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Resonance Fluorescence with
Quantum Stochastic Calculus**

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Constructing the Davies process of Resonance Fluorescence with Quantum Stochastic Calculus

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Abstract

Starting point is a given semigroup of completely positive maps on the 2×2 matrices. This semigroup describes the irreversible evolution of a decaying 2-level atom. Using the integral-sum kernel approach [11] to quantum stochastic calculus we couple the 2-level atom to an environment, which in our case will be interpreted as the electromagnetic field. The irreversible time evolution of the 2-level atom then stems from the reversible time evolution of atom and field together. Mathematically speaking, we have constructed a Markov dilation [9] of the semigroup.

The next step is to drive the atom by a laser and to count the photons emitted into the field by the decaying 2-level atom. For every possible sequence of photon counts we construct a map that gives the time evolution of the 2-level atom inferred by that sequence. The family of maps that we obtain in this way forms a so-called Davies process [4], [16]. In his book Davies describes the structure of these processes, which brings us into the field of quantum trajectories [2]. Within our model we calculate the jump operators and we briefly describe the resulting counting process.

1 Introduction

In this paper we want to illustrate that quantum stochastic calculus together with the processes studied by Davies in his book [4], and explained in his paper with Srinivas [16], form a suitable mathematically rigorous framework for doing quantum trajectory theory [2]. As an example we consider here the case of resonance fluorescence.

Our starting point is a semigroup of transition operators $\{T_t\}_{t \geq 0}$ on the algebra M_2 of all 2×2 -matrices. This semigroup describes the irreversible evolution of a spontaneously decaying 2-level atom in the Heisenberg picture. By coupling the atom to a quantum noise, we construct a stationary quantum Markov process having precisely these transition operators. If we impose the requirements that the external noise be a Bose field, and the quantum Markov process be minimal, then the latter is uniquely determined. It is called the *minimal Bose dilation* of (M_2, T_t, g) [10], where g is the ground state of the 2-level atom.

Since this dilation is uniquely determined, any other reversible dynamical model which couples (M_2, T_t, g) to some Bose field necessarily contains this Bose dilation as a subsystem. Therefore, without deriving our model from an explicit Schrödinger equation

(by performing a Markovian limit) we may safely assume it to be a physically correct way to describe the interaction of the 2-level atom with the electromagnetic field.

We will couple the 2-level atom to the electromagnetic field by using quantum stochastic calculus [15], [14]. We use a version of quantum stochastic calculus based on integral-sum kernels [11], [12], [14], which has the advantage that we have an explicit construction for the solution of the quantum stochastic differential equation with which we will describe the coupling of atom and field. Having this explicit construction in our hands is important for doing the actual calculations we encounter later on.

To be able to discuss resonance fluorescence we have to use a dilation where we have two channels in the electromagnetic field. On one of them we will put a laser state to drive the 2-level atom. We will call this field the *forward channel* and the other one the *side channel*. We will then count photons in both channels. We need the side channel, because there we know that all detected photons are fluorescence photons. In the forward channel a detected photon could just as well be coming directly from the laser.

For every event that can occur in the photon counters we construct a map giving the evolution of the 2-level atom inferred by that event. We will see that the family of maps we obtain, fulfills the axioms for the processes discussed by Davies [4]. We have constructed the Davies process of resonance fluorescence.

Using the structure theory for Davies processes [4] we can decompose the process into its trajectories [2]. Within our model we calculate the expression for the jump operators and for the time evolution in between jumps. Note that a jump in the system occurs the moment we detect a photon, since our knowledge concerning the system changes.

Using the above apparatus we show that the resulting counting process in the side channel is a so-called *renewal process*.

2 The dilation

Let M_2 , the algebra of 2×2 -matrices, stand for the algebra of observables of a 2-level atom. On this algebra we are given a (continuous) semigroup $\{T_t\}_{t \geq 0}$ of completely positive maps. This semigroup describes the, generally irreversible, evolution of the 2-level atom. Lindblad's Theorem [13] then says that $T_t = \exp tL$ where $L : M_2 \rightarrow M_2$ can be written as: for $A \in M_2$:

$$L(A) = i[H, A] + \sum_{j=1}^k V_j^* A V_j - \frac{1}{2} \{V_j^* V_j, A\}, \quad (1)$$

where the V_j and H are fixed 2×2 -matrices, H being Hermitian. In this paper we will restrict to the simpler case where $H = 0$ and there are just two V_j 's. This means there is dissipation only into two channels, the forward channel described by V_f , and

the side channel described by V_s . We choose V_f and V_s such that:

$$V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad V_f = \kappa_f V, \quad V_s = \kappa_s V, \quad |\kappa_f|^2 + |\kappa_s|^2 = 1.$$

This exactly gives the time evolution for spontaneous decay to the ground state of the 2-level atom into two decay channels, where the decay rates are given by $|\kappa_f|^2$ and $|\kappa_s|^2$.

We want to see this irreversible evolution of the 2-level atom as stemming from a reversible evolution of the atom coupled to, in this case, two decay channels in the field. So let us first construct the algebra of observables for these fields. Let \mathcal{F} be the symmetric Fock space over the Hilbert space $L^2(\mathbb{R})$ of square integrable wave functions on the real line, i.e. $\mathcal{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2(\mathbb{R})^{\otimes n}$. The electromagnetic field is given by creation and annihilation operators on \mathcal{F} , generating the algebra of all bounded operators. We need two copies of this algebra, which we denote by \mathcal{W}_f , the field which will be the forward channel, and \mathcal{W}_s , the field which will be the side channel.

The evolution over a time t of a free field is given by the second quantization of the left shift, i.e. the second quantization of the operator on $L^2(\mathbb{R})$ which maps $f(\cdot)$ into $f(\cdot + t)$. We denote the second quantization of this operator by S_t . This means that in the Heisenberg picture we have an evolution on $\mathcal{W}_f \otimes \mathcal{W}_s$ mapping A into $(S_t^* \otimes S_t^*)A(S_t \otimes S_t)$ ($= (S_{-t} \otimes S_{-t})A(S_t \otimes S_t)$), also denoted by $\text{Ad}[S_t \otimes S_t](A)$.

