Etropic uncertainty in finite systems

Hans Maassen

Singapore, August 27, 2013.

Institute for Mathematical Sciences,
Center for Quantum Technologies

My personal history concerning this subject

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The inequality has been applied in quantum key distribution, entanglement distillation, has been improved upon in special cases, and is generally well-known in quantum information.

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- 4. Extension to conditional entropies.

  Berta, Christandl, Colbeck, Renes, Renner 2010.

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One outcome is certain, the other completely uncertain.

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Let $\pi = (\pi_1, \ldots, \pi_d)$ be a probability distribution. For $\alpha > 0$ let H_α denote the Rényi entropy

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  In fact, the derivative has a meaning: it is $-(1-\alpha)^{-2}$ times the relative entropy of the normalised power distribution π^α with respect to π.

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Of course, taking $\alpha \to 1$ we obtain the ordinary entropic uncertainty relation.

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Of course, taking $\alpha \to 1$ we obtain the ordinary entropic uncertainty relation. Also, since $\alpha \mapsto H_\alpha(\pi)$ is decreasing, we have for all $\alpha \leq 1$:

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So our raw data are now a unitary $d \times d$ matrix U and a unit vector $\psi \in \mathbb{C}^n$, and we have $\widehat{\psi} = U\psi$.

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Equivalently:

$$\log \|\psi\|_p - \log \|\widehat{\psi}\|_{\widehat{p}} \geq \left(\frac{1}{\widehat{p}} - \frac{1}{p} \right) \log c .$$

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Theorem (Riesz-Thorin)

For all $d \times d$-matrices T the function

$$[0,1] \times [0,1] \to \mathbb{R} : \qquad \left(\frac{1}{p}, \frac{1}{q}\right) \mapsto \log \|T\|_{p \to q}$$

is convex.

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Let U be a unitary $d \times d$-matrix, $\psi \in \mathbb{C}^d$ a vector of unit 2-norm: $\|\psi\|_2 = 1$, and let $c := \max_{j,k} |\langle \widehat{e}_j, e_k \rangle|$.

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Since $f_U\left(\frac{1}{2}\right) = \log \|U\|_{2\to 2} = 0$ and $f_U(1) = \log \|U\|_{1\to\infty} = \log c$, we conclude that

$$f'\left(\frac{1}{2}\right) \leq \frac{f(1) - f(\frac{1}{2})}{1 - \frac{1}{2}} \leq 2\log c \,.$$

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Since $f_U'(\frac{1}{2}) \leq 2\log c$, it follows that $H(|\widehat{\psi}|^2) + H(|\psi|^2) \geq \log(1/c^2)$. □

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$$\begin{aligned} F'(0) &= -\log c - \sum_{j=1}^{d} \log |\widehat{\psi}_j| \overline{\widehat{\psi}}_j (U\psi)_j - \sum_{k=1}^{d} \log |\psi_k| \overline{(U^* \widehat{\psi})_k} \psi_k \\ &= -\log c - \frac{1}{2} \left(H(|\widehat{\psi}|^2 + H(|\psi|^2) \right) . \end{aligned}$$

The statement follows. □

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Theorem (3)

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$H(\pi) + H(\widehat{\pi}) = \log(\#D) + \log(\#\widehat{D}) = \log(\#D \cdot \#\widehat{D}) = \log \frac{1}{c^2}$.

Necessity:

$F'(0) = 0$ implies $F(z) = 1$ for all $z \in S$:

$$c^{-z} \sum_{j,k} \overline{\widehat{\psi}_j} |\widehat{\psi}_j|^z \, u_{jk} \, \psi_k |\psi_k|^z = 1 \; .$$

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