neumann algebra of a discrete group

We now turn to the classification of factors, following the original work of Murray and von Neumann [20]. First, we show that there is actually something worth classifying by displaying a class of nontrivial examples of factors. In contrast, 'trivial' examples of factors are those of the form $M = B(H)$, or, slightly more involved (whence the name 'factor'): $M = B(H_1)\otimes 1_{H_2} \subset B(H_1\otimes H_2)$ or its commutant $M' = 1_{H_1}\otimes B(H_2)$. The first is isomorphic to $B(H_1)$ by the map $a\otimes 1_{H_2} \mapsto a$, and similar for the second.

To get something new, define the **group algebra** $\mathbb{C}G$ of a discrete group G as the *-algebra of functions $f : G \to \mathbb{C}$ with finite support, with operations

$$f * g(x) = \sum_{y\in G} f(xy^{-1})g(y); \qquad f^*(x) = \overline{f(x^{-1})}. \tag{4.41}$$

Note that $\mathbb{C}G$ has a unit, namely δ_e (where $\delta_e(x) = \delta_{e,x}$). Next, represent $\mathbb{C}G$ on the Hilbert space $H = \ell^2(G)$ by $f \mapsto \pi_L(f) = \sum_x f(x)U_L(x)$, where the (left) regular representation U_L of G is defined by $U_L(x)\psi(y) = \psi(x^{-1}y)$. The **von Neumann algebra of** G is defined as $W^*(G) = \pi_L(\mathbb{C}G)''$, so that $W^*(G) \subset B(\ell^2(G))$.

To see what is going on here, consider the case where G is finite,[60] with an associated finite set $\hat{G}$ of (unitary) equivalence classes of irreducible representations.[61] Take some representative U_γ of each $\gamma \in \hat{G}$, realized on $H_\gamma \cong \mathbb{C}^{d_\gamma}$ (with $d_\gamma < \infty$), and define a Hilbert space $H_{\hat{G}} = \bigoplus_{\gamma\in\hat{G}} M_{d_\gamma}(\mathbb{C})$ consisting of matrix-valued functions $\hat{\psi}$ on $\hat{G}$, $\hat{\psi}(\gamma) \in M_{d_\gamma}(\mathbb{C})$, with inner product $(\hat{\psi}, \hat{\varphi}) = \sum_\gamma \mathrm{Tr}\,(\hat{\psi}(\gamma)^*\hat{\varphi}(\gamma))$. Then U_L and $W^*(G)$ are block-diagonalized by a unitary[62] transformation $u : \ell^2(G) \to H_{\hat{G}}$,

$$u\psi(\gamma) \equiv \hat{\psi}(\gamma) = \sqrt{d_\gamma}\sum_{x\in G} \psi(x)U_\gamma(x); \tag{4.42}$$

$$u^{-1}\hat{\psi}(x) \equiv \psi(x) = \frac{1}{|G|}\sum_{\gamma\in\hat{G}} \sqrt{d_\gamma}\mathrm{Tr}\,(U_\gamma(x)^*\hat{\psi}(\gamma)). \tag{4.43}$$

Indeed, writing $\hat{U}_L(x) \equiv uU_L(x)u^{-1}$, we easily find $\hat{U}_L(x)\hat{\psi}(\gamma) = U_\gamma(x)\hat{\psi}(\gamma)$, and similarly $\hat{\pi}_L(f)\psi(\gamma) = (1/\sqrt{d_\gamma})\hat{f}(\gamma)\psi(\gamma)$, where $\hat{\pi}_L(f) = u\pi_L(f)u^{-1}$. Using Schur's lemma (i.e., $\pi_\gamma(G)'' = M_{d_\gamma}(\mathbb{C})$), we obtain $uW^*(G)u^{-1} = H_{\hat{G}}$, seen as a *-algebra, acting on itself (now in its original guise as a Hilbert space) by $\hat{f}\hat{\psi}(\gamma) = \hat{f}(\gamma)\hat{\psi}(\gamma)$. Thus $W^*(G) \cong \bigoplus_{\gamma\in\hat{G}} M_{d_\gamma}(\mathbb{C})$ is isomorphic to a finite direct sum of matrix algebras, whose center $W^*(G)\cap W^*(G)'$ is isomorphic to $\mathbb{C}(\hat{G})$. In fact, $\tilde{f} : \hat{G} \to \mathbb{C}$ corresponds to $\hat{\psi}(\gamma) = \tilde{f}(\gamma)\cdot 1_{d_\gamma}$ in $H_{\hat{G}}$ and hence to $f(x) = (1/|G|)\sum_\gamma \sqrt{d_\gamma}\tilde{f}(\gamma)\overline{\chi_\gamma(x)}$ in $W^*(G)$. In particular, *f lies in the center of $W^*(G)$ iff it is a class function.* This also follows from (4.41), since $g * g = g * f$ for all g iff $f(yxy^{-1}) = f(x)$ for all $x, y \in G$.

---

[60] See, for example, [31] for the finite group theory used here (all based on the Peter–Weyl theory).

[61] Recall that the cardinality of $\hat{G}$ equals the number k_G of conjugacy classes in G. This is because the class functions on G by definition form a k_G-dimensional subspace $C(G)^c$ of $\ell^2(G)$, whilst the characters $x \mapsto \chi_\gamma(x) = \mathrm{Tr}\,U_\gamma(x)$, $\gamma \in \hat{G}$, form a basis of $C(G)^c$.

[62] Unitarity follows from the fact that the matrix elements $x \mapsto (e_i, U_\gamma(x)e_j)$, where (e_i) is an o.n.b. of H_γ, form a basis of $\ell^2(G)$, combined with Schur's orthogonality relations. This implies in turn that each irreducible representation U_γ of G, is contained in U_L with multiplicity d_γ.

4.9 Nontrivial examples of factors

As we have seen, for finite G the group von Neumann algebra $W^*(G)$ is not a factor (unless $G = \{e\}$). However, our computation of the center $W^*(G) \cap W^*(G)'$ suggests:

Theorem 4.23 *The group von Neumann algebra $W^*(G)$ of a countable group is a factor iff all nontrivial conjugace classes in G (i.e., all except $\{e\}$) are infinite.*

Since $\pi_L(\mathbb{C}G)$ consists of all *finite* sums $\sum_x f(x)U_L(x)$, it is likely that $W^*(G) = \pi_L(\mathbb{C}G)''$ incorporates all *strongly convergent* sums of this kind. Indeed, the map

$$\tau(f) = f(e) \tag{4.44}$$

from $\mathbb{C}G$ to $\mathbb{C}$ extends to $W^*(G)$ by (strong) continuity and has the property that for any $a \in W^*(G)$ one has $a = \sum_x f(x)U_L(x)$ with $f(x) = \tau(aU_L(x^{-1}))U_L(x)$, where the sum over G converges strongly (exercise). We identify such a with the corresponding function f, and compute $\tau(f^* * f) = \|f\|_2^2$, so that $W^*(G) \subset \ell^2(G)$. The computation at the very end of the previous section still holds, from which we infer that $f \in W^*(G) \cap W^*(G)'$ iff f is constant on each conjugacy class of G. If so, and if the condition in the theorem holds, then $f \in \ell^2(G)$ iff $f = \lambda\delta_e$, or $f \in \mathbb{C} \cdot 1$. Conversely, any f that is constant on some finite conjugacy class (different from $\{e\}$) and zero elsewehere is central without being a multiple of the unit. ∎

Do such "icc" groups actually exist? In fact, there are infinitely many of them: each free group on $n > 1$ generators is an example. Another example is the group S_∞ of finite permutations of $\mathbb{N}$.[63] In any case, we would now like to determine if such a factor $W^*(G)$ is "trivial" or not. The simples way to do so involves the *trace*.

Definition 4.24 *A **trace** on a von Neumann algebra M is a map $\tau : M_+ \to [0, \infty]$ satisfying $\tau(\lambda \cdot a + b) = \lambda \cdot \tau(a) + \tau(b)$ for all $a, b \in M_+$ and $\lambda \geq 0$, and $\tau(aa^*) = \tau(a^*a)$ for all $a \in M$ (equivalently $\tau(uau^*) = \tau(a)$ for all $a \in M_+$ and unitary $u \in M$).*

*A trace is **finite** if $\tau(a) < \infty$ for all $a \in M_+$, **semifinite** if for any $a \in M_+$ there is a nonzero $b \leq a$ in M_+ for which $\tau(b) < \infty$, and **infinite** otherwise.*

*A von Neumann algebra is called **(semi)finite** if it admits a faithful σ-weakly continuous (semi)finite trace,[64] and **purely infinite** otherwise.*

The usual trace Tr is a trace on $M = B(H)$ in this sense; $B(H)$ is finite iff $\dim(H) < \infty$ and semifinite otherwise. So is the map τ in (4.44), making $W^*G)$ finite.

