Quantum information and stabilization of quantum states by feedback control

Hans Maassen

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1. Finite quantum systems
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2. Flow diagrams
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for some subset $A \subset \Omega$. 
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\[ \mathcal{A} = (\mathbb{C} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}) \oplus (M_2 \oplus M_2 \oplus \cdots \oplus M_2) \oplus (M_3 \oplus M_3 \oplus \cdots \oplus M_3) \oplus \cdots \]
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\( \mathcal{A} \) is **purely quantum** if

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The elementary matrix algebras are \( M_1, M_2, M_3, \ldots \).
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The system $A$ is classical iff it is composed of $\mathbb{C}$'s ($= M_1$'s) only:

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All other matrix algebras are composed of these.
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The system \( A \) is classical iff it is composed of \( \mathbb{C}'s \) \((= M_1's)\) only:

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\( \mathcal{Z}(A) := A \cap A' \) is the center of \( A \), having \( \mathbb{C} \cdot \mathbb{I} \) in every component.
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Operations on quantum systems
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\end{itemize}

The adjoint of such a \textit{completely positive unit preserving} map sends states to states; we write

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(This result already holds in the commutative case.)
Operations on classical systems
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The first we call the *Heisenberg picture*, the second the *Schrödinger picture*. 
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Information flows from left to right, but the map \( T \) goes from right to left!
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[Diagram of quantum operations]

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Preparation and destruction
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\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\times
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\sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sin
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The product map \( a \otimes b \mapsto ab \) is not positive if \( A \) is noncommutative.
The "no cloning" theorem
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It follows that \( a \otimes 1 \mathbb{I} \) is multiplicative for \( C \).
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General quantum measurement
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$\varphi \xrightarrow{S(A)} M^* \xrightarrow{\pi} S(\Omega)$
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A state $\varphi$ on $A$ is mapped to a probability distribution $\pi$ on $\Omega$. 
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$m_\omega \in \mathcal{A}$ positive; $\sum_{\omega \in \Omega} m_\omega = 1_\mathcal{A}$. 

$$M^*$$

$S(\mathcal{A})$ $\pi$

$S(\Omega)$

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Positive Operator Valued Measure (POVM)
von Neumann Measurement
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Suppose $m_\omega = p_\omega$, mutually orthogonal projections in $\mathcal{A}$. 
von Neumann Measurement

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\[ \mathcal{A} \quad \xrightarrow{j} \quad C \]
von Neumann Measurement

Suppose $m_\omega = p_\omega$, mutually orthogonal projections in $\mathcal{A}$.

Then the operation $j : \mathcal{C} \to \mathcal{A}$ is a *-homomorphism:

$$j(f)j(g) = \left( \sum_\omega f(\omega)p_\omega \right) \left( \sum_y g(y)p_y \right) = \sum_\omega f(\omega)g(\omega)p_\omega = j(fg).$$
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The connection with observables (random variables) is:
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The connection with observables (random variables) is:
If $p_j$ is the event that $a$ takes the value $\alpha_j$, then we can associate to the von Neumann measurement the self-adjoint operator
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a := \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k .
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Every self-adjoint operator is of this form.
Quantum Instruments
We do not have to throw our original quantum system away when measuring.
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Opening up the environment
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$$T(x) = V \circ (x \otimes 1)$$

*for some* $k \in \mathbb{N}$, *and some compression* $V : M_n \otimes M_k \rightarrow M_n$:

![Diagram](image)

An operation $V : M_n \rightarrow M_m$ ($n \geq m$) is called a **compression** if there exists an isometry $\mathbb{C}^m \rightarrow \mathbb{C}^n$ such that for all $x \in \mathbb{C}^m$:

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Opening up the environment

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We shall denote compressions by triangular boxes:

---

\[
\begin{array}{c}
M_n \\
T \\
M_n
\end{array}
\]

\[
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M_n \\
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M_k
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$$
\begin{array}{ccc}
M_n & T & M_n \\
\cdots & & \cdots \\
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    x & x & 0 \\
    & \ddots & \\
    0 & x & \ddots
\end{pmatrix}
\begin{pmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_k
\end{pmatrix}
\]

\[
= v^*(x \otimes 1_k) v
\]

\[
= V(x \otimes 1_k).
\]
The conjugate channel

\[ T \quad M_n \quad M_n \quad = \quad V \quad M_n \quad M_n \quad M_k \quad \]

\[ T^C \quad M_n \quad M_k \quad = \quad V \quad M_n \quad M_n \quad M_k \quad \]
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Information leaking away into the environment.

