Quantum information and stabilization of quantum states by feedback control

Hans Maassen

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Luc Bouten (Nijmegen)
Karol Życzkowski (Krakow)

Overview

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   - Stabilization of a given state

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(This result already holds in the commutative case.)

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The first we call the *Heisenberg picture*, the second the *Schrödinger picture*.

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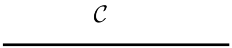

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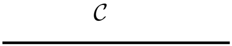

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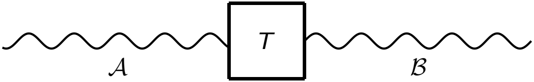

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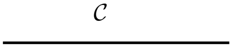

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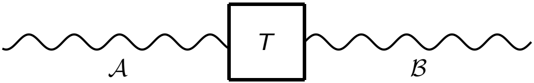

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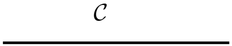

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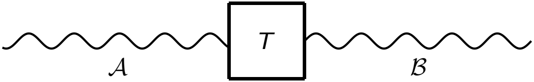

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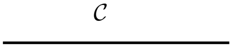

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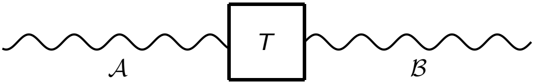

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The tensor product $\mathcal{A} \otimes \mathcal{B}$ is denoted by parallel lines:

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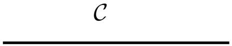

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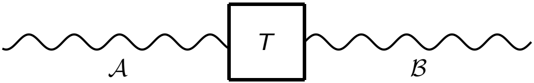

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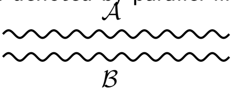

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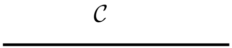

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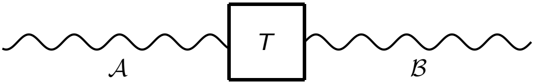

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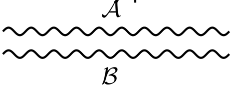

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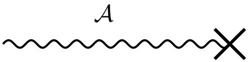

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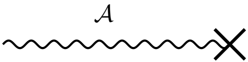

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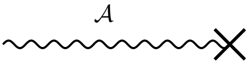

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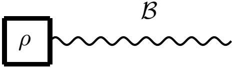

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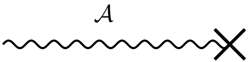

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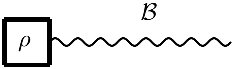

Interpretation: there are many ways to prepare a system, but only one way to destroy (or just ignore) it.

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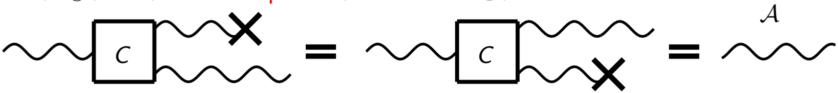

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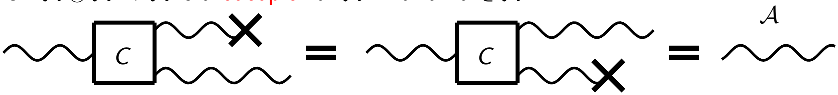

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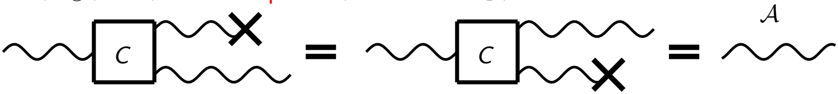

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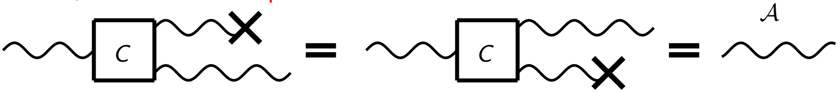

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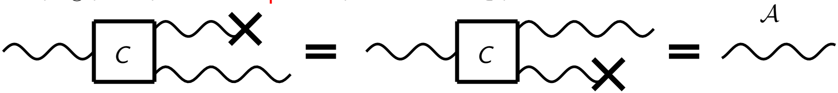

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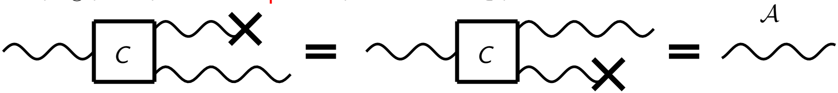

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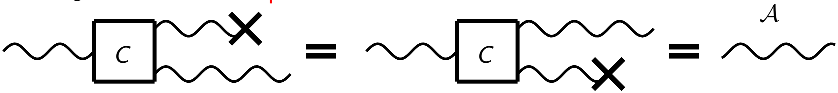

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The product map $a \otimes b \mapsto ab$ is not positive if $\mathcal{A}$ is noncommutative.

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An Ω-valued measurement

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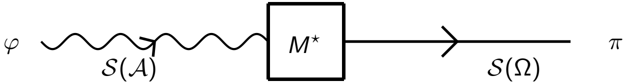

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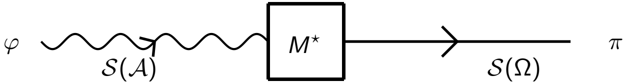

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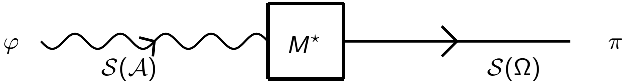

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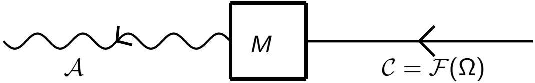

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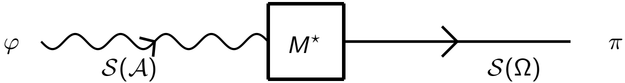

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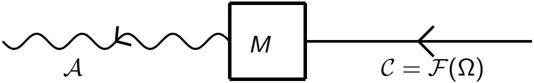

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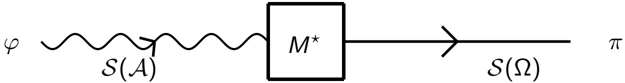

