Quantum Information, Probability and Statistics

Hans Maassen

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Overview of the talk

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   1.1 Axioms

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   4.2 Product states on the center; the cases $n = 2, 3, 4, 5$
5. Quantum statistics: Young diagrams as estimators

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This is indeed a generalization since the multiplication operators by 1_A with $A \in \Sigma$ are a full set of projections on $L^2(\Omega, \Sigma, \mathbb{P})$.

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It can be experimentally falsified.

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$$\mathbb{P}_\phi[p_\alpha = q_\beta] = \sin^2(\alpha - \beta) \ .$$

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$$\begin{array}{ccc} p_1(\omega) & - & q_1(\omega) \\ | & & | \\ q_2(\omega) & - & p_2(\omega) \end{array}$$

meets an even number of equality signs. $\square$

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Clearly

$$1 > \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\ .$$

So in this case:

$$\varphi[p_1 = q_1] > \varphi[p_1 = q_2] + \varphi[q_2 = p_2] + \varphi[p_2 = q_1]\ .$$

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Challenge: produce programs $\mathcal{O}, \mathcal{X}, \mathcal{Y}$ in such a way that for randomly chosen $a_1, \ldots, a_n; b_1, \ldots b_n \in \{0, 1\}$ the outputs $X_1(a), \ldots, X_n(a); Y_1(b), \ldots, Y_n(b)$ make the score

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the following has been obtained experimentally*:

$$\mathbb{E}(Z_n) = \frac{n}{4}(\sqrt{2} - 1) \approx 10.4\% \text{ of } n \ .$$

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Note that, by Gleason's theorem, under φ the expectation of the observable is

$$\mathbb{E}_\varphi(a) = \sum_{j=1}^{k} \alpha_j \mathbb{P}_\varphi[a \text{ takes value } \alpha_j] = \sum_{j=1}^{k} \alpha_j \varphi(p_j) = \varphi\left(\sum_{j=1}^{k} \alpha_j p_j\right) = \varphi(a) \ .$$

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where a and b are (nondegenerate) observables with spectral projections p_i and q_j, and $c^2 := \max_{i,j} \mathrm{tr}(p_i q_j)$.

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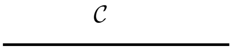

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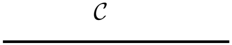

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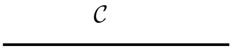

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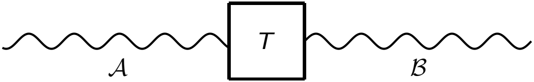

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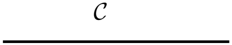

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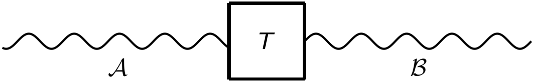

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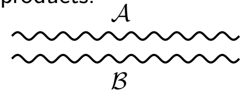

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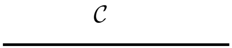

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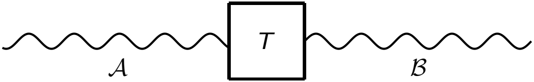

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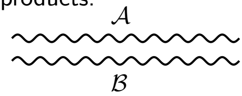

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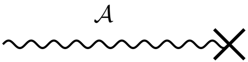

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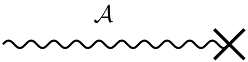

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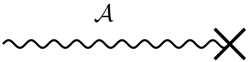

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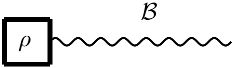

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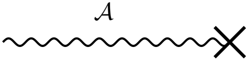

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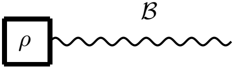

Interpretation: there are many ways to prepare a system, but only one way to destroy (or just ignore) it.

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(These results already hold in the commutative case.)

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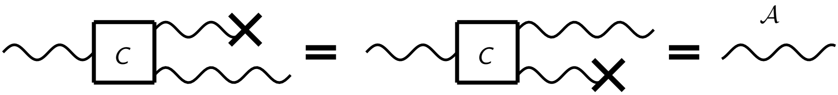

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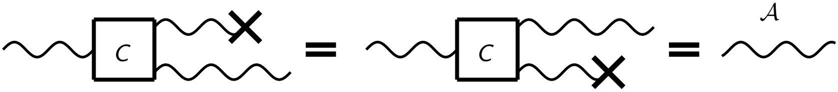

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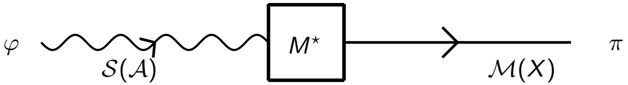

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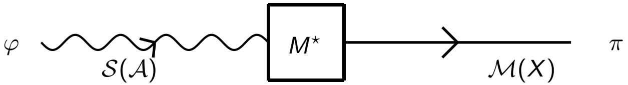

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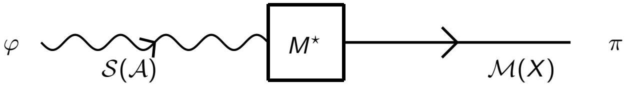

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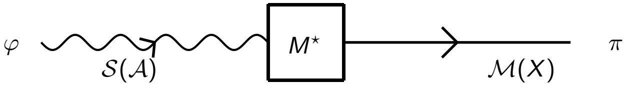

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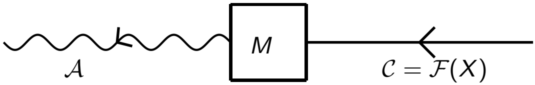

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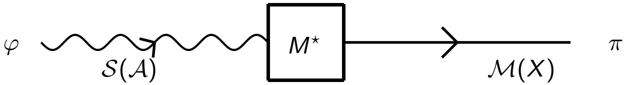

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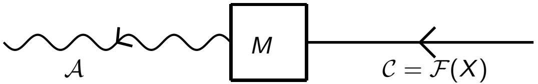

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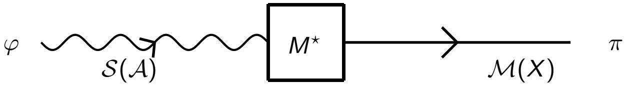

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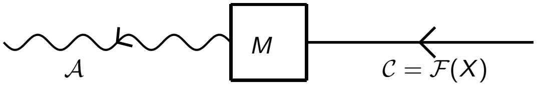

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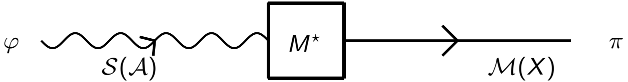

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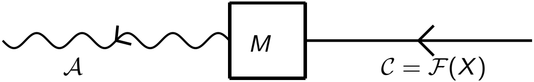

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Positive Operator Valued Measure (POVM)

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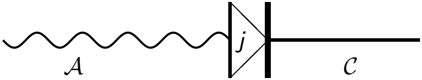

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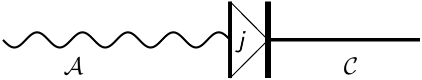

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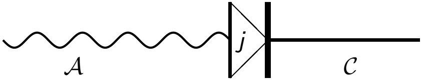

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We conclude that a von Neumann measurement is a right-invertible morphism in QProb from an arbitrary object to an abelian object.

