The discrete entropic uncertainty relation

Hans Maassen

Goodbye, Jos! July 15, 2011.

History

History

Spring 1986.

History

Spring 1986. 'Quantum Club':

History

Spring 1986. ‘Quantum Club’:

Jan Hilgevoord,

Dennis Dieckx,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

. . .?

History

Spring 1986. 'Quantum Club':

Jan Hilgevoord,

Dennis Dieckx,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

. . .?

Subject:

History

Spring 1986. 'Quantum Club':

Jan Hilgevoord,

Dennis Dieckx,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

...?

Subject: 'Entropic Uncertainty'

History

Spring 1986. 'Quantum Club':

Jan Hilgevoord,

Dennis Diecks,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

...?

Subject: 'Entropic Uncertainty'

following Byalinicki-Birula, Heidelberg, October 1984

History

Spring 1986. 'Quantum Club':

Jan Hilgevoord,

Dennis Dieckx,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

. . .?

Subject: 'Entropic Uncertainty'

following Byalinicki-Birula, Heidelberg, October 1984

Remark Jos:

History

Spring 1986. 'Quantum Club':

Jan Hilgevoord,

Dennis Dieckx,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

. . .?

Subject: 'Entropic Uncertainty'

following Byalinicki-Birula, Heidelberg, October 1984

Remark Jos: Conjecture Karl Kraus:

History

Spring 1986. 'Quantum Club':

Jan Hilgevoord,

Dennis Dieks,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

...?

Subject: 'Entropic Uncertainty'

following Byalinicki-Birula, Heidelberg, October 1984

Remark Jos: Conjecture Karl Kraus:

For complementary observables A and B:

$$H(A) + H(B) \geq \log d \ .$$

History

Spring 1986. 'Quantum Club':

Jan Hilgevoord,

Dennis Dieckx,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

...?

Subject: 'Entropic Uncertainty'

following Byalinicki-Birula, Heidelberg, October 1984

Remark Jos: Conjecture Karl Kraus:

For complementary observables A and B:

$$H(A) + H(B) \geq \log d \; .$$

During that meeting:

History

Spring 1986. ‘Quantum Club’:

Jan Hilgevoord,

Dennis Diecks,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

. . .?

Subject: ‘Entropic Uncertainty’

following Byalinicki-Birula, Heidelberg, October 1984

Remark Jos: Conjecture Karl Kraus:

For complementary observables A and B:

$$H(A) + H(B) \geq \log d \ .$$

During that meeting: Proof of conjecture by the same means:

History

Spring 1986. 'Quantum Club':

Jan Hilgevoord,

Dennis Dieckx,

Michiel van Lambalgen

Dick Hoekzema,

Lou-Fé Feiner,

Jos & myself.

...?

Subject: 'Entropic Uncertainty'

following Byalinicki-Birula, Heidelberg, October 1984

Remark Jos: Conjecture Karl Kraus:

For complementary observables A and B:

$$H(A) + H(B) \geq \log d \ .$$

During that meeting: Proof of conjecture by the same means:

Riesz-Thorin interpolation.

Success story

Success story

Jos wrote down the result in a Phys. Rev. Letter, and elaborated on it in his Ph. D. thesis.

Success story

Jos wrote down the result in a Phys. Rev. Letter, and elaborated on it in his Ph. D. thesis.

The Letter still is, for both of us, by far the most cited item on our publication lists.

Success story

Jos wrote down the result in a Phys. Rev. Letter, and elaborated on it in his Ph. D. thesis.

The Letter still is, for both of us, by far the most cited item on our publication lists.

The inequality has been applied in quantum key distribution, entanglement distillation, has been improved upon in special cases, and is generally well-known in quantum information.

Success story

Jos wrote down the result in a Phys. Rev. Letter, and elaborated on it in his Ph. D. thesis.

The Letter still is, for both of us, by far the most cited item on our publication lists.

The inequality has been applied in quantum key distribution, entanglement distillation, has been improved upon in special cases, and is generally well-known in quantum information.

Aim of the talk

Success story

Jos wrote down the result in a Phys. Rev. Letter, and elaborated on it in his Ph. D. thesis.

The Letter still is, for both of us, by far the most cited item on our publication lists.

The inequality has been applied in quantum key distribution, entanglement distillation, has been improved upon in special cases, and is generally well-known in quantum information.

Aim of the talk

But it still eludes intuition. The question is rarely asked why it holds.

Success story

Jos wrote down the result in a Phys. Rev. Letter, and elaborated on it in his Ph. D. thesis.

The Letter still is, for both of us, by far the most cited item on our publication lists.

The inequality has been applied in quantum key distribution, entanglement distillation, has been improved upon in special cases, and is generally well-known in quantum information.

Aim of the talk

But it still eludes intuition. The question is rarely asked why it holds.

I would like to address this question today.

This talk:

This talk:

- 1. The discrete entropic uncertainty relation.

This talk:

- 1. The discrete entropic uncertainty relation.
- 2. Three proofs.

This talk:

- 1. The discrete entropic uncertainty relation.
- 2. Three proofs.
  - Riesz 1928;

This talk:

- 1. The discrete entropic uncertainty relation.
- 2. Three proofs.
  - Riesz 1928;
  - Riesz-Thorin interpolation;

This talk:

- 1. The discrete entropic uncertainty relation.
- 2. Three proofs.
  - Riesz 1928;
  - Riesz-Thorin interpolation;
  - A magic holomorphic function.

This talk:

- 1. The discrete entropic uncertainty relation.
- 2. Three proofs.
  - Riesz 1928;
  - Riesz-Thorin interpolation;
  - A magic holomorphic function.
- 3. When do we have equality?

The situation.

The situation.

INPUT:

The situation.

INPUT:

$\mathcal{H}$: Hilbert space of dimension d ;

The situation.

INPUT:

$$\mathcal{H} : \quad \text{Hilbert space of dimension } d \; ;$$

One unit vector:

$$\psi \in \mathcal{H} \; .$$

The situation.