*Heisenberg’s Principle:* Information that leaks away disappears from the channel.
The Heisenberg Principle:
No Information Extraction Without Perturbation
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4. Protecting information by ignorance
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Let $V : M_n \rightarrow \mathcal{B}(\mathcal{L})$ denote a compression to a subspace $\mathcal{L} \subset \mathbb{C}^n$.
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A necessary and sufficient condition for the subspace $\mathcal{L} \subset \mathbb{C}^n$ to be protected against the operation
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A necessary and sufficient condition for the subspace $\mathcal{L} \subset \mathbb{C}^n$ to be protected against the operation

$$T : x \mapsto \sum_{i=1}^{k} a_i^* x a_i$$

is that for some complex $k \times k$ matrix $(\lambda_{ij})$:

$$p_{\mathcal{L}} a_i^* a_j p_{\mathcal{L}} = \lambda_{ij} p_{\mathcal{L}}.$$
Proof of necessity of Knill-Laflamme condition by diagrams
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\[ B(\mathcal{L}) \backslash P \quad M_n \quad T \quad M_n \]
Proof of necessity of Knill-Laflamme condition by diagrams

\[ B(\mathcal{L}) \rightarrow P \rightarrow M_n \rightarrow T \rightarrow M_n \]
Proof of necessity of Knill-Laflamme condition by diagrams
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$B(\mathcal{L}) \xrightarrow{P} M_n \xrightarrow{T} M_n \xrightarrow{\exists D} B(\mathcal{L}) \quad \equiv \quad B(\mathcal{L})$

STINESPRING!
Proof of necessity of Knill-Laflamme condition by diagrams

\[ B(\mathcal{L}) \xrightarrow{P} M_n \xrightarrow{T} M_n \xrightarrow{\exists D} B(\mathcal{L}) = B(\mathcal{L}) \]

STINESPRING!
Proof of necessity of Knill-Laflamme condition by diagrams

\[ \exists D \]

\[ \exists \text{STINESPRING !} \]
Proof of necessity of Knill-Laflamme condition by diagrams

$B(\mathcal{L})$ $P$ $M_n$ $T$ $M_n$ $\exists D$ $B(\mathcal{L})$ $= B(\mathcal{L})$

$B(\mathcal{L})$ $P$ $M_n$ $V$ $M_n$ $\times$

STINESPRING!
Proof of necessity of Knill-Laflamme condition by diagrams

\[ \mathcal{B}(\mathcal{L}) \xrightarrow{P} M_n \xrightarrow{T} M_n \xrightarrow{\exists D} \mathcal{B}(\mathcal{L}) = \mathcal{B}(\mathcal{L}) \]

\[ \mathcal{B}(\mathcal{L}) \xrightarrow{P} M_n \xrightarrow{V} M_n \xrightarrow{\exists D} \mathcal{B}(\mathcal{L}) \]

STINESPRING!
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\[ B(\mathcal{L}) \xrightarrow{P} M_n \xrightarrow{T} M_n \xrightarrow{\exists D} B(\mathcal{L}) = \]

\[ B(\mathcal{L}) \xrightarrow{P} M_n \xrightarrow{V} M_n \xrightarrow{\exists D} B(\mathcal{L}) = \]

STINESPRING!

