Probab. Th. Rel. Fields 83, 489–508 (1989)

Probability Theory and Related Fields

Quantum Poisson Processes and Dilations of Dynamical Semigroups

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Summary. The notion of a quantum Poisson process over a quantum measure space is introduced. This process is used to construct new quantum Markov processes on the matrix algebra M_n with stationary faithful state ϕ. If $(\mathscr{M}, \mu)$ is the quantum measure space in question ($\mathscr{M}$ a von Neumann algebra and μ a faithful normal weight), then the semigroup e^{tL} of transition operators on (M_n, ϕ) has generator

$$L:\ M_n \to M_n:\ a \to i[h, a] + (id \otimes \mu)(u^*(a \otimes \mathbb{1})\, u - a \otimes \mathbb{1}),$$

where u is an arbitrary unitary element of the centraliser of $(M_n \otimes \mathscr{M}, \phi \otimes \mu)$.

1. Introduction

In probability theory, semigroups of transition operators on a probability space describe Markov processes. Constructing the full Markov process from such a semigroup is called *dilating* the semigroup.

In quantum mechanics irreversible behaviour of physical systems is described by dynamical semigroups on operator algebras. "Explaining" such irreversible behaviour by finding possible embeddings into a reversibly evolving world again corresponds to *dilating* these semigroups. Whereas the dilation of a Markovian semigroup is uniquely determined in the commutative (i.e., classical probabilistic) situation, in the generalised non-commutative (quantum probabilistic) setting it is not. It is therefore a challenge to find new constructions of dilations. A general theory of dilations can be found in the work of B. Kümmerer [Küm 1], [Küm 2], [Küm 3].

Many dilations consist of coupling a system to some external "quantum noise", which is the subject of the quantum stochastic calculus of Hudson and Parthasarathy ([HuP 1], [HuP 2], [ApH], [HuL], [BSW], [Lin]). In the present paper we apply this calculus to construct new dilations, as was done in several preceding publications ([HuP 2], [Fri 1, 2], [Maa], [LiM]). We adopt the approach of Kümmerer and only admit semigroups and dilations which leave

* Supported by the Netherlands Organization for Scientific Research NWO

some faithful normal state invariant. This restriction brings the theory close in spirit to probability theory where all states are faithful. Physically it corresponds to the assumption of a positive temperature. As a consequence of this restriction only certain combinations of the known noises dA, dA^*, $d\Lambda$, dF and dF^* of [HuP1], [ApH] are allowed. Besides the finite temperature Brownian motion ([HuL], [Lin]) and its fermion counterpart ([ApH], [BSW]), we found an (apparently!) new non-commutative example, which may be called a *"quantum Poisson process"*. It is a generalisation of the classical Poisson process, as occurring in Fock space in the description in [HuP1]. It can be used as a noise source in order to dilate semigroups which were covered neither by the construction in [KüM], nor by constructions based on quantum Brownian motion ([FrG], [LiM], [ApH]). A concrete version of such a "new" dilation was found back in the work of Dümcke [Düm] as a description of an N-level system interacting with a quantum gas in the low density limit.

While the present work was being prepared, Kümmerer [Küm2] superseded our main theorem by a much stronger result, characterising the class of all dilatable dynamical semigroups in continuous time in terms of those in discrete time. In fact, he first introduced the term "non-commutative Poisson process" for a mathematical object quite closely related to ours. Nevertheless we feel that the method of the present paper is sufficiently different from Kümmerer's to justify a separate presentation; in particular the contact made with quantum stochastic calculus, and the actual occurrence of Poisson-distributed noise operators may appeal to some readers.

It occurred to us during the preparation that our quantum Poisson process is in fact so canonical, that it should be known already. And indeed we learned later that the same object has been investigated in the late sixties by Streater and Wulfsohn [StW], and Araki [Ara] in their study of infinitely divisible representations of the current algebra. Obviously, at that time no connection with Tomita-Takesaki theory or with quantum stochastic calculus could be laid. And finally, the idea of something like our quantum Poisson process being constructable by stochastic calculus, has recently occurred to almost all groups of investigators in this field; we wish to mention an article of Evans and Hudson [EvH].

This paper is organised as follows. In Sect. 2 we define the quantum Poisson process over a W^*-algebra $\mathcal{M}$ with a finite faithful normal weight μ, and then we generalise the construction to $L^\infty(\mathbb{R})\otimes\mathcal{M}$. The connection with quantum stochastic calculus is discussed in Sect. 3, where quantum stochastic differential equations involving Poisson noise are studied. This calculus is applied to construct dilations of a class of quantum dynamical semigroups on the matrix algebra M_n in Sect. 4. Some examples are given in Sect. 5, where also the question of detailed balance is briefly considered.

2. The Quantum Poisson Process

By a *quantum measure space* we mean a pair $(\mathcal{M}, \mu)$, where $\mathcal{M}$ is a W^* algebra and μ is a faithful normal semifinite weight on $\mathcal{M}$. A quantum measure space is called *finite* if $\mu(\mathbb{1}) < +\infty$, and a *quantum probability space* if $\mu(\mathbb{1}) = 1$ (i.e., if μ is a state on $\mathscr{M}$).

Via the well-known Gel'fand-Naimark-Segal (GNS) construction, a finite quantum measure space admits a canonical representation π as operators on a Hilbert space $\mathscr{H}$, with a cyclic and separating vector ξ inducing the weight μ as $\mu(a) = \langle \xi, \pi(a)\,\xi \rangle$, $a \in \mathscr{M}$; the triple $(\mathscr{H}, \pi, \xi)$ will be called the GNS triple associated with $(\mathscr{M}, \mu)$. There is a canonical commutant anti-representation π' of $\mathscr{M}$ on $\mathscr{H}$, with the same cyclic and separating vector ξ, inducing the same weight μ on $\mathscr{M}$. The Tomita-Takesaki theory [ToT] associates with a finite quantum measure space $(\mathscr{M}, \mu)$ a positive self-adjoint *modular operator* Δ_μ on $\mathscr{H}$, defined by the relation $\langle \Delta_\mu^{1/2} \pi(a)\,\xi, \Delta_\mu^{1/2} \pi(b)\,\xi \rangle = \langle \pi(b^*)\,\xi, \pi(a^*)\,\xi \rangle$, $a, b \in \mathscr{M}$, and the *modular automorphism group* $\{\sigma_t^\mu \colon t \in \mathbb{R}\}$ of $(\mathscr{M}, \mu)$, defined by $\pi \circ \sigma_t^\mu(a) = \Delta_\mu^{it} \pi(a)\, \Delta_\mu^{-it}$, $a \in \mathscr{M}$, $t \in \mathbb{R}$.

Most of the above statements admit generalisations to the situation where μ is not finite but only semifinite; however, a cyclic and separating vector ξ in $\mathscr{H}$ inducing the weight no longer exists in this situation.

Let $(\mathscr{M}, \mu)$ be a finite quantum measure space, with GNS triple $(\mathscr{H}, \pi, \mu)$. We construct a quantum probability space $(\mathscr{N}, \nu)$ associated with $(\mathscr{M}, \mu)$, and a linear map N of $\mathscr{M}$ into the unbounded operators affiliated with $\mathscr{N}$, to be called the *quantum Poisson process* over $(\mathscr{M}, \mu)$, as follows.

Let $\mathscr{F}(\mathscr{H})$ be the symmetric Fock space over $\mathscr{H}$, and let for $\eta \in \mathscr{H}$ the *exponential vector* $\psi(\eta) \in \mathscr{F}(\mathscr{H})$ be given by

$$\psi(\eta) = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, \eta^{\otimes n}.$$

For $\zeta \in \mathscr{H}$, define the unitary *Weyl operator* $W(\zeta)$ on $\mathscr{F}(\mathscr{H})$ by

$$W(\zeta)\,\psi(\eta) = \exp\{-\tfrac{1}{2}\|\zeta\|^2 - \langle \zeta, \eta \rangle\}\,\psi(\eta + \zeta), \qquad \eta \in \mathscr{H}.$$

For $x \in \mathscr{B}_1(\mathscr{H})$ (the unit ball of $\mathscr{B}(\mathscr{H})$), a bounded operator $\Gamma(x)$ is defined on $\mathscr{F}(\mathscr{H})$ by

$$\Gamma(x)\,\psi(\eta) = \psi(x\eta), \qquad \eta \in \mathscr{H}.$$

We note that $\|\Gamma(x)\| = 1$, $\Gamma(x)^* = \Gamma(x^*)$ and that, if u is unitary on $\mathscr{H}$, $\Gamma(u)$ is unitary on $\mathscr{F}(\mathscr{H})$. Moreover, $x_n, x \in \mathscr{B}_1(\mathscr{H})$ and $x = \text{w-}\lim_{n\to\infty} x_n$ imply that $\Gamma(x) = \text{w-}\lim_{n\to\infty} \Gamma(x_n)$.