Quantum Instruments

Quantum Instruments

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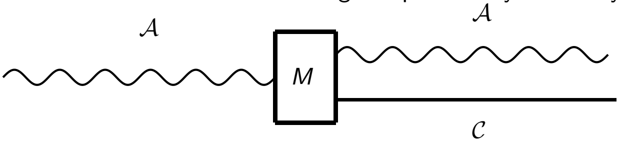

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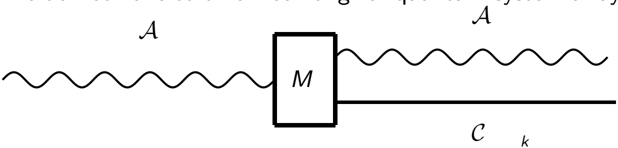

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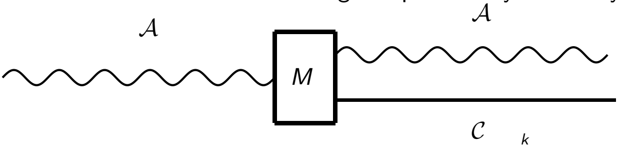

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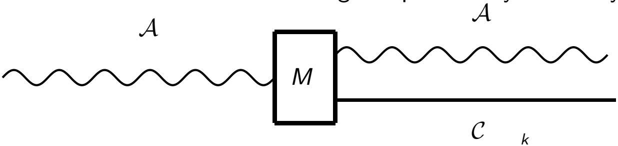

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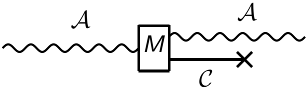

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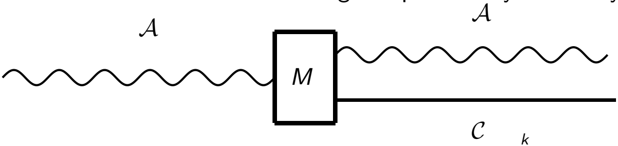

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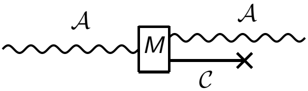

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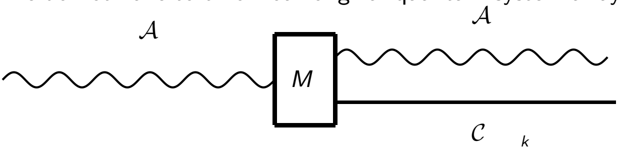

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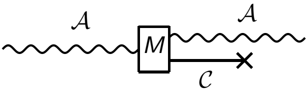

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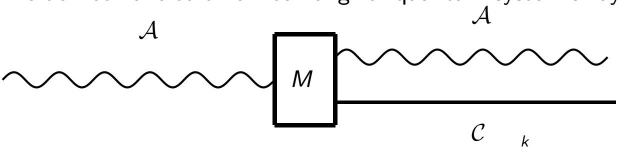

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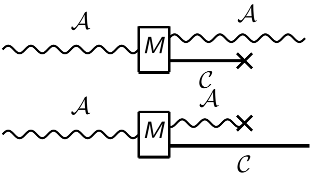

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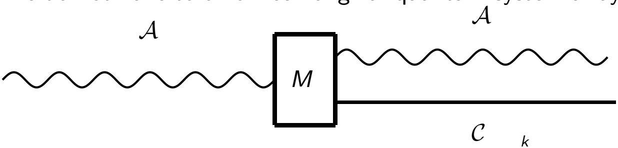

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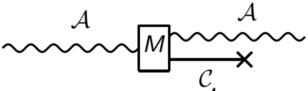

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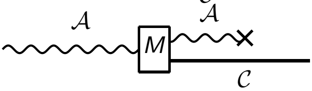

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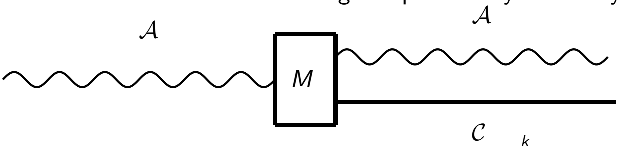

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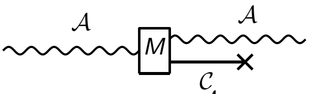

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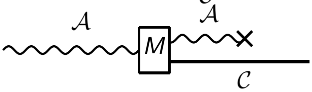

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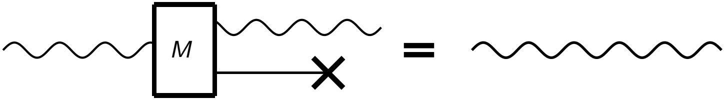

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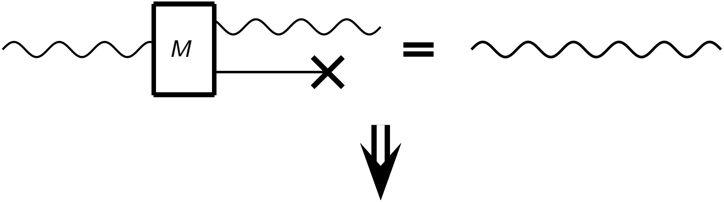

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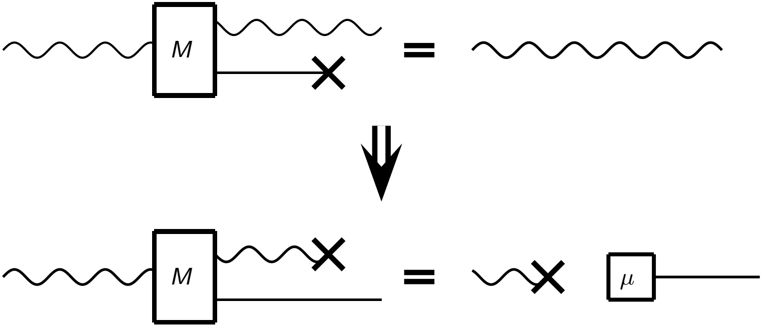

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Here $\widehat{\mathcal{A}}_I$ denotes the von Neumann algebra generated by $\widehat{T}_t \circ i(\mathcal{A})$, $t \in I$, and $\mathbb{E}(a|\mathcal{A}_I)$ the associated conditional expectation.

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I.e., a quantum system in a classical environment.

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$$L(x) = i[h, x] + \sum_{j=1}^{k} \big(a_j x a_j - \frac{1}{2}(a_j^2 x + x a_j^2)\big) + \sum_{i=1}^{l} \kappa_i (u_i^* x u_i - x) \ ,$$

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where h and $a_1, a_2, \ldots, a_k$ are self-adjoint, and $u_1, u_2, \ldots, u_l$ unitary elements of M_n, and $\kappa_1, \kappa_2, \ldots, \kappa_l$ are positive numbers.

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$$L(x) := c^* x c - \frac{1}{2}(c^* c x + x c^* c) \, , \qquad (x \in M_n),$$

with $c \in M_n$, then e^{tL} has an essentially commutative dilation iff c is a normal matrix (i.e.: $c^* c = c c^*$), and its spectrum lies on a circle or on a straight line.

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An important example is resonance fluorescence, the solutions of which can be explicitly given in terms of sum-integral kernels (Maassen 1985).