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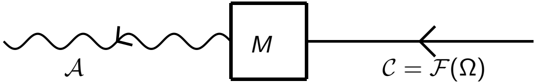

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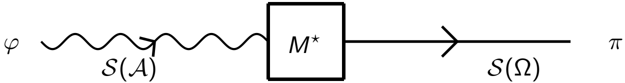

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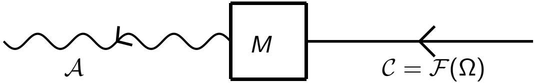

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Positive Operator Valued Measure (POVM)

von Neumann Measurement

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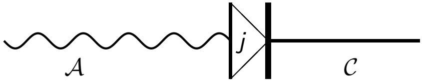

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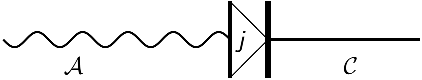

Then the operation $j : \mathcal{C} \to \mathcal{A}$ is a *-homomorphism:

$$j(f)j(g) = \left(\sum_{\omega} f(\omega) p_\omega \right) \left(\sum_{y} g(y) p_y \right) = \sum_{\omega} f(\omega) g(\omega) p_\omega = j(fg) \ .$$

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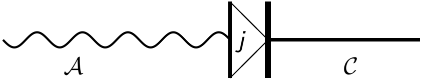

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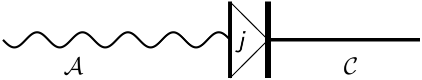

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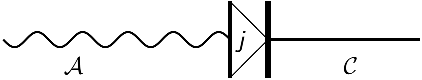

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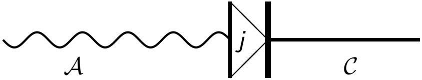

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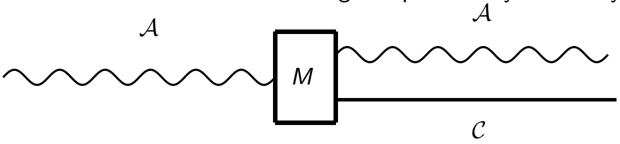
Every self-adjoint operator is of this form.

Quantum Instruments

Quantum Instruments

We do not have to throw our original quantum system away when measuring.



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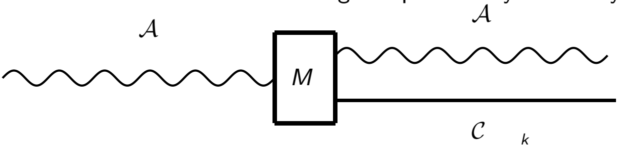

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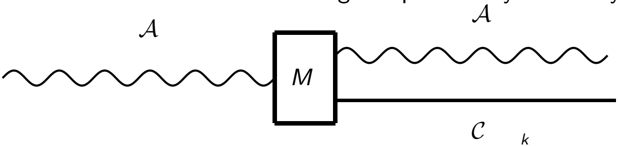

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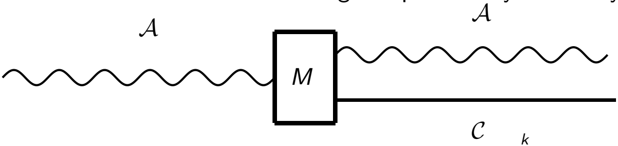

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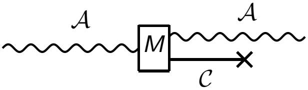

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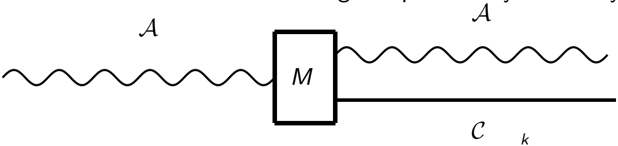

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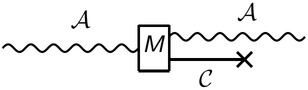

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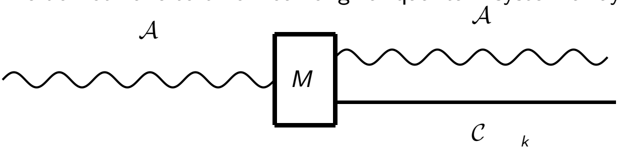

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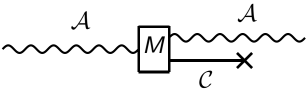

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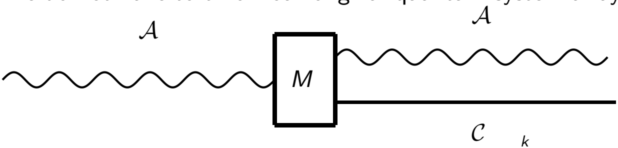

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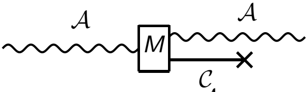

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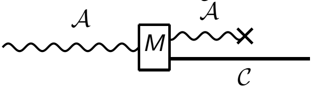

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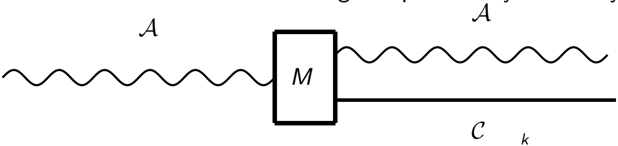

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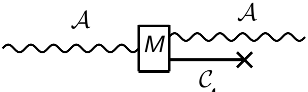

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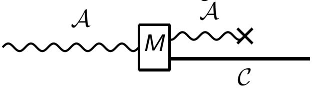

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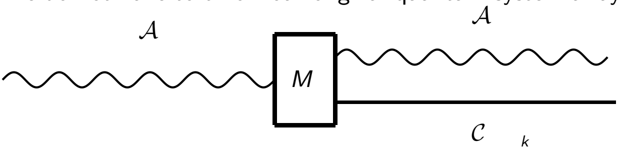

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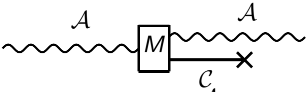

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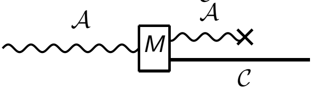

$Q : \mathcal{C} \to \mathcal{A} : \quad f \mapsto M(\mathbb{1} \otimes f)$ (POVM)