HEISENBERG!
Proof of necessity of Knill-Laflamme condition by diagrams

\[ \mathcal{B}(\mathcal{L}) \overset{\mathcal{P}}{\longrightarrow} M_n \overset{T}{\longrightarrow} M_n \overset{\exists D}{\longrightarrow} \mathcal{B}(\mathcal{L}) \]

STINESPRING !

\[ \mathcal{B}(\mathcal{L}) \overset{\mathcal{P}}{\longrightarrow} M_n \overset{\mathcal{V}}{\longrightarrow} M_n \overset{\exists D}{\longrightarrow} \mathcal{B}(\mathcal{L}) \]

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Proof of necessity of Knill-Laflamme condition by diagrams

\[ B(\mathcal{L}) \xrightarrow{P} M_n \xrightarrow{T} M_n \xrightarrow{\exists D} B(\mathcal{L}) \]

\[ \exists D_B(\mathcal{L}) \]

STINESPRING!

\[ B(\mathcal{L}) \xrightarrow{P} M_n \xrightarrow{V} \] HEISENBERG!

\[ B(\mathcal{L}) \]

\[ B(\mathcal{L}) \]
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\[ B(\mathcal{L}) \quad P \quad M_n \quad T \quad M_n \quad \exists D \quad B(\mathcal{L}) = \]

\[ B(\mathcal{L}) \]

\[ B(\mathcal{L}) \quad P \quad M_n \quad V \quad M_n \quad \exists D \quad B(\mathcal{L}) = \]

\[ B(\mathcal{L}) \]

STINESPRING!

\[ B(\mathcal{L}) \quad P \quad M_n \quad V \quad \]
Proof of necessity of Knill-Laflamme condition by diagrams

\[ B(\mathcal{L}) \]  
\[ \xrightarrow{P} \]  
\[ M_n \]  
\[ \xrightarrow{T} \]  
\[ M_n \]  
\[ \exists D \]  
\[ B(\mathcal{L}) \]

\[ \cong \]  
\[ B(\mathcal{L}) \]

\[ B(\mathcal{L}) \]  
\[ \xrightarrow{P} \]  
\[ M_n \]  
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\[ M_n \]  
\[ \exists D \]  
\[ B(\mathcal{L}) \]

\[ \cong \]  
\[ B(\mathcal{L}) \]

\[ B(\mathcal{L}) \]  
\[ \xrightarrow{P} \]  
\[ M_n \]  
\[ \xrightarrow{V} \]  
\[ \times \]  
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\[ STINESPRING ! \]

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\[ B(\mathcal{L}) \] \[ P \] \[ M_n \] \[ T \] \[ M_n \] \[ \exists D \] \[ B(\mathcal{L}) \] \[ = \] \[ B(\mathcal{L}) \]

STINESPRING !

\[ B(\mathcal{L}) \] \[ P \] \[ M_n \] \[ V \] \[ \times \] \[ \exists D \] \[ B(\mathcal{L}) \] \[ = \] \[ B(\mathcal{L}) \]

HEISENBERG !

\[ B(\mathcal{L}) \] \[ P \] \[ M_n \] \[ V \] \[ \times \] \[ = \] \[ B(\mathcal{L}) \] \[ \times \] \[ \lambda \] \[ M_k \]
Proof of necessity of Knill-Laflamme condition by diagrams

The last line says that the space $L$ is dark: no information leaks out and only some random output $\lambda$ results:
Proof of necessity of Knill-Laflamme condition by diagrams

The last line says that the space $\mathcal{L}$ is dark: no information leaks out and only some random output $\lambda$ results:

$$P \circ V(\mathbb{1}_n \otimes y) = \lambda(y) \cdot \mathbb{1}_\mathcal{L}.$$
Translating the darkness condition back
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We note that this is precisely the Knill-Laflamme condition:
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LHS: \quad P \circ V(\mathbb{1}_n \otimes y)
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We note that this is precisely the Knill-Laflamme condition:

\[ \text{LHS} : \quad P \circ V(1_n \otimes y) = p(a_1^*, \cdots, a_k^*) \begin{pmatrix} y_{11} \cdot 1 & \cdots & y_{1k} \cdot 1 \\ \vdots & & \vdots \\ y_{k1} \cdot 1 & \cdots & y_{kk} \cdot 1 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} \]
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\[ = \sum_{i,j=1}^{k} p a_i^* a_j p \cdot y_{ij} \]

\[ \text{RHS} : \quad p \cdot \lambda(y) = p \cdot \sum_{i,j=1}^{k} \lambda_{ij} y_{ij} \]
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\[
= \sum_{i,j=1}^k p a_i^* a_j p \cdot y_{ij}
\]

\[
RHS : \quad p \cdot \lambda(y) = p \cdot \sum_{i,j=1}^k \lambda_{ij} y_{ij}
\]