The presence of the atom in the fields introduces a perturbation on the evolution of the free fields. We let this perturbation be given by a certain family of unitary operators $\{U_t\}_{t \in \mathbb{R}}$ on $\mathbb{C}^2 \otimes \mathcal{F} \otimes \mathcal{F}$, which will be specified later, that forms a *cocycle* with respect to the shift $S_t \otimes S_t$, i.e. for all $t, s \in \mathbb{R}$: $U_{t+s} = (S_{-s} \otimes S_{-s})U_t(S_s \otimes S_s)U_s$. Given this cocycle, we let the time evolution of the atom and the fields together be given by the following one-parameter group $\{\hat{T}_t\}_{t \in \mathbb{R}}$ (i.e. the evolution is now *reversible*) of *-automorphisms on $M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s$: for all $A \in M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s$:

$$\hat{T}_t(A) = \begin{cases} U_t^{-1} S_{-t} A S_t U_t & \text{if } t \geq 0 \\ S_{-t} U_{-t} A U_{-t}^{-1} S_t & \text{if } t < 0 \end{cases},$$

The solution of the following quantum stochastic differential equation [8], [15] provides us with a cocycle of unitaries with respect to the shift:

$$dU_t = \{V_f dA_{f,t}^* - V_f^* dA_{f,t} + V_s dA_{s,t}^* - V_s^* dA_{s,t} - \frac{1}{2} V^* V dt\} U_t, \quad U_0 = I. \quad (2)$$

In the next section we will give an explicit construction for the solution U_t of this equation. It can be shown ([8], [5], [11], [15]) that if the cocycle satisfies equation (2) we have constructed a so-called *quantum Markov dilation* $(M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s, \{\hat{T}_t\}_{t \in \mathbb{R}}, \text{id} \otimes \phi \otimes \phi)$ of the quantum dynamical system $(M_2, \{T_t\}_{t \geq 0}, g)$ [9], [10], where ϕ is the vector state on $\mathcal{W}_{f,s}$ given by the vacuum vector. This means that the following *dilation*

diagram commutes for all $t \geq 0$ (and that the resulting quantum process is Markov):

$$\begin{array}{ccc}
M_2 & \xrightarrow{T_t} & M_2 \\
\text{Id}_{\otimes I \otimes I} \downarrow & & \uparrow \text{Id}_{\otimes \phi \otimes \phi} \\
M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s & \xrightarrow{\hat{T}_t} & M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s
\end{array} \tag{3}$$

i.e. for all $A \in M_2$: $T_t(A) = (\text{Id} \otimes \phi \otimes \phi)(\hat{T}_t(A \otimes I \otimes I))$.

Let us here look briefly in the Schrödinger picture at the above diagram. If we start with a state ρ of the 2-level atom (i.e. we are now in the upper right hand corner of the diagram), then this state undergoes the following sequence of maps:

$$\begin{aligned}
\rho(\cdot) &\mapsto \rho(\text{Id} \otimes \phi \otimes \phi(\cdot)) = \rho \otimes \phi \otimes \phi(\cdot) \mapsto \rho \otimes \phi \otimes \phi(\hat{T}_t(\cdot)) \mapsto \\
&\rho \otimes \phi \otimes \phi(\hat{T}_t(\cdot \otimes I \otimes I)) = \text{Tr}_{\mathcal{F}_f \otimes \mathcal{F}_s}(\hat{T}_{t*}(\rho \otimes \phi \otimes \phi))(\cdot),
\end{aligned}$$

i.e. ρ maps to $\text{Tr}_{\mathcal{F}_f \otimes \mathcal{F}_s}(\hat{T}_{t*}(\rho \otimes \phi \otimes \phi))$. This means that ρ is first coupled to the two fields both in the vacuum state, then they are time evolved together and then there is a partial trace taken over the two fields.

3 Guichardet space and integral-sum kernels

Let us now turn to giving the explicit construction for the solution of equation (2). For this we need the *Guichardet space* Ω [7] of \mathbb{R} , which is the space of all finite subsets of \mathbb{R} , i.e. $\Omega := \bigcup_{n \in \mathbb{N}} \Omega_n$, where $\Omega_n := \{\sigma \subset \mathbb{R}; |\sigma| = n\}$. Let us denote by λ_n the Lebesgue measure on \mathbb{R}^n . If, for $n \in \mathbb{N}$, we let $j_n : \mathbb{R}^n \rightarrow \Omega_n$ denote the map that maps an n -tuple (t_1, t_2, \dots, t_n) into the set $\{t_1, t_2, \dots, t_n\}$, then we can define a measure μ_n on Ω by: $\mu_n(E) := \frac{1}{n!} \lambda_n(j_n^{-1}(E))$ for all E in the sigma field Σ_n of Ω_n induced by j_n and the Borel sigma field of \mathbb{R}^n . Now we define a measure μ on Ω such that $\mu(\{\emptyset\}) = 1$ and $\mu = \mu_n$ on Ω_n . This means we have now turned the Guichardet space into the measure space (Ω, Σ, μ) .

The key to constructing the solution of equation (2) is to identify the symmetric Fock space \mathcal{F} with the space of all quadratically integrable functions on the Guichardet space $L^2(\Omega, \mu)$. To see this identification note that $L^2(\Omega_n, \mu_n)$ is, in the canonical way, unitarily equivalent with the space of all quadratically integrable functions on \mathbb{R}^n invariant under permutations of coordinates, denoted $L^2_{\text{Sym}}(\mathbb{R}^n)$. It is now obvious how to identify $\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2_{\text{Sym}}(\mathbb{R}^n)$ with $L^2(\Omega, \mu) = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2(\Omega_n, \mu_n)$.

For every $f \in L^2(\mathbb{R})$ we define the *exponential vector* $e(f) \in \mathcal{F}$ in the following way: $e(f) := 1 \oplus f \oplus \frac{1}{\sqrt{2}} f^{\otimes 2} \oplus \frac{1}{\sqrt{6}} f^{\otimes 3} \oplus \dots$. Note that the linear span of all exponential vectors forms a dense subspace of \mathcal{F} . For every $f \in L^2(\mathbb{R})$ we define the *coherent vector* $\psi(f)$ to be the exponential vector of f normalised to unity, i.e. $\psi(f) = \exp(-\frac{1}{2}\|f\|^2)e(f)$. Under the above identification of \mathcal{F} with $L^2(\Omega, \mu)$, the exponential vector (of an $f \in L^2(\mathbb{R})$) $e(f)$ is mapped into an element of $L^2(\Omega, \mu)$ which we denote by $\pi(f)$ and which is given by: $\pi(f) : \Omega \rightarrow \mathbb{C} : \omega \mapsto \prod_{s \in \omega} f(s)$,

where the empty product $\prod_{s \in \emptyset} f(s)$ is defined to be 1. We will often choose for f the *indicator function* of a certain interval $I \subset \mathbb{R}$, which we denote by χ_I . This is the function which is 1 on I and 0 elsewhere.