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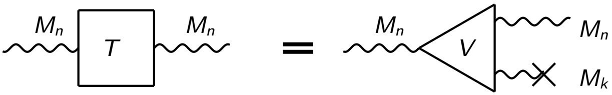

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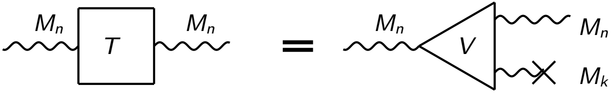

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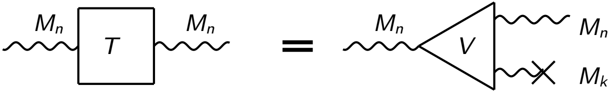

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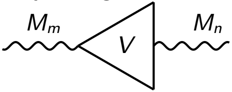

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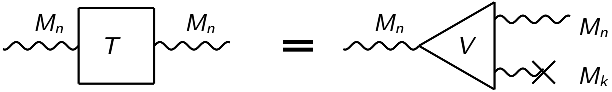

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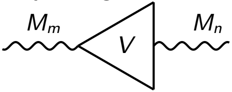

Stinespring versus Kraus representation

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Stinespring versus Kraus representation

$$\begin{aligned} T(x) &= \sum_{i=1}^{k} a_i^* x a_i \\ &= (a_1^*, a_2^*, \cdots, a_k^*) \begin{pmatrix} x & & & & \\ & x & & 0 & \\ & & \ddots & & \\ & 0 & & x & \\ & & & & x \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} \end{aligned}$$

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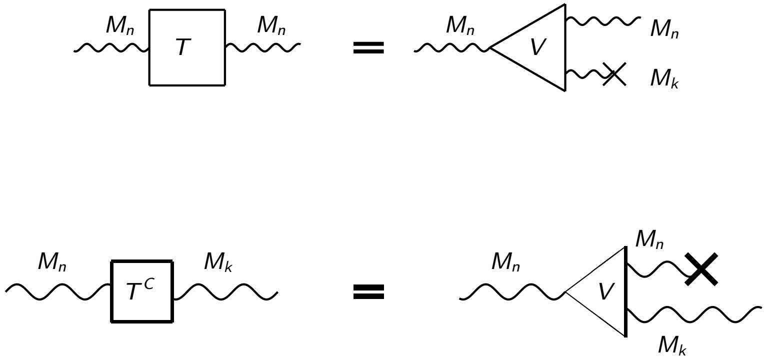
The conjugate channel



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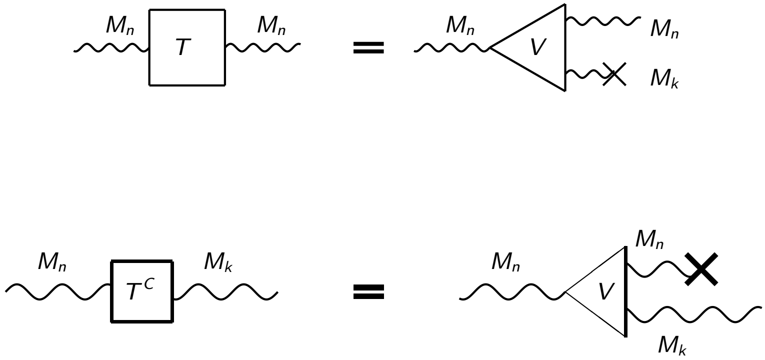

Information leaking away into the environment.

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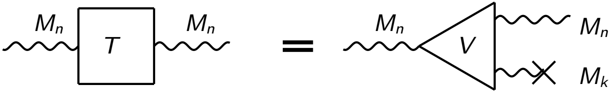

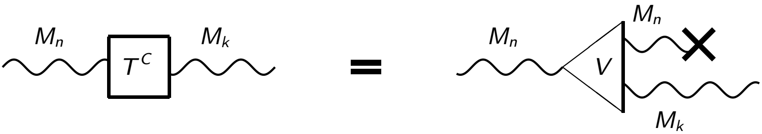

Information leaking away into the environment.
Heisenberg's Principle: Information that leaks away disappears from the channel

The Heisenberg Principle: No Information Extraction Without Perturbation

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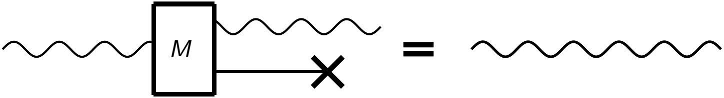

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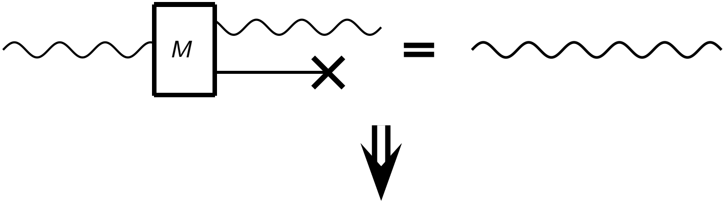

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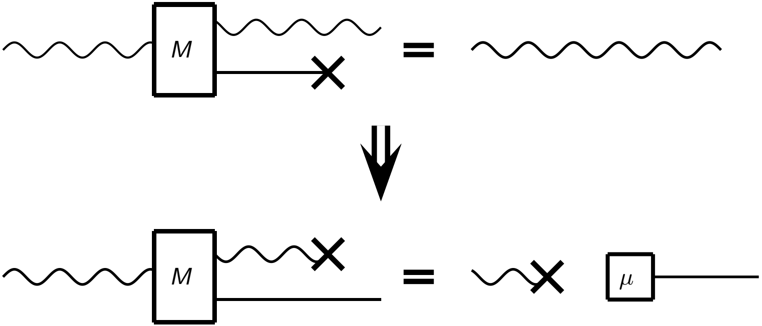

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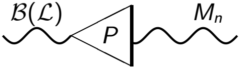

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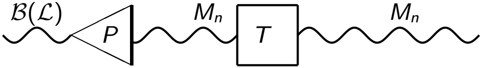