To distinguish $W^*(G)$ from $B(H)$, we state without proof [33, Vol. I, Cor. v.2.32]:

Theorem 4.25 *Any two nonzero σ-weakly continuous (semi)finite traces τ, τ' on a (semi)finite factor are proportional, i.e., $\tau' = \lambda\tau$ for some $\lambda \in \mathbb{R}^+$.*

Since $W^*(G)$ is infinite-dimensional for icc G, $W^*(G) \not\cong M_n(\mathbb{C})$. For infinite-dimensional H, however, we have $\mathrm{Tr}\,(1) = \infty$ on $B(H)$, whereas $\tau(1) = 1$ on $W^*(G)$. Hence, as Tr and τ are σ-weakly continuous, $W^*(G) \not\cong B(H)$ for any H.

[63] A *j-cycle* is a cyclic permutation of j objects (called the *carrier* of the cycle in question). Any element p of $S_\infty = \cup_n S_n$ is a finite product of j-cycles with disjoint carriers, and for each $j \in \mathbb{N}$ the number of j-cycles in such a decomposition of p is uniquely determined by p. Two permutations in S_∞, then, are conjugate iff they have the same number of j-cycles, for all $j \in \mathbb{N}$ (cf., e.g., [31]).

[64] A finite trace on a factor is automatically σ-weakly continuous [33, Vol. I, Prop. v.2.5].

4.10 Equivalence of projections

Murray and von Neumann actually started with the projection lattice $P(M)$. In Proposition 4.14 we have already introduced a *partial* ordering $\leqslant$ on $P(M)$, but another ordering exists, which turns out to be *total* whenever M is a factor.

Definition 4.26 *Let $P(M)$ be the projection lattice of a von Neumann algebra M. We say that $p \sim q$ in $P(M)$ when there exists $u \in M$ such that $u^*u = p$ and $uu^* = q$, and that $p \lesssim q$ if there is $p' \in P(M)$ with $p \sim p'$ and $p' \leqslant q$.*

It is an exercise to show that $\sim$ is an equivalence relation. Through its stated properties, the operator u in this definition is unitary from pH to qH, vanishes on $(pH)^{\perp}$, and has range qH. Such an operator is called a **partial isometry**, with **initial projection** p and **final projection** q. It follows that a *necessary* condition for $p \sim q$ is that $\dim(pH) = \dim(qH)$, but (unless $M = B(H)$) this is by no means *sufficient*, since the unitary mapping pH to qH is required to lie in M. For example, if $H = \mathbb{C} \oplus \mathbb{C}$ then $p = \mathrm{diag}(1, 0)$ is equivalent to $q = \mathrm{diag}(0, 1)$ with respect to $M = M_2(\mathbb{C})$, but not with respect to $M = \mathbb{C} \oplus \mathbb{C}$ (i.e., the diagonal 2×2 matrices).

To see how natural this definition is, consider a unitary representation U of a group G on H. If $H_i \subset H$ is stable under $U(G)$, $i = 1, 2$, then the restrictions U_i of U to H_i are unitarily equivalent precisely when $[H_1] \sim [H_2]$ with respect to $M = U(G)'$ (where $[H_i]$ is the projection onto H). Furthermore, U_1 is unitarily equivalent to a subrepresentation of U_2 iff $[H_1] \lesssim [H_2]$. More generally, if $N \subset B(H)$ is a von Neumann algebra, with stable subspaces H_i, $i = 1, 2$, then the restrictions N_i to H_i are unitarily eqiuvalent iff $[H_1] \sim [H_2]$ with respect to $M = N'$, et cetera.

One may compare projections in M with sets and compare $\leqslant$, $\sim$, and $\lesssim$ with $\subseteq$ (inclusion), $\cong$ (isomorphism), and $\hookrightarrow$ (the existence of an injective map), respectively. The Schröder–Bernstein Theorem of set theory states that if $X \hookrightarrow Y$ and $Y \hookrightarrow X$, then $X \cong Y$. Similarly (with the proof as an exercise):

Proposition 4.27 *If $p \lesssim q$ and $q \lesssim p$, then $p \sim q$.*

The special role of factors with respect to the partial ordering $\lesssim$ now emerges.

Proposition 4.28 *If M is a factor, then $\lesssim$ is a total ordering (i.e., $p \lesssim q$ or $q \lesssim p$).*

The property of a factor that leads to this result (with the proof as an exercise) is:

Lemma 4.29 *Let M be a factor. For any nonzero projections $p, q \in P(M)$ there are nonzero projections $p', q' \in P(M)$ such that $p' \leqslant p$, $q' \leqslant q$, and $p' \sim q'$.*

In other words, any pair p, q of nonzero projections in a factor has a pair of equivalent subprojections. Accordingly, from Zorn's Lemma (brr ...) there exist maximal orthogonal families (p_i) and (q_j) with $p_i \leqslant p$, $q_j \leqslant q$, and $p_i \sim q_j$, for all i, j. Defining $p_0 = \vee_i p_i = \sum_i p_i$ and $q_0 = \vee_j q_j = \sum_j q_j$, one has $p_0 \leqslant p$, $q_0 \leqslant q$, and $p_0 \sim q_0$ (exercise). But then, by maximality, $p - p_0$ and $q - q_0$ have no equivalent subprojections. To avoid a contradiction with Lemma 4.29, at least one of these must vanish. If $p - p_0 = 0$, then $p \sim q_0 \leqslant q$, so $p \lesssim q$. If $q - q_0 = 0$, one similarly has $q \lesssim p$ (both may be zero, in which case $p \sim q$ by Proposition 4.27). ■

4.11 The Murray–von Neumann classification of factors [20]

Definition 4.30 *A projection p in M is called* ***finite*** *if $q \sim p$ and $q \leqslant p$ for some $q \in P(M)$ implies $q = p$, and* ***minimal*** *if $q \leq p$, $q \in P(M)$, implies $q = p$ or $q = 0$.*

As an alternative to Definition 4.24, we then have [33, Vol. I, Thms. V.2.4, V.2.15]:

Proposition 4.31 *A factor M is finite iff 1 is finite, semifinite iff 1 majorizes a finite projection, and purely infinite iff all projections are infinite.*

For $M = B(H)$, we have p (in)finite iff $\dim(pH) = \mathrm{Tr}\,(p)$ is (in)finite. However, if G is icc, all projections $p \in W^*(G)$ are finite, despite $\mathrm{Tr}\,(p) = \infty$. All minimal projections in $B(H)$ are one-dimensional, whereas $W^*(G)$ has no minimal projections.

Definition 4.32 *A factor M is said to be of type:*

- I *if it has at least one minimal projection, subdivided into:*
  - *type* I_n *($n \in \mathbb{N}$) if M is finite and 1 is the sum of n minimal projections;*
  - *type* I_∞ *if M is type* I *and semifinite but not finite;*
- II *if it has no minimal projections but some nonzero finite projection, with:*
  - *type* II_1 *if M is type* II *and finite;*
  - *type* II_∞ *if M is type* II *and semifinite but not finite;*
- III *if all nonzero projections are infinite.*

A nice understanding of these types arises from a construction similar to the trace.