We are now ready to start the construction of the solution U_t of equation (2). Define the *integral-sum kernel* of U_t (name will become apparent in a minute) to be the map u_t that maps four disjoint finite subsets of \mathbb{R} , $\sigma_f, \sigma_s, \tau_f, \tau_s$ (where f and s stand for "forward" and "side") to the following 2×2 -matrix, where we write $\sigma_f \cup \sigma_s \cup \tau_f \cup \tau_s$ also as $\{t_1, t_2, \dots, t_k\}$ such that $t_1 < t_2 < \dots < t_k$ and $k \in \mathbb{N}$:

$$u_t(\sigma_f, \sigma_s, \tau_f, \tau_s) := \pi(\chi_{[0,t]})(\sigma_f \cup \sigma_s \cup \tau_f \cup \tau_s) \exp\left(-\frac{t-t_k}{2} V^* V\right) V_k \times \\ \exp\left(-\frac{t_k-t_{k-1}}{2} V^* V\right) V_{k-1} \dots V_1 \exp\left(-\frac{t_1}{2} V^* V\right),$$

where

$$V_j = \begin{cases} V_f & \text{if } t_j \in \sigma_f \\ -V_f^* & \text{if } t_j \in \tau_f \\ V_s & \text{if } t_j \in \sigma_s \\ -V_s^* & \text{if } t_j \in \tau_s \end{cases}.$$

Then we have the following theorem of Maassen, see [11], [12]:

Theorem 3.1: *After identifying $\mathbb{C}^2 \otimes \mathcal{F} \otimes \mathcal{F}$ with $L^2_{\mathbb{C}^2}(\Omega \times \Omega, \mu \times \mu)$, the space of all square integrable functions on $\Omega \times \Omega$ with values in \mathbb{C}^2 , the solution $U_t : L^2_{\mathbb{C}^2}(\Omega \times \Omega, \mu \times \mu) \rightarrow L^2_{\mathbb{C}^2}(\Omega \times \Omega, \mu \times \mu)$ of equation (2) is given by:*

$$(U_t f)(\omega_f, \omega_s) = \sum_{\substack{\sigma_f \subset \omega_f \\ \sigma_s \subset \omega_s}} \int_{\Omega \times \Omega} u_t(\sigma_f, \sigma_s, \tau_f, \tau_s) f((\omega_f \setminus \sigma_f) \cup \tau_f, (\omega_s \setminus \sigma_s) \cup \tau_s) d\tau_f d\tau_s.$$

Now we have an explicit expression for the time evolution $\hat{T}_t = \text{Ad}[\hat{U}_t]$, where \hat{U}_t is given by $S_t U_t$ if $t \geq 0$ and $U_{-t}^{-1} S_t$ if $t < 0$. The family $\{\hat{U}_t\}_{t \in \mathbb{R}}$ forms a group of unitary operators on $\mathbb{C}^2 \otimes \mathcal{F} \otimes \mathcal{F}$ describing the time evolution of the 2-level atom and the two fields together. Stone's Theorem says that there must be a Hamiltonian associated to this time evolution. This Hamiltonian has been calculated recently [6].

4 The Davies process

We now return to the situation in figure 3. We wish to make some changes in this diagram and for this we need to introduce some more notation regarding Guichardet spaces. Let $I \subset \mathbb{R}$ be an interval. Then the *Guichardet space of I* is the set $\Omega(I) = \bigcup_{n=0}^{\infty} \Omega_n(I)$, where $\Omega_n(I) = \{\sigma \subset I; |\sigma| = n\}$. In a similar way as for Ω , which is $\Omega(\mathbb{R})$, we can give these sets a measure structure: $(\Omega(I), \Sigma(I), \mu)$. Given a subset E of $\Omega(I)$ in the sigma field $\Sigma(I)$, we can construct the projection $M_{\chi_E} : L^2(\Omega, \mu) \rightarrow L^2(\Omega, \mu) : f \mapsto \chi_E f$.

Let I be $[-t, 0)$, then the events in $\Sigma([-t, 0))$, which we abbreviate to Σ_t , are events

in the output field of the atom up to time t . Remember that the evolution of a free field was given by the left shift and that the atom is sitting in the origin. Since the Guichardet space representation corresponds to the photon number picture, we can give concrete interpretations to the subsets in Σ_t . For instance, the subsets $\Omega_n([-t, 0))$, correspond to the events "there are n photons in the output of the atom into this field up to time t ".

Now back to the situation in figure 3. Suppose we have been observing the output in the forward and side channel of the atom up to time t with two photon counters. Then we are given two events E_f and E_s in Σ_t . Since we know the outcome of the measurements we have to change the time evolution of the 2-level atom, i.e. we have to project onto the observed events (see also [1]). This is summarized in the following figure:

$$\begin{array}{ccc}
M_2 & \xrightarrow{\mathcal{E}_0^t[E_f, E_s]} & M_2 \\
\text{Id}_{\otimes \chi_{E_f} \otimes \chi_{E_s}} \downarrow & & \uparrow \text{Id}_{\otimes \phi \otimes \phi} \\
M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s & \xrightarrow{\hat{T}_t} & M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s
\end{array}$$

where we have suppressed the capital letters M in the projections. The map $\mathcal{E}_0^t[E_f, E_s] : M_2 \rightarrow M_2 : A \mapsto \text{Id} \otimes \phi \otimes \phi(\hat{T}_t(A \otimes \chi_{E_f} \otimes \chi_{E_s}))$ is the unnormalized time evolution of the 2-level atom in the Heisenberg picture given that we see event E_f in the output of the forward channel and event E_s in the output of the side channel. If we are given a state on M_2 , i.e. a 2×2 density matrix ρ , then the probability of seeing event E_f in the forward channel and E_s in the side channel after t seconds of observation is given by: $\mathbb{P}_\rho^t[(E_f, E_s)] = \text{Tr}(\rho \mathcal{E}_0^t[E_f, E_s](I))$.