Repeated Instruments

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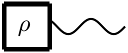

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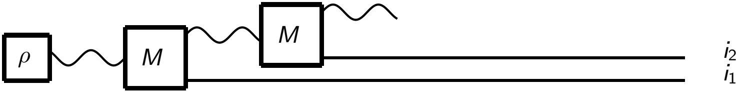

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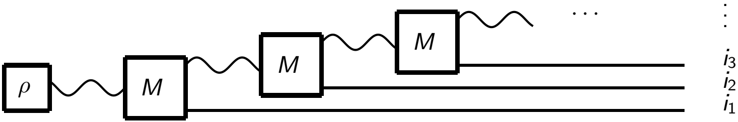

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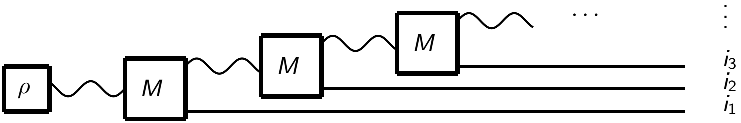

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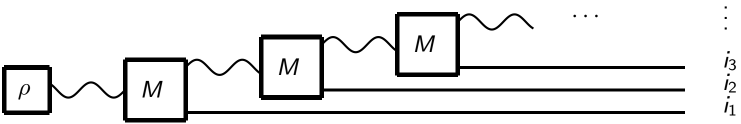

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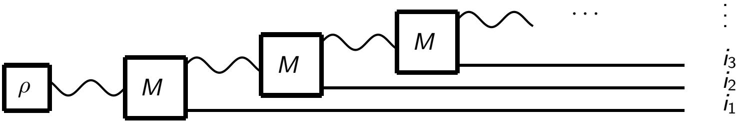

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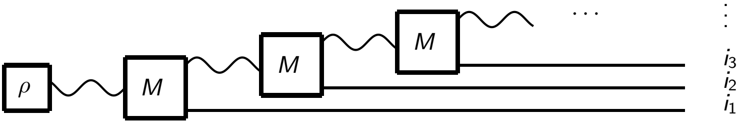

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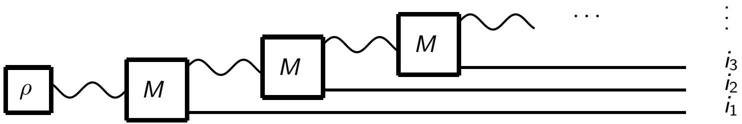

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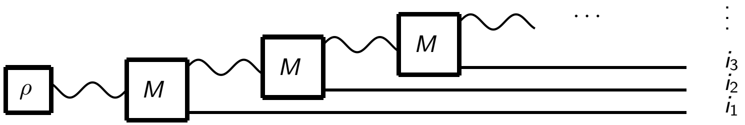

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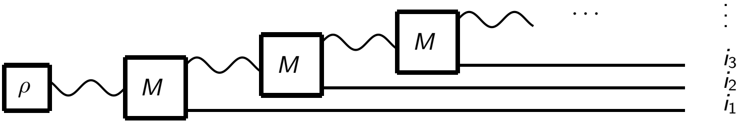

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Left shift: $\quad \sigma : \Omega \to \Omega : (\sigma\omega)_j := \omega_{j+1}$.

Ergodicity of measurement outcomes

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Theorem

If $T = M(\cdot \otimes \mathbb{1}) = \sum_i T_i$ *has a unique invariant state, then* Q_∞ *is ergodic, i.e.:*

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Corollary

If ρ is the unique invariant state, then the sequence of measurement outcomes is ergodic under $\mathbb{P}_\rho$.

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This extends to all $A \in \Sigma$. In particalar to the case $A = E$:

$$\lambda \cdot \mathbb{1} = Q_\infty(E) = Q_\infty(E \cap E) = \lambda Q_\infty(E)$$

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Suppose $\sigma^{-1}(E) = E$. Then

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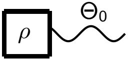
So $\lambda = 0$ or 1.

$\square$

Quantum Trajectories

Quantum Trajectories



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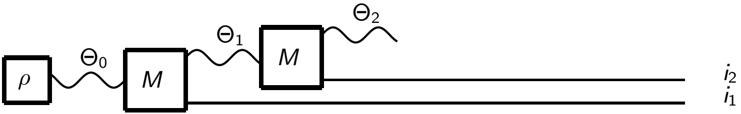

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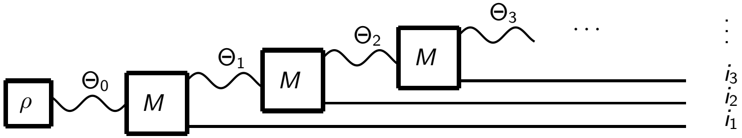

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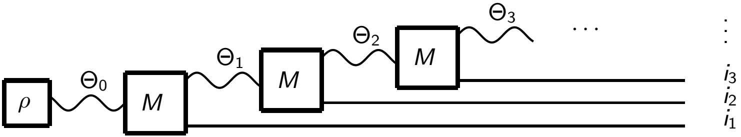

$$\Theta_n : \Omega \to \mathcal{S}(\mathcal{A}) : \qquad \Theta_n(\omega) : x \mapsto \frac{\rho(T_{\omega_1} \circ \cdots \circ T_{\omega_n}(x))}{\rho(T_{\omega_1} \circ \cdots \circ T_{\omega_n}(\mathbb{1}))} .$$

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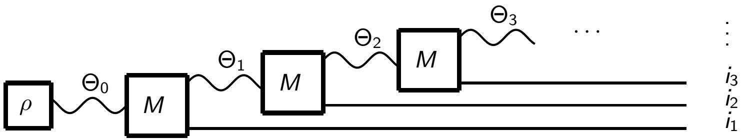

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Theorem

For any state ρ on $\mathcal{A}$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Theta_j = \Theta_\infty \qquad \mathbb{P}_\rho\text{-a.s.} ,$$

where the random variable Θ_∞ takes values in the T-invariant states on $\mathcal{A}$.

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Averaging over m yields: $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}\big(\Theta_j(x) - \Theta_j(Px)\big) \longrightarrow 0.$ $\square$

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For a given completely symmetric state, we want to find out if it is entangled or not, and, if so, to quantify how entangled it is.

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But also some recent work in pure mathematics turns out to be surprisingly relevant to our question.

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The minimal projections in this center cut both group representations into their irreducible components, and they are labeled by Young diagrams.

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$$\mathcal{Z} := \mathcal{Z}(n,d) := \pi(S_n)' \cap \pi'(SU(d))' \, .$$

The minimal projections in this center cut both group representations into their irreducible components, and they are labeled by Young diagrams.

For example

$$\begin{aligned} 3 \otimes 3 \otimes 3 \quad &= \quad (10 \otimes 1_+) \oplus (8 \otimes 2) \oplus (1 \otimes 1_-) \, . \\ &= \quad \ydiagram{3} \quad \oplus \quad \ydiagram{2,1} \quad \oplus \quad \ydiagram{1,1,1} \, . \end{aligned}$$

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$$\pi(f) : \quad \psi_1 \otimes \ldots \otimes \psi_n \mapsto \sum_{\sigma \in S_n} f(\sigma)\psi_{\sigma^{-1}(1)} \otimes \ldots \otimes \psi_{\sigma^{-1}n} \ .$$

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The trace on this "Hilbert space" is of a particularly simple form:

$$\mathrm{trreg}(f) := \sum_{\sigma \in S_n} \langle \delta_\sigma \ , f * \delta_\sigma \rangle = \sum_{\sigma \in S_n} (f * \delta_\sigma)(\sigma) = n! \cdot f(e) \ ,$$

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The normalized version $\tau_{\mathrm{reg}} := \frac{1}{n!}\mathrm{trreg}$ is the regular trace state.

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The states on the center form a simplex with extreme points ρ_i given by

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The character $\chi_i(\sigma)$ is the trace of σ in its irreducible representation labelled by i.

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For example:

$$d\left(\begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2} \end{array} \right) = \frac{5!}{4 \times 3 \times 2} = 5 \quad \text{hook lengths:} \quad \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 1 \\ \cline{1-2} \end{array} \, .$$

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Let $n, d \in \mathbb{N}$. Let Y denote a Young frame with n boxes and height $h(Y)$. Then

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For example, the symmetric subspace, having Young frame $\ydiagram{4}$, is nonzero in $(\mathbb{C}^d)^{\otimes 4}$ for every one-particle dimension d,
but, according to Pauli's exclusion principle, the antisymmetric subspace, with Young frame $\ydiagram{1,1,1,1}$, needs $d \geq 4$.
Hence the above theorem generalizes this exclusion principle.

