The Quantum Monty Hall Problem

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We consider a quantum version of a well-known statistical decision problem, whose solution is, at first sight, counter-intuitive to many. In the quantum version a continuum of possible choices (rather than a finite set) has to be considered. It can be phrased as a two person game between a player P and a quiz master Q. Then P always has a strategy at least as good as in the classical case, while Q's best strategy results in a game having the same value as the classical game. We investigate the consequences of Q storing his information in classical or quantum ways. It turns out that Q's optimal strategy is to use a completely entangled quantum notepad, on which to encode his prior information.

I. INTRODUCTION

The well-known classical Monty Hall problem, also known under various other names [1], is set in the context of a television game show. It can be seen as a two person game, in which a player P tries to win a prize, but a show master (or Quiz master) Q tries to make it difficult for her [2]. Like other games of this sort [3, 4] it can be "quantized", i.e., its key elements can be formulated in a quantum mechanical context, allowing new strategies and new solutions. However, such quantizations are rarely unique, and depend critically on what is seen as a "key element", and also on how actions which might change the system are formalized, corresponding to how, in the classical version, information is gained by "looking at something". The Monty Hall problem is no exception, and there are already quantizations [5, 6]. The version we present in this paper was drafted independently, and indeed we come to a quite different conclusion. We discuss the relation between the two approaches in more detail in Sec. VII below.

Our paper is organized as follows. In Sec. II we review the classical game, whose various steps are translated to the quantum context in Sec. III. In Sec. VII we discuss some alternative ideas to quantize the game. As in the classical version, computer simulations help to understand the structure of the game and its basic strategic options. We provide a Java simulation [7] of our version from the point of view of player P. There is a basic strategy for P in our version, which we call the *classical strategy* (Sec. IV), which guarantees a gain for her at least as good as in the classical game. The further analysis depends on how Q records the information about the initial preparation, which he needs later on in the game: if he uses a classical notepad, he has to choose his strategy very carefully, or P can improve her odds beyond the classical expectation (Sec. V). These possibilities for P are illustrated especially nicely by the simulation [7]. The case of a quantum notepad is discussed in Sec. VI, and gives Q a simpler way to restrict P's expectations to the classical value.

II. THE CLASSICAL MONTY HALL PROBLEM

The classical Monty Hall problem is set in the context of a television game show. In the last round of the show, the candidates were given a chance to collect their prize (or lose it) in the following game:

1. Before the show the prize is hidden behind one of three closed doors. The show master knows where the prize is but, of course, the candidate does not.

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2. The candidate is asked to choose one of the three doors, which is, however, not opened at this stage.

3. The show master opens another door, and shows that there is no prize behind it. (He can do this, because he knows where the prize is).

4. The candidate can now open one of the remaining doors to either collect her prize or lose.

Of course, the question is: should the candidate stick to her original choice or "change her mind" and pick the other remaining door? As a quick test usually shows, most people will stick to their first choice. After all, before the show master opened a door the two doors were equivalent, and they were not touched (nor was the prize moved). So they should still be equivalent. This argument seems so obvious that trained mathematicians and physicists fall for it almost as easily as anybody else [8].

However, the correct solution by which the candidates can, in fact, double their chance of winning, is to always choose the other door. The quickest way to convince people of this is to compare the game with another one, in which the show master offers the choice of either staying with your choice or *opening both other doors*. Anybody would prefer that, especially if the show master courteously offers to open one of the doors for you. But this is precisely what happens in the original game when you always change to the other door.

III. THE QUANTUM MONTY HALL PROBLEM

We will "quantize" only the key parts of the problem. That is, the prize and the players, as well as their publicly announced choices, will remain classical. The quantum version can even be played in a game show on classical TV.