These must be equal for all \( y \in M_k \):

\[
p a_i^* a_j p = \lambda_{ji} \cdot p.
\]
5. Partial darkness and protection by feedback
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Sometimes the subspace $\mathcal{D}$ is not completely dark, but yet a certain measurement on the conjugate channel reveals nothing about the state in $\mathcal{D}$. 
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$B(\mathcal{D}) \xrightarrow{P} M_n$
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\[ B(\mathcal{D}) \quad P \quad M_n \quad V \quad j \quad M_n \]
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By the Heisenberg principle, together with the existence of a copier for the straight line, the weaker darkness property is a necessary condition for this weaker protection property:
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$$B(\mathcal{D}) \xrightarrow{P} \mathcal{M}_n \xrightarrow{V} \mathcal{M}_n \xrightarrow{\lambda} \mathbb{C}^k$$
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\[ \begin{array}{c}
\mathcal{B}(\mathcal{D}) \\
\text{P} \\
\text{M}_n \\
\text{V} \\
\text{j} \\
\end{array} \quad \begin{array}{c}
\mathcal{B}(\mathcal{D}) \\
\text{X} \\
\mathcal{C}^k \\
\end{array} \]

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By the Heisenberg principle, together with the existence of a copier for the straight line, the weaker darkness property is a necessary condition for this weaker protection property: the existence of a decoding operation $D_\omega$ such that:

$$B(\mathcal{D}) \xrightarrow{P, V} B(\mathcal{D}) \xrightarrow{D_\omega} B(\mathcal{D})$$
5. Partial darkness and protection by feedback

Sometimes the subspace $D$ is not completely dark, but yet a certain measurement on the conjugate channel reveals nothing about the state in $D$.

This darkness property reads:

$$pa_i^*a_ip = \lambda_ip$$

In that case the subspace may still be protected, but now the measurement outcome has to be fed back into the system.

By the Heisenberg principle, together with the existence of a copier for the straight line, the weaker darkness property is a necessary condition for this weaker protection property: the existence of a decoding operation $D_\omega$ such that:

$$B(D) \overset{P}{\xrightarrow{M_n}} V \overset{M_n}{\xrightarrow{j}} B(D) \overset{\lambda}{\xrightarrow{\mathbb{C}^k}}$$

$$B(D) \overset{D_\omega}{\xrightarrow{B(D)}} B(D) \overset{\mathbb{C}^k}{\xrightarrow{\lambda}}$$
5. Partial darkness and protection by feedback

Sometimes the subspace $\mathcal{D}$ is not completely dark, but yet a certain measurement on the conjugate channel reveals nothing about the state in $\mathcal{D}$.

$$B(\mathcal{D}) P M_n V j = B(\mathcal{D}) \lambda \mathbb{C}^k$$

This darkness property reads:

$$p a_i^* a_i p = \lambda_i p$$

In that case the subspace may still be protected, but now the measurement outcome has to be fed back into the system.

By the Heisenberg principle, together with the existence of a copier for the straight line, the weaker darkness property is a necessary condition for this weaker protection property: the existence of a decoding operation $D_\omega$ such that:

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5. Partial darkness and protection by feedback

Sometimes the subspace $\mathcal{D}$ is not completely dark, but yet a certain measurement on the conjugate channel reveals nothing about the state in $\mathcal{D}$.

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In that case the subspace may still be protected, but now the measurement outcome has to be fed back into the system.

