Noncommutative Geometry and the Integer Quantum Hall Effect

Michelangelo Vargas Rivera

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Supervisor: Prof. Dr. N. P. Landsman

Institute for Theoretical Physics
Valckenierstraat 65
1018 XE Amsterdam
The Netherlands

University of Amsterdam

Abstract

Bloch theory has been very successful for describing electronic transport in solids. In the integer quantum Hall effect, however, Bloch theory fails, due to the breakdown of translation invariance. In this master's thesis we discuss an attempt of Jean Bellissard to generalise Bloch theory when translation invariance is broken. This theory gives a noncommutative C^*-algebra as the generalisation of the Brillouin zone. Therefore we introduce noncommutative geometry, developed by Alain Connes. We show how this theory succeeds in giving a complete description of the integer quantum Hall effect.

Contents

Chapter 1

Introduction

In the eighties Jean Bellissard introduced the use of noncommutative geometry in solid state physics to generalise Bloch theory to other materials than periodic ones. In the present master's thesis we discuss this mathematical framework, developed by Alain Connes, and show how this tool is used. We do this using the most spectacular example of noncommutative geometry in physics, also due to Bellissard: the integer quantum Hall effect.

Bloch theory has proven to be an excellent tool for describing electronic transport in crystalline solids. The main reason for this success is that in most crystals the electronic transport can be read of the motion of one electron, neglecting the interactions with other electrons and phonons (lattice vibrations). Making explicit use of the periodicity of the crystal, the Bloch theorem says that the electron in this configuration is an extended wavefunction. With this information one is able to determine the continuous spectrum of the Hamiltonian, which shows the typical band structure for crystalline solids. This band structure determines conductivity properties of the solid.

Because of the explicit use of the periodicity condition of Bloch theory, this theory is not applicable to non-periodic substances as amorphous solids or quasicrystals. If the periodicity is broken by disorder or the presence of a magnetic field the Bloch theory fails also. Remark that it is not true that in the presence of a magnetic field the periodic atomic structures of a crystal suddenly become non-periodic. The motion of an electron through such a solid, however, loses its translation invariance, which causes the failure of Bloch theory in this case.

Many theorists developed different tools to describe the electronic motions through these solids where the translation invariance breaks down. One can think of, finite scaling and the β-function of renormalisaion group for disordered systems, curved space representation for amorphous materials, or the cut-and-project method for quasicrystals. All these techniques, however, are specific for certain kind of sub-families of materials. The motivation of Bellissard to introduce noncommutative geometry in solid state physics, was the strong believe that this mathematical framework could generate a general theory for all kinds of solids. Replacing the old Brillouin zone by a generalisation of it, a noncommutative manifold, one is able to describe some other than periodic solids through a noncommutative Bloch theory. One of the most fascinating applications of noncommutative geometry in solid state physics is the complete description of the integer quantum Hall effect through this procedure. This is the subject of the following Chapters.

In 1980 Klaus von Klitzing discovered the integer quantum Hall effect. In short one can describe this effect as follows. Under a particular arrangement of a strong magnetic field and an electric current in some sample, the resistance normal to the current, the so-called Hall resistance, shows quantisation when plotted against the strength of the magnetic field. This quantisation is, given in certain units, integral and is measured to be very robust. Moreover, the current is dissipationless.

Almost immediately after the discovery of the integer quantum Hall effect, Laughlin proposed a gauge invariant argument to describe the quantisation of the conductance in this effect. This showed the topological character if the integer quantum Hall effect. Soon afterwards, other authors showed that the Hall resistance is indeed a topological invariant.

The problem with these early theories for the integer quantum Hall effect is that they are incomplete. They make the assumption that the amount of quantum fluxes through the unit cell is rational. This assumption is not physical and discards exactly the aperiodic character of the material (or more exactly, its breakdown of translation invariance). An introduction to the integer quantum Hall effect and a discussion of its early theories will be given in Chapter 2.

One can intuitively see why the Hall resistance is a topological invariant. Remember that a topological invariant is a quantity that stays the same under a continuous deformation. The robustness of the Hall resistance is exactly this invariance. One can change the geometry of the sample, the electron density and even the material of the sample, the quantisation of the Hall resistance (if any) stays the same.

The Hall resistance can be related to the first Chern class, which is a member of the well-known family of characteristic classes. These classes are maps from vector bundles over compact spaces to the de Rham cohomology groups. Using the specific features of these classes one can show the integrality of the Hall resistance. Characteristic classes and how one can relate this to the integer quantum Hall effect, will be the subject of Chapter 3.

To give a complete description of the integer quantum Hall effect, Bellissard generalised the old theories through noncommutative geometry. The basic concept of this mathematical framework is the relation between spaces and algebras. It is known that every compact Hausdorff space M can be written in terms of its associative algebra $C(M)$ of continuous functions on M. This commutative algebra is a so called commutative C^*-algebra. A theorem of Gelfand and Naimark states that every commutative algebra can be written in terms of the continuous functions over some compact Hausdorff space. Hence, working with C^*-algebras, one is always implicitly referring to the corresponding compact space.

This theory is extended, also by Gelfand and Naimark, to the noncommutative case. So, every noncommutative C^*-algebra corresponds to some space. Because the underlying space of a noncommutative algebra is hard to work with, one often does not try to recover it and just work with the algebra itself, referring to it as a noncommutative space. This is exactly what one does, when generalising the Brillouin zone. One considers some (noncommutative) C^*-algebra following from the theory, which in the commutative case is the algebra $C(B)$, with B the Brillouin zone. This noncommutative algebra will then be referred to as the noncommutative Brillouin zone. C^*-algebras and both Gelfand-Naimark theorems are the subject of Chapter 4.

We already mentioned that the Hall resistance is related to a Chern class. Actually the Hall resistance is the integral of the first Chern class over the Brillouin zone. Therefore, if one wants to express this quantity in the C^*-algebraic context, it is necessary to define the Chern class algebraically and to define a calculus on a noncommutative C^*-algebra. The intuitive manner to do this, is to start with the commutative case, and try to generalise it to the noncommutative framework. This generalisation is not straightforward and is the subject of Chapter 5.

The generalisation of the Chern class to the noncommutative setting will be done through the generalisation of the Chern character. This character is build out of the Chern classes and as such, is a map from vector bundles over compact spaces to the de Rham cohomology groups. The power of the Chern character lies in the fact that is a an isomorphism from equivalence classes of vector bundles to the cohomology groups. This equivalence classes form a group called the K^0-group. In Chapter 5, the C^*-algebraic version of the Chern character is given.

To complete the total picture of noncommutative geometry through the Chern character one also needs a algebraic counterpart for the de Rham cohomology. These cohomology groups are topological invariant, and define in a sense the calculus on a space, or in this case on an algebra. On algebras one could define different kind of cohomologies. The cohomology we will be needing is the a generalisation of the de Rham cohomology, that makes the noncommutative algebraic Chern character an isomorphism between the C^*-algebraic K-theory and this cohomology. This is the subject of Chapter 6.

When one has the tools needed to express the Chern character in C^*-algebraic terms, one could try to apply it to the integer quantum Hall effect. A crucial ingredient is the noncommutative C^*-algebra replacing the Brillouin zone. In Chapter 7 we introduce the algebra proposed by Bellissard, which generalises the Brillouin zone. Using the same physical arguments but applying the generalised tools one can describe the integer quantum Hall effect, without making the rationality assumption.

Before we continue the discussion of the integer quantum Hall effect, we would like to mention that noncommutative geometry has also other applications in solid state physics. A beautiful example is the use of noncommutative topology in the theory of electronic motion in quasicrystals (see [84], for an insightful overview, and references therein). Besides the applications in solid state physics, noncommutative geometry is also used in other parts of theoretical physics. We mention the standard model ([32], [57]) and string theory ([81]). We refer to [31], [39] and [52] for more on noncommutative geometry in physics.

Chapter 2

The integer quantum Hall effect

2.1 The quantum Hall effects

Consider a two-dimensional piece of conducting material placed in the xy-plane, with a current density J in the y-direction and a magnetic field B in the z-direction (see Figure 2.1). The Lorentz force will tend to push the charge carriers in the x-direction, orthogonal to the magnetic field and the electric current. In 1879 Hall discovered (see [42]) that this force was opposed by an induced electric potential E_H, such that the charge carriers would be undeflected in their motion. Hall did this by measuring a resistance in the y-direction, and refuted hereby an earlier statement of Maxwell that a magnetic field does not influence an electric current in a fixed conductor. This phenomenon that Hall discovered is called the Hall effect.

Let σ_0 be the contribution of the electric field to the velocity of a charge carrier with a resulting current J, i.e. $J = \sigma_0 E$. This contribution is given by (see [66])

$$\sigma_0 = ne^2\tau_0/m, \tag{2.1}$$

with n the charge carrier density, e the elementary charge unit, τ_0 the mean free time, and m the mass of the charge carrier. Consider the Lorentz relation (in SI units)

$$eE_H + ev \times B = 0. \tag{2.2}$$

We stress the fact that we are using SI units, since many textbooks, including the standard reference [66], use cgs units or both. Together with the relation

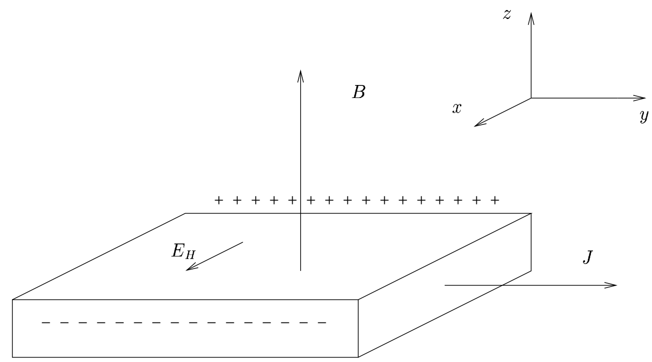

Figure 2.1: The Hall effect

for the resistivity tensor ρ

$$E = \begin{pmatrix} E_H \\ E_y \end{pmatrix} = \rho J, \tag{2.3}$$

we find

$$\rho = \begin{pmatrix} \rho_0 & \frac{B}{ne} \\ -\frac{B}{ne} & \rho_0 \end{pmatrix}, \tag{2.4}$$

where $\rho_0 = 1/\sigma_0$. We have used here Onsager's relations $\rho_{xx} = \rho_{yy}$ and $\rho_{xy} = -\rho_{yx}$, and $\vec{v} = J/(-ne)$. The Hall resistance R_H, which is defined as the resistance orthogonal to the current and magnetic field, equals $\rho_{xy} = B/(ne)$. Remark that in two dimensions resistance and resistivity are the same. We see that the Hall resistance is linear in the magnetic field.

In 1980 von Klitzing (see Figure 2.2) and coworkers (see [47]) discovered that for high magnetic fields and low temperatures, the responses are completely different. First of all, the Hall resistance is not linear in the magnetic field but gets stuck at values $h/(ie^2)$, with i an integer. Secondly, at these plateaux on the values $h/(ie^2)$ the current J in the y-direction is dissipationless i.e., $\rho_0 = 0$. This may not seem very surprising, for if we just take the limit of τ_0 to infinity (since the temperature is small), ρ_0 goes to zero. The difference, in the experiments such as von Klitzing did however, is that ρ_0 is not zero but *goes* to zero at the plateaux. Furthermore, as we will see later, disorder plays a very important role in this experiment such that τ_0 can not just go to infinity. Because of the relation between the conductivity σ and the resistivity ρ

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} = \frac{1}{\rho_{xx}^2 + \rho_{xy}^2} \begin{pmatrix} \rho_{xx} & -\rho_{xy} \\ \rho_{xy} & \rho_{xx} \end{pmatrix} = \rho^{-1} \tag{2.5}$$

we see that we could equivalently say that the conductance in the x-direction would be quantised at values ie^2/h and the conductance in the y-direction would go to zero. These two responses, that are shown in Figure 2.3, make up the Integer Quantum Hall Effect (IQHE).

Figure 2.2: Klaus von Klitzing

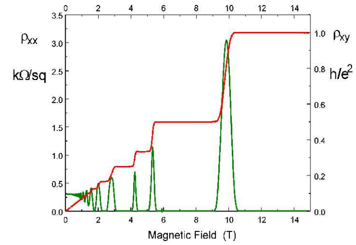

Figure 2.3: The IQHE

The most remarkable of this effect is its accuracy and its universality. The quantisation of the resistance is determined at least to a part per 10^7 accuracy and is independent of the purity of the material, its geometric details and even of the specific material used (see for instance [44] and [83]). Hence the fine structure constant $\alpha = e^2/(4\pi\epsilon_0\hbar c)$ can be determined very precise, since the speed of light c and the permittivity ϵ_0 are exact defined (see [65]). In his article von Klitzing already mentioned this important implication. For this discovery, he was rewarded the Nobel prize for physics in 1985.

The two-dimensional electron system that von Klitzing used, was constructed with a Metal-Oxide-Semiconductor Field Effect Transistor (MOSFET). This device is made out of an insulator fixed to a semiconductor on one side, and a metal on the other side. The two dimensional system is created at the interface of the semiconductor and the insulator by setting a voltage between the metal and the semiconductor. The MOSFET von Klitzing used was made out of Silicon (semiconductor) and Silicon-oxide. He did his experiments at a temperature of 1.5 K and a constant magnetic field of 18 T. Instead of varying the magnetic field he varied the voltage between the metal and the semiconductor, which has the same effect but is simpler to realise.

The IQHE can be understood if the system is incompressible (i.e., there is an energy gap between the ground state and the first excited state) and if there is some kind of disorder that causes a range of localised states. The incompressibility makes sure that the current in the y-direction flows dissipationless. In our system dissipation is given by scattering of occupied states below the Fermi energy with unoccupied states above the Fermi energy. Therefore, if the Fermi energy lies in a gap there can be now dissipation in our system. See Section 2.6 for more on this. The localised states causes the formation of the plateaux of Figure 2.3. When the Fermi energy happens to be in the region of these localised states, there will be no contribution to the current, hence no contribution to the resistance. In the IQHE the contribution of the electron-electron interactions to the dynamics of the system is much smaller than the contribution of the electron-disorder interactions. Consequently it is sufficient to study the single-electron Hamiltonian to explain the IQHE. See, again, Section 2.6 for more on the precise approximation we made in our system.

In 1982 Tsui, Stormer and Gossard (see [78]) discovered the Fractional Quantum Hall Effect (FQHE). In this effect the conductance in the y-direction does not only take values at integer multiples of e^2/h, but also at fractional multiples p/q (with p and q integers). Like in the IQHE, the conductance in the x-direction vanishes at the plateaux. For their experiment Tsui and Stormer used a GaAs-AlGaAs hetero-junction, which is a device made out of two semiconductors. It works just like a MOSFET, where one of the semiconductors (AlGaAs) takes the place of the insulator because of its larger energy gap. In the FQHE the electron-electron interactions do have a significant contribution to the dynamics, with respect to the electron-disorder interactions. This makes the FQHE much more difficult to understand and we will not elaborate on this. We only mention that for their discovery Tsui and Stormer were rewarded the Nobel prize for physics in 1998 together with Laughlin for his work on the explanation of the FQHE. For more on the FQHE (and IQHE) see for instance [26], [36], [66] and [76].

2.2 2D-electron gas in a strong magnetic field

In this section we discuss the basics of the 2D-electron gas in a strong magnetic field, which we use later to explain the IQHE.

Consider a two-dimensional conducting piece of material with area $\mathbb{L}^2$, and a magnetic field $\vec{B}$ perpendicular to it. Consider an electron gas without any interactions in this material. To describe the physics of this system it is sufficient to consider the single-electron Hamiltonian

$$H_0 = (\pi_x^2 + \pi_y^2)/2m, \tag{2.6}$$

where $\vec{\pi} = \vec{p} + e\vec{A}$, $\vec{A}$ is the magnetic potential (such that $\vec{B} = \vec{\nabla} \times \vec{A}$) and m is the effective mass. The Hamiltonian works on the Hilbert space $L^2(\mathbb{L}^2)$ of square integrable functions on the subset $\mathbb{L}^2$ of the real plane $\mathbb{R}^2$ trough the usual representation

$$\vec{p} = (p_x, p_y) \to -(\imath\hbar\partial_x, \imath\hbar\partial_y). \tag{2.7}$$

Following Kubo [49], we introduce the relative coordinates of the cyclotron motion,

$$\xi = \frac{1}{eB}\pi_y, \quad \eta = -\frac{1}{eB}\pi_x \tag{2.8}$$

with centre coordinates

$$X = x - \xi, \quad Y = y - \eta\,. \tag{2.9}$$

In these new coordinates the Hamiltonian is

$$H_0 = \frac{\hbar\omega_c}{2l_0^2}(\xi^2 + \eta^2)\,, \tag{2.10}$$

with $\omega_c = \frac{eB}{m}$ the cyclotron frequency and $l_0 = (\frac{\hbar}{eB})^{1/2}$ the so-called magnetic length, which is the characteristic length of this system. From the commutation relation $[\xi, \eta] = -\imath l_0^2$ we see that the Hamiltonian can be seen as a harmonic Hamiltonian

$$H_0 = \hbar\omega_c(a_+ a_- + \frac{1}{2}) \tag{2.11}$$

with the correspondences

$$a_+ = \frac{1}{\sqrt{2}l_0}(\xi - \imath\eta) \quad \text{and} \quad a_- = \frac{1}{\sqrt{2}l_0}(\xi + \imath\eta). \tag{2.12}$$

Thus the energy spectrum is given by the so-called Landau levels

$$E_n = (n + \frac{1}{2})\hbar\omega_c, \quad n = 0, 1, 2, \ldots, \quad . \tag{2.13}$$

The equations of motion for the centre coordinates are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}X &= \frac{\imath}{\hbar}[H_0, X] = 0\,, \\ \frac{\mathrm{d}}{\mathrm{d}t}Y &= \frac{\imath}{\hbar}[H_0, Y] = 0\,. \end{aligned} \tag{2.14}$$

This justifies the identification of (X, Y) with the centre coordinates of the cyclotron motion with frequency ω_c.

Let us take the Landau gauge, i.e., $\vec{A} = (0, A_y, 0) = (0, Bx, 0)$. The Hamiltonian can now be written as

$$H_0 = \frac{1}{2m}\left[p_x^2 + \left(p_y + eBx\right)^2\right]. \tag{2.15}$$

We see that the Hamiltonian is still translational invariant in the y direction and we can describe the single-electron in that direction in plane waves. We write any eigenfunction $\phi_{kn}(x, y)$ of H_0 in the form

$$\phi_{kn}(x, y) = \phi_{kn}(x)e^{\imath ky}\,. \tag{2.16}$$

If we apply periodic boundary conditions for the y-axis, the values for k are $k = \frac{2\pi}{L_y}i_k$ with $i_k = 0, \pm1, \pm2, \ldots,$ and L_y the length of our sample. If we let the operator H_0 act on the function $\phi_{kn}(x, y)$ we get

$$-\frac{\hbar^2}{2m}\phi''_{kn} + \frac{1}{2}m\omega_c(x - x_k)^2\phi_{kn} = E_{kn}\phi_{kn}\,, \tag{2.17}$$

with $x_k = -l_0^2 k$ and ϕ''_{kn} the second derivative to x. For each k this equation is the Schrödinger equation for the harmonic oscillator, with frequency ω_c and equilibrium $x = x_k$ As mentioned before the eigenvalues are the Landau eigenvalues given in (2.13) at Landau level n. The energy eigenvalues are independent of the momentum $\hbar k$ along the y-axis. Therefore, we say that the states are localised in x and extended in y. If the width of the sample is L_x, the degeneracy of a Landau level is the number of k's, such that the equilibrium point x_k lies in between 0 and L_x. The space between two consecutive centre points is $\frac{2\pi}{L_y}l_0^2$, hence the degeneracy g_n of a Landau level is

$$g_n = \frac{L_x L_y}{2\pi l_0^2}\,. \tag{2.18}$$

We can see this in a different way. Every Landau state occupies an area $2\pi l_0^2$, and the degeneracy is nothing but the total area $\mathbb{L}^2 = L_x L_y$ of our system divided by the area of a single state.

A useful quantity will be the filling factor ν

$$\nu = 2\pi l_0^2 n_0, \tag{2.19}$$

with n_0 the electron density. This filling factor is the dimensionless electron density. For instance, if $\nu = 1$ the first Landau level ($n = 0$) will be exactly filled.

As we mentioned in the previous Section we need localised states to explain the plateaux in the IQHE. This localisation is brought into our system by disorder, in the form of pollution of our material. Let us bring the disorder in our system and consider the effective Hamiltonian

$$H = H_0 + U(\vec{r}) \tag{2.20}$$

with $U(\vec{r})$ the electron-disorder interaction, and $\vec{r}$ the position vector. The equations of motion are now

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}X &= \frac{\imath}{\hbar}[H, X] = \frac{\imath}{\hbar}[U, X] = -\frac{l_0^2}{\hbar}\frac{\partial U}{\partial y}, \\ \frac{\mathrm{d}}{\mathrm{d}t}Y &= \frac{\imath}{\hbar}[H, Y] = \frac{\imath}{\hbar}[U, Y] = -\frac{l_0^2}{\hbar}\frac{\partial U}{\partial x}\,. \end{aligned} \tag{2.21}$$

Because of the disorder potential translation invariance is broken and (X, Y) is not constant anymore. The degeneracy of the states with different (X, Y) is therefore also broken, and the Landau levels, instead of a series of δ-functions, are broadened to bands. In these bands we can make a distinction between two kind of regions. First the regions of extended states that contribute to our Hall current and are called mobility edges. And secondly the regions of localised states, which are called mobility gaps (see Figure 2.4).

2.3 Laughlin's argument

In this section we discuss a gauge-invariant argument to explain the IQHE, introduced by Laughlin [54] and extended by Halperin [43]. We closely follow the discussion in [76].

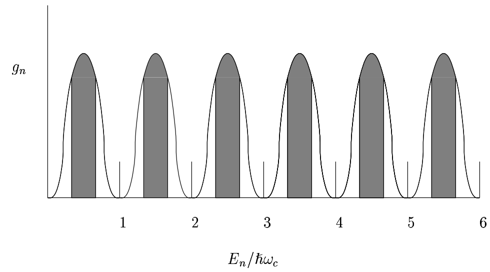

Figure 2.4: The broadening of the Landau levels. The shaded regions around $E_n/\hbar\omega_c = n + \frac{1}{2}$ are the extended states.

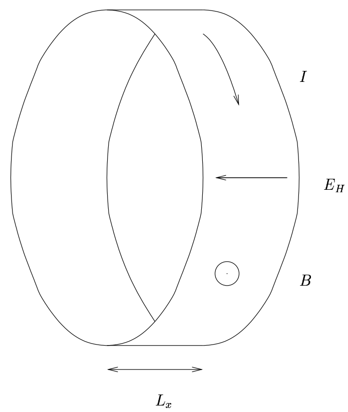

Figure 2.5: Diagram of the metallic loop that Laughlin used for his gedanken experiment.

Consider once again the two-dimensional sample of Figure 2.1 discussed before, but now bent such that if forms a ring in the (y, z)-plane, with circumference L_y and width L_x (see Figure 2.5). The magnetic field B pierces the ring everywhere orthogonally to the surface. One may take this geometry if we assume that the size, shape or edge conditions of the material do not effect the IQHE results. Considering the robustness of the experimental data, these assumptions are not unreasonable.

In this Section we give a relation between the current I and the flux dependence (trough the ring) of the Hamiltonian. This relates I with the electric potential E_H, and therefore gives the Hall resistance. Consider the auxiliary vector potential

$$\vec{a} = -\frac{q\phi_0}{L_y}\hat{y}\,, \tag{2.22}$$

where q is some dimensionless parameter and ϕ_0 is the flux quantum $\frac{h}{e}$. In the original setting where our sample is just a plane, the potential does not contribute to the magnetic field because $\vec{\nabla} \times \vec{a} = 0$. In the new geometry of Figure 2.5 however the potential corresponds to a magnetic field trough the ring. With this extra vector potential the unperturbed Hamiltonian of equation (2.6) becomes

$$H_0(q) = \frac{1}{2m}(\vec{p} + e\vec{A} - qe\frac{\phi_0}{L_y}\hat{y})^2 , \tag{2.23}$$

and the current operator is

$$I_y = -\frac{e}{m}(\vec{p} + e\vec{A} - qe\frac{\phi_0}{L_y}\hat{y}) \cdot \hat{y} = \frac{eL_y}{h}\frac{\partial H_0}{\partial q} . \tag{2.24}$$

We see that, if there is a current, the Hamiltonian is dependent on q and therefore on the vector potential $\vec{a}$. This is a consequence of the phase coherence of the wave functions around the ring. The phase coherence plays a role whenever the wave function is larger than the circumference of the circle. As the wave functions evolve around the ring, they pick up an extra phase factor ϕ_{AB} after a whole round, the so-called Aharonov-Bohm phase $\phi_{AB} = -\frac{e}{\hbar}\int \vec{a} \cdot \mathrm{d}\vec{r}$. Integrating over the loop gives $\phi_{AB} = 2\pi q$. This is exactly what happens if one should pierce the ring through the centre with a flux $\phi = -q\phi_0$. If q is an integer the wave functions are single-valued and there is no problem. If, however, q is not an integer the wave functions become multiple-valued, and therefore not physical. To overcome this problem we should change the original wave numbers $k = 2\pi i_k/L_y$, with $i_k = 0, \pm 1, \pm 2, \ldots$, to eliminate the extra Aharonov-Bohm phase. These new phase numbers are $k = 2\pi(i_k - q)/L_y$. This q-dependence makes the current non-zero. If the wave function vanishes within a region smaller than the circumference of the ring, the function is always single-valued and there is no necessity to change the boundary conditions. Such states do not carry any current and are called localised states.

We want to calculate the Hall resistance, as we place an electric field E_H in the x-direction We use the trick with the auxiliary vector potential to relate the current to the fraction q of flux quanta. The single particle Hamiltonian, in the Landau gauge without disorder, is now

$$H(q) = \frac{1}{2m}\left[\vec{p} + eB(x - \frac{q\phi_0}{BL_y})\hat{y}\right]^2 + eE_x x . \tag{2.25}$$

We see that if we slowly add some flux q to our system, the centre points of the states move in the x-direction from x_k to $x_k + q\phi_0/(BL_y)$. If we add precisely one flux quantum the set of centre points of the states is mapped to itself. The electrical field E_x is now responsible for the q-dependence of H. Consider the single particle wave function $\phi_{kn;q}(x)e^{\imath ky}$ and let the Hamiltonian act on it.

$$\begin{aligned} H(q)\phi_{kn;q}(x)e^{\imath ky} = & \left\{ \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_c^2\left[x - (x_k + \frac{q\phi_0}{BL_y} - \frac{v_d}{\omega_c})\right]^2 \right. \\ & \left. - \frac{1}{2}mv_d - \hbar k v_d + e\frac{E_x}{B}\frac{q\phi_0}{L_y} \right\} \phi_{kn;q}(x)e^{\imath ky} , \end{aligned} \tag{2.26}$$

where $v_d = E_x/B$ is the classical drift velocity. The energy is now k-dependent and therefore x_k-dependent as well. If we add some fraction q of flux quanta, the energy of our system changes. If we add exactly one flux quantum, all the wave functions and their energies remain the same. But during the process all occupied states have moved one step in the x-direction. Remember that the degeneracy of a Landau level equals the number of k's such that x_k lies in between 0 and L_x. Therefore we can conclude that, during the process of adding one flux quantum, we moved one electron per Landau level across the width L_x. The energy change is now $\Delta\mathcal{E} = neV = neE_xL_x$, where n is the number of filled Landau levels.

Consider Faraday's law

$$\frac{d\phi}{dt} = \int d\vec{S} \cdot \frac{d\vec{B}}{dt} = \int_C dlE_y = \int_C dl\rho_{yx}J_x \,, \tag{2.27}$$

where C is a contour enclosing the flux and J_x is the current density in the x-direction equal to I_x/L_y. Integrating this equation over time we obtain

$$\Delta\phi = \rho_{yx} \int dtI_x \,. \tag{2.28}$$

If we now choose for the flux change the flux quantum, the net change in charge is $-ne$. So

$$\frac{h}{e} = -\rho_{yx}ne \,, \tag{2.29}$$

and therefore

$$\rho_{yx} = -\frac{h}{ne^2} \,. \tag{2.30}$$

We now want to use this gauge-invariance argument for a system with disorder. This disorder helps us to explain the plateaux as we already mentioned. Another reason for putting disorder in our system is the robustness of the IQHE. From experiments it is known that the quantisation of the conductance is very precise and insensitive to changes of the material, size, number of charge carriers etcetera. So we also want to explain the quantisation with disorder in our system. The ingredients we used in Laughlin's argument in the system of Figure 2.5 were

1. Only the extended states can carry current, due to the flux changes.
2. For the system to be dissipationless, such that the diagonal terms of the resistivity tensor vanish, there must be a mobility gap.
3. If we insert one flux quantum, the wave functions and their energies will be the same.

Remember the density of states we discussed in Section 2.2, which ensures us that there is indeed a mobility gap of localised states between different regions of extended states. As long as the Fermi energy is in such a mobility gap, there cannot enter more current in our system. The current is then dissipationless, because dissipation is caused by scattering between occupied current-carrying states and unoccupied ones, near the Fermi level. See [76] and Section 2.6 for a more thorough consideration of the dissipation in our system.

If the energy of the system changes due to the insertion of one flux quantum into our system, this can only mean that there is a change in occupation of the states in the same Landau level. It cannot be caused by excitation to a higher Landau level, because the state would have to jump over the mobility gap, and this cannot be done without some heat change. Since heat changes causes dissipation this is not possible. The change of energy, then, must be the result of moving n electrons over the width of the ring. The final question, now, is how we can match n to the number of filled Landau levels in our system.

We are going to make this reasoning plausible by an example. Suppose that the extended states are restricted to the region $|x| > L_x/2 - \delta x$ and the localised states are restricted to the region $-L_x + \delta x < x < L_x - \delta x$. When we (adiabatically) insert one flux quantum into our system, all occupied extended states will move one step in the x-direction. Because no extended state below the Fermi energy can move into the region of the occupied localised states, there must be effectively one electron per Landau level that moves across the region of localised states to the other side of the ring. Thus in this example n is indeed the number of filled Landau levels.

Although we mentioned that the size and geometry of the material we were considering did not matter, Laughlin's argument still seems quite artificial. The value of Laughlin's article [54] lies particularly in the fact that it was the first article that tried to explain the IQHE geometrically. In Section 2.5 we discuss the continuation of this argument. Here the toroidal geometry will not lie in the physical space as in Laughlin's article but in the momentum space (the Brillouin zone).

2.4 Bloch electrons in a uniform magnetic field

In the previous Section we calculated the Hall resistance of an electron gas with disorder, making use of a special geometry. In the next Section we make explicit use of the fact that our conducting material is a crystal, to calculate the Hall resistance (conductance) via the Kubo-formula (see [49] or [26]). Normally we would use Bloch theory to describe a free electron gas in a crystal. The magnetic field, however, breaks down the translation invariance so crucial for this theory. In this Section we introduce the theory that overcomes this problem, the so-called magnetic Bloch theory. We follow the first part of the article by Kohmoto [48].

Consider again an electron gas in the (x, y)-plane, with a magnetic field B perpendicular to it. Again we only consider the single-electron Hamiltonian as we neglect the electron-electron interactions. To take into account the ionic lattice of the crystal we add a light potential U to our Hamiltonian.

$$H = \frac{1}{2m}\left(\vec{p} + e\vec{A}\right)^2 + U(x, y), \tag{2.31}$$

where the potential is periodic in two directions, such that

$$U(x + a, y) = U(x, y + b) = U(x, y). \tag{2.32}$$

Due to the vector potential A our Hamiltonian is not invariant under translations anymore.

The fundamental idea of magnetic Bloch theory is to find some kind of translation that keeps the system as well as the Hamiltonian invariant. These translations are called magnetic translations (see [85]). These translations are constructed as follows.