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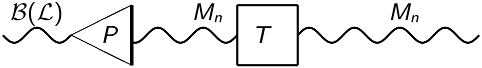

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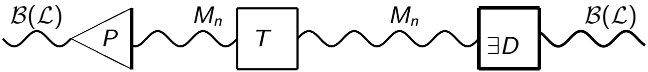

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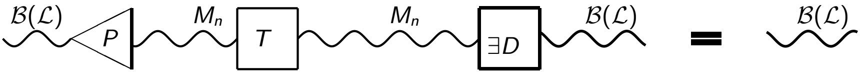

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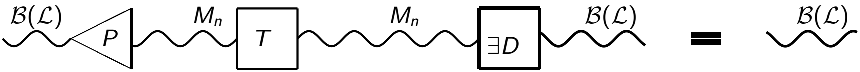

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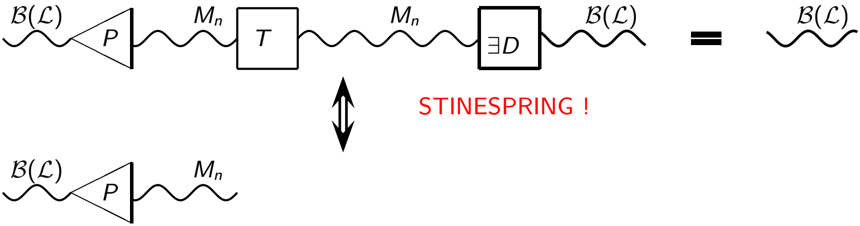

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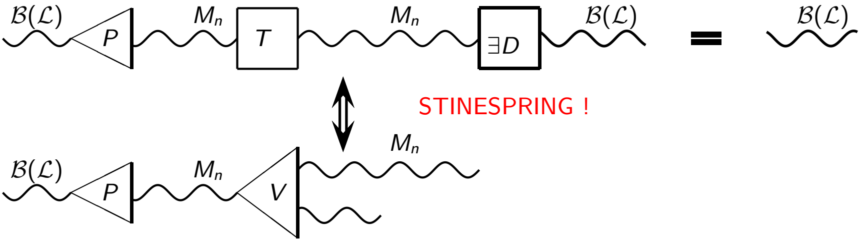

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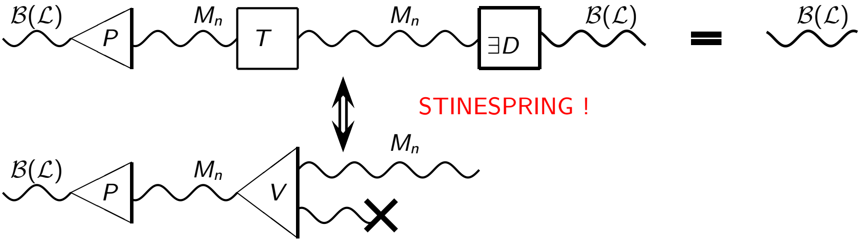

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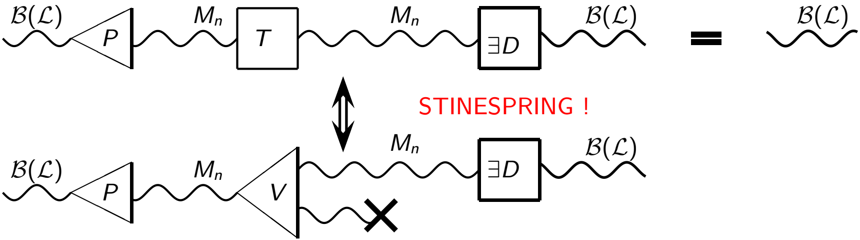

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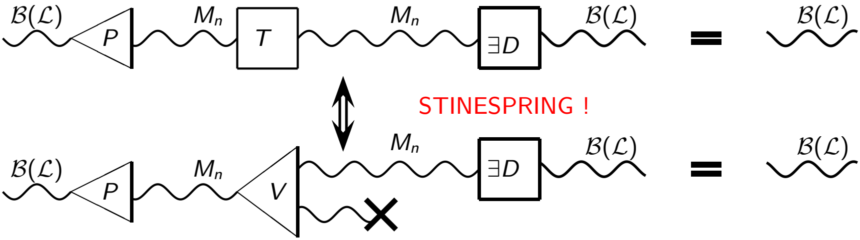

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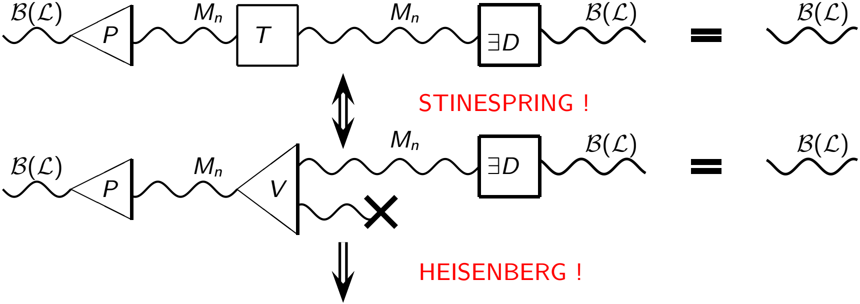

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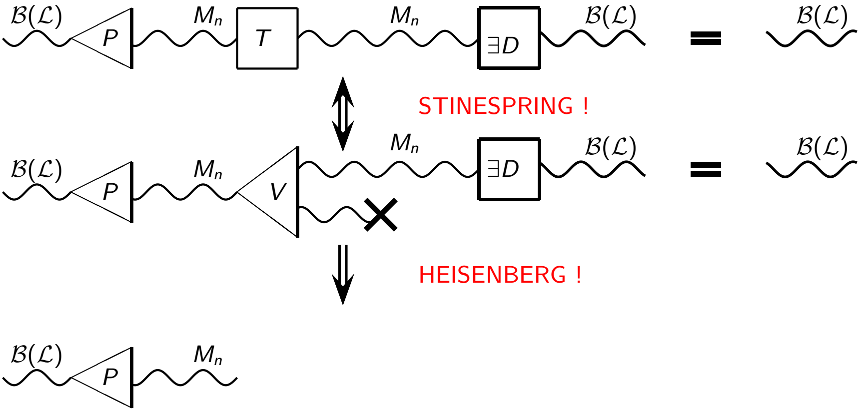

