

An ergodic theorem for repeated and continuous measurement

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paper in preparation.

Abstract: We prove an ergodic theorem for repeated measurement, indicating its significance for quantum trajectories in discrete time. We roughly sketch the extension to continuous time, and some connections to the algebraic theory of quantum Markov processes.

1. Measurement in an operational approach.

A measurement, whether quantummechanical or not, is an operation performed on a physical system which results in the extraction of information from that system, while possibly changing its state.

So before the measurement there is the physical system, described by a state ρ , (a probability measure in the classical case, a density matrix in the quantum case), and afterwards there is a piece of information, say an outcome $i \in \{1, 2, \dots, k\}$, and there is the system itself, in some new (or *posterior*) state θ_i :

$$\rho \longrightarrow (i, \theta_i) .$$

Now, a probabilistic theory rather than predicting an outcome i , gives a probability distribution $(\pi_1, \pi_2, \dots, \pi_k)$ on the possible outcomes. In fact the measurement operation is described by an affine map

$$M_* : \rho \mapsto (\pi_1 \theta_1, \pi_2 \theta_2, \dots, \pi_k \theta_k) ,$$

taking a state ρ on the algebra \mathcal{A} of observables of the system to a state on the tensor product of $\mathcal{C} := \mathbb{C}^k$ with \mathcal{A} . In the literature on measurement theory this is called an *operation valued measure* or *instrument* [Dav, Hel, Hol1, BGL]. We shall call the i 'th component $\pi_i \theta_i$ of the right hand side: $(T_i)_*(\rho)$. The maps M_* and $(T_i)_*$ are the (pre)duals of completely positive maps

$$M : \mathcal{C} \otimes \mathcal{A} \rightarrow \mathcal{A} \quad \text{and} \quad T_i : \mathcal{A} \rightarrow \mathcal{A} .$$

T_i describes the effect on the system's observables of the occurrence of an outcome i . The effect of the measurement on the system, when we ignore the outcome, is given by the map

$$T : \mathcal{A} \rightarrow \mathcal{A} : x \mapsto M(\mathbf{1} \otimes x) = \sum_{i=1}^k T_i(x) .$$

On the other hand, if we ignore the system after the measurement, we obtain the map

$$Q : \mathcal{C} \rightarrow \mathcal{A} : f \mapsto M(f \otimes \mathbf{1}) = \sum_{i=1}^k f(i) T_i(\mathbf{1}) ,$$

which is known as a *positive operator valued measure* or *generalised observable*. We note that M , T and Q are all completely positive and identity preserving linear operators on C^* -algebras. Such maps are called *operations*.

Example 1: Classical measurement with error.

We measure the length X of a bar by means of a measuring stick. The length X is a random variable having distribution ρ with support $[0, K]$, say. Its algebra of observables \mathcal{A} is $L^\infty([0, K], \rho)$. After the measurement the bar has the same length X as before, but a second random variable Y has arisen whose values depend in a stochastic way on X . Let us assume that Y is the result of measuring the length X with some random error, rounded off to an integer number of millimeters. Then Y takes values from 0 to k , where k is number of millimeters in the upper bound K .

This example is described by

$$((T_i)_*(\rho))(d\xi) = \pi_i(\xi)\rho(d\xi) ,$$

where $\pi_i(\xi)$ is the probability that the length $\xi \in [0, K]$ will be ‘measured’ as i millimeters. In the dual ‘Heisenberg’ picture the measurement is given by

$$M : \mathcal{C} \otimes \mathcal{A} \rightarrow \mathcal{A} : \quad M(f \otimes g)(\xi) = \sum_{i=1}^k f(i)\pi_i(\xi)g(\xi) .$$

In this example the measurement has no effect on the system, as is expressed by the relation

$$(Tg)(\xi) = \sum_{i=1}^k (T_i g)(\xi) = \left(\sum_{i=1}^k \pi_i(\xi) \right) g(\xi) = g(\xi) .$$

The generalised random variable Q is given by

$$Q(f)(\xi) = \sum_{i=1}^k f(i)\pi_i(\xi) .$$

Example 2: von Neumann measurement.