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$$\begin{aligned} Ta &:= \int_{SU(d)} (u \otimes \ldots \otimes u)^* \, a \, (u \otimes \ldots \otimes u) \, du \; ; \\ Ma &:= \frac{1}{n!} \sum_{\sigma \in S_n} \pi(\sigma) \, a \, \pi(\sigma) \; . \end{aligned}$$

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Theorem (**Separability of completely symmetric states)**

Let ϑ be a completely symmetric state on $\mathcal{B}\big((\mathbb{C}^d)^{\otimes n}\big)$. Then ϑ is separable iff its restriction to $\mathcal{Z}$ lies in the convex hull of the restricted product states.

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Conversely, if for all $z \in \mathcal{Z}$ we have

$$\vartheta(z) = \sum_i \mu_i \langle \psi_i \, , z\psi_i \rangle \, ,$$

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for some positive weights μ_i with sum 1 and unit product vectors ψ_i, then since ϑ is completely symmetric, we have for all $x \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} \vartheta(x) &= \vartheta(Px) = \sum_i \mu_i \langle \psi_i \, , Px\psi_i \rangle \\ &= \frac{1}{n!} \sum_i \sum_{\sigma \in S_n} \int_{SU(d)} \mu_i \big\langle \pi(\sigma)(u \otimes \ldots \otimes u)\psi_i, x\, \pi(\sigma)(u \otimes \ldots \otimes u)\psi_i \big\rangle \, du \; , \end{aligned}$$

which is a convex inegral of product states, hence separable. $\square$

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\mathrm{tr}_d^{\otimes n}(\pi(\sigma)) &= \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \langle e_{i_1} \otimes \cdots \otimes e_{i_n} \,, \pi(\sigma)\, e_{i_1} \otimes \cdots \otimes e_{i_n} \rangle \\
&= \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} \delta_{i_1 i_{\sigma^{-1}(1)}} \cdots \delta_{i_n i_{\sigma^{-1}(n)}} \,. \\
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$$
\tau_d^{\otimes n}(p_y) = \frac{d(Y)}{n!} \sum_{\sigma \in S_n} \chi_Y(\sigma) \frac{1}{d^n} \mathrm{tr}_d^{\otimes n}(\pi(\sigma)) \to \frac{d(Y)^2}{n!} \, , \qquad (d \to \infty) \, .
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$$w_\psi(Y) := \langle \psi_1 \otimes \ldots \otimes \psi_n \, , \pi(p_Y) \, \psi_1 \otimes \ldots \otimes \psi_n \rangle \ .$$

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$$\rho_Y : \quad p_{Y'} \mapsto \delta_{YY'} \ .$$

The product states throw their shadow on this simplex:
the affine components of the product state $\psi_1 \otimes \ldots \otimes \psi_n$ are the weights

$$w_\psi(Y) := \langle \psi_1 \otimes \ldots \otimes \psi_n \ , \pi(p_Y)\, \psi_1 \otimes \ldots \otimes \psi_n \rangle \ .$$

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Theorem

The density of a product state $\psi_1 \otimes \ldots \otimes \psi_n$ with respect to the regular trace is the normalized immanant of the Gram matrix of $\psi_1, \psi_2, \ldots, \psi_n$.

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The first inequality was proved by Schur in 1918, the second was conjectured by Elliott Lieb in 1967, and is still open!

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&= \frac{d(Y)}{n!} \mathrm{Imm}_Y(G(\psi)) = \frac{d(Y)^2}{n!} \widetilde{\mathrm{Imm}}_Y(G(\psi)) \,.
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The simplex $\mathcal{S}(\mathcal{Z}_n)$ for $n = 3$

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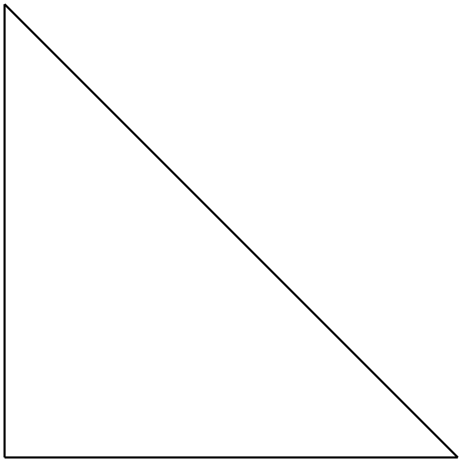

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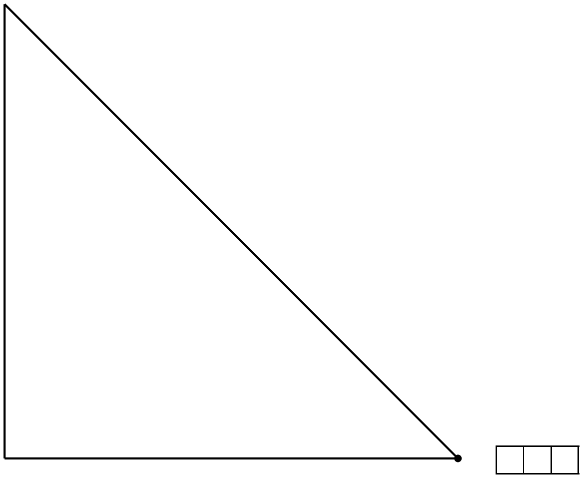

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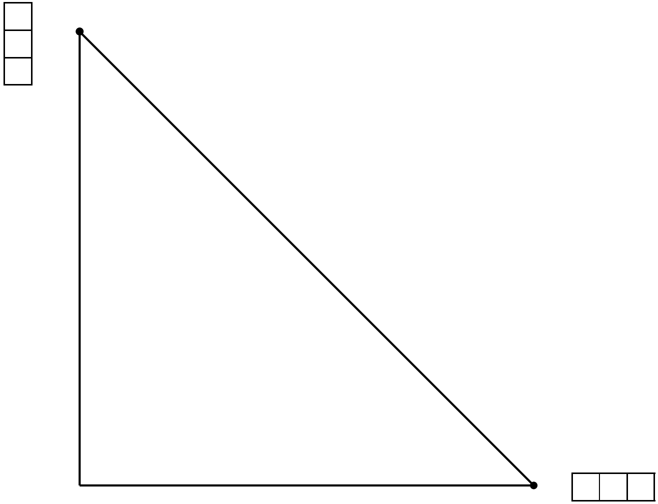

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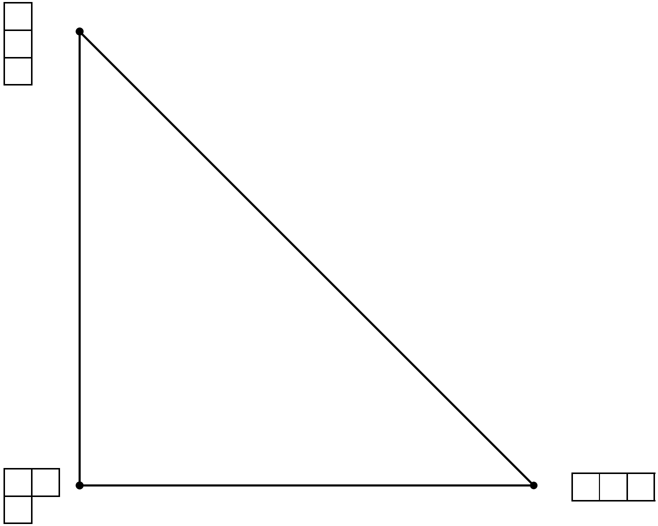