Definition 4.33 *A* ***dimension function*** *on a von Neumann algebra M is a function $d : P(M) \to [0, \infty]$ such that $d(p) < \infty$ iff p is finite, $d(p + q) = d(p) + d(q)$ if $pq = 0$ (i.e., $pH \perp qH$), and $d(p) = d(q)$ if $p \sim q$.*

Paraphrasing results in Murray and von Neumann's papers [20, 25], we now have:

Theorem 4.34 *For any von Neumann algebra M, the restriction of a trace to $P(M)$ is a dimension function. If $M \subset B(H)$ is a factor, with H separable:*

1. *Any σ-weakly continuous trace on M restricts to a completely additive dimension function with the additional property that $p \sim q$ if $d(p) = d(q)$.*
2. *Any dimension function with this additional property arises from a σ-weakly continuous trace, and hence is completely additive and unique up to scaling.*
3. *In that case, the dimension function d induces an isomorphism between $P(M)/\sim$ and some subset of $[0, \infty]$. Suitably normalizing d, this subset must be one of:*
   - $\{0, 1, 2, \ldots, n\}$ *for some $n \in \mathbb{N}$, in which case M is type* I_n*;*
   - $\mathbb{N} \cup \infty$ *(type* I_∞*);*
   - $[0, 1]$ *(type* II_1*);*
   - $[0, \infty]$ *(type* II_∞*);*
   - $\{0, \infty\}$ *(type* III*).*

Exercises for Lecture 7

1. This exercise sharpens the **polar decomposition** [8, VIII.3.11]:

   Lemma 4.35 *For any $a \in B(H)$ there exists a unique partial isometry w such that $a = w|a| = |a^*|w$, where $|a| = \sqrt{a^*a}$ (defined by the continuous functional calculus), the initial projection w^*w is $[(\ker a)^\perp] = [\overline{\mathrm{ran}(a^*)}]$, and the final projection ww^* is $[\overline{\mathrm{ran}(a)}]$.*

   Prove that if $a \in M \subset B(H)$, where M is a von Neumann algebra, then $w \in M$ and $|a| \in M$ (answer: [33, Vol. I, Prop. II.3.14]).

   In all exercises below, M is a factor on a separable Hilbert space.

2. Prove Lemma 4.29. *Hint* (assuming $p, q \in P(M)$ nonzero):
   (a) First show that $pMq \neq 0$.
   (b) Then show that the polar decomposition of some nonzero $a \in pMq$ gives an equivalence between nonzero subprojections of p and q.

3. Prove the following statements (relative to M):
   (a) If p is finite and $q \sim p$, then q is finite.
   (b) If p is finite and $q \precsim p$, then q is finite.
   (c) If $P(M)$ contains some infinite (i.e., not finite) p, then $\dim(H) = \infty$.
   (d) A minimal projection is finite.

4. (a) Show that for all $p, q \in P(M)$, $q \neq 0$, there exists an index set I and an orthogonal family $(p_i)_{i\in I}$ in $P(M)$ with $p_i \sim q$ for all i, as well as some $r \in P(M)$ with $r \precsim q$, and $r \neq q$, such that $p = \sum_{i\in I} p_i + r$.
   (b) Show that if p is finite, then $|I| < \infty$, and the cardinality $|I|$ of I is independent of the choices of the p_i and r.
   (c) Show that if q is minimal, then $r = 0$.

5. Show that if p and q are infinite, then $p \sim q$. *Hint*: use the previous exercise.

6. Let $M = B(H)$. Show from Theorem 4.34 that M is type $\mathrm{I}_{\dim(H)}$.
   Hint: prove that $d(p) := \dim(pH)$ defines a dimension function on M.

7. Let M be a type I factor with some minimal projection q. Define $d(p) = |I|$, where I is defined as in exercise 4.
   (a) Show that d is a dimension function.
   (b) Show that the range $d(P(M))$ of this function d is of the form listed in Theorem 4.34 (part 3) either as type I_n, or as type I_∞.

8. Let G be icc. Show that $W^*(G)$ is type II_1.

4.12 Classification of hyperfinite factors

Murray and von Neumann completely classified two sorts of factors. We say that a von Neumann algebra M is **hyperfinite** if $M = (\cup_n M_n)''$, where each von Neumann subalgebra $M_n \subset M$ is finite-dimensional and $M_n \subset M_{n+1}$. For example, if H is separable, then $M = B(H)$ is hyperfinite. Also, if some countable group G is the union of an increasing sequence of finite subgroups G_n, i.e., $G = \cup_n G_n$ with $G_n \subset G_{n+1}$, then the associated von Neumann algebra $W^*(G)$ is hyperfinite. This applies, for example, to the (icc) group $S_\infty = \cup_n S_n$ of finite permutations of $\mathbb{N}$.

Theorem 4.36 *Let $M \subset B(H)$ be a factor on a separable Hilbert space.*

- *If M is type* I*, then $M \cong B(H)$, for some Hilbert space H.*
- *If M is type* II_1 *and hyperfinite, then $M \cong W^*(S_\infty)$, henceforth called R.*

The proof of the first claim is a nontrivial exercise. The second is much more difficult [21]; it follows that $W^*(G) \cong R$ for any finitely generated icc group G. An example of a II_∞ factor is also quickly found, namely $M = N \otimes B(\ell^2)$, where N is II_1. In fact, any II_∞ factor on a separable Hilbert space is of this sort (exercise), but if M is hyperfinite, it is *a priori* unclear if N is, too (see below). Hence Murray and von Neumann were unable to classifiy even hyperfinite II_∞ factors. About type III they knew almost nothing, except for a couple of examples from ergodic theory [25].

Between 1971–1975, Alain Connes made two decisive contributions [3, 4, 5]:

1. Dividing type III factors into III_λ, $\lambda \in [0, 1]$, by means of a new invariant;
2. Completely classifying *hyperfinite* type II_∞ and type III factors, as follows:
   - There is a unique hyperfinite II_∞ factor, namely $R \otimes B(\ell^2)$;
   - There is a unique hyperfinite III_1 factor;
   - There is a unique hyperfinite III_λ factor for each $\lambda \in (0, 1)$;
   - There is an infinite family of hyperfinite III_0 factors, completely classified by the so-called *flow of weights* of Connes and Takesaki [33, Vol. II].

We list III_1 separately from III_λ for $\lambda \in (0, 1)$ for two reasons: first, "hyperfinite III_1" turns out to be *the* factor occurring in quantum field theory and quantum statistical mechanics of infinite systems [13] (while III_λ for $\lambda \in (0, 1)$ seems artificial), and second, the proof of uniqueness of the hyperfinite III_1 factor is much more difficult.[65]

Apart from the Tomita–Takesaki theory (cf. §4.13), an important technical tool of Connes was his own profound discovery that a von Neumann algebra $M \subset B(H)$, H separable, is hyperfinite iff it is **injective** in that there exists a σ-weakly continuous **conditional expectation** $E : B(H) \to M$, that is, a linear map $E : B(H) \to B(H)$ such that $E(a) \in M$ and $E(a^*) = E(a)^*$ for all $a \in B(H)$, $E^2 = E$, and $\|E\| = 1$.[66]

[65] There is an entire book about this proof [35]. In his review **MR1030046 (91a:46059)** of this book for *Mathematical Reviews* in 1991, E. Størmer wrote: 'At the time of writing this review, by far the deepest and most difficult proof in von Neumann algebra theory is the one of Connes and Haagerup on the uniqueness of the injective factor of type III_1 with separable predual.'

[66] It follows that $E(abc) = aE(b)c$ for all $a, c \in M$, $b \in B(H)$. The equivalence of hyperfiniteness and injectivity implies, for example, that if $M = N \otimes B(\ell^2)$ is hyperfinite, then so is N.

4.13 Tomita–Takesaki Theory

At a conference 1967, the Japanese mathematician Tomita and the German-Dutch mathematical physics trio Haag–Hugenholtz–Winnink (HHW) independently distributed two preprints that 'would later completely change the scope of operator algebra' [32, p. 235]. Tomita's original work was not quite correct, but after elaboration by Takesaki it formed the basis of both Connes's subsequent work on the classification of type III factors and, through HHW, of a powerful mathematically rigorous approach to the quantum statistical mechanics of infinite systems [13, 18].

Definition 4.37 *A von Neumann algebra $M \subset B(H)$ is in* ***standard form*** *if H contains a unit vector Ω that is cyclic and separating for M.*

Here we say that Ω is *separating* for M if $a\Omega \neq 0$ for all nonzero $a \in M$ (being cyclic for M, Ω is also separating for M'). The point is that *any von Neumann algebra can be brought into standard form* [33, Vol. II, §IX.1]. For separable H, this follows by picking an injective density operator $\hat{\rho}$ on H, whose associated state $\rho : a \mapsto \mathrm{Tr}\,(\hat{\rho}a)$ is faithful in that $\rho(a^*a) > 0$ for all nonzero $a \in M$, and passing to $\pi_\rho(M) \cong M$.

