Open problems of the conference
Automorphisms of Affine Spaces
July 6-10 2009
Radboud University
Nijmegen, The Netherlands

(See http://www.math.ru.nl/%7Emaubach/AAS/index.html for more info on this conference).

Notations: $\mathrm{GA}_n(\mathbb{C})$ is the set of polynomial automorphisms of $\mathbb{C}^n$.
$\mathrm{Aff}_n(\mathbb{C})$ is the set of affine automorphisms, i.e. compositions of linear maps and translations.
$\mathrm{TA}_n(\mathbb{C})$ is the subset of tame polynomial automorphisms, i.e. generated by triangular and affine automorphisms.

Marek Karas

Problem 1. Let F, G be Keller maps (i.e. polynomial endomorphisms satisfying $\det(Jac(F)) = \det(Jac(G)) = 1$). Suppose

$$F|_{xy=0} = G|_{xy=0}$$

Does this imply $F = G$?

Vladimir Bavula

Problem 2. Is it true that $\mathrm{GA}_n(k)$ (k a field of char. zero) is generated by tame automorphisms and finitely many one-parameter subgroups of automorphisms?

Wlodzimierz Danielewski

Problem 3. Find an intrinsic algebraic invariant of normal finitely generated k-algebras, that would capture algebraic rigidity of “tubular neighborhood of infinity” and could be thought of as adding a “continuous” dimension to the homotopy type at infinity. This invariant used in a straightforward way, although requiring perhaps complicated calculations, must differentiate all isomorphism classes of the algebras $k[x, y, z]/(x^n z - y^2 - f(x)y)$.

Problem 4. Classify normal affine equivariant embeddings of connected solvable linear algebraic groups for which all isotropy groups are semisimple. It seems to be easy when maximal tori have dimension one.

Gene Freudenburg

Problem 5. Let $R = \mathbb{C}[a, b] = \mathbb{C}^{[2]}$. Let $D : R[x, y, z] \to R[x, y, z]$ be a locally nilpotent R-derivation which is triangular, i.e. $D(x) \in R, D(y) \in R[x], D(z) \in R[x, y]$. Suppose D has a slice. Does it imply that $\ker(D) = R^{[2]}$?
Note: $\ker(D)$ is an $\mathbb{A}^2$-fibration over $\mathbb{A}^2$.

Yuriy Bodnarchuk

Problem 6. Let k be a field. Is the group $\mathrm{Aff}_n(k), n > 2$, a maximal subgroup of $\mathrm{TA}_n(k)$ (tame transformation group)?

Eric Edo

Problem 7. Let $R := \mathbb{Z}[z]/(z^3)$, $d \in R$. $P_1, Q_1 \in R[y]$ such that $P_1(Q_1(y)) = dy$. Do there exist $a, b, c \in R$ and $P, Q \in R[y]$ such that $P(Q(y)) = y$ and $P_1(y) = aP(\frac{1}{b}y), Q_2(y) = bQ(\frac{1}{c}y)$, and $a = dc$.

David Wright

Problem 8. Let G_i be the subgroup of $\mathrm{GA}_n(k)$ that stabilises $k \oplus kx_1 \oplus \ldots \oplus kx_i$. Note that $G_n = \mathrm{Aff}_n(k)$. If $n = 2$ then $G_2 = \mathrm{Aff}_2(k)$, and G_1 is the set of triangular automorphisms. Now $\mathrm{GA}_2(k) = G_1 *_{G_1 \cap G_2} G_2$, the Jung-van der Kulk theorem. Question: $\mathrm{GA}_n(k) =< G_1, \ldots, G_n >$. And if yes, is

$$\mathrm{TA}_n(k) = *G_i$$

the amalgamated product along pairwise intersections?

Problem 9. Same question, now for the tame automorphism subgroup: define $H_i = G_i \cap \mathrm{TA}_n(k)$. Question: $\mathrm{TA}_n(k) =< H_1, \ldots, H_n >$. And if yes, is

$$\mathrm{TA}_n(k) = *H_i$$

the amalgamated product along pairwise intersections?

Wenhua Zhao

Problem 10. Let $x = (x_1, x_2, \ldots, x_n)$ and $C_n := [0,1]^{\times n}$, the n-cube in $\mathbb{R}^n$. Suppose that $f(x) \in \mathbb{C}[x]$ and

$$\int_{C_n} f^m(x)\, dx = 0$$

for any $m \geq 1$.

Does this imply that $f = 0$? (Note that, when $n = 1$, this is true.)

A much weaker version of the problem above is given by the next open problem. But, first let us recall the following notion.

Definition *Let R be any commutative ring and $\mathcal{A}$ a commutative R-algebra. A R-subspace M of $\mathcal{A}$ is said to be a Mathieu subspace of $\mathcal{A}$ if the following property holds: for any $a, b \in \mathcal{A}$ with $a^m \in M$ when $m >> 0$, we have, $a^m b \in M$ when $m >> 0$.*

Note that, equivalently, one may replace the first "$m >> 0$" in the definition above by "$m \geq 1$".

Problem 11. Let $M := \{f \in \mathbb{C}[x] \mid \int_{C_n} f(x)dx = 0\}$. Is M a Mathieu subspace of the polynomial algebra $\mathbb{C}[x]$?

The next open problem asks if Mathieu subspaces are closed under the addition. More precisely,

Problem 12. Let $\mathcal{A}$ be a finitely generated k-algebra, where k is any field. Let M_1 and M_2 be any two Mathieu subspaces of $\mathcal{A}$. Is $M_1 + M_2$ also a Mathieu subspace of $\mathcal{A}$?

For more backgrounds and motivations of the notion of Mathieu subspaces and also the open problems above, see arXiv:0902.0212 [math.CV].

Leonid Makar-Limanov

Problem 13. Let A, B be commutative rings. How are $ML(A), ML(B)$ and $ML(A \otimes B)$ related?

Problem 14. Let A, B be rings. Suppose that $A \otimes B = F_n$, a free associative algebra of rank n. Does it imply that A, B are both free associative algebras too?