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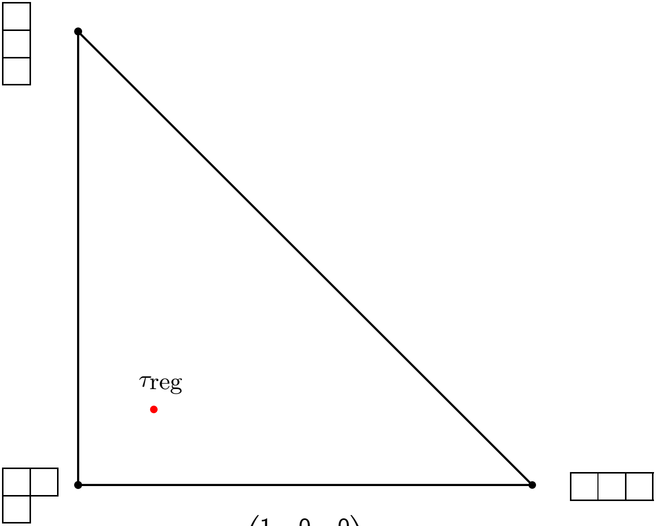

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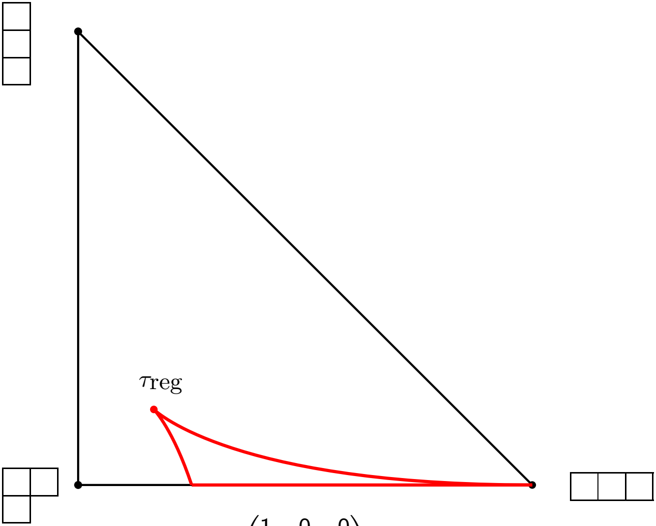

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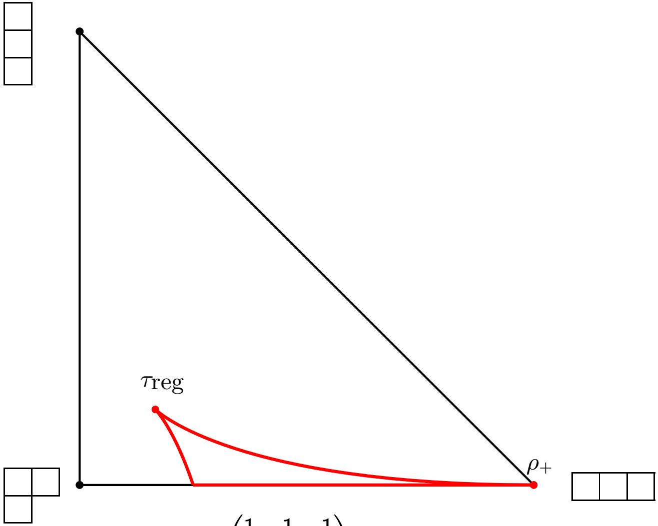

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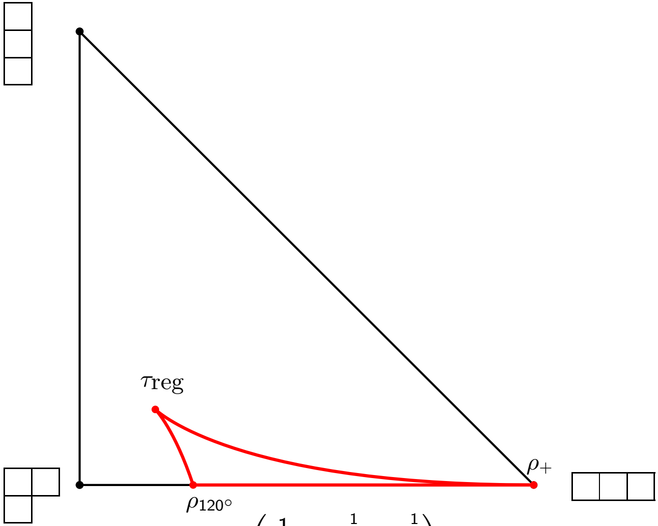

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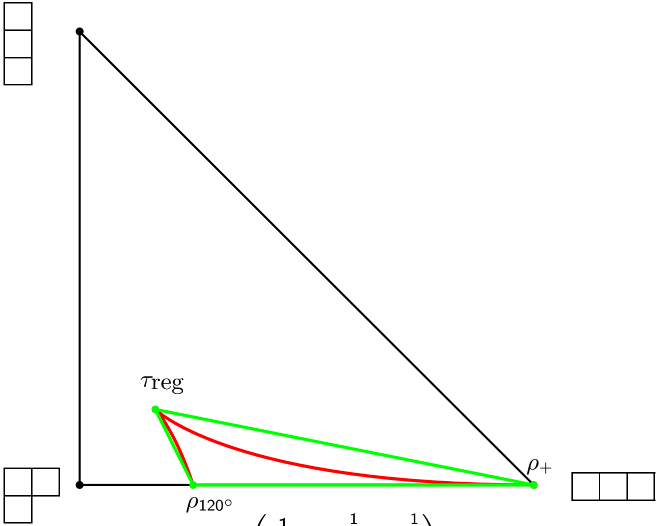

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- Does is grow or shrink with increasing n?

The Schur and Lieb inequalities

We have $2\mathcal{P}(n) - 3$ inequalities, which divide the state space $\mathcal{S}(\mathcal{Z}_n)$ into compartments, and claim the the shadow of the product states falls into one of them.

Schur's 1918 inequality states that for all separables states ρ and all Young frames $Y \neq \{-\}$:

$$\rho(p_Y) \geq d(Y)^2 \, \rho(p_-) \; .$$

Lieb's 1967 conjecture hopes that for all separable ρ and all Young frames

$Y \neq \{+\}$:

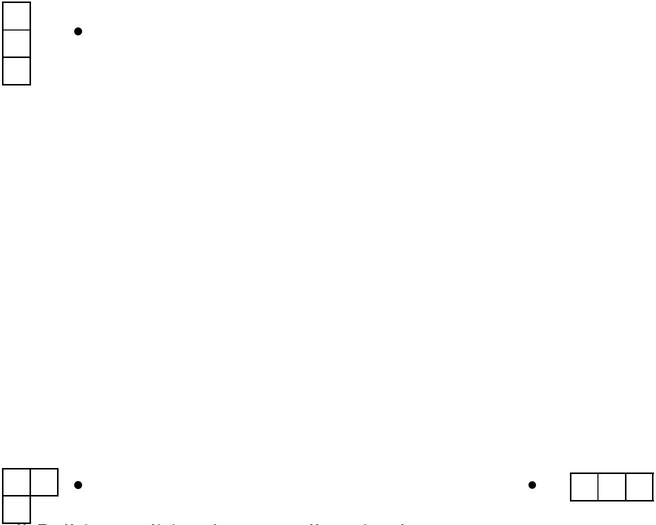
$$\rho(p_Y) \leq d(Y)^2 \, \rho(p_+) \; .$$

The last trivial inequality says that for all separable ρ:

$$\rho(p_-) \leq \rho(p_+) \; .$$

These are all Bell inequalities.