By the Heisenberg principle, together with the existence of a copier for the straight line, the weaker *darkness property* is a necessary condition for this weaker *protection property*: the existence of a decoding operation $D_\omega$ such that:

The two conditions are actually equivalent.
State stabilisation by feedback control
State stabilisation by feedback control

Theorem
State stabilisation by feedback control

Theorem
Let $T : M_n \rightarrow M_n$ have Stinespring decomposition

$$T(x) = V(x \otimes 1_l) = v^*(x \otimes 1_l)v ,$$
State stabilisation by feedback control

**Theorem**

Let $T : M_n \to M_n$ have Stinespring decomposition

$$T(x) = V(x \otimes \mathbb{1}) = v^*(x \otimes \mathbb{1})v,$$

with $v : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^k$ isometric.
State stabilisation by feedback control

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Let $T : M_n \to M_n$ have Stinespring decomposition

$$T(x) = V(x \otimes 1) = v^*(x \otimes 1)v,$$

with $v : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^k$ isometric. Let $M : \mathcal{F}(\Omega) \to M_k$ be a POVM, satisfying

$$\forall \omega \in \Omega : \quad m(\omega) := M(\delta_\omega) \text{ has rank } 1.$$
Theorem

Let $T : M_n \to M_n$ have Stinespring decomposition

$$T(x) = V(x \otimes 1) = \nu^*(x \otimes 1)\nu,$$

with $\nu : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^k$ isometric. Let $M : \mathcal{F}(\Omega) \to M_k$ be a POVM, satisfying

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Then there exists an actuator map
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Then there exists an actuator map

$$w : \Omega \to \text{unitaries in } M_n$$

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State stabilisation by feedback control

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\[ \rho \xrightarrow{V} W. \]
State stabilisation by feedback control

**Theorem**

Let $T : M_n \to M_n$ have Stinespring decomposition

$$T(x) = V(x \otimes 1) = v^* (x \otimes 1) v,$$

with $v : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathbb{C}^k$ isometric. Let $M : \mathcal{F}(\Omega) \to M_k$ be a POVM, satisfying

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\[ T(x) = \sum_{\omega \in \Omega} b(\omega)^*_xb(\omega), \quad \text{hence} \quad \sum_{\omega \in \Omega} b(\omega)^*b(\omega) = \mathbb{1}. \]
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Now define unitaries \( w(\omega) \in M_n \) by the polar decomposition
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\[ b(\omega) \sqrt{\rho} = w(\omega)^{-1} |b(\omega)\sqrt{\rho}|. \]
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\[ \rho \circ V \circ (\text{id} \otimes M) \circ W(x) \]
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\[ \rho \circ V \circ \left( \text{id} \otimes M \right) \circ W(x) = \sum_\omega \sum_{ij} \rho \left( a_i^* w(\omega)^* x w(\omega) a_j \right) m_{ij}(\omega). \]
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\]
\[
= \sum_{\omega} \rho \left( b(\omega)^* w(\omega)^* x w(\omega) b(\omega) \right)
\]
Proof.

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\[ = \sum_{\omega} \rho \left( b(\omega)^* w(\omega)^* x w(\omega) b(\omega) \right) = \sum_{\omega} \text{tr} \left( x |b(\omega)\sqrt{\rho}|^2 \right) \]
Proof.

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\[
\begin{align*}
\rho \circ V \circ \left( \text{id} \otimes M \right) \circ W(x) &= \sum_{\omega} \sum_{ij} \rho \left( a_i^* w(\omega)^* x w(\omega) a_j \right) m_{ij}(\omega) \\
&= \sum_{\omega} \rho \left( b(\omega)^* w(\omega)^* x w(\omega) b(\omega) \right) \\
&= \sum_{\omega} \text{tr} \left( x \sqrt{\rho} b(\omega)^* b(\omega) \sqrt{\rho} \right) \\
&= \sum_{\omega} \text{tr} \left( x b(\omega)^* b(\omega) \sqrt{\rho} \right) 
\end{align*}
\]
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Let 
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\quad = \sum_{\omega} \rho \left( b(\omega)^* w(\omega)^* x w(\omega) b(\omega) \right)
\quad = \sum_{\omega} \text{tr} \left( x |b(\omega) \sqrt{\rho}|^2 \right)
\quad = \sum_{\omega} \text{tr} \left( x \sqrt{\rho} b(\omega)^* b(\omega) \sqrt{\rho} \right)
\quad = \text{tr}(x \rho).
\]
Repeated Instruments
Repeated Instruments