Consider the Bravais lattice (see [76]) with a and b its unit vectors. Define now for each Bravais lattice vector $\vec{R}$, a translation operator T_R which, acting on a smooth function $f(\vec{r})$, translates the argument with $\vec{R}$:

$$T_R f(\vec{r}) = f(\vec{r} + \vec{R}). \tag{2.33}$$

We can, using the sequence for the exponential, write this translation operator as

$$T_R = e^{\frac{i}{\hbar}\vec{R}\cdot\vec{p}} \tag{2.34}$$

The translated vector potential $A(\vec{r} + \vec{R})$ does not always equal $A(\vec{r})$. Because the magnetic field is uniform, the potential and the translated potential can differ a gradient of a scalar function g:

$$\vec{A}(\vec{r}) = \vec{A}(\vec{r} + \vec{R}) + \vec{\nabla} g(\vec{r}). \tag{2.35}$$

Consider the operators

$$\begin{aligned} \hat{T}_R &= e^{\frac{i}{\hbar}\vec{R}\cdot[\vec{p} + e(\vec{r}\wedge B)/2]} \\ &= T_R e^{\frac{ie}{\hbar}(B\wedge\vec{R})\cdot\vec{r}/2}. \end{aligned} \tag{2.36}$$

If we take the gauge such that $\vec{A} = (B \wedge \vec{r})/2$, we see that to compute

$$[\hat{T}_R, H] \tag{2.37}$$

we only need to know

$$[p_i + eA_i, p_k - eA_k] \quad \text{for} \quad i, k = x, y\,, \tag{2.38}$$

which is zero. We conclude that $\hat{T}_R$ commutes with H. Thus the Hamiltonian is invariant under these operators which are called **magnetic translations**. To use Bloch theory, we need to find functions that at the same time are eigenfunctions of H and $\hat{T}_R$. Note however that the magnetic translations do not commute with each other:

$$\hat{T}_a \hat{T}_b = e^{2\pi\imath\phi} \hat{T}_b \hat{T}_a \tag{2.39}$$

where ϕ is the quantity of flux quanta piercing the unit cell. When $\phi = p/q$ is a rational number (with largest common divisor 1, for p and q), there is a subset of translations that do commute. To find that subset we enlarge the unit cell such that a integer number of flux quanta pierces the unit cell. We can take a Bravais lattice with vectors of the form

$$\vec{R}' = n(q\vec{a}) + m\vec{b}. \tag{2.40}$$

We see that p flux quanta pierce this unit cell, and we call this cell the **magnetic unit cell**. The magnetic translation operators that belong to this new Bravais lattice, commute with each other. Let ψ be a eigenfunction of H and $\hat{T}_{R'}$ simultaneously. The eigenvalues corresponding to $\hat{T}_{qa}$ and $\hat{T}_b$ are given by

$$\begin{aligned} \hat{T}_{qa}\psi &= e^{\imath k_1 qa}\,\psi \\ \hat{T}_b\psi &= e^{\imath k_2 b}\,\psi \end{aligned} \tag{2.41}$$

where k_1 and k_2 are the so-called generalised crystal momenta and make up the **magnetic Brillouin zone**: $0 \leq k_1 \leq 2\pi/qa$, $0 \leq k_2 \leq 2\pi/b$. The eigenfunctions are labelled with the two crystal momenta k_1, k_2 and with a band index α, and they can be expressed as Bloch functions

$$\psi^{(\alpha)}_{k_1 k_2}(x,y) = e^{\imath k_1 x + \imath k_2 y}\, u^{(\alpha)}_{k_1 k_2}(x,y). \tag{2.42}$$

These functions u, satisfy the so-called generalised Bloch conditions

$$\begin{aligned} u^{(\alpha)}_{k_1 k_2}(x+qa,y) &= e^{-\imath\pi py/b}\, u^{(\alpha)}_{k_1 k_2}(x,y) \\ u^{(\alpha)}_{k_1 k_2}(x,y+b) &= e^{-\imath\pi px/qa}\, u^{(\alpha)}_{k_1 k_2}(x,y)\,, \end{aligned} \tag{2.43}$$

and are eigenfunctions of the Hamiltonian

$$\hat{H} = \frac{1}{2m}(\vec{p} + \hbar\vec{k} + e\vec{A})^2 + U(x,y), \tag{2.44}$$

such that

$$H\psi_{k_1 k_2} = e^{\imath k_1 x + k_2 y} \hat{H} u_{k_1 k_2}. \tag{2.45}$$

We now continue as we would continue with Bloch theory. That is, we determine the physics of our system through the spectrum of a single-particle Hamiltonian. This is done in the following Section, making use of the Kubo formula.

2.5 The topological invariance approach

In this section we discuss the approach introduced by Thouless et al. (TKNN) [77], which links the Hall conductance to a topological invariant. A main ingredient in this theory is the use of a Kubo formula for the conductance. See the following Section for the validity of this formula. We do not follow TKNN exactly, but we make use of the conceptually very clear articles of Kohmoto [48] and Watson [79].

Let $U(x, y)$ be a potential, periodic in both dimensions. Consider again the Hamiltonian

$$\hat{H} = \frac{1}{2m}(\vec{p} + \hbar\vec{k} + e\vec{A})^2 + U(x, y), \tag{2.46}$$

corresponding to the generalised Bloch functions of equation (2.43). Note that its eigenvalues depend continuously on $\vec{k}$. For a fixed band index α, the eigenvalues form a band as one varies $\vec{k}$ through the magnetic Brillouin zone. This is called a magnetic sub band. If one applies a small electric field on our system, we can use the linear response theory. In this linear response theory the Hall conductance is given by the Kubo formula (see one of the original papers [49] or, for instance, [26] for a derivation),

$$\sigma_{xy} = -\imath \frac{e^2\hbar}{L_x L_y} \sum_{E^\alpha < E_F < E^\beta} \frac{(v_y)_{\alpha\beta}(v_x)_{\beta\alpha} - (v_x)_{\alpha\beta}(v_y)_{\beta\alpha}}{(E^\alpha - E^\beta)^2}, \tag{2.47}$$

with $L_x L_y$ the area of the system and with E_F the Fermi energy and where the sum is over all states under and above this Fermi level. Note that the sum is over all states, which means that we also have to integrate over $\vec{k}$ within each band. Let us consider the matrix elements of the velocity operator $\vec{v} = (-\imath\hbar\vec{\nabla} + e\vec{A})/m$ by integrating over one magnetic unit cell, i.e.,

$$(\vec{v})_{\alpha\beta} = \delta_{k_1 k_1'} \delta_{k_2 k_2'} \int_0^{qa} dx \int_0^b dy\, u^{\alpha^*}_{k_1 k_2} \vec{v} u^{\beta}_{k_1' k_2'}, \tag{2.48}$$

where the states are normalised such that, $\int_0^{qa} dx \int_0^b dy |u|^2 = 1$. These matrix elements can be expressed in term of partial derivatives of the Hamiltonian $\hat{H}$ by

$$\begin{aligned} (v_x)_{\alpha\beta} &= \frac{1}{\hbar}\langle\alpha|\frac{\partial \hat{H}}{\partial k_1}|\beta\rangle, \\ (v_y)_{\alpha\beta} &= \frac{1}{\hbar}\langle\alpha|\frac{\partial \hat{H}}{\partial k_2}|\beta\rangle\,. \end{aligned} \tag{2.49}$$

First-order perturbation theory gives the change of the wave function u^α due to a change $\delta k_i\, (\partial\hat{H}/\partial k_i)$ $(i = 1, 2)$ of the Hamiltonian as

$$\delta u^\alpha = \sum_\beta (E^\alpha - E^\beta)^{-1} \delta k_i (\frac{\partial \hat{H}}{\partial k_i})_{\alpha\beta} u^\beta\,. \tag{2.50}$$

Thus,

$$\langle\alpha|\frac{\partial\hat{H}}{\partial k_j}|\beta\rangle = (E^\beta - E^\alpha)\langle\alpha|\frac{\partial u^\beta}{\partial k_j}\rangle = -(E^\beta - E^\alpha)\langle\frac{\partial u^\alpha}{\partial k_j}|\beta\rangle\,. \tag{2.51}$$

And we can write the Kubo formula as follows

$$\sigma_{xy} = -\frac{\imath}{L_1L_2}\frac{e^2}{\hbar}\sum_{E_\alpha<E_F<E_\beta}\left[\langle\frac{\partial u^\alpha}{\partial k_2}|\beta\rangle\langle\beta|\frac{\partial u^\alpha}{\partial k_1}\rangle - \langle\frac{\partial u^\alpha}{\partial k_1}|\beta\rangle\langle\beta|\frac{\partial u^\alpha}{\partial k_2}\rangle\right]. \tag{2.52}$$

Changing the sum $\sum_{E_F<E^\beta}$ into $\sum_{E^\beta} - \sum_{E^\beta<E_F}$ and using the fact that $\sum_{E_\beta}|\beta\rangle\langle\beta| = 1$ we obtain

$$\sigma_{xy} = \frac{\imath}{L_1L_2}\frac{e^2}{\hbar}\sum_{E_\alpha<E_F}\left[\langle\frac{\partial u^\alpha}{\partial k_1}|\frac{\partial u^\alpha}{\partial k_2}\rangle - \langle\frac{\partial u^\alpha}{\partial k_2}|\frac{\partial u^\alpha}{\partial k_1}\rangle\right]. \tag{2.53}$$

If the Fermi level is in a subband gap this reduces to

$$\sigma_{xy}^{(\alpha)} = \frac{\imath}{2\pi}\frac{e^2}{h}\int d^2k\int d^2r\left[\frac{\partial u^{\alpha^*}}{\partial k_2}\frac{\partial u^\alpha}{\partial k_1} - \frac{\partial u^{\alpha^*}}{\partial k_1}\frac{\partial u^\alpha}{\partial k_2}\right] \tag{2.54}$$

for the contribution to the Hall conductance of a filled band α. The integrals are over the magnetic Brillouin zone and over the magnetic unit cell respectively.

Introduce now a vector field in the magnetic Brillouin zone by

$$\hat{A}(k_1,k_2) = \int d^2r u^*_{k_1k_2}\vec{\nabla}_k u_{k_1k_2} = \langle u_{k_1k_2}|\vec{\nabla}_k|u_{k_1k_2}\rangle \tag{2.55}$$

where $\vec{\nabla}_k$ is the vector operator with the components $\partial/\partial k_1$ and $\partial/\partial k_2$. The contribution of the band index α can be written as

$$\sigma_{xy}^{(\alpha)} = \frac{\imath}{2\pi}\frac{e^2}{h}\int d^2k\left[\vec{\nabla}_k\times\hat{A}(k_1,k_2)\right]_3, \tag{2.56}$$

where $[\cdot]_3$ stands for the third component.Remember that the magnetic Brillouin zone is actually a torus $\mathbb{T}^2$, because we can identify $(k_1,0)$ with $(k_1,2\pi/b)$ and $(0,k_2)$ with $(2\pi/(qa),k_2)$.

Using Stokes' theorem, and remarking that a torus has no boundary, we could conclude from equation (2.56) that the contribution $\sigma_{xy}^{(\alpha)}$ is zero. The reason that this is not the case comes from that fact that the vector field $\hat{A}$ can not be uniquely defined on the whole torus. In fact, the vector field $\hat{A}$ can only be defined locally. In Section 3.4 we give an explicit vector bundle over the torus $\mathbb{T}^2$ and show how $\hat{A}$ transforms given a change of frame. Thereafter we show that the contribution of equation (2.55) must be an integer times $\frac{e^2}{h}$. This is done by giving the relation between the integrand $\vec{\nabla}_k\times\hat{A}(k_1,k_2)$ and the first Chern class defined in Chapter 3. This relation was first proposed by Avron et. al. in [6]. In their article, they introduce the techniques of algebraic topology in the integer quantum Hall effect. This approach is the starting point of the generalisation proposed by Bellissard. For now we give a more intuitive argument for the quantisation of equation (2.56).

The fact that the vector field $\hat{A}$ can not be defined uniquely is a consequence of the generalised Bloch conditions of equation (2.43). The wave functions pick up an extra phase $2\pi p$ as they travel around the magnetic unit cell defined in Section 2.4. The quantity p is exactly the number of flux quanta piercing the magnetic unit cell. Hence p is a real physical quantity and therefore gauge invariant. This extra phase factor implies that the wave function is zero in at least one point in the magnetic unit cell. If that is the case, $\hat{A}$ is indeed not well defined over the entire torus.

The trick is to define the vector field $\hat{A}$ locally within some regions $\{U_i\}$. If the regions U_i and U_j overlap we need to have some transition functions ϕ_{ij} such that the $\hat{A}_i$ in U_i equals $\hat{A}_j$ in U_j times the transition function. Such a transition function is given by

$$\phi_{ij}(k_1, k_2) = e^{\imath\chi(k_1,k_2)}. \tag{2.57}$$

Let us assume there is only one such zero point of $u_{k_1k_2}$. The torus is now split in two regions U_1 and U_2 such that the zero point lies only in one of them. The difference between the vector fields defined on the different regions is given by

$$\hat{A}_1(k_1, k_2) = \hat{A}_2(k_1, k_2) + \imath\vec{\nabla}_k\chi(k_1, k_2). \tag{2.58}$$

The contribution of the band α to the Hall conductance can consequently be written as

$$\begin{aligned} \sigma_{xy}^{(\alpha)} &= \frac{e^2}{h}\frac{1}{2\pi\imath}\Big\{\int_{U_1} d^2k\big[\vec{\nabla}_k \times \hat{A}_1\big]_3 + \int_{U_2} d^2k\big[\vec{\nabla}_k \times \hat{A}_2\big]_3\Big\} \\ &= \frac{e^2}{h}\frac{1}{2\pi\imath}\int_{\partial U} d\vec{k}\cdot\big[\hat{A}_1 - \hat{A}_2\big], \end{aligned} \tag{2.59}$$

where the minus sign in the second equation comes from the orientation of the boundary ∂U for U_2 compared to U_1. Now we have

$$\sigma_{xy}^{(\alpha)} = \frac{e^2}{h}n, \tag{2.60}$$

with

$$n = \frac{1}{2\pi}\int_{\partial H} d\vec{k}\cdot\vec{\nabla}_k\chi(k_1, k_2), \tag{2.61}$$

where n must be an integer because, after a complete trip around the boundary, the phase difference must be a integer value times 2π.

This explains the integral quantisation of the conductance in the integer quantum Hall effect. Note however, that we did not included disorder in this model. As we mentioned, disorder is necessary to explain the plateaux and the robustness of the integer quantum Hall picture. See [51] for a discussion of this model with disorder.

2.6 Gaps and assumptions in the theories

In this Section we discuss the assumptions we made explicitly or implicitly in the theory of Sections 2.2 to 2.5. Furthermore, we discuss the gaps of the theory in this Sections.

As we mentioned in Section 2.1 the IQHE occurs at high magnetic field and low temperatures. In the experiment of von Klitzing these quantities were respectively 18 T and 1.5 K. With these figures one can calculate that the energy $\hbar\omega_c$ is about a hundred times greater than the thermal energy $k_B T$. If the system also satisfies the inequality $\omega_c \tau_0 >> 1$, we say that the magnetic field is extremely strong (see [49] and [66]). We always assume this situation.

In Section 2.4 we made explicit use of the fact that the system we are considering is an electron gas in a crystal. The problem with such a system is that, due to the ion lattice, there can exist configurations which are lower in energy than the ground state without the lattice. A consequence is that the states of the system are hard to calculate. The system can be simplified in the following way.

The first simplification is that we neglect the motion of the ions in the lattice. A consequence is that an electron can not lose thermal energy to the ion, so a collision between them is elastic. Dissipation, that is, the loss of energy through heat, can only be caused by scattering between the electrons themselves. This approximation is the so-called adiabatic or Born-Oppenheimer approximation.

The second is that we assume the electrons in the core of an atom to be hardly contributing to the dynamics of our system. Their contribution to the system is mainly the excluding factor due to electrostatic forces and the Pauli principle. So the dynamics of our gas is governed by the electrons at the Fermi level.

The third simplification of our system we already assumed in the previous Sections. It is the use of the single-electron Hamiltonian to describe our system. This approximation is known as the density functional theory.

Assuming these simplifications we can use Bloch theory, that is, the states of our system can be written in the form $u_k(r)e^{\imath kr}$ with u having the periodicity of the lattice, $u_k(r+l) = u_k(r)$. The band structure of the system is given by the graph of the energy E_k against k. Using the Brillouin zone one can define the different band indices. One advantage of this theory is that E_k is always continuous within a band index. This is also valid for the magnetic Bloch theory discussed in Section 2.4.

Due to the first simplification of our system we concluded that dissipation is caused by scattering between electrons. The other two simplifications make sure that dissipation can only be caused by scattering between an occupied and an unoccupied state near the Fermi level. In Section 2.3 this was already the case, because in that Section we discussed an electron gas (without ion lattice) within the third simplification.

Although we know that the IQHE is insensitive to geometrical changes of the sample, one could dispute whether the Laughlin argument of Section 2.3 is somewhat far-fetched. There are experimental realisations of this geometry, called Corbino discs (see for instance [37] and [72]), but most of the experimental samples of the IQHE are just planes. In the Laughlin argument one relates to every flux quantum through the ring a displacement of n electrons from one side of the ring to the other. In quantum theory however there is no reason why this number n should always be the same (see for instance [4]). The conductance is related to the average of electrons transferred per flux quantum through the ring. This quantity is represented by the Kubo formula.

This Kubo formula is one of the main ingredients of the theory discussed in Section 2.5 based on [77]. The other one is the gauge invariance on the toroidal geometry in momentum space further developed in [6]. Although the domain of validity is unclear, we know from the theory and experiments that it is valid in the case of the IQHE. See for instance [9], [20], [35], [50], [61] and [74]

An advantage of this formulation of the IQHE is that it can be generalised to multi-particle Hamiltonians (see for instance [5], [62] and [75]). Although the generalisation to multi-particle Hamiltonians is not really necessary to explain the IQHE, it is very useful from the fundamental physics point of view. Fundamental in the sense that the theory is more complete, and that the theory could as well explain other observations.

There are, however, two unsatisfactory elements of the method introduced in Section 2.5. First, the domain of validity of the magnetic Bloch theory discussed in Section 2.4. This magnetic Bloch theory is only valid with the assumption that the flux through the unit cell is rational. This restriction is not too strong, since the rational numbers lie dense in the reals, but it is quite an unphysical assumption. Second, we did not discuss disorder in this model. Disorder is required to form the plateaux and ensures us of the robustness of the conductance. Without disorder, the model predicts only the quantisation and the numerical value of the conductance at isolated values of the magnetic field. For a discussion of disorder in this model we refer to [51].

The remainder of this thesis discusses a theory based on the Kubo formula of Section 2.5 and noncommutative geometry of Connes (see [29] and [31]) introduced by Bellissard (see [11]-[13]).

The early works on the IQHE discussed in the former Sections are chosen such that the they form a 'natural' history towards the theory we discuss in this thesis. Of course physics does not work that way, and there are some theories we do not discuss and still are relevant in modern theories. One of them is a field theoretical approach. See, for instance, the contribution of Pruisken in [66] Chapter 5.

Chapter 3

The classical Chern character

In this Chapter we discuss a topological invariant called the (classical) Chern character. With this character we are able to express the Kubo formula, discussed in Section 2.5, in terms of a topological invariant. This is done in the last Section of this Chapter. In the Chapters to come we introduce a manner to generalise this Chern character to the noncommutative case. This so-called noncommutative Chern character is a key ingredient for the noncommutative theory of the IQHE.

3.1 Invariant polynomials and Chern classes

In this section we discuss the theory of invariant polynomials, to define characteristic classes. These classes are defined on vector bundles $E \xrightarrow{\pi} M$ and lie in cohomology classes of the base space M. Cohomology classes are topological invariants, i.e. they are equal for spaces M and N if there exist a homeomorphism $f : M \to N$. One such characteristic class will be given explicitly, the Chern class ([27]). For more on characteristic classes see for instance [34], [40] and [59].

Let $A \in M_n(\mathbb{C})$, the set of all $n \times n$-matrices in $\mathbb{C}$. Let $P(A)$ be a polynomial in the components of A. We call $P(A)$ an invariant or characteristic polynomial if for every $g \in GL(n, \mathbb{C})$, one has

$$P(g^{-1}Ag) = P(A). \tag{3.1}$$

Examples of such polynomials are $\mathrm{Det}(\mathbb{1} + A)$ and $\mathrm{Tr}A$, which are used to define the Chern classes and Chern character. Consider the set of eigenvalues

$\{\lambda_1, \ldots, \lambda_n\}$ of a matrix A. Because of property (3.1), every invariant polynomial $P(A)$ is a symmetric function of these λ_i, and we can write $P(A)$ as a polynomial in $S_j(A)$, defined by

$$S_j(A) = \sum_{i_1 < i_2 < \cdots < i_j} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_j}. \tag{3.2}$$

Every invariant polynomial P defines a k-linear symmetric form $\widetilde{P} : M_n(\mathbb{C}) \times \cdots \times M_n(\mathbb{C}) \to \mathbb{C}$, called the **polarisation** of P, in the following manner. Consider for every $(A_1, \ldots, A_k)$, the expansion $(t_1 A_1 + \cdots + t_k A_k)^k$. Denote the coefficient of $t_1 \ldots t_k$ by $[A_1 \cdots A_k]$. Then

$$\widetilde{P}(A_1, \ldots, A_k) := \frac{1}{k!} P([A_1 \cdots A_k]) \tag{3.3}$$

is symmetric and invariant, by construction.

Example 3.1.1. Consider the polynomial $P(A) = \mathrm{Tr}(A)$. The polarisation $\widetilde{P}(A_1, A_2, A_3)$ is then equal to

$$\frac{1}{6}\mathrm{Tr}([A_1 A_2 A_3]) = \frac{1}{2}\mathrm{Tr}(A_1 A_2 A_3 + A_2 A_1 A_3) \tag{3.4}$$

due to the cyclicity of the trace.

Consider a complex (smooth) vector bundle E over M. Let $\Gamma^\infty(M, E)$ be the set of (smooth) sections over the vector bundle, and $\Omega^k(M)$ the set of k-forms on M. Remember that an connection ∇ on E is an operator

$$\nabla : \Gamma^\infty(M, E) \to \Gamma^\infty(M, E) \otimes \Omega^1(M) \tag{3.5}$$

that satisfies the relation

$$\nabla(\sigma f) = (\nabla\sigma) f + \sigma \otimes df, \tag{3.6}$$

for $f \in C^\infty(M)$ and $\sigma \in \Gamma^\infty(M, E)$. We can extend this connection to an operator

$$\nabla : \Gamma^\infty(M, E) \otimes \Omega^p(M) \to \Gamma^\infty(M, E) \otimes \Omega^{p+1}(M) \tag{3.7}$$

by enforcing the Leibniz rule

$$\nabla(\sigma \otimes \phi) = (\nabla\sigma) \otimes \phi + \sigma \otimes d\phi, \tag{3.8}$$

for ϕ a p-form and σ a section.

If ω is the matrix-valued 1-form, corresponding to the connection ∇, we define the curvature Θ as the matrix-valued 2-form

$$\Theta = d\omega + \omega \wedge \omega. \tag{3.9}$$

A simple calculation shows that $\Theta = \nabla^2$.

We are able to define characteristic polynomials on matrix-valued p-forms $A\eta$ through the relation

$$P(A\eta) = \eta P(A) \tag{3.10}$$

for η a p-form and A a $n\times n$-matrix. The polarised polynomials on matrix-valued p-forms are

$$\widetilde{P}(\eta_1 A_1, \ldots, \eta_k A_k) \tag{3.11}$$

with η_i the p-forms and A_i the matrices. The invariant polynomials defined on the curvature 2-form Θ lie in cohomology classes, as we see in the following two propositions.

Proposition 3.1.2. *If P is an invariant polynomial and Θ the curvature, then $P(\Theta)$ is closed.*

Proof. Consider the infinitesimal transformation $g = \mathbb{1}+g'$. Because $P(g\Theta g^{-1}) = P(\Theta)$, and using a power series expansion of g, we obtain

$$\sum_i \widetilde{P}(\Theta, \ldots, g'\Theta_i - \Theta_i g', \ldots \Theta) = 0, \tag{3.12}$$

for the polarised polynomial of $P(\Theta)$. By multi-linearity of the polynomial we find

$$\sum_i \widetilde{P}(\Theta, \ldots, \omega \wedge \Theta_i, \ldots, \Theta) = \sum_i \widetilde{P}(\Theta, \ldots, \Theta_i \wedge \omega, \ldots, \Theta). \tag{3.13}$$

Hence

$$d\widetilde{P}(\Theta, \ldots, \Theta) = \sum_i \widetilde{P}(\Theta, \ldots, \nabla\Theta, \ldots, \Theta), \tag{3.14}$$

where

$$\nabla\Theta = d\Theta + \omega \wedge \Theta - \Theta \wedge \omega = 0 \tag{3.15}$$

is the covariant differential of Θ, which is zero because of the Bianchi identity. We conclude that

$$dP(\Theta) = 0. \tag{3.16}$$

□

Proposition 3.1.3. *Let Θ and Θ' be two curvatures corresponding to, respectively, the connections ω and ω' on the vector bundle E. Let P be an invariant polynomial. The difference $P(\Theta) - P(\Theta')$ is exact.*

Proof. Let ω and ω' be two different connections on the bundle and let their curvatures be Θ and Θ' respectively. Consider the interpolation between the two connections

$$\omega_t = \omega + t\eta \quad 0 \leq t \leq 1, \tag{3.17}$$

where $\eta = \omega' - \omega$. Then

$$\begin{aligned}\Theta_t &= d\omega_t + \omega_t \wedge \omega_t \\ &= \Theta + t\nabla\eta + t^2\eta \wedge \eta.\end{aligned} \tag{3.18}$$

Let $\widetilde{P}(A, \ldots, A)$ be the polarisation of P and define

$$T(B, A) = l\widetilde{P}(B, \underbrace{A, \ldots, A}_{l-1 \text{ times}}). \tag{3.19}$$

Then

$$\frac{d}{dt}\widetilde{P}(\Theta_t, \ldots, \Theta_t) = T(\nabla\eta, \Theta_t) + 2tT(\eta \wedge \eta, \Theta_t). \tag{3.20}$$

On the other hand, we have

$$\begin{aligned}\nabla\Theta_t &= t\nabla^2\eta + t^2\nabla(\eta \wedge \eta) \\ &= t(\Theta \wedge \eta - \eta \wedge \Theta) + t^2(\nabla\eta \wedge \eta - \eta \wedge \Theta) \\ &= t(\Theta_t \wedge \eta - \eta \wedge \Theta_t).\end{aligned} \tag{3.21}$$

So, using (3.14),

$$\begin{aligned}dT(\eta, \Theta_t) &= T(\nabla\eta, \Theta_t) - l(l-1)\widetilde{P}(\eta, \nabla\Theta_t, \Theta_t, \ldots, \Theta_t) \\ &= T(\nabla\eta, \Theta_t) - l(l-1)t\widetilde{P}(\eta, (\Theta_t \wedge \eta - \eta \wedge \Theta_t), \Theta_t, \ldots, \Theta_t)\end{aligned} \tag{3.22}$$

And again using the multi-linearity as in (3.13) we obtain

$$2T(\eta \wedge \eta) = -l(l-1)\widetilde{P}(\eta, (\Theta_t \wedge \eta - \eta \wedge \Theta_t), \Theta_t, \ldots, \Theta_t)\,. \tag{3.23}$$

Combining the last two equations gives

$$dT(\eta, \Theta_t) = T(\nabla\eta, \Theta_t) + 2tT(\eta \wedge \eta, \Theta_t) = \frac{d}{dt}P(\Theta_t). \tag{3.24}$$

Thus

$$P(\Theta_t') - P(\Theta_t) = d\int_0^1 T(\eta, \Theta_t)dt \equiv dT(\omega', \omega). \tag{3.25}$$

□

These invariant polynomials defined on the curvature of a vector bundle $E \to M$, modulo exact forms, are called **characteristic** classes, and are denoted by

$$P(E) = \Big[P(\Theta)\Big] \in H^{2j}_{dR}(M). \tag{3.26}$$

Propositions 3.1.2 and 3.1.3, make sure that these classes are well-defined and lie in the de Rham cohomology classes.

Proposition 3.1.4. *Let P be an invariant polynomial and $E \to M$ a vector bundle.*

1. *The map*
$$P \to P(E) \in H^{2j}_{dR}(M) \tag{3.27}$$
is a homomorphism, called the **Weil homomorphism**.

2. *Let $f : N \to M$ be a differentiable map. With f^*E the pullback bundle of E, we have*
$$P(f^*E) = f^*P(E). \tag{3.28}$$

Proof.

1. Let P_k and Q_l be two invariant polynomials, and denote the curvature Θ as $T_i\Theta^i$. Then,
$$\begin{aligned}(P_kQ_l)(\Theta) = &\Theta^{i_1} \wedge \cdots \wedge \Theta^{i_k}\Theta^{j_1} \wedge \cdots \wedge \Theta^{j_l} \times \\ &\frac{1}{(k+l)!}\widetilde{P}_k(T_{i_1}, \ldots, T_{i_k})\, \widetilde{Q}_l(T_{j_1}, \ldots, T_{j_l}) \\ = &P_k(\Theta) \wedge P_l(\Theta).\end{aligned} \tag{3.29}$$

2. Let ω_i and ω_j be two local connections in the intersecting charts U_i and U_j of M. Let ϕ_{ij} be a transition function on $U_i \cap U_j$. The pullback $f^*\phi_{ij} = \phi_{ij}f$ is a transition function on the vector bundle f^*E. The corresponding $f^*\omega_i$ and $f^*\omega_j$ are related as
$$\begin{aligned}f^*\omega_j &= f^*({\phi_{ij}}^{-1}\omega_i\phi_{ij} + {\phi_{ij}}^{-1}d\phi_{ij}) \\ &= (f^*{\phi_{ij}}^{-1})(f^*\omega_i)(f^*\phi_{ij}) + (f^*{\phi_{ij}}^{-1})(f^*d\phi_{ij}).\end{aligned} \tag{3.30}$$
Hence $f^*\omega$ is a (local) connection on the bundle f^*E. The corresponding curvature is given by
$$d(f^*\omega_i) + f^*\omega_i \wedge f^*\omega\wedge = f^*(d\omega_i + \omega_i \wedge \omega_i) = f^*\Theta_i. \tag{3.31}$$
Therefore $f^*P(\Theta_i) = P(f^*\Theta_i)$, and equation (3.28) follows.

□

Remark 3.1.5. In equation (3.31) we made use of the fact that for a differentiable map f, its pullback f^* commutes with the exterior differential d. Due to this fact $H^k_{dR}(M) \simeq H^k_{dR}(N)$, whenever there exists a diffeomorphism $f : N \to M$. Moreover, $H^k_{dR}(M)$ is isomorphic to the cohomology with compact support vector space $H^k(M, \mathbb{R})$ (Alexander-Spanier cohomology). This ensures us that $H^k_{dR}(M) \simeq H^k_{dR}(N)$, whenever there exists a homeomorphism $f : N \to M$. This makes $P(E)$ a topological invariant for the base space M.

The Chern classes are defined through the Chern forms. We define the Chern forms as follows.

Definition 3.1.6. *The* **total Chern form** $c(\Theta)$ *is the polynomial*

$$c(\Theta) = \mathrm{Det}(\mathbb{1} + \frac{\imath}{2\pi}\Theta) = 1 + c_1(\Theta) + c_2(\Theta) + \cdots , \tag{3.32}$$

where the $c_j(\Theta)$*, the polynomials of degree* j*, are called the* **individual Chern forms**.

If $\{\lambda_1, \ldots, \lambda_k\}$ are the eigenvalues of Θ, we have the equation,

$$\mathrm{Det}(\mathbb{1} + \frac{\imath}{2\pi}\Theta) = \prod_{j=1}^{k}(1 + \frac{\imath}{2\pi}\lambda_j). \tag{3.33}$$

So the explicit expression of $c_j(\Theta)$ is given by

$$c_j(\Theta) = S_j(\frac{\imath}{2\pi}\Theta). \tag{3.34}$$

Because $c_j(\Theta) \in \Lambda^{2j}T^*M$,

$$c_j = 0 \quad \text{if} \quad 2j > \mathrm{dim}M \tag{3.35}$$

so $c(\Theta)$ will always be a finite sum.

Propositions 3.1.2 and 3.1.3 give a cohomology class $c_j(E)$

$$c_j(E) = \Big[c_j(\Theta)\Big] \in H^{2j}_{dR}(M). \tag{3.36}$$

These classes are called **Chern classes**. They are independent of the connection, because $c_j(\Theta) - c_j(\Theta')$ is exact. We define the **total Chern class** as $c(E) = c_0(E) + c_1(E) + \cdots$.

3.2 The Chern character

In this section we discuss some properties of the Chern classes and define the Chern character.

Working with vector bundles, we would like to have some operations on them. We define here two such operations.

Definition 3.2.1. *Let* E *and* F *be two vector bundles over* M*, with projections* π_E *and* π_F *, respectively. The* **Whitney sum** $E \oplus F$ *over* E *and* F *is defined as the pullback bundle* $f^*(E \times F)$*, with* $f : M \to M \times M$ *defined by* $f(x) = (x, x)$*. Hence,* $f^*(E \times F) = \{\, (x, u, v) \in M \times E \times F \,|\, f(x) = \big(\pi_E(u), \pi_F(v)\big) \,\}$ *and* *the projections $\pi_f : f^*(E \times F) \to E \times F$ and $\pi : f^*(E \times F) \to M$ defined by $\pi_f(x, u, v) = (u, v)$ and $\pi(x, u, v) = x$ make the following diagram commute.*

$$\begin{array}{ccc} f^*(E \times F) & \xrightarrow{\pi_f} & E \times F \\ \pi \downarrow & & \downarrow (\pi_E, \pi_F) \\ M & \xrightarrow{f} & M \times M \end{array} \tag{3.37}$$

The fibre $(E \oplus F)_x$ is given by $\pi^{-1}(x) = \pi_E{}^{-1}(x) \oplus \pi_F{}^{-1}(x) \simeq E_x \oplus F_x$, where $E_x \oplus F_x$ is the direct sum between the vector spaces E_x and F_x. Let $\phi_E \in GL(m, \mathbb{C})$ and $\phi_F \in GL(n, \mathbb{C})$ be transition functions for, respectively, E and F. The matrix

$$\phi_{E \oplus F} = \begin{pmatrix} \phi_E & 0 \\ 0 & \phi_F \end{pmatrix} \in GL(m + n, \mathbb{C}) \tag{3.38}$$

is then an transition function for the vector bundle $E \oplus F$. We can write the connection $\nabla_{E \oplus F}$ of $E \oplus F$ also in this form:

$$\nabla_{E \oplus F} = \nabla_E \oplus \nabla_F = \begin{pmatrix} \nabla_E & 0 \\ 0 & \nabla_F \end{pmatrix} \tag{3.39}$$

Definition 3.2.2. *Let E and F be two vector bundles over M, with projections π_E and π_F, respectively. The* **tensor product bundle** *$E \otimes F \xrightarrow{\pi} M$ is defined as*

$$E \otimes F = \{(e \otimes f, x) \in E_x \otimes F_x \times M\}, \tag{3.40}$$

with projection $\pi(e \otimes f, x) = x = \pi_E(e) = \pi_F(f)$, where $E_x \otimes F_x$ is the tensor product of the vector spaces E_x and F_x.

Lemma 3.2.3.

1. *If $f : M \to N$ is some C^∞ map and $E \to N$ a complex vector bundle, then*
$$c_j(f^*E) = f^* c_j(E). \tag{3.41}$$

2. *Let $E \to M$ and $F \to M$ be two vector bundles. We have*
$$c(E \oplus F) = c(E) \wedge c(F). \tag{3.42}$$

3. *For E a vector bundle of rank r and L a line bundle, we have*
$$c_1(E \otimes L) = c_1(E) + r c_1(L). \tag{3.43}$$

Proof.