The above notions are well-known. But now we wish to consider the von Neumann subalgebra $\mathscr{N}$ of $\mathscr{B}(\mathscr{F}(\mathscr{H}))$ generated by the bounded operators

$$V(a) = W(\xi)^{-1}\,\Gamma(\pi(a))\,W(\xi), \qquad a \in \mathscr{M}_1, \tag{2.1}$$

where $\mathscr{M}_1$ is the unit ball of $\mathscr{M}$, and the state ν on $\mathscr{N}$ defined as the restriction to $\mathscr{N}$ of the vector state determined by $\psi(0)$. We note first that $\mathscr{N}$ coincides with the weak closure of the linear span of $\{V(a)\colon a \in \mathscr{M}_1\}$, since $V(a)\,V(b) = V(ab)$. Next, we have

Theorem 2.1. *The Fock vacuum vector $\psi(0)$ is cyclic and separating for $\mathcal{N}$, so that $(\mathcal{N}, \nu)$ is a quantum probability space. The associated modular automorphism group $\{\sigma_t^\nu : t \in \mathbb{R}\}$ satisfies*

$$\sigma_t^\nu(V(a)) = V(\sigma_t^\mu(a)), \qquad a \in \mathcal{M}_1, \ t \in \mathbb{R}. \tag{2.2}$$

Proof. Using the canonical commutant anti-representation π' of $\mathcal{M}$, we may define

$$V'(b) = W(\xi)^{-1} \Gamma(\pi'(b)) W(\xi) : \ b \in \mathcal{M}_1,$$

and consider the von Neumann subalgebra $\tilde{\mathcal{N}}$ of $\mathcal{B}(\mathcal{F}(\mathcal{H}))$ generated by $\{V'(b): b \in \mathcal{M}_1\}$. Then $\tilde{\mathcal{N}} \subset \mathcal{N}'$, since

$$\begin{aligned} V(a)\, V'(b) &= W(\xi)^{-1} \Gamma(\pi(a)) \Gamma(\pi'(b)) W(\xi) \\ &= W(\xi)^{-1} \Gamma(\pi(a) \pi'(b)) W(\xi) \\ &= W(\xi)^{-1} \Gamma(\pi'(b) \pi(a)) W(\xi) \\ &= W(\xi)^{-1} \Gamma(\pi'(b)) \Gamma(\pi(a)) W(\xi) = V'(b)\, V(a). \end{aligned}$$

In order to prove that $\psi(0)$ is cyclic and separating for $\mathcal{N}$, it suffices therefore to prove that $\psi(0)$ is cyclic for both $\mathcal{N}$ and $\tilde{\mathcal{N}}$. Now, we have

$$\begin{aligned} V(a)\, \psi(0) &= \exp[-\tfrac{1}{2} \|\xi\|^2]\, W(-\xi)\, \psi(\pi(a)\, \xi) \\ &= \exp[-\|\xi\|^2 + \langle \xi, \pi(a)\, \xi \rangle]\, \psi([\pi(a) - \mathbb{1}]\, \xi). \end{aligned}$$

In particular, let a be unitary in $\mathcal{M}$, say $a = \exp[i\theta b]$ with $b = b^* \in \mathcal{M}$, $\theta \in \mathbb{R}$. Then we see that the closure of the linear span of the vectors $\{V(a)\, \psi(0): a \in \mathcal{M}_1\}$ contains the vectors of the form

$$\left(-i \frac{d}{d\theta}\right)^n \psi((\exp[i\theta \pi(b)] - \mathbb{1})\, \xi) \Big|_{\theta = 0}, \qquad b = b^* \in \mathcal{M},$$

whose n-particle component is $(n!)^{1/2} [\pi(b)\, \xi]^{\otimes n}$, and whose p-particle components for $p > n$ vanish identically. Since ξ is cyclic in $\mathcal{H}$ for $\pi(\mathcal{M})$, the set of such vectors is total in $\mathcal{F}(\mathcal{H})$; so $\psi(0)$ is cyclic in $\mathcal{F}(\mathcal{H})$ for $\mathcal{N}$. A similar argument applies to $\tilde{\mathcal{N}}$, since ξ is also cyclic in $\mathcal{H}$ for $\pi'(\mathcal{M})$. This proves the first part of the theorem.

To prove the second part, we define a group τ_t of automorphisms of $\mathcal{N}$ by

$$\tau_t(V(a)) = V(\sigma_t^\mu(a)),$$

and show that $\tau_t = \sigma_t^\nu$. Now, the modular group σ_t^α of a quantum probability space $(\mathcal{A}, \alpha)$ is characterized by the *KMS-condition*, [ToT] which requires the existence for every pair $a, b \in \mathcal{A}$ of a *KMS-function* $F_{a,b}$, i.e., a bounded and continuous function on the strip $\{z \in \mathbb{C}: 0 \leq \operatorname{Im} z \leq 1\}$, analytic on its interior and with boundary values

$$F_{a,b}(t) = \mu(\sigma_t^\alpha(a)\, b) \quad \text{and} \quad F_{a,b}(t+i) = \mu(b\, \sigma_t^\alpha(a)), \qquad (t \in \mathbb{R}).$$

Now, given $a, b \in \mathcal{M}_1$ a KMS-function for $V(a)$, $V(b)$ with respect to $(\mathcal{N}, \nu)$ and τ_t is given by

$$F_{V(a), V(b)}(z) = \exp(F_{a,b}(z) - \mu(\mathbb{1})).$$

This follows from the facts that $\nu(\tau_t(V(a))\, V(b)) = \nu(V(\sigma_t^\mu(a)\, b))$ and $\nu(V(b)\, \tau_t(V(a))) = \nu(V(b\, \sigma_t^\mu(a)))$, together with the computation

$$\begin{aligned} \nu(V(x)) &= \langle \psi(0), W(\xi)^{-1} \Gamma(x)\, W(\xi)\, \psi(0)\rangle \\ &= e^{-\|\xi\|^2} \langle \psi(\xi), \Gamma(x)\, \psi(\xi)\rangle = e^{-\|\xi\|^2} \langle \psi(\xi), \psi(x\xi)\rangle \\ &= e^{\langle \xi, x\xi\rangle - \|\xi\|^2} = e^{\mu(x) - \mu(\mathbb{1})}. \end{aligned} \tag{2.3}$$

Since the span of the operators $\{V(a): a \in \mathcal{M}_1\}$ is strongly dense in $\mathcal{N}$, the existence of an $(\mathcal{N}, \nu)$ KMS-function $F_{x,y}$ follows for all $x, y \in \mathcal{N}$ [BrR] and the theorem is proved. □

The remaining part of the construction of the quantum Poisson process is actually contained in some papers of the late sixties [StW], [Ara] dealing with infinitely divisible representations of the current algebra, well before the births of Tomita-Takesaki theory and of quantum stochastic calculus. We shall nevertheless include proofs of the theorems below, to make the present paper self-contained.

Definition 2.2. For a self-adjoint element a of $\mathcal{M}$, let $N(a)$ denote the (unbounded self-adjoint) generator of the strongly continuous unitary group $V(e^{i\alpha a})$, $(\alpha \in \mathbb{R})$. For an arbitrary element x of $\mathcal{M}$, let $N(x)$ be the closure of the restriction of $N\left(\frac{x+x^*}{2}\right) + iN\left(\frac{x-x^*}{2i}\right)$ to the linear span of the exponential vectors.

The second half of the above definition makes sense because the span of $\{\psi(\eta): \eta \in \mathcal{H}\}$ is a core for $N(a)$, $(a^* = a \in \mathcal{M})$.

We call $(\mathcal{N}, \nu; N)$ the *quantum Poisson process* over $(\mathcal{N}, \nu)$. This name is motivated by the following proposition.