The main quantum variable will be the position of the prize. It lies in a 3-dimensional complex Hilbert space $\mathcal{H}$, called the *game space*. We assume that an orthonormal basis is fixed for this space so that vectors can be identified by their components, but apart from this the basis has no significance for the game. A second important variable in the game is what we will call the show master's notepad. This might be classical information describing how the game space was prepared, or it might be a quantum system, entangled with the prize. In the latter case, the show master is able to do a quantum measurement on his notepad, providing him with classical information about the prize, without moving the prize, in the sense that the player's information about the prize is not changed by the mere fact that the show master "consults his notepad". A measurement on an auxiliary quantum system, even if entangled with a system of interest, does not alter the reduced state of the system of interest. After the show master has consulted his notepad, we are in the same situation as if the notepad had been a classical system all along. As in the classical game, the situation for the player might change when the show master, by opening a door, reveals to some extent what he saw in his notepad. Opening a door corresponds to a measurement along a one dimensional projection on $\mathcal{H}$.

The game proceeds in the following stages, closely analogous to the classical game:

1. Before the show the game space system is prepared quantum mechanically. Some information about this preparation is given to the show master Q. This can be in the form of another system, called the notepad, which is in a state correlated with the game space.

2. The candidate chooses some one dimensional projection p on $\mathcal{H}$.

3. The show master opens a door, i.e., he chooses a one dimensional projection q, and makes a Lüders/von Neumann measurement with projections q and $(\mathbb{1}-q)$. In order to do this, he is allowed first to consult his notebook. If it is a quantum system, this means that he carries out a measurement on the notebook. The joint state of prize and notebook then change, but the traced out or reduced state of the prize does not change, as far as the player is concerned. Two rules constrain the show master's choice of q: he must choose "another door" in the sense that $q \perp p$; and he must be *certain not to reveal the prize.* The purpose of his notepad is to enable him to do this. After these steps, the game space is effectively collapsed to the two-dimensional space $(\mathbb{1} - q)\mathcal{H}$.

4. The player P can now choose a one dimensional projection p' on $(\mathbb{1} - q)\mathcal{H}$, and the corresponding measurement on the game space is carried out. If it gives "yes" she collects the prize.

As in the classical case, the question is: how should the player choose the projection p' in order to maximize her chance of winning? Perhaps it is best to try out a few options in a simulation, for which a Java applet is available [7]. For the input to the applet, as well as for some of the discussion below it is easier to use unit vectors rather than one-dimensional projections. As standard notation we will use for $p = |\Phi\rangle\langle\Phi|$ for the door chosen by the player, $q = |\chi\rangle\langle\chi|$ for the door opened by Q, and $r = |\Psi\rangle\langle\Psi|$ for the initial position of the prize, if that is defined.

From the classical case it seems likely that choosing $p' = p$ is a bad idea. So let us say that the *classical strategy* in this game consists of always switching to the orthogonal complement of the previous choice, i.e., to take $p' = \mathbb{1} - q - p$. Note that this is always a projection because, by rule 3, p and q are orthogonal one dimensional projections. We will analyze this strategy in Sec. IV, which turns out to be possible without any specification of how the show master can guarantee not to stumble on the prize in step 3.

There are two main ways the show master can satisfy the rules. The first is that he chooses randomly the components of a vector in $\mathcal{H}$, and prepares the game space in the corresponding pure state. He can then just take a note of his choice on a classical pad, so that in stage 3 he can compute a vector orthogonal to both the direction of the preparation and the direction chosen by the player. Q's strategies in this case are discussed in Sec. IV. The second and more interesting way is to use a quantum notepad, i.e., another system with three dimensional Hilbert space $\mathcal{N}$, and to prepare a "maximally entangled state" on $\mathcal{H}\otimes\mathcal{N}$. Then until stage 3 the position of the prize is completely undetermined in the strong sense only possible in quantum mechanics, but the show master can find a safe door to open on $\mathcal{H}$ by making a suitable measurement on $\mathcal{N}$. Q's strategies in this case are discussed in Sec. VI.