INPUT:

$$\mathcal{H} : \quad \text{Hilbert space of dimension } d \ ;$$

One unit vector:

$$\psi \in \mathcal{H} \ .$$

Two orthonormal bases in $\mathcal{H}$:

$$e_1, e_2, \ldots, e_d \ ; \qquad \widehat{e}_1, \widehat{e}_2, \ldots, \widehat{e}_d \ .$$

The situation.

INPUT:

$\mathcal{H}$: Hilbert space of dimension d ;

One unit vector:

$$\psi \in \mathcal{H} .$$

Two orthonormal bases in $\mathcal{H}$:

$$e_1, e_2, \ldots, e_d \; ; \qquad \widehat{e}_1, \widehat{e}_2, \ldots, \widehat{e}_d \; .$$

Largest scalar product:

$$c := \max_{i,j} |\langle \widehat{e}_i, e_j \rangle| \; .$$

The situation.

INPUT:

$$\mathcal{H} : \quad \text{Hilbert space of dimension } d \; ;$$

One unit vector:

$$\psi \in \mathcal{H} \; .$$

Two orthonormal bases in $\mathcal{H}$:

$$e_1, e_2, \ldots, e_d \; ; \qquad \widehat{e}_1, \widehat{e}_2, \ldots, \widehat{e}_d \; .$$

Largest scalar product:

$$c := \max_{i,j} |\langle \widehat{e}_i, e_j \rangle| \; .$$

OUTPUT:

The situation.

INPUT:

$$\mathcal{H} : \quad \text{Hilbert space of dimension } d \; ;$$

One unit vector:

$$\psi \in \mathcal{H} \; .$$

Two orthonormal bases in $\mathcal{H}$:

$$e_1, e_2, \ldots, e_d \; ; \qquad \widehat{e}_1, \widehat{e}_2, \ldots, \widehat{e}_d \; .$$

Largest scalar product:

$$c := \max_{i,j} |\langle \widehat{e}_i, e_j \rangle| \; .$$

OUTPUT: Two probability distributions:

$$\pi_j := |\langle e_j, \psi \rangle|^2 \; ; \qquad \widehat{\pi}_k := |\langle \widehat{e}_k, \psi \rangle|^2 \; .$$

The discrete entropic uncertainty relation

The discrete entropic uncertainty relation

Definition

The entropy $H(\pi)$ of a discrete probability distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_d)$ is defined as

$$H(\pi) := -\sum_{j=1}^{d} \pi_j \log \pi_j \ .$$

The discrete entropic uncertainty relation

Definition

The entropy $H(\pi)$ of a discrete probability distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_d)$ is defined as

$$H(\pi) := -\sum_{j=1}^{d} \pi_j \log \pi_j \ .$$

This is the expected amount of information which the measurement will give, or equivalently, the amount of uncertainty which we have before the measurement.

The discrete entropic uncertainty relation

Definition

The entropy $H(\pi)$ of a discrete probability distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_d)$ is defined as

$$H(\pi) := -\sum_{j=1}^{d} \pi_j \log \pi_j \ .$$

This is the expected amount of information which the measurement will give, or equivalently, the amount of uncertainty which we have before the measurement.

Theorem (1)

The sum of the two uncertainties satisfies:

$$H(\pi) + H(\widehat{\pi}) \geq \log \frac{1}{c^2} \ .$$

Extreme cases:

Extreme cases:

- $e_1 = \widehat{e_1}$:

Extreme cases:

- $e_1 = \widehat{e}_1$:   Then $c = 1$ and the inequality becomes vacuous:

$$H(\pi) + H(\widehat{\pi}) \geq 0 \ .$$

Extreme cases:

- $e_1 = \widehat{e}_1$: Then $c = 1$ and the inequality becomes vacuous:

$$H(\pi) + H(\widehat{\pi}) \geq 0 \ .$$

In fact, equality can be reached by putting $\psi = e_1 = \widehat{e}_1$.

Extreme cases:

- $e_1 = \widehat{e}_1$: Then $c = 1$ and the inequality becomes vacuous:

$$H(\pi) + H(\widehat{\pi}) \geq 0 \; .$$

In fact, equality can be reached by putting $\psi = e_1 = \widehat{e}_1$.

Both outcomes are completely certain.

Extreme cases:

- $e_1 = \widehat{e}_1$: Then $c = 1$ and the inequality becomes vacuous:

$$H(\pi) + H(\widehat{\pi}) \geq 0 \; .$$

  In fact, equality can be reached by putting $\psi = e_1 = \widehat{e}_1$.

  Both outcomes are completely certain.

- Mutually Unbiased Bases:

$$|\langle e_j, \widehat{e}_k \rangle|^2 = \frac{1}{d} : \qquad c = \frac{1}{\sqrt{d}} \; .$$

Extreme cases:

- $e_1 = \widehat{e}_1$: Then $c = 1$ and the inequality becomes vacuous:

$$H(\pi) + H(\widehat{\pi}) \geq 0 \ .$$

  In fact, equality can be reached by putting $\psi = e_1 = \widehat{e}_1$.

  Both outcomes are completely certain.

- Mutually Unbiased Bases:

$$|\langle e_j, \widehat{e}_k \rangle|^2 = \frac{1}{d} : \qquad c = \frac{1}{\sqrt{d}} \ .$$

  Then we obtain Karl Kraus's conjecture:

$$H(\pi) + H(\widehat{\pi}) \geq \log d \ .$$

Extreme cases:

- $e_1 = \widehat{e}_1$: Then $c = 1$ and the inequality becomes vacuous:

$$H(\pi) + H(\widehat{\pi}) \geq 0 \ .$$

  In fact, equality can be reached by putting $\psi = e_1 = \widehat{e}_1$.

  Both outcomes are completely certain.

- Mutually Unbiased Bases:

$$|\langle e_j, \widehat{e}_k \rangle|^2 = \frac{1}{d} : \qquad c = \frac{1}{\sqrt{d}} \ .$$

  Then we obtain Karl Kraus's conjecture:

$$H(\pi) + H(\widehat{\pi}) \geq \log d \ .$$

  Again equality can be reached by putting $\psi = e_1$. Then $\pi = \delta_1$ and $\widehat{\pi}$ is the uniform distribution.