Proposition 2.3. *Let $p \in \mathcal{M}$ be an orthogonal projection. Then the spectrum of $N(p)$ is $\mathbb{N} = \{0, 1, 2, \ldots\}$. If $N(p) = \sum_{n \in \mathbb{N}} nP_n$ is its spectral decomposition, then*

$$\nu(P_n) = e^{-\mu(p)} \frac{\mu(p)^n}{n!}. \tag{2.4}$$

Proof. If $N(p) = \int \lambda E(d\lambda)$, then the probability measure $\nu \circ E$ on $\mathbb{R}$ has the characteristic function

$$\alpha \mapsto \nu(e^{i\alpha N(p)}) = \nu \circ V(e^{i\alpha p}) = \exp(\mu(e^{i\alpha p} - \mathbb{1})) = \exp((e^{i\alpha} - 1)\, \mu(p)).$$

This is the characteristic function of the Poisson distribution (2.4) of expectation $\mu(p)$. Since ν is faithful, the spectrum of $N(p)$ is the support of $\nu \circ E$, namely $\mathbb{N}$. □

Remarks. 1. The conclusion that $\mathrm{sp}(N(p)) \subset \mathbb{N}$ can also be drawn from the explicit form

$$N(a) = W(\xi)^{-1}(0 \oplus \pi(a) \oplus (\pi(a) \oplus \mathbb{1} + \mathbb{1} \otimes \pi(a)) \oplus \ldots)\, W(\xi).$$

2. An orthogonal projection $p \in \mathcal{M}$ in the quantum probability space $(\mathcal{M}, \mu/\mu(\mathbb{1}))$ can be interpreted as an *event* which is true with probability $\mu(p)/\mu(\mathbb{1})$ and false with probability $\mu(\mathbb{1}-p)/\mu(\mathbb{1})$. The quantum probability space $(\mathcal{N}, \nu)$ describes a collectivity of an expected number of $\mu(\mathbb{1})$ draws out of $(\mathcal{M}, \mu/\mu(\mathbb{1}))$. The event $P_n \in \mathcal{N}$ mentioned in Proposition 2.3 is the event that p "takes place" n times.

If $\mathcal{M}$ is commutative, we may write $\mathcal{M} = L^\infty(\Omega, \tilde{\mu})$ for some ordinary finite measure space $(\Omega, \tilde{\mu})$, where $\tilde{\mu}$ induces the weight μ on $\mathcal{M}$. If $\tilde{\mu}$ is non-atomic, then $(\mathcal{N}, \nu; N)$ has the following explicit representation:

$$\mathcal{N} = L^\infty(\Psi, \tilde{\nu}),$$

where $\Psi = \{\omega \subset \Omega \colon \omega \text{ finite}\}$, and $\tilde{\nu}$ is given by its value on the subsets $S_{n,B} \subset \Psi$ (B a measurable subset of Ω), $S_{n,B} = \{\omega \in \Psi \colon |\omega \cap B| = n\}$ by $\tilde{\nu}(S_{n,B}) = \exp\{-\tilde{\mu}(B)\}\, \tilde{\mu}(B)^n/n!$ The Poisson process N is then given by $N(1_B) = |\omega \cap B|$. This is the ordinary Poisson process over $(\Omega, \tilde{\mu})$.

If $\mathcal{M}$ is noncommutative, we may say that the quantum Poisson process over $(\mathcal{M}, \mu)$ "lumps together in a noncommutative way" a collection of (infinitely many) ordinary Poisson processes.

If Λ denotes the map of $\mathcal{M}$ into the unbounded operators in $\mathcal{F}(\mathcal{H})$ defined by linear extension of

$$\Lambda(x) = \text{infinitesimal generator of } \Gamma(\exp[i\alpha x])$$

for $x = x^* \in \mathcal{B}(\mathcal{H})$, the above construction identifies $N(a)$ with $W(\xi)^{-1}\Lambda(\pi(a))\, W(\xi)$. We might also use a unitarily equivalent representation and identify $N(a)$ with $\Lambda(\pi(a))$, upon replacing the Fock vacuum $\psi(0)$ by the coherent vector $W(\xi)\,\psi(0) = \exp\{-\frac{1}{2}\|\xi\|^2\}\,\psi(\xi)$. The latter representation displays more clearly the nature of $N(a)$ as a number operator, but the former is more convenient for our further developments, and we shall keep to it.

Next we would like to construct a quantum Poisson process over the quantum measure space $(L^\infty(\mathbb{R}) \otimes \mathcal{M}, \lambda \otimes \mu)$, where λ is the weight on $L^\infty(\mathbb{R})$ induced by the Lebesgue measure on $\mathbb{R}$. (This indeed brings us to the question of *infinitely divisible* representations, as in [StW], [Ara].) However, since this quantum measure space is not finite, we cannot apply the above construction directly. We consider first the von Neumann subalgebras $L^\infty([S, T]) \otimes \mathcal{M}$ of $L^\infty(\mathbb{R}) \otimes \mathcal{M}$ for $S < T \in \mathbb{R}$, and the corresponding restrictions $\lambda_{[S,T]} \otimes \mu$ of the weight $\lambda \otimes \mu$. For all $S, T \in \mathbb{R}$ with $S < T$, $(L^\infty([S, T]) \otimes \mathcal{M}, \lambda_{[S,T]} \otimes \mu)$ is a finite quantum measure space, and we may consider the quantum Poisson process $(\mathcal{N}_{[S,T]}, \nu_{[S,T]}; N_{[S,T]})$ over it. Fortunately, all the operators $N_{[S,T]}(x)$: $x \in L^\infty([S, T]) \otimes \mathcal{M}$, $S < T \in \mathbb{R}$, may be regarded as acting on the same Hilbert space $\mathcal{F}$, as the following proposition shows.

Proposition 2.4. *Let $(\mathcal{M}, \mu)$ be a finite quantum measure space, and let $\mathcal{X}$ be the weakly dense *-subalgebra of $L^\infty(\mathbb{R}) \otimes \mathcal{M}$ defined by*

$$\mathcal{X} = \bigcup_{S<T\in\mathbb{R}} L^\infty([S,T]) \otimes \mathcal{M}. \tag{2.5}$$

*Then there exists a Hilbert space $\mathcal{F}$, a von Neumann algebra $\mathcal{N}$ of operators on $\mathcal{F}$, a faithful normal state ν on $\mathcal{N}$ and a linear map N from $\mathcal{X}$ into the unbounded operators on $\mathcal{F}$ affiliated with $\mathcal{N}$ such that, for all $S<T$ in $\mathbb{R}$, the quantum Poisson process $(\mathcal{N}_{[S,T]}, \nu_{[S,T]}; N_{[S,T]})$ over $(L^\infty([S,T]) \otimes \mathcal{M}, \lambda_{[S,T]} \otimes \mu)$ may be realized as the restriction of $(\mathcal{N}, \nu; N)$ to $L^\infty([S,T]) \otimes \mathcal{M} \subset \mathcal{X}$. By "restriction of $(\mathcal{N}, \nu; N)$" to a *-subalgebra $\mathcal{Y}$ of $\mathcal{X}$, we mean that N is restricted to act on $\mathcal{Y}$, $\mathcal{N}$ is replaced by its von Neumann subalgebra $\mathcal{N}_{\mathcal{Y}}$ generated by $N(y)$: $y \in \mathcal{Y}$, and ν is restricted to $\mathcal{N}_{\mathcal{Y}}$.*

Proof. (Sketch.) Let $\mathcal{F}$ be the Fock space over $L^2(\mathbb{R}) \otimes \mathcal{H}$. For all $S<T$ in $\mathbb{R}$, we may write

$$\begin{aligned}\mathcal{F} &= \mathcal{F}(L^2((-\infty, S]) \otimes \mathcal{H}) \otimes \mathcal{F}(L^2([S,T]) \otimes \mathcal{H}) \otimes \mathcal{F}(L^2([T,+\infty)) \otimes \mathcal{H}) \\ &\equiv \mathcal{F}_{S]} \otimes \mathcal{F}_{[S,T]} \otimes \mathcal{F}_{[T}.\end{aligned}$$

The quantum Poisson process $(\mathcal{N}_{[S,T]}, \nu_{[S,T]}; N_{[S,T]})$ is naturally defined on $\mathcal{F}_{[S,T]}$ via Definition 2.2. For x in $L^\infty([S,T]) \otimes \mathcal{M}$, let

$$N(x) = \mathbb{1}_{\mathcal{F}_{S]}} \otimes N_{[S,T]}(x) \otimes \mathbb{1}_{\mathcal{F}_{[T}}. \tag{2.6}$$

If $S \leqq s < t \leqq T$ and $x \in L^\infty([s,t]) \otimes \mathcal{M}$, then x may be also regarded as an element of $L^\infty([S,T]) \otimes \mathcal{M}$. Note that, for $x \in L^\infty([s,t]) \otimes \mathcal{M}$, (2.6) is actually independent of S and T so long as $S \leqq s$ and $T \geqq t$. Then, let $\mathcal{N} = \{N(x)\colon x \in \mathcal{X}\}''$, and let ν be the restriction to $\mathcal{N}$ of the vector state determined by $\psi(0)$. It is easy to check that the triple $(\mathcal{N}, \nu; N)$ has all the required properties. □

With some abuse of language, the triple $(\mathcal{N}, \nu; N)$ defined in Proposition 2.4 will be called *the quantum Poisson process over* $(L^\infty(\mathbb{R}) \otimes \mathcal{M}, \lambda \otimes \mu)$.