IV. THE CLASSICAL STRATEGY

To explain why the classical strategy works almost as in the classical version of the problem, we look more closely at the situation, when Q has just reached the decision which door q he wants to open. His method of arriving at that decision is irrelevant. It must take into account the player's choice p, as well as Q's information about the preparation, and may involve some further randomness or even a quantum measurement on an auxiliary quantum system called the notepad. The choice of q is announced publicly by "opening" that door, and needs to be known as classical information to P for her to choose a door p' orthogonal to it. So we shall consider q as a classical random variable. What we need to study are the correlations of this variable with outcomes of later measurements made on the game system. That is to say, we have a *hybrid system* described by a combination of classical and quantum variables.

There is a standard form for states on such systems, which we will need to use here. Let us consider first the classical part. Since p was arbitrary and since the procedure used by the show master might involve classical randomization or quantum randomness (through measurement of a quantum notepad), q has a probability distribution, denoted by w, which typically will depend on p. Thus the expectation of any real valued function f of q is to be computed as $\int w(dq)\ f(q)$. Next we are interested in the joint response of a classical detector (described by a characteristic function f) and a quantum detector, e.g., given by a one dimensional projection p. By the theorem of Ozawa[13] the expectation can be expressed now in terms of the *conditional density operator* ρ_q (which is uniquely determined) as

$$\int w(dq)\ \mathrm{tr}(\rho_q p')f(q). \tag{1}$$

This too will typically depend on p, but we suppress that dependence also from the notation. Loosely speaking ρ_q is the density matrix P has to use for the game space after Q has announced his intention of opening door q. This is usually not the same conditional density operator as the one used by Q: Since Q has more classical information about the system, he may condition on that, leading to finer predictions. In contrast, ρ_q is conditioned only on the publicly available information.

From w and ρ_q we can compute the marginal density operator for the quantum subsystem, describing measurements without consideration of the classical variable q. This is the mean density operator

$$\overline{\rho} = \int w(dq)\ \rho_q\ . \tag{2}$$

It will not depend on p and it will be the same as the reduced density operator for the game space before the show master consults his notepad (he is not allowed to touch the prize), and even before the player chooses p (which cannot affect the prize).

From the rules alone we know two things about the conditional density operators: firstly, that $\mathrm{tr}(\rho_q\ q) = 0$: the show master must not hit the prize. Secondly, q and p must commute, so it does not matter which of the two we measure first. Thus for p we get on average $\int w(dq)\ \mathrm{tr}(\rho_q\ p) = \mathrm{tr}(\overline{\rho}\ p)$. Combining these two we get the overall probability w_c for winning with the classical strategy as

$$w_c = \int w(dq)\ \mathrm{tr}\big(\rho_q(\mathbb{1} - p - q)\big) = 1 - \mathrm{tr}(\overline{\rho}\ p)\ . \tag{3}$$

If we assume that $\overline{\rho}$ is known to P, from watching the show sufficiently often, the best strategy for P is to choose initially the p with the smallest expectation with respect to $\overline{\rho}$, just as in the classical game with uneven prize distribution it is best to choose initially the door least likely to contain the prize. If Q on the other hand wants to minimize P's gain [10], he will choose $\overline{\rho}$ to be uniform, which in the quantum case means $\overline{\rho} = \frac{1}{3}\mathbb{1}$, and hence $w_c = 2/3$.

V. STRATEGIES AGAINST CLASSICAL NOTEPADS

In this section we consider the case that the show master records the prepared direction of the prize on a classical notepad. We will denote the one dimensional projection of this preparation by r. Then when he has to open a door q, he needs to choose $q \perp r$ and $q \perp p$. This is always possible in a three dimensional space. But unless $p = r$, he has no choice: q is uniquely determined. This is the same as in the classical case, only that the condition "$p = r$", i.e., that the player chooses *exactly* the prize vector typically has probability zero. Hence Q's strategic options are not in the choice of q, but rather in the way he randomizes the prize positions r, i.e., in the choice of a probability measure v on the set of pure states. In order to safeguard against the classical strategy he will make certain that the mean density operator $\overline{\rho} = \int v(dr)\ r$ is unpolarized ($= \frac{1}{3}\mathbb{1}$). It seems that this is about all he has to do, and that the best the player can do is to use the classical strategy, and win 2/3 of the time. However, this turns out to be completely wrong.