Let $\mathcal{A} := M_n$, the algebra of all complex $n \times n$ -matrices. We think of \mathcal{A} as the observable algebra of some finite quantum system. Let p_1, p_2, \dots, p_k be mutually orthogonal projections in \mathcal{A} adding up to $\mathbf{1}$.

If some physical quantity is described by a self-adjoint matrix in \mathcal{A} whose eigenspaces are the ranges of the p_i , then according to von Neumann’s projection postulate a measurement of this quantity is described by

$$(T_i)_*(\rho) = p_i \rho p_i, \quad \text{so} \quad M(f \otimes x) = \sum_{i=1}^k f(i)p_i x p_i .$$

Example 3: von Neumann measurement followed by unitary evolution.

Modify the above example by taking

$$M(f \otimes x) := \sum_{i=1}^k p_i u^* x u p_i .$$

Each von Neumann measurement is now followed by a fixed unitary time evolution. This will have the effect of making repetitions of this operation more interesting.

Example 4: Kraus measurement.

Couple the finite quantum system with observable algebra \mathcal{A} to an finite ‘apparatus’ with observable algebra \mathcal{B} in the initial state β . Let the two systems evolve for a while, say according to a unitary matrix $u \in \mathcal{B} \otimes \mathcal{A}$, and then perform a von Neumann measurement on \mathcal{B} described by the mutually orthogonal projections $p_1, \dots, p_k \in \mathcal{B}$. Then obtain

$$(T_i)_*(\rho) : x \mapsto (\beta \otimes \rho)(u^*(p_i \otimes x)u) ,$$

or, in the ‘Heisenberg picture’,

$$T_i(x) = (\beta \otimes \text{id})(u^*(p_i \otimes x)u) .$$

Let us call this indirect von Neumann measurement *perfect* if β is a pure state and the p_i are one-dimensional projections. (This corresponds to maximal information concerning the apparatus, and maximally efficient measurement.) If this is the case, let us write $\beta(y) = \langle v, yv \rangle_{\mathcal{B}}$ and $p_i = |e_i\rangle\langle e_i|$. Then T_i is of the form

$$T_i(x) = a_i^* x a_i ,$$

where the *Kraus matrices* a_1, \dots, a_k [Kra] are given by

$$a_i = \sum_{j=1}^k \langle e_i, u e_j \rangle_{\mathcal{B}} \langle e_j, v \rangle$$

Here we have used the notation

$$\langle e_i, (y \otimes x) e_j \rangle_{\mathcal{B}} := \langle e_i, y e_j \rangle x \quad (x \in \mathcal{A}, y \in \mathcal{B}).$$

2. Repeated measurement.

By repeating the measurement operation of the previous section indefinitely, we obtain for every initial state ρ of the finite quantum system a stochastic process in discrete time, taking values in the outcome space $\mathcal{X} := \{1, 2, \dots, k\}$. We shall prove an ergodic theorem for this type of process.

Let $\Omega := \mathcal{X}^{\mathbb{N}}$, and let for $m \in \mathbb{N}$ and $i_1, \dots, i_m \in \mathcal{X}$ the cylinder sets $\Lambda_{i_1, \dots, i_m} \subset \Omega$ be given by

$$\Lambda_{i_1, \dots, i_m} := \{\omega \in \Omega \mid \omega_1 = i_1, \dots, \omega_m = i_m\} .$$

Denote by Σ the σ -algebra generated by all these Σ_m .

Let \mathcal{A} be a finite-dimensional von Neumann algebra, and let T_i ($i = 1, \dots, k$) be completely positive operators $\mathcal{A} \rightarrow \mathcal{A}$ such that their sum maps $\mathbf{1}_{\mathcal{A}}$ to itself.