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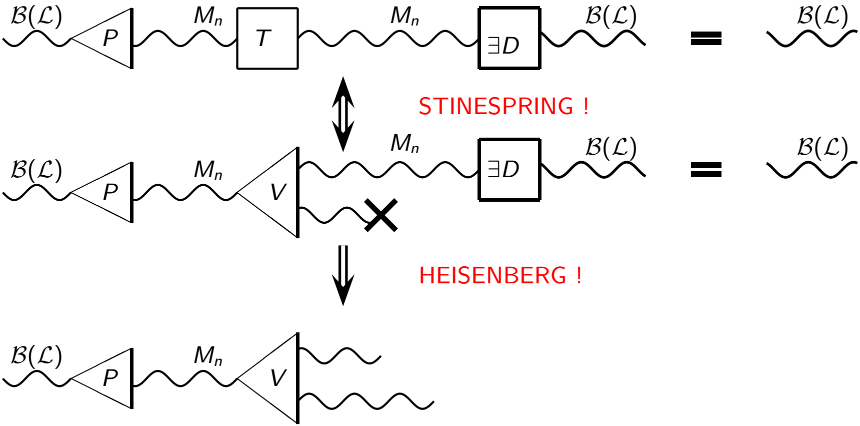

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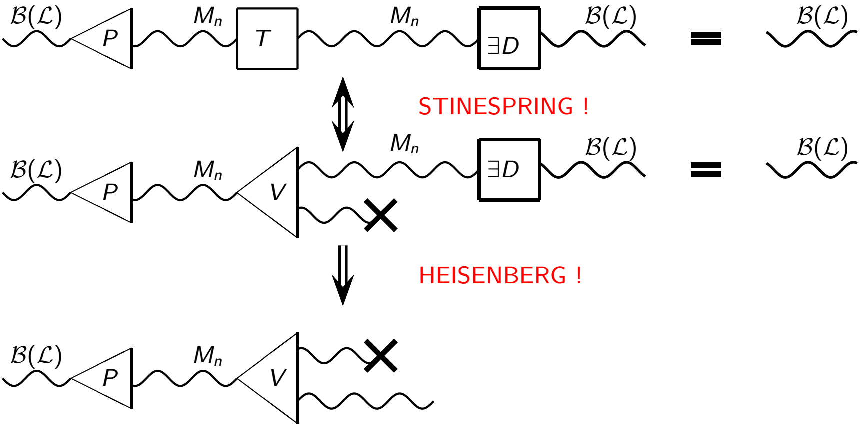

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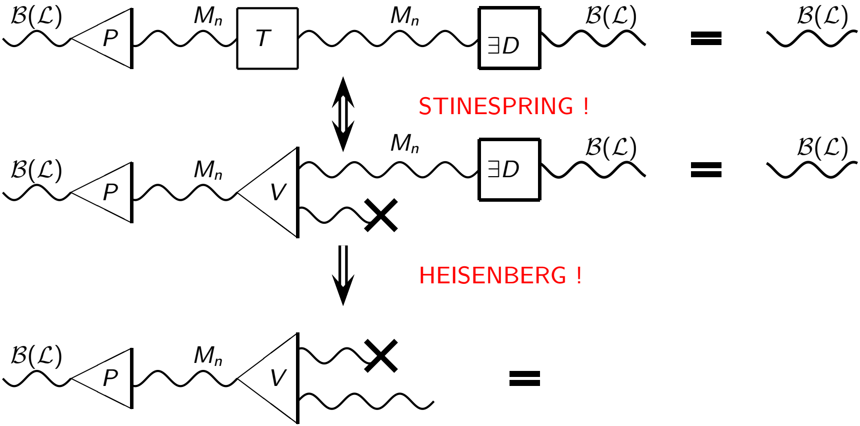

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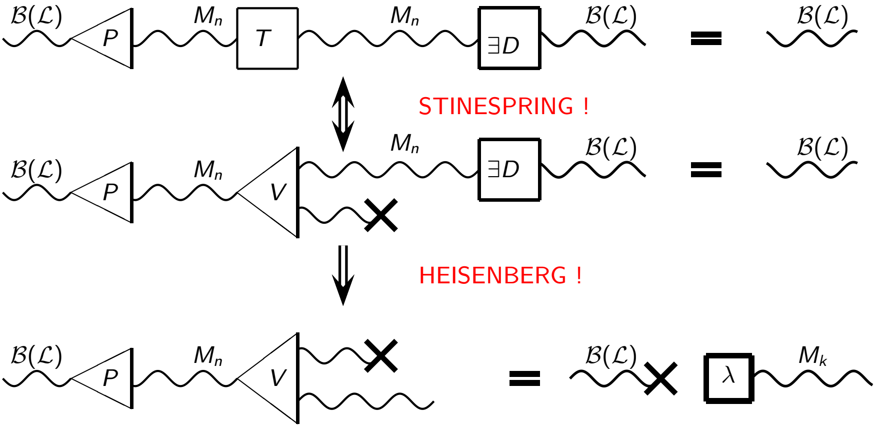

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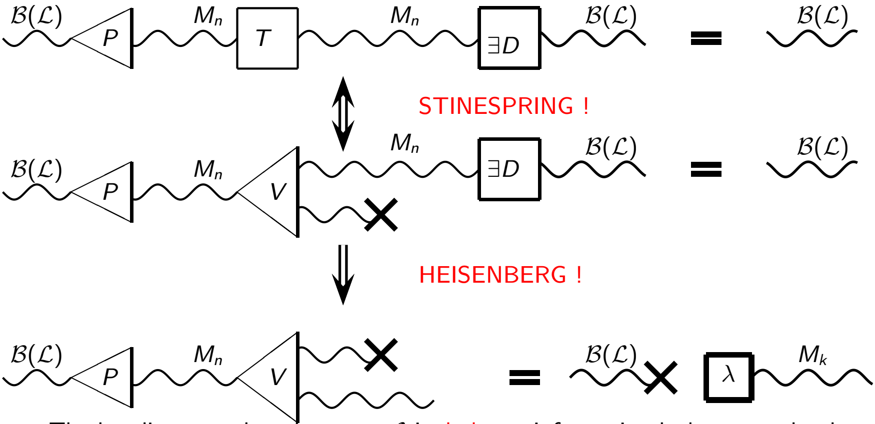

