QUANTUM PROBABILITY APPLIED TO THE DAMPED HARMONIC OSCILLATOR

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Abstract. In this introductory course we sketch the framework of quantum probability in order to discuss open quantum systems, in particular the damped harmonic oscillator.

Contents

1. The framework of quantum probability

Noncommutative probability theory (or 'quantum probability') generalises Kolmogorov's classical probability theory in a way that allows the inclusion of quantum mechanical models. For a discussion of its motivation we refer to [KüM] in this series. Basic sources on quantum probability outside the present series are [Bia], [Dav], [Hol], [Gud], [Mac], [Mey], [Neu], [Par], [Var]. An independent introduction is given here.

Making probability noncommutative. In the last two decades a succesful strategy has become popular in mathematics: the generalisation of classical mathematical structures by noncommutative algebraic constructions. The most widely known example where this strategy was applied is doubtlessly the noncommutative version of geometry, as explained in the imaginative book of A. Connes ([Con]). There the classical structures of a topological space and of a differential manifold are the pillars on which the K-theory of C*-algebras and a variety of cohomological algebras are built. Another application is the field of 'quantum groups', where the classical structure of a Lie group leads into new areas in the theory of Hopf algebras.

However, the oldest case by far is von Neumann's and Segal's 'noncommutative integration theory', which has developed into noncommutative measure theory and probability theory.

The general strategy consists of the following three steps.

(1) Encode the information contained in the classical structure into an appropriate algebra of functions on it.
(2) Characterise the resulting algebra axiomatically. One of the axioms will be commutativity.
(3) Drop the commutativity axiom.

Classical probability. Let us apply this strategy to the structure of a probability space.

We remind the reader that a probability space is a triple $(\Omega, \Sigma, \mathbb{P})$, where Ω is a set, Σ is a σ-algebra of subsets of Ω, containing Ω itself, and $\mathbb{P}$ is a σ-additive function $\Sigma \to [0,1]$ with the property that $\mathbb{P}(\Omega) = 1$ ([Kol]).

In applications Ω is interpreted as the set of all possible outcomes of a certain stochastic experiment. Σ consists of 'events', statements about the outcome of the experiment that can be tested by observation. When E is such an event, then $\mathbb{P}(E)$ is the probability that E will occur.

Applying the strategy. **Step 1**. We choose to consider the algebra $L^\infty(\Omega, \Sigma, \mathbb{P})$, consisting of all bounded measurable functions $f\colon \Omega \to \mathbb{C}$, where two such functions f and g are identified if $f - g$ vanishes $\mathbb{P}$-almost everywhere. On this algebra we consider the linear functional φ given by

$$\varphi(f) := \int_\Omega f \, d\mathbb{P} \; .$$

We have chosen measurable functions because we want to encode a measurable structure. (Had we been interested in the topological structure, we would have chosen an algebra of continuous functions.)

Bounded functions are an appropriate choice since they allow for unlimited multiplication of elements, while keeping φ well-defined.

The identification of functions which are almost everywhere equal is a technical simplification, standard in integration theory.

Now we must check whether all the relevant information in $(\Omega, \Sigma, \mathbb{P})$ has been faithfully encoded. Clearly, the triple $(\Omega, \Sigma, \mathbb{P})$ determines $L^\infty(\Omega, \Sigma, \mathbb{P})$ uniquely. In the converse direction, we recover a σ-algebra $\widetilde{\Sigma}$ by putting

$$\widetilde{\Sigma} := \left\{ \, p \in L^\infty(\Omega, \Sigma, \mathbb{P}) \, \middle| \, p = p^2 = p^* \, \right\} ,$$

which, however, is not isomorphic to Σ, since we have identified functions that are equal almost everywhere. In fact, $\widetilde{\Sigma}$ is the *measure algebra*, that is the quotient of Σ by the equivalence

$$S \sim T, \text{ meaning: } \quad \mathbb{P}\big((S \setminus T) \cup (T \setminus S)\big) = 0 \; .$$

This simplification is a gain rather than a loss.

Finally the probability measure $\mathbb{P}$ is regained by putting

$$\widetilde{\mathbb{P}} : \widetilde{\Sigma} \to [0,1] : p \mapsto \varphi(p) \; .$$

Step 2. $L^\infty(\Omega, \Sigma, \mathbb{P})$ is characterised as a commutative von Neumann algebra. A few definitions should now be given.

Let $\mathcal{H}$ be a Hilbert space and let $A_1, A_2, A_3, \cdots$ be a sequence of bounded operators on $\mathcal{H}$. This sequence is said to converge to a bounded operator A *in the strong operator topology* if for all $\psi \in \mathcal{H}$:

$$\lim_{n\to\infty} \| A_n\psi - A\psi \| = 0 \ .$$

It *increases to* A if, moreover, $A_j \leq A_{j+1}$ in the sense that $A_{j+1} - A_j$ is a positive operator.

A *von Neumann algebra* $\mathcal{A}$ is an algebra of bounded operators on some Hilbert space $\mathcal{H}$ which is closed in the strong operator topology. We shall always assume that $\mathcal{A}$ contains the identity operator $\mathbf{1}$, and we only consider separable Hilbert spaces.

A *state* on a von Neumann algebra $\mathcal{A}$ is a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ mapping $\mathbf{1}$ to the number 1, and which is *positive* in the sense that $\varphi(A^*A) \geq 0$ for all $A \in \mathcal{A}$. If $\varphi(A^*A) = 0$ only for $A = 0$, then φ is called *faithful*. If $\lim_\lambda \varphi(A_\lambda) = \varphi(A)$ for every net (A_λ) of positive operators increasing to A, then φ is called *normal*.

Finally, for a bounded function f on a probability space $(\Omega, \Sigma, \mathbb{P})$ we denote by M_f the operator of multiplication by f on the Hilbert space $L^2(\Omega, \Sigma, \mathbb{P})$.

Proposition 1.1. *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. Then the algebra*

$$\mathcal{A} := \left\{ \ M_f \ \middle| \ f \in L^\infty(\Omega, \Sigma, \mathbb{P}) \ \right\}$$

is a (commutative) von Neumann algebra of operators on $L^2(\Omega, \Sigma, \mathbb{P})$, and the map $\varphi : M_f \mapsto \int f d\mathbb{P}$ is a faithful normal state on $\mathcal{A}$. Conversely, every commutative *von Neumann algebra $\mathcal{A}$ with a faithful normal state φ is of the above form for some classical probability space.*

Proof. We only prove the first part of the theorem. The point is to show that $\mathcal{A}$ is strongly closed. So let (f_λ) be a net of L^∞-functions such that M_{f_λ} tends strongly to some bounded operator X on $\mathcal{H} := L^2(\Omega, \Sigma, \mathbb{P})$, that is for all $\psi \in \mathcal{H}$ we have

$$L^2\text{-}\lim_\lambda f_\lambda\psi = X\psi \ .$$

Without loss of generality we may assume that $\| X \| = 1$. We must show that $X = M_f$ for some $f \in L^\infty$. Put $f := X1$. Then for all $g \in L^\infty$ we have

$$Xg = L^2\text{-}\lim_\lambda f_\lambda g = L^2\text{-}\lim_\lambda M_g f_\lambda = M_g \left(L^2\text{-}\lim_\lambda f_\lambda \cdot 1 \right) = M_g X1 = M_g f = fg \ .$$

Now let the event E_ε for $\varepsilon > 0$ be defined by

$$E_\varepsilon := \left\{ \ \omega \in \Omega \ \middle| \ |f(\omega)|^2 \geq 1 + \varepsilon \ \right\} \ .$$

Then, since $\| X \| \leq 1$,

$$\mathbb{P}(E_\varepsilon) = \| 1_{E_\varepsilon} \|^2 \geq \| X1_{E_\varepsilon} \|^2 = \| f1_{E_\varepsilon} \|^2 = \int_{E_\varepsilon} |f|^2 d\mathbb{P} \ \geq (1+\varepsilon)\mathbb{P}(E_\varepsilon) \ ,$$

and it follows that $\mathbb{P}(E_\varepsilon) = 0$. Since this holds for all $\varepsilon > 0$, we have $|f| \leq 1$ almost everywhere with respect to $\mathbb{P}$. So $f \in L^\infty(\Omega, \Sigma, \mathbb{P})$. Finally, since the operators X and M_f are both bounded and coincide on the dense subspace L^∞ of $\mathcal{H}$, they are equal. $\square$

Step 3. We now drop the commutativity requirement to arrive at the following definition.

Definition. By a *noncommutative probability space* we mean a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a von Neumann algebra of operators on some Hilbert space $\mathcal{H}$, and φ is a normal state on $\mathcal{A}$. If φ is faithful, the probability space is called *non-degenerate*.

Events and random variables. Let us carry some important concepts of probability theory over from the structure $(\Omega, \Sigma, \mathbb{P})$ to the generalised probability space $(\mathcal{A}, \varphi)$. (Comparable discussions are found in [Mac], [KüM], [Par], [Mey].)

Classically an event is an element S of Σ. In Step 1 of the preceeding section this is replaced by the projection 1_S in $L^\infty(\Omega, \Sigma, \mathbb{P})$. In Step 3 the concept of an event is generalised to that of an arbitrary orthogonal projection in $\mathcal{A}$, that is an element satisfying $E^2 = E = E^*$. The state φ associates to this event the probability $\varphi(E)$. The operator 0 is the impossible event, and $\mathbf{1}$ is the sure event. Two events E and F are called *compatible* if EF is also an event, equivalently E and F commute:

$$EF = (EF)^* \Leftrightarrow EF(= (EF)^*) = FE \Rightarrow (EF)^2 = E^2F^2 = EF \ .$$

If this is the case, then the event EF stands for the occurrence of both E and F and the event $E \vee F := E + F - EF$ for the occurrence of either E or F or both. If $EF = 0$, then the occurrences of E and F exclude each other. So mutually exclusive events are described by orthogonal subspaces of $\mathcal{H}$.

A classical random variable on a probability space $(\Omega, \Sigma, \mathbb{P})$ is a measurable function X from Ω to some other measure space (Ω', Σ'). Such a function induces an embedding of Σ' into Σ given by

$$J_X : S \mapsto X^{-1}(S),$$

containing the same information as X itself. The probability distribution of X is given by

$$\mathbb{P}_X := \mathbb{P} \circ J_X : S \mapsto \mathbb{P}(X^{-1}(S)) \ .$$

Let us see what our program does with this structure. In Step 1 the embedding J_X is replaced by the mapping

$$j_X : L^\infty(\Omega', \Sigma', \mathbb{P}_X) \to L^\infty(\Omega, \Sigma, \mathbb{P}) : f \mapsto f \circ X \ ,$$

the natural extension of the map $1_S \mapsto E(S) := 1_{X^{-1}(S)}$ to an (injective) *-homomorphism $L^\infty(\Omega', \Sigma', \mathbb{P}_X) \to L^\infty(\Omega, \Sigma, \mathbb{P})$. Still the projection $E(S)$ stands for the classical event $X^{-1}(S)$ that the random variable X takes a value in $S \in \Sigma'$.

In Step 3 this is now generalised to the following notion.

Definition. By a *generalised random variable* on a noncommutative probability space $(\mathcal{A}, \varphi)$ we mean a *-homomorphism from some other von Neumann algebra $\mathcal{B}$ into $\mathcal{A}$ mapping $\mathbf{1}_\mathcal{B}$ to $\mathbf{1}_\mathcal{A}$. The *probability distribution* of j is the state $\psi := \varphi \circ j$ on $\mathcal{B}$.

We denote this state of affairs briefly by

$$j : (\mathcal{B}, \psi) \to (\mathcal{A}, \varphi) \ .$$

If $\mathcal{B}$ is commutative, say $(\mathcal{B}, \psi) = L^\infty(\Omega', \Sigma', \mathbb{P}')$, then the random variable j is said to *take values in* Ω', and j can be written

$$j(f) = \int_{\Omega'} f(\lambda) E(d\lambda) \ ,$$

where E denotes the projection-valued measure given by

$$E(S) := j(1_S), \quad (S \in \Sigma').$$

In the particular case that $\Omega' = \mathbb{R}$, j determines a unique self-adjoint operator on the representation space $\mathcal{H}$ of $\mathcal{A}$:

Theorem 1.2 (Spectral Theorem, von Neumann)**.** *There is a one-to-one correspondence between self-adjoint operators A on a Hilbert space $\mathcal{H}$ and projection-valued measures $E : \Sigma(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ such that*

$$A = \int_{\mathbb{R}} \lambda E(d\lambda) \ .$$

When $E(S) \in \mathcal{A}$ for all S in the Borel σ-algebra $\Sigma(\mathbb{R})$, then A is said to be *affiliated* to $\mathcal{A}$. Moreover:

Theorem 1.3 (Stone's Theorem)**.** *There is a one-to-one correspondence between strongly continuous unitary representations $t \mapsto U_t$ of the abelian group $\mathbb{R}$ into $\mathcal{A}$ and self-adjoint operators A affiliated to $\mathcal{A}$ such that*

$$U_t = e^{itA} \ .$$

Here the right hand side is to be read as the strongly convergent integral

$$e^{itA} := \int_{\mathbb{R}} e^{it\lambda} E(d\lambda) \ ,$$

where E is given by the spectral theorem. If we put $e_t : \mathbb{R} \to \mathbb{C} : x \mapsto e^{itx}$, then the connection with j can be written

$$j(e_t) = e^{itA} \ .$$

So altogether we can characterise a real-valued random variable or *observable* in any one of four ways:

(1) by a self-adjoint operator A affiliated to $\mathcal{A}$;
(2) by a projection-valued measure E in $\mathcal{A}$;
(3) by a normal injective *-homomorphism $j : L^\infty(\mathbb{R}, \Sigma(\mathbb{R}), \mathbb{P}) \to \mathcal{A}$; and
(4) by a one-parameter unitary group $(U_t)_{t \in \mathbb{R}}$ in $\mathcal{A}$.