1. This follows immediately from Proposition 3.1.4.

2. If Θ_E and Θ_F are the curvatures of respectively E and F, equation (3.39) gives us for the curvature $\Theta_{E\oplus F}$ of $E \oplus F$
$$\Theta_{E\oplus F} = \Theta_E \oplus \Theta_F = \begin{pmatrix} \Theta_E & 0 \\ 0 & \Theta_F \end{pmatrix}. \tag{3.44}$$
Therefore,
$$\begin{aligned} \mathrm{Det}(\mathbb{1} + \frac{\imath}{2\pi}\Theta_{E\oplus F}) &= \mathrm{Det}(\mathbb{1} + \frac{\imath}{2\pi}\Theta_E) \cdot \mathrm{Det}(\mathbb{1} + \frac{\imath}{2\pi}\Theta_F) \\ &= c(\Theta_E) \cdot c(\Theta_F). \end{aligned} \tag{3.45}$$
Hence (3.42)follows.

3. If Θ_E and Θ_L are the curvatures of respectively E and L, one can show that the curvature $\Theta_{E\otimes L}$ of $E \otimes L$ is given by
$$\Theta_{E\otimes L} = \Theta_E \otimes \mathbb{1} + \mathbb{1} \otimes \Theta_L. \tag{3.46}$$
Hence
$$c_1(E \otimes L) = \left[\frac{\imath}{2\pi}\mathrm{Tr}\Theta_{E\otimes L}\right] = c_1(E) + rc_1(L). \tag{3.47}$$

□

Note that property 3 of Lemma 3.2.3 cannot be extended to every vector bundle L. Inspired by the Chern classes we define a invariant polynomial that is well defined under the tensor product of vector bundles.

Definition 3.2.4. *The* **Chern character** *is defined as*
$$ch(E) = \mathrm{Tr}e^{\frac{\imath}{2\pi}\Theta} = \sum_j \frac{1}{j!}\mathrm{Tr}(\frac{\imath}{2\pi}\Theta)^j, \tag{3.48}$$
with Θ*, again, the curvature* 2*-form and* Tr *the normal trace over matrices.*

If we use the eigenvalue expression for Ω we find
$$ch(E) = \sum_{i=1}^{k} e^{\frac{\imath}{2\pi}\lambda_i} = \sum_{i=1}^{k}\sum_j \frac{1}{j!}(\frac{\imath}{2\pi}\lambda_i)^j, \tag{3.49}$$
and we can express the Chern character in terms of the Chern classes:
$$ch(E) = k + c_1(E) + \frac{1}{2}(c_1^2 - 2c_2)(E) + \frac{1}{6}(c_1^3 - 3c_1c_2 - 3c_3)(E) + \cdots. \tag{3.50}$$
With this fact, Lemma 3.2.3 and Proposition 3.1.4, one sees that the Chern character lies in a de Rham cohomology class. Hence the Chern character is indeed a topological invariant of the base space M.

Because the transition functions lie in $GL(k, \mathbb{C})$, the connection (and curvature) takes values in the corresponding Lie algebra $gl(k, \mathbb{C})$. The Chern character is then well defined under the tensor product of vector bundles.

Lemma 3.2.5.

1. *If $f : M \to N$ is some C^∞ map and $E \to N$ a complex vector bundle, then*
$$ch(f^*E) = f^*ch(E). \tag{3.51}$$
2. *Let $E \to M$ and $F \to M$ be two vector bundles. We have*
$$\begin{aligned} ch(E \oplus F) &= ch(E) + ch(F). \\ ch(E \otimes F) &= ch(E) \wedge ch(F) \end{aligned} \tag{3.52}$$

Proof.

1. This follows directly from Lemma 3.2.3.
2. This follows directly from the construction of the Chern character:
$$\begin{aligned} ch(E \oplus F) = \mathrm{Tr} e^{\frac{\imath}{2\pi}\Theta_{E\oplus F}} &= \sum_j \frac{1}{j!}\mathrm{Tr}(\frac{\imath}{2\pi}\Theta_{E\oplus F})^j \\ &= \sum_j \frac{1}{j!}\mathrm{Tr}\big[(\frac{\imath}{2\pi}\Theta_E)^j + (\frac{\imath}{2\pi}\Theta_F)^j\big] = ch(E) + ch(F). \end{aligned} \tag{3.53}$$
And
$$\begin{aligned} ch(E \otimes F) &= \sum_j \frac{1}{j!}\mathrm{Tr}(\frac{\imath}{2\pi}\Theta_E \otimes \mathbb{1} + \mathbb{1} \otimes \Theta_F)^j \\ &= \sum_j \frac{1}{j!}\big(\frac{\imath}{2\pi}\big)^j \sum_i \binom{j}{i} \mathrm{Tr}{\Theta_E}^i \mathrm{Tr}{\Theta_F}^{j-i} \\ &= ch(E) \wedge ch(F). \end{aligned} \tag{3.54}$$

□

3.3 The group $K^0(M)$

In this Section we discuss the semigroup structure on the set of all vector bundles over a compact Hausdorff space M, and introduce the Grothendieck construction to make a group structure.

Let E and F be two vector bundles over M. We say that they are isomorphic, and write $E \cong F$, if there exists a homeomorphism $\varphi : E \to F$, such that each $\varphi|_{E_x}$ is a linear isomorphism between vector spaces.

Consider $V(M)$, the set of all vector bundles over M, modulo isomorphisms. This set is a Abelian semigroup with the Whitney sum as addition,

$$[E] + [F] := [E \oplus F]. \tag{3.55}$$

It is easy to see that $V(M)$ is indeed Abelian and associative. Note that $V(M)$ is really a semigroup and not a group, for we do not have an inverse.

Example 3.3.1. Consider the two-sphere S^2. Let E be the tangent bundle $T(S^2)$ over S^2 and E' be the trivial bundle I^2 over S^2. Then

$$\begin{aligned} T(S^2) \oplus I^1 &\simeq I^3, \\ I^2 \oplus I^1 &\simeq I^3, \end{aligned} \tag{3.56}$$

but $T(S^2)$ is not isomorphic with I^2.

We see that summing with trivial bundles helps to cancel inequalities, due to low dimension. With this in mind we define an equivalence class.

Definition 3.3.2. *The vector bundles E and E' are called* **stably equivalent**, *written $E \overset{s}{\simeq} E'$ or $[E]_s = [E']_s$, if*

$$E \oplus I^j \simeq E' \oplus I^k \quad \textit{for some} \quad j, k \in \mathbb{N}. \tag{3.57}$$

We see that in our previous example, $T(S^2) \overset{s}{\simeq} I^2$. We know, by a theorem of Swan, that for each vector bundle E over a compact Hausdorff space M, we can find an F over M such that $E \oplus F$ is a trivial bundle. Because the trivial bundles act as units in the stable equivalence, we can introduce a formal difference between the vector bundles,

$$E \ominus E' := E \oplus E'', \tag{3.58}$$

whenever $E' \oplus E''$ is a trivial bundle. Thus we can define an inverse and we have a group structure on the set of stable equivalence classes of vector bundles over M. We denote this group as $K^0(M)$.

To every Abelian semigroup S belongs an associated Abelian group $G(S)$, called the **Grothendieck group**. The Grothendieck group of $V(M)$ is isomorphic to $K^0(M)$ and will later be used in the construction of K-theory. The Grothendieck group of any semigroup S, is constructed as follows. Define an equivalence relation $\sim$ on $S \times S$ as

$$(x, y) \sim (x', y') \quad \text{if} \quad \exists\, z \in S : x + y' + z = x' + y + z. \tag{3.59}$$

The equivalence class of (x, y) is written as $[x, y]_0$. The set $G(S) := S \times S/\sim$, is an Abelian group with group operation

$$[x, y]_0 + [x', y']_0 = [x + x', y + y']_0, \tag{3.60}$$

and inverse and identity

$$[x, y]_0^{-1} = [y, x]_0, \qquad 0 = [x, x]_0. \tag{3.61}$$

Proposition 3.3.3. *The groups $K^0(M)$ and $G(V(M))$ are isomorphic.*

Proof. Consider the map $\phi : K^0(M) \to G(V(M))$, defined by

$$[E]_s \mapsto \big[[E]_s + [H]_s, [H]_s\big]_0, \tag{3.62}$$

for some $[H]_s \in K^0(M)$. Note that ϕ is independent of the choice of $[H]_s$, because $\big[[E]_s + [H]_s, [H]_s\big]_0 = \big[[E]_s + [G]_s, [G]_s\big]_0$.

1. ϕ is a group homomorphism:

$$\begin{aligned} \phi\big([E]_s + [F]_s\big) &= \phi\big([E \oplus F]_s\big) = \big[[E \oplus F]_s + [H]_s, [H]_s\big]_0 \\ &= \big[[E]_s + [F]_s + 2[H]_s, 2[H]_s\big]_0 \\ &= \big[[E]_s + [H]_s, [H]_s\big]_0 + \big[[F]_s + [H]_s, [H]_s\big]_0 \\ &= \phi\big([E]_s\big) + \phi\big([F]_s\big). \end{aligned} \tag{3.63}$$

2. ϕ is injective:
If $\phi\big([E]_s\big) = 0$, then

$$\big[[E]_s + [H]_s, [H]_s\big]_0 = 0 = \big[[F]_s, [F]_s\big]_0. \tag{3.64}$$

Thus

$$[E]_s + [H]_s + [F]_s + [G]_s = [F]_s + [H]_s + [G]_s, \tag{3.65}$$

for some $[G]_s \in K^0(M)$. And

$$[E]_s = 0. \tag{3.66}$$

3. ϕ is surjective
Consider $\big[[E]_s, [F]_s\big]_0 \in G(V(M))$.

$$\begin{aligned} \big[[E]_s, [F]_s\big]_0 &= \big[[E]_s + 2[H]_s, [F]_s + 2[H]_s\big]_0 \\ &= \big[[E]_s + [H]_s, [H]_s\big]_0 + \big[[H]_s, [F]_s + [H]_s\big]_0 \\ &= \phi\big([E]_s\big) - \big[[F]_s + [H]_s, [H]_s\big]_0 \\ &= \phi\big([E]_s\big) - \phi\big([F]_s\big) = \phi\big([E]_s - [F]_s\big). \end{aligned} \tag{3.67}$$

□

Proposition 3.3.4. *The Chern character ch is a ring isomorphism*

$$ch : K^0(M) \otimes_{\mathbb{Z}} \mathbb{C} \to H^{even}_{dR}(M). \tag{3.68}$$

See [39] for proof.

3.4 Back to the IQHE

In this section we link the Hall conductance given in equation (2.56) in the previous Chapter to an expression in terms of the first Chern class. This confirms the claim made in Section 2.5 that the Hall conductance is a topological invariant.

Consider the functions $u_{k_1k_2}$ of equation (2.42), where we omitted the band index α and integrated over all space $\mathbb{R}^2$. These functions can be seen as sections of a complex vector bundle of rank 1 in the following manner. Consider the magnetic Brillouin zone we discussed in Section 2.4. As we mentioned in Section 2.5 this space is a torus $\mathbb{T}^2$. Hence, the states $u_{k_1k_2}$ can be seen as smooth functions $u(k_1, k_2)$ from $\mathbb{T}^2$ to $\mathbb{C}$, with inner product

$$(u, v) = \int_{\mathbb{T}^2} u^* v \, dk_1 dk_2 \quad \forall\, u, v \in C^\infty(\mathbb{T}^2, \mathbb{C}). \tag{3.69}$$

Taking the canonical projection $\pi : \mathbb{T}^2 \times \mathbb{C} \to \mathbb{T}^2$, we have constructed the trivial vector bundle over the base space $\mathbb{T}^2$, where the sections of the vector bundle are just the smooth functions from $\mathbb{T}^2$ to $\mathbb{C}$, hence the states $u_{k_1k_2}$.

The states u are normalised such that

$$||u|| = (u, u)^{\frac{1}{2}} = 1. \tag{3.70}$$

We want this expectation value of a state u to be invariant under the action of a transition function ϕ on such a vector state, therefore we demand these functions to be elements of $U(1)$. Note that these transition functions are just phase transformations.

In the same manner we can describe vector fields and differential forms as sections on a vector bundle over $\mathbb{T}^2$. These constructions however, are defined locally and the vector bundle is not necessarily trivial. A differential form of degree 2 for instance, is a section of a vector bundle that looks locally like $U_i \times \Lambda^2(T^*\mathbb{T}^2) \simeq U_i \times \mathbb{C}$ for $\{U_i\}$ an open covering of $\mathbb{T}^2$. We still want our transition functions to be elements of $U(1)$. This statement is the fundamental principle of gauge-theory (of electro-magnetism). For more on this matter we refer to [8].

We now construct an explicit vector bundle over $\mathbb{T}^2$, and relate the Kubo formula (2.47) to the first Chern class of this vector bundle. Consider the following open subsets of $\mathbb{T}^2$.

$$\begin{aligned}
U_1' &= \{(k_1, k_2)\,|\, 0 < k_1 < \pi/qa\,,\ 0 < k_2 < \pi/b\},\\
U_2' &= \{(k_1, k_2)\,|\, \pi/qa < k_1 < 2\pi/qa\,,\ 0 < k_2 < \pi/b\},\\
U_3' &= \{(k_1, k_2)\,|\, \pi/qa < k_1 < 2\pi/qa\,,\ \pi/b < k_2 < \pi/b\},\\
U_4' &= \{(k_1, k_2)\,|\, 0 < k_1 < \pi/qa\,,\ \pi/b < k_2 < \pi/b\}.
\end{aligned} \tag{3.71}$$

Construct an open covering $\{U_i\}$ of $\mathbb{T}^2$ by letting U_i be an open subset of $\mathbb{T}^2$, slightly bigger then U_i', such that U_i' fits entirely in U_i. Define the transition functions $\phi_{ij} : (U_i \cap U_j) \to U(1)$ as

$$\phi_{ij} = e^{\imath f^{ij}} = e^{\imath(\theta^i - \theta^j)}, \tag{3.72}$$

with θ^i and θ^j some smooth functions. The transition functions obey the compatibility conditions

$$\phi_{ij}\phi_{jk} = \phi_{ik} \quad \text{and} \quad \phi_{ii} = 1. \tag{3.73}$$

The covering $\{U_i\}$ of $\mathbb{T}^2$ and the transition functions (3.72) define a $U(1)$ vector bundle E over $\mathbb{T}^2$. This is done by gluing together the trivial vector bundles $U_i \times \mathbb{C}$ and choosing a representation ρ of $U(1)$ on $\mathbb{C}$. The vector bundle E is now the union $\bigcup_j U_j \times \mathbb{C}$ where we identify the points (p, v) and $(p, \rho(\phi_{ij}(p))v)$. This procedure is the same procedure one uses when constructing the corresponding $U(1)$ vector bundle of a $U(1)$ principle bundle (see for instance [8] or [34].

Consider now a connection ∇ on a vector bundle $E \to \mathbb{T}^2$ with projection π. This is a transformation from sections to sections, hence it is defined locally by a frame of a neighbourhood U_i in $\mathbb{T}^2$. Such a frame is given by a set of sections $\{e_\mu\}$ such that $\{e_\mu(k_1, k_2)\}$ is a basis for the fibre $V_{(k_1,k_2)}$ of the point (k_1, k_2) in U_i. This set of sections is supposed to change continuously with respect to (k_1, k_2) in U_i such that this frame gives the local trivialisation $U_i \times V \simeq \pi^{-1}(U_i)$ of the vector bundle. This is done with the identification

$$((k_1, k_2), z^\mu) \mapsto e_\mu(k_1, k_2) z^\mu(k_1, k_2) = z, \tag{3.74}$$

with $z = (z_1, z_2) \in V_{(k_1,k_2)}$ and z^μ the coordinate in V. In terms of equations (3.7) and (3.8) the connection ∇ can be written as

$$\begin{aligned} \nabla(e_\mu z^\mu) &= (\nabla e_\mu) \otimes z^\mu + e_\mu \otimes dz^\mu \\ &= e_\mu \otimes \omega^\mu_\nu z^\nu + e_\mu \otimes dz^\mu, \end{aligned} \tag{3.75}$$

where ω^μ_ν is the matrix valued 1-form. The dependence with respect to (k_1, k_2) gives

$$\nabla_{k_a}(s) = e_\mu (\omega^\mu_\nu)_a z^\nu + e_\mu \frac{\partial}{\partial k_a} z^\mu, \tag{3.76}$$

where we have put $s = e_\mu z^\mu$.

We show that the vector field $\hat{A}(k_1, k_2)$ of equation (2.55) can be written in terms of the matrix valued 1-form ω^μ_ν. Consider the 1-form $A(k_1, k_2)$ given by

$$A(k_1, k_2) = \hat{A}_a(k_1, k_2) dk_a = \langle u_{k_1 k_2} | \frac{\partial}{\partial k_a} | u_{k_1 k_2} \rangle dk_a. \tag{3.77}$$

The states $u_{k_1 k_2}$ can be seen as sections on our vector bundle, as we already mentioned. Remember that the fiber of our vector bundle equals $\mathbb{C}$, hence the frame is given by a set of one element $e_\mu = u$. This frame can be transformed in the region $U_i \cap U_j$ through the transition functions (3.72) in the following manner

$$u'(k_1, k_2) = u(k_1, k_2)\phi_{ij}^{-1}(k_1, k_2) = u(k_1, k_2)e^{-\imath f^{ij}(k_1,k_2)}. \tag{3.78}$$

This change of frame gives for $A(k_1, k_2)$

$$\begin{aligned} A'(k_1, k_2) &= A(k_1, k_2) - \imath \frac{\partial}{\partial k_a} f^{ij}(k_1, k_2) dk_a \\ &= \phi_{ij} A \phi_{ij}^{-1}(k_1, k_2) + \phi_{ij} d\phi_{ij}^{-1}(k_1, k_2). \end{aligned} \tag{3.79}$$

The collection of 1-forms obeying this transformation 3.79 defines the connection ∇.

The corresponding 2-form Θ of equation (3.9) is given by

$$\begin{aligned} F = dA + A \wedge A &= (\frac{\partial}{\partial k_a} \hat{A}_b + \hat{A}_a \hat{A}_b) dk_a \wedge dk_b \\ &= \frac{\partial}{\partial k_a} \hat{A}_b dk_a \wedge dk_b \end{aligned} \tag{3.80}$$

Consider now the contribution of the band index α to the conductance $\sigma_{xy}^{(\alpha)}$ of equation (2.56). We can express this contribution in terms of the curvature 2-form Θ

$$\begin{aligned} \int_{\mathbb{T}^2} d^2k \Big[\vec{\nabla}_k \times \vec{A}(k_1, k_2)\Big]_3 &= \int_{\mathbb{T}^2} \frac{\partial}{\partial k_a} \hat{A}_b dk_a \wedge dk_b \\ &= \int_{\mathbb{T}^2} \Theta = -\imath 2\pi \int_{\mathbb{T}^2} c_1, \end{aligned} \tag{3.81}$$

where c_1 is the individual Chern 1-form (see Definition 3.1.6 and equation (3.34)).

The first Chern class of $\mathbb{T}^2$ equals the Euler class of the underlying real bundle and can be related to the Euler characteristic χ, through the index theorem for the de Rham complex (see for instance the references [19], [34] or [59])

$$\int_{\mathbb{T}^2} c_1 = \int_{\mathbb{T}^2} e(\mathbb{T}^2) = \chi(\mathbb{T}^2). \tag{3.82}$$

The Euler characteristic is always an integer as it is the following sum over the dimensions of the de Rham cohomology groups

$$\chi(\mathbb{T}^2) = \sum_i (-1)^i \dim H^i_{dR}(\mathbb{T}^2). \tag{3.83}$$

Combining this with the expression for the contribution of the conductance $\sigma_{xy}^{(\alpha)}$ and with the equations (3.81) and (3.82) the integral quantisation of the conductance follows immediately:

$$\sigma_{xy} = \sum_\alpha \sigma_{xy}^{(\alpha)} = \sum_\alpha \frac{e^2}{h} n, \tag{3.84}$$

with n an integer.

There is another manner to look at the integrality of the Euler characteristic. Namely through the Hopf index theorem. This theorem states that the Euler characteristic $\chi(\mathbb{T}^2)$ equals the indices of a vector field (as a section of the unit tangent bundle) on $\mathbb{T}^2$. These indices count the amount of zeros in a vector field and the amount of times this vector field rotates around such a zero. This is obviously an integer.

This closes the discussion of the early theories of the IQHE. The goal of this thesis is to explain how one can generalise the theory discussed in Section 2.5 and finished in this Section. That is, how one can generalise it to the case where we do *not* make the assumption that the quantity of flux ϕ, through the unit cell, is rational. Remember equation (2.39),

$$\hat{T}_a \hat{T}_b = e^{2\pi\imath\phi} \hat{T}_b \hat{T}_a. \tag{3.85}$$

The reason for taking $\phi = p/q$ rational was that we could magnify the unit cell and define translation operators $\hat{T}_{qa}$ and $\hat{T}_b$ that did commute. The fact that these operators commute was crucial in the theory we discussed in the previous Sections. In the remainder of this thesis we discuss a tool called noncommutative geometry, to generalise Bloch theory to the case where ϕ is not rational, and therefore the translations operators do not commute.

Chapter 4

From spaces to algebras

In this Chapter we give a relation between spaces and algebras, via the Gelfand-Naimark theorem. We also introduce noncommutative topology, linking vector bundles to finitely generated projective modules. With this information we are able to define the Chern character in algebraic terms.

Looking at the set of observables in a physical system, we expect to find two kinds of structures. First, an algebraic one. This allows us to use a product on the observables, as to build up, for example, operators. We then, would also like that some sequences of observables can converge. So the second structure we expect on our algebra of observables, is a topological structure. We want this topology to have a algebraic link, such that the operations on the observables are continuous. An example of such a set, is a Banach algebra.

Definition 4.0.1. *A* **Banach algebra** A *is both an algebra (over* $\mathbb{C}$*) and a Banach space, such that*

$$||ab|| \leq ||a|| \cdot ||b||. \tag{4.1}$$

From now on, we will always assume that an algebra A is defined over $\mathbb{C}$. The measured observables in our physical system are all real, so we need to distinguish real from complex numbers. We can do this by introducing an involution, on our algebra A. An involution on A is defined as a map $* : A \to A$, such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$

$$a^{**} = a, \quad (ab)^* = b^* a^*, \quad (\lambda a)^* = \overline{\lambda} a^*. \tag{4.2}$$

An element $a \in A$, with the property $a^* = a$ is called self-adjoint.

Examples of Banach algebras with involution are Von Neumann algebras and C^*-algebras. Both can be seen as a *-subalgebra of the bounded operators $\mathcal{L}(H)$ on some Hilbert space H. The bounded operators on a Hilbert space form a

Banach space in the operator norm (see for example [53]). A von Neumann algebra is a closed subspace of $\mathcal{L}(H)$ in the weak topology, while a C^*-algebra is closed in the norm. The weak topology on the algebra of observables, can often generate the whole algebra $\mathcal{L}(H)$. This is often too large for our purposes. The C^*-algebra is a better candidate for the algebra of observables. These C^*-algebras are exactly the algebraic constructions we use in the Gelfand-Naimark theorem: the link between spaces and algebras.

4.1 C^*-algebras

In this Section we discuss some basics of C^*-algebras. These constructions play a fundamental role in the Gelfand-Naimark theorem, which gives a link between spaces and algebras. We mainly used [53] for this Section.

Definition 4.1.1. *A* **C*-algebra** *A is a Banach algebra with a involution, such that*

$$||a^*a|| = ||a||^2. \tag{4.3}$$

One of the best-known examples of a C^*-algebra is the set of continuous functions, over a compact Hausdorff space.

Proposition 4.1.2. *Let $C(M)$ be the set of continuous functions over the compact Hausdorff space M. Define the pointwise product and the supremum norm $||\cdot||_\infty$,*

$$||f||_\infty := \sup_{x\in M}|f(x)|. \tag{4.4}$$

on $C(M)$. If we define the involution as the complex conjugate, $C(M)$ is a C^*-algebra.

Proof. A closed bounded subset of a complete set is also complete, hence $C(M)$ is a Banach space, since M is compact. With the pointwise product, and the complex conjugate as involution, the norm satisfies the following properties

$$\begin{aligned} ||fg||_\infty &= \sup_{x\in M}|f(x)g(x)| \leq \sup_{x\in M}|f(x)||g(x)| \leq ||f||_\infty ||g||_\infty, \\ ||f^*f||_\infty &= \sup_{x\in M}|\overline{f(x)}f(x)| = \sup_{x\in M}\left||f(x)|^2\right| = ||f||^2. \end{aligned} \tag{4.5}$$

We conclude that $C(M)$ is, indeed, a C^*-algebra. □

The main idea of algebraic geometry is the investigation of geometries that come from an algebra or, vice versa, the study of a space M, trough its associative algebra $C(M)$. We do this by the following construction. Consider the set $s(A)$ of an algebra A,

$$s(A) = \{\phi : A \to \mathbb{C} \,|\, \phi(fg) = \phi(f)\phi(g),\, \phi \neq 0\}. \tag{4.6}$$

Whenever M is compact and Hausdorff, there exists, using the right topology, an homeomorphism between M and $s(C(M))$, given by the evaluation function,

$$\phi_x(f) = f(x) \quad \forall x \in M. \tag{4.7}$$

We can follow this idea also the other way around, and try to find an associative (compact and Hausdorff) space M, such that for given commutative algebra A, $A = C(M)$. Gelfand and Naimark proved in 1943 that this can be done if and only if A is a commutative C^*-algebra. Before we prove this, we first give some definitions.

Definition 4.1.3. *A* ***-homomorphism** *between two C^*-algebra A and B, is a linear map $\varphi : A \to B$ such that for all $a, b \in A$*

$$\begin{aligned} \varphi(ab) &= \varphi(a)\varphi(b), \\ \varphi(a^*) &= \varphi(a)^*. \end{aligned} \tag{4.8}$$

An *-isomorphism is a bijective *-homomorphism.

Definition 4.1.4. *The* **structure space** Δ *of a C^*-algebra A, is given by*

$$\Delta(A) = \{\omega : A \to \mathbb{C} \,|\, \omega \text{ linear}, \, \omega(fg) = \omega(f)\omega(g), \, \omega \neq 0\}. \tag{4.9}$$

This space is the one we are looking for. We remark that $\omega(\mathbb{I}) = 1$ (we are assuming here that A is unital), due to the homomorphism property and the fact that ω is nonzero. The functionals ω are then continuous in the operator norm. It is clear that $\Delta(A) \subset A^*$, the dual of A. We can define the weak*-topology on this set, i.e. the topology such that

$$\varphi_n \to \varphi \quad \text{iff} \quad \varphi_n(a) \to \varphi(a) \quad \forall\, a \in A. \tag{4.10}$$

The topological space then created, is compact and Hausdorff (see [53]).

Following the procedure we used for $C(M)$, we define the **Gelfand transform** $\hat{}: A \to C(\Delta(A))$ as the map

$$\widehat{a}(\omega) := \omega(a). \tag{4.11}$$

It is easily seen that this transform is a homomorphism. Moreover, this transform is exactly the isomorphism we are looking for.

Theorem 4.1.5. *Let A be a unital commutative C^*-algebra. Then there exists a compact Hausdorff space M such that $C(M)$ is isomorphic to A.*

Proof. Consider the structure space $M = \Delta(A)$ and the Gelfand transform from A to $C(M)$. We show that this is an isomorphism.

1. A simple calculation shows that the Gelfand transform is a homomorphism

2. We show that it is a *-homomorphism, i.e. $\widehat{a^*} = \widehat{a}^* = \overline{a}$. Because every $a \in A$ can be written as a linear span of self-adjoint elements, it suffice to proof that $\omega(a) \in \mathbb{R}$ for $a^* = a$. Write $\omega(a) = \alpha + \imath\beta$, with $\alpha, \beta \in \mathbb{R}$. The self-adjoint element $b = a - \alpha\mathbb{I} \in A$ gives $\omega(b) = \imath\beta$, because of $\omega(\mathbb{I}) = 1$. For $t \in \mathbb{R}$ we now have
$$|\omega(b) + \imath t\mathbb{I}|^2 = \beta^2 + 2\beta t + t^2. \tag{4.12}$$
On the other hand we also have
$$\begin{aligned} |\omega(b) + \imath t\mathbb{I}|^2 &\leq ||b + \imath t\mathbb{I}||^2 \quad (\ \omega \text{ is continuous}) \\ &= ||(b + \imath t\mathbb{I})(b - \imath t\mathbb{I})|| = ||b^2 + t^2\mathbb{I}|| \\ &\leq ||b||^2 + t^2. \end{aligned} \tag{4.13}$$
Comparing these two expression for all $t \in \mathbb{R}$, gives $\beta = 0$. And we proved that the Gelfand transform is a *-homomorphism.

3. We show that the Gelfand transform is an isometry. Injectivity follows. Assume $a^* = a$. Then $||a^2|| = ||a||^2$, and $||a^{2^m}|| = ||a||^{2^m}$ for all $m \in \mathbb{N}$. One can show ([53]), that the spectral radius $r(a)$ is given by
$$r(a) = \lim_{n\to\infty} ||a^n||^{\frac{1}{n}}, \tag{4.14}$$
while the spectrum $\sigma(a)$ of a can be written as the set
$$\sigma(a) = \{\widehat{a}(\omega) \,|\, \omega \in \Delta(A)\}. \tag{4.15}$$
So we have $r(a) = ||a||$, and
$$||\widehat{a}||_\infty = ||a||. \tag{4.16}$$
The element a^*a is self-adjoint for any $a \in A$ hence
$$||a||^2 = ||a^*a|| = ||\widehat{a^*a}||_\infty = ||\widehat{a^*}\widehat{a}||_\infty = ||\widehat{a}||_\infty^2. \tag{4.17}$$
Thus the Gelfand transformation is an isometry and injectivity follows.

4. Surjectivity follows from the Stone-Weierstrass theorem, which we give without proof (see [64]):

   Theorem 4.1.6. *Let M be a compact Hausdorff space. Every C^*-subalgebra of $C(M)$, which separates points on M, and contains $\mathbb{1} \in C(M)$, coincides with $C(M)$*

   We saw that $\widehat{A} := \{\widehat{a}|a \in A\}$ is closed under multiplication and complex conjugation. Together with the isometry this gives that $\widehat{A}$ is a C^*-subalgebra of $C(\Delta(A))$. If $\omega_1 \neq \omega_2$, there is an $a \in A$ such that $\omega_1(a) \neq \omega_2(a)$, and that gives us $\widehat{a}(\omega_1) \neq \widehat{a}(\omega_2)$. We know therefore that $\widehat{A}$ separates points on $C(\Delta(A))$. Because $\widehat{\mathbb{1}} = \mathbb{1} \in C(\Delta(A))$, we have $\mathbb{1} \in \widehat{A}$. We conclude that $\widehat{A} = C(\Delta(A))$.

We have showed that the Gelfand transformation is indeed a (isometric) isomorphism between unital commutative C^*-algebras and the continuous function algebras of a corresponding compact and Hausdorff space. □

Remark 4.1.7. We can reformulate this in a categorical manner. Let CH be the category of compact Hausdorff spaces and CCA the category of unital commutative C^*-algebras. Consider the cofunctor $C : CH \to CCA$, which maps an object M to an object $C(M)$, and an arrow $f : M \to N$ to an arrow $f^* : C(N) \to C(M)$ defined by $f^*(g) = g \circ f$. Theorem 4.1.5 is equivalent to saying that the cofunctor C is essentially surjective. The cofunctor C is even fully faithful (see for instance [53]), i.e. for every pair of objects (M, N) the map

$$C : \mathrm{Hom}_{CH}(M, N) \to \mathrm{Hom}_{CCA}\big(C(N), C(M)\big) \tag{4.18}$$

is bijective. These two properties make the functor an equivalence of the categories CH and CCA. This means that there exists a cofunctor $\Delta : CCA \to CH$, such that $\Delta \circ C \simeq \mathbb{1}_{CH}$ and $C \circ \Delta \simeq \mathbb{1}_{CCA}$ are natural isomorphisms. This cofunctor is exactly the cofunctor that maps an object A to the structure space $\Delta(A)$ of (4.9), and maps an arrow $\phi : A \to B$ to the arrow $\phi^* : \Delta(B) \to \Delta(A)$ defined by $\phi^*(\omega) = \omega \circ \phi$.

Remark 4.1.8. We assumed all the time, that the space M was compact and Hausdorff. If the space M is not compact, but locally compact, the algebra $C(M)$ will be too large to say something about M. In this case we prefer to look at a smaller algebra, namely $C_c(M)$ the algebra of continuous functions (over M) with compact support. This C^*-algebra does not contain a unit, and is exactly the algebra we use for the non-unital version of Theorem 4.1.5. We have, for a non-unital commutative algebra A, $A \simeq C_c(\Delta(A))$, with $M = \Delta(A)$ a locally compact Hausdorff space. Adding a unit to A (see end of the following Section below), is similar to the one-point compactification of M. For more information about this matter see, for instance, [39] and [53].

4.2 The Gelfand-Naimark theorem

In this Section we focus on noncommutative (unital) C^*-algebras and give a noncommutative version of Theorem 4.1.5. This theorem links C^*-algebras to the set of bounded linear operators on a Hilbert space. For this Section we made use of [53], [39] and [63].

Proposition 4.2.1. *Consider the bounded linear operators $\mathcal{L}(H)$ on a Hilbert space H, with the normal operator product and operator norm. If we define the following involution on it*

$$\langle u | a^* v \rangle = \langle au | v \rangle \quad \forall u, v \in H, a \in \mathcal{L}(H), \tag{4.19}$$

$\mathcal{L}(H)$ is a C^*-algebra.