For example, $M = B(H)$ acting on H is not in standard form, but acting on $B_2(H)$ by *left* multiplication it is, where $B_2(H)$ is the Hilbert space of Hilbert–Schmidt operators on H with inner product $(a, b) = \mathrm{Tr}\,(a^*b)$. If $\hat{\rho} \in B_1(H)$ is an injective density operator on H, then $\Omega = \sqrt{\hat{\rho}} \in B_2(H)$ brings M in standard form. In this case, $M' \cong B(H)^{op}$ (where the suffix "*op*" means that multiplication is done in the opposite order, i.e. ab in $B(H)^{op}$ is equal to ba in $B(H)$), which acts on $B_2(H)$ by *right* multiplication. If $H = \mathbb{C}^n$, one simply has $B(H) = B_2(H) = M_n(\mathbb{C})$.

Let $M \subset B(H)$ be in standard form. Tomita introduced the (unbounded) *antilinear* operator S with initial domain $D(S_0) = M\Omega$ and action $S_0(a\Omega) = a^*\Omega$; this domain is dense because Ω is cyclic for M, and the action is well defined since Ω is separating for M. This operator turns out to be closable, with closure S. Any closed operator a has a polar decomposition $a = v|a|$, where v is a partial isometry and $|a| = \sqrt{a^*a}$. For S, we write $S = J\Delta^{1/2}$, where J is an *antilinear* partial isometry and $\Delta = S^*S$. Since S is injective with dense range, J is actually anti-unitary, satisfying $J^* = J$ and $J^2 = 1$. Furthermore, $\Delta \geq 0$, so that Δ^{it} is well defined for $t \in \mathbb{R}$: writing $\Delta = \exp(h)$ for the self-adjoint operator $h = \log\Delta$, we have $\Delta^{it} = \exp(ith)$.

Theorem 4.38 (Tomita and Takesaki) *Let $M \subset B(H)$ be a von Neumann algebra in standard form, let S be the closure of the antilinear operator S_0 defined by $S_0(a\Omega) = a^*\Omega$, and let $S = J\Delta^{1/2}$ be the polar decomposition of S. Then:*

- $M' = JMJ \equiv \{JaJ \mid a \in M\}$.
- *For each $t \in \mathbb{R}$ and $a \in M$, the operator $\alpha_t(a) := \Delta^{it}a\Delta^{-it}$ lies in M.*
- *The map $t \mapsto \alpha_t$ is a group homomorphism from $\mathbb{R}$ to $\mathrm{Aut}(M)$ (i.e., the group of all automorphisms of M), which is continuous in that for each $a \in M$ the function $t \mapsto \alpha_t(a)$ from $\mathbb{R}$ to M is continuous w.r.t. the σ-weak topology.*

This theorem was originally stated and proved in the language of Hilbert algebras [33, Vol. II]; for a relatively short proof that avoids this formalism, see [1, §2.5].

The exercises illustrate this deep result in the context of some simple examples.

4.14 Connes's "cocycle Radon–Nikodym Theorem"

The image of $\mathbb{R}$ in Aut(M) by α is called the **modular group** of M associated with the cyclic and separating vector Ω (or rather, with the corresponding σ-weakly continuous faithful state ω). Simple examples (as in the exercises) show that the modular group explicitly depends on the vector Ω. In his thesis [3], Connes analyzed the dependence of α on Ω. To state the simplest version of his result (to which the case involving two different Hilbert spaces can be easily reduced), assume that H contains two different vectors Ω_1 and Ω_2, each of which is cyclic and separating for M. We write $\alpha_t^{(i)}$ for the time-evolution derived from Ω_i, $i = 1, 2$.

Theorem 4.39 *There is a family U_t of unitary operators in M, $t \in \mathbb{R}$, such that*

$$\alpha_t^{(1)}(a) = U_t \alpha_t^{(2)}(a) U_t^*; \tag{4.45}$$

$$U_{t+s} = U_s \alpha_s^{(2)}(U_t). \tag{4.46}$$

The proof of this theorem (Connes's favourite [7]) is based on the following idea. Extend M to Mat$_2(M)$, i.e., the von Neumann algebra of 2×2 matrices with entries in M, and let Mat$_2(M)$ act on $H_2 = H \oplus H$ in the obvious way. Subsequently, let Mat$_2(M)$ act on $H_4 = H\oplus H\oplus H\oplus H = H_2 \oplus H_2$ by simply doubling the action on H_2. The vector $\Omega = (\Omega_1, 0, 0, \Omega_2) \in H_4$ is then cyclic and separating for Mat$_2(M)$, with corresponding $\Delta = \mathrm{diag}(\Delta_1, \Delta_4, \Delta_3, \Delta_2)$ (exercise). Here Δ_1 and Δ_2 are just the operators on H originally defined by Ω_1 and Ω_2, respectively, and Δ_3 and Δ_4 are certain operators on H. Denoting elements of Mat$_2(M)$ by $\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$,

$$\Delta^{it} \begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{a} \end{pmatrix} \Delta^{-it} = \begin{pmatrix} \tilde{\alpha}_t^{(1)}(\mathbf{a}) & \mathbf{0} \\ \mathbf{0} & \tilde{\alpha}_t^{(2)}(\mathbf{a}) \end{pmatrix}, \tag{4.47}$$

$$\tilde{\alpha}_t^{(1)}(\mathbf{a}) := \begin{pmatrix} \Delta_1^{it} a_{11} \Delta_1^{-it} & \Delta_1^{it} a_{12} \Delta_4^{-it} \\ \Delta_4^{it} a_{21} \Delta_1^{-it} & \Delta_4^{it}\ a_{22} \Delta_4^{-it} \end{pmatrix}; \tag{4.48}$$

$$\tilde{\alpha}_t^{(2)}(\mathbf{a}) := \begin{pmatrix} \Delta_3^{it} a_{11} \Delta_3^{-it} & \Delta_3^{it} a_{12} \Delta_2^{-it} \\ \Delta_2^{it} a_{21} \Delta_3^{-it} & \Delta_2^{it}\ a_{22} \Delta_2^{-it} \end{pmatrix}. \tag{4.49}$$

But by Theorem 4.38, the right-hand side of (4.47) must be of the form diag$(\mathbf{b}, \mathbf{b})$ for some $\mathbf{b} \in$ Mat$_2(M)$, so that $\tilde{\alpha}_t^{(1)}(\mathbf{a}) = \tilde{\alpha}_t^{(2)}(\mathbf{a})$. This allows us to replace $\Delta_4^{it}\ a_{22}\Delta_4^{-it}$ in (4.48) by $\Delta_2^{it}\ a_{22}\Delta_2^{-it}$. We then put $U_t = \Delta_1^{it}\Delta_4^{-it}$, which, unlike either Δ_1^{it} or Δ_4^{-it}, lies in M, because each entry in $\tilde{\alpha}_t^{(1)}(\mathbf{a})$ must lie in M if all the a_{ij} do, and here we have taken $a_{12} = 1$. All claims of the theorem may then be verified using elementary computations with 2×2 matrices. For example, combining

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with the property $\tilde{\alpha}_t^{(1)}(\mathbf{ab}) = \tilde{\alpha}_t^{(1)}(\mathbf{a})\tilde{\alpha}_t^{(1)}(\mathbf{b})$, we recover (4.45). Using the identity

$$\begin{pmatrix} 0 & U_t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & U_t, \end{pmatrix}$$

and evolving each side to time s, we arrive at (4.46). A 'Proof from the Book'!

4.15 Connes's parametrization of type III factors

Theorem 4.39 shows that the modular group of M is independent of the vector Ω used to define it, "up to inner transformations". To make this precise, we say that an automorphism $\gamma : M \to M$ is **inner** if there exists a unitary element $u \in M$ such that $\gamma(a) = uau^*$ for all $a \in M$. The inner automorphisms of M form a normal subgroup $\mathrm{Inn}(M)$ of the group $\mathrm{Aut}(M)$ of all automorphisms, with quotient $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$. Hence the image $\pi(\alpha(\mathbb{R}))$ of the modular group in $\mathrm{Out}(M)$ under the canonical projection $\pi : \mathrm{Aut}(M) \to \mathrm{Out}(M)$ is independent of Ω, and invariants of this image will be invariants of M itself. Such invariants are trivial if M is a factor of type I or II, since in that case $\pi(\alpha(\mathbb{R})) = \{e\}$; to see this in the finite case (i.e., type I_n or type II_1), take a finite trace τ on M and check that $\Delta = 1$ for $\pi_\tau(M) \cong M$. For the semifinite but not finite case (i.e., type I_∞ or type II_∞), a slight generalization of the GNS-construction leads to the same conclusion.