Extreme cases:

- $e_1 = \widehat{e}_1$: Then $c = 1$ and the inequality becomes vacuous:

$$H(\pi) + H(\widehat{\pi}) \geq 0 \ .$$

In fact, equality can be reached by putting $\psi = e_1 = \widehat{e}_1$.

Both outcomes are completely certain.

- Mutually Unbiased Bases:

$$|\langle e_j, \widehat{e}_k \rangle|^2 = \frac{1}{d} : \qquad c = \frac{1}{\sqrt{d}} \ .$$

Then we obtain Karl Kraus's conjecture:

$$H(\pi) + H(\widehat{\pi}) \geq \log d \ .$$

Again equality can be reached by putting $\psi = e_1$. Then $\pi = \delta_1$ and $\widehat{\pi}$ is the uniform distribution.

One outcome is certain, the other completely uncertain.

Rényi entropies

Rényi entropies

Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

Rényi entropies

Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

In particular:

$$H_1(\pi) := \lim_{\alpha \to 1} H_\alpha(\pi) = H(\pi) \ .$$

Rényi entropies

Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

In particular:

$$H_1(\pi) := \lim_{\alpha \to 1} H_\alpha(\pi) = H(\pi) \ .$$

This can be calculated as follows:

Rényi entropies

Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

In particular:

$$H_1(\pi) := \lim_{\alpha \to 1} H_\alpha(\pi) = H(\pi) \ .$$

This can be calculated as follows:

$$H_\alpha(\pi) \quad =$$

Rényi entropies

Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

In particular:

$$H_1(\pi) := \lim_{\alpha \to 1} H_\alpha(\pi) = H(\pi) \ .$$

This can be calculated as follows:

$$H_\alpha(\pi) \quad = \quad - \frac{\log \sum_{j=1}^{d} \pi_j^\alpha - \log \sum_{j=1}^{d} \pi_j^1}{\alpha - 1}$$

Rényi entropies

Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

In particular:

$$H_1(\pi) := \lim_{\alpha \to 1} H_\alpha(\pi) = H(\pi) \ .$$

This can be calculated as follows:

$$\begin{aligned} H_\alpha(\pi) &= -\frac{\log \sum_{j=1}^{d} \pi_j^\alpha - \log \sum_{j=1}^{d} \pi_j^1}{\alpha - 1} \\ &\xrightarrow{\alpha \to 1} -\frac{d}{d\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \Big|_{\alpha=1} \end{aligned}$$

Rényi entropies

Let $\pi = (\pi_1, \dots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

In particular:

$$H_1(\pi) := \lim_{\alpha \to 1} H_\alpha(\pi) = H(\pi) \ .$$

This can be calculated as follows:

$$\begin{aligned} H_\alpha(\pi) \quad &= \quad - \frac{\log \sum_{j=1}^{d} \pi_j^\alpha - \log \sum_{j=1}^{d} \pi_j^1}{\alpha - 1} \\ &\overset{\alpha \to 1}{\longrightarrow} \quad -\frac{d}{d\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \Big|_{\alpha=1} = -\frac{d}{d\alpha} \sum_{j=1}^{d} \pi_j^\alpha \Big|_{\alpha=1} \end{aligned}$$

Rényi entropies

Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

In particular:

$$H_1(\pi) := \lim_{\alpha \to 1} H_\alpha(\pi) = H(\pi) \ .$$

This can be calculated as follows:

$$\begin{aligned} H_\alpha(\pi) \quad &= \quad - \frac{\log \sum_{j=1}^{d} \pi_j^\alpha - \log \sum_{j=1}^{d} \pi_j^1}{\alpha - 1} \\ &\overset{\alpha \to 1}{\longrightarrow} \quad -\frac{d}{d\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \Big|_{\alpha=1} = -\frac{d}{d\alpha} \sum_{j=1}^{d} \pi_j^\alpha \Big|_{\alpha=1} \\ &= \quad - \sum_{j=1}^{d} \pi_j \log \pi_j \end{aligned}$$

Rényi entropies

Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

$$H_\alpha(\pi) := \frac{1}{1-\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \ .$$

In particular:

$$H_1(\pi) := \lim_{\alpha \to 1} H_\alpha(\pi) = H(\pi) \ .$$

This can be calculated as follows:

$$\begin{aligned} H_\alpha(\pi) \quad &= \quad - \frac{\log \sum_{j=1}^{d} \pi_j^\alpha - \log \sum_{j=1}^{d} \pi_j^1}{\alpha - 1} \\ \stackrel{\alpha \to 1}{\longrightarrow} \quad & -\frac{d}{d\alpha} \log \sum_{j=1}^{d} \pi_j^\alpha \Big|_{\alpha=1} = -\frac{d}{d\alpha} \sum_{j=1}^{d} \pi_j^\alpha \Big|_{\alpha=1} \\ = \quad & -\sum_{j=1}^{d} \pi_j \log \pi_j = H(\pi) \ . \end{aligned}$$

Generalized entropic uncertainty relations

Generalized entropic uncertainty relations

Maybe it is not so well known that in our 1988 Phys. Rev. Letter Jos proved the inequality for all the Rényi entropies:

Generalized entropic uncertainty relations

Maybe it is not so well known that in our 1988 Phys. Rev. Letter Jos proved the inequality for all the Rényi entropies:

Theorem (2)

Let $\alpha, \widehat{\alpha}$ be such that $\frac{1}{\alpha} + \frac{1}{\widehat{\alpha}} = 2$. Then

$$H_{\alpha}(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) \geq \log \frac{1}{c^2} \ .$$

Generalized entropic uncertainty relations

Maybe it is not so well known that in our 1988 Phys. Rev. Letter Jos proved the inequality for all the Rényi entropies:

Theorem (2)

Let $\alpha, \widehat{\alpha}$ be such that $\frac{1}{\alpha} + \frac{1}{\widehat{\alpha}} = 2$. Then

$$H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) \geq \log \frac{1}{c^2} \ .$$

Of course, taking $\alpha \to 1$ we obtain the ordinary entropic uncertainty relation.

