PROBLEM 1
THE ŁOJASIEWICZ EXPONENTS OF NONDEGENERATE SINGULARITIES AND POLYNOMIAL FUNCTIONS

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Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with isolated critical point at 0 i.e. the mapping $\operatorname{grad} f = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ has an isolated zero at 0 (as usual we identify a function-germ with its representatives). In this case f is called a *singularity*. One of interesting invariants of the singularity f is *the Łojasiewicz exponent* $\mathcal{L}_0(f)$ *of* f *at* 0 which is defined by

$$\mathcal{L}_0(f) = \inf\{\theta : |\operatorname{grad} f(z)| \geqslant C\,|z|^\theta \quad \text{in a ngh. of 0 for some constant } C > 0\}.$$

There are many known properties and effective formulas for $\mathcal{L}_0(F)$ (see [CK], [L-JT], [L1], [P]). The basic property of $\mathcal{L}_0(F)$ is that it is a rational positive number

The simplest among singularities are *non-degenerate singularities*. They are defined via *Newton polyhedrons* – a combinatorial object connected with f. Let us define it. For simplicity we consider the case $n = 2$ (in the general case definitions are similar).

Let

$$f(x, y) = \sum_{\alpha,\beta} a_{\alpha,\beta} x^\alpha y^\beta$$

be the expansion of f in a neighbourhood of $0 \in \mathbb{C}^2$ in a convergent Taylor series. We put

$$\operatorname{supp} f := \{(\alpha, \beta) \in \mathbb{N}_0^2 : \ a_{\alpha,\beta} \neq 0\}.$$

The *Newton diagram* $\Gamma_+(f)$ *of* f is the convex hull of $(\operatorname{supp} f) + \mathbb{R}_+^2$ (where $\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 : x \geqslant 0, \ y \geqslant 0\}$). The boundary of $\Gamma_+(f)$ is the union of two half-lines and a finite number of compact and pairwise non-parallel segments. *The Newton polygon* $\Gamma(f)$ is the set of these compact segments (in 2-dimensional case "Newton

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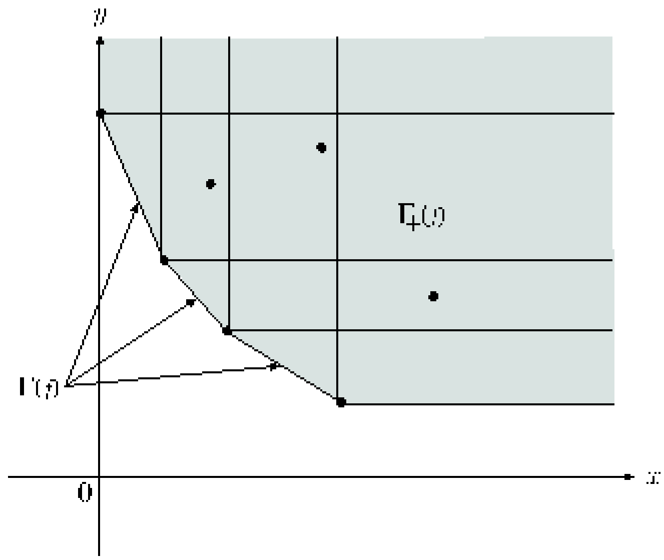
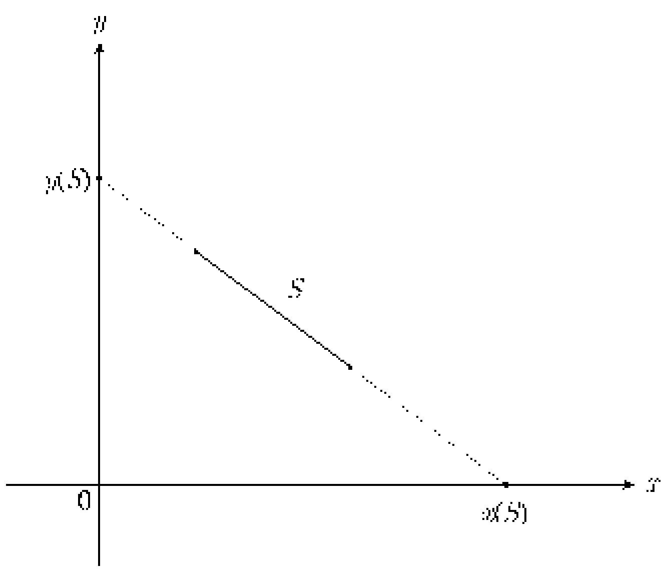
polyhedron" = "Newton polygon") .