The setting is still not complete for describing resonance fluorescence. Since we are not driving the atom, both the forward and the side channel fields are in the vacuum state, at most one photon can appear in the output. We change this by putting on the forward channel a coherent state with amplitude $z \in \mathbb{C}$, defined by: $\gamma_{z_t} : \mathcal{W} \rightarrow \mathbb{C} : A \mapsto \exp(-t|z|^2) \langle \pi(z\chi_{[0,t]}), A\pi(z\chi_{[0,t]}) \rangle$. Note that γ_0 is the vacuum state. Putting a coherent state on the forward channel mimics a laser driving the atom. We have suppressed its oscillations for the sake of simplicity. Now we are ready to do resonance fluorescence, i.e. the diagram has changed into:

$$\begin{array}{ccc}
M_2 & \xrightarrow{\mathcal{E}_z^t[E_f, E_s]} & M_2 \\
\text{Id}_{\otimes \chi_{E_f} \otimes \chi_{E_s}} \downarrow & & \uparrow \text{Id}_{\otimes \gamma_{z_t} \otimes \gamma_0} \\
M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s & \xrightarrow{\hat{T}_t} & M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s
\end{array}$$

where the map $\mathcal{E}_z^t[E_f, E_s] : M_2 \rightarrow M_2$ is now defined by $\mathcal{E}_z^t[E_f, E_s](A) := \text{Id} \otimes \gamma_{z_t} \otimes \gamma_0(\hat{T}_t(A \otimes \chi_{E_f} \otimes \chi_{E_s}))$. It describes the unnormalized time evolution of the laser-driven atom given that we see event E_f in the output of the forward channel and event E_s in the output of the side channel. Given a state ρ of the atom, the probability of seeing event E_f in the forward channel and E_s in the side channel after t seconds of observation is now given by: $\mathbb{P}_\rho^t[(E_f, E_s)] = \text{Tr}(\rho \mathcal{E}_z^t[E_f, E_s](I))$. To make

the notation lighter we suppress the z in \mathcal{E}_z^t in the following.

Since $L^2(\Omega, \Sigma, \mu) \otimes L^2(\Omega, \Sigma, \mu)$ is canonically isomorphic to $L^2(\Omega \times \Omega, \Sigma \otimes \Sigma, \mu \times \mu)$ we can simplify our notation even a bit further. By identifying these spaces we can write $\mathcal{E}^t[E_f, E_s] = \mathcal{E}^t[E_f \times E_s]$, where the righthandside is defined by: for all $E \in \Sigma_t \otimes \Sigma_t, A \in M_2, t \geq 0$: $\mathcal{E}^t[E](A) := \text{Id} \otimes \gamma_{z_t, 0}(\hat{T}_t(A \otimes \chi_E))$, where $\gamma_{z_t, 0}$ is an abbreviation for $\gamma_{z_t} \otimes \gamma_0$. We will now study the properties of the family of maps we defined.

Theorem 4.1: *The family of maps $\{\mathcal{E}^t[E]\}_{t \geq 0, E \in \Sigma_t \otimes \Sigma_t}$ satisfies the axioms of a Davies process, [4]:*

1. For all $t \geq 0$ and $E \in \Sigma_t \otimes \Sigma_t$, $\mathcal{E}^t[E]$ is completely positive.
2. For all $t \geq 0$ and all countable collections of disjoint sets $\{E_n\}$ in $\Sigma_t \otimes \Sigma_t$ and for all $A \in M_2$: $\mathcal{E}^t\left[\bigcup_n E_n\right](A) = \sum_n \mathcal{E}^t[E_n](A)$.
3. For all $t \geq 0$ we have $\mathcal{E}^t\left[\Omega([-t, 0]) \times \Omega([-t, 0])\right](I) = I$.
4. For all $A \in M_2$: $\lim_{t \rightarrow 0} \mathcal{E}^t\left[\Omega([-t, 0]) \times \Omega([-t, 0])\right](A) = A$.
5. For all $t, s \geq 0$ and $E \in \Sigma_s \otimes \Sigma_s, F \in \Sigma_t \otimes \Sigma_t$ and all $A \in M_2$ we have:
 $\mathcal{E}^t[F] \circ \mathcal{E}^s[E](A) = \mathcal{E}^{s+t}[F - s\tilde{U}E](A)$,
 where $F - s \in \Sigma([-t-s, -s]) \otimes \Sigma([-t-s, -s])$ is given by:
 $F - s = \{(f_f - s, f_s - s); (f_f, f_s) \in F\}$ and \tilde{U} is defined by:
 $A\tilde{U}B = \{(\omega_f \cup \sigma_f, \omega_s \cup \sigma_s); (\omega_f, \omega_s) \in A, (\sigma_f, \sigma_s) \in B\}$.

Proof. The only point where there is really something to prove is point 5. Let us first introduce some short notation which we shall only use in this proof. Let $\pi(z_t, 0)$ denote $\pi(z\chi_{[0, t]}) \otimes \pi(0)$ and denote $S_t \otimes S_t$ just by S_t . Further we use the notation $\sigma_t(U_s)$ for $S_{-t}U_sS_t$. Then for all $A \in M_2, s, t \geq 0, E \in \Sigma_s \otimes \Sigma_s$ and $F \in \Sigma_t \otimes \Sigma_t$ we have:

$$\begin{aligned} \frac{\mathcal{E}^t[F] \circ \mathcal{E}^s[E](A)}{\exp(-(s+t)|z|^2)} &= \mathcal{E}^t[F]\left(\left\langle \pi(z_s, 0), \hat{T}_s(A \otimes \chi_E)\pi(z_s, 0) \right\rangle\right) \exp(t|z|^2) = \\ &\left\langle \pi(z_t, 0), \hat{T}_t\left(\left\langle \pi(z_s, 0), \hat{T}_s(A \otimes \chi_E)\pi(z_s, 0) \right\rangle \otimes \chi_F\right)\pi(z_t, 0) \right\rangle = \\ &\left\langle \pi(z_t, 0), U_t^*\left\langle \pi(z_s, 0), \hat{T}_s(A \otimes \chi_E)\pi(z_s, 0) \right\rangle \otimes \chi_{F+t}U_t\pi(z_t, 0) \right\rangle = \\ &\left\langle \pi(z_t, 0), U_t^*\left\langle S_{-t}\pi(z_s, 0), S_{-t}\hat{T}_s(A \otimes \chi_E)S_tS_{-t}\pi(z_s, 0) \right\rangle \otimes \chi_{F+t}U_t\pi(z_t, 0) \right\rangle = \\ &\left\langle \pi(z_t, 0), U_t^*\left\langle S_{-t}\pi(z_s, 0), \sigma_t(U_s)^*A \otimes \chi_{E+t+s}\sigma_t(U_s)S_{-t}\pi(z_s, 0) \right\rangle \otimes \chi_{F+t}U_t\pi(z_t, 0) \right\rangle. \end{aligned}$$