Notation

Notation

We shall indicate the components of ψ in the two bases by

$$\psi_k := \langle e_k, \psi \rangle \ ;$$

$$\widehat{\psi}_j := \langle \widehat{e}_j, \psi \rangle \ .$$

Notation

We shall indicate the components of ψ in the two bases by

$$\psi_k := \langle e_k, \psi \rangle \ ;$$

$$\widehat{\psi}_j := \langle \widehat{e}_j, \psi \rangle \ .$$

If we define the unitary matrix $U = (u_{jk})_{j,k=1}^{d}$ by

$$u_{jk} := \langle \widehat{e}_j, e_k \rangle \ ,$$

Notation

We shall indicate the components of ψ in the two bases by

$$\psi_k := \langle e_k, \psi \rangle \; ;$$

$$\widehat{\psi}_j := \langle \widehat{e}_j, \psi \rangle \; .$$

If we define the unitary matrix $U = (u_{jk})_{j,k=1}^{d}$ by

$$u_{jk} := \langle \widehat{e}_j, e_k \rangle \; ,$$

then we may write

$$\widehat{\psi}_j = \langle \widehat{e}_j, \psi \rangle = \sum_{k=1}^{d} \langle \widehat{e}_j, e_k \rangle \langle e_k, \psi \rangle = \sum_{k=1}^{d} u_{jk} \psi_k \; .$$

Notation

We shall indicate the components of ψ in the two bases by

$$\psi_k := \langle e_k, \psi \rangle \ ;$$

$$\widehat{\psi}_j := \langle \widehat{e}_j, \psi \rangle \ .$$

If we define the unitary matrix $U = (u_{jk})_{j,k=1}^{d}$ by

$$u_{jk} := \langle \widehat{e}_j, e_k \rangle \ ,$$

then we may write

$$\widehat{\psi}_j = \langle \widehat{e}_j, \psi \rangle = \sum_{k=1}^{d} \langle \widehat{e}_j, e_k \rangle \langle e_k, \psi \rangle = \sum_{k=1}^{d} u_{jk} \psi_k \ .$$

So our raw data are now a unitary $d \times d$ matrix U and a unit vector $\psi \in \mathbb{C}^n$, and we have $\widehat{\psi} = U\psi$.

Our 1988 proof

Our 1988 proof

Our 1988 proof

Theorem (Marcel Riesz 1928)

For $1 \le p \le 2 \le \widehat{p} \le \infty$ *with* $\frac{1}{p} + \frac{1}{\widehat{p}} = 1$*:*

$$\left(c \sum_{j=1}^{d} |\widehat{\psi}_j|^{\widehat{p}} \right)^{1/\widehat{p}} \le \left(c \sum_{j=1}^{d} |\psi_k|^{p} \right)^{1/p} .$$

Our 1988 proof

Theorem (Marcel Riesz 1928)

For $1 \le p \le 2 \le \widehat{p} \le \infty$ *with* $\frac{1}{p} + \frac{1}{\widehat{p}} = 1$*:*

$$\left(c \sum_{j=1}^{d} |\widehat{\psi}_j|^{\widehat{p}} \right)^{1/\widehat{p}} \le \left(c \sum_{j=1}^{d} |\psi_k|^{p} \right)^{1/p} .$$

More briefly this can be stated as follows:

$$c^{1/\widehat{p}} \|\widehat{\psi}\|_{\widehat{p}} \le c^{1/p} \|\psi\|_p .$$

Our 1988 proof

Theorem (Marcel Riesz 1928)

For $1 \leq p \leq 2 \leq \widehat{p} \leq \infty$ *with* $\frac{1}{p} + \frac{1}{\widehat{p}} = 1$*:*

$$\left(c \sum_{j=1}^{d} |\widehat{\psi}_j|^{\widehat{p}} \right)^{1/\widehat{p}} \leq \left(c \sum_{j=1}^{d} |\psi_k|^{p} \right)^{1/p} .$$

More briefly this can be stated as follows:

$$c^{1/\widehat{p}} \|\widehat{\psi}\|_{\widehat{p}} \leq c^{1/p} \|\psi\|_p .$$

Equivalently:

$$\log \|\psi\|_p - \log \|\widehat{\psi}\|_{\widehat{p}} \geq \left(\frac{1}{\widehat{p}} - \frac{1}{p} \right) \log c .$$

Proof.
(of the uncertainty relation)

Proof.

(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

Proof.

(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

$$H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) \quad =$$

Proof.

(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

$$H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) \quad = \quad \frac{\alpha}{1-\alpha} \log \|\pi\|_\alpha + \frac{\widehat{\alpha}}{1-\widehat{\alpha}} \log \|\widehat{\pi}\|_{\widehat{\alpha}}$$

Proof.

(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

$$\begin{aligned} H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) &= \frac{\alpha}{1-\alpha} \log \|\pi\|_\alpha + \frac{\widehat{\alpha}}{1-\widehat{\alpha}} \log \|\widehat{\pi}\|_{\widehat{\alpha}} \\ &= \frac{2\alpha}{1-\alpha} \log \left(\|\psi\|_{2\alpha} - \log \|\widehat{\psi}\|_{2\widehat{\alpha}} \right) \end{aligned}$$

Proof.

(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

$$\begin{aligned}
H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) &= \frac{\alpha}{1-\alpha}\log\|\pi\|_\alpha + \frac{\widehat{\alpha}}{1-\widehat{\alpha}}\log\|\widehat{\pi}\|_{\widehat{\alpha}} \\
&= \frac{2\alpha}{1-\alpha}\log\left(\|\psi\|_{2\alpha} - \log\|\widehat{\psi}\|_{2\widehat{\alpha}}\right) \\
&\geq \frac{2\alpha}{1-\alpha}\left(\frac{1}{2\widehat{\alpha}} - \frac{1}{2\alpha}\right)\log c
\end{aligned}$$

Proof.