A. Preparing along the axes

Suppose the show master decides that since the player can win as in the classical case, he might as well play classical as well, and save the cost for an expensive random generator. Thus he fixes a basis and chooses each one of the basis vectors with probability 1/3. Then $\overline{\rho} = \frac{1}{3}\mathbb{1}$, and there seems to be no giveaway. In fact, the two can now play the classical version, with P choosing likewise a projection along a basis vector.

But suppose she does not, and chooses instead the projection along the vector $\Phi = (1, 1, 1)/\sqrt{3}$. Then if prize happens to be prepared in the direction $\Psi = (1, 0, 0)$, the show master has no choice but to choose for q the unique projection orthogonal to these two, which is along $\chi = (0, 1, -1)$. So when Q announces his choice, P only has to look which component of the vector is zero, to *find the prize with certainty!*

This might seem to be an artifact of the rather minimalistic choice of probability distribution. But suppose that Q has settled for any *arbitrary finite collection of vectors* Ψ_α and their probabilities. Then P can choose a vector Φ which lies in none of the two dimensional subspaces spanned by two of the Ψ_α. This is possible, even with a random choice of Φ, because the union of these two dimensional subspaces has measure zero. Then, when Q announces the projection q, P will be able to reconstruct the prize vector with certainty: at most one of the Ψ_α can be orthogonal to q. Because if there were two, they would span a two dimensional subspace, and together with Φ they would span a three dimensional subspace orthogonal to q, which is a contradiction.

Of course, any choice of vectors announced with floating point precision is a choice from a finite set. Hence the last argument would seem to allow P to win with certainty in every realistic situation. However, this only works if she is permitted to ask for q at any desired precision. So by the same token (fixed length of floating point mantissa) this advantage is again destroyed.

This shows, however, where the miracle strategies come from: by announcing q, the show master has not just given the player $\log_2 3$ bits of information, but an infinite amount, coded in the digits of the components of q (or the vector χ).

B. Preparing real vectors

The discreteness of the probability distribution is not the key point in the previous example. In fact there is another way to economize on random generators, which proves to be just as disastrous for Q. The vectors in $\mathcal{H}$ are specified by three complex numbers. So what about choosing them real for simplicity? An overall phase does not matter anyhow, so this restriction does not seem to be very dramatic.

Here the winning strategy for P is to take $\Phi = (1, i, 0)/\sqrt{2}$, or another vector whose real and imaginary parts are linearly independent. Then the vector $\chi \perp \Phi$ announced by Q will have a similar property, and also must be orthogonal to the real prize vector. But then we can simply compute the prize vector as the outer product of real and imaginary part of χ.

For the vector Φ specified above we find that if the prize is at $\Psi = (\Psi_1, \Psi_2, \Psi_3)$, with $\Psi_k \in \mathbb{R}$, the unique vector χ orthogonal to Φ and Ψ is connected to Ψ via the transformations

$$\chi \ \propto \ (\Psi_3,\ -i\Psi_3,\ -\Psi_1 + i\Psi_2) \tag{4}$$

$$\Psi \ \propto \ (-\Re e\chi_3,\ \Im m\chi_3,\ \chi_1)\ , \tag{5}$$

where "$\propto$" means "equal up to a factor", and it is understood that an overall phase for χ is chosen to make χ_1 real. This is also the convention used in the simulation [7], so Eq. (5) can be tried out as a universal cheat against show masters using only real vectors.