Proof. The bounded operators on a Banach space form again a Banach space. This, together with the computation

$$||ab|| = \sup_{||v||=1} ||(ab)v|| \leq ||a|| \, ||bv|| \leq ||a|| \, ||b||, \tag{4.20}$$

make of $\mathcal{L}(H)$ a Banach algebra. Consider now the following calculation where we use the Cauchy-Schwartz inequality

$$||av||^2 = \langle av|av\rangle = \langle v|a^*av\rangle \leq ||v|| \, ||a^*av|| \leq ||a^*a|| \, ||v||. \tag{4.21}$$

Due to this inequality and the definition of the operator norm we have

$$||a|| \leq ||a^*a|| \leq ||a^*|| \, ||a||, \tag{4.22}$$

and we see that $||a|| \leq ||a^*||$. With a similar calculation we can show that $||a^*|| \leq ||a||$, therefore $||a^*|| = ||a||$. Plugging this back into (4.22), we get the C^*-condition

$$||a||^2 = ||a^*a||, \tag{4.23}$$

and the bounded linear operators $\mathcal{L}(H)$ on a Hilbert space H make up a C^*-algebra. □

Remark that if the dimension n of the Hilbert space is smaller than infinity, we can identify H with $\mathbb{C}^n$, and therefore $\mathcal{L}(H)$ with $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices over $\mathbb{C}$.

This C^*-algebra is probably the best-known noncommutative C^*-algebra, and it plays a fundamental role in the noncommutative version of Theorem 4.1.5. Before we give and prove this Theorem we give some definitions.

Definition 4.2.2. *A* **state** *on a unital C^*-algebra $\mathcal{A}$, is a functional $\omega : \mathcal{A} \to \mathbb{C}$, with the following properties*

$$\begin{aligned} &\omega(\mathbb{1}) = 1, \\ &\omega(a^*a) \geq 0 \quad \forall a \in \mathcal{A}. \end{aligned} \tag{4.24}$$

We see that the characters, defined in the previous Section in (4.9), are all states. Both properties following from the homomorphism property. We say that the states are the generalisation of the characters, and form the so-called noncommutative structure space.

Definition 4.2.3. *A* **representation** *of a C^*-algebra $\mathcal{A}$ on a Hilbert space H is *-homomorphism between $\mathcal{A}$ and $\mathcal{L}(H)$.*

Again we see here a generalisation of the commutative case. The Gelfand transform (4.11) can be seen as a representation of A on the one-dimensional Hilbert space $\Delta(A) \simeq \mathbb{C}$. We now generalise two equivalent properties of the Gelfand transform: non-degeneracy and cyclicity.

- We say that a representation π is **non-degenerate** if $\pi(\mathcal{A})$ is non-degenerate on H, i.e. if for each non-zero vector $u \in H$ there is an element $a \in \mathcal{A}$ such that $\pi(a)u \neq 0$.
- We say that a representation π is **cyclic** if there is a cyclic vector $v \in H$ for $\pi(\mathcal{A})$, i.e. if there is a element $v \in H$ such that the $\pi(\mathcal{A})$-invariant subspace generated by v is dense in H.

In the previous Section we found for every state $\omega \in \Delta(A)$ a representation $\wedge : A \to C(\Delta(A))$, such that $A \simeq C(\Delta(A))$. We want to do the same for the generalised case. For every non-degenerate representation π, we can find a normalised vector $u \in H$ such that u defines a state on $\mathcal{A}$, with

$$u(a) = \langle u|\pi(a)u\rangle. \tag{4.25}$$

Every cyclic representation is immediate non-degenerate. Hence we only need to concentrate on the cyclic representations, and ask ourself if we can construct such a representation of $\mathcal{A}$, given any state on $\mathcal{A}$. We can do this, using the **Gelfand-Naimark-Segal (GNS) construction**.

Proposition 4.2.4 (GNS-construction). *For every state ω on the (unital) C^*-algebra $\mathcal{A}$, there is a cyclic representation π_ω of $\mathcal{A}$, with cyclic vector u_ω such that*

$$\langle \pi_\omega(a)u_\omega|u_\omega\rangle = \omega(a), \tag{4.26}$$

for all $a \in \mathcal{A}$.

Proof. In the same manner we made a Hilbert space out of the set of characters, we want to make a Hilbert space out of the set of states on $\mathcal{A}$. We can do this in two steps.

1. Given a state ω on the C^*-algebra $\mathcal{A}$, define a sesquilinear form $\langle\cdot|\cdot\rangle_\omega$ on $\mathcal{A}$ as
$$\langle a|b\rangle_\omega := \omega(a^*b). \tag{4.27}$$
Because this form is positive semi-definite, it obeys the Cauchy-Schwarz inequality
$$|\langle a|b\rangle_\omega|^2 \leq \langle a|a\rangle_\omega \langle b|b\rangle_\omega. \tag{4.28}$$
To make this form a inner product we define the nullspace N_ω as
$$N_\omega := \{a \in \mathcal{A} \,|\, \langle a|a\rangle_\omega = 0\} = \{a \in \mathcal{A} \,|\, \langle a|b\rangle_\omega = 0 \quad \forall b \in \mathcal{A}\}, \tag{4.29}$$
where the second equality comes from (4.28). This nullspace is a left ideal in $\mathcal{A}$.
2. The sesquilinear form (4.27) is an inner product on the space $\mathcal{A}/N_\omega$, making it a pre-Hilbert space. The equivalence class of a in $\mathcal{A}/N_\omega$ is denoted by $\underline{a}$. The completion H_ω of $\mathcal{A}/N_\omega$, is the Hilbert space we are looking for.

Having constructed a Hilbert space for every state ω, we are now going to construct a cyclic representation on $\mathcal{A}$, over this Hilbert space.

3. Consider the maps $\pi_\omega(a) : \mathcal{A}/N_\omega \to \mathcal{A}/N_\omega$, defined by

$$\pi_\omega(a) : \underline{b} \mapsto \underline{ab}. \tag{4.30}$$

If π_ω is continuous we can extend these maps to

$$\pi_\omega : \mathcal{A} \to \mathcal{L}(H_\omega). \tag{4.31}$$

Continuity of π_ω follows from $||\pi_\omega(a)|| \leq ||a||$, due to the inequality (see for instance [53])

$$\omega(b^* a^* ab) \leq ||a^* a|| \omega(b^* b). \tag{4.32}$$

The fact that these maps are *-homomorphisms makes them representations of $\mathcal{A}$ over H_ω.

4. Remember that we are working with unital C^*-algebras. We can write then $u_\omega = \underline{\mathbb{1}} \in \mathcal{A}/N_\omega$. With this we have $\pi_\omega(b)u_\omega = \underline{b}$ for $b \in \mathcal{A}$, and $\pi_\omega(\mathcal{A})u_\omega = \mathcal{A}/N_\omega$. This makes of u_ω a cyclic vector for the representation π_ω.

Now it is easy to see that, indeed,

$$\langle \pi_\omega(a)u_\omega | u_\omega \rangle = \omega(a). \tag{4.33}$$

□

With this construction we are able to proof the **Gelfand-Naimark Theorem**.

Theorem 4.2.5 (Gelfand-Naimark). *Every (unital) C^*-algebra $\mathcal{A}$ has an isometric representation to a closed subalgebra of the algebra $\mathcal{L}(H)$ of bounded operators on some Hilbert space.*

Proof. Using the Theorem of Hahn-Banach one can show ([53]) that for each non-zero element $b \in \mathcal{A}$, there exists a state $\omega = \omega_b$ such that

$$\omega(b^* b) = ||b||^2. \tag{4.34}$$

With the GNS-construction of Proposition 4.2.4 (and in particular (4.26)), one has

$$||\pi_\omega(b)u_\omega|| = ||b||. \tag{4.35}$$

Take now the direct sum $\pi := \oplus_\omega \pi_\omega$ of all the GNS-representations of these states $\omega = \omega_b$. This map is a representation on the direct sum $H := \oplus_\omega H_\omega$. And, as we saw from (4.35), this representation is isometric. □

The representation π we constructed in this proof of the Gelfand-Naimark Theorem, is called the **universal representation**. In C^*-algebras, for morphisms, isometry is equivalent to injectivity. Thus, we could equivalently say that every (unital) C^*-algebra is isomorphic to a subalgebra of $\mathcal{L}(H)$, for some Hilbert space H.

Both Proposition 4.2.4 as Theorem 4.2.5 are given for a unital C^*-algebra. It is possible to generalise these statements to any C^*-algebra. We do this by unitisation of the algebra.

Let $\mathcal{A}$ be a C^*-algebra without unit. We can add a unit to $\mathcal{A}$, by the following construction. Consider the vector space $\mathcal{A}_{\mathbb{1}} := \mathcal{A} \oplus \mathbb{C}$, with multiplication

$$(a, \mu)(b, \lambda) = ab + \lambda a + \mu b + \lambda\mu, \tag{4.36}$$

and involution $(a, \lambda)^* = (a^*, \overline{\lambda})$. One can treat $(a, \lambda) \in \mathcal{A}_{\mathbb{1}}$ as operators on $\mathcal{A}$, and define a operator norm on it. This norm is a C^*-norm, and the inclusion $\mathcal{A} \subset \mathcal{A}_{\mathbb{1}}$ is a *-homomorphism. The C^*-algebra thus created, is unique such that $\mathcal{A}_{\mathbb{1}}/\mathcal{A} \simeq \mathbb{C}$. The C^*-algebra $\mathcal{A}_{\mathbb{1}}$ is called the **unitisation** of $\mathcal{A}$. Using an approximate unit one can make a unique extension to a state $\omega_{\mathbb{1}}$ on $\mathcal{A}_{\mathbb{1}}$, for given state ω on $\mathcal{A}$. With this one can generalise the GNS-construction (and therefore also Theorem 4.2.5) for any C^*-algebra.

4.3 Noncommutative topology

In Section 4.1 we showed that any unital commutative C^*-algebra is isomorphic to $C(M)$, with M some compact Hausdorff space. One can even show ([53]) that this space M must be unique (up to homeomorphism), and is homeomorphic to the structure space $\Delta(C(M))$. This is due to the fact that the category of compact Hausdorff spaces is equivalent to the category of (unital) commutative C^*-algebras, as we remarked after the proof of Theorem 4.1.5. It implies

$$M \simeq N \text{ (homeomorphic)} \quad \Longleftrightarrow \quad C(M) \simeq C(N) \text{ (isomorphic)}. \tag{4.37}$$

This means that all the topological information of a compact Hausdorff space can be translated into some algebraic structure on a unital commutative C^*-algebra.

In this Section we give an algebraic analogue of a vector bundle E over M. In Chapter 3 we looked at smooth vector bundles over a manifold M. The corresponding function algebra is $C^\infty(M)$, the smooth functions over M. This algebra is not a C^*-algebra, but we know that it lies dense in the algebra $C(M)$. In this and the following Section we concentrate on the C^*-algebra $C(M)$. In Section 5.1 we mention whenever and how the theory holds for the algebra $C^\infty(M)$.

The corresponding vector bundles of the algebra $C(M)$ are the continuous vector bundles. For these bundles we define a vector bundle morphism as follows.

Definition 4.3.1. *Let E and F be two vector bundles over M, with projections π_E, π_F respectively. A continuous map $\phi : E \to F$ is a* **vector bundle morphism** *if $\pi_F\phi = \pi_E$, and if the restriction of ϕ to the vector space E_x is linear. If ϕ is also a homeomorphism, and if the restriction of ϕ to E_x is an isomorphism between vector spaces, we call ϕ a* **vector bundle isomorphism**

The algebraic analogue of a vector bundle, that is invariant under these isomorphisms, is some kind of module. A **module** is a generalisation of a vector space. Where vector spaces are defined over fields, modules are defined over rings. Hence, a right A-module $\mathcal{E}$, is a Abelian group with an action of the ring A on $\mathcal{E}$ such that

$$\sigma(ab) = (\sigma a)(b), \quad \sigma(a+b) = \sigma(a) + \sigma(b), \quad (\sigma + \tau)(a) = \sigma(a) + \tau(a), \tag{4.38}$$

for all $\sigma, \tau \in \mathcal{E}$, $a, b \in A$. We suggestively used the notation A for our ring, because we want to work with modules over the algebra $A = C(M)$, for some compact Hausdorff space M. Remark that all our algebras are associative, hence rings.

Consider now the set $\Gamma(M, E)$ of all sections over the vector bundle E. We denote this set as $\Gamma(E)$ if there can be no confusion about the base space M. This set is a (right) $C(M)$-module, with the pointwise product

$$\sigma f(x) = \sigma(x) f(x), \quad \forall\, \sigma \in \Gamma(M, E),\ f \in C(M), \tag{4.39}$$

as action. This module contains exactly the topological information of the vector bundle. This means that $\Gamma(M, E)$ is isomorphic to $\Gamma(M, F)$ as $C(M)$-modules if and only if E is isomorphic to F as vector bundles. We prove this in the following.

1. Let $\phi : E \to F$ be a vector bundle morphism. Consider the associative map $\Gamma\phi : \Gamma(E) \to \Gamma(F)$, defined by $\Gamma\phi(s) = \phi \circ s$. Because the map $\Gamma\phi$ is $C(M)$-linear and conserves the group structure of $\Gamma(E)$, it is a $C(M)$-module morphism. Whenever ϕ is a vector bundle isomorphism we can construct the $C(M)$-module morphism $\Gamma\psi := \Gamma\phi^{-1} : F \to E$. Because of the property
$$\Gamma\psi\phi = \Gamma\psi\,\Gamma\phi, \tag{4.40}$$
we have, for every $s \in \Gamma(E)$ and $x \in M$
$$\begin{aligned} \Gamma\psi\,\Gamma\phi(s)(x) &= \Gamma\psi\phi(s)(x) = \psi\phi \circ s(x) \\ &= \phi^{-1}\phi \circ s(x) = s(x). \end{aligned} \tag{4.41}$$
In the same way we can show that $\Gamma\phi\,\Gamma\psi t = t$ for every $t \in \Gamma(F)$, and conclude that $\Gamma\phi$ is a $C(M)$-module isomorphism, whenever ϕ is an isomorphism between vector bundles.

2. Conversely, let Φ be a $C(M)$-module morphism between $\Gamma(E)$ and $\Gamma(F)$. Then

$$\begin{aligned}\Phi \in Hom_A(\Gamma(E), \Gamma(F)) &\simeq \Gamma(E^*) \otimes_A \Gamma(F) \\ &\overset{\theta}{\simeq} \Gamma(E^* \otimes_A F)\end{aligned} \tag{4.42}$$

   where $E^* \to M$ is the dual vector bundle of $E \to M$ (see for the isomorphisms for instance [39]). Every such element can be identified with an element $\Gamma\phi$ for some vector bundle morphism $\phi : E \to F$. Using (4.41), we see that, if $\Gamma(E)$ is isomorphic to $\Gamma(F)$ as $C(M)$-modules, E is isomorphic to F as vector bundles.

Remark 4.3.2. Let $\mathcal{V}(M)$ be the category of vector bundles over M and $\mathit{\Gamma}(M)$ the category of the sets $\Gamma(M, E)$ of the vector bundles E. The transformation Γ discussed above is a functor between these two categories, due to the first part of the proof. The second part of the proof makes the functor Γ even fully faithful, hence an equivalence of categories. The fact that the functor is essentially surjective lies in the construction of the set $\Gamma(M, E)$.

The $C(M)$-modules $\Gamma(M, E)$ we are looking at, have more structure then we already mentioned. In fact they are finitely generated projective modules.

Definition 4.3.3. *A A-module $\mathcal{E}$ is called a* **finitely generated projective module** *if there exist an element $n \in \mathbb{N}$ and a $p \in M_n(A)$, with $p^2 = p$ such that $\mathcal{E} \simeq pA^n$ as A-modules.*

Before we really proof that every $\Gamma(M, E)$ is a finitely generated projective modules, and vice versa, we first give the following definition.

Definition 4.3.4. *A* **short exact sequence** *(a* **SES** *for short) is a collection of maps*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{4.43}$$

such that α is injective, β is surjective, and $im\,\alpha = \ker\beta$.

Consider now a SES $E \to F \to G$ of vector bundles over the space M, where the maps are vector bundle morphisms. If M is paracompact and Hausdorff such a sequence **splits**, i.e. $F \simeq E \oplus G$. Such a sequence is called split exact. Remember that the spaces we use are compact Hausdorff, hence paracompact and Hausdorff. We will not mention this always, but assume it implicitly. We are now able to prove the statement earlier made.

Proposition 4.3.5. *The $C(M)$-modules $\Gamma(M, E)$ are finitely generated projective modules.*

Proof. By a Theorem of Serre and Swan we can find for every vector bundle $E \to M$ another vector bundle $F \to M$ such that $E \oplus F$ is a trivial vector bundle, i.e. $E \oplus F = M \times \mathbb{C}^n$. (This Theorem also uses the fact that M is compact and Hausdorff.) Therefore we are able to construct a split SES $F \to M \times \mathbb{C}^n \to E$. Because every homeomorphism between vector bundles gives rise to an isomorphism between sections, the sequence $\Gamma(F) \to \Gamma(M \times \mathbb{C}^n) \to \Gamma(E)$ will also be split exact. We can identify the set of continuous functions from $M \times \mathbb{C}^n$ to M, with the module $C(M)^n$. The set $\Gamma(E)$ is then a direct summand of $C(M)^n$, hence a finitely generated projective $C(M)$-module. □

To complete the proof of our statement, we use the next proposition.

Proposition 4.3.6. *Every finitely generated projective $C(M)$-module $\mathcal{E}$ is of the form $\Gamma(M, E)$ for some vector bundle E.*

Proof. By definition $\mathcal{E} \simeq p\,C(M)^n$, for some $p \in M_n(C(M))$. Consider the following exact sequence

$$0 \longrightarrow \ker p \longrightarrow C(M)^n \xrightarrow{\;p\;} \mathcal{E} \longrightarrow 0 \tag{4.44}$$

We can split $C(M)^n$ into $pC(M)^n \oplus (\mathbb{1} - p)C(M)^n$. The kernel of p is then exactly $(\mathbb{1} - p)C(M)^n$ and the sequence splits. We can identify $C(M)^n$ with the set of sections $\Gamma(M, M \times \mathbb{C}^n)$ and then identify $p : C(M)^n \to C(M)^n$ with Γe, where $e : M \times \mathbb{C}^n \to M \times \mathbb{C}^n$ is a vector bundle morphism. The image of e is a subbundle $E(p)$ of $M \times \mathbb{C}^n$. We then have

$$\Gamma(M, E(p)) = \{e \circ s : s \in \Gamma(M \times \mathbb{C}^n)\} = \operatorname{im} p = \mathcal{E}. \tag{4.45}$$

□

The equivalence of categories between $\mathcal{V}(M)$ and $\mathit{\Gamma}(M)$ gives us the following correspondence between vector bundles and finitely generated projective modules.

$$E \simeq F \quad \Longleftrightarrow \quad p\,C(M)^n \simeq q\,C(M)^n \tag{4.46}$$

where $p, q \in M_n(C(M))$ are related to the vector bundles E, F as explained in Propositions 4.3.5 and 4.3.6. Note that every vector bundle of rank n can then be identified with a idempotent $p \in M_n(C(M))$. In the following section we start with this notion, and try to make a equivalence class for the idempotents, to make an analogue for the group $K^0(M)$ of Section 3.3.

4.4 C^*-algebraic K-theory

In this Section we discuss C^*-algebraic K-theory, a generalisation of the group $K^0(M)$, (the vector bundles modulo isomorphisms and stable equivalence), we introduced in Section 3.3.

Remember that for every vector bundle over M we have a corresponding finitely generated projective module, characterised by a idempotent $p = p^2 \in M_n(C(M)$. Consider the Whitney sum $E \oplus F$ between two vector bundles E and F, with corresponding finitely generated projective modules $pC(M)^n$ and $qC(M)^m$ respectively. This gives a $C(M)$-module isomorphism $pC(M)^n \oplus qC(M)^m \simeq rC(M)^{n+m}$. We can always rearrange the basis of $C(M)^{n+m}$ such that we get $r \simeq p \oplus q$ with the identity

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{n+m}(C(M)) \tag{4.47}$$

for $p \in M_n(C(M))$ and $q \in M_m(C(M))$. We are now able to find for every vector bundle over M a corresponding idempotent $p \in M_n(C(M))$, with the following identification of morphisms.

Lemma 4.4.1. *Let $p \in M_n(C(M))$ and $q \in M_m(C(M))$ be two idempotents. Let an element 0_N stand for the $N \times N$-zero matrix. The finitely generated projective modules $pC(M)^n$ and $qC(M)^m$ are isomorphic if and only if there exist an invertible matrix $z \in M_N(C(M))$ such that $z(p \oplus 0_{N-n})z^{-1} = q \oplus 0_{N-m}$ for some $N \in \mathbb{N}$.*

Proof. Let $\phi : pC(M)^n \to qC(M)^m$ be a $C(M)$-module isomorphism. We can construct the maps $\psi : C(M)^n \to C(M)^m$ and $\eta : C(M)^m \to C(M)^n$ by extending respectively ϕ to 0 on $(1-p)C(M)^n$ and ϕ^{-1} to 0 on $(1-q)C(M)^m$. There exist $g \in M_{n,m}(C(M))$ and $h \in M_{m,n}(C(M))$ such that $\psi(s) = gs$ and $\eta(t) = ht$, with the identifications $gh = q, hg = pg = gp = qg$ and $h = ph = hq$. Take $N := n + m$ and compute

$$\begin{pmatrix} g & 1-q \\ 1-p & h \end{pmatrix} \begin{pmatrix} h & 1-p \\ 1-q & g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.48}$$

and

$$\begin{pmatrix} g & 1-q \\ 1-p & h \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h & 1-p \\ 1-q & g \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \tag{4.49}$$

The other way around we have, for $zpz^{-1} = q$, $zpC(M)^N = qzC(M)^N$. □

Consider now the set $P(C(M))$ of idempotents in the disjoint union $\bigcup_n M_n(C(M))$, and define the following equivalence relations on it.

1. $p \sim p \oplus 0_m$ for $m \in \mathbb{N}$
2. $p \overset{u}{\sim} q$ if $zpz^{-1} = q$ for some invertible $z \in M_N(C(M))$,

We denote the second equivalence relation as $\overset{u}{\sim}$, because it is equal to the unitary equivalence for projections (see Lemma 4.4.3 below). Denote the set

$P(C(M))$ modulo these two equivalences as $\mathcal{P}(C(M))$. Every $p \in \mathcal{P}(C(M))$ will characterise a vector bundle, modulo isomorphisms. Like the set $V(M)$ of vector bundles, modulo isomorphisms, the set $\mathcal{P}(C(M))$ is also a semigroup. The Grothendieck group (see Section 3.3) of this semigroup is called the K_0-group of $C(M)$ and is denoted as $K_0(C(M))$.

Corollary 4.4.2. *There is a isomorphism θ_M between $K^0(M)$ and $K_0(C(M))$ natural in M, i.e. for every compact Hausdorff spaces M and N and continuous function $f : M \to N$, the diagram*

$$\begin{array}{ccc} K^0(M) & \xrightarrow{\theta_M} & K_0(C(M)) \\ {\scriptstyle K^0 f}\uparrow & & \uparrow{\scriptstyle K_0 \circ C f} \\ K^0(N) & \xrightarrow{\theta_N} & K_0(C(N)) \end{array} \tag{4.50}$$

commutes, and θ_M and θ_N are isomorphisms between Abelian groups.

Proof. This follows directly from the equivalence of categories between $\mathcal{V}(M)$ and $\Gamma(M)$, and Propositions 4.3.5 and 4.3.6. ☐

We now want to extend the notion of a K_0-group to any C^*-algebra $\mathcal{A}$. In order to do that we construct for any C^*-algebra $\mathcal{A}$ a C^*-algebra $M_n(\mathcal{A})$ of $n \times n$-matrices with entries in $\mathcal{A}$. The involution is transposing the matrix and involute every entry. Thus, we have for example

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \in M_2(\mathcal{A}). \tag{4.51}$$

For the norm of the C^*-algebra $M_n(\mathcal{A})$, we first represent $\mathcal{A}$ as a subalgebra of $\mathcal{L}(H)$, the bounded operators on some Hilbert space (see Theorem 4.2.5). We define the norm as the operator norm in $\mathcal{L}(H^n)$, where $H^n = \oplus_n H$.

Just as for $C(M)$ we take the disjoint union $\cup_n M_n(\mathcal{A})$ for a C^*-algebra $\mathcal{A}$ and look at the idempotents $p = p^2 \in M_n(\mathcal{A})$, for any n. We denote the space of these idempotents in $M_n(\mathcal{A})$ as $P_n(\mathcal{A})$. An element of $P_n(\mathcal{A})$ is called a **projection in $\mathcal{A}$**.

In the case of $C(M)$ we then defined an equivalence relation on the projections. Remark however, that we made use of the fact that $C(M)$ is unital. Because our C^*-algebra $\mathcal{A}$ is not necessary unital we have to redefine the equivalence relation on our projections $p \in P_n(\mathcal{A})$. To do that we first study different kind of equivalences of projections on a unital C^*-algebra $\mathcal{A}$.

Consider a unital C^*-algebra $\mathcal{A}$, with p, q projections in $\mathcal{A}$. We can define three equivalence relations on these projections.

1. (**Murray-von Neumann equivalence**), $p \overset{MvN}{\sim} q$ if there exists a $v \in \mathcal{A}$ such that $p = v^*v$ and $q = vv^*$.
2. (**Unitary equivalence**), $p \overset{u}{\sim} q$ if there exists a unitary element $u \in \mathcal{A}$ with $p = uqu^*$.
3. (**Homotopy equivalence**) $p \overset{h}{\sim} q$ if there exists a norm-continuous path $r(t)$ of projections, such that $p = r(0)$ and $q = r(1)$.

Lemma 4.4.3. *Let $\mathcal{A}$ be a unital C^*-algebra. Two projections p, q in $\mathcal{A}$ are unitary equivalent if and only if there exist an invertible element $z \in \mathcal{A}$ such that $zpz^{-1} = q$.*

Proof.

1. Let $p = uqu^*$. Take $z = u^*$, then $zpz^{-1} = q$
2. Let $zpz^{-1} = q$, and let $z = u|z|$ be the polar decomposition of z, with u a unitary element of $\mathcal{A}$. (See, for the existence of the polar decomposition, [73].) The relation $zp = qz$ and the fact that projections in $\mathcal{A}$ are self-adjoint, gives $pz^* = z^*q$. Hence
$$|z|^2 p = (z^*z)p = z^*qz = pz^*z = p|z|^2 \tag{4.52}$$
and p commutes with $|z|^2$ and therefore also with all elements in $C^*(1, |z|^2)$ the C^*-algebra generated by the elements 1 and $|z|^2$ In particular, p commutes with $|z|^{-1}$. Thus
$$q = quu^* = qz|z|^{-1}u^* = zp|z|^{-1}u^* = z|z|^{-1}pu^* = upu^*. \tag{4.53}$$
Hence p and q are unitary equivalent.

□

Thus, as we already mentioned, the second equivalence relation we defined for idempotents in $M_n(C(M))$ is indeed the unitary equivalence.

We have the following dependencies for the three equivalence relations we just defined.

Proposition 4.4.4. *Let p, q be projections in the unital C^*-algebra $\mathcal{A}$. The following statements hold.*

1. *If $p \overset{h}{\sim} q$ then $p \overset{u}{\sim} q$.*
2. *If $p \overset{u}{\sim} q$ then $p \overset{MvN}{\sim} q$.*

Proof.

1. Let $p \stackrel{h}{\sim} q$, and set $z = pq + (1-p)(1-q)$. One can show (see for instance [73]) that the elements z and zz^* are invertible. Consider the unitary element $u = z(zz^*)^{-\frac{1}{2}} = z|z|^{-1}$. We have the relations $pz = pq = zq$ and therefore $p = uqu^*$.

2. Let $p = uqu^*$, with u a unitary element in $\mathcal{A}$. Set $v = up$. This is clearly an element in $\mathcal{A}$. The relations $v^*v = p$ and $vv^* = q$ give the Murray-von Neumann equivalence.

□

Consider the set $P_n(\mathcal{A})$ of a unital C^*-algebra $\mathcal{A}$, and the disjoint union $P(\mathcal{A}) = \cup_n P_n(\mathcal{A})$. We put an addition $\oplus$ on $P(\mathcal{A})$ like we did for $P(C(M))$ in equation (4.47).

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in P_{n+m}(\mathcal{A}), \tag{4.54}$$

for $p \in P_n(\mathcal{A})$ and $q \in P_m(\mathcal{A})$. Define, as in the case of $P(C(M))$, the equivalence relation $p \sim p \oplus 0_m$ on it. Modulo this passing to matrix algebras, the three equivalence relations ($\stackrel{h}{\sim}$, $\stackrel{u}{\sim}$ and $\stackrel{MvN}{\sim}$) are actually equal in $P(\mathcal{A})$. This is a consequence of the following Proposition.

Proposition 4.4.5. *Let p and q be projections in a unital C^*-algebra $\mathcal{A}$. The following statements hold.*

1. *If $p \stackrel{MvN}{\sim} q$ in $\mathcal{A}$ then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \stackrel{u}{\sim} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(\mathcal{A})$.*

2. *$p \stackrel{u}{\sim} q$ in $\mathcal{A}$ then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \stackrel{h}{\sim} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(\mathcal{A})$.*

Proof.

1. Let there be a $v \in \mathcal{A}$ such that $p = v^*v$ and $q = vv^*$. Use the relation $v = qv = vp = qvp$ (see [73]), to show that the elements

$$u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix} \tag{4.55}$$

are unitary in $M_2(\mathcal{A})$. The relation

$$wu \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^*w^* = w \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \tag{4.56}$$

proves the first statement.

2. Let there be a unitary element in $\mathcal{A}$ such that $q = upu^*$. With the relation ([73])
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{h}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.57}$$
we can show that
$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{h}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.58}$$
Hence, there is a continuous path $t \mapsto \omega_t$ of unitary elements in $M_2(\mathcal{A})$, such that
$$\omega_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \omega_1 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}. \tag{4.59}$$
Put $e_t = \omega_t \operatorname{diag}(p, 0)\, \omega_t{}^*$. Every e_t is a projection in $M_2(\mathcal{A})$. The map $t \mapsto e_t$ is continuous, with $e_0 = \operatorname{diag}(p, 0)$ and $e_1 = \operatorname{diag}(q, 0)$. This proves the second statement.

□

Let $\mathcal{A}$ be a unital C^*-algebra. Define $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ as the set $P(\mathcal{A})$ modulo the equivalence relation $p \sim p \oplus 0_m$ and the Murray von Neumann relation. Denote the equivalence class of p in $\mathcal{P}(\mathcal{A})$ as $[p]_P$. Define an addition on this set by $[p]_P + [q]_P = [p \oplus q]_P$. One can show that this addition makes the set of projections $\mathcal{P}(\mathcal{A})$, an Abelian semigroup. we only show the commutativity of the addition. The other properties can be showed in a similar way and are proved in [73]. Let $p \in P_n(\mathcal{A})$ and $q \in P_m(\mathcal{A})$. Consider the element
$$v = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \in P_{n+m}(\mathcal{A}). \tag{4.60}$$
Then $p \oplus q = v^* v \overset{MvN}{\sim} vv^* = q \oplus p$.

The K_0-group is now defined, using the Grothendieck construction (see Section 3.3).

Definition 4.4.6. *The* $\mathbf{K}_0$ **-group** $K_0(\mathcal{A})$*, of a unital* C^**-algebra* $\mathcal{A}$*, is defined as the Grothendieck construction of the Abelian semigroup* $\mathcal{P}(\mathcal{A})$*. The equivalence class of an element* $[p]_P$ *in* $K_0(\mathcal{A})$ *is denoted as* $[p]_0$.

As we showed in Propositions 4.4.4 and 4.4.5, we could have equivalently used one of the other relations to define $\mathcal{P}(\mathcal{A})$, and $K_0(\mathcal{A})$. Thus, if we want to generalise the K_0-group to any (not necessarily unital) C^*-algebras, we have to choose the Murray-von Neumann or the homotopy equivalence relation. One of the problems of these constructions is however, that the functor K_0 is not necessarily half exact in this case. That is, if
$$0 \longrightarrow \mathcal{A} \xrightarrow{\psi} \mathcal{B} \xrightarrow{\phi} \mathcal{C} \longrightarrow 0 \tag{4.61}$$

is an exact sequence of C^*-algebras, the sequence

$$0 \longrightarrow K_0(\mathcal{A}) \xrightarrow{K_0\psi} K_0(\mathcal{B}) \xrightarrow{K_0\phi} K_0(\mathcal{C}) \longrightarrow 0 \tag{4.62}$$

does not need to obey the property $\operatorname{im} K_0\,\psi = \ker K_0\,\phi$. The functor $K_0\phi : K_0(\mathcal{A}) \to K_0(\mathcal{B})$ on an arrow $\phi : \mathcal{A} \to \mathcal{B}$ is here given by $K_0\phi([p]_0) = [\phi(p)]_0$, where ϕ is extended to a map from $M_n(\mathcal{A})$ to $M_n(\mathcal{B})$ by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{pmatrix}. \tag{4.63}$$

For more on the functoriality of K_0 see, for instance, [73].

A consequence of the fact that K_0 need not be half exact, is that the functor does not necessarily respect the direct sum, in the sense that $K_0(\mathcal{A} \oplus \mathcal{B})$ does not need to equal $K_0(\mathcal{A}) \oplus K_0(\mathcal{B})$. As is mentioned later we would like the functor K_0 to have this property.

To ensure that the functor K_0 is half exact, we construct it in a different way, using the unitisation $\mathcal{A}_{\mathbb{1}}$ of $\mathcal{A}$ discussed at the end of Section 4.2.

Definition 4.4.7. *Let $\mathcal{A}$ be a C^*-algebra without unit, and $\mathcal{A}_{\mathbb{1}}$ its unitisation. Consider the split exact sequence*

$$0 \longrightarrow \mathcal{A} \xrightarrow{i} \mathcal{A}_{\mathbb{1}} \xrightarrow{\pi} \mathbb{C} \longrightarrow 0 \tag{4.64}$$

with i the inclusion and π the projection $\pi(a, \lambda) = \lambda$. The $\mathbf{K_0}$**-group** *$K_0(\mathcal{A})$ of the C^*-algebra $\mathcal{A}$ is given by the kernel of the homomorphism $K_0\pi : K_0(\mathcal{A}_{\mathbb{1}}) \to K_0(\mathbb{C})$.*

If $\mathcal{A}$ is unital, $K_0 i$ is just the identity and $K_0\pi$ maps to zero. In this case, both definitions of the K_0-group are equivalent, hence Definition 4.4.7 is indeed a generalisation of Definition 4.4.6.