3. Stochastic Calculus

Here we develop a quantum stochastic calculus for the quantum Poisson process, add an initial space and study unitary solutions of quantum stochastic differential equations. Similar results have been obtained independently by Evans and Hudson [EvH].

The following propositions make contact with the quantum stochastic calculus of Hudson and Parthasarathy. We shall denote by A^*, Λ and A the creation, preservation (gauge) and annihilation operators, given in [HuP 1]:

$$A^*(\zeta)\,\psi(\eta) = \frac{d}{d\varepsilon}\,\psi(\eta + \varepsilon\zeta)\Big|_{\varepsilon=0}: \ \zeta, \eta \in \mathcal{H};$$

$$\Lambda(x)\,\psi(\eta) = \frac{d}{d\varepsilon}\,\psi(e^{\varepsilon x}\eta)\Big|_{\varepsilon=0}: \ x \in \mathcal{B}(\mathcal{H}), \eta \in \mathcal{H};$$

$$A(\zeta)\,\psi(\eta) = \langle \zeta, \eta \rangle\,\psi(\eta): \ \zeta, \eta \in \mathcal{H}.$$

Proposition 3.1. *For all exponential vectors $\psi(\eta)$: $\eta\in\mathscr{H}$, the quantum Poisson process $N(a)$: $a\in\mathscr{M}$ over $(\mathscr{M}, \mu)$ satisfies*

$$N(a)\,\psi(\eta)=[A^*(\pi(a)\,\xi)+A(\pi(a))+A(\pi(a^*)\,\xi)+\mu(a)]\,\psi(\eta). \tag{3.1}$$

Proof. Since the maps $a\mapsto N(a)$, $\zeta\mapsto A^*(\zeta)$, $x\mapsto A(x)$ are linear and $\zeta\mapsto A(\zeta)$ is conjugate-linear, it suffices to prove (3.1) for $a=a^*\in\mathscr{M}$. We calculate

$$\begin{aligned}
N(a)\,\psi(\eta)&=-i\frac{d}{d\alpha}\,W(\xi)^{-1}\,\Gamma(\exp[i\alpha\pi(a)])\,W(\xi)\,\psi(\eta)\Big|_{\alpha=0}\\
&=-i\frac{d}{d\alpha}\,[\exp\{-\tfrac{1}{2}\|\xi\|^2-\langle\xi,\eta\rangle\}\,W(-\xi)\,\psi(\exp[i\alpha\pi(a)]\,(\xi+\eta))]\Big|_{\alpha=0}\\
&=-i\frac{d}{d\alpha}\exp\{-\|\xi\|^2-\langle\xi,\eta\rangle-\langle-\xi,e^{i\alpha\pi(a)}(\xi+\eta)\rangle\}\,\psi(e^{i\alpha\pi(a)}(\xi+\eta)-\xi)\Big|_{\alpha=0}\\
&=-i\frac{d}{d\alpha}\exp\langle\xi,(e^{i\alpha\pi(a)}-\mathbb{1})\,(\xi+\eta)\rangle\,\psi(e^{i\alpha\pi(a)}\eta+(e^{i\alpha\pi(a)}-\mathbb{1})\,\xi)\Big|_{\alpha=0}\\
&=\langle\xi,\pi(a)\,(\xi+\eta)\rangle\,\psi(\eta)\\
&\quad+\lim_{\alpha\to 0}\frac{1}{i\alpha}\,[\psi(e^{i\alpha\pi(a)}\eta+(e^{i\alpha\pi(a)}-\mathbb{1})\,\xi)-\psi(e^{i\alpha\pi(a)}\eta)+\psi(e^{i\alpha\pi(a)}\eta)-\psi(\eta)]\\
&=\langle\xi,\pi(a)\,\xi\rangle\,\psi(\eta)+\langle\pi(a^*)\,\xi,\eta\rangle\,\psi(\eta)\\
&\quad-i\frac{d}{d\alpha}\,\psi(\eta+i\alpha\pi(a)\,\xi)\Big|_{\alpha=0}-i\frac{d}{d\alpha}\,\psi(e^{i\alpha\pi(a)}\eta)\Big|_{\alpha=0}\\
&=[\mu(a)+A(\pi(a^*)\,\xi)+A^*(\pi(a)\,\xi)+A(\pi(a))]\,\psi(\eta). \qquad\square
\end{aligned}$$

Now we consider the quantum Poisson process over $(L^\infty(\mathbb{R})\otimes\mathscr{M}, \lambda\otimes\mu)$. Accordingly, the Fock space over $\mathscr{H}$ will be replaced by the Fock space over $L^2(\mathbb{R})\otimes\mathscr{H}$. Following Hudson and Parthasarathy [HuP2], we shall consider creation, preservation and annihilation operators given by

$$\begin{aligned}
A_t^*(\zeta)&=A^*(1_{[0,t]}\otimes\zeta):\ t\in\mathbb{R}^+,\ \zeta\in\mathscr{H},\\
A_t(x)&=A(1_{[0,t]}\otimes x):\ t\in\mathbb{R}^+,\ x\in\mathscr{B}(\mathscr{H}),\\
A_t(\zeta)&=A(1_{[0,t]}\otimes\zeta):\ t\in\mathbb{R}^+,\ \zeta\in\mathscr{H},
\end{aligned}$$

where $1_{[0,t]}\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ denotes the indicator function of the interval $[0, t]$. With reference to Proposition 2.4 we introduce the notation

$$N_t(a)=N(1_{[0,t]}\otimes a),\qquad t\in\mathbb{R}^+,\ a\in\mathscr{M}. \tag{3.2}$$

As an immediate consequence of Proposition 3.1 we have, for all a in $\mathscr{M}$,

$$N_t(a)=A_t^*(\pi(a)\,\xi)+A_t(\pi(a))+A_t(\pi(a^*)\,\xi)+\mu(a)\,t, \tag{3.3}$$

in the sense that both sides have the same action on exponential vectors. We shall write symbolically

$$dN_t(a)=N_{t+dt}(a)-N_t(a), \qquad t\in\mathbb{R}^+,\ a\in\mathcal{M}.$$

The precise meaning of the stochastic differential $dN_t(a)$ is as in the quantum stochastic calculus of Hudson and Parthasarathy (see also Accardi and Parthasarathy [AcP]).

Proposition 3.2. *In the sense of the stochastic calculus of Hudson and Parthasarathy, we have the Itô formula*

$$dN_t(a)\,dN_t(b)=dN_t(ab), \qquad a, b\in\mathcal{M}. \tag{3.4}$$

Proof. Since for all $T>t$ we have

$$N_t(a)=W(1_{[0,T]}\otimes\xi)^{-1}\Lambda_t(\pi(a))\,W(1_{[0,T]}\otimes\xi),$$

where $\Lambda_t(x)=\Lambda(1_{[0,t]}\otimes x)$: $x\in\mathcal{B}(\mathcal{H})$ is as in Hudson and Parthasarathy, this follows directly from their relation [HuP 1]

$$d\Lambda_t(x)\,d\Lambda_t(y)=d\Lambda_t(xy), \qquad x, y\in\mathcal{B}(\mathcal{H}). \quad \square$$

In order to construct dilations of dynamical semigroups, we shall consider stochastic differential equations for unitary operators in $M_n\otimes\mathcal{N}$, where M_n is the algebra of all complex $n\times n$ matrices. An element X of $M_n\otimes\mathcal{N}$ may be regarded as an $n\times n$ matrix $(X_{ij})_{i,j=1,\dots,n}$ with entries X_{ij} in $\mathcal{N}$. Similarly, an element x of $M_n\otimes\mathcal{M}$ may be regarded as an $n\times n$ matrix $(x_{ij})_{i,j=1,\dots,n}$ with entries x_{ij} in $\mathcal{M}$. Let $(\mathcal{N}, v; N)$ be the quantum Poisson process over $(L^\infty(\mathbb{R})\otimes\mathcal{M}, \lambda\otimes\mu)$ introduced in Proposition 2.4, so that for all $t\geqq 0$ and $a\in\mathcal{M}$ the operator $N_t(a)$ is defined in $\mathcal{F}=\mathcal{F}(L^2(\mathbb{R})\otimes\mathcal{H})$. Then, for $x=(x_{ij})$ in $M_n\otimes\mathcal{M}$, let

$$N_t(x)=(N_t(x_{ij}))_{i,j=1,\dots,n}.$$

The matrix $N_t(x)$ is an unbounded operator on $\mathbb{C}^n\otimes\mathcal{F}$, affiliated with $M_n\otimes\mathcal{N}$. It should be clear that, for all a in M_n $(a_{ij}\in\mathbb{C})$ we have by linearity

$$\begin{aligned}N_t(x)\,(a\otimes\mathbb{1}_{\mathcal{N}})&=\left(\sum_{j=1}^n N_t(x_{ij})\,a_{jk}\right)_{i,k=1,\dots,n}\\&=\left(N\left(\sum_{j=1}^n (x_{ij}\,a_{jk})\right)\right)_{j,k=1,\dots,n}=N_t(x(a\otimes\mathbb{1}_{\mathcal{M}}))\end{aligned} \tag{3.5}$$

and similarly

$$(a\otimes\mathbb{1}_{\mathcal{N}})\,N_t(x)=N_t((a\otimes\mathbb{1}_{\mathcal{M}})\,x). \tag{3.6}$$