C. Uniform distribution

The previous two examples have one thing in common: the probability distribution of vectors employed by the show master is concentrated on a rather small set of pure states on $\mathcal{H}$. Clearly, if the distribution is more spread out, it is no longer possible for P to get the prize every time. Hence it is a good idea for Q to choose a distribution which is as uniform as possible. There is a natural definition of "uniform" distribution in this context, namely the unique probability distribution on the unit vectors, which is invariant under arbitrary unitary transformations [11]. Is this a good strategy for Q?

Let us consider the conditional density operator ρ_q, which depends on the two orthogonal projections p, q. It implicitly contains an average over all prize vectors leading to the same q, given p. Therefore, ρ_q must be invariant under all unitary rotations of $\mathcal{H}$ fixing these two vectors, which means that it must be diagonal in the same basis as $p, q, (\mathbb{1} - p - q)$. Moreover, the eigenvalues cannot depend on p and q, since every pair of orthogonal one dimensional projections can be transformed into any other by a unitary rotation. Since we know the average eigenvalue in the p-direction to be 1/3, we find

$$\rho_q = \frac{1}{3}p + \frac{2}{3}(\mathbb{1} - p - q)\ . \tag{6}$$

Hence the classical strategy for P is clearly optimal. In other words, the pair of strategies: "uniform distribution for Q and classical strategy for P" is an equilibrium point of the game. We do not know yet, whether this equilibrium is unique, in other words: If Q does not play precisely by the uniform distribution: can P always improve on the classical strategy? We suppose that the answer to this question is yes, to find a proof of this conjecture has turned out, however, to be a hard problem.

VI. STRATEGIES FOR QUANTUM NOTEPADS

A. Quantum notepad and entanglement

As briefly described in Sec. III, a notepad is quite generally another physical system, which by its initial preparation is brought into a state correlated with the game system, so that measurements on the notepad provide information about the game system as well. The classical notepad we were considering in the previous section can be formalized in this way, too, although it may appear a bit artificial to introduce a physical system on which some random variable is "written". For the quantum notepads considered in this section, however, this view is mandatory.

The simplest form of a quantum notepad meeting Q's requirements is another system with three dimensional Hilbert space $\mathcal{N}$. Denoting some bases in $\mathcal{H}$ and $\mathcal{N}$ by ket vectors $|1\rangle, |2\rangle, |3\rangle$, and abbreviating $|k\rangle \otimes |\ell\rangle \equiv |k\ell\rangle \in \mathcal{H} \otimes \mathcal{N}$, we introduce the "maximally entangled vector"

$$\Omega = \frac{1}{\sqrt{3}} \sum_{k=1}^{3} |kk\rangle \ . \tag{7}$$

It clearly depends on the bases we have chosen, but not as much as one might expect. Its crucial property is the equation

$$(X \otimes \mathbb{1})\Omega = (\mathbb{1} \otimes X^T)\Omega \ , \tag{8}$$

where the transpose X^T is defined by $\langle k|X^T|\ell\rangle = \langle \ell|X|k\rangle$. Note that in this way an operator X on $\mathcal{H}$ becomes an operator X^T on $\mathcal{N}$, and this becomes possible by using the same labels for the basis vectors in both spaces. If we now change the basis in $\mathcal{H}$ by a unitary operator U, say, we find that there is a corresponding basis change on $\mathcal{N}$, leaving the vector Ω invariant, namely $(U^{-1} \otimes U^T)\Omega = \Omega$. Hence Ω does not depend on the individual bases, but represents a particular way of identifying $\mathcal{H}$ with $\mathcal{N}$ [12], and all bases accordingly. This is underlined by the observation that the restriction of this state to either factor is the completely unpolarized state, i.e.,

$$\langle \Omega|A \otimes \mathbb{1}|\Omega\rangle = \langle \Omega|\mathbb{1} \otimes A|\Omega\rangle = \frac{1}{3}\mathrm{tr}(A) \ , \tag{9}$$

i.e., the unique state invariant under all unitary rotations.