Now we use the cocycle identity and the continuous tensor product structure of the

symmetric Fock space to obtain:

$$\begin{aligned} \frac{\mathcal{E}^t[F] \circ \mathcal{E}^s[E](A)}{\exp(-(s+t)|z|^2)} &= \left\langle \pi(z_{t+s}, 0), (\sigma_t(U_s)U_t)^* A \otimes \chi_{F+t\dot{\cup}E+t+s} \sigma_t(U_s)U_t \pi(z_{t+s}, 0) \right\rangle = \\ &\left\langle \pi(z_{t+s}, 0), U_{t+s}^* A \otimes \chi_{F+t\dot{\cup}E+t+s} U_{t+s} \pi(z_{t+s}, 0) \right\rangle = \\ &\left\langle \pi(z_{t+s}, 0), \hat{T}_{t+s}(A \otimes \chi_{F-s\dot{\cup}E}) \pi(z_{t+s}, 0) \right\rangle = \frac{\mathcal{E}^{s+t}[F - s\tilde{U}E](A)}{\exp(-(s+t)|z|^2)}. \end{aligned}$$

□

Define maps $Y_t : M_2 \rightarrow M_2 : A \mapsto \mathcal{E}^t[\{(\emptyset, \emptyset)\}](A)$. They represent the evolution of the atom when it is observed that no photons entered the decay channels. Then we have that for all $t, s \geq 0$: $Y_t Y_s = \mathcal{E}^t[\{(\emptyset, \emptyset)\}] \circ \mathcal{E}^s[\{(\emptyset, \emptyset)\}] = \mathcal{E}^{t+s}[\{(\emptyset, \emptyset)\} - s\dot{\cup}\{(\emptyset, \emptyset)\}] = \mathcal{E}^{t+s}[\{(\emptyset, \emptyset)\}] = Y_{t+s}$, i.e. the family $\{Y_t\}_{t \geq 0}$ forms a semigroup. Now observe that for $A \in M_2$ and $t \geq 0$ we have:

$$\begin{aligned} Y_t(A) &= \mathcal{E}^t[\{(\emptyset), \{\emptyset\}\}](A) = \text{Id} \otimes \gamma_{z_t} \otimes \gamma_0(\hat{T}_t(A \otimes \chi_{\{\emptyset\}} \otimes \chi_{\{\emptyset\}})) = \\ &\left\langle \pi(z_t) \otimes \pi(0), U_t^* A \otimes \chi_{\{\emptyset\}} \otimes \chi_{\{\emptyset\}} U_t \pi(z_t) \otimes \pi(0) \right\rangle \exp(-t|z|^2) = \\ &(U_t \pi(z_t) \otimes \pi(0))^* (\emptyset, \emptyset) A (U_t \pi(z_t) \otimes \pi(0)) (\emptyset, \emptyset) \exp(-t|z|^2). \end{aligned}$$

If we define $B_t : \mathbb{C}^2 \rightarrow \mathbb{C}^2 : v \mapsto \left(\exp(-\frac{1}{2}t|z|^2) (U_t \pi(z_t) \otimes \pi(0)) \right) (\emptyset, \emptyset) v$, then we see, using Theorem 3.1, that B_t is the following semigroup of contractions:

$$B_t = \exp\left(-\frac{1}{2}(|z|^2 I_2 + V^* V + 2zV_f^*)t\right), \quad (4)$$

and for all $A \in M_2$: $Y_t(A) = B_t^* A B_t$. We say that the Davies process \mathcal{E}^t is *ideal*, see [4].

Furthermore Theorem 4.1 point 2, leads to:

$$\mathcal{E}^t[\Omega[-t, 0] \times \Omega[-t, 0] \setminus \{(\emptyset, \emptyset)\}](I) = I - B_t^* B_t.$$

If we use this and the expression for B_t (4), then we can do some estimations which in the end lead to:

$$\mathcal{E}^t[\Omega[-t, 0] \times \Omega[-t, 0] \setminus \{(\emptyset, \emptyset)\}](I) \leq tKI,$$

with $K = (2|z|^2|\kappa_f|^2 + 1)$. This property can be summarized by saying that the Davies process \mathcal{E}^t has *bounded interaction rate*, see [4].

5 Quantum trajectories

In the seventies Davies studied the structure of what we now call Davies processes [4]. Let us first state his results, as far as relevant, in the context of the process we are studying.

Lemma 5.1: (*Davies [4]*) Given any ideal Davies process \mathcal{E}^t with bounded interaction rate, as defined in the previous section, we have existence of the following limits:

$$\mathcal{J}_f := \lim_{t \downarrow 0} \frac{1}{t} \mathcal{E}^t[\Omega_1[-t, 0), \{\emptyset\}] \quad \text{and} \quad \mathcal{J}_s := \lim_{t \downarrow 0} \frac{1}{t} \mathcal{E}^t[\{\emptyset\}, \Omega_1[-t, 0)].$$

These completely positive maps represent the action we have to apply on the 2-level atom the moment we see one photon appear in the forward, respectively side channel. They are the *jump operations* for these channels. We will explicitly calculate these limits later on, but first we turn our attention to decomposing the Davies process into its trajectories [2]. For this we need the following definition.

Definition 5.2: Let $Y_t : M_2 \rightarrow M_2$ be the maps from the previous section, i.e. $Y_t = \mathcal{E}^t[\{\emptyset\}, \{\emptyset\}]$ and let \mathcal{J}_f and \mathcal{J}_s be the maps from lemma 5.1. Let ω_f and ω_s be disjoint elements of $\Omega[-t, 0)$ and denote $\omega_f \cup \omega_s$ also as $\{t_1, \dots, t_k\}$ where $-t \leq t_1 < t_2 < \dots < t_k \leq 0$ for a $k \in \mathbb{N}$. Then we define:

$$W_{Y, \mathcal{J}_f, \mathcal{J}_s}(\omega_f, \omega_s) := Y_{t_1+t} \mathcal{J}^{t_1} Y_{t_2-t_1} \mathcal{J}^{t_2} \dots \mathcal{J}^{t_k} Y_{-t_k},$$

where \mathcal{J}^{t_i} denotes \mathcal{J}_s if $t_i \in \omega_s$ and \mathcal{J}_f if $t_i \in \omega_f$.

Since Y_t is the time evolution of the system when, both in the forward and the side channels, no photons are detected and \mathcal{J}_f and \mathcal{J}_s are the jump operations that we have to apply when a photon in the corresponding channels appears, it is clear that the string of maps $Y_{t-t_1} \mathcal{J}^{t_1} Y_{t_1-t_2} \mathcal{J}^{t_2} \dots \mathcal{J}^{t_k} Y_{t_k}$ represents the *trajectory* of an observable A in M_2 when we find the outcomes ω_f in the forward and ω_s in the side channel during our counting experiment. The following theorem of Davies [4] shows how to decompose the Davies process into its trajectories.