For any fixed x in $M_n\otimes\mathcal{M}$, the family $\{N_t(x)\colon t\geqq 0\}$ is an example of an *adapted process* in the sense of Hudson and Parthasarathy. Roughly, a family $\{X_t\colon t\geqq 0\}$ of operators in $\mathbb{C}^n\otimes\mathcal{F}(L^2(\mathbb{R})\otimes\mathcal{H})$ is called adapted if X_t is of the form $\mathbb{1}\otimes\hat{X}_t\otimes\mathbb{1}$ when the Hilbert space in which X_t acts is identified with $\mathscr{F}(L^2((-\infty,0])\otimes\mathscr{H})\otimes[\mathbb{C}^n\otimes\mathscr{F}(L^2([0,t])\otimes\mathscr{H}]\otimes\mathscr{F}(L^2([t,+\infty))\otimes\mathscr{H})$.

The following theorem is a result of the same kind as Sect. 7 of [HuP1]. It has been obtained independently also by Evans and Hudson [EvH].

Theorem 3.4. *For x in $M_n\otimes\mathscr{M}$ and $h=h^*$ in M_n, consider the quantum stochastic differential equation*

$$dU_t=[dN_t(x)-ih\otimes\mathbb{1}\,dt]\,U_t, \tag{3.7}$$

with initial condition $U_0=\mathbb{1}$. Then there exists a unique adapted solution U_t $(t\geqq 0)$ affiliated with $M_n\otimes\mathscr{N}$. The operators U_t are unitary for all t in $\mathbb{R}^+$ if and only if x is of the form

$$x=u-\mathbb{1}:\ u \text{ unitary in } M_n\otimes\mathscr{M}. \tag{3.8}$$

Proof. The existence and uniqueness of the adapted solution is proved as in Proposition 7.1 of [HuP1], see also Evans and Hudson [EvH]. It is clear from the construction that the solution is affiliated with $M_n\otimes\mathscr{N}$. To discuss unitarity, we imitate Theorem 7.1 of [HuP1] and consider the operators

$$X_t=U_t^*\,U_t,\qquad Y_t=U_t\,U_t^*:\ t\in\mathbb{R}^+,$$

satisfying the quantum stochastic differential equations

$$\begin{aligned}dX_t&=(dU_t^*)\,U_t+U_t^*(dU_t)+(dU_t^*)(dU_t)\\&=U_t^*[dN_t(x^*)+dN_t(x)+dN_t(x^*)\,dN_t(x)]\,U_t\\&=U_t^*\,dN_t(x^*+x+x^*x)\,U_t\end{aligned} \tag{3.9}$$

and (with similar manipulations)

$$dY_t=[dN_t(x)-ih\otimes\mathbb{1}\,dt]\,Y_t+Y_t[dN_t(x^*)+ih\otimes\mathbb{1}\,dt]+dN_t(x)\,Y_t\,dN_t(x^*). \tag{3.10}$$

If $x=u-\mathbb{1}$, u unitary, then $dX_t=0$ for all t and X_t is identically equal to its initial value $\mathbb{1}$; on the other hand, the constant $\mathbb{1}$ is the (unique) solution of (3.10). Hence $U_t^*\,U_t=\mathbb{1}=U_t\,U_t^*$ for all t. Conversely, if U_t is unitary for all t, we must have $dX_t=0=dY_t$ for all t. It follows that

$$x^*+x+x^*x=0=x+x^*+xx^*,$$

which means that x is of the form (3.8). □

In the next section, we shall also need the following.

Lemma 3.5. *Let σ and $\tilde{\sigma}$ be automorphisms of $M_n\otimes\mathscr{M}$ and of $M_n\otimes\mathscr{N}$ respectively, satisfying*

$$N_t(\sigma(x))=\tilde{\sigma}(N_t(x)),\qquad x\in M_n\otimes\mathscr{M},\ t\in\mathbb{R}^+ \tag{3.11}$$

(where $\tilde{\sigma}$ is extended to the operators affiliated with $M_n\otimes\mathscr{N}$ in the obvious way), and let U_t be the solution of the quantum stochastic differential equation (3.7). Then $\tilde{\sigma}(U_t)=U_t$, for all t if and only if $\sigma(x)=x$ and $\tilde{\sigma}(h\otimes\mathbb{1})=h\otimes\mathbb{1}$.

Proof. Suppose first that $\sigma(x)=x$ and $\tilde{\sigma}(h\otimes\mathbb{1})=h\otimes\mathbb{1}$, and let $Z_t=\tilde{\sigma}(U_t)$. Then

$$\begin{aligned} dZ_t=\tilde{\sigma}(dU_t)&=\tilde{\sigma}([dN_t(x)-ih\otimes\mathbb{1}\,dt]\,U_t)\\ &=\tilde{\sigma}(dN_t(x)-ih\otimes\mathbb{1}\,dt)\,\tilde{\sigma}(U_t)\\ &=[dN_t(\sigma(x))-i\tilde{\sigma}(h\otimes\mathbb{1})\,dt]\,Z_t\\ &=[dN_t(x)-i(h\otimes\mathbb{1})\,dt]\,Z_t. \end{aligned}$$

So Z_t satisfies the same quantum stochastic differential equation as U_t. It follows from (3.11) that $\tilde{\sigma}(U_t)$ is adapted if U_t is; therefore $\tilde{\sigma}(U_t)=Z_t=U_t$.

Conversely, if U_t is known to be invariant under $\tilde{\sigma}$ for all t, we must have

$$dN_t(\sigma(x))-i\tilde{\sigma}(h\otimes\mathbb{1})\,dt=dN_t(x)-ih\otimes\mathbb{1}\,dt,$$

that is

$$dN_t(\sigma(x)-x)=-i[h\otimes\mathbb{1}-\tilde{\sigma}(h\otimes\mathbb{1})]\,dt.$$

But dN_t and dt are linearly independent, hence $\sigma(x)=x$ and $h\otimes\mathbb{1}=\tilde{\sigma}(h\otimes\mathbb{1})$. □

4. Dilations of Dynamical Semigroups on M_n

We apply the quantum Poisson process N_t to construct Markov dilations of dynamical semigroups on M_n in the sense of Kümmerer [Küm 1]. Such dilations are non-commutative generalisations of stationary Markov processes. For convenience we recall the relevant definitions. A *morphism* T: $(\mathscr{A}_1,\phi_1)\to(\mathscr{A}_2,\phi_2)$ between the quantum probability spaces $(\mathscr{A}_1,\phi_1)$ and $(\mathscr{A}_2,\phi_2)$ is a completely positive map T: $\mathscr{A}_1\to\mathscr{A}_2$ satisfying $T(\mathbb{1}_{\mathscr{A}_1})=\mathbb{1}_{\mathscr{A}_2}$ and $\phi_2\circ T=\phi_1$. By a *dynamical system* one means a triple $(\mathscr{A},\phi,T_t)$, where $(\mathscr{A},\phi)$ is a quantum probability space and $(T_t)_{\geqq 0}$ is a semigroup of morphisms $(\mathscr{A},\phi)\to(\mathscr{A},\phi)$ with $T_0=id_{\mathscr{A}}$. Such a dynamical system is said to be *reversible* if T_t is an automorphism for all $t\geqq 0$. In a reversible dynamical system the semigroup can be extended to a group $(T_t)_{t\in\mathbb{R}}$ by defining $T_t:=(T_{-t})^{-1}$ for $t<0$.