Proposition 1. *There exists a unique \mathcal{A} -valued probability measure Q_∞ on (Ω, Σ) such that*

$$Q_\infty(\Lambda_{i_1, \dots, i_m}) = T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m}(\mathbf{1}) .$$

In particular, if ρ is a state on \mathcal{A} , then $\mathbb{P}_\rho := \rho \circ Q_\infty$ is an ordinary $[0, 1]$ -valued probability measure on (Ω, Σ) .

Proof. By the reconstruction theorem of Kolmogorov and Daniel it suffices to prove consistency: for all $i_1, \dots, i_m \in \mathcal{X}$,

$$\sum_{i=1}^k Q_\infty(\Lambda_{i_1, \dots, i_m, i}) = Q_\infty(\Lambda_{i_1, \dots, i_m}) .$$

Indeed, since $T(\mathbf{1}) = \mathbf{1}$, the l.h.s. is equal to

$$\begin{aligned} \sum_{i=1}^k T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m} \circ T_i(\mathbf{1}) &= T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m} \circ T(\mathbf{1}) \\ &= T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m}(\mathbf{1}) , \end{aligned}$$

which is equal to the r.h.s. □

We shall consider the left shift σ on Ω given by

$$(\sigma\omega)_j := \omega_{j+1} .$$

A probability measure μ on (Ω, Σ) is called *stationary* if for all $B \in \Sigma$ we have

$$\mu(\sigma^{-1}(B)) = \mu(B) .$$

Proposition 2. *If $\rho \circ T = \rho$, then \mathbb{P}_ρ is stationary.*

Proof. Since any probability measure on (Ω, Σ) is determined by its values on the cylinder sets $\Lambda_{i_1, \dots, i_m}$, it suffices to prove the equality

$$\mathbb{P}_\rho(\sigma^{-1}(\Lambda_{i_1, \dots, i_m})) = \mathbb{P}_\rho(\Lambda_{i_1, \dots, i_m}) .$$

Now,

$$\sigma^{-1}(\Lambda_{i_1, \dots, i_m}) = \bigcup_{i=1}^k \Lambda_{i, i_1, \dots, i_m} .$$

Therefore, if $\rho \circ T = \rho$, the l.h.s. of the equality to be proved is equal to

$$\begin{aligned} \sum_{i=1}^k \mathbb{P}_\rho(\Lambda_{i, i_1, \dots, i_m}) &= \sum_{i=1}^k \rho \circ T_i \circ T_{i_1} \circ \dots \circ T_{i_m}(\mathbf{1}) \\ &= \rho \circ T \circ T_{i_1} \circ \dots \circ T_{i_m}(\mathbf{1}) \\ &= \rho \circ T_{i_1} \circ \dots \circ T_{i_m}(\mathbf{1}) , \end{aligned}$$

which is equal to the r.h.s. □

In preparation of our ergodic theorem we prove the following lemma.

Lemma 3. For all $B \in \Sigma$, $m \in \mathbb{N}$, and $i_1, i_2, \dots, i_m \in \mathcal{X}$ we have

$$Q_\infty(\Lambda_{i_1, \dots, i_m} \cap \sigma^{-m}(B)) = T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m} \circ Q_\infty(B),$$

in particular,

$$Q_\infty(\sigma^{-1}(B)) = T \circ Q_\infty(B).$$

Proof. It suffices to prove the first equality for $B = \Lambda_{j_1, \dots, j_l}$ (some $l \in \mathbb{N}$, $j_1, \dots, j_l \in \mathcal{X}$). But then

$$\Lambda_{i_1, \dots, i_m} \cap \sigma^{-m}(B) = \Lambda_{i_1, \dots, i_m, j_1, \dots, j_l},$$

so that both sides of the first equality are equal to

$$T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m} \circ T_{j_1} \circ T_{j_2} \circ \dots \circ T_{j_l}(\mathbf{1}).$$

The second equality follows from the first since its l.h.s. side is equal to

$$\sum_{i=1}^k Q_\infty(\Lambda_i \cap \sigma^{-1}(B)).$$

□

We shall call an \mathcal{A} -valued probability measure R on (Ω, Σ) *ergodic* if for all $E \in \Sigma$ we have

$$\sigma^{-1}(E) = E \quad \Rightarrow \quad R(E) \in \{0, \mathbf{1}\}.$$

Theorem 4. If T_* has a unique invariant state, then Q_∞ is ergodic.