To find invariants for type III factors, we therefore need to extract information from the modular group $t \mapsto \alpha_t$ "up to inner automorphisms". Recall Def. 2.15.

Definition 4.40 *Let $\alpha : \mathbb{R} \to \mathrm{Aut}(M)$ be an action of $\mathbb{R}$ on a von Neumann algebra.*

- *The **Arveson spectrum** $\mathrm{sp}(\alpha)$ of α consists of all $k \in \mathbb{R}$ for which there is a sequence (x_n) in M with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|\alpha_t(x_n) - e^{ikt}x_n\| = 0\, \forall t \in \mathbb{R}$.*
- *Let $M^\alpha = \{x \in M \mid \alpha_t(x) = x\, \forall t \in \mathbb{R}\}$. If $e \in P(M^\alpha)$ and $M_e := \{x \in M \mid xe = ex = x\}$, then $\alpha_t : M \to M$ restricts to $\alpha_t^e : M_e \to M_e$, defining a (group) homomorphism $\alpha^e : \mathbb{R} \to \mathrm{Aut}(M_e)$, $t \mapsto \alpha_t^e$. The **Connes spectrum** of α is $\Gamma'(\alpha) := \bigcap_{0\neq e\in P(M^\alpha)} \mathrm{sp}(\alpha^e) \subset \mathbb{R}$, or, equivalently, $\Gamma(\alpha) := \exp(\Gamma'(\alpha)) \subset \mathbb{R}_*^+$.*

The Connes spectrum $\Gamma(\alpha)$ is a closed subgroup of $\mathbb{R}_*^+$, which has the great virtue that if $\pi(\alpha(\mathbb{R})) = \pi(\alpha'(\mathbb{R}))$, then $\Gamma(\alpha) = \Gamma(\alpha')$. So if α is the modular group of M with respect to some state ω, then $\Gamma(\alpha)$ is independent of ω and may therefore be called $\Gamma(M)$. This invariant can also be defined through the usual spectrum of self-adjoint operators on Hilbert space. To this effect, Connes defined and proved

$$S(M) := \bigcap_{\omega} \sigma(\Delta_\omega) = \bigcap_{0\neq e\in P(M^\alpha)} \sigma(\Delta_{\varphi_e}), \tag{4.50}$$

where the first intersection is over all σ-weakly continuous faithful states ω on M, whereas in the second one takes *a fixed* σ-weakly continuous faithful state φ on M, and restricts it to $\varphi_e := \varphi_{|M_e}$. Furthermore, Δ_ω denotes the operator Δ on H_ω, defined w.r.t. the usual cyclic unit vector Ω_ω of the GNS-construction, etc. If M is a type I or II factor (on a separable Hilbert space) one has $S(M) = \{1\}$, whereas $0 \in S(M)$ iff M is type III. Connes showed that $\Gamma(M) = S(M) \cap \mathbb{R}_*^+$, and the known classification of closed subgroups of $\mathbb{R}_*^+$ yields his parametrization of type III factors:

Definition 4.41 *Let M be a type* III *factor. Then M is said to be of type:*

- III_0 *if $\Gamma(M) = \{1\}$;*
- III_λ, *where $\lambda \in (0,1)$, if $\Gamma(M) = \lambda^{\mathbb{Z}}$;*
- III_1 *if $\Gamma(M) = \mathbb{R}_*^+$.*

Bonus exercises for Lecture 8.[67]

1. Let $M \subset B(H)$ be a von Neumann algebra. Show that $\Omega \in H$ is cyclic for M' iff it is separating for M (equivalently, Ω is cyclic for M iff it is separating for M'). Hence M is in standard form iff Ω is cyclic for both M and M'.

2. Let the von Neumann algebra $M = M_n(\mathbb{C})$ act on the Hilbert space $H = M_n(\mathbb{C})$ (with inner product $(a, b) = \mathrm{Tr}\,(a^*b)$) by left multiplication.

   (a) Take $\Omega = 1_n/\sqrt{n}$ as a unit vector in H. Show that Ω is cyclic and separating for M, that $Sa = a^*$, $J = S$, and $\Delta = 1$. Conclude that $\alpha_t(a) = a$ for all $t \in \mathbb{R}$, $a \in M$ (so that $\alpha : \mathbb{R} \to \mathrm{Aut}(M)$ is trivial).

   (b) Now take an arbitrary injective density matrix $\hat{\rho}$, and define $\Omega = \hat{\rho}^{1/2}$. In other words (diagonalizing $\hat{\rho}$), there is an o.n.b. (e_i) of $\mathbb{C}^n$ and there are numbers $p_i \in (0, 1)$ with $\sum_i p_i = 1$ so that $\Omega = \sum_{i=1}^n \sqrt{p_i}|e_i\rangle\langle e_i|$ as a unit vector in H (the choice $p_i = 1/n$ for all i gives the previous case).

   Show that Ω is cyclic and separating for M, that $S\Psi = \hat{\rho}^{-1/2}\Psi^*\hat{\rho}^{1/2}$, $J\Psi = \Psi^*$, and $\Delta\Psi = \hat{\rho}\Psi\hat{\rho}^{-1}$, where $\Psi \in H$. For $a \in M$, check that $Ja^*J\Psi = \Psi a$, so that, given $\hat{\rho}^* = \hat{\rho}$, we have $\Delta = \hat{\rho}J\hat{\rho}^{-1}J$.

   As in physics [18], write $\hat{\rho} = \exp(h)$ for some self-adjoint $h \in M$, and write $\tilde{a}$ for Ja^*J, so that $\tilde{a} \in M'$ if $a \in M$. Hence $\Delta = \exp(h - \tilde{h}) = \exp(h)\exp(-\tilde{h})$, since h and $\tilde{h}$ commute. Conclude that $\alpha_t(a) = e^{ith}ae^{-ith}$ for all $t \in \mathbb{R}$, $a \in M$ (so that $\alpha : \mathbb{R} \to \mathrm{Aut}(M)$ is nontrivial, but inner).

3. Suppose a von Neumann algebra M admits a faithful finite trace τ. Show that $S = J$ (and hence $\Delta = 1$) for $\pi_\tau(M) \subset B(H_\tau)$ with respect to $\Omega = \Omega_\tau$.

4. In the proof of Theorem 4.39, check that the operator Δ takes the diagonal form $\Delta = \mathrm{diag}(\Delta_1, \Delta_4, \Delta_3, \Delta_2)$, where Δ_1 and Δ_2 are the operators on H originally defined by Ω_1 and Ω_2.

   In the exercises below, M is a factor on a separable Hilbert space.

5. Prove that a type I factor $M \subset B(H)$ is isomorphic to $B(K)$, for some Hilbert space K. *Hint.* Show that:

   (a) $pMp = \mathbb{C} \cdot p$ whenever p is a minimal projection.

   (b) Any two minimal projections in M are equivalent.

   (c) $1 = \sum_{i \in I} p_i$ for some orthogonal family $(p_i)_{i \in I}$ of minimal projections;

   (d) If $u_i^*u_i = p_1$ and $u_iu_i^* = p_i$, then $p_ixp_j = \lambda_{ij}u_iu_j^*$ for some $\lambda_{ij} \in \mathbb{C}$, $x \in M$.

   Finally, let K be $\mathbb{C}^n$ if $|I| = n$ or $K = \ell^2$ if $I \cong \mathbb{N}$, with o.n.b. $(p_i)_{i \in I}$, and let $H' = p_1H$. Show that there is a unique unitary $u : K \otimes H' \to H$ such that $u(e_i \otimes \psi) = u_i\psi$ for all $\psi \in H'$ and $i \in I$, and finally prove that $u^*Mu = B(K) \otimes 1_{H'}$.

[67] If you like, submit nos. 5 and 6 by mail to A.J. Lindenhovius, IMAPP, FNWI, RU, Heyendaalseweg 135, 6525 AJ Nijmegen, in order to replace your lowest mark for the earlier weeks.

6. Show that a II_∞ factor M is isomorphic to $N \otimes B(\ell^2)$, where N is type II_1:

   (a) Pick a nonzero finite projection p in M and show that there exists an orthogonal family $(p_i)_{i\in\mathbb{N}}$ of projections with $p_i \sim p$ for all i and $\sum_i p_i = 1$. In what follows, u_i is a partial isometry in M such that $u_i^* u_i = p$ and $u_i u_i^* = p_i$.

   (b) Let $H' = pH$ and consider $N = pMp$ as a von Neumann algebra on $B(H')$. Prove that N is a factor, type II_1 factor.