Let us consider now a measurement on the notepad. It is described, like any measurement, by a positive operator valued measure. Let us take it discrete-valued for the moment to simplify the explanation. Thus the measurement is given by a collection of positive operators F_x on $\mathcal{N}$, such that $\sum_x F_x = \mathbb{1}$. Here the labels "x" are the (classical) outcomes of the measurement, and $\mathrm{tr}(\rho\ F_x)$ is interpreted as the probability of getting this result, when measuring on systems prepared according to ρ. How could Q now infer from this result a safe door q for him to open in the game? This would mean that F_x measured on $\mathcal{N}$, and q measured on $\mathcal{H}$ never give a simultaneous positive response, when measured in the state Ω, i.e.,

$$0 = \langle \Omega|q \otimes F_x|\Omega\rangle = \langle \Omega|\mathbb{1} \otimes F_x q^T|\Omega\rangle = \frac{1}{3}\mathrm{tr}(q^T\ F_x) \ . \tag{10}$$

Since F_x and q^T are both positive, this equivalent to $F_x q^T = 0$. Of course, Q's choice must also satisfy the constraint $q \perp p$. There are different ways of arranging this, which we discuss in the following two subsections.

B. Equivalence if observable is chosen beforehand

Suppose Q chooses the measurement beforehand, and let us suppose it is discrete, as before. Then for every outcome x and every p he must be able to find a one dimensional projection satisfying both constraints $F_x^T q = 0$ and $qp = 0$. Clearly, this requires that F_x has at least a two dimensional null space, i.e., $F_x = |\phi_x\rangle\langle\phi_x|$, with a possibly non-normalized vector $\phi_x \in \mathcal{N}$. It will be convenient to take the vectors ϕ_x to be normalized, and to define $F_x = v_x|\phi_x\rangle\langle\phi_x|$ with factors v_x summing to 3, the dimension of the Hilbert space (take the trace of the normalization condition for F). We can further simplify this structure, by identifying outcomes x with the same ϕ_x, since for these the same projection q has to be chosen anyhow. We can therefore drop the index "x", and consider the measure to be defined directly on the set of one dimensional projections. But this is precisely the structure we had used to describe a classical notepad. This is not an accidental analogy: apart from taking transposes as in equation (8) this measure has precisely the same strategic meaning as the measure of a classical notepad.

This is not surprising: if the observable is chosen beforehand, it does not matter whether the show master actually performs the measurement before or after the player's choice. But if he does it before P's choice, we can just as well consider this measurement with its classical output as part of the preparation of a classical notepad, in which the result is recorded.

C. Simplified strategy for Q

Obviously the full potential of entanglement is used only, when Q chooses his observable after P's choice.

Since the position of the prize is "objectively undetermined" until then, it might seem that there are now ways to beat the 2/3 limit. However, the arguments for the classical strategy hold in this case as well. So the best Q can hope for are some simplified strategies. For example, he can now get away with something like measuring along axes only, even though for classical notepads using "axes only" was a certain loss for Q.

We can state this in a stronger way, by introducing tougher rules for Q: In this variant P not only picks the direction p, but also two more projections p' and p'' such that $p + p' + p'' = \mathbb{1}$. Then Q is not only required to open a door $q \perp p$, but we require that either $q = p'$ or $q = p''$. It is obvious how Q can play this game with an entangled notepad: he just uses the transposes of p, p', p'' as his observable. Then everything is as in the classical version, and the equilibrium is again at 2/3.

VII. ALTERNATIVE VERSIONS AND QUANTIZATIONS OF THE GAME

A. Variants arising already in the classical case

Some variants of the problem can also be considered in the classical case, and they tend to trivialize the problem, so that P's final choice becomes equivalent to "Q has prepared a coin, and P guesses heads or tails". Here are some possibilities, formulated in a way applying both to the classical and the quantum version.

- *Q is allowed to touch the prize after P made her first choice.* Clearly, in this case Q can reshuffle the system, and equalize the odds between the remaining doors. So no matter what P chooses, there will be a 50% chance for getting the prize.