Interpretation of quantum probability. It is a surprising fact that nature — at least on small scales — appears to be governed by noncommutative probability.

Quantum probability describes manipulations performed on physical systems by certain mappings between generalised probability spaces called *operations*. (The same has been said about classical probability see, e.g. [Kam].) These mappings will be treated in some detail in Section 3. The generalised random variable which we just saw is such a mapping. It represents the operation of restricting attention to a subsystem. Another such mapping is the conditional expectation, describing the

immersion of a physical system into a larger one. Yet other operations are the time evolution and the transition operator: they represent the act of waiting for some time while the system evolves on its own, or in interaction with something else respectively.

At the end of a chain of operations we land in some probability space $(\mathcal{A}, \varphi)$, and we need a way to interpret it in terms of the outcome of the physical experiment which is being described. The rules for interpretation are as follows.

- Some of the orthogonal projections in $\mathcal{A}$ have an interpretation as observable events.

In the language of Mackey ([Mac]) events may be considered as *questions*, which can be asked to the system. It answers by saying 'yes' or 'no'.

- The experiment can be repeated arbitrarily often. Each time we are free to choose new questions, that is we are allowed to adjust observation equipment without our whole experiment becoming a different one. (This is the main distinction with the stochastic experiments envisaged by Kolmogorov in [Kol].)
- Compatible questions can be asked together in the same trial.
- Incompatible questions can be asked in different trials.
- Inside one single trial it is sometimes possible to ask incompatible questions one *after* the other. The order will then influence the probabilities: if the questions $E_1, E_2, \ldots, E_k$ are asked in each trial and in this order, the asymptotic fraction of the trials in which they are all answered 'yes' is

$$\varphi(E_1 E_2 \cdots E_{k-1} E_k E_{k-1} \cdots E_2 E_1) \ .$$

Remark. It is sometimes difficult to say where the operation ends and the observation begins. For example, posing the question E after the question F can alternatively be viewed as an *operation* $l^\infty(\{0,1\} \times \{0,1\}) \to \mathcal{A}$, followed by the observation of the compatible events $\{1\} \times \{0,1\}$ and $\{0,1\} \times \{1\}$.

The quantum coin toss: 'spin'. The simplest noncommutative von Neumann algebra is M_2, the algebra of all 2×2 matrices with complex entries. And the simplest noncommutative probability space is $(M_2, \frac{1}{2}\mathrm{tr}\,)$, the 'fair quantum coin toss'.

The events in this probability space are the orthogonal projections in M_2: the complex 2×2 matrices E satisfying

$$E^2 = E = E^* \ .$$

Let us see what these projections look like. Since E is self-adjoint, it must have two real eigenvalues, and since $E^2 = E$ these must both be 0 or 1. So we have three possibilities.

(0) Both are 0; that is $E = 0$, the impossible event.
(1) One of them is 0 and the other is 1.
(2) Both are 1; that is $E = \mathbf{1}$, the sure event.

In Case (1), E is a one-dimensional projection satisfying

$$\mathrm{tr}\, E = 0 + 1 = 1 \ \text{ and } \ \det E = 0 \cdot 1 = 0 \ .$$

As $E^* = E$ and $\operatorname{tr} E = 1$ we may write

$$E = \tfrac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix} , \quad \text{with } (x,y,z) \in \mathbb{R}^3 .$$

Then $\det E = 0$ implies that

$$\tfrac{1}{4}((1-z^2)-(x^2+y^2)) = 0 \quad \Longrightarrow \quad x^2+y^2+z^2 = 1 .$$

So the one-dimensional projections in M_2 are parametrised by the unit sphere S_2.

Notation. For $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ let us write

$$\sigma(a) := \begin{bmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{bmatrix} = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 ,$$

where σ_1, σ_2 and σ_3 are the *Pauli matrices*

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

We note that for all $a, b \in \mathbb{R}^3$ we have

$$\sigma(a)\sigma(b) = \langle a, b\rangle \cdot \mathbf{1} + i\sigma(a \times b) . \tag{1.1}$$

Let us write

$$E(a) := \tfrac{1}{2}(\mathbf{1} + \sigma(a)), \quad (\| a \| = 1) . \tag{1.2}$$

In the same way the possible states on M_2 can be calculated. We find that

$$\varphi(A) = \operatorname{tr}(\rho A) \text{ where } \rho = \rho(a) := \tfrac{1}{2}(\mathbf{1} + \sigma(a)), \quad \| a \| \leq 1 . \tag{1.3}$$

The situation is summarised by the following proposition.

Proposition 1.4. *The states on M_2 are parameterised by the unit ball in $\mathbb{R}^3$, as in* (1.3)*, and the one-dimensional projections in M_2 are parametrised by the unit sphere as in* (1.2)*. The probability of the event $E(a)$ in the state $\rho(b)$ is given by*

$$\operatorname{tr}(\rho(b)E(a)) = \tfrac{1}{2}(1 + \langle a, b\rangle) .$$

The events $E(a)$ and $E(b)$ are compatible if and only if $a = \pm b$. Moreover we have for all $a \in S_2$:

$$E(a) + E(-a) = \mathbf{1} , \quad E(a)E(-a) = 0 .$$

Interpretation. The probability distribution of the quantum coin toss or *'qubit'* is given by a unit vector b in three-dimensions. For every a on the unit sphere we can say with probability one that of the two events $E(a)$ and $E(-a)$ exactly one will occur, $E(a)$ having probability $\frac{1}{2}(1 + \langle a, b\rangle)$. We therefore have, for each direction a, a classical coin toss with probability for heads equal to $\frac{1}{2}(1 + \langle a, b\rangle)$. The coin tosses in different directions are incompatible.

Particular case: the quantum fair coin is modelled by $(M_2, \frac{1}{2}\operatorname{tr})$.

The quantum coin toss is realised in nature: the spin direction of a particle with total spin $\frac{1}{2}\hbar$ behaves in this way.

Positive definite kernels. In this section we introduce a useful tool for the construction of Hilbert spaces, used heavily in quantum probability.

Let $\mathcal{S}$ be a set and let K be a *kernel* on $\mathcal{S}$, that is a function $\mathcal{S} \times \mathcal{S} \to \mathbb{C}$. Then K is called *positive definite* if for all $n \in \mathbb{N}$ and all n-tuples $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \overline{\lambda_i} \lambda_j K(x_i, x_j) \geq 0 \; .$$

Theorem 1.5 (Kolmogorov's dilation theorem)**.** *Let K be a positive definite kernel on a set $\mathcal{S}$. Then up to unitary equivalence there exists a unique Hilbert space $\mathcal{H}$ and a unique embedding $V : \mathcal{S} \to \mathcal{H}$ such that*

$$\forall_{x,y \in \mathcal{S}} : \langle V(x), V(y) \rangle = K(x, y) \tag{1.4}$$

$$\text{and } \overline{\bigvee V(\mathcal{S})} = \mathcal{H} \; . \tag{1.5}$$

A map $V : \mathcal{S} \to \mathcal{H}$ is called a (*Kolmogorov*) *dilation* if (1.4) holds. It is called *minimal* if (1.5) holds.

Proof. Consider the space $\mathcal{L}$ of all functions $\mathcal{S} \to \mathbb{C}$ with finite support. Then $\mathcal{L}$ becomes a pre-Hilbert space if we define the (pre-)inner product

$$\langle \lambda, \mu \rangle := \sum_{x \in \mathcal{S}} \sum_{y \in \mathcal{S}} \overline{\lambda(x)} K(x, y) \mu(y) \; .$$

Dividing out the null space

$$\mathcal{N} := \left\{ \, \lambda \in \mathcal{L} \, \middle| \, \langle \lambda, \lambda \rangle = 0 \, \right\}$$

and forming the completion $\mathcal{H}_K$ of $\mathcal{L}/\mathcal{N}$, we let $V_K : \mathcal{S} \to \mathcal{H}_K$ be given by

$$V_K(x) := \delta_x + \mathcal{N} \; .$$

Then for all $x, y \in \mathcal{S}$:

$$\langle V_K(x), V_K(y) \rangle = \langle \delta_x + \mathcal{N}, \delta_y + \mathcal{N} \rangle_{\mathcal{L}/\mathcal{N}} = \langle \delta_x, \delta_y \rangle_{\mathcal{L}} = K(x, y) \; .$$

Now let $V : \mathcal{S} \to \mathcal{H}$ be a second minimal Kolmogorov dilation of K. Then we define a map

$$U_0 : \mathcal{L} \to \mathcal{H} : \quad \lambda \mapsto \sum_{x \in \mathcal{S}} \lambda(x) V(x) \; .$$

This map vanishes on $\mathcal{N}$: for $\lambda \in \mathcal{N}$ we have

$$\| \, U_0 \lambda \, \|^2 = \left\| \sum_{x \in \mathcal{S}} \lambda(x) V(x) \right\|^2 = \sum_{x \in \mathcal{S}} \sum_{y \in \mathcal{S}} \overline{\lambda(x)} K(x, y) \lambda(y) = \langle \lambda, \lambda \rangle_{\mathcal{L}} = 0 \; .$$

So U_0 may be considered as a map $\mathcal{L}/\mathcal{N} \to \mathcal{H}$. By the same calculation we find that U_0 is isometric. Since $\bigvee V(\mathcal{S})$ is dense in $\mathcal{H}$ and $\bigvee V_K(\mathcal{S})$ is dense in $\mathcal{H}_K$, U_0 extends to a unitary map $U : \mathcal{H}_K \to \mathcal{H}$ mapping $V_K(x)$ to $V(x)$. □

Examples 1.6.

(a) Let $\mathcal{S}$ be any set and let $K(x, y) := \delta_{x,y}$. Then $\mathcal{H} = l^2(\mathcal{S})$ and V maps the elements of $\mathcal{S}$ to the standard orthonormal basis of $\mathcal{H}$.

(b) Let $\mathcal{S} := \mathcal{H}_1 \times \mathcal{H}_2$, the Cartesian product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Let

$$K\big((\psi_1,\psi_2),(\chi_1,\chi_2)\big) := \langle\psi_1,\chi_1\rangle \cdot \langle\psi_2,\chi_2\rangle \ .$$

Then $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$, and $V(\psi_1,\psi_2) = \psi_1 \otimes \psi_2$.

(c) Let $\mathcal{S}$ be a Hilbert space; call it $\mathcal{K}$ for the occasion. Let $K(\psi,\chi) := \langle\psi,\chi\rangle^2$. Then $\mathcal{H}$ is the *symmetric* tensor product $\mathcal{K} \otimes_{\mathrm{s}} \mathcal{K}$ and $V(\psi) = \psi \otimes \psi$.

(d) Let $\mathcal{S} := \mathcal{K}$ as in Example (c). Let K be the positive definite kernel

$$K(\psi,\chi) := e^{\langle\psi,\chi\rangle} \ .$$

Then the Kolmogorov dilation is the *Fock space* $\mathcal{F}(\mathcal{K})$ over $\mathcal{K}$, defined as

$$\mathcal{F}(\mathcal{K}) := \mathbb{C} \oplus \mathcal{K} \oplus \tfrac{1}{2}\big(\mathcal{K} \otimes_{\mathrm{s}} \mathcal{K}\big) \oplus \tfrac{1}{6}\big(\mathcal{K} \otimes_{\mathrm{s}} \mathcal{K} \otimes_{\mathrm{s}} \mathcal{K}\big) \oplus \cdots\cdots$$

and $V(\psi)$ is the so-called *exponential vector* or *coherent vector*

$$\mathrm{Exp}\,(\psi) := 1 \oplus \psi \oplus (\psi \otimes \psi) \oplus (\psi \otimes \psi \otimes \psi) \oplus \cdots$$

(e) Let $\mathcal{S} = \mathbb{R}$ and let $K : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be given by

$$K(s,t) := e^{-\eta|s-t| + i\omega(s-t)} \ , \qquad (\eta > 0, \omega \in \mathbb{R}).$$

The Kolmogorov dilation of this kernel can be cast in the form

$$\mathcal{H} = L^2(\mathbb{R}, 2\eta dx) \ ; \qquad V : t \mapsto v_t \in L^2(\mathbb{R}) : \quad v_t(x) := \begin{cases} e^{(\eta - i\omega)(x-t)} & \text{if } x \le t; \\ 0 & \text{if } x > t. \end{cases}$$

2. Some quantum mechanics

Quantum mechanics is a physical theory that fits in the framework of noncommutative probability, but which has much more structure. It deals with particles and fields, using observables like position, momentum, angular momentum, energy, charge, spin, isospin, etc. All these observables develop in time according to a certain dynamical rule, namely the Schrödinger equation.

In this section we shall pick out a few elements of this theory that are of particular interest to our main example: the damped harmonic oscillator considered as a quantum Markov chain.