\[ \rho \]
Repeated Instruments

\[
\rho \quad M \quad i_1
\]
Repeated Instruments

\[ \rho \rightarrow M \rightarrow M \rightarrow i_2 \]

\[ i_1 \]
Repeated Instruments

\[ \rho M_1 M_2 M_3 \ldots \]

\[ \vdots \]

\[ i_3 \]

\[ i_2 \]

\[ i_1 \]
Repeated Instruments

\[ \rho, M_1, M_2, M_3, \ldots \]

\[ \Omega := \{1, 2, \ldots, k\}^N ; \]
Repeated Instruments

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\( \Sigma_m \) generated by cilinder sets:
Repeated Instruments

$\Omega := \{1, 2, \ldots, k\}^\mathbb{N}$;

$\Sigma_m$ generated by cylinder sets:

$\Lambda_{i_1, \ldots, i_m} := \{\omega \in \Omega | \omega_1 = i_1, \ldots, \omega_m = i_m\}$.
Repeated Instruments

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POVM:
Repeated Instruments

\[ \Omega := \{1, 2, \ldots, k\}^N ; \]

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POVM: \[ Q_m(\Lambda_{i_1, \ldots, i_m}) := T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m}(\mathbb{I}) . \]
Repeated Instruments

\[ \Omega := \{1, 2, \ldots, k\}^N ; \]

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\[ \mathbb{P}_\rho := \rho \circ Q_\infty . \]
Repeated Instruments

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\( \Sigma_m \) generated by cylinder sets:

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POVM:

\[ Q_m(\Lambda_{i_1, \ldots, i_m}) := T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m}(\mathbb{1}) . \]

\[ \mathbb{P}_\rho := \rho \circ Q_\infty . \]

Left shift:

\[ \sigma : \Omega \rightarrow \Omega : (\sigma \omega)_j := \omega_{j+1} . \]
Quantum Trajectories
Quantum Trajectories

\[ \rho \xrightarrow{\Theta_0} \]
Quantum Trajectories

\[ \rho \rightarrow M \rightarrow i_1 \]
Quantum Trajectories
Quantum Trajectories
Quantum Trajectories

$\Theta_n : \Omega \rightarrow S(A) : \Theta_n(\omega) : x \mapsto \frac{\rho(T_{\omega_1} \circ \cdots \circ T_{\omega_n}(x))}{\rho(T_{\omega_1} \circ \cdots \circ T_{\omega_n}(\mathbb{I}))}$. 
Theorem

For any state $\rho$ on $\mathcal{A}$:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Theta_j = \Theta_\infty \quad \mathbb{P}_\rho\text{-a.s.},$$

where the random variable $\Theta_\infty$ takes values in the $T$-invariant states on $\mathcal{A}$. 

$$\Theta_n : \Omega \to S(\mathcal{A}) : \Theta_n(\omega) : x \mapsto \frac{\rho(T_{\omega_1} \circ \cdots \circ T_{\omega_n}(x))}{\rho(T_{\omega_1} \circ \cdots \circ T_{\omega_n}(\mathbb{I}))}.$$
Proof.
Proof.
First note that for all $x \in A$: 

□
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$$\mathbb{E}_\rho(\Theta_{n+1}(x)|\Sigma_n)$$
Proof.
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$$\mathbb{E}_\rho(\Theta_{n+1}(x)|\Sigma_n) = \sum_{i=1}^{k} \Theta_n(T_i \mathbb{1})$$
Proof.
First note that for all $x \in \mathcal{A}$:

$$
\mathbb{E}_\rho(\Theta_{n+1}(x)|\Sigma_n) = \sum_{i=1}^{k} \Theta_n(T_i \mathbb{1}) \cdot \frac{\Theta_n(T_i(x))}{\Theta_n(T_i(\mathbb{1}))}
$$
Proof.
First note that for all $x \in \mathcal{A}$:

$$
\mathbb{E}_\rho(\Theta_{n+1}(x)|\Sigma_n) = \sum_{i=1}^{k} \Theta_n(T_i 1) \cdot \frac{\Theta_n(T_i(x))}{\Theta_n(T_i(1))} = \Theta_n(T(x)).
$$
Proof.  
First note that for all $x \in \mathcal{A}$:

$$
\mathbb{E}_\rho (\Theta_{n+1}(x) | \Sigma_n) = \sum_{i=1}^{k} \Theta_n(T_i \mathbb{1}) \cdot \frac{\Theta_n(T_i(x))}{\Theta_n(T_i(\mathbb{1}))} = \Theta_n(T(x)) .
$$

Let $P : \mathcal{A} \to \mathcal{A}$ denote the (ergodic) projection of $T$. 

\[ \square \]
Proof.
First note that for all $x \in \mathcal{A}$:

$$\mathbb{E}_\rho(\Theta_{n+1}(x) | \Sigma_n) = \sum_{i=1}^{k} \Theta_n(T_i \mathbb{1}) \cdot \frac{\Theta_n(T_i(x))}{\Theta_n(T_i(\mathbb{1}))} = \Theta_n(T(x)).$$

Let $P : \mathcal{A} \to \mathcal{A}$ denote the (ergodic) projection of $T$. Then $(\Theta_n(Px))_{n \in \mathbb{N}}$ is a $\mathbb{P}_\rho$-martingale. 

□
Proof. First note that for all $x \in \mathcal{A}$:

$$
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$$

Let $P : \mathcal{A} \to \mathcal{A}$ denote the (ergodic) projection of $T$. Then $(\Theta_n(Px))_{n \in \mathbb{N}}$ is a $\mathbb{P}_\rho$-martingale since $\mathbb{E}_\rho(\Theta_{n+1}(P(x))|\Sigma_n) = \Theta_n(TP(x)) = \Theta_n(P(x))$. 

\hfill \Box
Proof.
First note that for all \( x \in \mathcal{A} \):

\[
\mathbb{E}_\rho(\Theta_{n+1}(x)|\Sigma_n) = \sum_{i=1}^{k} \Theta_n(T_i\mathbb{1}) \cdot \frac{\Theta_n(T_i(x))}{\Theta_n(T_i(\mathbb{1}))} = \Theta_n(T(x)) .
\]

Let \( P : \mathcal{A} \rightarrow \mathcal{A} \) denote the (ergodic) projection of \( T \). Then \( (\Theta_n(Px))_{n \in \mathbb{N}} \) is a \( \mathbb{P}_\rho \)-martingale since \( \mathbb{E}_\rho(\Theta_{n+1}(P(x))|\Sigma_n) = \Theta_n(TP(x)) = \Theta_n(P(x)) \).

Say \( \Theta_n(Px) \rightarrow \Theta_{\infty}(x) \) as \( n \rightarrow \infty \).
Proof.

First note that for all $x \in \mathcal{A}$:

$$
\mathbb{E}_\rho(\Theta_{n+1}(x)|\Sigma_n) = \sum_{i=1}^{k} \Theta_n(T_i(1)) \cdot \frac{\Theta_n(T_i(x))}{\Theta_n(T_i(1))} = \Theta_n(T(x)) .
$$

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Say $\Theta_n(Px) \rightarrow \Theta_\infty(x)$ as $n \rightarrow \infty$.

But also the innovations $V_n(x) := \Theta_{n+1}(x) - \Theta(Tx)$ form a martingale $Y_n(x)$ by weighted addition:

$$
Y_n(x) := \sum_{j=1}^{n} \frac{1}{j} V_j \quad \text{with} \quad \mathbb{E}_\rho(|Y_n(x)|^2) \leq 4\|x\|^2 \frac{\pi^2}{6} .
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Proof.
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Averaging over $m$ yields: $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} (\Theta_j(x) - \Theta_j(Px)) \rightarrow 0$. 

\boxed{\square}