(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

$$
\begin{aligned}
H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) &= \frac{\alpha}{1-\alpha} \log \|\pi\|_\alpha + \frac{\widehat{\alpha}}{1-\widehat{\alpha}} \log \|\widehat{\pi}\|_{\widehat{\alpha}} \\
&= \frac{2\alpha}{1-\alpha} \log \left(\|\psi\|_{2\alpha} - \log \|\widehat{\psi}\|_{2\widehat{\alpha}} \right) \\
&\geq \frac{2\alpha}{1-\alpha} \left(\frac{1}{2\widehat{\alpha}} - \frac{1}{2\alpha} \right) \log c \\
&= -2 \log c \; .
\end{aligned}
$$

Proof.

(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

$$\begin{aligned} H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) &= \frac{\alpha}{1-\alpha} \log \|\pi\|_\alpha + \frac{\widehat{\alpha}}{1-\widehat{\alpha}} \log \|\widehat{\pi}\|_{\widehat{\alpha}} \\ &= \frac{2\alpha}{1-\alpha} \log \left(\|\psi\|_{2\alpha} - \log \|\widehat{\psi}\|_{2\widehat{\alpha}} \right) \\ &\geq \frac{2\alpha}{1-\alpha} \left(\frac{1}{2\widehat{\alpha}} - \frac{1}{2\alpha} \right) \log c \\ &= -2 \log c \ . \end{aligned}$$

Taking $\alpha \to 1$ we also obtain the ordinary entropic uncertainty relation. □

Proof.

(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

$$\begin{aligned} H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) &= \frac{\alpha}{1-\alpha}\log\|\pi\|_\alpha + \frac{\widehat{\alpha}}{1-\widehat{\alpha}}\log\|\widehat{\pi}\|_{\widehat{\alpha}} \\ &= \frac{2\alpha}{1-\alpha}\log\left(\|\psi\|_{2\alpha} - \log\|\widehat{\psi}\|_{2\widehat{\alpha}}\right) \\ &\geq \frac{2\alpha}{1-\alpha}\left(\frac{1}{2\widehat{\alpha}} - \frac{1}{2\alpha}\right)\log c \\ &= -2\log c\,. \end{aligned}$$

Taking $\alpha \to 1$ we also obtain the ordinary entropic uncertainty relation. □

Proof.
(of the uncertainty relation)

From here it is just a few steps to the entropic uncertainty relation:

$$\begin{aligned} H_\alpha(\pi) + H_{\widehat{\alpha}}(\widehat{\pi}) &= \frac{\alpha}{1-\alpha} \log \|\pi\|_\alpha + \frac{\widehat{\alpha}}{1-\widehat{\alpha}} \log \|\widehat{\pi}\|_{\widehat{\alpha}} \\ &= \frac{2\alpha}{1-\alpha} \log \left(\|\psi\|_{2\alpha} - \log \|\widehat{\psi}\|_{2\widehat{\alpha}} \right) \\ &\geq \frac{2\alpha}{1-\alpha} \left(\frac{1}{2\widehat{\alpha}} - \frac{1}{2\alpha} \right) \log c \\ &= -2 \log c \ . \end{aligned}$$

Taking $\alpha \to 1$ we also obtain the ordinary entropic uncertainty relation. □

I proved the entropic uncertainty relation!

Riesz-Thorin interpolation

Riesz-Thorin interpolation

The above theorem of Riesz is a special case of the following.

Riesz-Thorin interpolation

The above theorem of Riesz is a special case of the following.

For $p, q \in [1, \infty]$ and a $d \times d$-matrix T, let $\|T\|_{p \to q}$ denote the norm of T seen as an operator from $\mathbb{C}^d$ with p-norm to $\mathbb{C}^d$ wit h q-norm:

Riesz-Thorin interpolation

The above theorem of Riesz is a special case of the following.

For $p, q \in [1, \infty]$ and a $d \times d$-matrix T, let $\|T\|_{p \to q}$ denote the norm of T seen as an operator from $\mathbb{C}^d$ with p-norm to $\mathbb{C}^d$ wit h q-norm:

$$\|T\|_{p \to q} := \max_{\|\psi\|_p = 1} \|T\psi\|_q \ .$$

Riesz-Thorin interpolation

The above theorem of Riesz is a special case of the following.

For $p, q \in [1, \infty]$ and a $d \times d$-matrix T, let $\|T\|_{p \to q}$ denote the norm of T seen as an operator from $\mathbb{C}^d$ with p-norm to $\mathbb{C}^d$ wit h q-norm:

$$\|T\|_{p \to q} := \max_{\|\psi\|_p = 1} \|T\psi\|_q \ .$$

Theorem (Riesz-Thorin)

Riesz-Thorin interpolation

The above theorem of Riesz is a special case of the following.

For $p, q \in [1, \infty]$ and a $d \times d$-matrix T, let $\|T\|_{p \to q}$ denote the norm of T seen as an operator from $\mathbb{C}^d$ with p-norm to $\mathbb{C}^d$ wit h q-norm:

$$\|T\|_{p \to q} := \max_{\|\psi\|_p = 1} \|T\psi\|_q \ .$$

Theorem (Riesz-Thorin)

For all $d \times d$-matrices T the function

$$[0,1] \times [0,1] \to \mathbb{R} : \qquad \left(\frac{1}{p}, \frac{1}{q}\right) \mapsto \log \|U\|_{p \to q}$$

is convex.

Entropic uncertainty by interpolation

Entropic uncertainty by interpolation

Let U be a unitary $d \times d$-matrix, $\psi \in \mathbb{C}^d$ a vector of unit 2-norm: $\|\psi\|_2 = 1$, and let $c := \max_{j,k} |\langle \widehat{e}_j, e_k \rangle|$.