An important consequence of the above ergodicity theorem is that path averages are equal to quantummechanical expectations:

Corollary 5. (Ergodic theorem for repeated measurement.) If T_* has a unique invariant state $\rho \in \mathcal{A}_*$, then for any initial state $\theta \in \mathcal{A}_*$ and any sequence $i_1, \dots, i_m \in \{1, 2, \dots, k\}$, we have almost surely with respect to \mathbb{P}_θ

$$\lim_{n \rightarrow \infty} \frac{1}{n} \cdot \#\{j < n \mid \omega_{j+1} = i_1, \omega_{j+2} = i_2, \dots, \omega_{j+m} = i_m\} = \rho \circ T_{i_1} \circ \dots \circ T_{i_m}(\mathbf{1}).$$

Proof of the Corollary. By Proposition 2, \mathbb{P}_ρ is stationary. By Birkhoff's individual ergodic theorem, the path average on the l.h.s., $F(\omega)$ say, exists for almost all $\omega \in \Omega$. Since $F = F \circ \sigma$, the events $E_{[a,b]} := \{\omega \in \Omega \mid a \leq F(\omega) \leq b\}$ are σ -invariant, hence by Theorem 4 they all have Q_∞ -measure either 0 or $\mathbf{1}$. This implies that for some $c \in \mathbb{R}$ we have $Q_\infty(E_{\{c\}}) = \mathbf{1}$, hence $\mathbb{P}_\theta(E_{\{c\}}) = 1$ for all states $\theta \in \mathcal{A}_*$. In particular c must be the expectation $\mathbb{E}_\rho(F)$ of F under \mathbb{P}_ρ . Using the stationarity of ρ we may calculate:

$$c = \mathbb{E}_\rho(F) = \mathbb{E}_\rho(1_{\Lambda_{i_1, \dots, i_m}}) = \mathbb{P}_\rho(\Lambda_{i_1, \dots, i_m}) = \rho \circ T_{i_1} \circ \dots \circ T_{i_m}(\mathbf{1}).$$

□

Proof of Theorem 4. As \mathcal{A} is finite-dimensional, uniqueness of the T_* -invariant state ρ implies that all T -invariant elements of \mathcal{A} are multiples of $\mathbf{1}$. Now let $E \in \Sigma$ be such that $\sigma^{-1}(E) = E$. Then by Lemma 3,

$$Q_\infty(E) = Q_\infty(\sigma^{-1}(E)) = T \circ Q_\infty(E) ,$$

so $Q_\infty(E) = \lambda \cdot \mathbf{1}$. It remains to show that $\lambda = 0$ or 1 . For this purpose, define an \mathcal{A} -valued measure Q_E on (Ω, Σ) by

$$Q_E(B) := Q_\infty(B \cap E), \quad (B \in \Sigma).$$

By Lemma 3 we have for all $m \in \mathbb{N}$, $i_1, i_2, \dots, i_m \in \mathcal{X}$,

$$\begin{aligned} Q_E(\Lambda_{i_1, \dots, i_m}) &= Q_\infty(\Lambda_{i_1, \dots, i_m} \cap E) \\ &= Q_\infty(\Lambda_{i_1, \dots, i_m} \cap \sigma^{-m}(E)) \\ &= T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m}(Q_\infty(E)) \\ &= \lambda \cdot T_{i_1} \circ T_{i_2} \circ \dots \circ T_{i_m}(\mathbf{1}) \\ &= \lambda Q_\infty(\Lambda_{i_1, \dots, i_m}) . \end{aligned}$$

And since a measure on (Ω, Σ) is determined by its values on the cylinder sets, we conclude that for all $B \in \Sigma$:

$$Q_E(B) = \lambda Q_\infty(B) .$$

Applying this relation to E itself, we find that

$$\lambda \cdot \mathbf{1} = Q_\infty(E) = Q_\infty(E \cap E) = Q_E(E) = \lambda Q_\infty(E) = \lambda^2 \cdot \mathbf{1} .$$

Therefore $\lambda = 0$ or 1 . □

3. Application to the examples.

Example 1: Classical measurement with error.