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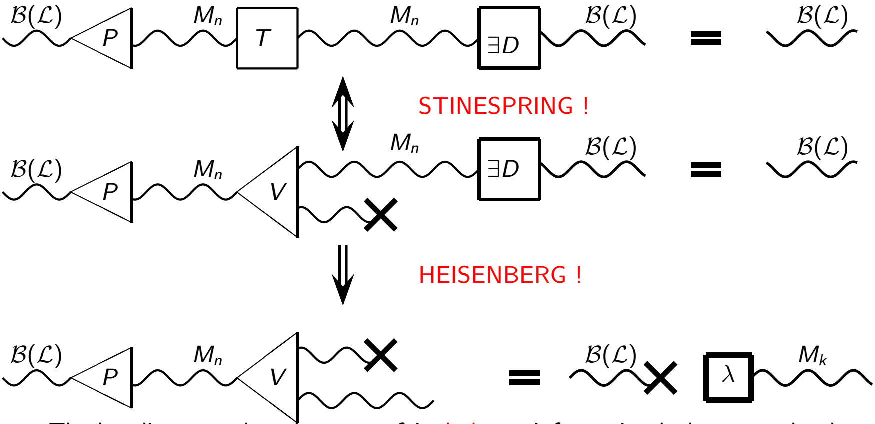

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These must be equal for all $y \in M_k$:

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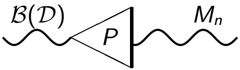

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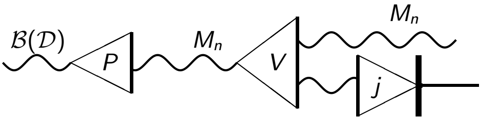

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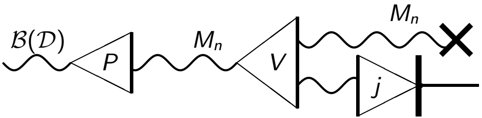

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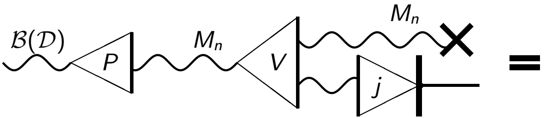

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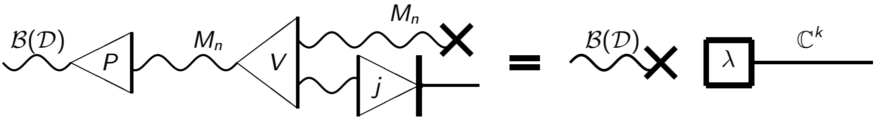

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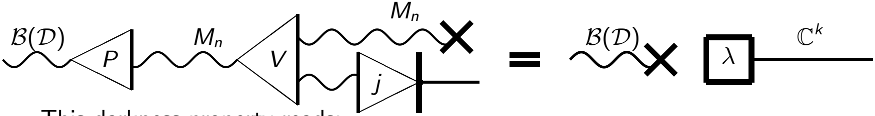

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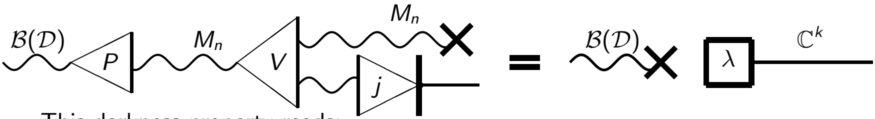

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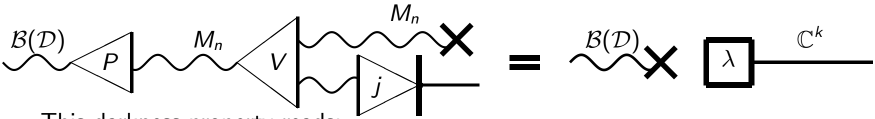

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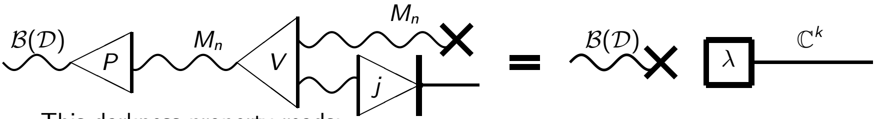

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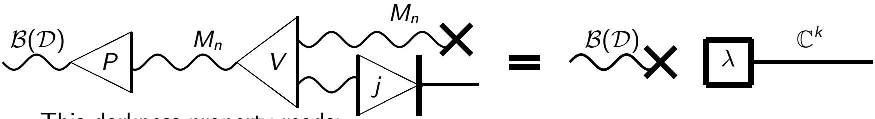

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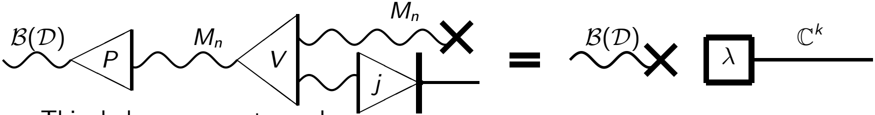

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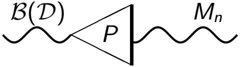

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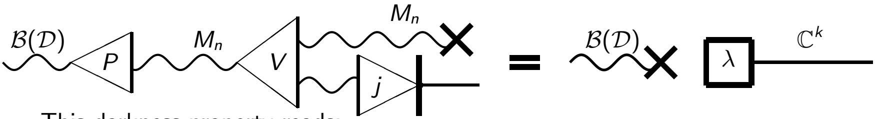

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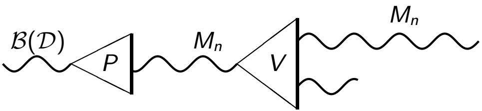