Entropic uncertainty by interpolation

Let U be a unitary $d \times d$-matrix, $\psi \in \mathbb{C}^d$ a vector of unit 2-norm: $\|\psi\|_2 = 1$, and let $c := \max_{j,k} |\langle \widehat{e}_j, e_k \rangle|$. Then we have

$$\|U\|_{2\to 2} = 1 \qquad \text{since } U \text{ is unitary;}$$

Entropic uncertainty by interpolation

Let U be a unitary $d \times d$-matrix, $\psi \in \mathbb{C}^d$ a vector of unit 2-norm: $\|\psi\|_2 = 1$, and let $c := \max_{j,k} |\langle \widehat{e}_j, e_k \rangle|$. Then we have

$$\begin{aligned} \|U\|_{2\to 2} &= 1 \qquad \text{since } U \text{ is unitary;} \\ \|U\|_{1\to\infty} &= c \qquad \text{since} \quad |(U\psi)_j| = \left| \sum_{k=1}^{d} u_{jk}\psi_k \right| \le c \sum_{k=1}^{d} |\psi_k| \, . \end{aligned}$$

Entropic uncertainty by interpolation

Let U be a unitary $d \times d$-matrix, $\psi \in \mathbb{C}^d$ a vector of unit 2-norm: $\|\psi\|_2 = 1$, and let $c := \max_{j,k} |\langle \widehat{e}_j, e_k \rangle|$. Then we have

$$\begin{aligned} \|U\|_{2\to 2} &= 1 && \text{since } U \text{ is unitary;} \\ \|U\|_{1\to\infty} &= c && \text{since} \quad |(U\psi)_j| = \left| \sum_{k=1}^{d} u_{jk}\psi_k \right| \le c \sum_{k=1}^{d} |\psi_k| \, . \end{aligned}$$

According to the Riesz-Thorin interpolation theorem the function

$$f_U : [0,1] \to [0,1] : \quad \frac{1}{p} \mapsto \log \|U\|_{p\to\widehat{p}}$$

is convex.

Entropic uncertainty by interpolation

Let U be a unitary $d \times d$-matrix, $\psi \in \mathbb{C}^d$ a vector of unit 2-norm: $\|\psi\|_2 = 1$, and let $c := \max_{j,k} |\langle \widehat{e}_j, e_k \rangle|$. Then we have

$$\begin{aligned} \|U\|_{2\to 2} &= 1 \qquad \text{since } U \text{ is unitary;} \\ \|U\|_{1\to\infty} &= c \qquad \text{since} \quad |(U\psi)_j| = \left| \sum_{k=1}^{d} u_{jk}\psi_k \right| \le c \sum_{k=1}^{d} |\psi_k| \; . \end{aligned}$$

According to the Riesz-Thorin interpolation theorem the function

$$f_U : [0,1] \to [0,1] : \quad \frac{1}{p} \mapsto \log \|U\|_{p\to\widehat{p}}$$

is convex.

Since $f_U\left(\frac{1}{2}\right) = \log \|U\|_{2\to 2} = 0$ and $f_U(1) = \log \|U\|_{1\to\infty} = \log c$, we conclude that

$$f'\left(\frac{1}{2}\right) \le \frac{f(1) - f(\frac{1}{2})}{1 - \frac{1}{2}} \le 2\log c \; .$$

On the other hand, for all $\psi \in \mathbb{C}^d$ and $p \in [0,, \infty]$:

On the other hand, for all $\psi \in \mathbb{C}^d$ and $p \in [0,,\infty]$:

$$f_U\left(\frac{1}{p}\right) = \log \|U\|_p \geq \log \|U\psi\|_{\widehat{p}} - \log \|\psi\|_p \ .$$

On the other hand, for all $\psi \in \mathbb{C}^d$ and $p \in [0,,\infty]$:

$$f_U\left(\frac{1}{p}\right) = \log \|U\|_p \geq \log \|U\psi\|_{\widehat{p}} - \log \|\psi\|_p \ .$$

Since we have equality at $\frac{1}{p} = \frac{1}{2}$, we may differentiate the above inequality:

On the other hand, for all $\psi \in \mathbb{C}^d$ and $p \in [0, , \infty]$:

$$f_U\left(\frac{1}{p}\right) = \log \|U\|_p \geq \log \|U\psi\|_{\widehat{p}} - \log \|\psi\|_p \ .$$

Since we have equality at $\frac{1}{p} = \frac{1}{2}$, we may differentiate the above inequality:

$$f_U'\left(\frac{1}{2}\right) \geq -H(|\widehat{\psi}|^2) - H(|\psi|^2) \ .$$

On the other hand, for all $\psi \in \mathbb{C}^d$ and $p \in [0,,\infty]$:

$$f_U\left(\frac{1}{p}\right) = \log \|U\|_p \geq \log \|U\psi\|_{\widehat{p}} - \log \|\psi\|_p \ .$$

Since we have equality at $\frac{1}{p} = \frac{1}{2}$, we may differentiate the above inequality:

$$f_U'\left(\frac{1}{2}\right) \geq -H(|\widehat{\psi}|^2) - H(|\psi|^2) \ .$$

Since $f_U'(\frac{1}{2}) \leq 2\log c$, it follows that $H(|\widehat{\psi}|^2) + H(|\psi|^2) \geq \log(1/c^2)$. □

Thorin's proof of Riesz convexity.

Thorin's proof of Riesz convexity.

If we believe Riesz' convexity result, then we are done.

Thorin's proof of Riesz convexity.

If we believe Riesz' convexity result, then we are done. By why is it true?

Thorin's proof of Riesz convexity.

If we believe Riesz' convexity result, then we are done. By why is it true?

Our third proof will be more basic. It starts from the following.

Thorin's proof of Riesz convexity.

If we believe Riesz' convexity result, then we are done. By why is it true?

Our third proof will be more basic. It starts from the following.

Let S denote the strip $\{ z \in \mathbb{C} \,|\, 0 \leq \mathrm{Re}\, z \leq 1 \}$.

Thorin's proof of Riesz convexity.

If we believe Riesz' convexity result, then we are done. By why is it true?

Our third proof will be more basic. It starts from the following.

Let S denote the strip $\{ z \in \mathbb{C} \mid 0 \leq \mathrm{Re}\, z \leq 1 \}$.

Theorem (Phragmén-Lindelöf)

Thorin's proof of Riesz convexity.