Position and momentum. Let us start with a simple example: a particle on a line. This particle must have a *position* observable, a projection valued measure on the Borel σ-algebra $\Sigma(\mathbb{R})$ of the real line $\mathbb{R}$:

$$E : \Sigma(\mathbb{R}) \to \mathcal{B}(\mathcal{H}) \ .$$

The easiest choice (valid when the particle is alone in the world and has no further degrees of freedom) is

$$\mathcal{H} := L^2(\mathbb{R}) \ ;$$
$$E(S) : \psi \mapsto 1_S \cdot \psi \ .$$

In this example the Hilbert space $\mathcal{H}$ naturally carries a second real-valued random variable in the form of the group $(T_t)_{t\in\mathbb{R}}$ of spatial translations:

$$(T_t\psi)(x) := \psi(x - \hbar t) \ , \tag{2.1}$$

according to the remark following Stone's theorem (Theorem 1.3). This second observable is called the *momentum* of the particle. The constant $\hbar$ is determined by the units of length and of momentum which we choose to apply. The associated self-adjoint operators are Q and P given by

$$(Q\psi)(x) = x\psi(x) \; ;$$
$$(P\psi)(x) = -i\hbar \frac{\partial}{\partial x}\psi(x) \; . \tag{2.2}$$

Just as we have $T_t = e^{-itP}$, it is natural to introduce $S_s := e^{isQ}$ whose action on $\mathcal{H}$ is

$$S_s\psi(x) := e^{isx}\psi(x) \; . \tag{2.3}$$

The operators P and Q satisfy Heisenberg's *canonical commutation relation* (CCR)

$$[P, Q] = -i\hbar \cdot \mathbf{1} \; . \tag{2.4}$$

A pair of self-adjoint operators (P, Q) satisfying (2.4) is called a *canonical pair*.

Representations of the canonical commutation relations. What kinds of canonical pairs are there?

Before this question can be answered, it has to be reformulated. Relation (2.4) is not satisfactory as a definition of a canonical pair since the domains on the left and on the right are not the same. Worse than that, quite pathological examples can be constructed, even if (2.4) is postulated to hold on a dense stable domain, with the property that P and Q only admit unique self-adjoint extensions ([ReS]).

In order to circumvent such domain complications, Weyl proposed to replace (2.4) by a relation between the associated unitary groups (T_t) and (S_s), namely:

$$T_tS_s = e^{-i\hbar st}S_sT_t \; , \qquad (s, t \in \mathbb{R}) \; . \tag{2.5}$$

It was von Neumann's idea to combine the two into a two-parameter family

$$W(t, s) := e^{\frac{i\hbar}{2}st}T_tS_s \; , \tag{2.6}$$

forming a 'twisted' representation of $\mathbb{R}^2$, as expressed by the *Weyl relation*: for all $s, t, u, v \in \mathbb{R}$,

$$W(t, s)W(u, v) = e^{-\frac{i\hbar}{2}(tv-su)}W(t + u, s + v) \; . \tag{2.7}$$

This relation captures the group property of T_t and S_s together with the relation (2.5). Formally,

$$W(t, s) = e^{i(sQ-tP)} \; .$$

We shall call the representation on $L^2(\mathbb{R})$ of the CCR given by (2.1), (2.2), (2.3) and (2.6) the *standard representation* of the CCR.

Here and in the rest of the text we shall follow the quantum probabilist's convention ([Mey]), namely that

$$\hbar = 2 \; .$$

Theorem 2.1 (von Neumann's Uniqueness Theorem)**.** *Let $\big(W(t, s)\big)_{t,s\in\mathbb{R}}$ be a strongly continuous family of unitary operators on some Hilbert space $\mathcal{H}$ satisfying the Weyl relation* (2.7)*. Then $\mathcal{H}$ is unitarily equivalent with $L^2(\mathbb{R}) \otimes \mathcal{K}$, such that $W(t, s)$ corresponds to $W_S(t, s) \otimes \mathbf{1}$, where W_S is the standard representation of the CCR.*

Proof. Let $W : \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ satisfy the Weyl relation (2.7). For each integrable function $f : \mathbb{R}^2 \to \mathbb{C}$ with $\int\int |f(t,s)|dt\,ds < \infty$, define a bounded operator $A(f)$ on $\mathcal{H}$ by the strong sense integral

$$A(f) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,s)W(t,s)dt\,ds\ .$$

We find the following calculating rules for such operators $A(f)$ and their *kernels* f:

$$\begin{aligned} A(f)+A(g) &= A(f+g)\ ; \\ A(f)^* &= A(\widetilde{f}),\ \text{where } \widetilde{f}(t,s) := \overline{f(-t,-s)}\ ; \\ A(f)A(g) &= A(f*g)\ . \end{aligned}$$

Here the 'twisted convolution product' $*$ is defined by

$$(f*g)(t,s) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(tv-su)} f(t-u,s-v)g(u,v)du\,dv\ .$$

Moreover we claim that an operator (on a nontrivial Hilbert space) can have at most one kernel:

$$A(f)=0 \quad\Longrightarrow\quad \mathcal{H}=\{0\} \text{ or } f=0\ . \tag{2.8}$$

Indeed, if $A(f)=0$ then we have for all $a,b\in\mathbb{R}$,

$$0 = W(a,b)^*A(f)W(a,b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2i(as-bt)} f(t,s)W(t,s)dt\,ds\ .$$

Applying the linear functional $A \mapsto \langle\varphi, A\psi\rangle$ with $\varphi,\psi\in\mathcal{H}$ to both sides of this equation, we find that for all $\varphi,\psi\in\mathcal{H}$ the (integrable) function

$$(t,s)\mapsto f(t,s)\langle\varphi, W(t,s)\psi\rangle$$

has Fourier transform 0. By the separability of $\mathcal{H}$, either $W(t,s)=0$ for some (t,s), (that is $\mathcal{H}=\{0\}$), or $f(t,s)=0$ for almost all (t,s).

The key to the proof of uniqueness is the operator

$$E := \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(t^2+s^2\right)}W(t,s)dt\,ds\ .$$

It has the remarkable property that for all $a,b\in\mathbb{R}$, $EW(t,s)E$ is a scalar multiple of E:

$$EW(a,b)E = e^{-\frac{1}{2}\left(a^2+b^2\right)}E\ . \tag{2.9}$$

Indeed, E has kernel $g(t,s) := \frac{1}{\pi}e^{-\frac{1}{2}\left(t^2+s^2\right)}$, and the product $W(a,b)E$ has kernel

$$h(u,v) := \frac{1}{\pi}e^{-i(av-bu)}\cdot e^{-\frac{1}{2}\left((a-u)^2+(b-v)^2\right)}\ .$$

So $EW(a,b)E$ has kernel

$$\begin{aligned}
(g*h)(t,s) &= \int\int e^{-i(tv-su)}g(t-u,s-v)h(u,v)du\,dv \\
&= \frac{1}{\pi^2}\int\int e^{-i(tv-su)}e^{-\frac{1}{2}\left((t-u)^2+(s-v)^2\right)}e^{-i(av-bu)}e^{-\frac{1}{2}\left((a-u)^2+(b-v)^2\right)}du\,dv \\
&= \frac{1}{\pi^2}e^{-\frac{1}{2}\left(a^2+b^2\right)}e^{-\frac{1}{2}\left(t^2+s^2\right)}\int\int e^{-(u^2+v^2)}e^{(u-iv)(t+is+a+ib)}du\,dv \\
&= \frac{1}{\pi^2}e^{-\frac{1}{2}\left(a^2+b^2\right)}e^{-\frac{1}{2}\left(t^2+s^2\right)}\int\int e^{-\left(u-\frac{1}{2}(t+is+a+ib)\right)^2-\left(v-\frac{i}{2}(t+is+a+ib)\right)^2}du\,dv \\
&= \frac{1}{\pi}e^{-\frac{1}{2}\left(a^2+b^2\right)}\cdot e^{-\frac{1}{2}\left(t^2+s^2\right)} \\
&= e^{-\frac{1}{2}\left(a^2+b^2\right)}g(t,s)\ ,
\end{aligned}$$

which proves (2.9).

We conclude that $E^* = E$ (since $\widetilde{g} = g$), $E^2 = E$ (putting $a = b = 0$ in (2.9)), and that $E\mathcal{A}E = \mathbb{C}E$, where $\mathcal{A}$ is the von Neumann algebra generated by the Weyl operators. So E is a minimal projection in $\mathcal{A}$. Denote its range by $\mathcal{K}$. Then we have for all $\varphi,\psi\in\mathcal{K}$ and all $t,s,u,v\in\mathbb{R}$:

$$\begin{aligned}
\langle W(t,s)\varphi, W(u,v)\psi\rangle &= \langle W(t,s)E\varphi, W(u,v)E\psi\rangle \\
&= \langle\varphi, EW(-t,-s)W(u,v)E\psi\rangle \\
&= e^{i(tv-su)}\langle\varphi, EW(u-t,v-s)E\psi\rangle \\
&= e^{i(tv-su)}e^{-\frac{1}{2}\left((u-t)^2+(v-s)^2\right)}\langle\varphi, E\psi\rangle \\
&= e^{(t-is)(u+iv)}e^{-\frac{1}{2}(t^2+s^2+u^2+v^2)}\langle\varphi,\psi\rangle\ ,
\end{aligned}$$

Therefore the map

$$V:\mathbb{R}^2\times\mathcal{K}\to\mathcal{H}:\big((t,s),\varphi\big)\mapsto e^{\frac{1}{2}(t^2+s^2)}W(t,s)\varphi$$

is a Kolmogorov dilation (cf. Section 1) of the positive definite kernel

$$K:(\mathbb{R}^2\times\mathcal{K})\times(\mathbb{R}^2\times\mathcal{K})\to\mathbb{C},\quad \big((t,s),\varphi;(u,v),\psi\big)\mapsto e^{(t-is)(u+iv)}\langle\varphi,\psi\rangle\ . \tag{2.10}$$

By explicit calculation you will find that E_S is the orthogonal projection onto the one-dimensional subspace spanned by the unit vector $\Omega(x) := \sqrt{\gamma(x)}$, where

$$\gamma(x) := \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\ .$$

So in the standard case the dilation is

$$V_S:\mathbb{R}^2\to L^2(\mathbb{R}):(t,s)\mapsto e^{\frac{1}{2}(t^2+s^2)}e^{-\frac{i}{2}ts}e^{isx}\Omega(x-2t)\ .$$

By Kolmogorov's Dilation Theorem, there exists a unitary equivalence $U: L^2(\mathbb{R})\otimes\mathcal{K}\to\mathcal{H}$ such that for all $a,b\in\mathbb{R}$ and $\psi\in\mathcal{K}$:

$$U\big(W_S(a,b)\Omega\otimes\psi\big) = W(a,b)\psi\ ,$$

and therefore for all $a,b\in\mathbb{R}$:

$$W(a,b) = U\big(W_S(a,b)\otimes\mathbf{1}\big)U^{-1}\ ,$$

provided that the range of V is dense in $\mathcal{H}$. Let $\mathcal{L}$ denote the orthogonal complement of this range. Then $\mathcal{L}$ is invariant for the Weyl operators; let $W_0(t,s)$ be the restriction of $W(t,s)$ to $\mathcal{L}$. Construct $E_0 := A_0(g)$ in terms of W_0 in the same way as E was constructed from W. Then clearly $E_0 \leq E$, but also $E_0 \perp E$. So $E_0 = A_0(g) = 0$ and by (2.8) we have $\mathcal{L} = \{0\}$. □

Exercise. Calculate the minimal projection E_S in the standard representation.

Energy and time evolution. The evolution in time of a closed quantum system is given by a pointwise strongly continuous one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of *-automorphisms of the observable algebra $\mathcal{A}$.

Like in the case of a particle on a line, for a finite number n of (distinguishable) particles in d-dimensional space we take $\mathcal{A} = \mathcal{B}(\mathcal{H})$ with $\mathcal{H} = L^2(\mathbb{R}^{nd})$. Since all automorphisms of this algebra are implemented by unitary transformations of $\mathcal{H}$, the group (α_t) is of the form

$$\alpha_t(A) = U_t A U_t^{-1} .$$

It is possible to choose the unitaries so that $t \mapsto U_t$ is a strongly continuous unitary representation $\mathbb{R} \to \mathcal{U}(\mathcal{H})$. We denote its Stone generator by $H/\hbar$:

$$U_t = e^{itH/\hbar} .$$

The self-adjoint operator H corresponds to an observable of the system of particles, called its *energy*. The operator H itself is known as the *Hamilton operator* or *Hamiltonian* of the system. As the Hamiltonian commutes with the time evolution operators, energy is a conserved quantity:

$$\alpha_t(H) = U_t H U_t^{-1} = H .$$

The nature of a physical system is characterised by its *dynamical law* (a term of Hermann Weyl, see [Wey]). This is an equation which expresses the Hamiltonian in terms of other observables. For n interacting particles in $\mathbb{R}^d$ in the absence of magnetic fields the dynamical law takes the form

$$H = \sum_{j=1}^{nd} \frac{1}{2m_{k(j)}} P_j^2 + V(Q_1, Q_2, \ldots, Q_{nd})$$

for some function $V : (\mathbb{R}^d)^n \to \mathbb{R}$, called the *potential*. The positive constants m_k, $k = 1, \cdots, n$ are the *masses* of the particles. (Incidentally we put $k(j) := 1 + [(j-1)/d]$, where [ ] denotes integer part, in order to attach the same mass to the coordinates of the same particle.)