   (c) Show that the operator $u : H' \otimes \ell^2 \to H$ defined by $u(\psi \otimes e_i) = u_i\psi$ is unitary and satisfies $uN \otimes B(\ell^2)u^* = M$.

Appendix 2: Trace class and Hilbert–Schmidt operators

We now deepen our understanding of the **trace** on $B(H)$, introduced in §2.10. Our aim is to prove the duality $B_1(H)^* = B(H)$, which in detail reads as follows.[68]

Theorem 4.42 *1. As usual, for $\hat{\rho} \in B_1(H)$, define $\rho \in B(H)^*$ by*

$$\rho(a) := \mathrm{Tr}\,(\hat{\rho}a). \tag{4.51}$$

The map $\hat{\rho} \mapsto \rho$ from $B_1(H)$ to $B(H)^*$ is an isomorphism of Banach spaces onto its image,[69] which therefore is a norm-closed subspace of $B(H)^*$.

2. *The same map from $B_1(H)$ to $B_0(H)^*$ is an isomorphism of Banach spaces.*

3. *For $a \in B(H)$, define $\tilde{a} \in B_1(H)^*$ by*

$$\tilde{a}(\hat{\rho}) := \mathrm{Tr}\,(\hat{\rho}a). \tag{4.52}$$

One then has $B(H) \cong B_1(H)^*$ as Banach spaces.

The proof will be given at the end of this appendix, preceded by some preliminaries. Let $\{e_i\}_i$ be an o.n.b. in a Hilbert space H. For $a \in B(H)$ and $a \geq 0$, define

$$\mathrm{Tr}\,(a) := \sum_i (e_i, ae_i). \tag{4.53}$$

Elementary computations (cf. [28]) give

$$\mathrm{Tr}\,(a^*a) = \mathrm{Tr}\,(aa^*) \tag{4.54}$$

for all $a \in B(H)$ (whether or not these expressions are finite), and

$$\mathrm{Tr}\,(uau^*) = \mathrm{Tr}\,(a) \tag{4.55}$$

for $a \geq 0$ and u unitary. In particular, (4.53) is independent of the choice of the basis. For any $a \in B(H)$, define the **trace norm** of a by

$$\|a\|_1 := \mathrm{Tr}\,(|a|) = \mathrm{Tr}\,(\sqrt{a^*a}), \tag{4.56}$$

and the **trace class** $B_1(H) \subset B(H)$ by

$$B_1(H) := \{a \in B(H) \mid \|a\|_1 < \infty\}. \tag{4.57}$$

The **Hilbert–Schmidt norm** is defined by

$$\|a\|_2^2 := \mathrm{Tr}\,(|a|^2) = \mathrm{Tr}\,(a^*a), \tag{4.58}$$

with associated **Hilbert–Schmidt class** $B_2(H) \subset B(H)$ defined by

$$B_2(H) := \{a \in B(H) \mid \|a\|_2 < \infty\}. \tag{4.59}$$

With our usual (Dirac) notation $|v\rangle\langle w|$ for the operator $u \mapsto (w, u)v$, it follows that

$$\| |v\rangle\langle w| \, \|_1 = \| |v\rangle\langle w| \, \|_2 = \|v\| \, \|w\|. \tag{4.60}$$

[68] This may be seen as a noncommutative generalization of the dualities $\ell_0^* \cong \ell^1$ with $(\ell^1)^* \cong \ell^\infty$.
[69] An isomorphism of Banach spaces is by definition isometric.

Proposition 4.43 *1. For all $a \in B(H)$ one has*

$$\|a\| \leq |a\|_2 \leq \|a\|_1. \tag{4.61}$$

2. *Every trace-class operator and every Hilbert–Schmidt operator is compact, so that one has the inclusions*

$$B_1(H) \subset B_2(H) \subset B_0(H) \subset B(H). \tag{4.62}$$

3. *The Hilbert–Schmidt operators $B_2(H)$ form a Hilbert space in the inner product*

$$(a, b) := \mathrm{Tr}\,(a^* b); \tag{4.63}$$

in particular, $B_2(H)$ is a Banach space in the norm (4.58).

4. *$B_2(H)$ is a two-sided *-ideal in $B(H)$.*

5. *For $a \in B(H)$ and $b \in B_1(H)$ one has*

$$|\mathrm{Tr}\,(ab)| \leq \|a\|\|b\|_1. \tag{4.64}$$

6. *$B_1(H)$ is a Banach space in the norm (4.56).*

7. *$B_1(H)$ is a two-sided *-ideal in $B(H)$.*

8. *If $a \in B_1(H)$, then (4.53) converges absolutely and is independent of the basis chosen.*

9. *For either a and b in $B_2(H)$, or $a \in B_1(H)$ and $b \in B(H)$, or $a \in B(H)$ and $b \in B_1(H)$, one has*

$$\mathrm{Tr}\,(ab) = \mathrm{Tr}\,(ba). \tag{4.65}$$

Proof.

1. Although $\|b\| \geq \|bv\|$ for all $b \in B(H)$ and all unit vectors v, for every $\epsilon > 0$ there is a $v \in H$ of norm 1 such that $\|b\|^2 \leq \|bv\|^2 + \epsilon$. Put $b = (a^*a)^{1/4}$, and note that $\|(a^*a)^{1/4}\|^2 = \|a\|$. Completing v to a basis $\{e_i\}_i$, we have

$$\|a\| = \|(a^*a)^{1/4}\|^2 \leq \|(a^*a)^{1/4}v\|^2 + \epsilon \leq \sum_i \|(a^*a)^{1/4}e_i\|^2 + \epsilon = \|a\|_1 + \epsilon.$$

Since this holds for all $\epsilon \geq 0$, one has

$$\|a\| \leq |a\|_1. \tag{4.66}$$

Similarly, $\|a\| \leq |a\|_2$. The remaining inequality in (4.61) will follow from the next item.

2. Let $a \in B_1(H)$. Since $\sum_i (e_i, |a|e_i) < \infty$, for every $\epsilon > 0$ we can find n such that $\sum_{i>n}(e_i, |a|e_i) < \epsilon$. Let p_n be the projection onto the linear span of $\{e_i\}_{i=1,\dots,n}$. Using (4.66), we have

$$\|p_n^{\perp}|a|^{1/2}\|^2 = \|p_n^{\perp}|a|p_n^{\perp}\| \leq \|p_n^{\perp}|a|p_n^{\perp}\|_1 < \epsilon.$$

Since $p_n^{\perp} = 1 - p_n$, it follows that $p_n|a|^{1/2} \to |a|^{1/2}$ in the norm topology. Using continuity of the involution and of multiplication, it follows that $p_n|a|p_n \to |a|$ in norm. Since each operator $p_n|a|p_n$ obviously has finite rank, $|a|$ is compact. But a has polar decomposition $a = u|a|$, so that a is compact (since $B_0(H)$ is a two-sided ideal in $B(H)$). The proof for $B_2(H)$ is analogous.

From the spectral theorem for compact operators, one now has $\|a\|_p = \sum \lambda_i^p$ for $p = 1, 2$, where the $\lambda_i \geq 0$ are the eigenvalues of $|a| \geq 0$. The second inequality in (4.61) is now immediate. Hence (4.61) has been proved.

3. The polarization formula $(a+b)^*(a+b) + (a-b)^*(a-b) = 2(a^*a + b^*b)$ yields the inequality $(a+b)^*(a+b) \leq 2(a^*a + b^*b)$, since $c^*c \geq 0$ for all c, including $c = a - b$. Hence $a, b \in B_2(H)$ implies $a + b \in B_2(H)$. With the obvious $\lambda a \in B_2(H)$ when $a \in B_2(H)$ for all $\lambda \in \mathbb{C}$, this proves that $B_2(H)$ is a vector space. The sesquilinear form (4.63) is clearly positive semidefinite, so the Cauchy–Schwarz inequality holds, and $B_2(H)$ is a pre-Hilbert space. In fact, by (4.61) the form (4.63) is positive definite, since $\|\cdot\|$ is a norm. Finally, we show that $B_2(H)$ is complete. Pick a basis $\{e_i\}_{i\in I}$ in H (not necessarily countable), and note that $B_2(H)$ is the closure of the linear span of all operators of the form $a = \sum_{i,j} a_{ij}|e_i\rangle\langle e_j|$. This is because of the continuity of the inclusions in (4.62) and the fact that $B_0(H)$ is itself the closure of this linear span. An easy calculation gives

$$\|\sum_{i,j} a_{ij}|e_i\rangle\langle e_j|\|_2^2 = \sum_{i,j} |a_{ij}|^2.$$

Hence $B_2(H)$ is isomorphic to the space of square-summable sequences indexed by $J = I \times I$; this is well known to be a Hilbert space for any index set J.