In this example T is the identity map on \mathcal{A} . So the assumption of the Theorem is that $\dim(\mathcal{A})=1$. Since $\mathcal{A} = L^\infty([0, K], \rho)$, this means that $\rho = \delta_\xi$ for some length $\xi \in [0, K]$. In that case the measurement process $\omega_1, \omega_2, \dots$ is a sequence of independent random variables all with distribution $\pi(\xi)$. Such a sequence is indeed ergodic by the law of large numbers. Note however, that if different values ξ_1 and ξ_2 can occur with positive probability, then the path average would still exist, but could take different values according to chance.

Example 2: Repeated von Neumann measurement.

This is not an interesting case. The first measurement determines the outcome, and all later measurements confirm it. Uniqueness of ρ amounts to $k = 1$, i.e., we are measuring a sure observable without error.

Example 3: Alternating von Neumann measurement and Schrödinger evolution.

In this case the condition of uniqueness of the invariant state becomes

$$\{u, p_1, p_2, \dots, p_k\}' = \mathbb{C} \mathbf{1} .$$

The unique invariant state is the *trace* state on $\mathcal{A} = M_n$: $\rho(x) = \frac{1}{n} \text{tr}(x)$. If we take for p_i one-dimensional projections, say $p_i = |e_i\rangle\langle e_i|$, then the stochastic sequence of outcomes is a Markov chain with transition probabilities

$$|\langle e_i, u e_j \rangle|^2 .$$

This is a bistochastic transition matrix, indeed having equipartition as an equilibrium distribution. The condition that $\{u, p_1, p_2, \dots, p_k\}' = \mathbb{C} \cdot \mathbf{1}$ makes the transition matrix irreducible and the equilibrium distribution unique.

Example 4: Davies processes or quantum trajectories in discrete time.

This is our most interesting example. Let us take repeated Kraus measurements, i.e.

$$T_i(x) = a_i^* x a_i, \quad (i = 1, \dots, k) ,$$

for some $a_1, a_2, \dots, a_k \in \mathcal{A} = M_n$ with $\sum_i a_i^* a_i = \mathbf{1}$. Then, if $\rho \circ T = \rho$ we have a stationary measurement sequence satisfying

$$\mathbb{P}_\rho[\omega_1 = i_1, \omega_2 = i_2, \dots, \omega_m = i_m] = \rho(a_{i_1}^* a_{i_2}^* \cdots a_{i_m}^* a_{i_m} \cdots a_{i_2} a_{i_1}) .$$

In general, this is not a Markov chain. However, it is intimately connected with the following Hilbert space valued Markov chain.

On $(\Omega, \Sigma, \mathbb{P}_\rho)$, consider the stochastic process $\Psi_0, \Psi_1, \Psi_2, \dots$ with values in $\mathcal{H} := \mathbb{C}^n$ given by

$$\Psi_m(\omega) := \frac{a_{\omega_m} a_{\omega_{m-1}} \cdots a_{\omega_2} a_{\omega_1} \psi_0}{\|a_{\omega_m} a_{\omega_{m-1}} \cdots a_{\omega_2} a_{\omega_1} \psi_0\|} .$$

The process Ψ is called the *quantum trajectory* associated to this repeated Kraus measurement.