Chapter 5

The noncommutative Chern character

In the previous Chapter we discussed the link between compact Hausdorff spaces and C^*-algebras. In this Chapter we construct a Chern character defined on a C^*-algebra. If the C^*-algebra is noncommutative the character is called the noncommutative Chern character.

5.1 The algebraic Chern character

In this Section we give the algebraic structure of the Chern character we defined in Chapter 3. This Chern character is defined through a connection on a smooth vector bundle E over a manifold M. The corresponding algebra of a compact manifold M is $C^\infty(M)$, a dense subalgebra of $C(M)$. Although $C^\infty(M)$ is not a C^*-algebra, we can characterise M by it, in a Gelfand-Naimark way. Hence we also have a translation from spaces to algebras in the smooth case. Propositions 4.3.5 and 4.3.6 can be translated immediately into the smooth variant: the category of smooth vector bundles over a compact manifold M, is equivalent to the category of finitely generated projective $C^\infty(M)$-modules. Also, the C^*-algebraic K-theory of $C(M)$, has its smooth variant. One does this by replacing the set of projections in $C(M)$ with an open neighbourhood to which it is homotopy equivalent. If $i : C^\infty(M) \to C(M)$ is the inclusion map, $K_0 i : K_0(C^\infty(M)) \to K_0(C(M))$ is an isomorphism. Thus if we want to consider the K_0-group of the smooth algebra $C^\infty(M)$ it is sufficient to consider the C^*-algebraic K-theory of $C(M)$. For more on this construction, with $C^\infty(M)$ as algebra, see [46].

As we saw in Section 3.2, the Chern character is a map from the group $K^0(M)$ to the de Rham cohomology $H^*_{dR}(M)$. In Section 4.4 we saw that we can relate $K^0(M)$ to the K_0-group of the C^*-algebra $C(M)$, therefore also to the K_0-group of $C^\infty(M)$.

We defined the Chern character via a connection on a vector bundle. Such a connection can be generalised to a connection on a finitely generated projective module. It is then necessary to identify the ring of C^∞-differential forms $\Omega(M)$, as a differential graded algebra over $C^\infty(M)$, i.e., a graded algebra together with a derivation d of degree 1, such that d is a differential. See for a definition of these terms Section 5.2. To every associative algebra we define a corresponding differential graded algebra given in Section 6.1. With this information we are ready to define a connection from purely algebraic constructions.

Definition 5.1.1. *Let $\mathcal{A}$ be a commutative algebra, $\mathcal{E}$ a finitely generated projective module over $\mathcal{A}$, and $\Omega(\mathcal{A}) = \oplus_p \Omega^p(\mathcal{A})$ the corresponding differential graded algebra over $\mathcal{A}$, with derivation d. A connection is an operator*

$$\nabla : \mathcal{E} \otimes \Omega^p(\mathcal{A}) \to \mathcal{E} \otimes \Omega^{p+1}(\mathcal{A}), \tag{5.1}$$

that satisfies the relation

$$\nabla(s \otimes a) = (\nabla s) \otimes a + s \otimes da, \tag{5.2}$$

for $s \in \mathcal{E}$ and $a \in \Omega^p(\mathcal{A})$.

Because $\mathcal{E}$ is a finitely generated projective module over $\mathcal{A}$, it can be written as $e\mathcal{A}^n$, with e an idempotent in $M_n(\mathcal{A})$. The module $\mathcal{E}$ is then determined by this projection. If ∇ is a some connection on $\mathcal{A}^n$, then $(e \otimes 1)\nabla$ is a connection on $e\mathcal{A}^n$. Consider now the connection coming from the derivation d of the graded algebra $\Omega(\mathcal{A})$. Since the Chern character is invariant with respect to the connection on the vector bundle, as we saw in Chapter 3, it should also be invariant in the algebraic case (as we will see in the following Section). Hence we can define the algebraic Chern character in terms of the matrix valued one-form connection $\nabla = ed$ on $\mathcal{E}$.

The corresponding curvature $\Omega = \nabla^2$ is given by,

$$\Omega s = (ed)^2 s = e\, de\, ds. \tag{5.3}$$

Algebraically a Chern character defined on a finitely projective module $\mathcal{E} = e\mathcal{A}^n$ is equal to a Chern character defined on the corresponding idempotent e.

Definition 5.1.2. *The* **Chern character** *on projections $e \in M_n(\mathcal{A})$ is defined as*

$$ch(e) := \mathrm{Tr} e^{\Omega} = \sum_{k=0}^{\infty} ch_{2k} e := \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{Tr}(d_\nabla e)^{2k}. \tag{5.4}$$

The trace here is the matrix trace over the matrix valued two-form Ω.

In the next Section we are going to generalise this construction to arbitrary C^*-algebras $\mathcal{A}$.

5.2 Noncommutative geometry

In this Section we define a calculus on any unital, not necessarily commutative, C^*-algebra. With this calculus we are able to define a generalisation of the Chern character of Section 5.1.

In Section 5.1 we used an object (Ω, d), called a differential graded algebra, to describe the ring of differentials of a vector bundle algebraically. Such a construction is defined as a graded algebra Ω with a derivation d of degree 1, such that d is a differential. Hence, $\Omega = \oplus_{p\geq 0}\Omega^p$, with

$$d : \Omega^p \to \Omega^{p+1}, \quad d^2 = 0, \tag{5.5}$$

and

$$ab \in \Omega^{p+q} \quad \text{whenever} \quad a \in \Omega^p,\, b \in \Omega^q. \tag{5.6}$$

For a calculus on an algebra we need, besides the differentials in the form of the graded algebra, also an integral. Such an integral is defined through a trace on the algebra. A total calculus is defined by a cycle.

Definition 5.2.1. *A n-dimensional* **cycle** *is a triple $(\Omega, d, \int)$, where $\Omega = \oplus_{j=0}^{n}\Omega^n$ together with d is some differential graded algebra, and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace.*

Remark that we can consider (Ω, d) as a cochain complex (see Definition 6.0.6). And because the trace is closed we can consider the triple $(\Omega, d, \int)$ as a (co)cycle of this complex. This justifies the use of the term cycle.

A cycle $(\Omega, d, \int)$ together with a homomorphism $\mathcal{A} \stackrel{\rho}{\to} \Omega^0$ is called a **cycle over** $\mathcal{A}$. Such a cycle over $\mathcal{A}$ defines the calculus on $\mathcal{A}$.

The classical Chern character was defined through a connection on the vector bundle. The algebraic Chern character was defined through a connection on a $C(M)$-module. The noncommutative Chern character is in the same manner defined through a connection on a finite generated projective $\mathcal{A}$-module.

Definition 5.2.2. *Let $\mathcal{A} \stackrel{\rho}{\to} \Omega$ be a cycle over the unital C^*-algebra $\mathcal{A}$, and $\mathcal{E}$ be a finitely generated projective module over $\mathcal{A}$. Then a connection ∇ on $\mathcal{E}$ is an linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ such that*

$$\nabla(\xi x) = (\nabla\xi)x + \xi \otimes d\rho(x), \quad \forall \xi \in \mathcal{E}, x \in \mathcal{A}. \tag{5.7}$$

Note that this definition generalises the definitions given in Sections 3.1 and 5.1. Again, one can extend this connection to a map $\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$, by imposing

$$\nabla(\xi \otimes \omega) = (\nabla\xi)\omega + \xi \otimes d\omega, \quad \forall \xi \in \mathcal{E}, \omega \in \Omega. \tag{5.8}$$

In [29] Connes showed that every finitely generated projective module admits a connection. Furthermore he proved that the extension of a connection to a map $\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$ is unique. He proved this by considering $\mathcal{E} \otimes_{\mathcal{A}} \Omega$ as a finitely generated projective Ω-module.

As in the previous Sections we define the Chern character Ch over a projective module, in terms of the curvature $\Theta = \nabla^2$.

In Section 3.1 we denoted the curvature Θ as a matrix valued 2-form, so we could use the normal (matrix) trace on it, in Section 3.2. This was done by seeing the curvature as an element in the algebra

$$\begin{aligned} \Gamma^\infty\big(End(E) \otimes \Lambda^2 T^* M\big) &\simeq End_{\Omega(M)}\big(\Gamma^\infty(E) \otimes_{C^\infty(M)} \Omega^2(M)\big) \\ &\simeq End\big(\Gamma^\infty(E)\big) \otimes_{C^\infty(M)} \Omega^2(M), \\ &= End(\Gamma) \otimes \Omega^2, \end{aligned} \tag{5.9}$$

and choosing a base in $\Gamma^\infty(E)$. The last equation defines the shorthand notation we will use from now on.

Thereafter we showed that the Chern character was closed. We did this, using the Bianchi identity $\nabla(\Theta) = 0$ and the fact that the trace is an invariant polynomial. In the Bianchi identity (3.15), we implicitly used a connection $\widetilde{\nabla}$ on the algebra $End_\Omega\big(\Gamma \otimes \Omega\big)$. This connection is induced by ∇ through

$$\widetilde{\nabla}(T) = \nabla \circ T - (-1)^p T \circ \nabla, \tag{5.10}$$

where p is the degree of T in $End\big(\Gamma\big) \otimes \Omega$, defined as follows. Let $S \in \Gamma \otimes \Omega^k$. The endomorphism T is of degree p, if $TS \in \Gamma \otimes \Omega^{k+p}$.

To build the noncommutative Chern character, we repeat this procedure for the connection of definition 5.2.2 (that is, for the connection given in equation (5.8)). Inspired by equation (5.10) we construct a derivation on $End_\Omega(\mathcal{E} \otimes_{\mathcal{A}} \Omega)$.

Lemma 5.2.3. *For every connection $\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$, we have an induced derivation $\widetilde{\nabla}$ of degree 1 defined by*

$$\begin{aligned} &\widetilde{\nabla} : End_\Omega(\mathcal{E} \otimes_A \Omega) \to End_\Omega(\mathcal{E} \otimes_A \Omega), \\ &\widetilde{\nabla}(T) = \nabla \circ T - (-)^p T \circ \nabla, \end{aligned} \tag{5.11}$$

for every $T \in End_\Omega(\mathcal{E} \otimes_A \Omega)$ with degree p. This derivation obeys the Bianchi identity $\widetilde{\nabla}(\nabla^2) = 0$.

Proof.

1. The operator $\widetilde{\nabla}$ is Ω-linear. Let $\omega \in \Omega^k$ and $\eta \in \mathcal{E} \otimes_{\mathcal{A}} \Omega$. We have, for every $T \in End_\Omega(\mathcal{E} \otimes_A \Omega)$ of degree p,
$$\begin{aligned} \nabla T(\omega\eta) &= \nabla\big((-1)^{pk}\omega T(\eta)\big) \\ &= (-1)^{pk} d(\omega) T(\eta) + (-1)^{pk}(-1)^k \omega \nabla(T(\eta)), \end{aligned} \tag{5.12}$$
and
$$\begin{aligned} T(\nabla(\omega\eta)) &= T\big(d(\omega)\eta + (-1)^k \omega \nabla(\eta)\big) \\ &= (-1)^{p(k+1)} d(\omega) T(\eta) + (-1)^k (-1)^{pk} \omega T(\nabla(\eta)), \end{aligned} \tag{5.13}$$
hence
$$\widetilde{\nabla}(T)(\omega\eta) = (-1)^{pk+k}\omega\widetilde{\nabla}(T)(\eta) = (-1)^{k(p+1)}\omega\widetilde{\nabla}(T)(\eta). \tag{5.14}$$

2. The operator $\widetilde{\nabla}$ is a derivation. Let $T, T' \in End_\Omega(\mathcal{E} \otimes \Omega)$ with degree p and p' respectively.
$$\begin{aligned} \widetilde{\nabla}(T)T' + (-1)^p T\widetilde{\nabla}(T') &= \big(\nabla \circ T - (-1)^p T \circ \nabla\big)T' \\ &\quad + (-1)^p T\big(\nabla(T') - (-1)^{p'} T' \circ \nabla\big) \\ &= \nabla \circ (TT') - (-1)^{p+p'} TT' \circ \nabla. \end{aligned} \tag{5.15}$$

3. The operator $\widetilde{\nabla}$ obeys the Bianchi identity.
$$\widetilde{\nabla}(\nabla^2) = \nabla \circ \nabla^2 - \nabla^2 \circ \nabla = 0 \tag{5.16}$$

□

Having constructed a derivation on $End_\Omega(\mathcal{E} \otimes_{\mathcal{A}} \Omega)$, we want to put a trace on it. We use the ungraded trace
$$\mathrm{Tr}_{\mathcal{A}} : End_\Omega(\mathcal{E}) \to \mathcal{A}/[\mathcal{A}, \mathcal{A}] \tag{5.17}$$
from the endomorphisms on $\mathcal{E}$ to the algebra $\mathcal{A}$, modulo commutators. This trace is just the normal matrix trace, since $\mathcal{E}$ is a finitely generated projective module. We extend this trace canonically to a trace
$$\mathrm{Tr}_\Omega : End_\Omega(\mathcal{E} \otimes_{\mathcal{A}} \Omega) \to \Omega/[\Omega, \Omega]. \tag{5.18}$$
A more detailed construction of these traces can be found in [33].

We are now able to define the noncommutative Chern character.

Definition 5.2.4. *Let $\mathcal{A} \xrightarrow{\rho} \Omega$ be a cycle over the unital (not necessarily commutative) algebra $\mathcal{A}$, and let $\mathcal{E}$ be a finitely generated projective module over $\mathcal{A}$. The noncommutative Chern character Ch of $\mathcal{E}$ is defined by*
$$Ch(\mathcal{E}) = \sum_j \frac{1}{j}\mathrm{Tr}_\Omega(\frac{\imath}{2\pi}\nabla^2)^j, \tag{5.19}$$
where ∇ is a connection, associated to the cycle ρ and $\mathcal{E}$.

With the important property

$$d \circ \mathrm{Tr}_\Omega = \mathrm{Tr}_\Omega \circ \widetilde{\nabla}, \tag{5.20}$$

and the Bianchi identity one can show that this Chern character is closed in the calculus of the cycle $\mathcal{A} \overset{\rho}{\to} \Omega$ over $\mathcal{A}$ (see [33]). Therefore this character defines a cohomology class. More on this noncommutative cohomology can be found in the following Chapter. Again in reference [33], one shows that the cohomology classes retained from the Chern character are independent of the chosen connection ∇. Every finitely generated projective $\mathcal{A}$-module (modulo isomorphisms) can be associated with an element $p \in K_0(\mathcal{A})$ (see Chapter Sections 4.3 and 4.4). Consequently, given a cycle over an algebra $\mathcal{A}$ and a finitely generated projective module over that algebra, the Chern character is a well-defined map from the K_0-group of $\mathcal{A}$ to some cohomology corresponding to the cycle over $\mathcal{A}$.

Remark 5.2.5. We defined the noncommutative Chern character for a unital algebra. The character for a non-unital algebra $\mathcal{A}$ can be constructed in the following way. Consider the unitisation $\mathcal{A}_{\mathbb{1}}$ of the non-unital algebra $\mathcal{A}$ (see last paragraph of Section 4.2). As we saw in Definition 4.4.7, the K_0-group of $\mathcal{A}$ is given by the kernel of the map $K_0\pi : K_0(\mathcal{A}_{\mathbb{1}}) \to K_0(\mathbb{C})$. Consider the Chern character $Ch_{\Omega_{\mathbb{1}}}$ corresponding to the unitised cycle $\mathcal{A}_{\mathbb{1}} \overset{\rho_{\mathbb{1}}}{\to} \Omega_{\mathbb{1}}$. The restriction of $Ch_{\Omega_{\mathbb{1}}}$ to the kernel of $K_0\pi$ in the diagram

$$\begin{array}{ccc} K_0(\mathcal{A}_{\mathbb{1}}) & \xrightarrow{Ch_{\Omega_{\mathbb{1}}}} & H(\mathcal{A}_{\mathbb{1}}) \\ {\scriptstyle K_0\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ K^0(\mathbb{C}) & \xrightarrow{\dim} & \mathbb{C} \end{array} \tag{5.21}$$

defines the Chern character from $K_0(\mathcal{A})$ to the cohomology group $H(\mathcal{A})$.

Remark 5.2.6. In [29] Connes gives an explicit cycle over the algebra $End_{\mathcal{A}}(\mathcal{E})$ corresponding to the cycle $\rho : \mathcal{A} \to \Omega$. He does this through the following construction. Consider the graded algebra $End_\Omega(\mathcal{E} \otimes \Omega)$ with the graded derivation of Lemma 5.2.3 and the graded trace of equation (5.18). This is almost a cycle over $End_\Omega(\mathcal{E} \otimes \Omega)$, except that the derivation $\widetilde{\nabla}$ does not obey $\widetilde{\nabla}^2 = 0$. But using the Bianchi identity of Lemma 5.2.3 and the fact that $\widetilde{\nabla}^2(T) = \nabla^2 T - T\nabla^2$ for all $T \in End_\Omega(\mathcal{E} \otimes \Omega)$, Connes proves that one can construct a cycle over $End_\Omega(\mathcal{E} \otimes \Omega)$ by adding an element X of degree 1 to this algebra. Taking now the usual homomorphism from $End_{\mathcal{A}}(\mathcal{E})$ to $End_\Omega(\mathcal{E} \otimes \Omega)$ gives us a cycle over $End_{\mathcal{A}}(\mathcal{E})$. The character of this cycle (see equation (5.33)) defines the noncommutative Chern character of $\mathcal{E}$.

5.3 Fredholm modules

In this Section we will construct an explicit cycle over a algebra $\mathcal{A}$. This construction uses the notion of a Fredholm module, which is a noncommutative generalisation of an elliptic operator on a compact Hausdorff space M. An elliptic operator on a compact Hausdorff space (see [1]) is constructed as follows.

Let H_1 and H_2 be two Hilbert spaces and let π_1 and π_2 be two corresponding uniform continuous representations of the C^*-algebra $C(M)$ on H_1 and H_2. A bounded linear operator

$$P : H_1 \to H_2 \tag{5.22}$$

is an operator on M if for every element $f \in C(M)$, the commutator $P\pi_1(f) - \pi_2(f)P$ is a compact operator. If this operator is also Fredholm, i.e. $\ker P$ and $\operatorname{coker} P$ are finite-dimensional and P has closed range, the operator is an elliptic operator on M. One can show ([1]) that, to every elliptic operator on M, one can associate a K-cycle in the K-homology of M. This homology can be seen as the dual of (topological) K-theory. For more on K-homology see the overview [10] of Baum and Douglas or the reference [45].

An elliptic operator can be seen as an operator between the $C(M)$-modules H_1 and H_2. This motivates the definition of a Fredholm operator over any C^*-algebra $\mathcal{A}$.

Definition 5.3.1. *Let A be a C^*-algebra. An even* **Fredholm module** *over $\mathcal{A}$ is defined as a pair (H, F) such that*

1. *the set $H = H_1 \oplus H_2$ is a $\mathbb{Z}/2$-graded Hilbert space with grading operator γ, i.e. γh is h for $h \in H_1$ and $-h$ for $h \in H_2$,*
2. *there exists a representation π of $\mathcal{A}$ on the Hilbert space H,*
3. *the element F is an operator on H such that $F = F^*$, $F^2 = 1$, $\gamma F = -F\gamma$ and the graded commutator $[F, \pi(a)]$ is compact for every $a \in \mathcal{A}$.*

If we take H without any grading, (H, F) is called an *odd* Fredholm module. This is a special case of an even Fredholm module and generalises an elliptic operator from H to H. If there can be no confusion we often write $\pi(a) = a \in \mathcal{L}(H)$, for simplicity.

We are now able to define a **n-summable Fredholm module** over a C^*-algebra $\mathcal{A}$. This is an even Fredholm module (H, F) such that

$$[F, a] \in \mathcal{L}^n(H), \tag{5.23}$$

where $\mathcal{L}^n(H)$ is the nth Schatten class. This class is an ideal of the compact operators on H, with the ℓ^n-convergency property on the eigenvalue expansion of its operators. The most common Schatten classes are the trace class $\mathcal{L}^1(H)$

(Tr $|T| < \infty$) and the Hilbert-Schmidt class $\mathcal{L}^2(H)$ (Tr $T^*T < \infty$). Two important properties of these ideals are that $\mathcal{L}^p(H) \subset \mathcal{L}^r(H)$ for $p < r$ and that they obey Hölders inequality

$$\mathrm{Tr}|TS| \leq ||T||_p||S||_q \quad \text{whenever} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{5.24}$$

For more on Schatten classes see for instance [39].

With this n-summable Fredholm module over $\mathcal{A}$ we build up a cycle over $\mathcal{A}$. Let $\mathcal{A}$ be a unital algebra. We define a derivation d on $\mathcal{L}(H)$ by,

$$dT = \imath[F,T] \quad \text{for all} \quad T \in \mathcal{L}(H). \tag{5.25}$$

With this derivation we construct a differential graded algebra (Ω, d). Let, for each $j \in \mathbb{N}$, Ω^j be the linear span in $\mathcal{L}(H)$, of the operators

$$a_0 da_1 \cdots da_j, \quad a_k \in \mathcal{A}. \tag{5.26}$$

The following Lemma shows that d is indeed a (graded) derivation and that (Ω, d) is a differential graded algebra.

Lemma 5.3.2.

1. $d^2T = 0 \quad \forall T \in \mathcal{L}(H)$.
2. $d(T_1T_2) = (dT_1)T_2 + (-1)^{\partial T_1}T_1dT_2 \quad \forall T_1, T_2 \in \mathcal{L}(H)$.
3. $d\Omega^j \subset \Omega^{j+1}$.
4. $\Omega^j \times \Omega^k \subset \Omega^{j+k}$.
5. $\Omega^k \subset \mathcal{L}^{(n+1)/k}(H)$.

The proof of this Lemma (see [29]) uses some straightforward calculations and the Hölder inequality for part 5.

From this Lemma we conclude that $\Omega = \oplus_{j=0}^n \Omega^j$ is indeed a graded algebra, and together with the derivation d, it will form a differential graded algebra. We only need a closed graded trace to complete the cycle. This trace will be defined in terms of the supertrace Tr_s.

Definition 5.3.3. *Let $T \in \mathcal{L}(H)$ such that $[F,T] \in \mathcal{L}^1(H)$. The* **supertrace** Tr_s *on T will be defined by*

$$\mathrm{Tr}_s(T) = \frac{1}{2}\mathrm{Tr}(\gamma F[F,T]). \tag{5.27}$$

Remark that the trace on $\gamma F[F,T]$ is well-defined, because F is a bounded operator and $[F,T]$ is trace class, hence by Hölders inequality $F[F,T] \in \mathcal{L}^1(H)$.

We mention some useful properties of this supertrace without proof (see [29]).

Lemma 5.3.4.

1. *If* $T \in \Omega^i$, *with* i *odd, then* $\mathrm{Tr}_s(T) = 0$.
2. *If* $T \in \mathcal{L}^1(H)$, *then* $\mathrm{Tr}_s(T) = \mathrm{Tr}(\gamma T)$.

Due to Lemma 5.3.2, $[F, T]$ is an element of $\mathcal{L}^1(H)$ for every homogeneous $T \in \Omega^n$. This ensures us that the supertrace is well-defined on Ω^n.

Lemma 5.3.5. *The restriction of* Tr_s *to* Ω^n *defines a closed graded trace on the differential graded algebra* Ω.

Proof. Let $\omega \in \Omega^{n-1}$, then $\mathrm{Tr}_s(d\omega) = 0$ because $d^2 = 0$. So Tr_s is a closed trace.

We now want to show that it is a graded trace. Hence, we want to show that for $\omega_1 \in \Omega^{n_1}, \omega_2 \in \Omega^{n_2}, n_1 + n_2 = n$

$$\mathrm{Tr}_s(\omega_1\omega_2) = (-1)^{n_1 n_2}\mathrm{Tr}_s(\omega_2\omega_1). \tag{5.28}$$

If n is odd, equation (5.28) is an empty statement, due to Lemma 5.3.4 part 1. If we take n even, we have

$$(-1)^{n_1+n_2} = 1 \quad \text{and} \quad (-1)^{n_1} = (-1)^{n_2} = (-1)^{n_1 n_2}. \tag{5.29}$$

Because of the trace property and the fact that γF commutes with $d\omega_1$ and $d\omega_2$, we have

$$\begin{aligned}
\mathrm{Tr}_s(\omega_1\omega_2) &= \mathrm{Tr}(\gamma F d(\omega_1\omega_2)) \\
&= \mathrm{Tr}(\gamma F d\omega_1\omega_2) + (-1)^{n_1}\mathrm{Tr}(\gamma F \omega_1 d\omega_2) \\
&= \mathrm{Tr}(\gamma F \omega_2 d\omega_1) + (-1)^{n_1}\mathrm{Tr}(\gamma F d\omega_2\omega_1) \\
&= (-1)^{n_1}\mathrm{Tr}(\gamma F d(\omega_2\omega_1)) \\
&= (-1)^{n_1 n_2}\mathrm{Tr}_s(\omega_2\omega_1).
\end{aligned} \tag{5.30}$$

□

We are now able to associate a n-dimensional cycle over $\mathcal{A}$, to a given $(n+1)$-dimensional Fredholm module (H, F) over $\mathcal{A}$.

Definition 5.3.6. *Let* $n = 2m$ *an integer and* (H, F) *be a* $n+1$*-dimensional Fredholm module over* $\mathcal{A}$. *The* **associated cycle** *over* $\mathcal{A}$, *will be given by the differential graded algebra* (Ω, d) *following from Lemma 5.3.2, together with the integral*

$$\int \omega = (2\imath\pi)^m m! \mathrm{Tr}_s(\omega) \quad \forall \omega \in \Omega^n, \tag{5.31}$$

and the homomorphism $\pi : \mathcal{A} \to \Omega^0 \subset \mathcal{L}(H)$.

This cycle can be seen as the dual of an element in $K_0(\mathcal{A})$. In the following Section we will see how this cycle induces a functional on $\mathcal{A}^{\otimes n+1}$ that can be extended to a functional on $K_0(\mathcal{A})$.

5.4 The Chern character of a Fredholm module

We mentioned in the previous Section that a Fredholm module is a generalisation of an elliptic operator P on a compact Hausdorff space M. Because P is a Fredholm operator by definition, we can define

$$\mathrm{Index}P := \dim\ker P - \dim\mathrm{coker}P < \infty. \tag{5.32}$$

An important property of this Index map is that it is invariant under compact perturbations, i.e. $\mathrm{Index}(P+K) = \mathrm{Index}(P)$ for every compact operator K on the Hilbert space. This map classifies elliptic operators on M, by inducing an isomorphism between these operators and the dual of $K^0(M)$ (see [1]). The index map can be expressed in topological terms, using (topological) K-theory and in analytical (local) terms, using the symbol class $\sigma(P)$ of P. The equality of both expressions is known as the Atiyah-Singer Theorem ([2]). In reference [3] it is shown that this can be expressed in cohomological language, using the Chern character. In this Section we generalise this notion to the noncommutative language following Connes ([31]). To do that we first have to define a Chern character of a Fredholm module.

One can associate to every cycle $\mathcal{A} \xrightarrow{\rho} \Omega$ over an algebra $\mathcal{A}$, a unique character $\tau : A^{\otimes n+1} \to \mathbb{C}$. This character is determined by

$$\tau(a_0, \ldots, a_n) = \int \rho(a_0)\, d(\rho(a_1)) \cdots d(\rho(a_n)). \tag{5.33}$$

Proposition 5.4.1. *Let $n = 2m$ be an even integer and let (H, F) be a $(n+1)$-summable Fredholm module over $\mathcal{A}$. Consider the corresponding character τ_n of the associated cycle of Definition 5.3.6:*

$$\tau_n(a_0, \ldots, a_n) = (2\imath\pi)^m m! \mathrm{Tr}_s(a_0\, da_1 \cdots da_n). \tag{5.34}$$

The following equations hold:

1. $\tau_n(a_1, \ldots, a_n, a_0) = \tau_n(a_0, \ldots, a_n)$,
2. $\sum_{j=0}^{n}(-1)^j \tau_n(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1}\tau_n(a_{n+1}a_0, \ldots, a_n) = 0$
.

Proof. Assume $\mathcal{A} = \Omega^0$ so we can omit ρ.

1. Because Tr_s is a closed graded trace, we have

$$\begin{aligned}\tau_n(a_0, \ldots, a_n) &= \int a_0\, da_1 da_2 \cdots da_n = (-1)^{(n-1)1} \int da_2 \cdots da_n a_0 da_1 \\ &= (-1)^n \int da_2 \cdots da_n da_0 a_1 = \tau_n(a_1, \ldots, a_n, a_0).\end{aligned} \tag{5.35}$$

2. The Leibniz identity gives

$$\begin{aligned}da_1 \cdots da_n a_{n+1} = &\sum_{j=1}^{n} (-1)^{n-j} da_1 \cdots d(a_j a_{j+1}) \cdots da_{n+1} \\ &+ (-1)^n a_1 da_2 \cdots da_{n+1}.\end{aligned} \tag{5.36}$$

Consequently,

$$\begin{aligned}&\sum_{j=0}^{n} (-1)_j \tau_n(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \\ &\qquad = \sum_{j=0}^{n} (-1)^{n-j} \int a_0 da_1 \cdots d(a_j a_{j+1}) \cdots da_{n+1} \\ &\qquad = \int a_0 a_1 da_2 \cdots da_{n+1} + \int a_0 da_1 \cdots da_n a_{n+1} \\ &\qquad\quad - (-1)^n \int a_0 a_1 da_2 \cdots da_{n+1} \\ &\qquad = (-1)^n \tau_n(a_{n+1} a_0, a_1, \ldots, a_n).\end{aligned} \tag{5.37}$$

□

In the following Chapter we will see that the properties of τ_n, given in Proposition 5.4.1, make τ_n an element of $HC^n(\mathcal{A})$, where $HC^n(\mathcal{A}) \subset HC(\mathcal{A})$ is the nth cyclic cohomology group.

Remark 5.4.2. There seems to be an ambiguity in our construction of $\tau_n \in HC(\mathcal{A})$, because of the fact that $\mathcal{L}^{n+1}(H) \subset \mathcal{L}^{n+3}(H)$, i.e. whenever (H, F) is a $(n+1)$-summable Fredholm module it is also a $(n+3)$-summable Fredholm module. Hence we could as well define τ_{n+2} instead of τ_2 to represent the corresponding character in the cohomology $HC(\mathcal{A})$. There is however a map $S : HC^n \to HC^{n+2}$ (see Section 6.2) such that $S\tau_n = \tau_{n+2}$ in $HC(\mathcal{A})$. And in periodic cyclic cohomology $HP(\mathcal{A}) \subset HC(\mathcal{A})$ (see Definition 6.2.8) we even have the equality $\tau_n = \tau_{n+2}$. Consequently, in $HP(\mathcal{A})$ we have a unique character τ corresponding to a Fredholm module (H, F).

Motivated by equations (5.4) and (5.19) we define the Chern character of a Fredholm module in terms of the character of equation (5.34).

Definition 5.4.3. *Let (H, F) be a finite summable Fredholm module over $\mathcal{A}$. The Chern character $Ch(H, F)$ of this module, is the element of $H^*(\mathcal{A})$ given by one of the characters τ_{2m}, with m big enough.*

The next Theorem is a generalisation of the cohomological Index Theorem of Atiyah and Singer, due to Connes ([29]).

Theorem 5.4.4. *Let $n = 2m$ and let (H, F) be a $n+1$-summable Fredholm module over a unital algebra $\mathcal{A}$. Let $\langle \cdot, \cdot \rangle$ be the pairing between the K_0-group of $\mathcal{A}$ and the cyclic cohomology group H^n_λ (see Proposition 6.2.7 in the following Chapter). The Index map $K_0 \to \mathbb{Z}$ is given by*

$$\mathrm{Index} F_e^1 = \langle [e], [\tau_n] \rangle, \tag{5.38}$$

where $[e] \in K_0(\mathcal{A})$, with e a projection in $\mathcal{A}$, $[\tau_n]$ the class in H^n_λ of the corresponding character of the Fredholm module, and F_e^1 the Fredholm operator $e(F \otimes 1)e$ from $e(H_1 \otimes \mathbb{C})$ to $e(H_2 \otimes \mathbb{C})$.

Proof. If we replace $\mathcal{A}$ by $M_k(\mathcal{A})$ and (H, F) by $(H \otimes \mathbb{C}^k, F \otimes \mathbb{1})$, we may assume $k = 1$. Since $\gamma F = -F\gamma$ and $F^2 = 1$ we have P, Q such that

$$F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}, \tag{5.39}$$

with $PQ = \mathbb{1}_{H_1}$ and $QP = \mathbb{1}_{H_2}$. Let $H^+ = eH_1$ and $H^- = eH_2$. Define P' (resp. Q') as the operator eP (resp. eQ) restricted to H^+ (resp. H^-). Hence, $\mathbb{1}_{H^+} - Q'P'$ (resp. $\mathbb{1}_{H^-} - P'Q'$) is the restriction of $e - eFeFe$ to H^+ (resp. H^-). We give the following Proposition without proof (see [29] or [39]).