Free particles. If $V = 0$, then U_t factorises into a tensor product of nd one-dimensional evolution operators, all of the form

$$U_t = e^{itH/\hbar} = e^{i\frac{t}{2m\hbar}P^2} .$$

Since the Hamiltonian $H = P^2/2m$ now commutes with P, momentum is conserved:

$$\alpha_t(P) = P .$$

On a formal level the time development of the operator Q is found by solving the differential equation

$$\frac{d}{dt}\alpha_t(Q) = \frac{d}{dt}U_t Q U_t^{-1} = \frac{i}{2m\hbar}[P^2, \alpha_t(Q)], \tag{2.11}$$

a solution of which is

$$\alpha_t(Q) = Q + \frac{t}{m}P \ .$$

According to the Uniqueness Theorem the canonical pairs (P, Q) and $(P, Q + \frac{t}{m}P)$ are indeed unitarily equivalent. So we expect the evolution of the Weyl operators to be the following:

$$\begin{aligned}\alpha_t\big(W(x,y)\big) = \alpha_t\left(e^{-ixP+iyQ}\right) &= e^{-ixP+iy(Q+\frac{t}{m}P)} \\ &= e^{-i(x-\frac{t}{m}y)P+iyQ} = W\left(x - \frac{t}{m}y\ , y\right) \ .\end{aligned}$$

Proposition 2.2. *Let $P := -i\hbar\frac{\partial}{\partial x}$ denote the momentum operator on $\mathcal{H} := L^2(\mathbb{R})$, and let $W : \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ be given by* (2.7)*. Let*

$$U_t := e^{i\frac{t}{2m\hbar}P^2} \ .$$

Then

$$U_t W(x,y) U_t^{-1} = W\left(x - \frac{t}{m}y\ , y\right) \ .$$

Proof. From the definitions of T_t and E_Q it follows that for all measurable sets $B \subset \mathbb{R}$ and all $t \in \mathbb{R}$:

$$T_t E_Q(B) T_t^{-1} = E_Q(B + \hbar t) \ .$$

By the uniqueness theorem irreducible representations of the CCR have the symmetry $Q \to P$, $P \to -Q$. So we also have the exchanged imprimitivity relation

$$\forall_{B\in\Sigma(\mathbb{R})}\forall_{y\in\mathbb{R}} : \quad S_y E_P(B) S_y^{-1} = E_P(B + \hbar y) \ .$$

Hence for all $y, t \in \mathbb{R}$,

$$\begin{aligned}S_y U_{-t} S_y^{-1} &= S_y\left(\int_{-\infty}^{\infty} e^{-i\frac{t}{2m\hbar}\lambda^2} E_P(d\lambda)\right) S_y^{-1} \\ &= \int_{-\infty}^{\infty} e^{-i\frac{t}{2m\hbar}(\lambda-\hbar y)^2} E_P(d\lambda) \\ &= U_{-t}\cdot T_{-\frac{t}{m}y}\cdot e^{-i\frac{t\hbar}{2m}y^2} \ .\end{aligned}$$

Multiplying by U_t on the left and by S_y on the right we find

$$U_t W(0,y) U_t^{-1} = U_t S_y U_t^{-1} = e^{-i\frac{t\hbar}{2m}y^2} T_{-\frac{t}{m}y} S_y = W\left(-\frac{t}{m}y\ , y\right) \ .$$

As T_x commutes with U_t we may freely add $(x, 0)$ to the argument of W, and the proposition is proved. □

By imposing some state φ on $\mathcal{A} = \mathcal{B}(L^2(\mathbb{R}))$, all stochastic information on the model $(\mathcal{A}, \varphi, \alpha_t)$ can be obtained from the evolution equation $\alpha_t(Q) = Q + \frac{t}{m}P$. For example, at large times t the random variable $\frac{1}{t}\alpha_t(Q)$ approaches $\frac{1}{m}P$ in distribution, provided that φ does not favour large Q values too much. So a position measurement at a late time can serve as a measurement of momentum at time 0. This puts into perspective the well-known uncertainty principle for position and momentum at equal times.

The Schrödinger picture and the Schrödinger equation. The type of description of a system given so far, namely with random variables moving in time, and the state φ given once and for all, is called the *Heisenberg picture* of quantum mechanics. In probability theory this is common usage, and we shall adopt it also in quantum probability.

However, quantum mechanics is often thought of in a different way, where one lets the state move, and keeps the operators fixed. This is close to Schrödinger's 'wave mechanics', and is therefore called the *Schrödinger picture*:

If we take for φ a pure (that is extremal) state on the algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is, say, $L^2(\mathbb{R}^{nd})$:

$$\varphi(A) = \langle \psi, A\psi \rangle \ , \qquad (\psi \in \mathcal{H}, \| \psi \| = 1),$$

then we can express all probabilities at later times t in terms of the *wave function*

$$\psi(x_1, \ldots, x_{nd}; t) := (U_t^{-1}\psi)(x_1, \ldots, x_{nd}) \ .$$

This wave function satisfies the *Schrödinger equation*, a partial differential equation reflecting the dynamical law:

$$\begin{aligned} &- i\hbar \frac{\partial}{\partial t}\psi(x_1, \ldots, x_{nd}; t) \\ &= \sum_{j=1}^{nd} -\frac{1}{2m_{k(j)}\hbar^2}\frac{\partial^2}{\partial x_j^2}\psi(x_1, \ldots, x_{nd}; t) + V(x_1, \ldots, x_{nd})\psi(x_1, \ldots, x_{nd}; t) \ . \end{aligned}$$

If E is an orthogonal projection in $\mathcal{H}$, then the probability of the associated event can be calculated in the Schrödinger picture by

$$\begin{aligned} \varphi(\alpha_t(E)) &= \langle \psi, U_t E U_t^{-1}\psi \rangle = \langle U_t^{-1}\psi, E U_t^{-1}\psi \rangle \\ &= \int_{\mathbb{R}^{nd}} \overline{\psi_t(x_1, \ldots, x_{nd})}\big(E\psi_t\big)(x_1, \ldots, x_{nd}) dx_1, \ldots, dx_{nd} \ . \end{aligned}$$

The harmonic oscillator. A harmonic oscillator is a canonical pair (Q, P) of observables that under time evolution $(\alpha_t)_{t\in\mathbb{R}}$ performs a rotation such as

$$\begin{aligned} \alpha_t(Q) &= Q\cos t + P \sin t \ ; \\ \alpha_t(P) &= -Q \sin t + P \cos t \ . \end{aligned}$$

Since rotation in the plane is symplectic (preserves the area two-form), this evolution respects the canonical commutation relation $QP - PQ = i\hbar \cdot \mathbf{1}$. So by the Uniqueness Theorem it determines (up to a time-dependent phase) a group of unitary transformations $(U_t)_{t\in\mathbb{R}}$ of the Hilbert space on which it is represented. (For example, $U_{\frac{\pi}{2}}$ is a

unitary transformation of $L^2(\mathbb{R})$ that sends Q into P and P into $-Q$ in the standard representation (2.2). This is the Fourier transform.)

Making a formal calculation as in (2.11) by differentiating the equality

$$\alpha_t(A) = e^{itH/\hbar} A e^{-itH/\hbar}$$

we find that a Hamiltonian of the form

$$H = \tfrac{1}{2}(P^2 + Q^2) \tag{2.12}$$

can be expected to generate such a rotating evolution.

The textbook treatment of the harmonic oscillator (e.g. [Han]), follows the elegant algebraic reasoning of Dirac, who rewrote the Hamiltonian (2.12) as

$$H = \tfrac{1}{2}(Q - iP)(Q + iP) + \tfrac{1}{2}i[P, Q] =: \hbar a^* a + \tfrac{1}{2}\hbar \cdot \mathbf{1} \ .$$

The operators a and a^* are then seen to lower and raise the eigenvalue of H, and are called the *annihilation* and *creation* operators.

Here we choose to proceed more analytically, seizing the opportunity to introduce techniques which will be useful again later on for the treatment of free quantum fields and the damped oscillator.

Our goal is to describe H and U_t explicitly.

Heisenberg's matrix representation. First we note that, since α_t has period 2π, the differences between spectral points of H must be multiples of $\hbar$. On the grounds of (2.12) we suspect that H is bounded from below, so let us try

$$\mathrm{sp}(H) = \hbar\mathbb{N} + c \ .$$

We take as our Hilbert space $\mathcal{H}_H := l^2\left(\mathbb{N}, \frac{1}{n!}\right)$ with the Hamiltonian given by

$$(H\vartheta)(n) = (\hbar n + c)\vartheta(n) \ .$$

The subscript 'H' indicates that on this space we wish to stage matrix mechanics of the Heisenberg type. If we define on $\mathcal{H}_H$ the 'product' or 'coherent' vectors

$$\pi(z) := (1, z, z^2, z^3, \cdots), \quad (z \in \mathbb{C}) \ ,$$

then our intended time evolution takes the form

$$U_t^H \pi(z) = e^{itc/\hbar} \pi\left(e^{it} z\right) \ . \tag{2.13}$$

Now we want to represent a canonical pair (P, Q) in this space, or equivalently, Weyl operators $W(z)$, that rotate in the same way: $U_t W(z) U_t^{-1} = W(e^{it} z)$. We note that

$$\langle \pi(u), \pi(v) \rangle = \sum_{n=0}^{\infty} \frac{\overline{u}^n v^n}{n!} = e^{\overline{u}v} \ ,$$

so that we have here another dilation of the positive definite kernel (2.10) used in the proof of the Uniqueness Theorem. An irredicible representation of the CCR is close at hand. Put:

$$W_H(z)\pi(u) = e^{-\overline{z}u - \frac{1}{2}|z|^2} \pi(u + z) \ , \quad (z, u \in \mathbb{C}) \ .$$

These operators satisfy the Weyl relation

$$W_H(w) W_H(z) = e^{-i\mathrm{Im}\,(\overline{w}z)} W_H(w + z) \ , \tag{2.14}$$

the same as (2.7) if we identify $W(t,s)$ with $W_H(t+is)$. Clearly we have also obtained

$$U_t W(z) U_t^{-1} = W\left(e^{it} z\right) \ . \tag{2.15}$$

Let us summarise, again replacing $\hbar$ by 2.

Proposition 2.3. *The Heisenberg representation of the Harmonic oscillator is given by*

$$\begin{aligned} \mathcal{H}_H &= l^2(\mathbb{N}, \frac{1}{n!}) \ ; \\ (H_H \vartheta)(n) &= (2n+1)\vartheta(n); \qquad U_t^H \pi(z) = e^{\frac{i}{2}t} \pi(e^{it} z) \ ; \\ W_H(z)\pi(u) &= e^{-\overline{z}u - \frac{1}{2}|z|^2} \pi(u+z) \ . \end{aligned}$$

In concrete terms, on the standard orthonormal basis,

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \dots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad P = \frac{1}{i} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ -1 & 0 & \sqrt{2} & 0 & 0 & \dots \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & 0 & \dots \\ 0 & 0 & -\sqrt{3} & 0 & \sqrt{4} & \dots \\ 0 & 0 & 0 & -\sqrt{4} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

These matrices satisfy

$$QP - PQ = 2i \cdot \mathbf{1} \text{ and } \tfrac{1}{2}(Q^2 + P^2) = H \ ,$$

where

$$H = \begin{bmatrix} 1 & & & & & \\ & 3 & & & \emptyset & \\ & & 5 & & & \\ & & & 7 & & \\ & \emptyset & & & 9 & \\ & & & & & \ddots \end{bmatrix} .$$

Proof. It only remains to check the matrices for Q and P. We note that

$$e^{iyQ} \pi(u) = W_H(iy)\pi(u) = e^{iyu - \frac{1}{2}y^2} \pi(u+iy) \ ,$$

and we find by differentiation

$$Q\pi(u) = u\pi(u) + \pi'(u) \ .$$

Taking the coefficient of u^n the matrix of Q is found. The matrix for P is found in the same way. The choice of the ground state energy $c = \frac{1}{2}\hbar = 1$ in the definition of H fixes the relation with Q and P correctly. □

The Gaussian representation. Here is another useful representation of the harmonic oscillator algebra on a Hilbert space.

Let $\mathcal{H}_G := L^2(\mathbb{R}, \gamma)$, where γ is the standard Gauss measure on $\mathbb{R}$:

$$\gamma(dx) := \gamma(x)dx := \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx \ .$$

Define for $z \in \mathbb{C}$ the vector $\varepsilon(z)$ by

$$\varepsilon(z) : x \mapsto e^{zx - \frac{1}{2}z^2} \ .$$

Then $\varepsilon(z)$ with $z \in \mathbb{C}$ is a total set in $\mathcal{H}_G$. (Actually, $z \in i\mathbb{R}$ is already sufficient by the uniqueness of the Fourier transform.) Again we find

$$\langle \varepsilon(z), \varepsilon(u) \rangle = e^{\overline{z}u} \ , \qquad (z, u \in \mathbb{C}) \ .$$

Proposition 2.4. *There exists a unitary map $U_{HG} : \mathcal{H}_H \to \mathcal{H}_G$ such that for all $z \in \mathbb{C}$*

$$U_{HG}\, \pi(z) = \varepsilon(z) \ .$$

This map sends the vector $e_n := (0, \cdots, 0, 1, 0, \cdots)$ into the n-th Hermite polynomial, where these polynomials are given by the generating function

$$\sum_{n=0}^{\infty} z^n h_n(x) = e^{zx - \frac{1}{2}z^2} \ .$$

Consequently, this version of the Hermite polynomials satisfies

$$\int_{-\infty}^{\infty} h_n(x) h_m(x)\, \gamma(dx) = \frac{1}{n!}\delta_{nm} \ .$$

Proof. The map $\pi(z) \mapsto \varepsilon(z)$ extends to a unitary map since the linear spans of the ranges of π and ε are dense and both π and ε are minimal dilations of the positive definite kernel $(z, u) \mapsto e^{\overline{z}u}$. □

Let us carry over the relevant operators with this unitary transformation. We find:

$$\left(e^{isQ_G}\psi\right)(x) = e^{isx}\psi(x), \quad (Q_G\psi)(x) = x\psi(x) \ ;$$

$$\left(e^{-itP_G}\psi\right)(x) = \psi(x - 2t)\left(\frac{\gamma(x-2t)}{\gamma(x)}\right)^{1/2}, \quad (P_G\psi)(x) = ix\psi(x) - 2\psi'(x) \ ;$$

$$H_G\psi = (2N_G + \mathbf{1})\psi = -2\frac{\partial^2}{\partial x^2}\psi + 2x\frac{\partial}{\partial x}\psi + \psi \ .$$

The Schrödinger representation. Finally we get to the standard Schrödinger representation (2.1), (2.3) and (2.6) of the harmonic oscillator by dividing away a factor $\sqrt{\gamma(x)}$. Let $\mathcal{H}_S := L^2(\mathbb{R})$ and define

$$U_{GS} : \mathcal{H}_G \to \mathcal{H}_S : (U_{GS}\psi)(x) := \sqrt{\gamma(x)}\psi(x) \ .$$

The problem of damping. A damped harmonic oscillator is an evolution $(T_t)_{t\geq 0}$ on the real-linear span of a canonical pair (P, Q) that has the form

$$\begin{aligned} T_t(Q) &= e^{-\eta t}\left(Q\cos\omega t + P\sin\omega t\right) \ , \\ T_t(P) &= e^{-\eta t}\left(-Q\sin\omega t + P\cos\omega t\right) \ , \qquad (\eta > 0) \ . \end{aligned} \tag{2.16}$$

(We apologise for a clash of notation: T_t is not related to translations.) This spiralling motion in the plane compresses areas by a factor $e^{-2\eta t}$, so that for $t > 0$ the operators $T_t(Q)$ and $T_t(P)$ disobey the canonical commutation relation, and T_t cannot be extended to an automorphism of $\mathcal{B}(\mathcal{H})$.