4. With $a \in B_2(H)$ one has $a^* \in B_2(H)$ by (4.54). Furthermore, for a unitray u it is immediate from the definition of the Hilbert–Schmidt norm that $ua \in B_2(H)$. By the linearity of $B_2(H)$, this implies $ba \in B_2(H)$ for all $b \in B(H)$. Taking the star yields the same conclusion for ab.

5. Assume $a \in B(H)$ and $b \in B_1(H)$, and let $b = u|b|$ be the polar decomposition of b. We write $\text{Tr}\,(ab) = ((au|b|^{1/2})^*, |b|^{1/2})$ in the inner product (4.63). This makes sense, since by definition of $B_1(H)$ $b \in B_1(H)$ implies $|b| \in B_1(H)$. Hence $|b|^{1/2} \in B_2(H)$ by definition of $B_2(H)$. The latter being a *-ideal in $B(H)$, one the also has $(au|b|^{1/2})^* \in B_2(H)$. The Cauchy–Schwarz inequality then yields

$$|\text{Tr}\,(ab)|^2 \leq \||b|^{1/2}\|_2^2 \|au|b|^{1/2}\|_2^2 = \|b\|_1 \sum_i \|au|b|^{1/2}e_i\|^2.$$

Since $\|au|b|^{1/2}e_i\| \leq \|au\|\||b|^{1/2}e_i\|$ and $\|au\| \leq \|a\|$, the claim follows.

6. Let $a, b \in B_1(H)$, and let $a+b = u|a+b|$ be the polar decomposition. Then
$$\|a+b\|_1 = \text{Tr}\,(u^*(a+b)) = \text{Tr}\,(u^*a) + \text{Tr}\,(u^*b).$$
Applying (4.64) with $\|u^*\| \leq 1$, one has $\|a+b\|_1 \leq \|a\|_1 + \|b\|_1$. Hence $B_1(H)$ is a vector space and $\|\cdot\|_1$ is a norm; it is positive definite by (4.61). We now prove completeness of $B_1(H)$. Let $\{a_n\}$ be a Cauchy sequence in $\|\cdot\|_1$. By (4.61), this is also a Cauchy sequence for $\|\cdot\|$, which converges to some $a \in B_0(H)$. Let $a - a_n = u|a-a_n|$, so that $|a-a_n| = u^*(a-a_n)$. Writing p_N for the projection on the linear span of $e_1, \ldots, e_N$, one has, for $N < \infty$,
$$\sum_{i=1}^N (e_i, |a-a_n|e_i) = \lim_{m\to\infty} \text{Tr}\,(p_N u^*(a_m - a_n)) \leq \limsup_m \|a_m - a_n\|_1,$$
where we used (4.64) to derive the inequality. Since the right-hand side is independent of N, we can let $N \to \infty$ on the left, to obtain
$$\|a - a_n\|_1 \leq \limsup_m \|a_m - a_n\|_1.$$
Since $\{a_n\}$ is a Cauchy sequence, it follows that $\lim_n \|a-a_n\|_1 = 0$, so that the sequence converges to $a \in B_1(H)$ also in the trace norm. Note that a similar proof can be given for $B_2(H)$.

7. For u unitary, one has $|ua| = |a|$, so if $a \in B_1(H)$ then $ua \in B_1(H)$. By linearity and the fact that each bounded operator is a linear combination of at most four unitaires (exercise!), one has $ba \in B_1(H)$ for all $b \in B(H)$. Similarly, $|au| = u^{-1}|a|u'$, so by (4.55) one has $au \in B_1(H)$, etc. Finally, if $a = u|a|$ and $a^* = \tilde{u}|a^*|$ then $|a^*| = \tilde{u}^*|a|u^*$, which is in $B_1(H)$ by the previous argument in this item, so that $a^* \in B_1(H)$.

8. It is clear from (4.64) that $\text{Tr}\,(a) < \infty$. To prove absolute convergence, with $a = u|a|^{1/2}|a|^{1/2}$ one has
$$|(e_i, ae_i)| \leq \||a|^{1/2}u^*e_i\|\||a|^{1/2}e_i\|.$$
Taking the sum and using the Cauchy–Schwarz inequality, one obtains
$$\sum_i |(e_i, ae_i)| \leq \||a|^{1/2}u^*\|_2\||a|^{1/2}\|_2.$$
By the argument in the proof of (4.64) (with b replaced by a), the right-hand side is finite. Independence of the basis is now an easy exercise.

9. It is enough to consider the case where a or b (as appropriate) is unitary, in which case one uses (4.55) and item 4 or 7. ∎

At last, we are now in a position to prove Theorem 4.42.

1. For $\hat{\rho} \in B_1(H)$ and $\rho \in B(H)^*$, it is clear from (4.64) and (4.65) that

$$\|\rho\| \leq \|\hat{\rho}\|_1. \tag{4.67}$$

For the converse inequality, write $\hat{\rho} = u|a|$, and estimate

$$\|\hat{\rho}\|_1 = \mathrm{Tr}\,(|\hat{\rho}|) = \mathrm{Tr}\,(u^*\hat{\rho}) = \rho(u^*) \leq \|\rho\|\|u^*\| \leq \|\rho\|.$$

Hence $\|\hat{\rho}\|_1 = \|\rho\|$ for $\rho \in B(H)^*$.

2. For part 2 of the theorem we also need this equality for $\rho \in B_0(H)^*$. To that effect, and replace the above estimate by

$$\mathrm{Tr}\,(p_N|\hat{\rho}|) = \mathrm{Tr}\,(p_N u^*\hat{\rho}) = \rho(p_N u^*) \leq \|\rho\|\|p_N u^*\| \leq \|\rho\|. \tag{4.68}$$

The point is that, whereas $1 \notin B_0(H)$ in general, one now has $p_N u^* \in B_0(H)$. Letting $N \to \infty$ then yields the desired result.

Now take $\varphi \in B_0(H)^*$ and $a \in B_2(H) \subseteq B_0(H)$. Then

$$|\varphi(a)| \leq \|\varphi\|\|a\| \leq \|\varphi\|\|a\|_2.$$

Since $B_2(H)$ is a Hilbert space, by Riesz–Fischer there is an operator $\hat{\rho} \in B_2(H)$ such that $\varphi(a) = \mathrm{Tr}\,(\hat{\rho}a)$ for all $a \in B_2(H)$. Letting $N \to \infty$ in (4.68), this can be sharpened to $\hat{\rho} \in B_1(H)$. Hence the map $\hat{\rho} \mapsto \rho$ is surjective onto $B_0(H)^*$ and isometric, so it must be an isomorphism of Banach spaces.

3. Now let $a \in B(H)$ and $\tilde{a} \in B_1(H)^*$. From (4.64) and (4.65) one has $\|\tilde{a}\| \leq \|a\|$. For the converse, take $v, w \in H$. On the one hand, one has

$$\tilde{a}(|w\rangle\langle v|) = (v, aw),$$

and on the other one has

$$|\tilde{a}(|w\rangle\langle v|)| \leq \|\tilde{a}\|\||w\rangle\langle v|\|_1.$$

Combining these and using (4.60) yields

$$|(v, aw)| \leq \|\tilde{a}\|\,\|v\|\|w\|. \tag{4.69}$$

Taking v, w of unit length and using the identity

$$\|a\| = \sup\{|(v, aw)| \mid v, w \in H, \|v\| = \|w\| = 1\} \tag{4.70}$$

for any $a \in B(H)$ gives $\|a\| \leq \|\tilde{a}\|$, so that $\|\tilde{a}\| = \|a\|$.

A given $\varphi \in B_1(H)^*$ defines an operator a on H by $\varphi(|w\rangle\langle v|) = (v, aw)$. Since the operators $|w\rangle\langle v|$ span $B_1(H)$, one has $\varphi = \tilde{a}$, and by (4.69) and (4.70) it follows that $a \in B(H)$. The proof of Theorem 4.42 is now complete. ■

Appendix 3

Half of Theorem 4.9 evidently follows from Theorem 4.10. The converse ('if') implication uses a refinement of the GNS-construction, where the state ω is assumed to be σ-weakly continuous (such states are also called **normal**). In that case, using the theory of σ-weakly closed ideals of von Neumann algebras, it can be shown that $\pi_\omega(M)$ coincides with $\pi_\omega(M)''$ and hence is a von Neumann algebra [1, Thm. 2.4.24].