A dynamical system $(\mathscr{A},\phi,T_t)$ is said to possess a *dilation* $(\hat{\mathscr{A}},\hat{\phi},\hat{T}_t;j)$ if $(\hat{\mathscr{A}},\hat{\phi},\hat{T}_t)$ is a reversible dynamical system and j is a morphism $(\mathscr{A},\phi)\to(\hat{\mathscr{A}},\hat{\phi})$ such that for some morphism E: $(\hat{\mathscr{A}},\hat{\phi})\to(\mathscr{A},\phi)$ the following diagram commutes for all $t\geqq 0$:

$$\begin{array}{ccc} (\mathscr{A},\phi) & \xrightarrow{\;T_t\;} & (\mathscr{A},\phi)\\ {\scriptstyle j}\downarrow & & \uparrow{\scriptstyle E}\\ (\hat{\mathscr{A}},\hat{\phi}) & \xrightarrow{\;\hat{T}_t\;} & (\hat{\mathscr{A}},\hat{\phi}) \end{array} \tag{4.1}$$

If T_1 is any morphism of $(\mathscr{A},\phi)$, then $(\mathscr{A},\phi,T_1)$ is said to possess a *dilation of first order* $(\hat{\mathscr{A}},\hat{\phi},\hat{T}_1;j)$ if the diagram (4.1) commutes for $t=0,1$ (where we define $T_0=id_{\mathscr{A}}$ and $\hat{T}_0=id_{\hat{\mathscr{A}}}$). Such a dilation of first order is said to be *inner* if $\hat{T}_1$ is an inner automorphism of $(\hat{\mathscr{A}},\hat{\phi})$, i.e. $\hat{T}_1(a)=u^*au$ for all $a\in\hat{\mathscr{A}}$ and for some unitary u in the centraliser of $(\hat{\mathscr{A}},\hat{\phi})$. (The centraliser of a quantum probability space is the fixed point set of its modular group; u is fixed under $\sigma_t^{\hat{\phi}}$ if and only if $\hat{\phi}(u^* \cdot u) = \hat{\phi}(\cdot)$.) From the commuting diagram (4.1) with $t=0$ it follows that j is an injective *-homomorphism, and $j \circ E$ is a conditional expectation onto the W^*-subalgebra $j(\mathscr{A})$ of $\hat{\mathscr{A}}$ with respect to the state $\hat{\phi}$. From the existence of this conditional expectation it follows [Tak] that $\sigma_t^{\hat{\phi}}(j(\mathscr{A})) \subset j(\mathscr{A})$. For an interval $I \subset \mathbb{R}$, let $\mathscr{A}_I$ denote the W^*-subalgebra of $\hat{\mathscr{A}}$ generated by $\bigcup_{t \in I} \hat{T}_t \circ j(\mathscr{A})$. One can show that conditional expectations E_I onto these subalgebras exist in $(\hat{\mathscr{A}}, \hat{\phi})$. The dilation $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t)$ is said to be a *Markov dilation* if

$$E_{(-\infty, 0]} \mathscr{A}_{[0, \infty)} = \mathscr{A}_{\{0\}}. \tag{4.2}$$

We are now ready to formulate our main result.

Theorem 4.1. *Let ϕ be a faithful state on the algebra M_n of all complex $n \times n$ matrices. Let M: $(M_n, \phi) \to (M_n, \phi)$ be a morphism possessing an inner dilation of first order, and let L: $M_n \to M_n$ be given by*

$$L(a) = M(a) - a, \qquad (a \in M_n). \tag{4.3}$$

Then the dynamical system (M_n, ϕ, e^{tL}) possesses a Markov dilation.

Proof. Since M_n is a factor of type I, every dilation of first order of (M_n, ϕ, M) must be of the "tensor" form $(M_n \otimes \mathscr{M}, \phi \otimes \mu, \hat{M}; id \otimes \mathbb{1})$ for some quantum probability space $(\mathscr{M}, \mu)$ [Küm1]. Since $\hat{M}$ is an inner automorphism of $(M_n \otimes \mathscr{M}, \phi \otimes \mu)$ it is of the form $\hat{M} = \mathrm{Ad}\, u := u^* \cdot u$ with $\sigma_t^{\phi \otimes \mu}(u) = u$. So we may write for $a \in M_n$:

$$L(a) = (id \otimes \mu)(u^*(a \otimes \mathbb{1})\, u) - a. \tag{4.4}$$

Now let $(\mathscr{N}, \nu; N)$ be the quantum Poisson process over $(L^\infty(\mathbb{R}) \otimes \mathscr{M}, \lambda \otimes \mu)$ introduced in Sect. 3. Let $(U_t)_{t \geqq 0}$ be the solution of the quantum stochastic differential equation

$$dU_t = dN_t(u - \mathbb{1})\, U_t \tag{4.5}$$

with initial condition $U_0 = \mathbb{1}$. Since u is fixed under $\sigma_t^{\phi \otimes \mu}$, we may conclude from Lemma 3.5 and Theorem 2.1 that U_t is fixed under $\sigma_t^{\phi \otimes \nu}$, hence $\mathrm{Ad}\, U_t$ is an inner automorphism of $(M_n \otimes \mathscr{N}, \phi \otimes \nu)$. We now continue the construction as usual ([HuP2], [Fri], [Maa]): we introduce the right shift $(S_t)_{t \in \mathbb{R}}$ on $\mathscr{N}$ by

$$S_t(N(1_{[s,u]} \otimes a)) = N(1_{[s+t, u+t]} \otimes a)$$

and show in the usual way that

$$U_{t+s} = (id \otimes S_s)(U_t)\, U_s, \qquad (s, t \geqq 0),$$

so that, upon defining $\hat{T}_t$ on $M_n \otimes \mathscr{N}$ by

$$\hat{T}_t := \mathrm{Ad}\, U_t \circ (id \otimes S_t), \qquad (t \geqq 0),$$

$(M_n \otimes \mathcal{N}, \phi \otimes \nu, \hat{T}_t)$ becomes a reversible dynamical system. Now we define a family of morphisms $T_t: (M_n, \phi) \to (M_n, \phi)$ by the diagram

$$\begin{array}{ccc} (M_n, \phi) & \xrightarrow{T_t} & (M_n, \phi) \\ {\scriptstyle id \otimes \mathbb{1}} \downarrow & & \uparrow {\scriptstyle id \otimes \nu} \\ (M_n \otimes \mathcal{N}, \phi \otimes \nu) & \xrightarrow{\hat{T}_t} & (M_n \otimes \mathcal{N}, \phi \otimes \nu) \end{array} \tag{4.6}$$

It remains to prove that $T_t = e^{tL}$. We first need the following lemma.

Lemma 4.2. *For all* $x \in M_n \otimes \mathcal{M}$ *and* $t \in \mathbb{R}$ *we have*

$$(id \otimes \nu)(dN_t(x)) = (id \otimes \mu)(x)\, dt \tag{4.7}$$

and

$$(id \otimes \nu)(U_t^*\, dN_t(x)\, U_t) = (id \otimes \nu)(U_t^*((id \otimes \mu)(x) \otimes \mathbb{1})\, U_t)\, dt. \tag{4.8}$$

Proof. Equation (4.7) is an immediate consequence of (3.3). Then also (4.8) follows upon taking into account the continuous tensor product structure of the symmetric Fock space over $L^2(\mathbb{R}) \otimes \mathcal{H}$ and the adaptedness of U_t. □

We may now calculate T_t as follows. For all $a \in M_n$ we have

$$\begin{aligned} dT_t(a) &= (id \otimes \nu)\,[d(U_t^*(a \otimes \mathbb{1})\, U_t)] \\ &= (id \otimes \nu)\,[(dU_t^*)(a \otimes \mathbb{1})\, U_t + U_t^*(a \otimes \mathbb{1})(dU_t) + (dU_t^*)(a \otimes \mathbb{1})(dU_t)] \\ &= (id \otimes \nu)\,[U_t^*\{dN_t(u^* - \mathbb{1})(a \otimes \mathbb{1}) + (a \otimes \mathbb{1})\, dN_t(u - \mathbb{1}) \\ &\qquad + dN_t(u^* - \mathbb{1})(a \otimes \mathbb{1})\, dN_t(u - \mathbb{1})\}\, U_t]. \end{aligned}$$

Taking into account equations (3.5) and (3.6), the Itô formula (3.4) and the linearity of N_t, we obtain

$$\begin{aligned} dT_t(a) &= (id \otimes \nu)\,[U_t^*\, dN_t((u^* - \mathbb{1})(a \otimes \mathbb{1}) + (a \otimes \mathbb{1})(u - \mathbb{1}) \\ &\qquad + (u^* - \mathbb{1})(a \otimes \mathbb{1})(u - \mathbb{1}))\, U_t] \\ &= (id \otimes \nu)\,[U_t^*\, dN_t(u^*(a \otimes \mathbb{1})\, u - a \otimes \mathbb{1})\, U_t]. \end{aligned}$$

Applying the lemma we conclude that

$$\begin{aligned} dT_t(a) &= (id \otimes \nu)\,[U_t^*(id \otimes \mu)\,[u^*(a \otimes \mathbb{1}_{\mathcal{M}})\, u - a \otimes \mathbb{1}_{\mathcal{M}}] \otimes \mathbb{1}_{\mathcal{N}})\, U_t]\, dt \\ &= (id \otimes \nu) \circ \mathrm{Ad}\, U_t\,[L(a) \otimes \mathbb{1}_{\mathcal{N}}]\, dt = T_t(L(a))\, dt. \end{aligned}$$

So indeed $T_t = e^{tL}$, and $(M_n \otimes \mathcal{N}, \phi \otimes \nu, \hat{T}_t; id \otimes \mathbb{1})$ is a dilation of (M_n, ϕ, e^{tL}). The Markov property (4.2) follows from the adaptedness of U_t. □

Obviously, we may scale the time in the dynamical semigroup by changing the total mass $\mu(\mathbb{1})$ of the auxiliary quantum measure space $(\mathcal{M}, \mu)$ (which is 1 in the above construction). Also, the addition of a derivation to L does not affect the dilatability of e^{tL}. Hence we have the following.