For each segment $S \in \Gamma(f)$ we define:

1. $x(S)$ – the abscissa of the point, when the line determined by S intersects the horizontal axis,

2. $y(S)$ – the ordinate of the point, when the line determined by S intersects the vertical axis,

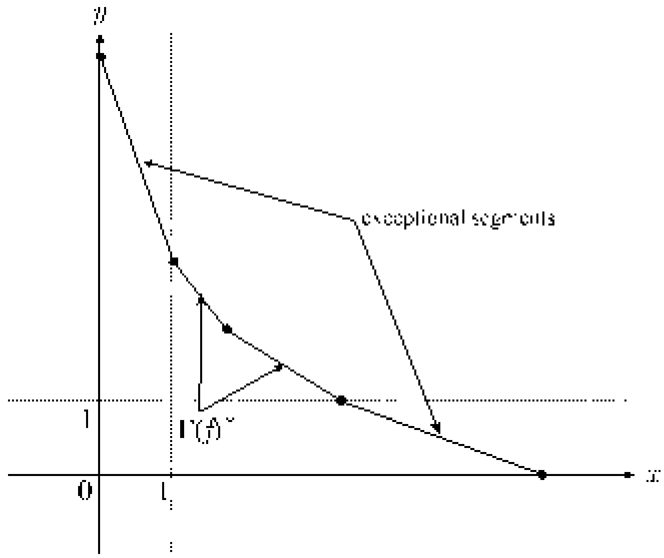


3. $\mathrm{in}(S) = \sum_{(\alpha,\beta)\in S} a_{\alpha,\beta} x^\alpha y^\beta$.

The *reduced Newton polygon* $\Gamma(f)^*$ is obtained from $\Gamma(f)$ by omitting the *exceptional segments* according to the rule:

(a) we omit the first segment if abscissas of their ends lie on the lines $x = 0$ and $x = 1$, respectively,

(b) we omit the last segment if ordinates of their ends lie on the lines $y = 0$ and $y = 1$, respectively.



In other words exceptional segments are segments in $\Gamma(f)$ which lie in the walls of thickness 1 around the axes.

Remark 1. *Since we assume that f has an isolated critical point at $0 \in \mathbb{C}^2$, then it is easy to show that $\Gamma(f)^* = \emptyset$ if and only if in an appropriate linear system of coordinates in $\mathbb{C}^2$*

$$f(x, y) = xy + \sum_{\substack{\alpha,\beta \\ \alpha+\beta \geqslant 3}} h_{\alpha,\beta} x^\alpha y^\beta.$$

For such f we have $\mathcal{L}_0(f) = 1$. Then, in the sequel, the assumption that $\Gamma(f)^* \neq \emptyset$ is not very restrictive.

We say f is *nondegenerate (in the Kouchnirenko's sense)* [K] if for every $S \in \Gamma(f)$ (equivalently for every $S \in \Gamma(f)^*$) the system of equations

$$\begin{aligned} \frac{\partial}{\partial x} \operatorname{in}(S) &= 0, \\ \frac{\partial}{\partial y} \operatorname{in}(S) &= 0 \end{aligned}$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$.

Theorem 1. *([L1], Thm 2.1) If f is a nondegenerate singularity at $0 \in \mathbb{C}^2$ and $\Gamma(f)^* \neq \emptyset$, then*

$$\mathcal{L}_0(f) = \max_{S \in \Gamma(f)^*} (x(S), y(S)) - 1. \tag{0.1}$$

By this theorem if f is nondegenerate and $\Gamma(f)^* \neq \emptyset$ then $\mathcal{L}_0(f)$ can be read of its Newton diagram.

In n-dimensional case one can analogously define the above notions for a singularity $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Namely,

1. $\mathcal{L}_0(f)$,

2. $\operatorname{supp} f$,
3. $\Gamma_+(f)$, $\Gamma(f)$,
4. $z_1(S), \ldots, z_n(S)$ for each face $S \in \Gamma(f)$, where $(z_1, \ldots, z_n)$ are coordinates in $\mathbb{C}^n$.
5. nondegeneratness of f.

Now, we may pose the problem

Problem 1. *Define appropriately exceptional faces of $\Gamma(f)$ such that*

$$\mathcal{L}_0(f) = \max_{S \in \Gamma(f)^*} (z_1(S), \ldots, z_n(S)) - 1.$$

If $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial one can similarly define *the Łojasiewicz exponent $\mathcal{L}_\infty(f)$ at infinity of f at* 0 which is defined by

$$\mathcal{L}_\infty(f) = \sup\{\theta : |\operatorname{grad} f(z)| \geqslant C\,|z|^\theta \quad \text{in a ngh. of infinity for some constant } C > 0\}$$

and similar notions connected with the Newton polyhedron of f at infinity.

Problem 2. *Find effective formulas for $\mathcal{L}_\infty(f)$ in nondegerate case in n-dimensional case.*

Remark 2. *As in local case the 2-dimensional case at infinity is completely solved* [L2].

References

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