The immanant inequalities for $n = 3$ in a picture



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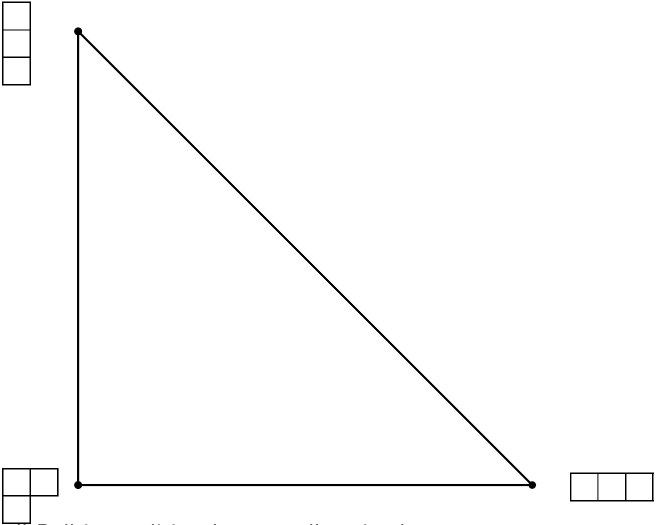

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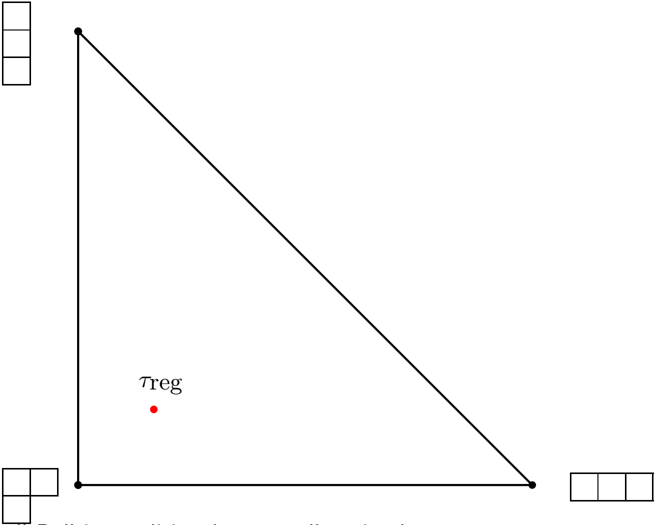

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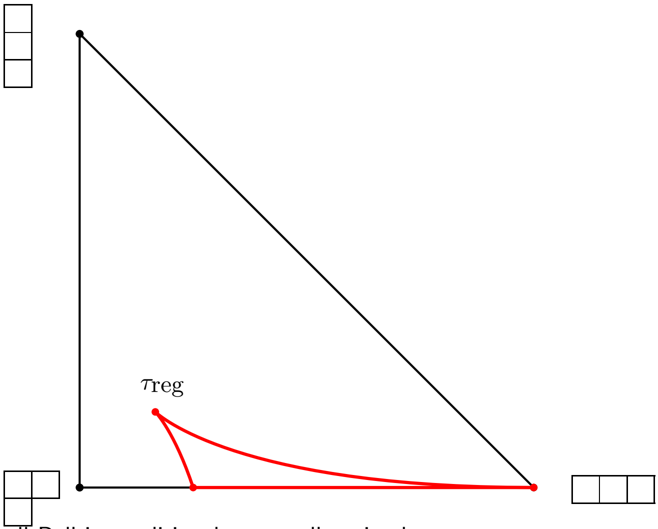

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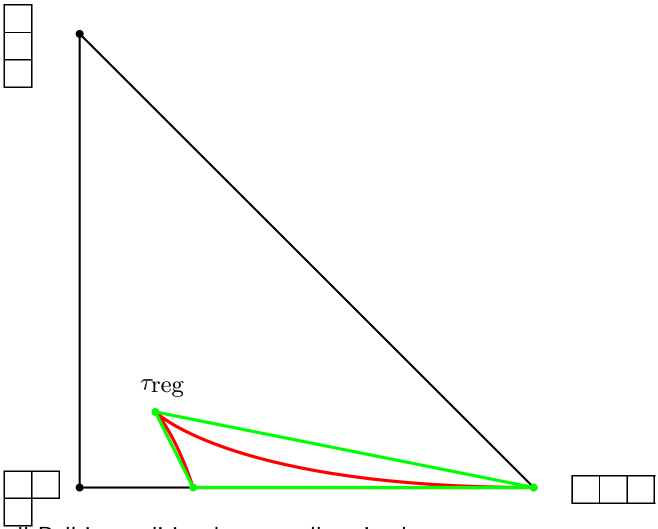

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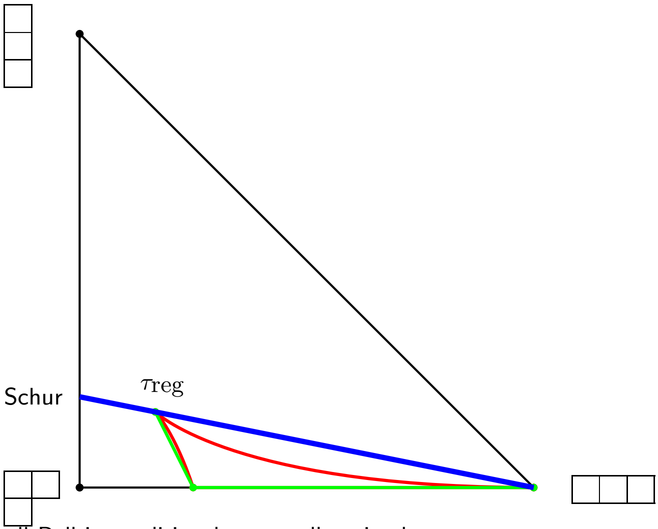

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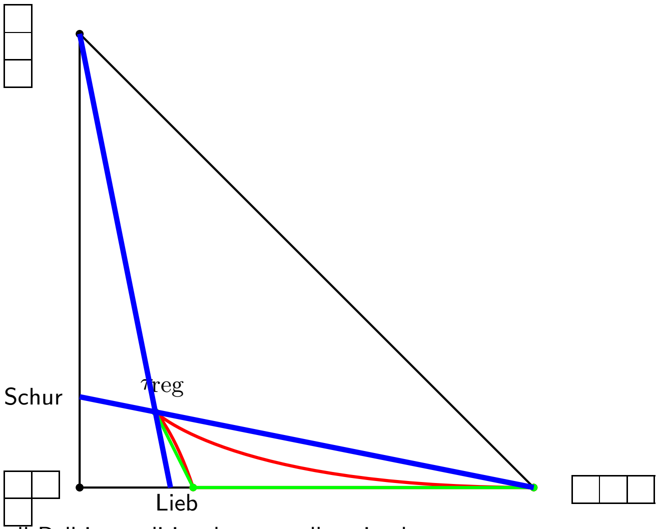