Since normal pure state on a von Neumann algebra may not exist, the 'crazy' Hilbert space H_c in the proof of Theorem 3.20 (see §3.10) must be replaced by the even crazier direct sum $H_{ec} = \bigoplus_{\omega \in S_n(M)} H_\omega$, where this time the sum is over all normal states on M. Similarly, in Lemma 3.8 one should now have a normal state instead of a pure state. Otherwise, the proof that M has a faithful representation as a von Neumann algebra on a Hilbert space is essentially follows the proof of Theorem 3.20 (see Sakai's own book [30, Thm. 1.16.7] for details). ■

We next prove (4.37). The inclusion $M \subset M^{\perp\perp}$ is trivial. For the converse, pick $a \notin M$; since M is a von Neumann algebra, it is σ-weakly closed, so its complement M^c in $B(H)$ is σ-weakly open. Hence there are $\varphi \in B(H)_*$ and $\epsilon > 0$ such that the open neighbourhood

$$\mathcal{O}(a) := \{b \in B(H) \mid |\varphi(a) - \varphi(b)| < \epsilon\}$$

of a entirely lies in M^c. So $|\varphi(a) - \varphi(b)| \geq \epsilon$ for all $b \in M$. This implies $\varphi(b) = 0$ by linearity in b. Hence $|\varphi(a)| \geq \epsilon$, so $a \notin M^{\perp\perp}$, hence $M^{\perp\perp} \subset M$.

For (4.38), first note that $M^\perp$ is a norm-closed subspace of $B(H)_* = B_1(H)$, which is a Banach space in the trace-norm (which coincides with the norm inherited from $B(H)^*$, since the injection $B_1(H) \hookrightarrow B(H)^*$ is an isometry). Hence the quotient $B(H)_*/M^\perp$ is a Banach space in the canonical norm

$$\|\dot{\varphi}\| := \inf\{\|\varphi + \psi\| \mid \psi \in M^\perp\}.$$

where $\dot{\varphi}$ is the image of $\varphi \in B(H)_*$ under the canonical projection, and the norm is the one in $B(H)^*$. Let $\varphi^\upharpoonright := \varphi \upharpoonright M$ be the restriction of $\varphi \in B(H)_*$ to M. It is clear that the map $\varphi^\upharpoonright \mapsto \dot{\varphi}$ is well defined and is a linear bijection from M_* to $B(H)_*/M^\perp$. In fact, this map is isometric. Firstly, one trivially has

$$\|\varphi^\upharpoonright\| = \sup\{|\varphi(a)| \mid a \in M_u\} = \inf_{\psi \in M^\perp} \sup\{|\varphi(a) + \psi(a)| \mid a \in M_u\},$$

since $\psi(a) = 0$. But this is clearly majorized by

$$\|\dot{\varphi}\| = \inf_{\psi \in M^\perp} \sup\{|\varphi(a) + \psi(a)| \mid a \in B(H)_1\},$$

since now the supremum is taken over a larger set. Hence $\|\varphi^\upharpoonright\| \leq \|\dot{\varphi}\|$. Conversely, for any $\varphi \in B(H)_*$ with $\|\dot{\varphi}\| = 1$ there exists, by a version of the Hahn–Banach theorem, an $a \in B(H)$ with $\hat{a} \in M^{\perp\perp}$, $\varphi(a) = 1$ and $\|a\| = 1$. From (4.37) one then infers that $\|\varphi^\upharpoonright\| \geq |\varphi(a)| = 1 = \|\dot{\varphi}\|$. This finishes the proof of Theorem 4.10. ■

References

[1] Bratteli, O. & Robinson, D.W. (1987). *Operator Algebras and Quantum Statistical Mechanics. Vol. I: C^*- and W^*-Algebras, Symmetry Groups, Decomposition of States.* 2nd Ed. Berlin: Springer.

[2] Bratteli, O. & Robinson, D.W. (1981). *Operator Algebras and Quantum Statistical Mechanics. Vol. II: Equilibrium States, Models in Statistical Mechanics.* Berlin: Springer.

[3] A. Connes, Une classification des facteurs de type III, *Ann. Sci. École Norm. Sup.* **6**, 133–252 (1973).

[4] A. Connes, Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$, *Ann. Math.* **104**, 73–115 (1976).

[5] A. Connes, *Noncommutative Geometry* (Academic Press, San Diego, 1994).

[6] A. Connes & M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives* (AMS, 2008).

[7] G. Cornelissen, N.P. Landsman, and W. van Suijlekom, 'The flashes of insight never came for free': Interview with Alain Connes, *Nieuw Archief voor Wiskunde* **5/11**, 250–256 (2010). Available at `www.math.ru.nl/~landsman/ConnesNAW.pdf`.

[8] J.B. Conway, *A Course in Functional Analysis* (Springer, New York, 1990).

[9] T. Coquand and B. Spitters, Constructive Gelfand duality for C*-algebras, *Math. Proc. Camb. Phil. Soc.* 147, 339–344 (2009).

[10] J. Dixmier, *Von Neumann Algebras* (North-Holland, Amsterdam, 1981).

[11] R. Dudely, *Real Analysis and Probability* (Wadsworth & Brooks/Cole, Pacific Grove, 1989).

[12] E.G. Effros and Z.-J. Ruan, *Operator spaces* (Oxford University Press, New York, 2000).

[13] R. Haag, *Local Quantum Physics* (Springer, Berlin, 1992).

[14] P. T. Johnstone, *Stone Spaces* (Cambridge University Press, 1982).

[15] V.F.R. Jones, *Subfactors and knots* (AMS, 1991).

[16] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol. I: Elementary Theory* (Academic Press, New York, 1983).

[17] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol. II: Advanced Theory* (Academic Press, New York, 1986).

[18] N.P. Landsman and Ch.G. van Weert, Real- and imaginary-time field theory at finite temperature and density, *Phys. Rep.* **145**, 141–249 (1987).

[19] J. Lurie, *Math261y: von Neumann algebras* (Harvard University), Lecture notes (in progress) available at `www.math.harvard.edu/~lurie/261y.html`.

[20] F. J. Murray and J. von Neumann, On rings of operators, *Ann. Math.* **37**, 116–229 (1936).

[21] F. J. Murray and J. von Neumann, On rings of operators, II, *Trans. Amer. Math. Soc.* **41**, 208–248 (1936).

[22] F. J. Murray and J. von Neumann, On rings of operators, IV, *Ann. Math.* **44**, 716–808 (1943).

[23] J. von Neumann, Zur Algebra der Funktionaloperatoren und theorie der normalen Operatoren, *Math. Ann.* **102**, 370–427 (1929).

[24] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932).

[25] J. von Neumann, On rings of operators, III, *Ann. Math.* **41**, 94–161 (1940).

[26] J. von Neumann, On rings of operators. Reduction theory, *Ann. Math.* **50**, 401–485 (1949).

[27] J. von Neumann, *Collected Works. Vol. III: Rings of Operators*, ed. A. H. Taub (Pergamon Press, New York, 1961).

[28] G.K. Pedersen, *Analysis Now*, 2nd printing (Springer, New York, 1995).

[29] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. I: Functional Analysis* (Academic Press, San Diego, 1980).

[30] S. Sakai, *C*-algebras and W*-algebras* (Springer, Berlin, 1971).

[31] B. Simon, *Representations of Finite and Compact Groups* (AMS, 1996).

[32] M. Takesaki, Twenty-five years in the theory of type III von Neumann algebras, in: R.S. Doran (ed.), *C^*-algebras: 1943–1993, A Fifty Year Celebration*, pp. 233–239 (AMS, Providence, 1994).

[33] M. Takesaki, *Theory of Operator Algebras, Vols. I–III* (Springer, Heidelberg, 2002–2003).

[34] J. Weidmann, *Lineare Operatoren in Hilberträumen*, Vol. I (Teubner, Stuttgart, 2000). (Teubner, Stuttgart, 2003).

[35] S. Wright, *Uniqueness of the injective* III_1 *factor* (Lecture Notes in Mathematics **1413**, Springer-Verlag, Berlin, 1989).