Yet this damped oscillatory behaviour occurs in nature, for instance when an atom is loosing its energy to its surroundings by emission of light. So it would be worth while to make sense of it. There are two basic questions related to this model.

Question 1. How should T_t be extended to $\mathcal{B}(\mathcal{H})$?
Question 2. Can $(T_t)_{t\geq 0}$ be explained as part of a larger whole that evolves by *-automorphisms of the form $a_t(A) = U_t A U_t^{-1}$, where U_t^{-1} satisfies a Schrödinger equation?

Spirals and jumps. In Heisenberg's matrix mechanics atoms were supposed to move in a mixture of two ways. Most of the time they were thought to rotate according to the evolution U_t^H as described above, but occasionally they made random jumps down the ladder of eigenvalues of the energy operator H. Each time an atom made such a jump, it emitted a quantum of light whose (angular) frequency ω was related to the size E of the jump by

$$E = \hbar\omega \ .$$

The probability per unit of time for the atom to jump was given by Fermi's 'Golden Rule', formulated in terms of the coupling between the atom and its surroundings, and it is proportional to the damping rate η.

In the following sections we shall describe this behaviour as a quantum Markov process. Both jumps and spirals will be visible in the extension of our T_t to the atom's full observable algebra. This will be our answer to Question 1, for which we shall need the notion of completely positive operators.

Our answer to Question 2 will be a reconstruction of the atom's surroundings: a *dilation*. There we shall see how the atom can absorb and emit quanta.

3. Conditional expectations and operations

We shall now give a sketch of the operational approach to quantum probability which was pioneered by Davies, Lewis and Evans ([Dav], [EvL]).

Conditional expectations in finite dimension. In this section we choose for definiteness: $\mathcal{A} := M_n$, the algebra of all complex $n \times n$ matrices, and

$$\varphi : \mathcal{A} \to \mathbb{C} : \quad A \mapsto \mathrm{tr}\left(\rho A\right) \ ,$$

where ρ is a symmetric $n \times n$ matrix with strictly positive eigenvalues and trace 1, so that φ is faithful.

Let A be a symmetric $n \times n$ matrix with the spectral decomposition

$$A = \sum_{\alpha \in \mathrm{sp}(A)} \alpha E_\alpha \ .$$

The orthogonal projections E_α, $\alpha \in \mathrm{sp}(A)$, form a partition of unity. *Measuring* the *observable* A means asking all the compatible questions E_α at the same time. Precisely one of the answers will be 'yes', as stipulated in the interpretation rules. If the answer to E_α is 'yes', then A is said to take the value α. This happens with probability $\varphi(E_\alpha)$.

It is natural to define the *expectation* of A as

$$\sum_{\alpha \in \mathrm{sp}(A)} \alpha \varphi(E_\alpha) = \varphi\left(\sum_{\alpha \in \mathrm{sp}(A)} \alpha E_\alpha \right) = \varphi(A) \ .$$

So the state φ not only plays the role of a probability measure, but naturally extends to the associated expectation.

Now let $B = B^* \in \mathcal{A}$ be a second observable with spectral decomposition

$$B = \sum_{\beta \in \mathrm{sp}(B)} \beta F_\beta \ .$$

If we first measure B and then A in each trial, in the limit of increasingly many trials we obtain a probability measure $\mathbb{P}$ on $\mathrm{sp}(A) \times \mathrm{sp}(B)$. By the discussion of interpretation of quantum probability in Section 1 the probabilities are given by

$$\mathbb{P}(\{(\alpha, \beta)\}) = \varphi(F_\beta E_\alpha F_\beta) \ .$$

It is then natural to define the conditional probability $\mathbb{P}[A = \alpha | B = \beta]$ as that proportion of the trials that have yielded $B = \beta$ which turn out to give $A = \alpha$ later:

$$\mathbb{P}[A = \alpha | B = \beta] := \frac{\mathbb{P}(\{(\alpha, \beta)\})}{\sum_{\alpha \in \mathrm{sp}(A)} \mathbb{P}(\{(\alpha, \beta)\})} = \frac{\varphi(F_\beta E_\alpha F_\beta)}{\varphi(F_\beta)} \ .$$

The associated conditional expectation is naturally defined as

$$\mathbb{E}(A | [B = \beta]) := \sum_{\alpha \in \mathrm{sp}(A)} \alpha \mathbb{P}[A = \alpha | B = \beta] = \frac{\varphi(F_\beta A F_\beta)}{\varphi(F_\beta)} \ ,$$

Note that this is a function, f say, of β. Seen as a quantum random variable this conditional expectation is described by the matrix $f(B)$:

$$\mathbb{E}(A|B) := f(B) = \sum_{\beta \in \mathrm{sp}(B)} f(\beta) F_\beta = \sum_{\beta \in \mathrm{sp}(B)} \frac{\varphi(F_\beta A F_\beta)}{\varphi(F_\beta)} F_\beta \ . \tag{3.1}$$

Note that

$$\varphi(\mathbb{E}(A|B)) = \sum_{\beta \in \mathrm{sp}(B)} \varphi(F_\beta A F_\beta) \ .$$

Remark. In general we do *not* have

$$\varphi(\mathbb{E}(A|B)) = \varphi(A)\ . \tag{3.2}$$

The left hand side is the expectation of A after measuring B. The right hand side is the expectation of A without any previous operation. The fact that these two expectation values can differ is typical for quantum probability.

Let us give a simple counterexample to (3.2) here: Let $\mathcal{A} := M_2$, choose $\lambda \in (0,1)$, and put

$$\rho = \begin{bmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\ .$$

It is readily checked that

$$\varphi(A) = \lambda, \qquad \varphi\big(\mathbb{E}(A|B)\big) = \tfrac{1}{2}\ ,$$

so that the equality (3.2) holds if and only if φ is the trace state.

The conditional expectation given a discrete random variable. Let $\mathcal{B}$ denote the (abelian) subalgebra of M_n generated by $\mathbf{1}$ and B, a hermitian matrix. In quantum probability theory it is accepted practice ([EvL]) to call $\mathbb{E}(A|B)$ the *conditional expectation of A given the algebra $\mathcal{B}$*, written $P_{\mathcal{B}}(A)$ *only* if the equality (3.2) *does* hold. The reason is that the definition as it stands does not generalise to observables B with continuous spectrum, or to noncommutative subalgebras $\mathcal{B}$ of $\mathcal{A}$. (The value of $\varphi(\mathbb{E}(A|B))$ changes if the possible values of B are not all distinguished while measuring B: if β_1 and β_2 are not distinguished, their eigenspaces group together into a single subspace, and in (3.1) the projections F_{β_1} and F_{β_2} are replaced by the projection $F_{\beta_1} + F_{\beta_2}$.)

Note that the projections in $\mathcal{B}$ are labeled by *subsets* of $\mathrm{sp}(B)$:

$$\mathcal{E}(\mathcal{B}) = \Big\{ \sum_{\beta\in V} F_\beta \,\Big|\, V \subset \mathrm{sp}(B) \Big\}\ ,$$

and that $\mathcal{B}$ is the linear span of the projections F_β.

The following is a finite dimensional version of Takesaki's theorem ([Tak]) on the existence of conditional expectations onto von Neumann subalgebras.

Theorem 3.1. *Let $B = B^* \in M_n$ and let $\mathcal{B}$ be the *-algebra generated by $\mathbf{1}$ and B. Let $\varphi : M_n \to \mathbb{C} : A \mapsto \mathrm{tr}\,(\rho A)$ with ρ strictly positive and $\mathrm{tr}\,(\rho) = 1$. Then the following are equivalent.*

(a) *There exists a linear map $P : M_n \to \mathcal{B}$ such that*

$$\forall_{A\in M_n}\forall_{F\in\mathcal{E}(\mathcal{B})} : \quad \varphi(FAF) = \varphi(FP(A)F)\ . \tag{3.3}$$

(b) *There exists a linear map P from M_n onto $\mathcal{B}$ such that*
  (i) *P maps positive definite matrices to positive definite matrices.*
  (ii) *$P(\mathbf{1}) = \mathbf{1}$;*
  (iii) *$\varphi \circ P = \varphi$;*
  (iv) *$P^2 = P$.*

(c) *$B\rho = \rho B$.*

If these equivalent conditions hold, then the linear maps P mentioned in (a) and (b) are the same. It is called the *conditional expectation onto $\mathcal{B}$ compatible with φ.*

Proof. (a) $\Longrightarrow$ (b): suppose $P : M_n \to \mathcal{B}$ is such that (3.3) holds.

Let $A \geq 0$ and decompose $P(A)$ as $\sum_{\beta\in\mathrm{sp}(B)} a_\beta F_\beta$ with $F_\beta \in \mathcal{E}(\mathcal{B})$. Then $a_\beta\varphi(F_\beta) = \varphi(F_\beta P(A)) = \varphi(F_\beta P(A)F_\beta) = \varphi(F_\beta A F_\beta) \geq 0$. So $a_\beta \geq 0$ for all β and $P(A) \geq 0$.

Putting $A = \mathbf{1}$ in (3.3) we find that for all $\beta \in \mathrm{sp}(B)$: $\varphi(F_\beta) = \varphi(F_\beta P(\mathbf{1})F_\beta) = \varphi(F_\beta P(\mathbf{1}))$. Writing $P(\mathbf{1}) = \sum_{\beta\in\mathrm{sp}(B)} e_\beta F_\beta$, we see that $e_\beta = 1$, hence $P(\mathbf{1}) = \mathbf{1}$.

By putting $F = \mathbf{1}$ in (3.3), (iii) is obtained.

Finally, given A, the element $P(A)$ of $\mathcal{B}$ is obviously uniquely determined by (3.3). But if $A \in \mathcal{B}$, then $P(A) := A$ clearly satisfies (3.3). It follows that P is an idempotent with range $\mathcal{B}$.

(b) $\Longrightarrow$ (c): Make a Hilbert space out of $\mathcal{A} = M_n$ by endowing it with the inner product

$$\langle X, Y\rangle_\varphi := \varphi(X^*Y) \ .$$

We claim that on this Hilbert space P is an orthogonal projection. Since P is idempotent by assumption (b)(iv), it suffices to show that P is a contraction:

$$\|P(A)\|_\varphi \leq \|A\|_\varphi \ . \tag{3.4}$$

Given $A \in M_n$, define numbers $a_\beta \in \mathbb{C}$ and $b_\beta \geq 0$ by

$$P(A) = \sum_{\beta\in\mathrm{sp}(B)} a_\beta F_\beta \ ; \qquad P(A^*A) = \sum_{\beta\in\mathrm{sp}(B)} b_\beta F_\beta \ .$$

Then from the positivity property (b)(i) it follows that

$$\forall_{\lambda\in\mathbb{C}} : \quad P\big((\lambda\cdot\mathbf{1} - A)^*(\lambda\cdot\mathbf{1} - A)\big) \geq 0 \ .$$

This implies that for all $\beta \in \mathrm{sp}(B)$ and all $\lambda \in \mathbb{C}$,

$$|\lambda|^2 - (\overline{\lambda}a_\beta + \lambda\overline{a_\beta}) + b_\beta \geq 0 \ ,$$

from which it follows that

$$|a_\beta|^2 \leq b_\beta \ , \quad \text{that is} \quad P(A)^*P(A) \leq P(A^*A) \ .$$

Applying φ to the last inequality and using (iii) yields the statement (3.4). So P is an orthogonal projection $M_n \to \mathcal{B}$, that is for all $A \in M_n$,

$$A - P(A) \perp_\varphi \mathcal{B} \ .$$

This means that for all $A \in M_n$:

$$\varphi(AB) = \varphi(P(A)B) \ \text{ and } \ \varphi(BA) = \varphi(BP(A)) \ .$$

But then, since $\mathcal{B}$ is commutative,

$$\varphi(BA) = \varphi(BP(A)) = \varphi(P(A)B) = \varphi(AB) \ .$$

It follows that

$$\mathrm{tr}\,(\rho BA) = \mathrm{tr}\,(\rho AB) = \mathrm{tr}\,(B\rho A) \ ,$$

and (c) is proved.