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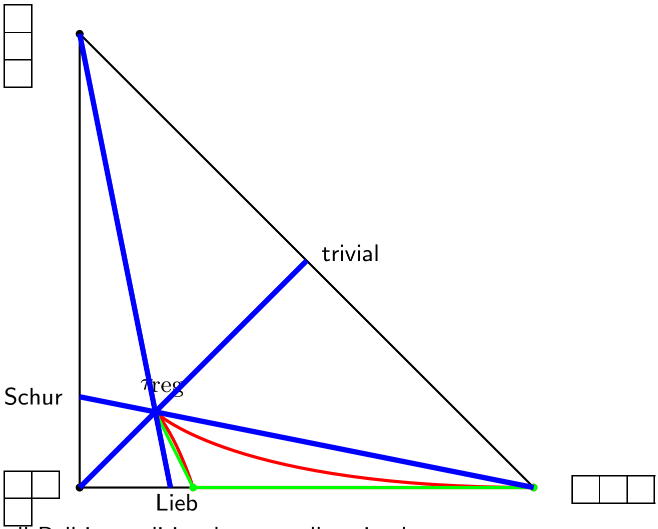

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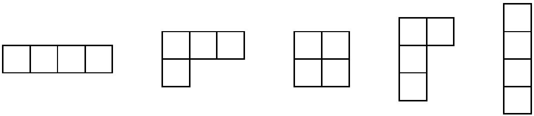
In the case $n = 4$ there are five Young diagrams:



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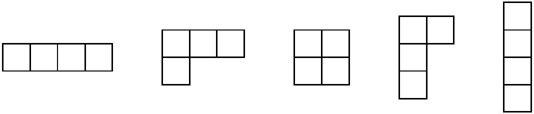

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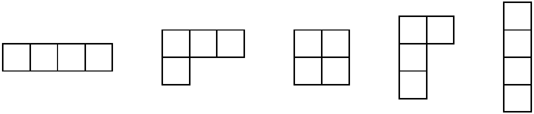

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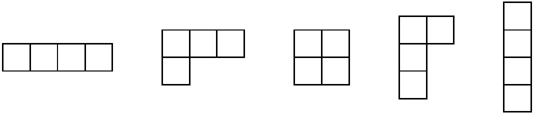

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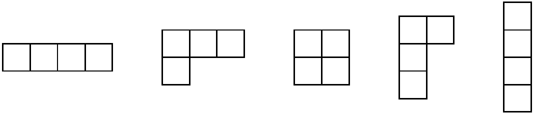

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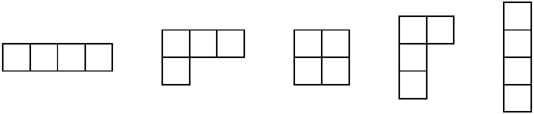

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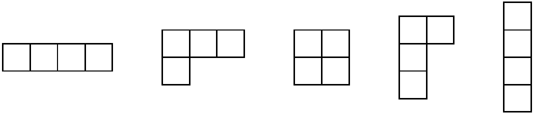

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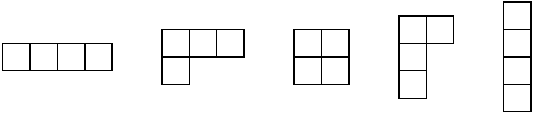

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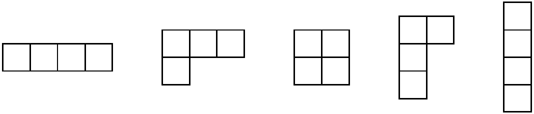

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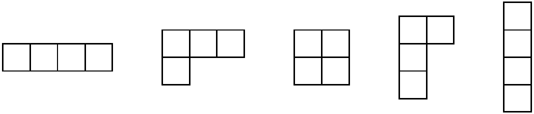

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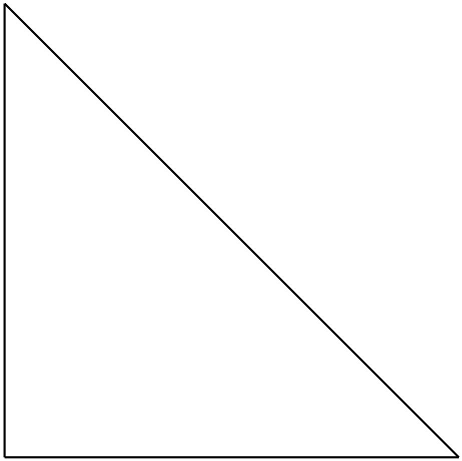

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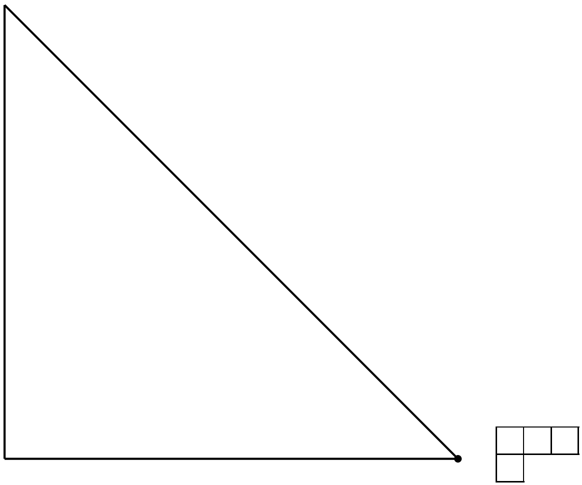

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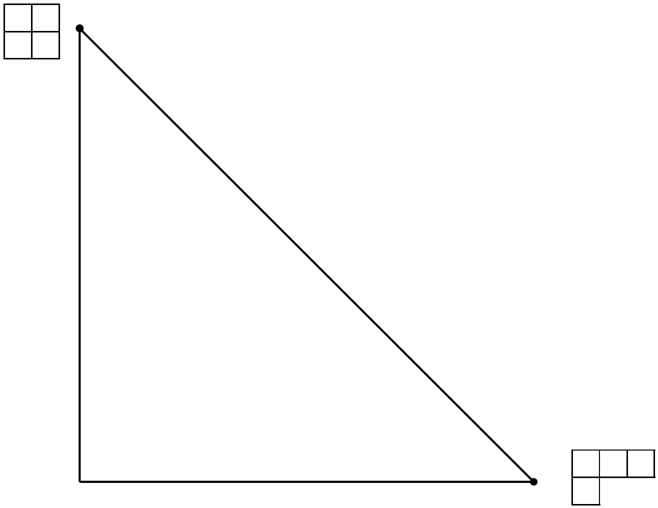

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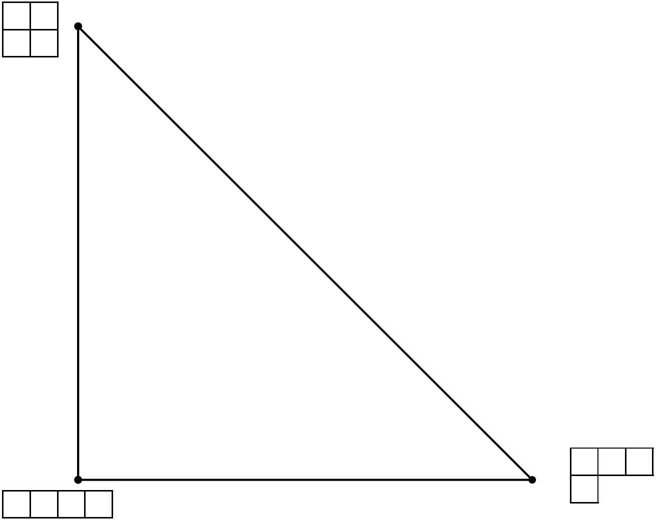