If we believe Riesz' convexity result, then we are done. By why is it true?

Our third proof will be more basic. It starts from the following.

Let S denote the strip $\{ z \in \mathbb{C} \mid 0 \leq \mathrm{Re}\, z \leq 1 \}$.

Theorem (Phragmén-Lindelöf)

Let F be a bounded holomorphic function on S such that $|F(z)| \leq 1$ on the boundary of S.

Thorin's proof of Riesz convexity.

If we believe Riesz' convexity result, then we are done. By why is it true?

Our third proof will be more basic. It starts from the following.

Let S denote the strip $\{\, z \in \mathbb{C} \mid 0 \le \mathrm{Re}\, z \le 1 \,\}$.

Theorem (Phragmén-Lindelöf)

Let F be a bounded holomorphic function on S such that $|F(z)| \le 1$ on the boundary of S. Then $|F(z)| \le 1$ on all of S.

Thorin's $\mathcal{MAGIC\ \ FUNCTION}$

Thorin's $\mathcal{MAGIC}$ $\mathcal{FUNCTION}$

$$F(z) := c^{-z} \sum_{j=1}^{d} \sum_{k=1}^{d} \overline{\widehat{\psi}_j} |\widehat{\psi}_j|^z \cdot u_{jk} \cdot \psi_k |\psi_k|^z \; .$$

Thorin's $\mathcal{MAGIC\ \ FUNCTION}$

$$F(z) := c^{-z} \sum_{j=1}^{d} \sum_{k=1}^{d} \overline{\widehat{\psi}}_j |\widehat{\psi}_j|^z \cdot u_{jk} \cdot \psi_k |\psi_k|^z .$$

- F is bounded: $|F(z)| \le c^{-1} \sum_{jk} |\widehat{\psi}_j| \cdot |\psi_k| = c^{-1} \|\widehat{\psi}\|_1 \cdot \|\psi\|_1$.

Thorin's $\mathcal{MAGIC\ \ FUNCTION}$

$$F(z) := c^{-z} \sum_{j=1}^{d} \sum_{k=1}^{d} \overline{\widehat{\psi}_j} |\widehat{\psi}_j|^z \cdot u_{jk} \cdot \psi_k |\psi_k|^z .$$

- F is bounded: $|F(z)| \leq c^{-1} \sum_{jk} |\widehat{\psi}_j| \cdot |\psi_k| = c^{-1} \|\widehat{\psi}\|_1 \cdot \|\psi\|_1$.
- $F(0) = 1$: $F(0) = \langle \widehat{\psi}, U\psi \rangle = \|\widehat{\psi}\|_2^2 = 1$.

Thorin's $\mathcal{MAGIC\ \ FUNCTION}$

$$F(z) := c^{-z} \sum_{j=1}^{d} \sum_{k=1}^{d} \overline{\widehat{\psi}_j} |\widehat{\psi}_j|^z \cdot u_{jk} \cdot \psi_k |\psi_k|^z .$$

- F is bounded: $|F(z)| \leq c^{-1} \sum_{jk} |\widehat{\psi}_j| \cdot |\psi_k| = c^{-1} \|\widehat{\psi}\|_1 \cdot \|\psi\|_1$.
- $F(0) = 1$: $F(0) = \langle \widehat{\psi}, U\psi \rangle = \|\widehat{\psi}\|_2^2 = 1$.
- $|F(iy)| \leq 1$: $F(iy) = c^{-iy} \langle \varphi, U\chi \rangle$,

  where $\varphi_j := |\widehat{\psi}_j|^{iy} \widehat{\psi}_j$ and $\chi_k := |\psi_k|^{iy} \psi_k$ are unit vectors;

Thorin's $\mathcal{MAGIC}$ $\mathcal{FUNCTION}$

$$F(z) := c^{-z} \sum_{j=1}^{d} \sum_{k=1}^{d} \overline{\widehat{\psi}_j} |\widehat{\psi}_j|^z \cdot u_{jk} \cdot \psi_k |\psi_k|^z .$$

- F is bounded: $|F(z)| \le c^{-1} \sum_{jk} |\widehat{\psi}_j| \cdot |\psi_k| = c^{-1} \|\widehat{\psi}\|_1 \cdot \|\psi\|_1$.
- $F(0) = 1$: $F(0) = \langle \widehat{\psi}, U\psi \rangle = \|\widehat{\psi}\|_2^2 = 1$.
- $|F(iy)| \le 1$: $F(iy) = c^{-iy} \langle \varphi, U\chi \rangle$,

  where $\varphi_j := |\widehat{\psi}_j|^{iy} \widehat{\psi}_j$ and $\chi_k := |\psi_k|^{iy} \psi_k$ are unit vectors;
- $|F(1+iy)| \le 1$: $|F(1+iy)| \le \frac{1}{c} \sum_{jk} |\widehat{\psi}_j|^2 \cdot |u_{jk}| \cdot |\psi_k|^2$.

Thorin's $\mathcal{MAGIC\ \ FUNCTION}$

$$F(z) := c^{-z} \sum_{j=1}^{d} \sum_{k=1}^{d} \overline{\widehat{\psi}}_j |\widehat{\psi}_j|^z \cdot u_{jk} \cdot \psi_k |\psi_k|^z .$$

- F is bounded: $|F(z)| \leq c^{-1} \sum_{jk} |\widehat{\psi}_j| \cdot |\psi_k| = c^{-1} \|\widehat{\psi}\|_1 \cdot \|\psi\|_1$.
- $F(0) = 1$: $F(0) = \langle \widehat{\psi}, U\psi \rangle = \|\widehat{\psi}\|_2^2 = 1$.
- $|F(iy)| \leq 1$: $F(iy) = c^{-iy} \langle \varphi, U\chi \rangle$,
  where $\varphi_j := |\widehat{\psi}_j|^{iy} \widehat{\psi}_j$ and $\chi_k := |\psi_k|^{iy} \psi_k$ are unit vectors;
- $|F(1+iy)| \leq 1$: $|F(1+iy)| \leq \frac{1}{c} \sum_{jk} |\widehat{\psi}_j|^2 \cdot |u_{jk}| \cdot |\psi_k|^2$.