Theorem 5.3: (*Davies [4]*) Given any ideal Davies process \mathcal{E}^t with bounded interaction rate, as defined in the previous section, we have for all $t \geq 0$, $E_f, E_s \in \Sigma_t$ and $A \in M_2$:

$$\mathcal{E}^t[E_f, E_s](A) = \int_{E_f \times E_s} W_{Y, \mathcal{J}_f, \mathcal{J}_s}(\omega_f, \omega_s)(A) d\mu(\omega_f) d\mu(\omega_s).$$

In the previous section we already found the expression for the time evolution in between jumps: Y_t . We now turn to the calculation of \mathcal{J}_f and \mathcal{J}_s . For all A in M_2 we have:

$$\mathcal{J}_f(A) = \lim_{t \downarrow 0} \frac{1}{t} \mathcal{E}^t[(\Omega_1[-t, 0), \{\emptyset\})](A) = \lim_{t \downarrow 0} \frac{\int_0^t \text{Ad}[U_t \pi(z) \otimes \pi(0)(\{s\}, \emptyset)](A) ds}{t \exp(-t|z|^2)}.$$

Now look at $U_t\pi(z) \otimes \pi(0)(\{s\}, \emptyset)$, we find by using Theorem 3.1:

$$\begin{aligned} U_t\pi(z) \otimes \pi(0)(\{s\}, \emptyset) &= \sum_{\sigma \subset \{s\}} \int_{\Omega} u_t(\sigma, \emptyset, \tau, \emptyset) z^{1-|\sigma|+|\tau|} d\tau = zu_t(\emptyset, \emptyset, \emptyset, \emptyset) + \\ & z^2 \int_0^t u_t(\emptyset, \emptyset, \{r\}, \emptyset) dr + u_t(\{s\}, \emptyset, \emptyset, \emptyset) + z \int_0^t u_t(\{s\}, \emptyset, \{r\}, \emptyset) dr + \\ & z^2 \int_0^t \int_0^{r_2} u_t(\{s\}, \emptyset, \{r_1, r_2\}, \emptyset) dr_1 dr_2 = \begin{pmatrix} z \exp(-\frac{t}{2}) & 2z^2 \bar{\kappa}_f \exp(-\frac{t}{2}) - 2z^2 \bar{\kappa}_f \\ \kappa_f \exp(-\frac{t}{2}) & z \end{pmatrix}. \end{aligned}$$

Therefore we get, for all $A \in M_2$:

$$\begin{aligned} \mathcal{J}_f(A) &= \lim_{t \downarrow 0} \frac{\int_0^t \text{Ad}[U_t\pi(z) \otimes \pi(0)(\{s\}, \emptyset)](A) ds}{t \exp(-t|z|^2)} = \text{Ad} \left[\begin{pmatrix} z & 0 \\ \kappa_f & z \end{pmatrix} \right] (A) = \\ & \text{Ad}[zI_2 + V_f](A). \end{aligned}$$

Let us now turn to the calculation of \mathcal{J}_s . We find for all $A \in M_2$:

$$\mathcal{J}_s(A) = \lim_{t \downarrow 0} \frac{1}{t} \mathcal{E}^t[(\{\emptyset\}, \Omega_1[-t, 0))] (A) = \lim_{t \downarrow 0} \frac{\int_0^t \text{Ad}[U_t\pi(z) \otimes \pi(0)(\emptyset, \{s\})](A) ds}{t \exp(-t|z|^2)}.$$

Taking a closer look at $U_t\pi(z) \otimes \pi(0)(\emptyset, \{s\})$, applying Theorem 3.1:

$$\begin{aligned} U_t\pi(z) \otimes \pi(0)(\emptyset, \{s\}) &= \int_{\Omega} u_t(\emptyset, \{s\}, \tau, \emptyset) z^{|\tau|} d\tau = u_t(\emptyset, \{s\}, \emptyset, \emptyset) + \\ & z \int_0^t u_t(\emptyset, \{s\}, \{r\}, \emptyset) dr + z^2 \int_0^t \int_0^{r_2} u_t(\emptyset, \{s\}, \{r_1, r_2\}, \emptyset) dr_1 dr_2 = \begin{pmatrix} 0 & 0 \\ \kappa_s \exp(-\frac{t}{2}) & 0 \end{pmatrix}. \end{aligned}$$

Therefore we get, for all $A \in M_2$:

$$\begin{aligned} \mathcal{J}_s(A) &= \lim_{t \downarrow 0} \frac{\int_0^t \text{Ad}[U_t\pi(z) \otimes \pi(0)(\emptyset, \{s\})](A) ds}{t \exp(-t|z|^2)} = \text{Ad} \left[\begin{pmatrix} 0 & 0 \\ \kappa_s & 0 \end{pmatrix} \right] (A) = \\ & = \text{Ad}[V_s](A). \end{aligned}$$

Since we are driving the atom with a laser now, the time evolution when we do not observe the side channel nor the forward channel is now given by $T_t^z := \mathcal{E}^t[\Omega[-t, 0), \Omega[-t, 0)]$ and no longer by T_t . We will now derive the Master equation for this new time evolution. For this we need the Dyson series: let L_0 and J be maps from $M_2 \rightarrow M_2$, then for all $t \geq 0$:

$$\exp(t(L_0 + J)) = \int_{\Omega[-t, 0)} \exp((\omega_1 + t)L_0) J \exp((\omega_2 - \omega_1)L_0) J \dots J \exp(-\omega_k L_0) d\omega,$$

where we have written ω as $\{\omega_1, \dots, \omega_k\}$ with $-t \leq \omega_1 < \dots < \omega_k \leq 0$.

Now remember that $\{Y_t\}_{t \geq 0}$ is a semigroup so we can write $Y_t = \exp(tL_0)$. Then,

using Theorem 5.3 and twice the Dyson series, we see that:

$$T_t^z = \mathcal{E}^t[\Omega[-t, 0), \Omega[-t, 0)] = \int_{\Omega[-t, 0) \times \Omega[-t, 0)} W_{Y, \mathcal{J}_f, \mathcal{J}_s}(\omega_f, \omega_s) d\omega_f d\omega_s = \exp(t(L_0 + \mathcal{J}_f + \mathcal{J}_s)).$$

This means we get the following Master equation:

$$\frac{d}{dt} T_t^z = L_0 + \mathcal{J}_f + \mathcal{J}_s = -\frac{1}{2}\{V^*V, \cdot\} + [zV^* - \bar{z}V, \cdot] + V^* \cdot V, \quad (5)$$

which is exactly the Master equation for resonance fluorescence (see [2]) if we take $z = -i\frac{\Omega}{2}$ with Ω , the *Rabi frequency*, real.