Proposition 5.4.5. *Let $P, Q \in \mathcal{L}(H)$ be such that, for positive integer p, $\mathbb{1} - PQ$ and $\mathbb{1} - QP$ are elements in $\mathcal{L}^p(H)$. Then P is a Fredholm operator and for any integer $N \geq p$ one has*

$$\mathrm{Index} P = \mathrm{Tr}(\mathbb{1} - QP)^N - \mathrm{Tr}(\mathbb{1} - PQ)^N. \tag{5.40}$$

Because $e - eFeFe = -e[F, e]^2 e$ and $[F, e] \in \mathcal{L}^{n+1}(H)$ we have $e - eFeFe \in \mathcal{L}^m(H)$ and Proposition 5.4.5 gives

$$\begin{aligned} \mathrm{Index} P' &= \mathrm{Tr}(\mathbb{1}_{H^+} - Q'P')^{m+1} - \mathrm{Tr}(\mathbb{1}_{H^-} - P'Q')^{m+1} \\ &= \mathrm{Tr}\gamma(e - eFeFe)^{m+1}. \end{aligned} \tag{5.41}$$

We show that the pairing (see equation (6.44))

$$\langle e, \tau_n \rangle = \frac{(-1)^m}{2} \mathrm{Tr}\big(\gamma F[F, e[F, e]^{2m}]\big) = \frac{(-1)^m}{2} \mathrm{Tr}\big(\gamma F[F, e]^{2m+1}\big) \tag{5.42}$$

equals equation (5.41).

Because of the identity $[F,e] = e[F,e] + [F,e]e$ we have

$$\mathrm{Tr}\big(\gamma F[F,e]^{2m+1}\big) = \mathrm{Tr}\big(\gamma Fe[F,e][F,e]^{2m}\big) + \mathrm{Tr}\big(\gamma F[F,e]e[F,e]^{2m}\big). \tag{5.43}$$

Now, using $\gamma F = -F\gamma$, $F[F,e]^{2m+1} = -[F,e]^{2m+1}F$ and the trace property, the first term becomes

$$\mathrm{Tr}\big(\gamma Fe[F,e]^{2m+1}\big) = \mathrm{Tr}\big(\gamma eF[F,e]^{2m+1}\big). \tag{5.44}$$

Because $e[F,e]^2 = [F,e]^2 e$ and $e^2 = e$ equation (5.43) equals

$$\begin{aligned} \mathrm{Tr}\big(\gamma F[F,e]^{2m+1}\big) &= 2\mathrm{Tr}\big(\gamma eF[F,e]e[F,e]^{2m}\big) \\ &= 2(-1)^m \mathrm{Tr}\big(\gamma(e - eFeF)^{m+1}\big). \end{aligned} \tag{5.45}$$

This concludes the proof. □

Chapter 6

The algebraic cohomology

De Rham cohomology is a topological invariant in de sense that, whenever two manifolds M and N are diffeomorphic, their de Rham cohomology groups $H_{dR}(M)$ and $H_{dR}(N)$ are isomorphic. Because the corresponding algebras $C^\infty(M)$ and $C^\infty(N)$ (see Chapter 4) are also isomorphic, the de Rham cohomology can be expressed in algebraic terms. This Chapter discusses two such algebraic constructions, which are direct generalisations of de Rham cohomology: Hochschild and cyclic cohomology. The area of mathematics that investigates these kind of algebraic constructions is called homological algebra. In the following we give a brief introduction. For more on this subject see, for instance [24] or [80].

Let $\mathcal{A}$ be an associative algebra.

Definition 6.0.6. *A* **(cochain) complex** $C = (C, d)$ *of $\mathcal{A}$-modules is a family $\{C^n\}_{n\in\mathbb{Z}}$ of $\mathcal{A}$-modules, together with $\mathcal{A}$-module maps $d = d^n : C^n \to C^{n+1}$ such that $d \circ d = 0$.*

A more illustrative way to denote a complex is the following sequence of maps

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots \tag{6.1}$$

A complex C is called positive (negative) if $C^n = 0$ for all $n < 0$ ($n > 0$). In this and the following sections we assume every complex to be positive . The maps d are called the **differentials** or **coboundary operators** of C. The kernel of d^n is the module of n-cocycles of C, denoted by $Z^n = Z^n(C)$, while the image of d^n is called the module of $n+1$-coboundaries, denoted by $B^{n+1} = B^{n+1}(C)$. Because $d \circ d = 0$, we have $B^n \subset Z^n$ for all n, so the quotient Z^n/B^n is well defined. This quotient, $H^n = H^n(C) = Z^n/B^n$, will be called the n^{th} **cohomology** module of C. The homology of a (chain) complex is the algebraic dual of the cohomology. That is, the differentials have degree -1 and the indices are lowered.

Let C and D be complexes of $\mathcal{A}$-modules. Denote for convenience both differentials with d. A cochain map $f : C \to D$ is a family of $\mathcal{A}$-module homomorphisms $f^n : C^n \to D^n$ such that $fd = df$, i.e. $f^{n+1}d^n = d^n f^n$. That is, such that the following diagram commutes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{n-1} & \xrightarrow{d^{n-1}} & C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & D^{n-1} & \xrightarrow{d^{n-1}} & D^n & \xrightarrow{d^n} & D^{n+1} & \longrightarrow & \cdots \end{array} \tag{6.2}$$

Note that the cochain map sends coboundaries to coboundaries, and cocycles to cocycles, hence maps $H^n(C) \to H^n(D)$.

Let f,g : $C \to D$ be cochain maps. A homotopy h between f and g is a sequence of morphisms $h^n : C^n \to D^{n-1}$ such that $dh + hd = f - g$, i.e. $d^{n-1}h^n + h^{n+1}d^n = f^n - g^n$. If f and g are homotopic and $x \in H^n(C)$, we get $f(x) - g(x) = d(h(x))$. We see that $f(x) = g(x)$ in $H^n(D)$ and conclude that $H^n f = H^n g$, whenever f and g are homotopic. If the identity map on $C^\bullet$ is homotopic with the zero map, the cochain C will be called **contractible**, and the homotopy h, satisfying $dh + hd = 1$, will be called a contracting homotopy.

A complex C is called **acyclic** if $H^n(C) = 0$ for all $n > 0$. Let $x \in H^n(C)$ and let C be a contractible complex, then $x = d(h(x))$, so $x \in B^n$. We can conclude that any contractible complex C is acyclic. Note that a complex is acyclic if its sequence of maps is exact, i.e. whenever $\ker d^{n+1} = \operatorname{im} d^n$ for all n.

A **grading** in a module C is defined by a family of submodules C^n such that C is the direct sum $\oplus_n C^n$. By this means we can consider a complex C as a graded module together with differentials $d^n : C^n \to C^{n+1}$.

6.1 Hochschild cohomology

In this section we construct the universal differential graded algebra, and define Hochschild cohomology, the canonical cohomology for a unital associative algebra $\mathcal{A}$. This cohomology is a generalisation of the de Rham cohomology, and serves to define cyclic cohomology in the following Section.

Let $\mathcal{A}$ be a unital algebra. The **universal differential graded algebra** $(\Omega(\mathcal{A}) = \oplus, d)$ over $\mathcal{A}$ is the unique differential graded algebra with the universal property. That is, for every differential graded algebra (Ω, δ) over $\mathcal{A}$ with $\mathcal{A} \xrightarrow{\rho} \Omega^0$ there exists a unique algebra morphism $f : \Omega(\mathcal{A}) \to \Omega$ such that f, restricted to $\Omega^0(\mathcal{A}) = \mathcal{A}$, is exactly ρ. Because of this universal property one can define unambiguously a differential graded algebra on $\mathcal{A}$, hence a cohomology on $\mathcal{A}$. For convenience we denote $\Omega^n(\mathcal{A}) = \Omega^n\mathcal{A}$.

To construct the universal differential graded algebra $(\Omega(\mathcal{A}), d)$, we only need to construct the derivation d, because we require $\Omega^0\mathcal{A} = \mathcal{A}$. This derivation is constructed as follows.

Let d be the linear map $d : \mathcal{A} \to \mathcal{A} \otimes A$ defined by $da := 1 \otimes a - a \otimes 1$. Because

$$\begin{aligned} d(ab) = 1 \otimes ab - ab \otimes 1 &= a \otimes b - ab \otimes 1 + 1 \otimes ab - a \otimes b \\ &= a\, db + da\, b, \end{aligned} \tag{6.3}$$

for every $a, b \in \mathcal{A}$, the map d is a derivation. Let $\Omega^1\mathcal{A}$ be the subset of $\mathcal{A} \otimes \mathcal{A}$ generated by the elements adb. We can turn $\Omega^1\mathcal{A}$ into a (bi)module over $\mathcal{A}$, defining a left and a right action on $c \in \mathcal{A}$, by

$$c(a\, db) := ca\, db, \qquad (a\, db)c := a\, d(bc) - ab\, dc, \tag{6.4}$$

respectively. Some straightforward calculations show that these actions are compatible in the sense that

$$c_1\big((a\, db)c_2\big) = \big(c_1(a\, db)\big)c_2\,, \tag{6.5}$$

hence $\Omega\mathcal{A}^1$ is indeed a bimodule. We call $\Omega^1\mathcal{A}$ the bimodule of universal 1-forms over $\mathcal{A}$. The universal property of this module is the following: for each derivation D of the algebra $\mathcal{A}$ into a bimodule $\mathcal{E}$, we can find a unique bimodule morphism $\iota_D : \Omega^1\mathcal{A} \to \mathcal{E}$, with $D = \iota_D \circ d$. This bimodule morphism is given by

$$\iota_D(a \otimes b) := aDb. \tag{6.6}$$

Remember that $D(1) = 0$ for every derivation D. Then ι_D is a bimodule morphism, because of the module structures of $\Omega^1\mathcal{A}$.

Remark 6.1.1. When A is a commutative algebra, a bimodule over A should be symmetric, i.e. the notions of the right and left module over A are identical. In this case it is reasonable to expect that $a\, db = db\, a$. The universal bimodule of 1-forms, denoted as $\Omega^1_{\mathrm{ab}}\mathcal{A}$, is consequently a subset of $\Omega(\mathcal{A})$. Let $(\Omega(\mathcal{A}))^2$ be the subbimodule of $\Omega(\mathcal{A})$ given by the elements $da\, db$. Because $\mathcal{A}$ is commutative we have the relation

$$da\, db = (1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1) = -(a\, db - db\, a) \tag{6.7}$$

Define $\Omega^1_{\mathrm{ab}}\mathcal{A}$ as $\Omega(\mathcal{A})/(\Omega(\mathcal{A}))^2$. From equation (6.7) follows that $\Omega^1_{\mathrm{ab}}\mathcal{A}$ is indeed a symmetric bimodule. To verify that $\Omega^1_{\mathrm{ab}}\mathcal{A}$ obeys the universal property one follows the same steps as we made for $\Omega(\mathcal{A})$ (see also [39]).

We now want to construct the universal differential graded algebra by extending the derivation d to a differential on $\Omega^i\mathcal{A}$. Denote $\bar{\mathcal{A}} := \mathcal{A}/\mathbb{C}$ and $\bar{a}$ the equivalence class of a in $\bar{\mathcal{A}}$. Using the map $a_0 \otimes \bar{a_1} \mapsto a_0\, da_1$ and the fact that $d1 = 0$, one sees that $\Omega^1\mathcal{A}$ is isomorphic to $\mathcal{A}\otimes\bar{A}$. The bimodule structures of equation (6.4) translate to

$$\begin{aligned} c(a_0\otimes\bar{a}_1) &= ca_0\otimes\bar{a}_1 \\ (a_0\otimes\bar{a}_1)c &= a_0\otimes\overline{a_1c} - a_0a_1\otimes\bar{c}. \end{aligned} \tag{6.8}$$

Define $\Omega^i\mathcal{A} := \Omega^1\mathcal{A}\otimes_A\Omega^1\mathcal{A}\otimes_A\cdots\otimes_A\Omega^1\mathcal{A}$ (i times), so that $\Omega^i\mathcal{A}\simeq\mathcal{A}\otimes\bar{\mathcal{A}}^{\otimes i}$. This isomorphism $A\otimes_A A\to A$ is given by the multiplication $a\otimes b\mapsto ab$. We now define the differential $d:\mathcal{A}\otimes\bar{\mathcal{A}}^{\otimes n}\to\mathcal{A}\otimes\bar{A}^{\otimes n+1}$ by

$$d(a_0\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_n) := 1\otimes\bar{a}_0\otimes\cdots\otimes\bar{a}_n. \tag{6.9}$$

Because $\bar{1}=0$, the map d is indeed a differential. Extending the isomorphism $a_0\otimes\bar{a}_1\mapsto a_0\,da_1$ to $\mathcal{A}\otimes\bar{\mathcal{A}}^{\otimes i}$ gives the identity $a_0\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_n = a_0da_1\cdots da_n$. The bimodule construction of $\Omega^i\mathcal{A}$ is a direct generalisation of equation (6.4) and is given by

$$\begin{aligned} c(a_0\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_n) &= c(a_0da_1\ldots da_n) \\ &= ca_0da_1\ldots da_n, \\ (a_0\otimes\bar{a}_1\otimes\cdots\otimes\bar{a}_n)c &= (a_0da_1\ldots da_n)c \\ &= a_0da_1\ldots da_{n-1}d(a_nc) - a_0da_1\ldots da_{n-1}a_ndc \\ &= (-1)^n a_0a_1da_2\ldots da_ndc \\ &\quad+\sum_{j=1}^{n-1}(-1)^{n-j}a_0da_1\ldots d(a_ja_{j+1})\ldots da_ndc \\ &\quad+ a_0da_1\ldots da_{n-1}d(a_nc). \end{aligned} \tag{6.10}$$

Consider now the algebra $\Omega(\mathcal{A}) = \oplus_{i=0}^n\Omega^n\mathcal{A}$ with the product defined by

$$(a_0da_1\cdots da_k)(b_0db_1\cdots db_l) := \big((a_0da_1\cdots da_k)b_0\big)db_1\cdots b_l. \tag{6.11}$$

With this product $\Omega(\mathcal{A})$ is clearly a graded algebra and d a derivation of degree 1. The differential graded algebra we just constructed is the universal differential graded algebra.

We verify the universal property. Let (Ω,δ) be a differential graded algebra over $\mathcal{A}$ with $\mathcal{A}\stackrel{\rho}{\to}\Omega^0$. Consider the map $f:\Omega(\mathcal{A})\to\Omega$ given by

$$f(a_0da_1\cdots da_n) := \rho(a_0)\,\delta\big(\rho(a_1)\big)\cdots\delta\big(\rho(a_n)\big). \tag{6.12}$$

Some computations show that f is an algebra morphism and is equal to ρ when restricted to $\mathcal{A}$.

Remark 6.1.2. This construction of the universal graded algebra can be extended to the non-unital case. If $\mathcal{A}$ is non-unital and $\mathcal{A}_{\mathbb{1}}$ its unitisation (see Section 4.2), we consider the bimodule $\Omega^i\mathcal{A}\simeq\mathcal{A}_{\mathbb{1}}\otimes\mathcal{A}^{\otimes i}$ instead of $\mathcal{A}\otimes\bar{\mathcal{A}}^{\otimes i}$. In some literature the universal differential graded algebra is always defined on $\mathcal{A}_{\mathbb{1}}$, whether the algebra $\mathcal{A}$ is unital or not (see for instance [29]).

In the commutative case the universal differential graded algebra, denoted by $\Omega_{\mathrm{ab}}\mathcal{A}$, is given by the exterior algebra $\Lambda_{\mathcal{A}}\Omega^1_{\mathrm{ab}}\mathcal{A}$ of $\Omega^1_{\mathrm{ab}}\mathcal{A}$ over $\mathcal{A}$. We can understand this from the general case. Let $a_0da_1da_2 \in \Omega^2\mathcal{A}$ and let $\mathcal{A}$ be commutative. Because $\mathcal{A}$ is commutative ($adb = dba$) we have

$$\begin{aligned} a_0da_1da_2 &= a_0d(a_1da_2) = a_0d\big(d(a_1a_2) - da_1a_2\big) \\ &= -a_0d(a_2da_1) = -a_0da_2da_1. \end{aligned} \tag{6.13}$$

Hence, the interchanging of i and j in an element

$$a_0da_1 \cdots da_i \cdots da_j \cdots da_k \in \Omega^k\mathcal{A} \tag{6.14}$$

with $\mathcal{A}$ commutative, gives rise to a minus sign, which is typical for an exterior algebra.

Remember that we can consider a differential graded algebra as a cochain complex. We use this notion in the following Proposition, which shows that the universal differential graded algebra is a generalisation of the de Rham cohomology. For the proof of this Proposition we refer to [39].

Proposition 6.1.3. *The exterior algebra $\Omega_{ab}C^\infty(M)$ can be identified with the de Rham complex $\Omega_{dR}(M)$ of differential forms on M.*

Consider the set $C^n(\mathcal{A})$ of $(n+1)$-linear functionals on $\mathcal{A}$ and put $C(\mathcal{A}) = \oplus C^n(\mathcal{A})$. Such a functional can be seen as a linear form on $\mathcal{A}^{\otimes(n+1)}$, or a n-linear form on $\mathcal{A}$ with values in the dual $\mathcal{A}^*$. The algebra $\mathcal{A}^*$ is a bimodule over $\mathcal{A}$, if we define the actions as $(a\varphi b)(c) = \varphi(bca)$ for every $\varphi \in \mathcal{A}^*$ and $a, b, c \in \mathcal{A}$. Consider the map $b : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ defined by

$$\begin{aligned} b\varphi(a_0, \ldots, a_{n+1}) = &\sum_{j=0}^{n}(-1)^j\varphi(a_0, \ldots, a_ja_{j+1}, \ldots, a_{n+1}) \\ &+ (-1)^{n+1}\varphi(a_{n+1}a_0, \ldots, a_n). \end{aligned} \tag{6.15}$$

Since each $C^n(\mathcal{A})$ is an $\mathcal{A}$-bimodule and $b^2 = 0$, $C(\mathcal{A})$ together with b is a cochain complex.

Definition 6.1.4. *The cohomology of $C(\mathcal{A})$ together with the coboundary of equation (6.15) is the* **Hochschild cohomology** *of $\mathcal{A}$, denoted by $HH(A)$.*

In particular is a Hochschild 0-cocycle τ on the algebra $\mathcal{A}$ a trace, because $\tau \in \mathcal{A}^* = Hom(\mathcal{A}, \mathbb{C})$ and $\tau(a_0a_1) - \tau(a_1a_0) = b\tau(a_0, a_1) = 0$.

Remark 6.1.5. Consider now the set $C^n(\mathcal{A})$ as the set of n-linear forms with values in $\mathcal{A}^*$, denoted by $C^n(\mathcal{A}, \mathcal{A}^*)$. The coboundary b of equation (6.15) transforms to a coboundary $b : C^n(\mathcal{A}, \mathcal{A}^*) \to C^{n+1}(\mathcal{A}, \mathcal{A}^*)$ given by

$$\begin{aligned}(b\widetilde{\varphi})(a_1, \ldots, a_{n+1}) = a_1\widetilde{\varphi}(a_2, \ldots, a_{n+1}) \\ + \sum_{j=1}^{n}(-1)^j\widetilde{\varphi}(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) \\ + (-1)^{n+1}\widetilde{\varphi}(a_1, \ldots, a_n)a_{n+1},\end{aligned} \tag{6.16}$$

where we use that $\widetilde{\varphi}(a_1, \ldots, a_n)(a_0) = \varphi(a_0, \ldots a_n)$ for every $\varphi \in C^n(\mathcal{A})$. The cohomology of $C(\mathcal{A}, \mathcal{A}^*) = \oplus C^n(\mathcal{A}, \mathcal{A}^*)$ with the coboundary of equation (6.16) is the Hochschild cohomology of $\mathcal{A}$ over $\mathcal{A}^*$, denoted by $HH(\mathcal{A}, \mathcal{A}^*)$. One can generalise this to any $\mathcal{A}$-bimodule $\mathcal{E}$ and define a corresponding Hochschild cohomology $HH(\mathcal{A}, \mathcal{E})$. Then, $HH^0(\mathcal{A}, \mathcal{E}) = \{s \in \mathcal{E} : as = sa \ \forall a \in \mathcal{A}\}$, and $HH^1(\mathcal{A}, \mathcal{E}) = \mathrm{Der}\,(\mathcal{A}, \mathcal{E})/\,\mathrm{Der}'\,(\mathcal{A}, \mathcal{E})$, where $\mathrm{Der}\,(\mathcal{A}, \mathcal{E})$ is the vector space of all $\mathcal{A}$-bimodule derivations with values in $\mathcal{E}$ and $\mathrm{Der}'\,(\mathcal{A}, \mathcal{E})$ are all inner derivations, i.e. all $sa - as$ with $s \in \mathcal{E}$ and $a \in \mathcal{A}$.

To extend the trace property of the 0-cocycles to higher orders we introduce cyclicity.

Definition 6.1.6. *A n-cochain $\varphi \in C^n(\mathcal{A})$ on A is* **cyclic** *if $\lambda\varphi = \varphi$, where*

$$(\lambda\varphi)(a_0, a_1, \ldots, a_n) := (-1)^n\varphi(a_n, a_0, \ldots, a_{n-1}). \tag{6.17}$$

A cyclic cocycle is a cyclic cochain with the property $b\varphi = 0$. This is indeed a generalisation of the trace property. For instance, a cyclic 1-cocycle $\varphi(a_0, a_1)$ satisfies $\varphi(a_0, a_1) = -\varphi(a_1, a_0)$ and

$$\varphi(a_0a_1, a_2) - \varphi(a_0, a_1a_2) + \varphi(a_2a_0, a_1) = 0 \tag{6.18}$$

As we already pointed out, the universal differential graded algebra over $\mathcal{A}$ gives our cohomology over $\mathcal{A}$. The next Proposition shows how the Hochschild cohomology (with cyclicity) follows naturally from the algebra $\mathcal{A}$ and its corresponding universal differential graded algebra.

Proposition 6.1.7. *Let τ be a $(n+1)$-linear functional on $\mathcal{A}$ that vanishes on $\mathbb{C} \oplus \mathcal{A}^n$. The following statements are equivalent.*

1. *There exists a n-dimensional cycle $(\Omega, \delta, \int)$ and a homomorphism $\rho : \mathcal{A} \to \Omega^0$ such that*

$$\tau(a_0, \ldots, a_n) = \int \rho(a_0)\delta\big(\rho(a_1)\big)\cdots\delta\big(\rho(a_n)\big) \quad \forall a_i \in \mathcal{A}. \tag{6.19}$$

2. *There exists a closed graded trace T of dimension n on $\Omega(\mathcal{A})$ such that*

$$\tau(a_0, \ldots, a_n) = T(a_0da_1 \cdots da_n) \quad \forall a_i \in \mathcal{A}. \tag{6.20}$$

3. The functional τ satisfies the cyclicity property

$$\tau(a_1,\dots,a_n,a_0) = (-1)^n \tau(a_0,\dots,a_n) \quad \forall a_i \in \mathcal{A} \tag{6.21}$$

and is a Hochschild cocycle, i.e. for all $a_i \in \mathcal{A}$

$$\sum_{j=0}^{n} (-1)^j \tau(a_0,\dots,a_j a_{j+1},\dots,a_{n+1}) + (-1)^{n+1}\tau(a_{n+1}a_0,\dots,a_n) = 0. \tag{6.22}$$

Proof. The equivalence between (1) and (2) is immediate from the universal property of $\Omega(\mathcal{A})$. Using the same arguments as the proof of Proposition 5.4.1, one can prove that (3) follows from (1). To complete the proof we show that (2) follows from (3). Let ϕ be a $(n+1)$-linear functional on $\mathcal{A}$, and define $\hat{\phi}$ as a linear functional on $\Omega^n(\mathcal{A})$ by

$$\hat{\phi}(a_0 da_1 \cdots da_n) = \phi(a_0,a_1,\dots,a_n). \tag{6.23}$$

This construction gives us $\hat{\tau}(d\omega) = 0$ for every $\omega \in \Omega^{n-1}(\mathcal{A})$, because τ vanishes on $\mathbb{C}\oplus\mathcal{A}^n$. Hence $\hat{\tau}$ is closed. We want to show that $\hat{\tau}$ is a graded trace. We have

$$\hat{\tau}\big((a_0 da_1\cdots da_k)(a_{k+1}da_{k+2}\cdots da_{n+1})\big) = \sum_{j=0}^{k}(-1)^{k-j}\tau(a_0,\dots a_j a_{j+1},\dots,a_{n+1}) \tag{6.24}$$

and

$$(-1)^{k(n-k)}\hat{\tau}\big((a_{k+1}da_{k+2}\cdots da_{n+1})(a_0 da_1\cdots da_k)\big) = \sum_{j=0}^{n-k}(-1)^{k(n-k)+n-k-j}\tau(a_{k+1},\dots a_{k+1+j}a_{k+1+j+1},\dots,a_k). \tag{6.25}$$

Because $\tau = \lambda^{k+1}\tau$ and the signature of λ^{k+1} is $(-1)^{n(k+1)}$ (see [39]), equation (6.25) becomes

$$-\sum_{j=k+1}^{n+1}(-1)^{k-j}\tau(a_0,\dots,a_j a_{j+1},\dots,a_{n+1}), \tag{6.26}$$

and equation (6.22) ensures us that $\hat{\tau}$ is indeed a graded trace. □

Remark 6.1.8. If we take the unitisation $\mathcal{A}_{\mathbb{1}}$ of $\mathcal{A}$, as in [29] (see Remark 6.1.2), to build $\Omega(\mathcal{A})$ it is not necessary to demand that τ vanishes on $\mathbb{C}\oplus\mathcal{A}^n$ in Proposition 6.1.7.

This Proposition gives a correspondence between cycles over $\mathcal{A}$ and cyclic cocycles. In the following Section we see that this correspondence is given by the (noncommutative) Chern character.

6.2 Cyclic cohomology

In Proposition 6.1.3 we saw how the exterior algebra $\Omega_{ab}C^\infty(M)$ can be identified with the de Rham complex. Thereafter we saw in Proposition 6.1.7 that the $\Omega(\mathcal{A})$, which is the generalisation of $\Omega_{ab}C^\infty(M)$, corresponds with Hochschild cohomology. To do that properly one has to introduce cyclicity. This is a direct consequence of the fact that the algebra $\mathcal{A}$ is noncommutative and the (graded) commutator $db + bd$ does not equal zero. This cyclicity defines cyclic cohomology, which is introduced by Connes in [29], and is the noncommutative generalisation of de Rham cohomology.

Consider the set of all cyclic n-cochains in $C^n(\mathcal{A})$, denoted by $C^n_\lambda(\mathcal{A})$. The following Proposition shows that the set $C_\lambda(\mathcal{A}) = \oplus C^n_\lambda(\mathcal{A})$ together with the Hochschild coboundary (6.15) is a subcomplex of the Hochschild complex.

Proposition 6.2.1. *Let b be the Hochschild coboundary. For every $\phi \in C_\lambda(\mathcal{A})$, the element $b\,\phi$ still lies in $C_\lambda(\mathcal{A})$.*

Proof. Consider the truncated Hochschild coboundary $b' : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$

$$b'\varphi(a_0, \ldots, a_{n+1}) := \sum_{j=0}^{n} (-1)^j \varphi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}), \tag{6.27}$$

and the difference $r := b - b'$, given by

$$r\varphi(a_0, \ldots, a_{n+1}) := (-1)^{n+1}\varphi(a_{n+1}a_0, a_1, \ldots, a_n). \tag{6.28}$$

For every $\phi \in C^n(\mathcal{A})$ and $j = 0, \ldots, n$ we have,

$$\begin{aligned} \lambda^{-j-1} r \lambda^j \phi(a_0, \ldots, a_{n+1}) &= (-1)^{(j+1)(n+1)} r \lambda^j \phi(a_{j+1}, \ldots, a_{n+1}, a_0, \ldots, a_j) \\ &= (-1)^{(j+2)(n+1)} \lambda^j \phi(a_j a_{j+1}, \ldots, a_{n+1}, a_0, \ldots, a_{j-1}) \\ &= (-1)^j \phi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}). \end{aligned} \tag{6.29}$$

Therefore

$$b' = \sum_{j=0}^{n} \lambda^{-j-1} r \lambda^j, \qquad b = b' + r = \sum_{j=0}^{n+1} \lambda^{-j-1} r \lambda^j, \tag{6.30}$$

because $\lambda^{n+1} = 1$ on C^n and $\lambda^{-n-2} = 1$ on C^{n+1}. For each $\phi \in C^n(\mathcal{A})$ one has

$$\begin{aligned} (1-\lambda)b &= (1-\lambda)\sum_{j=0}^{n+1} \lambda^{-j-1} r \lambda^j \\ &= \sum_{j=0}^{n} \lambda^{-j-1} r \lambda^j + r - \sum_{j=0}^{n} \lambda^{-j} r \lambda^j - \lambda^{-n-1} r \\ &= \sum_{j=0}^{n} \lambda^{-j-1} r \lambda^j - \sum_{j=1}^{n+1} \lambda^{-j} r \lambda^j = b'(1-\lambda). \end{aligned} \tag{6.31}$$

Hence for every $\phi \in C_\lambda(\mathcal{A})$, i.e. $(1-\lambda)\phi = 0$

$$(1-\lambda)b\phi = b'(1-\lambda)\phi = 0. \tag{6.32}$$

□

We can take the cohomology of this complex.

Definition 6.2.2. *Consider the subcomplex $(C_\lambda(\mathcal{A}), b)$ of the Hochschild complex of $\mathcal{A}$. The* **cyclic cohomology** *$HC(\mathcal{A})$ of the algebra $\mathcal{A}$ is the cohomology of this subcomplex.*

We give the following Proposition without proof.

Proposition 6.2.3.

1. *Let u be an invertible element in the unital algebra $\mathcal{A}$ and θ the corresponding inner automorphism defined by $\theta(x) = uxu^{-1}$. The induced map $\theta : HC(\mathcal{A}) \to HC(\mathcal{A})$ is the identity on $HC(\mathcal{A})$.*

2. *Let $\rho : \mathcal{A} \to \mathcal{A}$ be a homomorphism and X an invertible element in $M_2(\mathcal{A})$ such that*
$$X \begin{pmatrix} a & 0 \\ 0 & \rho(a) \end{pmatrix} X^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \rho(a) \end{pmatrix} \quad , a \in \mathcal{A}. \tag{6.33}$$
Then $HC^n(\mathcal{A}) = 0$ for all n.

Let $(\Omega, d, \int)$ be a cycle. We call this cycle **vanishing** if the algebra Ω^0 satisfies condition 2. of Proposition 6.2.3. Remember that we can associate to every cycle a character τ given in equation (5.33). Proposition 6.1.7 gives the following Corollary.

Corollary 6.2.4. *Let τ be a $(n+1)$-linear functional on $\mathcal{A}$. Then*

1. $\tau \in Z^n_\lambda(\mathcal{A})$ *if and only if* τ *is a character.*

2. $\tau \in B^n_\lambda(\mathcal{A})$ *if and only if* τ *is the character of a vanishing cycle.*

In the following we construct a product $HC^n(\mathcal{A}) \otimes HC^m(\mathcal{B}) = HC^{n+m}(\mathcal{A} \otimes \mathcal{B})$, called the cup product. Because of the universal property of $\Omega(\mathcal{A} \otimes \mathcal{B})$ one has a homomorphism $\pi : \Omega(\mathcal{A} \otimes \mathcal{B}) \to \Omega(\mathcal{A}) \otimes \Omega(\mathcal{B})$, where $\Omega(\mathcal{A}) \otimes \Omega(\mathcal{B})$ is the graded tensor product

$$(\omega_{a_1} \otimes \omega_{b_1})(\omega_{a_2} \otimes \omega_{b_2}) = (-1)^{\partial\omega_{b_1}\partial\omega_{a_2}}(\omega_{a_1}\omega_{a_2} \otimes \omega_{b_1}\omega_{b_2}). \tag{6.34}$$

Using the equivalences of Proposition 6.1.7 one can define for any $\phi \in C^n(\mathcal{A})$ and $\psi \in C^m(\mathcal{B})$ the cup product $\phi \vee \psi$ defined by

$$(\phi \vee \psi)\hat{} = (\hat{\phi} \otimes \hat{\psi}) \circ \pi, \tag{6.35}$$

where $(\phi \vee \psi)\hat{}$, $\hat{\phi}$ and $\hat{\psi}$ are the corresponding linear functionals (traces) on respectively $\Omega(\mathcal{A} \otimes \mathcal{B})$, $\Omega(\mathcal{A})$ and $\Omega(\mathcal{B})$.

Proposition 6.2.5.

1. *The character of the tensor product of two cycles is the cup product of their characters.*
2. *The cup product defines a homomorphism*

$$HC^n(\mathcal{A}) \otimes HC^m(\mathcal{B}) \to HC^{n+m}(\mathcal{A} \otimes \mathcal{B}). \tag{6.36}$$

Proof.

1. Given two cycles $(\Omega_1, d_1, \int_1)$ and $(\Omega_2, d_2, \int_2)$, with respectively the homomorphisms $\rho_1 : \mathcal{A} \to \Omega_1$ and $\rho_2 : \mathcal{A} \to \Omega_2$, one has by construction the commutative triangle

$$\begin{array}{ccc} \Omega(\mathcal{A} \otimes \mathcal{B}) & \xrightarrow{\pi} & \Omega(\mathcal{A}) \otimes \Omega(\mathcal{B}) \\ & \searrow^{\widetilde{(\rho_1 \otimes \rho_2)}} & \downarrow{\scriptstyle \tilde{\rho}_1 \otimes \tilde{\rho}_2} \\ & & \Omega_1 \otimes \Omega_2 \end{array} \tag{6.37}$$

   where $\tilde{\rho}$ is the corresponding algebra morphism between the universal differential graded algebra and the cycle.

2. Let $\phi \in Z^n_\lambda(\mathcal{A})$ and $\psi \in Z^m_\lambda(\mathcal{B})$. Due to Proposition 6.1.7 both $\hat{\phi}$ as $\hat{\psi}$ are closed graded traces on respectively $\Omega(\mathcal{A})$ and $\Omega(\mathcal{B})$, hence $\hat{\phi} \otimes \hat{\psi}$ is a closed graded trace on $\Omega(\mathcal{A}) \otimes \Omega(\mathcal{B})$ and $\phi \vee \psi \in Z^{n+m}_\lambda(\mathcal{A} \otimes \mathcal{B})$. We need to show that if $\phi \in B^n_\lambda(\mathcal{A})$, the cup product $\phi \vee \psi$ is also a boundary. This follows from 1., Corollary 6.2.4 and the fact that the tensor product of a cycle with a vanishing cycle is also vanishing.