It follows that $|F(z)| \leq 1$ for all $z \in S$. In particular: $\mathrm{Re}\, F'(0) \leq 0$, but...

Thorin's $\mathcal{MAGIC\ \ FUNCTION}$

$$F(z) := c^{-z} \sum_{j=1}^{d} \sum_{k=1}^{d} \overline{\widehat{\psi}}_j |\widehat{\psi}_j|^z \cdot u_{jk} \cdot \psi_k |\psi_k|^z \,.$$

- F is bounded: $|F(z)| \le c^{-1} \sum_{jk} |\widehat{\psi}_j| \cdot |\psi_k| = c^{-1} \|\widehat{\psi}\|_1 \cdot \|\psi\|_1$.
- $F(0) = 1$: $F(0) = \langle \widehat{\psi}, U\psi \rangle = \|\widehat{\psi}\|_2^2 = 1$.
- $|F(iy)| \le 1$: $F(iy) = c^{-iy} \langle \varphi, U\chi \rangle$,
  where $\varphi_j := |\widehat{\psi}_j|^{iy} \widehat{\psi}_j$ and $\chi_k := |\psi_k|^{iy} \psi_k$ are unit vectors;
- $|F(1+iy)| \le 1$: $|F(1+iy)| \le \frac{1}{c} \sum_{jk} |\widehat{\psi}_j|^2 \cdot |u_{jk}| \cdot |\psi_k|^2$.

It follows that $|F(z)| \le 1$ for all $z \in S$. In particular: $\mathrm{Re}\, F'(0) \le 0$, but...

$$\begin{aligned} F'(0) &= -\log c - \sum_{j=1}^{d} \log |\widehat{\psi}_j| \overline{\widehat{\psi}}_j (U\psi)_j - \sum_{k=1}^{d} \log |\psi_k| \overline{(U^* \widehat{\psi})_k \psi_k} \\ &= -\log c - \frac{1}{2} \left(H(|\widehat{\psi}|^2 + H(|\psi|^2) \right) . \end{aligned}$$

The statement follows. □

Application: spotting equality

We can have $H(\pi) + H(\widehat{\pi}) = \log(1/c^2)$, which means that $F'(0) = 0$, only if

$$F(z) = 1 \qquad \text{everywhere on the strip!}$$

(This is Hopf's theorem.) From this we deduce:

Theorem

We have equality in the discrete entropic uncertainty relation if and only if ψ and $\widehat{\psi}$ are supported by certain subsets D and $\widehat{D}$ of $\{1, 2, \ldots, d\}$, on which we have:

$$|\psi_k|^2 = \frac{1}{\#D} \; ; \quad |\widehat{\psi}_j|^2 = \frac{1}{\#\widehat{D}} \; ; \quad c^2 = \frac{1}{\#D \cdot \#\widehat{D}} \; .$$

In particular: $\#D \cdot \#\widehat{D} \leq d$: the supports are very small!

Examples of saturation

I Mutually unbiased bases: $c = \frac{1}{\sqrt{d}}$. We can take $D = \{k\}$ and $\widehat{D} = \{1, 2, \ldots, d\}$ or vice versa.

II Conjugate bases: $\langle \widehat{e}_j, e_k \rangle = \frac{1}{\sqrt{d}} e^{\frac{2\pi i}{d} jk}$.

Suppose $d = n\widehat{n}$. Then we can also take the pure state vector

$$\begin{aligned} \psi_k &= \frac{1}{\sqrt{n}} && \text{if } k \text{ is divisible by } \widehat{n}\text{; 0 otherwise;} \\ \widehat{\psi}_j &= \frac{1}{\sqrt{\widehat{n}}} && \text{if } j \text{ is divisible by } n\text{; 0 otherwise.} \end{aligned}$$

III And many others! For example

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}; \quad \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \widehat{\psi} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Proof of saturation theorem

Sufficiency: $H(\pi) + H(\widehat{\pi}) = \log(\#D) + \log(\#\widehat{D}) = \log(\#D \cdot \#\widehat{D}) = \log \frac{1}{c^2}$.

Necessity: $F'(0) = 0$ implies $F(z) = 1$ for all $z \in S$:

$$\sum_{j,k} \overline{\widehat{\psi}}_j |\widehat{\psi}_j|^z \, u_{jk} \, \psi_k |\psi_k|^z = c^z \ .$$

In particular for $z = 1$:

$$\sum_{j,k} |\widehat{\psi}_j|^2 \cdot |\psi_k|^2 \cdot \underbrace{\left(e^{-i\widehat{\theta}_j} \cdot e^{i\theta_k} \cdot \frac{1}{c} u_{jk} \right)}_{=1!} = 1 \ .$$

Let D, $\widehat{D}$ denote the supports of ψ and $\widehat{\psi}$. The we have for $j \in \widehat{D}$, $k \in D$:

$$u_{jk} = c \cdot e^{i(\widehat{\theta}_j - \theta_k)} \ .$$

Hence for $j \in \widehat{D}$:

$$\widehat{\psi}_j = \sum_{k \in D} u_{jk} \psi_k = c e^{i\widehat{\theta}_j} \sum_{k \in D} e^{-i\theta_k} \psi_k = c e^{i\widehat{\theta}_j} \sum_k |\psi_k| \ .$$

Proof of saturation theorem

We see that $|\widehat{\psi}_j| = c\|\psi\|_1$: the abolute value of $\widehat{\psi}$, (and also that of ψ) is constant on its support. By normalization it then follows that

$$|\psi_k|^2 = \frac{1}{\#D}\ , \quad |\widehat{\psi}_j|^2 = \frac{1}{\#\widehat{D}}\ .$$

And also:

$$\|\widehat{\psi}\|_1 = \#\widehat{D} \cdot c\|\psi\|_1 = \#\widehat{D} \cdot \#D \cdot c^2\|\widehat{\psi}\|_1\ ,$$

and we conclude that $1/c^2 = \#D \cdot \#\widehat{D}$. □