In the quantum optics literature (see for instance [2]), usually there is no photon counting measurement done in the forward channel, i.e. $E_f = \Omega[-t, 0)$. From here on we will do the same, we define for all $t \geq 0$ and $E_s \in \Sigma_t$: $\mathcal{M}^t[E_s] := \mathcal{E}^t[\Omega[-t, 0), E_s]$. In the following we will also suppress the index s on E_s . Using the Dyson series and Theorem 5.3 we find, for all $t \geq 0$ and $E \in \Sigma_t$:

$$\mathcal{M}^t[E] = \int_E W_{Z, \mathcal{J}_s}(\omega) d\mu(\omega), \quad (6)$$

where the time evolution in between side-channel-jumps Z_t is given by $Z_t = \exp(t(L_0 + \mathcal{J}_f))$ and W_{Z, \mathcal{J}_s} is defined in the obvious way analogous to Definition 5.2. Note that we have found exactly the same jump operator and time evolution in between jumps as in the usual quantum optics literature, see for instance [2], [3], i.e. we have succeeded in constructing the Davies process of resonance fluorescence with quantum stochastic calculus.

6 A renewal process

We will now look briefly at some features of the process \mathcal{M}^t we obtained. It is easily seen from the fact that $(\mathcal{J}_s)^2 = 0$ (i.e. $g_2(0) = 0$) that the photons in the side channel arrive *anti-bunched*: the probability to see two photons immediately after each other is 0. We will now show that the photon counting process in the side channel is a so-called *renewal process*.

We denote $\Sigma^t := \Sigma[0, t)$ and, via a shift, we let events E in Σ^t correspond to events $E - t$ in the output sigma field Σ_t . This means that an element $\omega = \{\omega_1, \dots, \omega_k\}$ in $E \in \Sigma^t$ with $0 \leq \omega_1 < \dots < \omega_k < t$ should be interpreted as seeing the first photon appear in the side channel at time ω_1 , the second at time ω_2 up to the k 'th photon at time ω_k .

Given that we start the photon counting measurement in the initial state ρ , we define on the sigma fields Σ^t ($t \geq 0$) probability measures in the natural way: for $E \in \Sigma^t$: $\mathbb{P}_\rho^t[E] := \text{Tr}(\rho \mathcal{M}^t[E - t](I))$. The family of sigma fields $\{\Sigma^t\}_{t \geq 0}$ generates a

sigma-field Σ^∞ of $\Omega[0, \infty)$. Using that $T_s^z(I) = I$, see equation (5), we find for all $E \in \Sigma^t$:

$$\begin{aligned}\mathbb{P}_\rho^{t+s}[E] &= \text{Tr}\left(\rho \mathcal{M}^{t+s}\left[(E\tilde{\cup}\Omega[t, t+s]) - (t+s)\right](I)\right) = \\ &= \text{Tr}\left(\rho \mathcal{M}^{t+s}[E - (t+s)\tilde{\cup}\Omega[-s, 0]](I)\right) = \text{Tr}\left(\rho \mathcal{M}^t[E - t]\mathcal{M}^s[\Omega[-s, 0]](I)\right) = \\ &= \text{Tr}\left(\rho \mathcal{M}^t[E - t]T_t^z(I)\right) = \text{Tr}\left(\rho \mathcal{M}^t[E - t](I)\right) = \mathbb{P}_\rho^t[E],\end{aligned}$$

so $\mathbb{P}_\rho^{t+s}[E]$ does not depend on s . Therefore the family $\{\mathbb{P}_\rho^t\}_{t \geq 0}$ on the sigma-fields $\{\Sigma^t\}_{t \geq 0}$ is consistent, hence by Kolmogorov's extension theorem it extends to a single probability measure \mathbb{P}_ρ on Σ^∞ .

In the following, when we write $\omega \in \Omega[0, \infty)$ as $\{\omega_1, \omega_2, \dots\}$, we imply that $0 \leq \omega_1 < \omega_2 < \dots$. For $j = 1, 2, \dots$ we define random variables:

$$X_j : \Omega[0, \infty) \rightarrow \overline{\mathbb{R}}^+ : \omega = \{\omega_1, \omega_2, \dots\} \mapsto \begin{cases} \omega_j - \omega_{j-1} & \text{if } |\omega| \geq j \\ \infty & \text{else} \end{cases},$$

where we take ω_0 to be 0. These random variables give the time elapsed between the $(j-1)$ th and j th detection of a photon. To prove that the counting process is a (*modified*) *renewal process* we have to show that for $i = 1, 2, \dots$ the random variables X_i are independent and for $i = 2, 3, \dots$ they are identically distributed. This means we have to show that for $i = 2, 3, \dots$ the distribution functions $F_{X_i}(x) := \mathbb{P}_\rho[X_i \leq x]$ are equal and for $i, j = 1, 2, \dots$ the joint probability distribution function $F_{X_i, X_j}(x, y) := \mathbb{P}_\rho[X_i \leq x \wedge X_j \leq y]$ factorizes: $F_{X_i, X_j}(x, y) = F_{X_i}(x)F_{X_j}(y)$.

Let us first introduce some convenient notation. Note that, using equation (6), we have for all $E \in \Sigma^t$:

$$\begin{aligned}\mathbb{P}_\rho[E] &= \mathbb{P}_\rho^t[E] = \text{Tr}\left(\rho \int_{E-t} W_{Z, \mathcal{J}_s}(\omega) d\mu(\omega)(I)\right) = \\ &= \text{Tr}\left(\rho \int_E Z_{\omega_1} \mathcal{J}_s Z_{\omega_2 - \omega_1} \mathcal{J}_s \dots \mathcal{J}_s Z_{t - \omega_k}(I) d\mu(\omega)\right).\end{aligned}$$

We will denote: $x_1 := \omega_1, x_2 := \omega_2 - \omega_1, \dots, x_{k+1} := t - \omega_k$, then we can write:

$$\mathbb{P}_\rho[E] = \int_E \text{Tr}\left(\rho Z_{x_1} \mathcal{J}_s Z_{x_2} \mathcal{J}_s \dots \mathcal{J}_s Z_{x_{k+1}}(I)\right) d\mu(\omega).$$