□

Corollary 6.2.6.

1. *$HC(\mathbb{C})$ is a polynomial ring with one generator σ of degree 2.*
2. *Every $HC(\mathcal{A})$ is a module over the ring $HC(\mathbb{C})$.*

Proof.

1. Consider the set $C^n(\mathbb{C})$ of $n+1$-functionals on $\mathbb{C}$. This set equals $\mathbb{C}$ with the identification

$$\phi(a_0,\ldots,a_{n+1}) = a_0\cdots a_{n+1}\phi(e,\ldots,e) \quad \forall\phi\in C^n(\mathbb{C}), \tag{6.38}$$

where e is the unity in $\mathbb{C}$. Hence every $\phi\in Z^n_\lambda(\mathbb{C})$ is characterised by $\phi(e,\ldots,e)$. This yields $HC^n(\mathbb{C})=0$ for n is odd, and $HC^n(\mathbb{C})=\mathbb{C}$ for n is even. Let $\phi\in Z^{2m}_\lambda(\mathbb{C})$ and $\psi\in Z^{2m'}_\lambda(\mathbb{C})$. We compute $\phi\vee\psi$.

$$\begin{aligned}(\phi\vee\psi)(e,\ldots,e) &= (\hat\phi\otimes\hat\psi)\pi\big((e\otimes e)d(e\otimes e)\cdots d(e\otimes e)\big)\\ &= \frac{(m+m')!}{m!m'!}\phi(e,\ldots,e)\psi(e,\ldots,e).\end{aligned} \tag{6.39}$$

One sees this using the identities

$$de = ede + (de)e, \quad e(de)e = 0, \quad e(de)^2 = (de)^2e. \tag{6.40}$$

For instance,

$$\pi\big((e\otimes e)\big(d(e\otimes e)\big)^4\big) = e(de)^4\otimes e + 2e(de)^2\otimes e(de)^2 + e\otimes e(de)^4. \tag{6.41}$$

Choose as generator of $HC(\mathbb{C})$ the 2-cocycle σ with $\sigma(e,e,e)=2\imath\pi$.

2. Let $\phi\in Z^n_\lambda(\mathcal{A})$. We prove that $\phi\vee\sigma=\sigma\vee\phi$ and we give an explicit formula of the corresponding map $S: HC^n(\mathcal{A})\to HC^{n+2}(\mathcal{A})$. One has

$$\begin{aligned}\frac{1}{2\imath\pi}(\phi\vee\sigma)(a_0,\ldots,a_{n+2}) = (\hat\phi\otimes\frac{1}{2\imath\pi}\hat\sigma)&\big(a_0\otimes ed(a_1\otimes e)\cdots d(a_{n+2}\otimes e)\big)\\ &\hat\phi(a_0a_1a_2da_3\cdots da_{n+2}) + \hat\phi(a_0da_1(a_2a_3)da_4\cdots da_{n+2}\\ &\hat\phi(a_0da_1\cdots da_{i-1}(a_ia_{i+1})da_{i+2}\cdots da_{n+2}\\ &\hat\phi(a_0da_1\cdots da_n(a_{n+1}a_{n+2})).\end{aligned} \tag{6.42}$$

Computing $\sigma\vee\phi$ gives the same result. Define for $\phi\in Z^n_\lambda(\mathcal{A})$, the element $S\phi=\sigma\vee\phi=\phi\vee\sigma\in Z^{n+2}_\lambda(\mathcal{A})$. Because of Proposition 6.2.5 we have $SB^n_\lambda(\mathcal{A})\subset B^{n+2}_\lambda(\mathcal{A})$. This makes $HC(\mathcal{A})$ a module over the ring $HC(\mathbb{C})$.

□

The following Proposition relates the K_0-group of $\mathcal{A}$ (see Section 4.4) to the cyclic cohomology of $\mathcal{A}$. To do that we consider, for given algebra $\mathcal{A}$, the algebra $M_k(\mathcal{A})$. We can extend an element $\phi\in Z^n_\lambda(\mathcal{A})$ to an element $\tilde\phi\in Z^n_\lambda(M_k(\mathcal{A}))$ by

$$\tilde\phi(a_0\otimes A_0,\ldots,a_n\otimes A_n) = \phi(a_0,\ldots,a_n)\mathrm{Tr}(A_0\cdots A_n), \tag{6.43}$$

where $a_i\in\mathcal{A}$, $A_i\in M_k(\mathbb{C})$ and where the trace is just the normal matrix trace.

Proposition 6.2.7.

1. *The following equation defines a bilinear pairing between the K_0-group of $\mathcal{A}$ and $HC^{even}(\mathcal{A})$*

$$\langle [e]_0, \phi \rangle = (2\imath\pi)^{-m}(m!)^{-1}(\phi \vee \mathrm{Tr})(e, \dots, e) \tag{6.44}$$

for $e \in P_k(\mathcal{A})$ and $\phi \in Z_\lambda^{2m}(\mathcal{A})$.

2. *One has $\langle [e]_0, S\phi \rangle = \langle [e]_0, \phi \rangle$.*

Proof.

1. To simplify the calculations, replace $\mathcal{A}$ by $M_k(\mathcal{A})$ and ϕ by $\tilde{\phi}$. If $\phi \in B_\lambda^{2m}(\mathcal{A})$ then $\phi \vee \mathrm{Tr}$ is also a coboundary, hence $\phi \vee \mathrm{Tr} = b\psi$ and

$$(\phi \vee \mathrm{Tr})(e, \dots, e) = b\psi(e, \dots, e) = \sum_{i=0}^{2m}(-1)^i \psi(e, \dots, e) = \psi(e, \dots, e) = 0, \tag{6.45}$$

because $\lambda\psi = -\psi$. If we take another representative of $[e]$, the expression $\phi \vee \mathrm{Tr}(e, \dots, e)$ will differ a coboundary (see [29]), hence it is only dependent of the equivalence class of e. We find

$$(\phi \vee \mathrm{Tr})(e, \dots, e) = (2\imath\pi)^m m! \phi(e, \dots, e). \tag{6.46}$$

2. One has

$$\begin{aligned} \frac{1}{2\imath\pi} S\phi(e, \dots, e) &= \sum_{j=1}^{2m} \hat{\phi}\big(e(de)^{j-1}e(de)^{n-j+1}\big) \\ &= (m+1)\phi(e, \dots, e), \end{aligned} \tag{6.47}$$

and the statement follows.

□

Part 2. of the last Proposition motivates a cohomology where $S\phi = \phi$, as it should be a generalisation of the de Rham cohomology. This cohomology is defined in the following.

Definition 6.2.8. *The* **periodic cyclic cohomology** $HP(\mathcal{A})$ *over $\mathcal{A}$ is defined by*

$$HP(\mathcal{A}) = HC(\mathcal{A}) \otimes_{HC(\mathbb{C})} \mathbb{C}. \tag{6.48}$$

Corollary 6.2.6 shows us that $HC(\mathbb{C})$ is isomorphic with the polynomial ring $\mathbb{C}[\sigma]$. This ring acts on $\mathbb{C}$ through the map $P(\sigma) \mapsto P(1)$, where P is some polynomial. We see that $HP(\mathcal{A})$ is just $HC(\mathcal{A})$ modulo the equivalence relation $S\phi \sim \phi$. The periodic cyclic cohomology obtains therefore a $\mathbb{Z}/2$-grading such that

$$HP(\mathcal{A}) = HP^0(\mathcal{A}) \oplus HP^1(\mathcal{A}). \tag{6.49}$$

This and the pairing of Proposition 6.2.7 gives the following Corollary.

Corollary 6.2.9. *There is a canonical pairing between $HP^{even}(\mathcal{A})$ and $K_0(\mathcal{A})$.*

Remark 6.2.10. One can also show (see [29]) that $HP^{odd}(\mathcal{A})$ pairs canonically with the $K_1(\mathcal{A})$-group of $\mathcal{A}$. In this way, the map S reflects the Bott periodicity of K-theory. For more on the K_1-group of $\mathcal{A}$ and Bott periodicity, see for instance [73].

Chapter 7

The noncommutative IQHE

In Chapter 2 we saw that classical Bloch theory failed to describe the Integer Quantum Hall Effect due to the breakdown of translation invariance. Because of the magnetic field applied, we replaced the translation operators by the magnetic translation operators $\hat{T}_R$ which commute as

$$\hat{T}_a\hat{T}_b = e^{2\pi\imath\phi}\hat{T}_b\hat{T}_a, \tag{7.1}$$

where ϕ is the quantity of flux piercing the unit cell. If ϕ is rational we can reduce the problem to the classical (translation invariant) case (see Section 2.4). In the case that ϕ is irrational this is not possible. In this Chapter we use the techniques of noncommutative geometry, discussed in the previous Chapter, to generalise the theory of the Integer Quantum Hall Effect for all ϕ.

7.1 The Hull

When constructing the algebra of observables one would normally take the Hamiltonian and its invariant transformations (in our case the magnetic translations). In this Section we follow a slightly different approach, using a C^*-dynamical system, called the Hull, to construct the algebra of observables for our system. This approach used on the so-called aperiodic solids, is due to Bellissard (see [12] and for some overviews [14], [15] and [17]).

Consider a first order differential equation $\dot{x} = f(x,t)$ with $f \in C^1(\mathbb{R}\times\mathbb{R})$ and initial condition $x(t=0) = x_0$. The (unique) solution of such a differential equation gives a flow $\phi(t, x_0)$ defined on $I_{x_0} \times x_0$ with I_{x_0} some interval dependent of $x_0 \in \mathbb{R}$. This flow has the following properties (see [41]),

i $\phi(0, x_0) = x_0$,

ii $\phi(t+s, x_0) = \phi(t, \phi(s, x_0))$ for $t+s, s \in I_{x_0}$ and $t \in I_{\phi(s,x_0)}$,

iii $\phi(t, x_0)$ is $C^1(\mathbb{R})$ in t and it has a $C^1(\mathbb{R})$ inverse given by $\phi(-t, x_0)$.

A map from $\mathbb{R}$ to $\mathbb{R}$ with these properties is called a dynamical system on $\mathbb{R}$. This notion can be generalised in the following manner.

Definition 7.1.1. *A* **C*-algebraic dynamical system** *is a triple* $(\mathcal{A}, G, \alpha)$ *with* $\mathcal{A}$ *a* C^**-algebra,* G *a locally compact group and* α *a continuous homomorphism from* G *to the group* $Aut(\mathcal{A})$ *of automorphisms of* $\mathcal{A}$ *with the topology of pointwise convergence.*

If the flow $\phi(t, x_0)$ is defined on the whole set $\mathbb{R} \times \mathbb{R}$, we see that it defines a C^*-algebraic dynamical system and that our definition is indeed a generalisation. We can see this by identifying G and $\mathcal{A}$ with $\mathbb{R}$ (as a group and a real C^*-algebra respectively) and identifying the map α with $t \to \phi(t, x)$.

Consider now a Hamiltonian, which stays conserved under some transformations (think of time evolution or space translation). This Hamiltonian generates a system of first order differential equations and one can associate a dynamical system to it. From this dynamical system we construct the algebra of observables of our system. So let us first consider our Hamiltonian.

As we mentioned in Section 2 we need disorder in our system that causes localised states. Let V_ω denote the potential that represents the configuration of the disorder. Our Hamiltonian with this disorder looks like

$$H_\omega = H_0 + V_\omega, \tag{7.2}$$

where H_0 is just the unperturbed Hamiltonian from equation (2.6). Note that V_ω is not anymore the periodic potential from equation (2.31), due to the disorder ω. But because we are assuming the disorder to be random distributed, we say that V_ω and therefore H, is globally (magnetic) translation invariant. This follows from the homogeneity of the medium we are considering. This means that in spite of the local differences it does not matter where we choose our origin because the physics will be the same. The exact definition of this homogeneity of the medium, hence of the operator H_ω follows.

Definition 7.1.2. *Let* U *be a unitary projective representation of* $\mathbb{R}^2$ *on the Hilbert space* $L^2(\mathbb{R}^2)$*, i.e., for every* $a \in \mathbb{R}^2$ *there exists a unitary operator* $U(a)$ *acting on* $L^2(\mathbb{R}^2)$ *such that*

1. $U(a)U(b) = U(a+b)e^{\imath\phi(a,b)}$ *for every* $a, b \in \mathbb{R}^2$*, with* $\phi(a,b)$ *some phase factor.*

2. *For every vector* $\psi \in L^2(\mathbb{R}^2)$ *the map given by* $a \in \mathbb{R}^2 \mapsto U(a)\psi \in L^2(\mathbb{R}^2)$ *is continuous.*

We call a self-adjoint operator H on $L^2(\mathbb{R}^2)$ **homogeneous** *with respect to U if the set*

$$S = \{R_a(z) = U(a)(z\mathbb{1} - H)^{-1}U(a)^{-1} | a \in \mathbb{R}^2\} \tag{7.3}$$

has a compact strong closure (for some $z \in \mathbb{C}$).

Let H be such a homogeneous operator and let $\Omega_H(z)$ be the strong closure of S in (7.3). The strong (operator) topology we used in this Definition is the weakest topology on $\mathcal{L}(L^2(\mathbb{R}^2))$ such that for all $\psi \in L^2(\mathbb{R}^2)$ the map $R_a(z) \mapsto R_a(z)\psi$ is continuous. Because $L^2(\mathbb{R}^2)$ has a countable base, $\Omega_H(z)$ is a compact metrisable space (see [67]). This implies that the full set of translates (in a) of $R_a(z)$ on some $\psi \in L^2(\mathbb{R}^2)$ can be approximated by a finite number of $\psi_i's$. One can show ([14]) that

$$\{\mathbb{1} + (z' - z)R_a(z)\}^{-1} = \{\mathbb{1} - (z' - z)R_a(z')\} \tag{7.4}$$

and that $\Omega_H(z)$ is homeomorphic to $\Omega_H(z')$ via the the action of some (finite) $a_i \in \mathbb{R}^2$ on the elements $R_a(z)$. By identifying these spaces we can just work with the abstract compact space Ω_H, with an action of $\mathbb{R}^2$ through the representation U on it. Let $T^a\omega$ denote the action of $a \in \mathbb{R}^2$ on $\omega \in \Omega_H$, and let $R_\omega(z)$ be the representative of ω in $\Omega_H(z)$. The triple $(\Omega_H, \mathbb{R}^2, U)$ is a dynamical system corresponding to the operator H and is called the **Hull** of an operator.

We are interested in the dynamical system defined by the Hamiltonian (7.2). Note however that this operator is not self-adjoint due to V_ω. We are however still able to define a Hull and dynamical system for H_ω if the potential satisfies some boundedness conditions. We demand our Hamiltonian H_ω to be measurable in the variable ω, which is reasonable since our disorder configuration is random. The following Theorem, which is proved in [14] and [60], gives the precise construction how to make a Hull, and a dynamical system associated to the Hamiltonian H_ω.

Theorem 7.1.3. *Consider the Hamiltonian H_ω of equation (7.2). Let V_ω be a real, measurable, essentially bounded function over $\mathbb{R}^2$. Then H_ω is homogeneous with respect to the magnetic translations $\hat{T}$ of equation (2.36).*

Remark 7.1.4. There is another manner (see [17]) to construct an associated dynamical system of the Hamiltonian H_ω. This is done by defining a Hull over a uniform discrete set $\mathcal{L}$ in $\mathbb{R}^2$, representing the disorder configuration ω of the system. In this construction one associates to every uniform discrete set $\mathcal{L}$ the so-called counting measure defined by

$$\nu^{\mathcal{L}}(x) = \sum_{y \in \mathcal{L}} \delta(x - y). \tag{7.5}$$

For every $a \in \mathbb{R}^2$ one defines the translation τ^a such that $\tau^a f(x) = f(x - a)$ for every continuous function on $\mathbb{R}^2$. The Hull of $\mathcal{L}$ is the dynamical system $(\Omega, \mathbb{R}^2, \tau)$ where Ω is the closure of the $\mathbb{R}^2$-orbit $\{\tau^a\nu^{\mathcal{L}} : a \in \mathbb{R}^2\}$ in $\mathcal{M}(\mathbb{R}^2)$. The space $\mathcal{M}(\mathbb{R}^2)$ is here the space of Radon measures, i.e., the linear functionals on $C_c(\mathbb{R}^2)$, the space of continuous functions with compact support, with the weak-* topology over $C_c(\mathbb{R}^2)$.

Although the two dynamical systems $(\Omega_{H_\omega}, \mathbb{R}^2, \hat{T})$ and $(\Omega, \mathbb{R}^2, \tau)$ are not exactly the same, one can show that they are semi-conjugate and that the physics behind these dynamical systems are the same. For the proofs of these statements and more on this matter see, again, [17].

To every C^*-algebraic dynamical system $(\mathcal{A}, G, \alpha)$ one can associate a C^*-algebra $\mathcal{A} \rtimes G$ (see [63]) called the crossed product. The main ingredient of the construction of this C^*-algebra is a non-zero left-invariant Radon measure μ_G on G, i.e. $\mu_G(sE) = \mu_G(E)$ for every Borel set E in G and every $s \in G$. This measure is called a left Haar measure and one can show that, up to multiplication with scalars, this measure is unique. Together with a left Haar measure exists an associated right Haar measure. This right Haar measure is associated to the left Haar measure trough the modular function Δ from G to the positive elements of $\mathbb{R}$ and the equation

$$\mu_G(Es) = \Delta(s)\mu_G(E). \tag{7.6}$$

With the identification $ds = d\mu_g(s)$ one has

$$d(ts) = ds, \quad d(st) = \Delta(t)ds, \quad d(s^{-1}) = \Delta(s)^{-1}ds. \tag{7.7}$$

The C^*-algebra corresponding to the dynamical system $(\mathcal{A}, G, \alpha)$ is now constructed through the space $C_c(G, \mathcal{A})$ of continuous functions from G to $\mathcal{A}$ with compact support. We make it a $*$-algebra by defining an involution and a convolution on it. Namely,

$$\begin{aligned} f^*(t) &= \Delta(t)^{-1}\alpha_t(f(t^{-1})^*), \\ (f * g)(t) &= \int f(s)\alpha_s(g(s^{-1}t))ds, \end{aligned} \tag{7.8}$$

for all $f, g \in C_c(G, \mathcal{A})$. We proceed by defining a norm by

$$||f||_1 = \int_G ||f(t)||_{\mathcal{A}} dt. \tag{7.9}$$

The completion of $C_c(G, \mathcal{A})$ in this norm is the algebra $L^1(G, \mathcal{A})$. Consider the universal representation (π_u, H_u), introduced in Section 4.2, of the space $L^1(G, \mathcal{A})$. It is the direct sum over all non-degenerate representations of $L^1(G, \mathcal{A})$. The **crossed product** of the dynamical system $(\mathcal{A}, G, \alpha)$ is the norm closure of $\pi_u(L^1(G, \mathcal{A}))$ in $\mathcal{L}(H_u)$, denoted by $\mathcal{A} \rtimes_\alpha G$ or just $\mathcal{A} \rtimes G$ for simplicity.

Remark 7.1.5. Often the crossed algebra of a dynamical system $(\mathcal{A}, G, \alpha)$ is called the covariance algebra, because of the correspondence between the covariant representations of $(\mathcal{A}, G, \alpha)$ and the non-degenerate representations of $L^1(G, \mathcal{A})$ (see [63]). Such a **covariant representation** of a dynamical system $(\mathcal{A}, G, \alpha)$ is a triple (π, u, H), where (π, H) is some representation of $\mathcal{A}$, and (u, H) is some unitary representation of G and

$$\pi(\alpha_t(x)) = u_t \pi(x) u_t{}^* \tag{7.10}$$

for all $x \in \mathcal{A}$ and $t \in G$. Having a covariant representation (π, u, H) one can represent (uniquely) $L^1(G, \mathcal{A})$ on H. This is done trough the equation

$$(\pi \times u)(y) = \int \pi(y(t)) u_t dt, \tag{7.11}$$

for every $y \in C_c(G, \mathcal{A})$.

Let $\mathcal{A}$ act faithfully on some Hilbert space $\mathcal{H}$ (through some representation π which will be omitted) and let $L^2(G, \mathcal{H}) = L^2(G) \otimes \mathcal{H}$ be the Hilbert space of square integrable functions from G to $\mathcal{H}$ with $\mathcal{H}$-valued inner product

$$\langle f | g \rangle = \int f(t)^* g(t) dt. \tag{7.12}$$

One can define a covariant representation $(\hat{\pi}, \lambda, \mathcal{H})$ of the dynamical system $(\mathcal{A}, G, \alpha)$ by

$$\big(\hat{\pi}(a) f\big)(t) = \alpha_{t^{-1}}(a) f(t), \quad (\lambda_t f)(s) = f(t^{-1}s), \tag{7.13}$$

for all $a \in \mathcal{A}$ and $t, s \in G$. The image of $(\hat{\pi} \times \lambda)$ given in equation (7.11) gives a C^*-algebra called the **reduced crossed product** which is denoted by $\mathcal{A} \rtimes_r G$. This reduced crossed product equals the crossed product if the group G of the dynamical system $(\mathcal{A}, G, \alpha)$ is amenable ([63]).

Definition 7.1.6. *Let $\mathcal{A}$ be a C^*-algebra in $L^\infty(G)$, invariant under left translation. We say that a state m of $\mathcal{A}$ is a* **left invariant mean** *if $m(\lambda_s f) = m(f)$ for every f in $\mathcal{A}$. We say that G is* **amenable** *if there is a left invariant mean on $L^\infty(G)$.*

One can show that every Abelian group is amenable ([63]). In our system the group G is given by $\mathbb{R}^2$ or $\mathbb{Z}^2$. The continuous case $G = \mathbb{R}^2$ corresponds to the Hamiltonian as we defined it in equation (7.2). The discrete case, $G = \mathbb{Z}^2$, corresponds to the model where we make use of the tight-binding approximation. We can make this approximation using the second simplification discussed in Section 2.6. In this case the Hamiltonian consists of hopping terms (see for instance [38]). Since in our system the group G is given by either $\mathbb{R}^2$ or $\mathbb{Z}^2$, G is commutative and therefore amenable.

In the following Section we construct the algebra of observables corresponding to the Hull of the Hamiltonian, with the magnetic translations as continuous homomorphisms.

7.2 The noncommutative algebra of observables

In this Section we construct the C^*-algebra corresponding to the Hamiltonian (7.2) and the magnetic translations. We show that this algebra can be seen as the algebra of observables of our system. We follow Bellissard (see for instance [15] and [16]).

Consider the dynamical system, denoted by $(\Omega_H, \mathbb{R}^2, T)$ associated to the Hamiltonian H_ω of Theorem 7.1.3. The Haar measure on the topological group $\mathbb{R}^2$ is just the squared Lebesgue measure, denoted by d^2s, and the modular function Δ is the identity. The group operation on $\mathbb{R}^2$ is additive and we denote the inverse t^{-1} of an element $t \in \mathbb{R}^2$ as $-t$. We follow Bellissard in [14] and we construct a C^*-algebra starting with $C_c(\Omega_H \times \mathbb{R}^2)$, the functions with compact support on $\Omega_H \times \mathbb{R}^2$. The reason why we start with this algebra can be made plausible by the following argument.

Consider an element A in our algebra of observables. This element is a measurable function on Ω_H and obeys the covariant condition

$$\hat{T}(x)A_\omega\hat{T}^*(x) = A_{\hat{T}^x\omega}, \tag{7.14}$$

where $\hat{T}$ is the magnetic translation of equation (2.36). Remember that this magnetic translation is given by

$$\hat{T}(y)f(x) = e^{\imath\lambda x\wedge y}T(y)f(x) = e^{\imath\lambda x\wedge y}f(x+y), \tag{7.15}$$

with λ the magnetic field strength times $\frac{e}{2\hbar}$. Therefore we can rewrite the matrix elements $\langle x|A_\omega|x'\rangle$ as follows

$$\langle x|A_\omega|x'\rangle = \langle x+y|A_{T^y\omega}|x'+y\rangle e^{-\imath\lambda(x-x')\wedge y} = a(T^y\omega, x-x'), \tag{7.16}$$

where a is a function of $\Omega_H \times \mathbb{R}^2$, measurable in the first variable.

Hence, starting with the algebra $C_c(\Omega_H \times \mathbb{R}^2)$ we define an involution and convolution on it similar to equation (7.8):

$$\begin{aligned} A^*(\omega, x) &= \overline{A(T^{-x}\omega, -x)}, \\ (A*B)(\omega, x) &= \int_{\mathbb{R}^2} A(\omega, y)B(T^{-y}\omega, x-y)e^{\imath\lambda x\wedge y}ds, \end{aligned} \tag{7.17}$$

for all $A, B \in C_c(\Omega_H \times \mathbb{R}^2)$, $\omega \in \Omega$ and $x \in \mathbb{R}^2$, where λ is $\frac{e}{2\hbar}$ times the magnetic field strength.

As for the crossed product, we proceed by defining a norm similar to equation (7.9). The completion of $C_c(\Omega \times \mathbb{R}^2)$ under this norm is denoted by $L^1(\Omega \times \mathbb{R}^2)$. The representation π_ω of this algebra on $L^2(\mathbb{R}^2)$, for given $\omega \in \Omega$, is given by

$$\pi_\omega(A)\psi(x) = \int_{\mathbb{R}^2} d^2y A(T^{-x}\omega, y-x)e^{\imath\lambda y\wedge x}\psi(y), \quad \psi \in L^2(\mathbb{R}^2). \tag{7.18}$$

The C^*-algebra denoted by $C^*(\Omega \times \mathbb{R}^2; \lambda)$ is now given by the completion of $L^1(\Omega \times \mathbb{R}^2)$ under the norm

$$||A|| = \sup_{\omega \in \Omega} ||\pi_\omega(A)||. \tag{7.19}$$

Remark 7.2.1. In [14] Bellissard uses the fact that the C^*-algebra of a groupoid is closely related to the C^*-algebra of the corresponding dynamical system (see [71]) .

Definition 7.2.2. *A* **groupoid** *is a small category with inverses, i.e., a family of two sets $\{G^0, G\}$ with two maps r and s from G (the arrows) to G^0 (the objects) and an inverse on G such that*

1. *there is an associative multiplication gh in G with $r(gh) = r(g)$, $s(gh) = s(h)$ if $r(h) = s(g)$ for $g, h \in G$,*
2. *$r(g^{-1}) = s(g)$, and $s(g^{-1}) = r(g)$ for $g \in G$,*
3. *G^0 can be seen as a subset of G with $r(x) = x = s(x)$ for all $x \in G^0$ and $gx = g$ for $s(g) = x$ and $xg = g$ for $r(g) = x$.*

The maps r and s on an element in G are called the range and the source of that element, respectively. The set G^0 is called the basis or the space of units of the groupoid. Mostly we will omit it and denote the groupoid as G.

The groupoid G defined by a dynamical system $(\Omega, \mathbb{R}^2, T)$ is given by $G = \Omega \times \mathbb{R}^2$ where Ω is the basis such that

$$r(\omega, x) = \omega, \qquad s(\omega, x) = T^{-x}\omega, \tag{7.20}$$

$$(\omega, x) = (\omega, y) \circ (T^{-y}\omega, x - y), \qquad (\omega, x)^{-1} = (T^{-x}\omega, -x). \tag{7.21}$$

See [70] and [71] for more on this.

Remark 7.2.3. In Remark 7.1.4 we considered the Hull Ω of the lattice $\mathcal{L}$ and we claimed that the physics of the corresponding dynamical system equalled the physics of the Hamiltonian H_ω. This is a consequence of the fact that H_ω is affiliated to the C^*-algebra $\mathcal{A}_\mathcal{L}$ corresponding to the lattice Hull (see [14], [17] and [16]).

Definition 7.2.4. *A covariant family (H_ω) of self adjoint operators is* **affiliated** *to $\mathcal{A}_\mathcal{L}$ if, for every $f \in C_0(\mathbb{R})$, the bounded operator $f(H_\omega)$ can be represented as $\pi_\omega(A_f)$ for certain $A_f \in \mathcal{A}_\mathcal{L}$ such that the map $A : f \in C_0(\mathbb{R}) \to A_f \in \mathcal{A}_\mathcal{L}$ is a bounded $*$-morphism.*

This means that the C^*-algebra coming from the operator H_ω is $*$-isomorphic to a subalgebra of the C^*-algebra corresponding to the lattice $\mathcal{L}$. Since both these algebras represent an algebra of observables, they give the same physics. It is possible that the Hamiltonian is not bounded and that it does not belong to our C^*-algebra of observables. The resolvent, however, does belong to this algebra and therefore also $f(H_\omega)$ for every $f \in C_0(\mathbb{R})$.

Consider the case $\lambda = 0$ of no magnetic field. The algebra of observables of our system corresponding to a perfect lattice is isomorphic to $C(\mathbb{B}) \otimes \mathcal{K}$, where $C(\mathbb{B})$ is the algebra of continuous functions over the Brillouin zone $\mathbb{B}$ and $\mathcal{K}$ is the algebra of compact operators (see [14] and [84] for the proof and the precise physical conditions of this statement). This is the reason to claim that the topological manifold corresponding to the C^*-algebra $C^*(\Omega \times \mathbb{R}^2; \lambda)$ is a generalisation of the Brillouin zone $\mathbb{B}$. As a topological manifold is completely described by its C^*-algebra, we call the C^*-algebra $C^*(\Omega \times \mathbb{R}^2; \lambda)$ the **noncommutative Brillouin zone** of our system.

Consider the tight-binding approximation mentioned in the previous Section. Simply said, this means that only the electrons with energy near the Fermi energy contribute to the physics. Bellissard introduces this approximation in the noncommutative framework to simplify the calculations. The Hamiltonian (7.2) can very well be an unbounded operator on the Hilbert space $L^2(\mathbb{R}^2)$. The corresponding discrete Hamiltonian H_{eff} is a bounded operator, acting on the Hilbert space $\ell^2(\mathbb{Z}^2)$, making the computations easier. In the tight-binding approximation, the case $G = \mathbb{Z}^2$, we can make the same constructions as in the previous two Sections 7.1 and 7.2 (see [11], [17] and [58]).

Having defined the algebra of observables of our system, we want to be able to work with it. What we need, is to define a calculus on our algebra, which will be the subject of our next Section.

7.3 The noncommutative calculus on the algebra of observables

In this Section we construct the calculus on the noncommutative algebra of observables $\mathcal{A} = C^*(\Omega \times G; \lambda)$, with G is $\mathbb{R}^2$. In a similar manner one can construct a calculus on the discrete noncommutative algebra of observables of the tight-binding approximation, i.e. $G = \mathbb{Z}^2$. Although we also use the discrete calculus in the following Sections, we do not construct it here explicitly, but refer to [15] and [16].

Consider a probability measure $\mathbb{P}$ on Ω such that $\mathbb{P}$ is invariant and ergodic under the action of G. Hence we can apply the Birkhoff ergodicity Theorem and we have for every $f \in L^1(\Omega, \mathbb{P})$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T_s^n x) = \int_\Omega f(y) d\mathbb{P}(y), \tag{7.22}$$

for almost all x (see [67]). We can always choose such a measure since G is amenable (see [11] and [63]).

Using this measure we define a trace $\mathcal{T}$ on $\mathcal{A}$ as follows

$$\mathcal{T}(A) = \int_\Omega A(\omega, 0) d\mathbb{P}(\omega), \quad \text{for every } A \in C_c(\Omega \times G) \tag{7.23}$$

Consider for every $p \geq 1$ the norm

$$||A||_{L^p} = \big(\mathcal{T}((AA^*)^{p/2})\big)^{2/p}, \tag{7.24}$$

and denote the completion of $C_c(\Omega \times G)$ under this norm as $L^p(\mathcal{A}, \mathcal{T})$. Consider the situation $p = 2$. Using the GNS-construction of Proposition 4.2.4 one can show that the trace defines a representation π of $\mathcal{A}$ on some Hilbert space such that

$$\langle \pi(a) u | u \rangle = \mathcal{T}(a), \tag{7.25}$$

where u is a cyclic vector of the Hilbert space for $\pi(\mathcal{A})$. This Hilbert space is exactly $L^2(\mathcal{A}, \mathcal{T})$.

Now that we have defined an integral on our algebra of observables we are able to express some physical quantities in algebraic terms. To do this we need the following Theorem, which is proven in [14].

Theorem 7.3.1. *Let A be an element in $C_c(\Omega \times G)$ and let Λ be a square in G around the zero. The trace of A can be obtained as the trace per volume of the operator $\pi_\omega(A)$ in the following manner. Namely for $\mathbb{P}$-almost all ω's we have*

$$\mathcal{T}(A) = \lim_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \mathrm{Tr}_\Lambda\big(\pi_\omega(A)\big), \tag{7.26}$$

where Tr_Λ is the restriction of the normal trace on $L^2(G)$ to $L^2(\Lambda)$.

The existence of a sequence Λ which converges to G is due to the fact that G is amenable. These Λ fulfill some finite volume properties, which are needed to prove the above Theorem. For more on this see [14] and [63].

Consider the density of states we discussed in Section 2.2. Due to some random disorder we obtain new states with energies in-between the Landau levels. These states are described by the density of states. We define this density through the integrated density of states $\mathcal{N}(E)$ which is the number of eigenvalues, of the Hamiltonian per volume, under the energy E:

$$\mathcal{N}(E) = \lim_{\Lambda \uparrow G} \frac{1}{|\Lambda|} \#\{\text{ eigenvalues of} H|_\Lambda \leq E\}, \tag{7.27}$$

where the degeneracy of the eigenvalues is counted with their multiplicity. This is a monotone function of E. Therefore the derivative $\rho(E) = d\mathcal{N}(E)/dE$ is a well-defined positive Lebesgue-Stieltjes measure (see for instance [67] and [69]). This quantity ρ is called the density of states. Thus, at $T = 0$ the electron density n in our system is given by

$$n = \int_{-\infty}^{E_F} dE\rho(E) = \mathcal{N}(E_F), \tag{7.28}$$

where E_F is the Fermi level.