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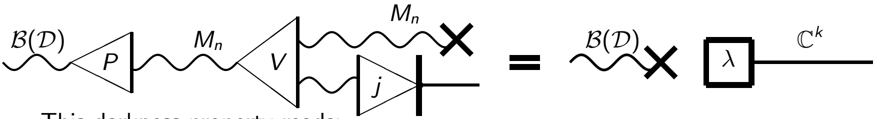

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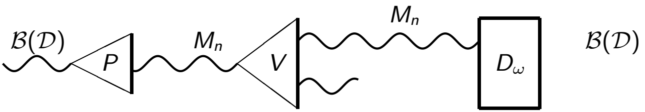

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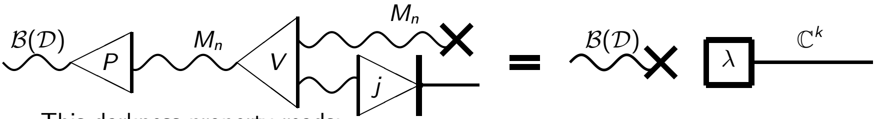

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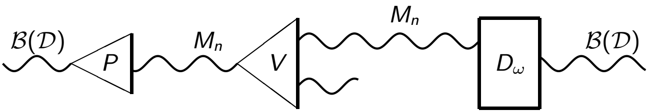

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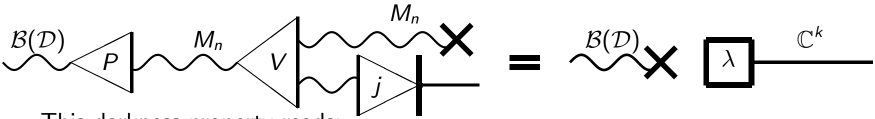

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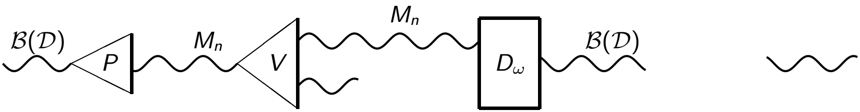

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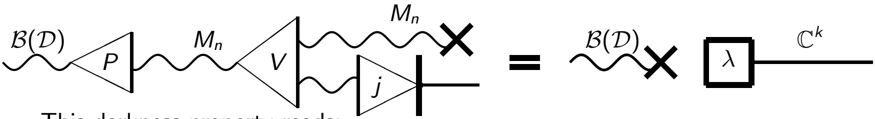

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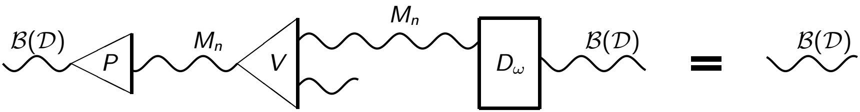

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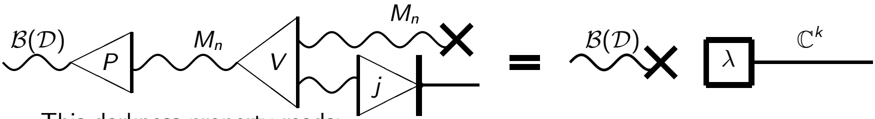

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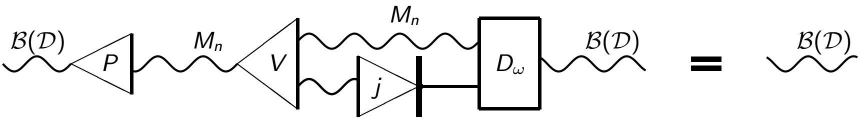

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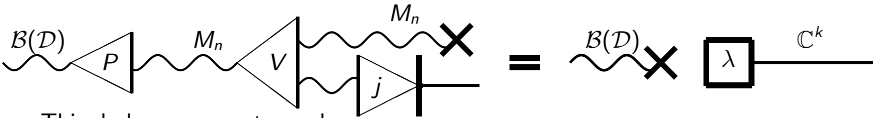

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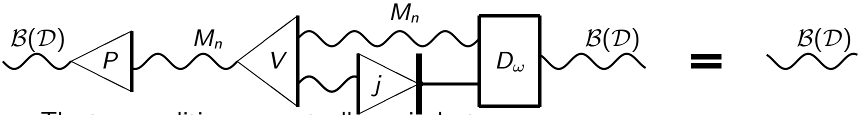

The two conditions are actually equivalent.

State stabilisation by feedback control

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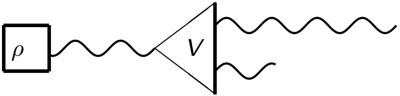

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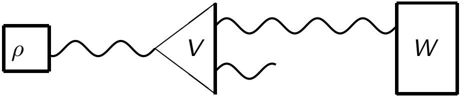

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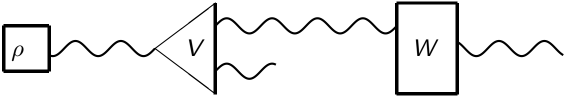

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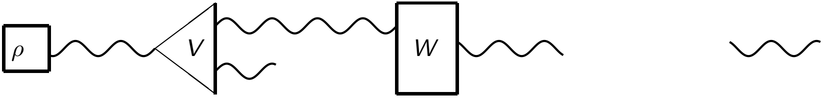

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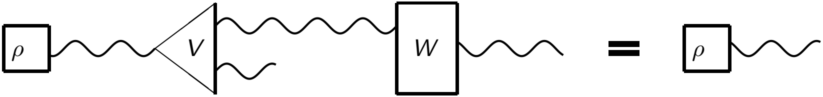

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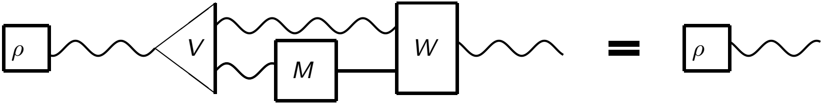

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□

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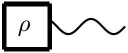

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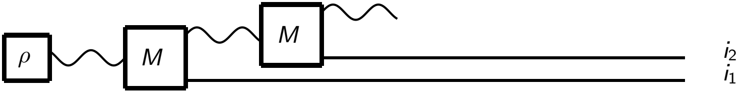