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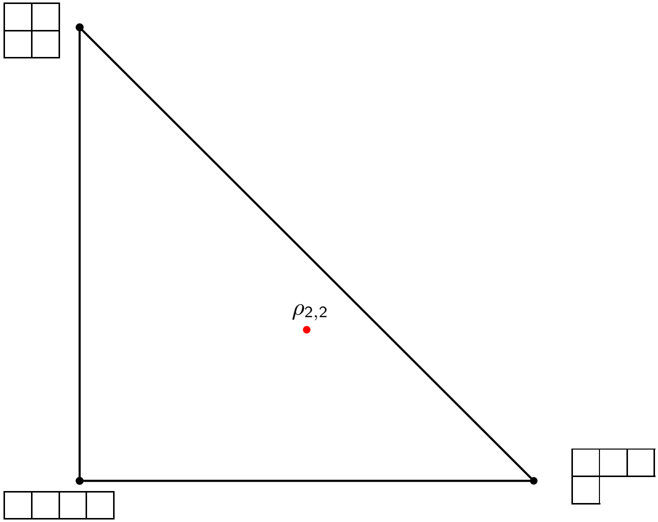

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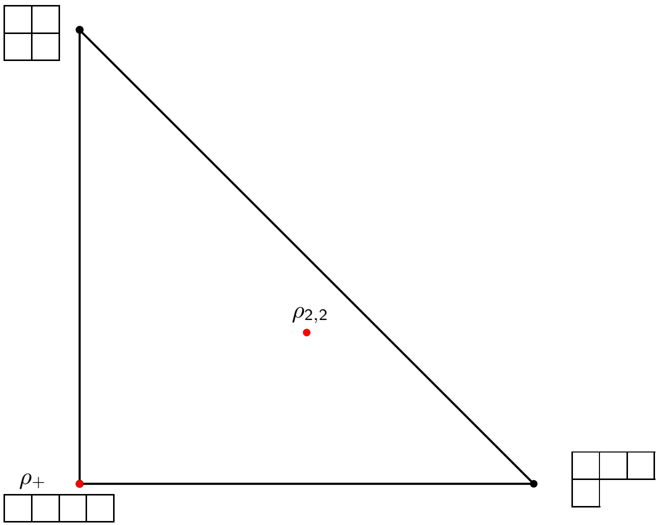

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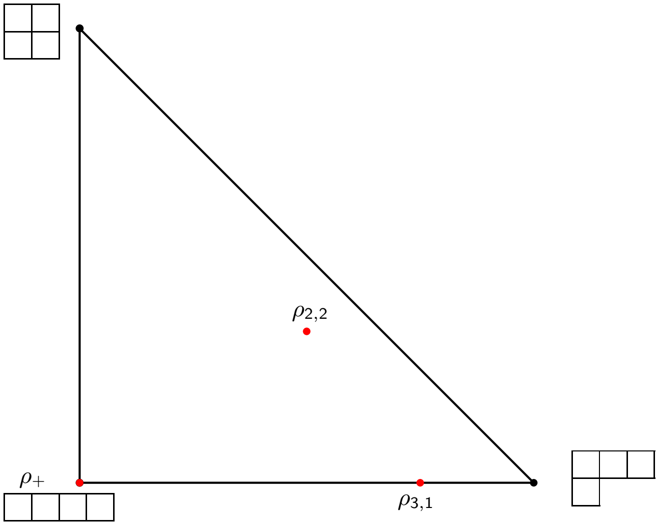

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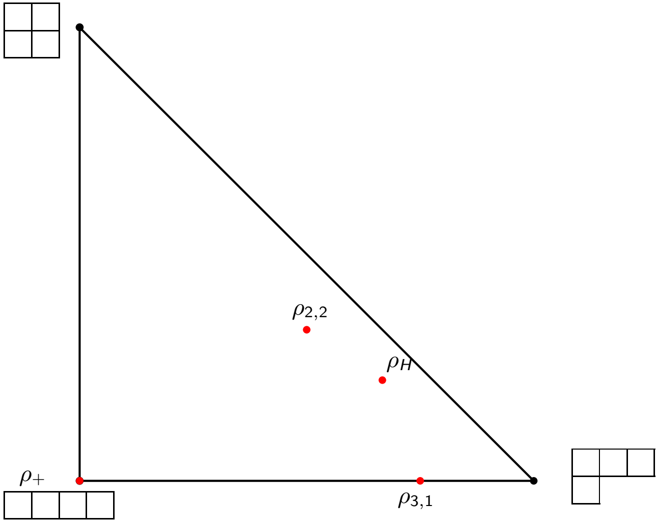

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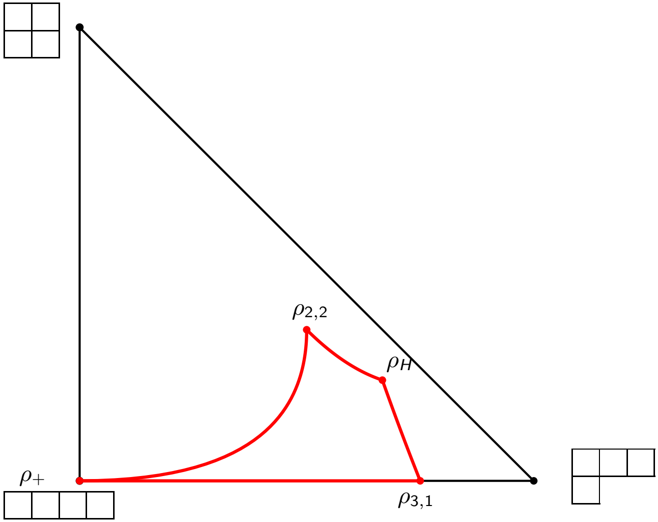

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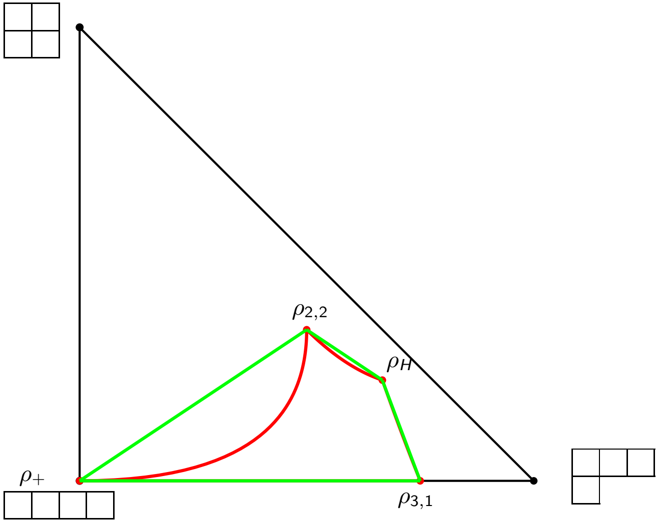

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In 1999 they showed that, already in the five qubit situation, the set of separable states on the center possesses a part that is bulging outward.

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Then a norm on the dual of $\mathcal{B}(\mathcal{H})$, is define by

$$\|\omega\|^V := \inf \left\{ \sum_{i=1}^{k} \lambda_i \,\middle|\, \omega = \sum_{i=1}^{k} \lambda_i \nu_i \, , k \in \mathbb{N}, \lambda_1, \lambda_2, \ldots, \lambda_k > 0, \nu_1, \nu_2, \ldots, \nu_k \in V \right\} ;$$

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When ρ is a state on $\mathcal{B}(\mathcal{H})$, we define its entanglement $E(\rho)$ by

$$E(\rho) := \|\rho\|^V \ .$$

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  (Here we would actually prefer equality!)

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In particular, the antisymmetric state has entanglement

$$E(\rho_-) = n! \ .$$

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Hence they define a von Neumann measurement on the system of n particles with Hilbert space $\mathbb{C}^d$.

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