- *Q is allowed to open the door first chosen by P.* Then there is no way P's first choice enters the rules, and we may analyze the game with stage 2 omitted, which is entirely trivial.

- *Q may open the door with the prize, in which case the game starts again.* Since Q knows where the prize is, this is the same as allowing him to abort the round, whenever he does not like what has happened so far, e.g., if he does not like the relative position of prize and P's choice. In the classical version he could thus cancel 50% of the cases, where P's choice is not the prize, thus equalizing the chances for P's two pure strategies. Similar possibilities apply in the quantum case.

B. Variants in which classical and quantum behave differently

- *Q may open the door with the prize, in which case P gets the prize.* In the classical version, revealing the prize is then the worst possible pure strategy, so mixing in a bit of it would seem to make things always worse for Q. Then although increasing Q's options in principle can only improve things for Q, one would advise him not to use the additional options. This is assuming, though, that in the remaining cases Q sticks to his old strategy. However, even classically, the relaxed rule gives him some new options: He can simply ignore the notepad, and open any door other than p. Then the game becomes effectively "P and Q open a door each, and P gets all prizes". Assuming uniform initial distribution of prizes this gives the same 2/3 winning chance as in the original game.

  The corresponding quantum strategy works in the same way. Assuming, for simplicity, a uniform mean density operator $\overline{\rho} = \frac{1}{3}\mathbb{1}$, Q's strategy of ignoring his prior information will give the classical 2/3 winning chance for P. But this is a considerable improvement for Q in cases where a non-uniform probability distribution of pure states previously gave Q a 100% chance of winning. So in the quantum case, doing two seemingly stupid things together amounts to a good strategy for Q: firstly, sometimes revealing the prize for P, and secondly ignoring all prior information.

  Note that this strategy is optimal for Q, because the classical strategy still guarantees the 2/3 winning chance for P. This can be seen with the same arguments as in Section IV. The only difference is that $\mathrm{tr}(\rho_q q)$ can be nonzero, since Q may open the door with the prize. However in this case P wins and we get instead of Equation (3)

$$w_c = \int w(dq)\, \mathrm{tr}\big(\rho_q(\mathbb{1} - p - q)\big) + \mathrm{tr}(\rho_q q) \qquad (11)$$

$$= 1 - \mathrm{tr}(\overline{\rho} p) = \frac{2}{3} \qquad (12)$$

- *As Q opens the door he is allowed to make a complete von Neumann measurement.* Classically, it would make no difference if the doors were completely transparent to the show master. He would not even need a pad then, because he could always look where the prize is. But "looking" is never innocent in quantum mechanics, and in this case it is tantamount to moving the prize around. So let us make it difficult for Q, by insisting that the initial preparation is along a fixed vector, known also to P, and that Q not only has to announce the direction q of the door he opens, but also the projections $q' \perp q$ and $q'' = \mathbb{1} - q - q'$ entering in the complete von Neumann measurement, which takes an arbitrary density operator ρ to

$$\rho \mapsto q\rho q + q'\rho q' + q''\rho q''. \qquad (13)$$

Moreover, we require as before, that q is orthogonal both to p and to the prize. The only thing remaining secret is which of the projections q' and q'' has detected the presence of the prize (This simply would allow P to open that door and collect). Q's simple strategy is now to choose q as before. The position of p is irrelevant for his choice of p' and p'': he will just take these directions at 45° to the prize vector. This will result in the unpolarized density operator $(q' + q'')/2$, and no matter what P chooses, her chances of hitting the prize will be 1/2. She will probably feel cheated, and she is, because even though she knows exactly where the prize was initially, the strategy "choose the prize, and stick to this choice" no longer works.