Proposition 6. *In the situation of Example 4 (perfect case), the stochastic process $\Psi_0, \Psi_1, \Psi_2, \dots$ is a classical Markov chain on the unit sphere of \mathcal{H} with initial condition $\Psi_0 = \psi_0$ and transition probabilities*

$$P(\psi, \theta) = \sum_{i=1}^k \|a_i \psi\|^2 \delta_\theta \left(\frac{a_i \psi}{\|a_i \psi\|} \right) ,$$

where

$$\delta_{\theta_1}(\theta_2) := \begin{cases} 1 & \text{if } \theta_1 = \theta_2, \\ 0 & \text{otherwise.} \end{cases}$$

The proof is a straightforward verification.

This Hilbert space valued version of the repeated Kraus measurement is very well suited for numerical simulation, and has been fruitfully employed in areas such as quantum optics [CSVR, WiM]. Our ergodic theorem implies that, if ρ is the unique T_* -invariant state, then the jump process of this quantum trajectory is ergodic, i.e. a single path reveals all the statistical properties of the whole jump process.

4. Continuous measurement.

In this Section we roughly sketch how the ergodic theorem of Section 2 can be extended to continuous measurement.

Making minimal assumptions, still allowing essentially the same proof, we arrive at the following structure.

For Σ we take a σ -algebra of subsets of some sample space Ω , and for all $0 \leq a \leq b$ we assume that we have a sub- σ -algebra $\Sigma_{[a,b]}$ of Σ such that, for $0 \leq a \leq b \leq c$,

$$\Sigma_{[a,b]} \cap \Sigma_{[b,c]} = \{\emptyset, \Omega\} \quad \text{and} \quad \Sigma_{[a,b]} \vee \Sigma_{[b,c]} = \Sigma_{[a,c]},$$

expressing the localisation in time of the measurement outcomes. We assume that for all $t \geq 0$ a (left) time shift $\sigma_t : \Omega \rightarrow \Omega$ is given, i.e. for all $t \geq 0$ and all a, b with $0 \leq a \leq b$ we must have:

$$\{\sigma_t^{-1}(A) \mid A \in \Sigma_{[a,b]}\} = \Sigma_{[a+t, b+t]}.$$

Let \mathcal{A} be our finite-dimensional von Neumann algebra, and for all $t \geq 0$ let a $\text{CP}(\mathcal{A})$ -valued measure M_t on $\Sigma_{[0,t]}$ be given such that for all $s, t \geq 0$,

(a) $T_t := M_t(\Omega)$ maps $\mathbf{1}_{\mathcal{A}}$ to itself;

(b) if $A \in \Sigma_{[0,t]}$ and $B \in \Sigma_{[0,s]}$, then $M_{t+s}(A \cap \sigma_t^{-1}(B)) = M_t(A) \circ M_s(B)$.

Then one proves along the same lines as in Section 2 that the family of \mathcal{A} -valued probability measures

$$Q_t : \Sigma_{[0,t]} \rightarrow \mathcal{A} : B \mapsto M_t(B)(\mathbf{1}_{\mathcal{A}})$$

is consistent and extends to a single \mathcal{A} -valued probability measure Q_∞ on Σ . Moreover, this measure is ergodic provided that the semigroup $((T_t)_*)_{t \geq 0}$ admits only a single invariant state on \mathcal{A} .

The above abstract scheme contains all the examples of continuous measurement termed ‘Markovian’, such as the jump processes of Srinivas and Davies [SrD], the diffusions of Gisin [Gis], and any infinitely divisible instrument in the sense of Holevo [Hol2, BaH].