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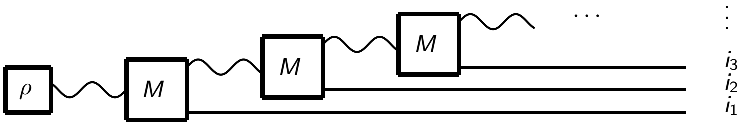

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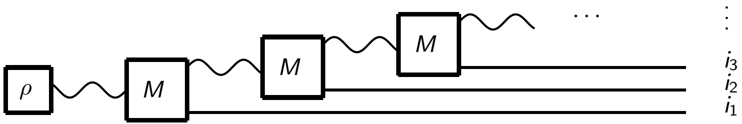

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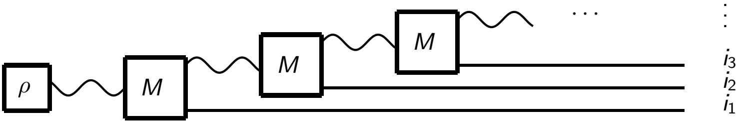

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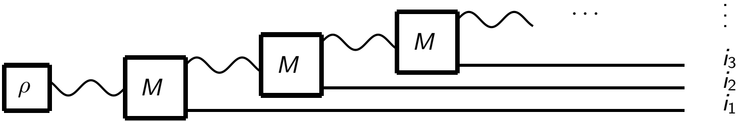

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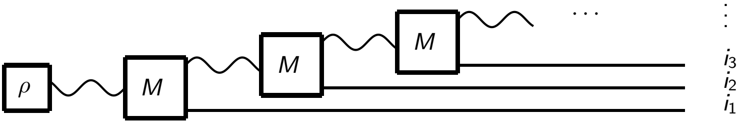

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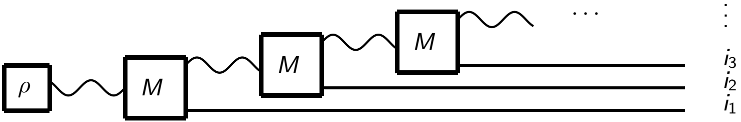
POVM:

Repeated Instruments



$$\Omega := \{1, 2, \ldots, k\}^{\mathbb{N}} ;$$

Σ_m generated by cilinder sets : $\quad \Lambda_{i_1,\ldots,i_m} := \{\omega \in \Omega | \omega_1 = i_1, \ldots, \omega_m = i_m\}$.

POVM: $\quad Q_m(\Lambda_{i_1,\ldots,i_m}) := T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_m}(\mathbb{1})$.

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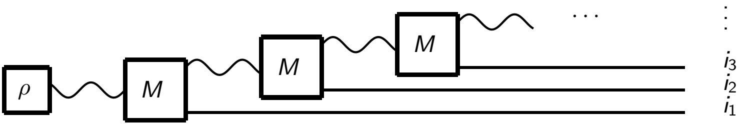

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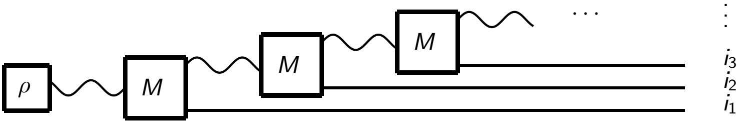

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Left shift: $\quad \sigma : \Omega \to \Omega : (\sigma\omega)_j := \omega_{j+1}$.

Quantum Trajectories

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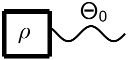

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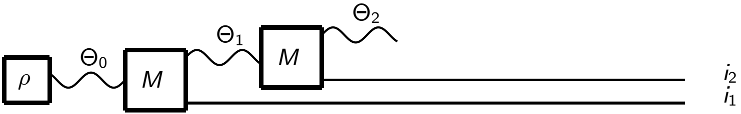

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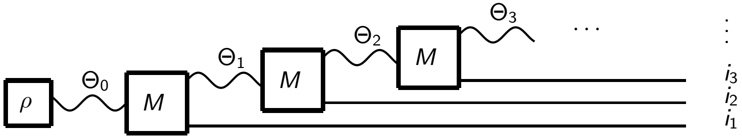

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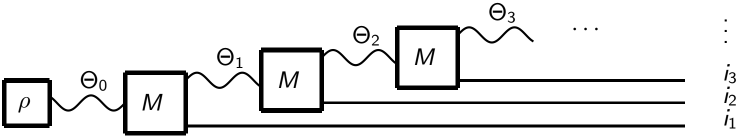

$$\Theta_n : \Omega \to \mathcal{S}(\mathcal{A}) : \qquad \Theta_n(\omega) : x \mapsto \frac{\rho(T_{\omega_1} \circ \cdots \circ T_{\omega_n}(x))}{\rho(T_{\omega_1} \circ \cdots \circ T_{\omega_n}(\mathbb{1}))} \,.$$

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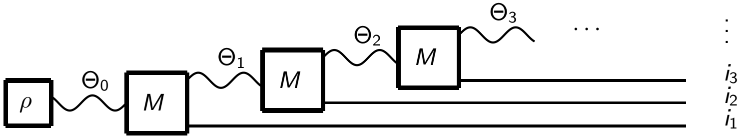

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Theorem

For any state ρ on $\mathcal{A}$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \Theta_j = \Theta_\infty \qquad \mathbb{P}_\rho\text{-a.s.} ,$$

where the random variable Θ_∞ takes values in the T-invariant states on $\mathcal{A}$.

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But also the innovations $V_n(x) := \Theta_{n+1}(x) - \Theta(Tx)$ form a martingale $Y_n(x)$ by weighted addition:

$$Y_n(x) := \sum_{j=1}^{n} \frac{1}{j} V_j \qquad \text{with} \qquad \mathbb{E}_\rho(|Y_n(x)|^2) \leq 4\|x\|^2 \frac{\pi^2}{6} \; .$$

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Averaging over m yields: $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}\big(\Theta_j(x) - \Theta_j(Px)\big) \longrightarrow 0.$ □