With Theorem 7.3.1 in mind we are able to define for every self-adjoint element $H \in \mathcal{A}$ the density of states as the positive real measure $d\mathcal{N}(E)/dE$ such that for every continuous function f with compact support on $\mathbb{R}$

$$\mathcal{T}(f(H)) = \int_{\mathbb{R}} d\mathcal{N}(E) f(E). \tag{7.29}$$

To complete the calculus on our algebra $\mathcal{A}$ we need a derivative ∂. This derivative is defined as the linear map from $C_c(\Omega \times G)$ to $C_c(\Omega \times G)$ and

$$\partial_i A(\omega, x) = \imath x_i A(\omega, x). \tag{7.30}$$

This defines a family of commuting $*$-derivations, which generates a 2-parameter group of $*$-automorphisms. Namely,

$$\rho_k(A)(\omega, x) = e^{\langle k|x\rangle} A(\omega, x), \tag{7.31}$$

where k lies in the dual of $\mathbb{R}^2$. This extends continuously to the C^*-algebra $\mathcal{A}$ and one can show that

$$\pi_\omega(\partial_i A) = \imath[X_i, \pi_\omega(A)], \quad \pi_\omega\big(\rho_k(A)\big) = e^{\langle k|X\rangle} \pi_\omega(A) e^{-\langle k|X\rangle}, \tag{7.32}$$

where X_i is the position operator, that means, multiplying with x_i in $L^2(G)$. Equation 7.32 can now be rewritten as

$$\pi_\omega(\partial_i A) = \imath \frac{\partial}{\partial k}\big(\pi_\omega(\rho_k(A))\big)|_{k=0}, \tag{7.33}$$

and we see that the derivative of our algebra is actually a derivative in the momentum space. Hence the derivative describes the geometry of our momentum space which is a noncommutative version of the Brillouin zone we are used to.

As usual we are interested in the elements of $A \in \mathcal{A}$ which are in a sense differentiable. We call an element $A \in \mathcal{A}$ N times differentiable if the map $k \in \mathbb{R}^2 \to \rho_k(A) \in \mathcal{A}$ is N times differentiable. This comes down to: A is N times differentiable if $||\partial_1^a \partial_2^b A|| < \infty$ for every a, b such that $a + b = N$. The set of this elements is denoted by $\mathcal{C}^N(\mathcal{A})$. This differentiability is actually a little too strict. It turns out that the physical quantities of our system are given in terms of $\mathcal{T}(Af(H))$ for some distribution f. The differentiability condition of $\mathcal{C}^N(\mathcal{A})$ can be relaxed and we just demand differentiability within the trace $\mathcal{T}$. Consider the Hilbert space $\mathcal{S}$, which is the completion of $C_c(\Omega_H, \mathbb{R}^2)$ under the norm

$$\langle A|B\rangle_{\mathcal{S}} = \mathcal{T}(A^*B) + \mathcal{T}(\partial_1 A^* \partial_1 B + \partial_2 A^* \partial_2 B). \tag{7.34}$$

This space $\mathcal{S}$ is called the **noncommutative Sobolev space** ([18] and [16]) after the commutative version (see [68] and [69]). An element in $\mathcal{S}$, hence any element $A \in C_0(\Omega_H, \mathbb{R}^2)$ such that $\mathcal{T}(|\nabla A|^2) < \infty$ with $\nabla = (\partial_1, \partial_2)$, is called Sobolev differentiable.

7.4 The noncommutative Kubo formula

In this Section we give an equation for the conductivity in the noncommutative setting. As in Section 2.5 we start from the linear response theory and give the Kubo formula for the conductivity. For the validity of this expression for the conductivity see Section 2.6 and the references [9], [35] and [61].

Consider the single-electron Hamiltonian with disorder of equation (7.2). This operator is represented on $L^2(\mathbb{R}^2)$ and the map $\omega \to H_\omega$ is a measurable function with respect to the probability space $(\Omega, \mathbb{P})$. See the former Section for the precise construction of this model. The linear response theory follows from averaging over time and using the relaxation time approximation. We will give here the basic construction of this approximation following [18], but will not get into details. For more information on this subject see [25] for the physical construction, or [20] for the more mathematical point of view.

As mentioned before, we can consider a single-electron system with disorder. This disorder is supposed to be random. The main idea of the relaxation time approximation is that the disorder can be described as collisions as the electron travels around our system. All information about the dissipative effects is therefore given by one unique parameter called the relaxation time τ. Normally one would define the relaxation time as the average time per collision. We, however, take an efficiency coefficient $0 \leq \kappa \leq 1$ into account. The smaller κ, the greater the efficiency of the collision. The relaxation time we use is given by $\tau/(1-\kappa)$. The Kubo formula relates dissipative coefficients to current correlation functions and gives us an expression for the conductivity since the conductivity is the differential of the current with respect to the electric field.

The conductivity of our system is given in terms of the current. To give an expression of the current we need the following observations. In our system the density matrix in consideration is just the Fermi-Dirac distribution

$$f_{\beta,\mu}(H) = (1 + e^{\beta(H-\mu)})^{-1}, \tag{7.35}$$

where $\beta = k_B T$, the inverse of the Boltzmann constant times the temperature, and μ is the chemical potential. The temperature averaged expectation value per volume of an observable $A \in \mathcal{A}$ is consequently given by

$$\langle A \rangle = \langle A \rangle_{\beta,\mu} = \mathcal{T}(A f_{\beta,\mu}(H)). \tag{7.36}$$

The conductivity tensor is now given (see [18] and [20]) by

$$\sigma_{ij} = \frac{e^2}{\hbar} \langle \partial_j f_{\beta,\mu}(H) | (\hbar(1-\hat{\kappa})/\tau - \hbar \mathcal{L}_H)^{-1} \partial_i H \rangle, \tag{7.37}$$

with $\mathcal{L}_H(A) = \imath/\hbar[H, A]$, and where $\hat{\kappa}$ is actually the operator, acting on $\mathcal{A}$, which represents the average change of an observable due to the collisions. We will not give the explicit form of this operator, but mention that the norm of this operator is proportional to κ.

As mentioned before, we take the following limits. The electrical field is vanishing small, the relaxation time goes to infinity and the temperature goes to zero. The distribution $f_{\beta,\mu}$ in this limit is

$$\lim_{\beta \to \infty} f_{\beta,\mu} = P_F, \tag{7.38}$$

which is the **Fermi projection**, i.e., the projection onto energy levels lower than the Fermi level. The limit is taken with respect to the norm in $L^2(\mathcal{A}, \mathcal{T})$ defined in equation (7.24). For this limit to exist we demand that the Fermi level is not a discontinuity of the DOS of the Hamiltonian H (see equation (7.29)). In this limit the conductivity of equation (7.37) becomes

$$\sigma_{ij} = \frac{e^2}{\hbar} \langle \partial_j P_F | - \hbar^{-1} \mathcal{L}_H^{-1} \partial_i H \rangle. \tag{7.39}$$

Let us consider the operator $-\hbar^{-1}\mathcal{L}_H^{-1}\partial_j H$. The matrix elements are given by

$$\langle E | - \hbar^{-1} \mathcal{L}_H^{-1} \partial_j H | E' \rangle = \frac{\imath \langle E | \partial_j H | E' \rangle}{E - E'}. \tag{7.40}$$

We can see this using $\langle E|\mathcal{L}_H(A)|E'\rangle = \frac{\imath}{\hbar}(E - E')\langle E|A|E'\rangle$, and the fact that the eigenvalues of an inverse of an operator is the inverse of the eigenvalues of that operator. If E approaches E' this will diverge. The reason that the conductance does not diverge, is that $\partial_j P_F$ is zero in this region (since $f_{\beta,\mu}$ is a bounded operator of the Hamiltonian). Indeed, due to the derivative properties of ∂_j and the projection properties of P_F we have

$$\partial_j P_F = (1 - P_F)\partial_j P_F P_F + P_F \partial_j P_F (1 - P_F), \tag{7.41}$$

and the matrix element $\langle E|\partial_j P_F|E'\rangle$ is only non zero, whenever $E < E_F < E'$ or $E' < E_F < E$. Thus instead of looking at $\mathcal{L}_H^{-1}\partial_j H$ it suffices to look only at the operators $(1 - P_F)\mathcal{L}_H^{-1}\partial_j H P_F$ and $P_F \mathcal{L}_H^{-1}\partial_j H(1 - P_F)$. We used here the properties of the inproduct.

The Theorem of this Section, which gives the noncommutative Kubo formula, is a consequence of the following Lemma. In this Lemma we demand the operator P_F to be Sobolev differentiable. Because P_F is an eigenprojections of an operator with resolvent in $\mathcal{A}$ and H is bounded from below, we know that it is trace class. In particular, we know that this operator is Hilbert-Schmidt. Now we only need to show that $\partial_i P_F$ is Hilbert-Schmidt. Using (7.23) we have

$$\begin{aligned}\mathcal{T}(|\nabla P_F|^2) &= \int_\Omega d\mathbb{P}(\omega)|\nabla P_F(\omega,0)|^2 \\ &= \int_\Omega d\mathbb{P}(\omega)\int dx\left|\langle 0|P_F|x\rangle\right|^2|x|^2 = \xi,\end{aligned} \tag{7.42}$$

where ξ measures the localisation length at the Fermi level (see [11], [18] and references therein). Hence, $\partial_i P_F$ is Hilbert-Schmidt whenever the Fermi level lies in a region of localised states.

Lemma 7.4.1. *If the Fermi level is not a discontinuity point of the density of states of H, and if the Fermi projection is Sobolev differentiable, the following formulas hold,*

$$P_F\mathcal{L}_H^{-1}\partial_i H(1-P_F) = -\imath\hbar P_F\partial_i P_F(1-P_F), \tag{7.43}$$

$$(1-P_F)\mathcal{L}_H^{-1}\partial_i P_F = \imath\hbar(1-P_F)\partial_i P_F P_F. \tag{7.44}$$

Proof. We prove only the first formula, because the second can be proved in the same way. Let B be the right hand side of formula 7.43. Then.

$$\begin{aligned}\mathcal{L}_H(B) &= HP_F\partial_i P_F(1-P_F) - P_F\partial_i P_F(1-P_F)H \\ &= P_F H\partial_i P_F(1-P_F) - P_F\partial_i P_F H(1-P_F) \\ &= P_F[H,\partial_i P_F](1-P_F)\end{aligned} \tag{7.45}$$

The last equation we got, by realizing that H and P_F commute. For the commutator in the last line we can write,

$$\begin{aligned}[H,\partial_i P_F] &= H\partial_i P_F - \partial_i P_F H \\ &= \partial_i(HP_F) - \partial_i HP_F - \partial_i(P_F H) + P_F\partial_i H \\ &= -[\partial_i H, P_F].\end{aligned} \tag{7.46}$$

We now have

$$\begin{aligned}\mathcal{L}_H(B) &= -P_F[\partial_i H, P_F](1-P_F) \\ &= -P_F\partial_i HP_F(1-P_F) + P_F\partial_i H(1-P_F) \\ &= P_F\partial_i H(1-P_F).\end{aligned} \tag{7.47}$$

Because B connects only energies above the Fermi level with energies below it, and since the Fermi level is not an eigenvalue of H, $\mathcal{L}_H(B)$ will never be zero and $\mathcal{L}_H$ is invertible on the subspace of such operators. We can conclude that,

$$\begin{aligned}B &= \mathcal{L}_H^{-1}(P_F\partial_i H(1-P_F)) \\ &= P_F\mathcal{L}_H^{-1}\partial_i H(1-P_F).\end{aligned} \tag{7.48}$$

□

The next Theorem is a consequence of this Lemma.

Corollary 7.4.2 (IQHE-Kubo formula). *If the Fermi level is not a discontinuity point of the DOS of H, and if the Fermi projection is Sobolev differentiable, the conductivity tensor is given by*

$$\sigma_{ij} = \frac{q^2}{\hbar} 2\imath\pi \mathcal{T}(P_F[\partial_i P_F, \partial_j P_F]), \tag{7.49}$$

in the zero temperature and infinite relaxation time limit. In particular, the direct conductivity vanishes.

7.5 Quantisation of the IQHE

In this section we prove the quantisation of the integer quantum Hall effect in the noncommutative framework. We also see how the plateaux appear in this picture. To do so, we introduce the necessary mathematics.

In Chapter 5 we saw that the geometry of an algebra $\mathcal{A}$ can be given by a Fredholm module (H, F). Let us introduce an even Fredholm module on the algebra $C_c(\Omega_H \times G)$. Consider the graded Hilbert space $\hat{\mathcal{H}} = \mathcal{H}_1 \oplus \mathcal{H}_2 = L^2(G) \oplus L^2(G)$ with grading operator γ

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.50}$$

Define a representation $\hat{\pi}_\omega$ of $C_c(\Omega_H \times G)$ on $\hat{\mathcal{H}}$ by

$$\hat{\pi}_\omega(A) = \hat{A}_\omega = \begin{pmatrix} A_\omega & 0 \\ 0 & a_\omega \end{pmatrix}, \quad A_\omega = \pi_\omega(A), \tag{7.51}$$

with π_ω the representation given in equation (7.18). The Fredholm module $(\hat{\mathcal{H}}, F)$ is now given by the operator

$$F = \begin{pmatrix} 0 & F^{+-} \\ F^{-+} & 0 \end{pmatrix} = \begin{pmatrix} 0 & u \\ u^* & 0 \end{pmatrix} \quad , \quad u = \frac{X_1 + \imath X_2}{|X_1 + \imath X_2|}, \tag{7.52}$$

with X_i the position operator on $L^2(G)$. See for the precise definition of a Fredholm module Definition 5.3.1. Just as in Section 5.3, equations (5.25) and (5.27), we define a derivation d and a supertrace Tr_S on $\mathcal{L}(\hat{\mathcal{H}})$, the bounded operators on $\hat{\mathcal{H}}$, as

$$dT = \imath[F, T], \quad \text{and} \quad \mathrm{Tr}_S(T) = \frac{1}{2}\mathrm{Tr}(\gamma F[F, T]) \quad \forall T \in \mathcal{L}(\hat{\mathcal{H}}). \tag{7.53}$$

Consider the Schatten classes $\mathcal{L}^p(\hat{\mathcal{H}})$ discussed in Section 5.3. These classes are ideals of the algebra of compact operators on $\hat{\mathcal{H}}$ such that $T \in \mathcal{L}^p(\hat{\mathcal{H}})$ if

$$\sum_{n=1}^{\infty} \mu_n^p(T) < \infty, \tag{7.54}$$

with $(\mu_n(T))$ the characteristic values of T, that is the eigenvalues of $(T^*T)^{\frac{1}{2}}$ in decreasing order. Consider now a slightly bigger ideal denoted by $\mathcal{L}^{p+}(\hat{\mathcal{H}})$, such that $T \in \mathcal{L}^{p+}(\hat{\mathcal{H}})$ if the characteristic values of T obey the property

$$\limsup_{N\to\infty} \frac{1}{\ln N} \sum_{n=1}^{N} \mu_n^p(T) < \infty. \tag{7.55}$$

These ideals are called the Dixmier ideals, and are the duals of the Macaev ideals (see for instance [30] and [31]). An element $T \in \mathcal{L}^{n+}(\hat{\mathcal{H}})$ is called n^+-summable. The advantage of these Dixmier ideals is that we can define a trace on it which vanishes when applied to a trace class operator. This property enables us to compute the Hochschild class of the character of a Fredholm module. This trace is called the Dixmier trace, and is constructed as follows (see [18] and [31]).

Consider a positive linear functional $\operatorname{Lim}_\omega$ on the space of bounded sequences $l^\infty(\mathbb{N})$, which has the following properties

1. $\operatorname{Lim}_\omega(\alpha_n) \leq 0$ if $\alpha_n \leq 0$,
2. $\operatorname{Lim}_\omega(\alpha_n) = \lim(\alpha_n)$ if α_n converges,
3. $\operatorname{Lim}_\omega(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \dots) = \operatorname{Lim}_\omega(\alpha_n)$,

for all $\alpha_n \in l^\infty(\mathbb{N})$. A Dixmier trace Tr_ω is defined as

$$\operatorname{Tr}_\omega(T) = \operatorname{Lim}_\omega \frac{1}{\ln N} \sum_{n=1}^{N} \mu_n(T). \tag{7.56}$$

If the so-called Cesáro means of the sequence $\frac{1}{\ln N} \sum_{n=1}^{N} \mu_n(T)$ converge (see [31] and [39]), $\operatorname{Tr}_\omega(T)$ is independent of ω. In that case we are able to define the **Dixmier trace** denoted by $\operatorname{Tr}_{\mathrm{Dix}}$. That this linear functional is indeed a trace, i.e. positive and vanishing on commutators, can be found in the references.

We are now able to relate the conductance (7.49) to the index of some Fredholm operator, hence an integer. We do this using two so-called Connes formulas and an applied version of Theorem 5.4.4. The first Connes theorem, based on a result of Connes in [30], gives a necessary summability condition to a Sobolev differentiable operator (see Section 7.3). See [18] for the proof of this Theorem.

Theorem 7.5.1 (First Connes formula). *Consider the Fredholm module $(\hat{\mathcal{H}}, F)$ over $C_c(\Omega_H \times G)$ defined by the equations (7.50)-(7.52). This module is 2^+-summable for $\mathbb{P}$-almost all ω's. Moreover for every $A \in C_c(\Omega_H \times G)$, the following formula holds:*

$$\mathcal{T}(|\nabla A|^2) = \frac{2}{\pi} Tr_{Dix}(|dA_\omega|^2), \quad \text{for } \mathbb{P}\text{-almost all } \omega. \tag{7.57}$$

This formula can be continued to elements A in the Sobolev space $\mathcal{S}$ associated to $\mathcal{T}$. In particular, if $A \in \mathcal{S}$, then $dA_\omega \in \mathcal{L}^{2+}$ for $\mathbb{P}$-almost all ω.

Although Bellissard proved this formula, in [18], only for the case $G = \mathbb{Z}^2$, leaving the case $G = \mathbb{R}^2$ for future work, we assume, in the case $G = \mathbb{R}^2$, the summability condition $dA_\omega \in \mathcal{L}^{2+}$ for $\mathbb{P}$-almost all ω. This condition is needed for the following Theorem to be well defined. This assumption is not unreasonable, but loses the connection with the localisation condition given in the previous Section.

In the following Theorem we relate the Kubo formula of Corollary 7.4.2 to the associated character of the Fredholm module $(\hat{\mathcal{H}}, F)$ over $C_c(\Omega \times G)$. We do this by considering the following expression (compare to (7.49)

$$\mathcal{T}_2(A_0, A_1, A_2) = 2\imath\pi \mathcal{T}(A_0 \partial_1 A_1 \partial_2 A_2 - A_0 \partial_2 A_1 \partial_1 A_2) \tag{7.58}$$

for $A_0, A_1, A_2 \in C_c(\Omega_H \times G)$. Remember that the associated character τ_2 of the Fredholm module $(\hat{\mathcal{H}}, F)$ is given by (see equation (5.33) and Proposition 5.4.1))

$$\tau_2(A_0, A_1, A_2) = 2\imath\pi \mathrm{Tr}_s(\hat{A}_{0,\omega} d\hat{A}_{1,\omega} d\hat{A}_{2,\omega}). \tag{7.59}$$

The conductance in terms of $\mathcal{T}_2$ can be related to the Fredholm module through the second Connes formula.

Theorem 7.5.2 (Second Connes formula). *For $A_0, A_1, A_2 \in C_c(\Omega \times G)$, we have the following formula:*

$$\mathcal{T}_2(A_0, A_1, A_2) = \int_\Omega d\mathbb{P}(\omega)\, Tr_s(\hat{A}_{0,\omega} d\hat{A}_{1,\omega} d\hat{A}_{2,\omega}). \tag{7.60}$$

We will not give the proof here but refer to [18], where the Theorem is proved for the case $G = \mathbb{Z}^2$. The continuous case where $G = \mathbb{R}^2$ is a direct generalisation which can be proved using [7] and [29]. Remark that we need Theorem 7.5.1 for the right hand side of equation (7.60) to be well defined.

Consider again equation (7.59) which is by Definition 5.4.3 a Chern character of the Fredholm module $(\hat{\mathcal{H}}, F)$, which pairs the K_0-group of $\mathcal{A}$ with the cyclic cohomology group (see Theorem 5.4.4 and Proposition 6.2.7). This pairing can be given in terms of the index of the Fredholm module, which is an integer. The following Proposition is similar to Theorem 5.4.4.

Proposition 7.5.3. *Consider the Fredholm module $(\hat{\mathcal{H}}, F)$ over $C_c(\Omega_H \times G)$ defined by the equations (7.50)-(7.52). Let $P \in C_c(\Omega \times G)$ be a 3-summable projection. Then $F_P^{+-} = PF^{+-}|_{P\mathcal{H}_2}$ is a Fredholm operator and*

$$\mathrm{Index}(F_P^{+-}) = 2\imath\pi \mathrm{Tr}_s(\hat{P}d\hat{P}d\hat{P}). \tag{7.61}$$

The proof of this Proposition is similar to the proof of Theorem 5.4.4 (with $P = e$). The pairing of equation (5.38) can be written as

$$\langle [e], [\tau_2] \rangle = \tau_2(e, e, e) = 2\imath\pi \mathrm{Tr}_s(\hat{e}\, d\hat{e}\, d\hat{e}), \tag{7.62}$$

where we used equations (6.44) and (6.46).

We are now able to state our main Theorem of this Section.

Theorem 7.5.4. *Let P be a projection belonging to the noncommutative Sobolev space $\mathcal{S}$. Then for $\mathbb{P}$-almost every $\omega \in \Omega$, P is 2^+-summable and*

$$2\imath\pi \mathcal{T}(P[\partial_1 P, \partial_2 P]) := \mathrm{Ch}(P) = \mathrm{Ind}(P_\omega u|_{P_\omega \mathcal{H}_2}), \tag{7.63}$$

where $P_\omega = \pi_\omega(P)$ and u as in equation (7.52). In particular, $\mathrm{Ch}(P)$ *is an integer.*

Proof. Consider the character $\mathrm{Ch}(P)$,

$$\begin{aligned} \mathrm{Ch}(P) &= 2\imath\pi \mathcal{T}(P[\partial_1 P, \partial_2 P]) \\ &= \mathcal{T}_2(P, P, P) \\ &= \int_\Omega d\mathbb{P}(\omega) \mathrm{Tr}_s(\hat{P}_\omega, d\hat{P}_\omega d\hat{P}_\omega), \end{aligned} \tag{7.64}$$

where we used Theorems 7.5.1 and 7.5.2 for the last equation. Proposition 7.5.3 gives us,

$$\mathrm{Ch}(P) = (2\imath\pi)^{-1} \int_\Omega d\mathbb{P}(\omega) \mathrm{Index}(P_\omega F^{+-}|_{P_\omega \mathcal{H}_2}). \tag{7.65}$$

We show now that the index is $\mathbb{P}$-almost independent of ω and therefore

$$\mathrm{Ch}(P) = \mathrm{Ind}(P_\omega F^{+-}|_{P_\omega \mathcal{H}_2}). \tag{7.66}$$

Using the ergodicity (and Birkhoff) of $\mathbb{P}$, we only have to show that the index is translation invariant. For indeed, if $f(\omega)$ is translation invariant we have,

$$\begin{aligned} \int d\mathbb{P}(\omega) f(\omega) &= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} dx f(T_x \omega) \\ &= \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} dx f(\omega) \\ &= 2\imath\pi f(\omega). \end{aligned} \tag{7.67}$$

The translation of P_ω by $a \in G$ is due to the covariance property $P_{T^a\omega}$. If we translate $F^{+-} = u$, we get $u + \mathcal{O}(\frac{1}{|X|})$. Thus, finally $P_\omega u|_{P_\omega \mathcal{H}_2}$ translates to $P_{T^{-a}\omega}(u + \mathcal{O}(\frac{1}{|X|}))|_{P_{T^{-a}\omega}\mathcal{H}_2}$, that is $P_{T^{-a}\omega}u|_{P_{T^{-a}\omega}\mathcal{H}_2}$ modulo a compact operator. Because a compact perturbation does not change the index of a Fredholm module, we can indeed conclude (7.66). □

This not only proves the quantisation of the conductance, but also the existence of the plateaux in the noncommutative framework. Namely, as long as the Fermi energy lies in a region of localised states, the character $\mathrm{Ch}(P_F)$ is constant. We refer to [18] for more on the relation between the localisation length and the Hall conductance

7.6 Further developments

In this Section we discuss a model introduced by Xia in [82], which has much resemblance with the work of Bellissard. This model starts with the same Kubo formula. But where Bellissard relates the Hall conductance to the (analytic) index of some Fredholm module, Xia relates this conductance to a topological index, as the difference in dimension between two vector bundles. Equating both indices of the different models gives an analogue of the classical Atiyah-Singer theorem in the sense that

$$\text{(analytical index)} = -\text{(topological index)}$$

We start by constructing an algebra of observables closely related to the one constructed in Section 7.2. Let Ω be a separable, connected, compact Hausdorff space. And let there be a group of homeomorphisms $\{\phi_{x,y} : (x, y) \in \mathbb{R}^2\}$ acting on Ω such that for every $f \in C(\Omega)$, the map $(x, y) \mapsto f \circ \phi_{x,y}$ from $\mathbb{R}^2$ to $C(\Omega)$ is continuous. Consider the algebra $C_c(\Omega \times \mathbb{R}^2)$ of continuous functions with compact support on the space $\Omega \times \mathbb{R}^2$. We define a representation π_ω for every $\omega \in \Omega$ on this algebra by

$$\big(\pi_\omega(A)f\big)(x, y) = \int_{\mathbb{R}^2} A\big(\omega + (x, y), \xi, \eta\big)e^{-\imath\beta x\eta} f(x + \xi, y + \eta) d\xi d\eta, \qquad (7.68)$$

with $A \in C_c(\Omega \times \mathbb{R}^2)$, $f \in L^2(\mathbb{R}^2)$ and β equal to the λ of equation (7.15). The algebra of observables of this Section is now given by all elements $\pi_\omega(A)$ with fixed ω and is denoted by $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$. Next Xia defines a convolution and an involution on the algebra $L^2(\Omega \times \mathbb{R}^2)$ very similar to the ones defined in equation (7.17). The C^*-algebra thus constructed is denoted by $C^*(\Omega, \mathbb{R}^2, \beta)$. This algebra is very similar to the algebra of observables constructed by Bellissard. Moreover they are unitarily equivalent to each other. For the precise construction of this algebra we refer to the original paper [82]. For the relation between both algebras we also refer to [58].

Xia proceeds by considering a natural isomorphism ϕ from $K_0(C^*(\Omega, \mathbb{R}^2, \beta))$ to $K_0(C(\Omega))$. He does this by applying twice the Thom isomorphism. The classical Thom isomorphism is a isomorphism from the K-group of a locally compact space X to the K-group of a (complex) vector bundle V over X (see for instance [46] for more details). A generalisation of this isomorphism (see [28]) is a map

$$K_i(\mathcal{A}) \to K_{i+1}(\mathcal{A} \rtimes_\alpha \mathbb{R}) \quad \text{with} \quad i \in \mathbb{Z}/2, \tag{7.69}$$

where $(\mathcal{A}, \mathbb{R}, \alpha)$ is a C^*-dynamical system. This gives the natural isomorphism ϕ:

$$\begin{aligned} K_0\big(C(\Omega)\big) &\simeq K_1\big(C(\Omega) \rtimes_\theta \mathbb{R}\big) \\ &\simeq K_0\big((C(\Omega) \rtimes_\theta \mathbb{R}) \rtimes_\gamma \mathbb{R}\big) \\ &\simeq K_0\big(C^*(\Omega, \mathbb{R}^2, \beta)\big), \end{aligned} \tag{7.70}$$

with θ and γ the corresponding homomorphisms.

Just as for the algebra of Bellissard we define a trace $\mathcal{T}$ and some derivations ∂_i on the algebra $C^*(\Omega, \mathbb{R}^2, \beta)$. They are similar to the constructions of Bellissard and we refer again to the article [82]. Consider now the conductance given by equation (7.49)

$$\sigma_{ij} = \frac{q^2}{\hbar} 2\imath\pi \mathcal{T}(P_F[\partial_i P_F, \partial_j P_F]). \tag{7.71}$$

Using the pairing (6.44) between K-theory and cyclic cohomology of Proposition 6.2.7 one can rewrite the expression for the conductance to

$$\sigma_{ij} = \frac{q^2}{\hbar} \langle [\psi], [P_F] \rangle, \tag{7.72}$$

where $[\psi]$ is the cocycle defined such as in (7.58)

$$\psi(a_0, a_1, a_2) = \mathcal{T}\big(a_0(\partial_1(a_1)\partial_2(a_2) - \partial_2(a_1)\partial_1(a_2))\big). \tag{7.73}$$

Xia subsequently shows that the pullback of ψ by the transposed of ϕ equals $-[\mu]$, where $[\mu]$ is the even cyclic cocycle defined by the measure

$$\mu(f) = \int_\Omega f(\omega) d\mu(\omega). \tag{7.74}$$

That is

$$\langle [\psi], [P_F] \rangle = -\langle [\mu], \phi^{-1}[P_F] \rangle \tag{7.75}$$

for $[P_F] \in K_0(C^*(\Omega, \mathbb{R}^2, \beta))$. Remember that every element of $K_0(C(\Omega))$ can be written as the difference between two vector bundle classes (see Sections 3.3 and 4.4). Hence $\phi^{-1}[P_F] = [V^+] - [V^-]$ for some vector bundles V^+ and V^- on Ω. And therefore one can write

$$\langle [\mu], \phi^{-1}[P_F] \rangle = \dim V^- - \dim V^+, \tag{7.76}$$

and, indeed, express the conductance σ_{ij} in topological terms, namely the difference in dimension of two vector bundles over the space Ω.

From equation (7.76) one sees immediately that the conductance takes the integer value times the known factor. In [82] Xia shows that this integer can be interpreted as the Landau band index only if $\phi^{-1}[P_F]$ lies in the subgroup $\mathbb{Z}[1]$ of $K_0(C(\Omega))$. Because the group $K_0(C(\Omega))$ is often bigger than $\mathbb{Z}[1]$, the Landau interpretation of the conductance does not need to be universally applicable. Xia suggests that therefore the K-theory class $\phi^{-1}[P_F]$ could contain more physical information of the integer quantum Hall effect than the conductivity itself. Examples of $\phi^{-1}[P_F] \notin \mathbb{Z}[1]$, however, have not yet been found. Therefore we can not say that this model is preferred over that of Bellissard. Moreover one could argue that Bellissard's model is more complete for the following reason. While in Xia's model we have to assume the Fermi energy to be in a gap, in Bellissard's model we can extend this assumption to the case where the Fermi energy lies in a region of localised states. This is a consequence of less restrictive smoothness conditions Bellissard uses on the projections.

Although we would prefer, on physical grounds, Bellissard's model, Xia's article is certainly not redundant. As a mathematical extension it can be very useful. For instance, the topological index is often easier to calculate than the analytical one. An other reason for studying Xia's model is its mathematical beauty. As we already mentioned, we can link both relations for the Hall conductivity to get the equation

$$\mathrm{Index}(F_P) = -\langle [\mu], \phi^{-1}[P_F] \rangle, \quad \forall P \in C^\infty_\omega(\Omega, \mathbb{R}^2, \beta), \tag{7.77}$$

which is an analogue of the classical Atiyah-Singer theorem.

About a decade after the first articles of Bellissard and Xia on the noncommutative geometry in the integer quantum Hall effect, these theories where extended to the hyperbolic geometry case ([23], [21], [55], [22] and [56]). In references [23] and [21] the hyperbolic geometry was introduced to describe the integer quantum Hall effect on real parabolic structures. It was believed that in this manner one could study the edge effects for the quantum Hall effect and the behaviour of electrons in quantum dots. In these articles the authors extended the discrete and continuous noncommutative Kubo formula to the parabolic geometry case and found an integer value for the Hall conductance. They did this by extending both approaches of Bellissard and Xia, finding an analytical and a topological index for the Hall conductance. Disorder, which generates the plateaux due to the localised states, is taken into account.

In the later references [55], [22] and [56] the hyperbolic structure was introduced with a different motivation. As we already mentioned in Section 2.1, the Coulomb interactions play an important role in the physics of the fractional quantum Hall effect. And exactly these interactions make it difficult to apply the Bloch theory to the system. The interpretation of the hyperbolic geometry was then, that it simulated the electron interactions. The single-electron Hamiltonian becomes an effective Hamiltonian when placed in a hyperbolic structure.

With this interpretation the authors could use a generalised Bloch theory on the fractional quantum Hall effect. They were able to reproduce the fractional values of the Hall conductance. Actually the model predicts too many fractions, which at present are not observed in the experimental setup. Because this limitation also exists for other models of the fractional quantum Hall effect, we are not (yet) urged to overthrow this model. An interesting physical implication of this model is that it predicts the existence of an absolute lower bound on the fractional values of the Hall conductance. This could be an excellent experimental test of the validity of the theoretical model.

Chapter 8

Conclusions

In this thesis we discussed some early theories of the integer quantum Hall effect. We saw that these theories relied on the somewhat unphysical assumption that the amount of flux through the magnetic unit cell should be rational. Another problem the theories encountered was combining disorder (to explain the plateaux) and the quantisation of the Hall resistance. Not soon after these theories were introduced, Bellissard proposed to use Connes' noncommutative geometry to give a complete description of the integer quantum Hall effect.

The noncommutative geometry approach to the integer quantum Hall effect uses a generalisation of the much celebrated Bloch theory. It does not assume other physics, but uses just other tools. We saw that with this approach one is able to describe the integer quantum Hall effect without making the rationality assumption. Also the quantisation and disorder go perfectly together.

While there are other explanations of the integer quantum Hall effect, that could be satisfying either, the beauty of the noncommutative geometry approach is its generality. It is a true generalisation of the Bloch theory and it is applicable to other non-periodic solids, such as quasicrystals. For a wide range of solids however, there is still a lot of work to be done, to fit this theory in.

Chapter 9

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