Let P denote the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then we have:

$$\begin{aligned}\mathcal{J}_s Z_{x_{k+1}}(I) &= \begin{pmatrix} |\kappa_s|^2 (Z_{x_{k+1}}(I))_{22} & 0 \\ 0 & 0 \end{pmatrix} = |\kappa_s|^2 (Z_{x_{k+1}}(I))_{22} P, \\ \mathcal{J}_s Z_{x_k}(P) &= \begin{pmatrix} |\kappa_s|^2 (Z_{x_k}(P))_{22} & 0 \\ 0 & 0 \end{pmatrix} = |\kappa_s|^2 (Z_{x_k}(P))_{22} P, \dots, \\ \mathcal{J}_s Z_{x_2}(P) &= \begin{pmatrix} |\kappa_s|^2 (Z_{x_2}(P))_{22} & 0 \\ 0 & 0 \end{pmatrix} = |\kappa_s|^2 (Z_{x_2}(P))_{22} P.\end{aligned}$$

Therefore, if we define $z(x) := |\kappa_s|^2 (Z_x(P))_{22}$, $z_{last}(x) := |\kappa_s|^2 (Z_x(I))_{22}$ and $z_{first}(x) := \text{Tr}(\rho Z_x(P))$, we can write (see also [3]):

$$\mathbb{P}_\rho[E] = \int_E z_{first}(x_1) \left(\prod_{l=2}^k z(x_l) \right) z_{last}(x_{k+1}) d\mu(\omega). \quad (7)$$

We would like to stress that this formula is only valid for events $E \in \Sigma^t$ and not for all events in Σ^∞ .

For $t \geq 0$ we introduce the following random variables:

$$N_t : \Omega[0, \infty) \rightarrow \mathbb{N} : \omega \mapsto |\omega \cap [0, t]|,$$

counting the number of photons appearing in the side channel up to time t . Since, for strictly positive driving field strengths, i.e. $|z| > 0$, the eigenvalues of the generator $L_0 + \mathcal{J}_f$ of the semigroup Z_t all have strictly negative real parts, we have $\lim_{t \rightarrow \infty} Z_t = 0$. Using this, formula (7) and the fact that the event $[N_t = 0]$ is an element of Σ^t , we obtain:

$$\lim_{t \rightarrow \infty} \mathbb{P}_\rho[N_t = 0] = \lim_{t \rightarrow \infty} z_{first}(t) = 0.$$

Now suppose we have that $\lim_{t \rightarrow \infty} \mathbb{P}_\rho[N_t \leq n] = 0$ for a certain $n \in \mathbb{N}$. For $s \leq t$ we use: $\mathbb{P}_\rho[N_t \leq n+1] = \mathbb{P}_\rho[N_t \leq n+1 | N_s \leq n] \mathbb{P}_\rho[N_s \leq n] + \mathbb{P}_\rho[N_t \leq n+1 | N_s > n] \mathbb{P}_\rho[N_s > n]$. Therefore we have:

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{P}_\rho[N_t \leq n+1] &= \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}_\rho[N_t \leq n+1] = \\ \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} \left(\mathbb{P}_\rho[N_t \leq n+1 | N_s \leq n] \mathbb{P}_\rho[N_s \leq n] + \mathbb{P}_\rho[N_t \leq n+1 | N_s > n] \mathbb{P}_\rho[N_s > n] \right) &= \\ \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}_\rho^t[N_t \leq n+1 | N_s > n] &= \lim_{s \rightarrow \infty} \lim_{t \rightarrow \infty} z_{last}(t-s) = 0. \end{aligned}$$

Now using induction, we get for $n \in \mathbb{N}$:

$$\lim_{t \rightarrow \infty} \mathbb{P}_\rho[N_t \leq n] = 0.$$

We are now ready to calculate the distribution functions F_{X_i} and F_{X_i, X_j} . The problem is that for instance the event $[X_i \leq x] \in \Sigma^\infty$ is not an element of Σ^t for a $t \in \mathbb{R}$. We solve this by conditioning on the event $[N_t \geq i]$ and taking the limit for t to infinity:

$$\begin{aligned} F_{X_i}(x) &= \mathbb{P}_\rho[X_i \leq x] = \\ \lim_{t \rightarrow \infty} \left(\mathbb{P}_\rho[X_i \leq x | N_t \geq i] \mathbb{P}_\rho[N_t \geq i] + \mathbb{P}_\rho[X_i \leq x | N_t < i] \mathbb{P}_\rho[N_t < i] \right) &= \\ \lim_{t \rightarrow \infty} \mathbb{P}_\rho^t[X_i \leq x \wedge N_t \geq i]. \end{aligned}$$

Now we use again formula (7) to obtain for $i \geq 2$:

$$\begin{aligned}
F_{X_i}(x) &= \lim_{t \rightarrow \infty} \sum_{k=i}^{\infty} \int_{\sum_{l=1}^{k+1} x_l = t} z_{first}(x_1) \left(\prod_{l=2}^k z(x_l) \right) z_{last}(x_{k+1}) dx_1 \dots dx_{k+1} = \\
&\lim_{t \rightarrow \infty} \int_0^x z(x_i) \left(\sum_{k=i}^{\infty} \int_{\sum_{l \neq i} x_l = t - x_i} z_{first}(x_1) dx_1 \left(\prod_{\substack{l=2 \\ l \neq i}}^k z(x_l) dx_l \right) z_{last}(x_{k+1}) dx_{k+1} \right) dx_i = \\
&\lim_{t \rightarrow \infty} \int_0^x z(x_i) \mathbb{P}_{\rho}^{t-x_i} [N_{t-x_i} \geq i-1] dx_i.
\end{aligned}$$

Then we use dominated convergence to interchange the limit and the integral to obtain:

$$F_{X_i}(x) = \int_0^x z(x') dx'.$$

When $i = 1$ we can repeat the whole calculation to find the same result when for z we substitute z_{first} . It is now obvious that for $i = 2, 3, \dots$ the random variables X_i are identically distributed.

In a similar fashion, only extracting two integrals now, we find that for $i, j = 2, 3, \dots$: $F_{X_i, X_j}(x, y) = \int_0^x \int_0^y z(x') z(y') dx' dy'$. If i or j is 1 we again have to substitute z_{first} for z . It is now obvious that the random variables X_i and X_j are independent. We conclude that the family of random variables $\{X_i\}_{i=1,2,\dots}$ is a (modified) renewal process.

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