Corollary 4.3. *If* $L\colon M_n \to M_n$ *is of the form*

$$L\colon\ a \to r(M(a)-a)+i[h,a], \tag{4.9}$$

with $h=h^*$ *in the centraliser of* (M_n, ϕ), $r>0$ *and* M *a morphism of* (M_n, ϕ) *possessing an inner dilation of first order, then* (M_n, ϕ, e^{tL}) *possesses a Markov dilation* ("of Poisson type").

Note that L may be written equivalently as

$$\begin{aligned} L(a) &= (id \otimes \mu)(u^*(a \otimes \mathbb{1})u - a \otimes \mathbb{1}) + i[h,a] \\ &= \tfrac{1}{2}(id \otimes \mu)([u^*, a \otimes \mathbb{1}]u + u^*[a \otimes \mathbb{1}, u]) + i[h,a], \end{aligned} \tag{4.10}$$

where u is a unitary element of the centraliser of $(M_n \otimes \mathcal{M}, \phi \otimes \mu)$.

Remarks. 1. It was shown by Kümmerer and Maassen [KüM] that $(M_n, \phi, \exp[tL])$ admits a Markov dilation using a *classical* Poisson process if and only if L is of the form (4.9) where M is a convex combination of $\mathrm{Ad}\,u_j$ with u_j unitary in M_n. Corollary 4.3 may be considered as a generalization of this result. A much more powerful generalization has been obtained by Kümmerer [Küm2], who can dispense with our conditions that $\mathcal{A}=M_n$ and that the tensor dilation of the morphism M be inner, and who has characterized, in addition, the class of quantum dynamical semigroups admitting a Markov dilation with a faithful normal invariant state $\hat{\phi}$.

2. The generators of the form (4.10) are essentially those found by Dümcke in his study of the low-density limit for an N-level system coupled to a heat bath which is a quantum gas. Then $\mathcal{M}$ is the algebra of observables for one bath particle, u is the collision operator (S-matrix) for the system interacting with one bath particle, and μ is the one-particle reduced state of the bath (strictly speaking, μ is a weight, and not even a finite weight after the thermodynamic limit for the bath has been taken). When the gas is dilute, collisions are infrequent, each collision takes a microscopic time to develop its asymptotic effect (the bath observable a is changed to $M(a)$, and reservoir particles hitting the system at different (macroscopic) times may be regarded as uncorrelated, thus allowing the replacement of $\mathcal{M}$ by $L^\infty(\mathbb{R}) \otimes \mathcal{M}$ as algebra of one-particle observables. Infrequent events tend to acquire a Poisson distribution in time, with some rate r. Then, neglecting the free time evolution determined by h, one should have some time evolution of the kind $a(t+dt)=rM(a(t))\,dt+(1-r)\,a(t)\,dt$, giving $\frac{d}{dt}a(t)=r[M(a(t))-a(t)]$. A more detailed comparison of the low-density limit with the quantum Poisson process can be found in [Fri3]. A similar physical comment could be applied to Kümmerer's construction of the "non-commutative Poisson process" in [Küm2].

5. Examples

In order to illustrate the results of Sect. 4, we have to construct examples of morphisms $M\colon (M_n, \phi) \to (M_n, \phi)$ admitting an inner tensor dilation of first order

$(M_n \otimes \mathcal{M}, \phi \otimes \mu, \mathrm{Ad}\, u; id \otimes \mathbb{1})$, u unitary in $M_n \otimes \mathcal{M}$ and satisfying $\sigma_t^{\phi \otimes \mu}(u) = u$ for all real t.

For the sake of concreteness we shall consider the case in which also $\mathcal{M}$ is the algebra M_p of all complex $p \times p$ matrices (p and n may be different). Without loss of generality, we may then assume that

$$\phi(a) = \mathrm{tr}[\rho a], \qquad a \in M_n,$$
$$\mu(b) = \mathrm{tr}[\xi^2 b], \qquad b \in M_p;$$

where ρ and ξ^2 are strictly positive *diagonal* matrices (with the appropriate dimensionalities) with unit trace:

$$\rho = \mathrm{diag}(\rho_1, \ldots, \rho_n), \qquad \rho_1, \ldots, \rho_n > 0, \qquad \sum_{j=1}^{n} \rho_j = 1,$$

$$\xi^2 = \mathrm{diag}(\xi_1^2, \ldots, \xi_p^2), \qquad \xi_1, \ldots, \xi_p > 0, \qquad \sum_{r=1}^{p} \xi_r^2 = 1.$$

The reason for the notation ξ^2 is that the GNS space for (M_p, μ) may be identified with M_p, with inner product $\langle \eta, \zeta \rangle = \mathrm{tr}[\eta^* \zeta]$ and with cyclic and separating vector (ξ_j).

Then the modular automorphism groups σ_t^ϕ and σ_t^μ are given by

$$\sigma_t^\phi(a) = \rho^{it} a \rho^{-it}, \qquad a \in M_n, \qquad t \in \mathbb{R},$$
$$\sigma_t^\mu(b) = \xi^{2it} b \xi^{-2it}, \qquad b \in M_p, \qquad t \in \mathbb{R}.$$

Now we have to characterize the unitary matrices u in $M_n \otimes M_p$ which are invariant under $\sigma_t^{\phi \otimes \mu} = \sigma_t^\phi \otimes \sigma_t^\mu$. Note first that, upon letting $(e_{rs})_{r,s=1,\ldots,p}$ be the canonical matrix units in M_p, any matrix u in $M_n \otimes M_p$ may be expanded as

$$u = \sum_{r,s=1}^{p} v_{rs} \otimes e_{rs} \tag{5.1}$$

where v_{rs} is a matrix in M_n for each r, s.

Lemma 5.1. *A matrix u in $M_n \otimes M_p$ with the expansion* (5.1) *is unitary if and only if*

$$\sum_{q=1}^{p} v_{qr}^* v_{qs} = \delta_{rs} \mathbb{1}, \qquad r, s = 1, \ldots, p, \tag{5.2}$$

and is invariant under $\sigma_t^{\phi \otimes \mu}$ if and only if

$$\rho v_{rs} \rho^{-1} = (\xi_s^2 \xi_r^{-2}) v_{rs}, \qquad r, s = 1, \ldots, p. \tag{5.3}$$

Proof. Straightforward calculation shows that (5.2) is equivalent to $u^* u = \mathbb{1}$ and that (5.3) is equivalent to $(\sigma_t^\phi \otimes \sigma_t^\mu)(u) = u$ for all real t. □

Proposition 5.2. *A map M of M_n into itself is a morphism $(M_n, \phi) \to (M_n, \phi)$ admitting an inner tensor dilation with the auxiliary algebra isomorphic to a matrix algebra if and only if, for some integer p, some p-tuple $(\xi_1, \ldots, \xi_p)$ of positive numbers with $\sum_{r=1}^{p} \xi_r^2 = 1$ and for some matrices $(v_{rs})_{r,s=1,\ldots,p}$ in M_n satisfying conditions* (5.2) *and* (5.3) *one has*

$$M(a) = \sum_{r,s=1}^{p} \xi_s^2 \, v_{rs}^* \, a v_{rs}: \; a \in M_n. \tag{5.4}$$

Proof. Immediate. □

We are now ready to produce some classes of examples.