(c) $\Longrightarrow$ (a): Suppose that $B\rho = \rho B$. Then for all $F \in \mathcal{E}(\mathcal{B})$ and all $A \in M_n$,

$$\varphi(FAF) = \mathrm{tr}\,(\rho FAF) = \mathrm{tr}\,(F\rho FA) = \mathrm{tr}\,(\rho F^2 A) = \mathrm{tr}\,(\rho FA) = \varphi(FA) \ .$$

Therefore, defining $P(A)$ by the r.h.s. of (3.1), and putting $F = \sum_{\beta\in V} F_\beta$ with $V \subset \mathrm{sp}(B)$:

$$\begin{aligned}\varphi(FP(A)F) &= \sum_{\beta\in\mathrm{sp}(B)} \frac{\varphi(F_\beta A F_\beta)}{\varphi(F_\beta)}\varphi(FF_\beta F) = \sum_{\beta\in V}\varphi(F_\beta A F_\beta)\\ &= \sum_{\beta\in V}\varphi(F_\beta A) = \varphi(FA) = \varphi(FAF)\ .\end{aligned}$$

□

Operations in finite dimension. Let $\mathcal{A}$ and $\mathcal{B}$ be finite dimensional von Neumann algebras, and let $\mathcal{A}^*$ and $\mathcal{B}^*$ denote their duals. A linear map $T : \mathcal{A} \to \mathcal{B}$ defines by duality a linear map $T^* : \mathcal{B}^* \to \mathcal{A}^*$.

The map T^* maps states into states if and only if T is *positive*, that is maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$, and is identity preserving.

The map T is said to be *n-positive* if $T \otimes \mathrm{id}$ maps positive elements of $\mathcal{A} \otimes M_n$ to positive elements of $\mathcal{B} \otimes M_n$:

$$(A_{ij})_{i,j=1}^n \geq 0 \quad \Longrightarrow \quad (T(A_{ij}))_{i,j=1}^n \geq 0\ .$$

T is called *completely positive* if it is n-positive for all $n \in \mathbb{N}$. In that case $T^*\otimes\mathrm{id}$ maps states on $\mathcal{B}\otimes M_n$ to states on $\mathcal{A}\otimes M_n$. T is called *identity preserving* if $T(\mathbf{1}_\mathcal{A}) = \mathbf{1}_\mathcal{B}$.

Definition. An *operation* $T : \mathcal{A} \to \mathcal{B}$ is a completely positive identity preserving map. Adjoints of operations will also be called operations.

The idea is that any physical procedure which takes as an input a state on some quantum system described by $\mathcal{B}$, and which turns out a state on a quantum system described by $\mathcal{A}$ must necessarily be of the above kind. Not all operations in the sense of the definition can actually be performed, but certainly nothing else is physically possible. Indeed any physical operation on a quantum system $\mathcal{A}$ should also define a physical operation on $\mathcal{A}\otimes\mathcal{R}$, where $\mathcal{R}$ stands for some quantum system not affected by the operation. The existence of such an 'innocent bystander' outside our quantum system $\mathcal{A}$ should never lead to the prediction by quantum theory of negative probabilities.

The following example shows that complete positivity is strictly stronger than positivity. Let

$$T : M_2 \to M_2 : \begin{bmatrix} a & b\\ c & d\end{bmatrix} \mapsto \begin{bmatrix} a & c\\ b & d\end{bmatrix}\ .$$

Then $T(A^*A) = T(A)T(A)^* \geq 0$ for all A, but

$$T\otimes\mathrm{id}\ : M_2\otimes M_2 \to M_2\otimes M_2 \text{ maps } \begin{bmatrix} 1&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&1\end{bmatrix} \text{ to } \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1\end{bmatrix};$$

i.e. it maps a one-dimensional projection to a matrix with eigenvalues 1 and -1.

Operations on quantum probability spaces. A quantum probability space $(\mathcal{A}, \varphi)$ has a canonical representation on a Hilbert space, called the GNS representation after Gel'fand, Naimark and Segal. It is the representation of $\mathcal{A}$ on $\mathcal{H}_\varphi$, the Kolmogorov dilation of the positive definite kernel

$$\mathcal{A} \times \mathcal{A} \to \mathbb{C} : \quad (A, B) \mapsto \varphi(A^* B) \; .$$

States which are given by density matrices on this space are called *normal states* on $\mathcal{A}$, and the set of all normal states is denoted by $\mathcal{A}_*$.

When we write $T : (\mathcal{A}, \varphi) \to (\mathcal{B}, \psi)$, we mean that T is a completely positive operator $\mathcal{A} \to \mathcal{B}$ such that $T(\mathbf{1}_\mathcal{A}) = \mathbf{1}_\mathcal{B}$ and *also* $\psi \circ T = \varphi$. The latter condition, which can equivalently be written as

$$T^* \psi = \varphi \; ,$$

ensures that T^* maps normal states to normal states. This property is only relevant for infinite dimensional von Neumann algebras. When speaking of operations between quantum probability spaces we shall always imply that the state is preserved.

Quantum stochastic processes. Let us now consider the category QP whose objects are quantum probability spaces and whose morphisms are operations.

Lemma 3.2 (Schwartz's inequality for completely positive operators)**.** *Let* $T : (\mathcal{A}, \varphi) \to (\mathcal{B}, \psi)$*. Then for all* $A \in \mathcal{A}$*,*

$$T(A^* A) \geq T(A)^* T(A) \; .$$

Proof. Let $\mathcal{A}$ be represented on $\mathcal{H}$. By the positivity of $T \otimes \mathrm{id}_{M_2}$ we have for all $A \in \mathcal{A}$,

$$\langle \psi \oplus T(A)\psi, \big(T \otimes \mathrm{id}\big) \left(\begin{bmatrix} A & -\mathbf{1} \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} A & -\mathbf{1} \\ 0 & 0 \end{bmatrix} \right) \psi \oplus T(A)\psi \rangle \geq 0 \; .$$

Writing this out we obtain

$$\langle \psi, \big(T(A^* A) - T(A)^* T(A) \big) \psi \rangle \geq 0 \; .$$

□

Corollary 3.3. *If* $T : (\mathcal{A}, \varphi) \to (\mathcal{B}, \psi)$ *then for all* $A \in \mathcal{A}$

$$\varphi(T(A)^* T(A)) \leq \varphi(A^* A) \; .$$

This inequality states that T is a contraction between the GNS Hilbert spaces of $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$.

Lemma 3.4. $T : (\mathcal{A}, \varphi) \to (\mathcal{B}, \psi)$ *is an isomorphism in the category QP if and only if* $T : \mathcal{A} \to \mathcal{B}$ *is a *-isomorphism.*

Proof. (Exercise:) Apply Schwartz's inequality to T and to T^{-1}. □

A *random variable* (cf. Events and random variables, in Section 1) is an injective *-homomorphism

$$j : (\mathcal{A}, \varphi) \to (\widehat{\mathcal{A}}, \widehat{\varphi}) \; .$$

A *quantum stochastic process* ([AFL]) is a family $(j_t)_{t\in\mathbb{T}}$ of random variables indexed by time $\mathbb{T}$. Here, $\mathbb{T}$ is a linearly ordered set such as $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{N}$ or $\mathbb{R}_+$. If $\mathbb{T}=\mathbb{R}$ or $\mathbb{R}_+$ we require that for all $A\in\mathcal{A}$ the curve $t\mapsto j_t(A)$ is strongly continuous.

If $\mathbb{T}$ is a group, say $\mathbb{Z}$ or $\mathbb{R}$, then the process is called *stationary* provided that $j_t=\widehat{T}_t\circ j_0$ for some representation $t\mapsto\widehat{T}_t$ of $\mathbb{T}$ into the automorphisms of $(\widehat{\mathcal{A}},\widehat{\varphi})$.

Open system interpretation. We are observing a subsystem with observable algebra $\mathcal{A}$ of a larger environment with algebra $\widehat{\mathcal{A}}$ that we cannot see. In the Heisenberg picture, the smaller algebra is moving inside the larger one. If $t_0\le t_1\le\cdots\le t_n$ is a sequence of times, and $E_1,E_2,\ldots,E_n$ a sequence of events in $\mathcal{A}$, then

$$\widehat{\varphi}\big(j_{t_1}(E_1)j_{t_2}(E_2)\cdots j_{t_{n-1}}(E_{n-1})j_{t_n}(E_n)j_{t_{n-1}}(E_{n-1})\cdots j_{t_2}(E_2)j_{t_1}(E_1)\big)$$

is the probability that E_1 occurs at time t_1, E_2 at time t_2, ..., and E_n at time t_n. Note the double role played here by the time ordering: Unless some of the questions $j_{t_k}(E_k)$ recur, that is they lie in $j_t(\mathcal{A})$ for different values of t, they must be asked in the order dictated by the times t_k.

Stochastic process interpretation. In a classical stochastic process $(X_t)_{t\in\mathbb{T}}$ the random variable X_t is a different one for different times t, so the events concerning X_t change in time accordingly. If the process is stationary, X_t and X_s differ by an automorphism of the underlying probability space. These observations generalise to the noncommutative situation.

Conditional expectations and transition operators. If we are to describe an open quantum system such as the damped harmonic oscillator by an internal dynamics, say $T_t:\mathcal{A}\to\mathcal{A}$, without reference to its surroundings, we need to be able to keep track of an observable A which starts in $\mathcal{A}$ at time zero, during its motion away from the algebra $\mathcal{A}$ at positive times. That is, we need its conditional expectation.

In view of the discussion of conditional expectations in finite dimensions, we give the following general definition.

Definition. Let $j:(\mathcal{A},\varphi)\to(\widehat{\mathcal{A}},\widehat{\varphi})$ be a random variable. The *conditional expectation* (if it exists) is the unique morphism $P:(\widehat{\mathcal{A}},\widehat{\varphi})\to(\mathcal{A},\varphi)$ for which

$$P\circ j=\mathrm{id}_{\mathcal{A}}\,.$$

Without proof we state some properties.

Proposition 3.5. *If $P:(\widehat{\mathcal{A}},\widehat{\varphi})\to(\mathcal{A},\varphi)$ is the conditional expectation with respect to $j:(\mathcal{A},\varphi)\to(\widehat{\mathcal{A}},\widehat{\varphi})$, then*

$$\forall_{B_1,B_2\in\mathcal{A}}\forall_{A\in\widehat{\mathcal{A}}}:B_1P(A)B_2=P(j(B_1)Aj(B_2))\,.$$

In particular

$$\forall_{F\in\mathcal{E}(\mathcal{A})}\forall_{A\in\mathcal{A}}:\widehat{\varphi}(j(F)Aj(F))=\varphi(FP(A)F)\,.$$

The second line indicates the connection with Theorem 3.1.

Markov processes. Let us now apply the above notion to an open quantum system.

Two-time-probabilities. Suppose that for all $s \in \mathbb{T}$ there exists a conditional expectation P_s with respect to j_s. Then the probability for F to occur at time s and E at time $t \geq s$ can be written as

$$\widehat{\varphi}(j_s(F)j_t(E)j_s(F)) = \varphi(FP_s(j_t(E))F) = \varphi(FT_{s,t}(E)F) \ ,$$

where $T_{s,t} = P_s \circ j_t$ is an operation on $(\mathcal{A}, \varphi)$, the *transition operator* from time s to time t.

Multi-time-probabilities. This reduction to the subsystem succeeds for more than two time points if there also exist conditional expectations $P_{(-\infty,t]}$ onto the algebras

$$\mathcal{A}_{(-\infty,t]} := \mathrm{vN}\big\{ \ j_s(\mathcal{A}) \ \big| \ s \leq t \ \big\} \ .$$

and moreover the *Markov property* holds:

$$t \leq s \quad \Longrightarrow \quad P_{(-\infty,t]}(j_s(\mathcal{A})) \subset j_t(\mathcal{A}) \ . \tag{3.5}$$

Proposition 3.6. *Let* $\big(j_t : (\mathcal{A}, \varphi) \to (\widehat{\mathcal{A}}, \widehat{\varphi})\big)_{t\in\mathbb{T}}$ *be a Markov process with conditional expectations* P_t*. Then the transition operators form a monoid:*

$$0 \leq s \leq t \leq u \quad \Longrightarrow \quad T_{s,t}T_{t,u} = T_{s,u} \ .$$

In particular, if the process is stationary, then $T_t := T_{0,t} = T_{s,s+t}$ *satisfies*

$$T_sT_t = T_{s+t} \qquad (s, t \geq 0).$$

In the latter case, $(T_t)_{t\geq 0}$ is known as the *dynamical semigroup* induced by the stationary Markov process. Conversely, the process $(j_t)_{t\in\mathbb{T}}$ is called a *Markov dilation* ([Küm]) of the dynamical semigroup $(T_t)_{t\in\mathbb{T}}$.

The situation is symbolised by the commutative diagram

$$\begin{array}{ccc} (\mathcal{A}, \varphi) & \xrightarrow{T_t} & (\mathcal{A}, \varphi) \\ {\scriptstyle j}\big\downarrow & & \big\uparrow{\scriptstyle P} \\ (\widehat{\mathcal{A}}, \widehat{\varphi}) & \xrightarrow{\widehat{T}_t} & (\widehat{\mathcal{A}}, \widehat{\varphi}) \end{array} \tag{3.6}$$

Our goal is to describe a Markov dilation of the damped harmonic oscillator.

4. Second quantisation

A quantum model of n harmonic oscillators is obtained by taking the n-fold tensor product of the representation $z \mapsto W(z) := \exp\big(i(\mathrm{Im}\, z)Q - i(\mathrm{Re}\, z)P\big)$ of the canonical commutation relation (CCR) over $\mathbb{C}$. This turns out to be equivalent to a single representation of the CCR over $\mathbb{C}^n$. An infinity of harmonic oscillators is obtained by replacing $\mathbb{C}$ with an infinite dimensional separable Hilbert space $\mathcal{K}$. It depends on the spectrum of the time evolution on $\mathcal{K}$ (discrete or continuous), whether a countable infinity of oscillators is obtained or a continuum, that is a quantum field. In our dilation of the damped harmonic oscillator we shall need a quantum field.

As in the case of a single oscillator we have the choice between different concrete representations: we may emphasise the field aspect of the construction, like in the Gaussian representation of the harmonic oscillator, or the particle aspect of it, like in its matrix representation. (The Schrödinger representation on $L^2(\mathbb{R})$ as in (2.2),

(2.6) has no analogue in infinite dimension, since there exists no Lebesgue measure on $\mathbb{R}^\infty$.)