C. Two published versions

- The quantization proposed in [5] is closely related to the second variant of the last Subsection: After Q has opened one door he is allowed to perform an arbitrary von Neumann measurement on the remaining two-dimensional subspace – i.e. he "looks" where the prize is. In the classical case this is an allowed (but completely superfluous) step. In the quantum case, however, the prize is shuffled around and we end up with a variant from Subsection VII A: Q is allowed to move the prize. In other words, the final result of the game is completely independent of the steps prior to this measurement and the whole game is reduced to coin tossing – which is not very interesting.
- A completely different quantization of the game is given in [6]. In contrast to our approach, the moves available to Q and P are here not preparations and mesurements but *operations* on a tripartite system which is initially in the pure state $\psi \in \mathcal{H}_Q \otimes \mathcal{H}_P \otimes \mathcal{H}_O$ (and different choices for ψ lead to different variants of the game). The Hilbert spaces $\mathcal{H}_Q, \mathcal{H}_P$ and $\mathcal{H}_O$ describe the doors where Q hides the prize, which P chooses in the second step and which Q opens afterwards and the gameplay is described by the unitary operator

$$U = (\cos\gamma U_S + \sin\gamma U_N) U_O (U_Q \otimes U_P \otimes \mathbb{1}). \qquad (14)$$

  U_Q and U_P are arbitrary unitaries, describing Q's and P's initial choice, U_O is the (fixed) opening box operator and U_S respectively U_N are P's "switching" and "not-switching" operators. The payoff is finally given as the expectation value of an appropriate observable \$ (for a precise definition of U_O, U_S, U_N and \$ see [6]). The basic idea behind this scheme is quite different from ours and a comparison of results is therefore impossible. Nevertheless, this is a nice example which shows that quantizing a classical game is very non-unique.

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[1] `www.cut-the-knot.com/hall.html`

[2] In this text, the show master is male, like Monty Hall, the host of the show, where the game first appeared. The player is female, like Marilyn vos Savant, who was the first to fight the public debate for the recognition of the correct solution, and had to take some sexist abuse for that.

[3] J. Eisert, M. Wilkens, and M. Lewenstein, Quantum Games and Quantum Strategies, *Phys. Rev. Lett.* **83** (1999) 307–3080.

[4] J. Eisert, and M. Wilkens, and M. Lewenstein, Quantum Games, *J. Mod. Opt.* **47** (2000) 2543–2556, preprint quant-ph/0004076.

[5] C.-F. Li, Y.-S. Zhang, Y.-F. Huang, G.-C. Guo, Quantum Monty Hall problem, quant-ph/0007120.

[6] A. P. Flitney and D. Abbott, *Quantum Monty Hall*, quant-ph/0109035 (2001)

[7] `http://www.imaph.tu-bs.de/qi/monty` .

[8] The only difference seems to be that professionals tend to pontificate about the obvious truth of the false solution. Embarassingly many have gone on record.

[9] J. von Neumann: Zur Theorie der Gesellschaftsspiele.

[10] Of course, real game show hosts don't care, they try to maximize ratings. Incidentally, this means that the quantum version will hardly ever be played on classical TV. For how can a black box containing some esoteric game space system compete with the dramatic potential of closed doors?

[11] Numerically the way to generate this distribution is to take the real and imaginary parts of all components of a vector as independent identically normally distributed random variables, and to normalize the resulting vector.

[12] More precisely, Ω is equivalent to an identification of $\mathcal{H}$ with the complex conjugate Hilbert space of $\mathcal{N}$. This structure routinely arises in the so-called GNS-representation of operator algebras.

[13] .A classical notepad is just a special case of a quantum notepad, on which we only consider commuting observables. Thus both for a classical and a quantum notepad, we can consider the show master's consultation of his notepad as the realization of c.p. instrument on the auxiliary system, which can be extended to a c.p. instrument on the joint system in the obvious way. Now apply the results of M. Ozawa, Conditional probability and a pos-

teriori states in quantum mechanics, *Publ. RIMS Kyoto Univ.* **21** (1985), 279–295, and reduce to the system of interest.