Example 1. (Essentially commutative dilations:) Suppose that in (5.1) one has, for all $r, s = 1, \ldots, p$.

$$v_{rs} = \delta_{rs} u_r: \; u_r \text{ unitary commuting with } \rho.$$

Then conditions (5.2) and (5.3) are satisfied, and M takes the form of a convex combination of automorphisms:

$$M(a) = \sum_{r=1}^{p} \xi_r^2 \, u_r^* \, a u_r: \; a \in M_n.$$

Note that the dilation is constructed with the use of the commuting processes $N_t(e_{rr})$: $r = 1, \ldots, p$, which are independent (classical) Poisson processes with intensities ξ_r^2, by Proposition 2.3. Then we are back to the situation studied in [KüM].

Example 2. (Compensating transitions.) An obvious way to satisfy conditions (5.2) and (5.3) is to let $p = n$, $\xi^2 = \rho$ and $v_{rs} = e_{sr}$. The resulting expression for M is

$$M(a) = \sum_{r,s=1}^{n} \xi_s^2 \, e_{rs} \, a e_{sr}.$$

The physical interpretation is as follows. The bath particles are identical to (but distinguishable from) the system of interest. When a bath particle hits the system, transitions between energy levels may occur; a transition from level s to level r for the system is compensated by the reverse transition for the bath particle.

Example 3. (Combination of Examples 1 and 2.) Let $n = kp$, k integer, and assume $(M_n, \phi) = (M_p, \mu) \otimes (M_k, \tau)$, where τ denotes the normalised trace. Put

$$v_{rs} = e_{sr} \otimes w_{rs}, \qquad w_{rs} \in M_k, \quad r, s = 1, \ldots, p;$$

then condition (5.3) is obviously satisfied, and condition (5.2) becomes

$$\sum_{q=1}^{p} e_{qr}\, e_{sq} \otimes w_{qr}^* \, w_{qs} = \delta_{rs}\, \mathbb{1} \Leftrightarrow w_{qr}^*\, w_{qr} = \mathbb{1} \quad \text{for all } q, r.$$

In words, each w_{rs} must be unitary. The resulting M is

$$M(a) = \sum_{r,s=1}^{p} \xi_s^2 \, [e_{rs} \otimes w_{rs}^*] \, a \, [e_{rs} \otimes w_{rs}], \qquad a \in M_n.$$

In particular, for $a = e_{ss} \otimes b$, $b \in M_k$, we have

$$M(e_{ss} \otimes b) = \xi_s^2 \sum_{r=1}^{p} e_{rr} \otimes w_{rs}^* \, b \, w_{rs}.$$

Example 4. (Generalisation of Example 2.) Let again $p = n$, $\xi^2 = \rho$. Operators v_{rs} satisfying (5.3) can be chosen as

$$v_{rs} = (1 - \delta_{rs})\, c_{sr}\, e_{sr} + \delta_{rs} \sum_{k=1}^{n} d_{rk}\, e_{kk},$$

where c_{sr} and d_{rk} are complex coefficients. In the case that ρ has a nondegenerate spectrum satisfying the additional condition that $\rho_i \, \rho_j^{-1} = \rho_k \, \rho_l^{-1}$ for $i \neq j$ implies $i = k$ and $j = l$ (i.e., the spectrum of $\sigma_t^\phi = \rho^{it}(\cdot)\, \rho^{-it}$ does not have accidental degeneracies), the above is the only way to satisfy (5.3). Imposing the requirement that also (5.2) is satisfied, we obtain, after some straightforward calculations,

$$(1 - \delta_{rq})\, |c_{rq}|^2 + |d_{rq}|^2 = 1 \qquad \text{for all } r, q;$$
$$c_{sr}\, \bar{d}_{rs} + \bar{c}_{rs}\, d_{sr} = 0 \qquad \text{for } r \neq s.$$

Note that it follows from the second equation that $|c_{rs}|^2/|c_{sr}|^2 = |d_{rs}|^2/|d_{sr}|^2$, and combining this with the first equation we obtain $|c_{rs}|^2 = |c_{sr}|^2$ $(r \neq s)$. Then, upon multiplying each v_{rs} by a suitable phase factor, which does not change the expression (5.4) of M, we may assume that $c_{rs} = c_{sr} \geqq 0$. Then $\bar{d}_{rs} = -d_{sr}$ if $c_{rs} \neq 0$. If $r = s$ or $c_{rs} = 0$, we have $d_{rs} = \exp[i\theta_{rs}]$ and there is no condition relating d_{rs} with d_{sr}.

The expression for M becomes

$$M(a) = \sum_{r \neq s = 1}^{n} \xi_s^2 \, c_{rs}^2 \, e_{rs}\, a\, e_{sr} + \sum_{k,l=1}^{n} D_{kl}\, e_{kk}\, a\, e_{ll},$$

where (D_{kl}) is the hermitian matrix given by $D_{kl}=\sum_{r=1}^{n} \bar{d}_{rk}\, d_{rl}$. Finally, we take up again the general discussion with some remarks on the property of detailed balance. The generator $L=M-id$ admits a ϕ-adjoint generator L^+ satisfying

$$\phi(L^+(a)\,b)=\phi(aL(b)), \qquad a, b\in M_n; \tag{5.5}$$

clearly, L^+ is given by M^+-id, where

$$\phi(M^+(a)\,b)=\phi(aM(b)), \qquad a, b\in M_n. \tag{5.6}$$

We compute explicitly

$$\begin{aligned}\phi(aM(b)) &= \sum_{r,s=1}^{p} \xi_s^2 \operatorname{tr}[\rho\, a v_{rs}^* b v_{rs}] \\ &= \sum_{r,s=1}^{p} \xi_s^2 \operatorname{tr}[\rho\rho^{-1} v_{rs}\, \rho\, a\, v_{rs}^* b] \\ &= \sum_{r,s=1}^{p} \xi_s^2 (\xi_s^2\, \xi_r^{-2})^{-1} \operatorname{tr}[\rho\, v_{rs}\, a v_{rs}^* b] \\ &= \sum_{r,s=1}^{p} \xi_r^2 \operatorname{tr}[\rho\, v_{rs}\, a v_{rs}^* b], \qquad a, b\in M_n,\end{aligned}$$

which proves that

$$M^+(a)=\sum_{r,s=1}^{p} \xi_r^2\, v_{rs}\, a v_{rs}^* = \sum_{r,s=1}^{p} \xi_s^2\, v_{sr}\, a v_{sr}^*, \qquad a\in M_n. \tag{5.7}$$

The semigroup $\exp[tL]$ is said to satisfy the *detailed balance* condition with respect to ϕ if $L-L^+$ (which in the present case is the same as $M-M^+$) is a derivation of M_n [KFGV].

We recall that the semigroups admitting a Markov dilation with a faithful normal invariant state constructed by means of quantum Brownian motion satisfy the detailed balance condition [Fri2]. This need not be true for dynamical semigroups admitting a Poisson dilation. In the case of coupling to a classical Poisson process, as in Example 1, this was already remarked by Kümmerer and Maassen [KüM]. In the case of Example 2, detailed balance holds, with $M=M^+$. In Example 3, detailed balance need not hold; we have in general

$$M^+(a)=\sum_{r,s=1}^{p} \xi_s^2 (e_{rs}\otimes w_{sr})\, a\, (e_{sr}\otimes w_{sr}^*),$$

and, for $a=e_{ss}\otimes b$,

$$M^+(e_{ss}\otimes b)=\xi_s^2 \sum_{r=1}^{p} e_{rr}\otimes w_{sr}\, b w_{sr}^*,$$

so that counterexamples to detailed balance can be constructed as in [KüM]. Finally, for Example 4 we have

$$M(a)-M^{+}(a)=\sum_{k,l=1}^{n}(2i\,\mathrm{Im}\,D_{kl})\,e_{kk}\,a\,e_{ll}$$

since the matrix c_{rs}^2 is symmetric; it follows, in particular, that the restriction of $\exp[Lt]$ to the diagonal matrices satisfies the detailed balance condition.

In conclusion, we may say that it is comparatively easy to construct examples of dilations (with an invariant state $\hat{\phi}$) of dynamical semigroups not satisfying the detailed balance condition, if the modular automorphism group σ_t^ϕ associated with the invariant state ϕ presents accidental degeneracies (Examples 1 and 3), whereas in the opposite case of no accidental degeneracies all known examples (like Examples 2 and 4) satisfy detailed balance. In particular, the Davies model of heat conduction [Dav], even in its extremely simplified version given in [Fri 2], which displays neither accidental degeneracies nor detailed balance (thus far) resists all attempts at providing it with a dilation having an invariant state.

Acknowledgements. We wish to thank B. Kümmerer for stimulating discussions, and R.F. Streater for bringing Refs. [StW], [Ara] to our attention.

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Received April 6, 1988; in revised form April 6, 1989