The following definition generalises the Weyl relation (2.14) over $\mathbb{C}$ to that over a general complex Hilbert space $\mathcal{K}$. If $\mathcal{K}$ is the L^2-space of some measure space (X, μ), then $\mathcal{K}$ may be considered as the 'quantisation' of X, and the construction below as its 'second quantisation'.

We refer to Mark Fannes' lectures in these volumes.

The functor Γ.

Definition. Let $\mathcal{K}$ be a complex Hilbert space. A *representation of the Canonical Commutation Relations* (*CCR*) *over* $\mathcal{K}$ is a map W from $\mathcal{K}$ to the unitary operators on some Hilbert space $\mathcal{H}$ such that for all $f, g \in \mathcal{K}$:

$$W(f)W(g) = e^{-i\mathrm{Im}\,\langle f,g\rangle} W(f+g)\ , \tag{4.1}$$

and $t \mapsto W(tf)\psi$ is continuous for all $f \in \mathcal{K}, \psi \in \mathcal{H}$. The map is called a *vacuum representation* if there is a unit vector $\Omega \in \mathcal{H}$ such that

$$\langle \Omega, W(f)\Omega\rangle = e^{-\frac{1}{2}\| f \|^2}\ .$$

A vacuum representation is called *cyclic* if the linear span of the vectors $W(f)\Omega$ is dense in $\mathcal{H}$.

A cyclic vacuum representation of the CCR over $\mathcal{K}$ can be constructed by a generalisation of the method used several times in the harmonic oscillator in Section 2: Let π be a minimal Kolmogorov decomposition of the positive definite kernel $\mathcal{K} \times \mathcal{K} \to \mathbb{C}$ mapping (f, g) to $e^{\langle f,g\rangle}$, and on the total set of 'coherent vectors' $\pi(g)$, define

$$W(f)\pi(g) := e^{-\langle f,g\rangle - \frac{1}{2}\| f \|^2}\pi(f+g)\ , \quad (f, g \in \mathcal{K})\ .$$

Then put $\Omega := \pi(0)$, and all the requirements in the above definition are met. On the other hand, given any cyclic vacuum representation of the CCR over $\mathcal{K}$ with vacuum vector Ω, a Kolmogorov decomposition π' of the above mentioned kernel is obtained by putting

$$\pi' : f \mapsto e^{\frac{1}{2}\| f \|^2} W(f)\Omega\ .$$

Indeed,

$$\begin{aligned}\langle \pi'(f), \pi'(g)\rangle &= e^{\frac{1}{2}(\| f \|^2 + \| g \|^2)} \cdot e^{i\mathrm{Im}\,\langle f,g\rangle} \langle \Omega, W(-f+g)\Omega\rangle \\ &= e^{\frac{1}{2}(\| f \|^2 + \| g \|^2)} \cdot e^{i\mathrm{Im}\,\langle f,g\rangle} e^{-\frac{1}{2}\| f-g \|^2} \\ &= e^{\langle f,g\rangle}\ .\end{aligned}$$

Thus all cyclic vacuum representations of the CCR over a Hilbert space $\mathcal{K}$ are unitarily equivalent. However, this can not be concluded from von Neumann's uniqueness theorem, since the latter breaks down for infinite dimesional $\mathcal{K}$. In this case there are indeed many inequivalent (non-vacuum) representations, for instance those associated to positive temperatures ([BrR]).

Since $e^{\langle f_1 \oplus f_2, g_1 \oplus g_2\rangle} = e^{\langle f_1, g_1\rangle} \cdot e^{\langle f_2, g_2\rangle}$, a representation of the CCR over a direct sum $\mathcal{K}_1 \oplus \mathcal{K}_2$ of Hilbert spaces is isomorphic to the tensor product of the representations of the CCR over $\mathcal{K}_1$ and $\mathcal{K}_2$.

Definition. Let $\Gamma_0(\mathcal{K})$ denote the linear span of the operators $W(f)$, $f \in \mathcal{K}$ in some representation of the CCR over $\mathcal{K}$. Let $\Gamma(\mathcal{K})$ be its strong closure. On the von Neumann algebra $\Gamma(\mathcal{K})$ we assume by default the vacuum state

$$\varphi_{\mathcal{K}}(W(f)) = e^{-\frac{1}{2}\| f \|^2} .$$

Thus $\Gamma(\mathcal{K}) = (\Gamma(\mathcal{K}), \varphi_{\mathcal{K}})$ is a quantum probability space.

If C is a contraction $\mathcal{K}_1 \to \mathcal{K}_2$, let $\Gamma_0(C) : \Gamma_0(\mathcal{K}_1) \to \Gamma_0(\mathcal{K}_2)$ be given by

$$\Gamma_0(C)\big(W(f)\big) := e^{\frac{1}{2}(\| Cf \|^2 - \| f \|^2)} W(Cf) . \tag{4.2}$$

Proposition 4.1. *The operator $\Gamma_0(C)$ has a unique strongly continuous extension to an operation $\Gamma(C)$ of $\Gamma(\mathcal{K})$.*

Proof. Cf. for instance [Tee]. □

Remark. Second quantisation is a functor Γ from the category of Hilbert spaces with contractions to the category of quantum probability spaces with operations.

Fields. From the Weyl relation (4.1) it follows that $\lambda \mapsto W(\lambda f)$ is a strongly continuous unitary representation of $\mathbb{R}$. By the spectral theorem there exists a self-adjoint operator $\Phi(f)$ on $\Gamma(\mathcal{K})$ such that

$$W(\lambda f) = e^{i\lambda\Phi(f)} .$$

The Weyl relations then imply that

$$[\Phi(f), \Phi(g)] = 2i\mathrm{Im}\,\langle f, g\rangle \cdot \mathbf{1} ,$$

and in the vacuum state $\varphi_{\mathcal{K}}$ the random variable $\Phi(f)$ has normal distribution with mean 0 and variance $\| f \|^2$. The random variables $\Phi(f)$ and $\Phi(g)$ are compatible if the inner product $\langle f, g\rangle$ is real, and independent if it is zero.

In particular, if $\mathcal{K} = L^2(\mathbb{R}, 2\eta dx)$ (as we shall need in Sections 5 and 6), then by putting

$$B_t := \begin{cases} \Phi(1_{[0,t]}) & \text{if } t \geq 0, \\ -\Phi(1_{[t,0]}) & \text{if } t < 0, \end{cases} \tag{4.3}$$

a stochastic process $(B_t)_{t\in\mathbb{R}}$ is defined with compatible normally distributed independent increments having variance

$$\varphi_{\mathcal{K}}\big((B_t - B_s)^2\big) = |t - s| .$$

Thus B_t is a classical Brownian motion.

Particles. A natural choice for the representation space of the CCR over $\mathcal{K}$ is the *Fock space* $\mathcal{F}(\mathcal{K})$ from Example (d) at the end of Section 1:

$$\mathcal{F}(\mathcal{K}) := \bigoplus_{n=0}^{\infty} \frac{1}{n!} \mathcal{K}^{\otimes_{\mathrm{symm}} n} ,$$

with $\pi(f)$ given by the exponential vectors. Given a contraction $C : \mathcal{K} \to \mathcal{K}$ let $\mathcal{F}(C)$ be the contraction $\mathcal{F}(\mathcal{K}) \to \mathcal{F}(\mathcal{K})$ mapping $\pi(f)$ to $\pi(Cf)$ for every $f \in \mathcal{K}$. This map can be written

$$\mathcal{F}(U) = \bigoplus_{n=0}^{\infty} C \otimes C \otimes \cdots \otimes C \ .$$

Given an orthogonal projection P on $\mathcal{K}$ let a self-adjoint operator $d\mathcal{F}(P)$ on $\mathcal{F}(\mathcal{K})$ be defined by

$$e^{i\lambda d\mathcal{F}(P)} := \mathcal{F}\left(e^{i\lambda P}\right) \ .$$

This operator $d\mathcal{F}(P)$ is interpreted as the random variable that counts for how many particles the 'question' P is anwered 'yes'. In particular the total number of particles N equals $d\mathcal{F}(\mathbf{1})$.

If $\mathcal{K} = L^2(\mathbb{R}, 2\eta dx)$, the Fock space $\mathcal{F}(\mathcal{K})$ can be written as $L^2(\Delta(\mathbb{R}), \mu_\eta)$ where $\Delta(\mathbb{R})$ is Guichardet's space over $\mathbb{R}$ ([Gui], [Maa] — see also the lectures by Martin Lindsay in these volumes).

$$\Delta(\mathbb{R}) := \left\{ \sigma \subset \mathbb{R} \,\middle|\, \#(\sigma) < \infty \right\};$$

and μ_η is the measure on Δ given by

$$\mu_\eta(\{\emptyset\}) = 1 \ ,$$

$$\mu_\eta(d\sigma) = (2\eta)^n \, ds_1 ds_2 \cdots ds_n \text{ if } \sigma = \{s_1, s_2, \cdots, s_n\}.$$

The coherent vectors are represented as the functions $\pi(f) : \Delta(\mathbb{R}) \to \mathbb{C}$ given by

$$\sigma \mapsto \prod_{s \in \sigma} f(s) \ .$$

Indeed,

$$\begin{aligned}
\langle \pi(f), \pi(g) \rangle &= \int_{\Delta(\mathbb{R})} \overline{\pi(f)(\sigma)} \pi(g)(\sigma) \mu_\eta(d\sigma) \\
&= \sum_{n=0}^{\infty} (2\eta)^n \int_{s_1 \le \cdots \le s_n} (\overline{f}g)(s_1) \cdots (\overline{f}g)(s_n) ds_1 \cdots ds_n \\
&= \sum_{n=0}^{\infty} \frac{(2\eta)^n}{n!} \int_{\mathbb{R}^n} (\overline{f}g)(s_1) \cdots (\overline{f}g)(s_n) ds_1 \cdots ds_n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(2\eta \int_{-\infty}^{\infty} \overline{f(s)} g(s) ds \right)^n = e^{\langle f, g \rangle} \ .
\end{aligned}$$

In this concrete representation the number operator $d\mathcal{F}(P)$, where P is the multiplication in $L^2(\mathbb{R})$ by 1_S, counting the number of particles in the region $S \subset \mathbb{R}$, is itself a multiplication operator, multiplying by the number $\#(\sigma \cap S)$ of quanta in S.

This is seen by the following calculation: for all $\lambda \in \mathbb{R}$, $f \in \mathcal{K}$ and $\sigma \in \Delta(\mathbb{R})$,

$$\begin{aligned}\left(e^{i\lambda d\mathcal{F}(P)}\pi(f)\right)(\sigma) &= \pi\left(e^{i\lambda P}f\right)(\sigma)\\ &= \prod_{t\in\sigma} e^{i\lambda 1_S(t)} f(t)\\ &= \exp\left(i\lambda \sum_{t\in\sigma} 1_S(t)\right)\cdot \pi(f)(\sigma) = e^{i\lambda \#(\sigma\cap S)}\cdot \pi(f)(\sigma)\ .\end{aligned}$$

5. Unitary dilations of spiraling motion

In preparation for the solution of the physical problem of damping posed at the end of Section 2, we now consider embeddings of the spiraling evolution (2.16) into a unitary one. Let us describe the spiral by

$$C_t : \mathbb{C} \to \mathbb{C} : z \mapsto e^{(-\eta+i\omega)t} z\ , \qquad (t \geq 0). \tag{5.1}$$

Theorem 5.1 ((Sz. Nagy, Foias 1953; special case.)**.** *Up to unitary equivalence there exists a unique Hilbert space $\mathcal{K}$ with a unit vector v and a one-parameter group of unitary transformations U_t on $\mathcal{K}$ such that the span of the vectors $U_t v$, $t \in \mathbb{R}$ is dense in $\mathcal{K}$ and*

$$\langle v, U_t v\rangle = e^{(-\eta+i\omega)t}\ , \quad (t \geq 0).$$

Proof. Existence: Take $\mathcal{K} := L^2(\mathbb{R}, 2\eta\, dx)$, let U_t be the shift to the right, and

$$v(x) := \begin{cases} 0 & \text{if } x > 0,\\ e^{(\eta-i\omega)t} & \text{if } x \leq 0.\end{cases}$$

Then we arrive at Example 1.6 (e) at the end of Section 1. Uniqueness follows from Theorem 1.5. □

The structure $(\mathcal{K}, J, U_t)$, illustrated in the diagram below, is called a *minimal unitary dilation* of $(C_t)_{t\geq 0}$.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{e^{(-\eta+i\omega)t}} & \mathbb{C} \\ {\scriptstyle J:z\mapsto zv}\downarrow & & \uparrow{\scriptstyle J^*:k\mapsto\langle v,k\rangle} \\ \mathcal{K} & \xrightarrow{U_t} & \mathcal{K} \end{array} \tag{5.2}$$

In practice several — unitarily equivalent — minimal unitary dilations of $(C_t)_{t\geq 0}$ can be useful. If $\mathcal{K} = L^2(\mathbb{R})$ and U_t is the shift, then we speak of *translation dilations* of $(C_t)_{t\geq 0}$. They differ only in the shape of $v \in L^2(\mathbb{R})$, which must satisfy

$$|\hat{v}(\lambda)|^2 = \frac{1}{(\lambda-\omega)^2+\eta^2}\ , \quad (\lambda \in \mathbb{R})\ .$$

Particular solutions are $\hat{v}_\pm(\lambda) := 1/(\lambda-\omega\pm i\eta)$. Here v_+, which occurred in the proof of Theorem 5.1, leads to the *incoming translation dilation* and v_- to the *outgoing translation dilation*:

$$v_-(x) = \begin{cases} e^{-(\eta+i\omega)x} & (x \geq 0);\\ 0 & (x < 0).\end{cases}$$

The former is more useful for the study of incoming fields and particles, the latter for outgoing ones. We shall have occasion to employ both below. The unitary equivalence of these two unitary dilations, asserted by Theorem 5.1, is implemented by the *scattering operator* S which in terms of the Fourier transform F can be written as

$$S := FM_sF^{-1} , \quad s(\lambda) := \frac{\lambda - \omega + i\eta}{\lambda - \omega - i\eta} .$$

Apart from these two translation dilations, the *interaction dilation* $(\mathcal{K}, J, U_t)$, where $\mathcal{K} = L^2(-\infty, 0] \oplus \mathbb{C} \oplus L^2[0, \infty)$, $J : z \mapsto 0 \oplus z \oplus 0$, and U_t describes a more complicated coupling of $\mathbb{C}$ to an *incoming* and an *outgoing channel*, is physically more enlightening, but too cumbersome to treat here. We refer to [KüS] for a thorough treatment.