5. Some algebraic connections.

In Section 1 we have seen that a *measurement* on a system with observable algebra \mathcal{A} can be viewed as an operation

$$M : \mathcal{C} \otimes \mathcal{A} \rightarrow \mathcal{A}.$$

with \mathcal{C} abelian. In the spirit of Example 4 (Kraus measurement) we may extend this idea somewhat by allowing the information extracted from the system to be quantum information: we replace the abelian algebra $\mathcal{C} \subset \mathcal{B}$ by \mathcal{B} itself, thus

postponing the choice of the abelian subalgebra to a later stage. So let us define a *generalised measurement operation* as an operation

$$M : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} .$$

Repeating this generalised measurement indefinitely leads to the scheme

$$\begin{array}{c}
 \mathcal{A} \\
 \nearrow \\
 \overbrace{\mathcal{B} \otimes \mathcal{A}} \\
 \nearrow \quad \nearrow \\
 \mathcal{B} \otimes \overbrace{\mathcal{B} \otimes \mathcal{A}} \\
 \nearrow \quad \nearrow \quad \nearrow \\
 \mathcal{B} \otimes \mathcal{B} \otimes \overbrace{\mathcal{B} \otimes \mathcal{A}} \\
 \vdots \quad \vdots \quad \vdots \quad \vdots
 \end{array}
 \begin{array}{l}
 M \\
 \text{id} \otimes M \\
 \text{id} \otimes \text{id} \otimes M \\
 \vdots
 \end{array}$$

In this way any state ρ on \mathcal{A} leads to a state on the infinite tensor product $\cdots \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{A}$. This is closely related to Accardi's early version of a Quantum Markov Process [Acc]. If ρ is invariant, i.e., if $\rho(M(\mathbf{1} \otimes x)) = \rho(x)$ for all $x \in \mathcal{A}$, then the above state naturally defines a shift-invariant state on $\bigotimes_{\mathbb{Z}} \mathcal{B}$, which was exploited by Fannes, Nachtegale and Werner to describe states on spin chains [FNW].

In the algebraic notation the m -fold measurement of Section 2 is described by the operation $M^{(m)} : \bigotimes_{i=1}^m \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A}$ given by

$$M^{(m)} := M \circ (\text{id} \otimes M) \circ \cdots \circ (\text{id} \otimes \cdots \otimes \text{id} \otimes M) .$$

For comparison we note that $M^{(m)}(p_1 \otimes \cdots \otimes p_m \otimes \mathbf{1}_{\mathcal{A}}) = Q_{\infty}(\Lambda_{i_1, \dots, i_m})$.

By attaching an infinite product of copies of (\mathcal{B}, β) to the right of the diagram above, which we interpret as a chain of measurement devices queuing up to be coupled to the system \mathcal{A} , we obtain a *dilation* in the sense of Kümmerer of the semigroup $(T^n)_{n \geq 0}$ to a group of automorphisms. This is indicated in the following diagram, which commutes for all $n \geq 0$.

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{T^n} & \mathcal{A} \\
 \mathbf{1} \otimes \text{id} \downarrow & & \uparrow (\bigotimes_{\mathbb{Z}} \beta) \otimes \text{id} \\
 (\bigotimes_{\mathbb{Z}} \mathcal{B}) \otimes \mathcal{A} & \xrightarrow{\widehat{T}^n} & (\bigotimes_{\mathbb{Z}} \mathcal{B}) \otimes \mathcal{A}
 \end{array} .$$

Here, \widehat{T} is given by

$$\widehat{T}(y \otimes x) := u^*(S y \otimes x) u ,$$

where S denotes the right shift on the infinite tensor power of \mathcal{B} , and $u \in \mathcal{B} \otimes \mathcal{A}$ is the unitary of Example 4, acting only on the 0-th component of this infinite tensor power.

The connection between the dilation and the repeated measurement is expressed by the following relation:

$$M^{(m)}(y_1 \otimes \cdots \otimes y_m \otimes x) = ((\otimes_{\mathbb{Z}} \beta) \otimes \text{id}) (\widehat{T}^m(\cdots \otimes \mathbf{1} \otimes y_1 \otimes \cdots \otimes y_m \otimes \overset{x}{\otimes} \mathbf{1} \otimes \mathbf{1} \otimes \cdots)).$$

Davies processes in discrete time are obtained by restriction to some abelian subalgebra $\otimes_{\mathbb{Z}} \mathcal{C}$ of $\otimes_{\mathbb{Z}} \mathcal{B}$.

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