6. The damped harmonic oscillator

Equipped with the notions introduced in Sections 1, 3, 4, and 5 we are now in a position to answer the questions posed at the end of Section 2.

We act with the second quantisation functor Γ of Section 4 on all four corners and all four arrows of the dilation diagram (5.2) of Section 5. The corners become quantum probability spaces (Section 1), and the arrows become operations (Section 3):

$$\begin{array}{ccc} \Gamma(\mathbb{C}) & \xrightarrow{\Gamma(C_t)} & \Gamma(\mathbb{C}) \\ {\scriptstyle \Gamma(J)}\downarrow & & \uparrow{\scriptstyle \Gamma(J^*)} \\ \Gamma(\mathcal{K}) & \xrightarrow{\Gamma(U_t)} & \Gamma(\mathcal{K}) \end{array} \tag{6.1}$$

The answers to Questions 1 and 2 can now be read off.

(1) $\Gamma(\mathbb{C}) = \mathcal{B}(\mathcal{H})$, where $\mathcal{H} = \mathcal{H}_H = l^2(\mathbb{N}, \frac{1}{n!})$ or equivalently $\mathcal{H} = \mathcal{H}_G = L^2(\mathbb{R}, \gamma)$ is the Hilbert space of the harmonic oscillator in the Heisenberg or in the Gaussian representation. The damped time evolution $T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is now given by

$$T_t := \Gamma(C_t) = \Gamma\left(e^{(-\eta + i\omega)t}\right) .$$

Then

$$T_t(W(z)) = e^{\frac{1}{2}(e^{-2\eta t} - 1)|z|^2} W(e^{(-\eta + i\omega)t} z) . \tag{6.2}$$

By substituting $W(t + is) = e^{-itP + isQ}$ and differentiating with respect to t and s respectively, we indeed obtain the equations (2.16).

(2) The diagram shows how T_t is embedded into a larger whole, where the time evolution is a one-parameter group of *-automorphisms, that is reversible. Here $j := \Gamma(J)$ is an injective *-homomorphism, $P = \Gamma(J^*)$ is a conditional expectation. By Theorem 5.1 this is the only quasifree dilation, that is in the range of the functor Γ. It is automatically Markov.

In this Section we shall discuss four aspects of the construction: the stochastic behaviour of the oscillator (spirals), its driving field (a quantum Brownian motion), the jumps between the levels of the oscillator (a death process), and the outgoing quanta (a point process). A complete picture would include the outgoing field and

the scattering of incoming particles as well. This can easily be achieved using the tools developed here.

Stochastic behaviour of the oscillator. By the functorial character of Γ we can split T_t as

$$T_t = \Gamma(e^{(-\eta+i\omega)t}) = \Gamma(e^{i\omega t})\Gamma(e^{-\eta t}) \ .$$

The operator $\Gamma(e^{-i\omega t})$ is the automorphism α_t studied in Section 3. So let us now look at the 'dissipative' part $\Gamma(e^{-\eta t})$.

Proposition 6.1. *For $0 \leq c < 1$ the operator $\Gamma(c)$ leaves invariant the abelian subalgebras generated by $\mathbf{1}$ and any of the operators $xP - yQ$ with $(x, y) \in \mathbb{R}^2$. In particular its action on the algebra*

$$\mathcal{Q} := \left\{ \ f(Q) \ \middle| \ f \in L^\infty(\mathbb{R}) \ \right\}$$

is given by

$$\Gamma(c)(f(Q)) = \frac{1}{\sqrt{2\pi(1-c^2)}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(1-c^2)}} \ f(cQ+x)\, dx \ . \tag{6.3}$$

Proof. Obviously, $\Gamma(c)$ leaves the linear span of $\left\{ W(\lambda z) \,\middle|\, \lambda \in \mathbb{R} \right\}$ invariant, and thus also its strong closure by Proposition 4.1. Putting $f(Q) = e^{iyQ}$ the r.h.s. of (6.3) equals

$$\left(\frac{1}{\sqrt{2\pi(1-c^2)}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2(1-c^2)}} e^{ixy} dx \right) \cdot e^{icyQ} = e^{-\frac{1}{2}(1-c^2)y^2} W(icy) \ ,$$

which is equal to the l.h.s. by the definition (4.2) of Γ. The theorem follows from the strong continuity of $\Gamma(c)$. □

We recognise the semigroup T_t of transition operators restricted to $\mathcal{Q}$ as the transition operators of a diffusion on $\mathbb{R}$ with a drift towards the origin proportional to the distance to the origin.

The driving field. Let us consider the dilation of the semigroup T_t for $\omega = 0$. We take the second quantised *incoming* translation dilation of Section 5, and substitute it into the diagram (6.1). Let B_t be the Brownian motion given by (4.3), and let $\hat{Q}_t$ denote the embedded oscillator $\Phi(v_t)$.

Proposition 6.2. *The embedded oscillator $\hat{Q}_t$ satisfied the integral equation*

$$\hat{Q}_t - \hat{Q}_s = -\eta \int_s^t \hat{Q}_u du + B_t - B_s \ . \tag{6.4}$$

This is the integral version of the stochastic differential equation

$$d\hat{Q}_t = -\eta \hat{Q}_t dt + dB_t \ .$$

So we find an embedded Ornstein-Uhlenbeck process in our Markov dilation.

Proof. ([LeT]) The following equality between functions in $\mathcal{K}$ holds:

$$U_t v_- - U_s v_- = -\eta \int_s^t v_u du + 1_{[s,t]} \ , \quad (s \leq t) \ .$$

Acting with Φ on both sides of the equation yields (6.4). □

Quanta. We now concentrate on another abelian subalgebra of $\Gamma(\mathbb{C})$, namely the algebra of all diagonal matrices in Heisenberg's matrix mechanics. In terms of the operator N denoting the number of excitations of the oscillator, this algebra can be written as

$$\mathcal{N} := \left\{ \ f(N) \ \middle| \ f \in l^\infty(\mathbb{N}) \ \right\} \sim l^\infty(\mathbb{N}) \ .$$

This time we need not put $\omega = 0$. Let φ_n denote the state $l^\infty(\mathbb{N}) \to \mathbb{C} : f \mapsto f(n)$.

Proposition 6.3. *The diagonal algebra $\mathcal{N}$ is invariant for T_t and*

$$\varphi_n \left(T_t \left(s^N \right) \right) = \left(1 - e^{-2\eta t}(1-s) \right)^n \ .$$

This is the probability generating function of a pure death process [GrS] with the generator

$$L = 2\eta \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 0 & 2 & -2 & 0 & \cdots \\ 0 & 0 & 3 & -3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \ .$$

Proof. (Sketch. Cf. [Tee] for the detailed proof.) We can write s^N as a weak integral over the operators $W(z)$:

$$s^N = \frac{1}{\pi(1-s)} \int_{\mathbb{C}} e^{-\frac{1}{2}\frac{1+s}{1-s}|z|^2} W(z) \lambda(dz) \ , \tag{6.5}$$

where λ denotes the two-dimensional Lebesgue measure on $\mathbb{C}$. This relation can be checked by taking matrix elements with respect to coherent vectors. Application of T_t to both sides of (6.5) yields for all $u, v \in \mathbb{C}$,

$$\langle \pi(u), T_t \left(s^N \right) \pi(v) \rangle = e^{\overline{u}v(1-e^{-2\eta t}(1-s))} \ .$$

Since this expression is not sensitive to the relative phase of u and v, the operator $T_t \left(s^N \right)$ lies in $\mathcal{N}$. The statement is proved by comparing the coefficients of $(\overline{u}v)^n$ on both sides. □

Emitted quanta. Finally, let us see what happens outside the oscillator while it is cascading down its energy spectrum. Since we are interested in outgoing quanta at positive times, let us now consider the *outgoing* translation dilation of $(T_t)_{t \geq 0}$ and represent $\Gamma(\mathcal{K})$ on the Fock space $L^2(\Delta(\mathbb{R}))$. We denote the number operator $d\mathcal{F}(P_{v_t})$ counting the excitations of the oscillator at time t by N_t.

From Proposition 6.3 it follows that the diagram (6.1) can be restricted to the subalgebra $\mathcal{N} \sim l^\infty(\mathbb{N})$

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{T_t} & \mathcal{N} \\ {\scriptstyle j}\downarrow & & \uparrow{\scriptstyle P} \\ \Gamma(\mathcal{K}) & \xrightarrow{\Gamma(U_t)} & \Gamma(\mathcal{K}) \end{array}$$

Here

$$\begin{aligned} j :&= \Gamma(J) : f \mapsto f(N_0)\ ; \\ P :&= \Gamma(J^*) : X \mapsto \left(\langle v^{\otimes n}, X v^{\otimes n}\rangle\right)_{n=0}^{\infty} \in l^\infty(\mathbb{N}). \end{aligned}$$

However, since for different times t and s the functions v_t and v_s are neither parallel nor orthogonal, the one-dimensional projections P_{v_t} and P_{v_s} do not commute. And since for $\lambda, \mu \in \mathbb{R}$

$$e^{i\lambda N_t} \cdot e^{i\mu N_s} = \mathcal{F}\left(e^{i\lambda P_{v_t}}\right) \mathcal{F}\left(e^{i\mu P_{v_s}}\right) = \mathcal{F}\left(e^{i\lambda P_{v_t}} e^{i\mu P_{v_s}}\right)\ ,$$

the number operators N_t and N_s do not commute either. So the embedded algebras $j_t(\mathcal{N})$ with $t \in \mathbb{R}$ do not generate an abelian subalgebra of $\Gamma(\mathcal{K})$, as was the case for the algebras $j_t(\mathcal{Q})$ above.

For every $t \in \mathbb{R}$ let us consider the following three number operators.

$$\begin{aligned} N_t &:= d\mathcal{F}(P_{v_t}), \quad \text{the number of quanta in the oscillator,} \\ M_t &:= d\mathcal{F}\left(M_{1_{[t,\infty)}}\right) \quad \text{the number of quanta that have not yet left the oscillator,} \\ K_t &:= d\mathcal{F}\left(M_{1_{(-\infty,t]}}\right) \quad \text{the number of outgoing quanta that have left the oscillator.} \end{aligned}$$

Note that the number $M_t - N_t$ of incoming quanta is not given by a multiplication operator, but the number K_t of outgoing quanta is. This is due to the fact that we are considering the outgoing translation dilation of T_t. Note furthermore that the operators M_t and K_s $(s, t \in \mathbb{R})$ all commute.

For positive times the number $M_t - N_t$ of incoming quanta has expectation 0 in the states of the form $\vartheta \circ P$ $(\vartheta \in l^1(\mathbb{N}))$ which we consider. So we may expect that replacing N_t by M_t would lead to an embedded classical Markov chain.

Proposition 6.4. *For $t \geq 0$ we have the following commuting diagram involving abelian von Neumann algebras.*

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{T_t} & \mathcal{N} \\ {\scriptstyle j:f\mapsto f(M_0)}\downarrow & & \uparrow{\scriptstyle P=\Gamma(J^*)} \\ L^\infty(\Delta, \mu_\eta) & \xrightarrow{\Gamma(U_t)} & L^\infty(\Delta, \mu_\eta) \end{array}$$

Proof. For all $z, u \in \mathbb{C}$, $t \geq 0$ and $s \in [0,1]$ we have

$$\begin{aligned}
\langle \pi(u), P \circ \Gamma(U_t) \circ j(s^N)\pi(z)\rangle &= \langle \pi(u), P\left(s^{M_t}\right)\pi(z)\rangle \\
&= \sum_{n=0}^{\infty} (\overline{u}z)^n \int_{\Delta_n(\mathbb{R})} s^{M_t(\sigma)} |v^{\otimes n}(\sigma)|^2 \mu_\eta(d\sigma) \\
&= \sum_{n=0}^{\infty} (2\eta\overline{u}z)^n \cdot \frac{1}{n!} \int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} s^{1_{[t,\infty)}(r_j)} |v(r_j)|^2 \right) dr_1 \cdots dr_n \\
&= \exp\left(2\eta\overline{u}z \int_0^\infty e^{-2\eta r} s^{1_{[t,\infty)}(r)} dr \right) \\
&= \exp\left(\overline{u}z \left(-e^{-2\eta r} \Big|_0^t - s e^{-2\eta r} \Big|_t^\infty \right) \right) \\
&= \exp\left(\overline{u}z (1 - e^{-2\eta t}(1-s)) \right) .
\end{aligned}$$

By Proposition 6.2 the latter expression is equal to $\langle \pi(u), T_t(s^N)\pi(z)\rangle$. □

Finally, since $K_t + M_t$ is equal to the total number of quanta, which is the same constant for all times, we conclude that a quantum is emitted at precisely the moment that the oscillator makes a downward jump. Moreover, these jumps are made at independent exponentially distributed random times.

These phenomena turn out to be natural consequences of damped harmonic motion in a noncommutative description.

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