DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN The Netherlands

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Joost Berson, Arno van den Essen, Stefan Maubach

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

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Abstract

In this paper it is proved that for any $\mathbb{Q}$-algebra R any locally nilpotent R-derivation D on $R[X, Y]$ having divergence zero and $1 \in (D(X), D(Y))$ (i) has a slice, and (ii) $A^D = R[P]$ for some P. Furthermore it is shown that any surjective R-derivation on $R[X, Y]$ having divergence zero is locally nilpotent. Connections with the Jacobian Conjecture are made.

1 Introduction

Locally nilpotent R-derivations on the polynomial ring $R[X, Y]$ where R is a UFD containing $\mathbb{Q}$ were studied by Daigle and Freudenburg in [1]. The more general situation where R is a (normal) noetherian domain containing $\mathbb{Q}$ was studied by Bhatwadekar and Dutta in [4]. They showed, amongst other things, that if D is a locally nilpotent derivation on $R[X, Y]$ such that the ideal generated by $D(X)$ and $D(Y)$ contains 1, then $R[X, Y]^D$ is a polynomial ring in one variable over R and $R[X, Y]$ is a polynomial ring in one variable over $R[X, Y]^D$. In particular this implies that D has a slice in $R[X, Y]$.

In this paper we generalise this result to arbitrary $\mathbb{Q}$-algebras R in the sense that we consider locally nilpotent derivations having divergence zero (in the domain case any locally nilpotent derivation has divergence zero).

Also we generalise a result of Stein in [2], asserting that any surjective k-derivation on $k[X, Y]$ (k a field of characteristic zero) is locally nilpotent, to surjective divergence zero R-derivations on $R[X, Y]$ where R is an arbitrary Noetherian $\mathbb{Q}$-algebra.

At the end of this paper we relate this result to the Jacobian Conjecture. In fact the importance of divergence zero derivations for this conjecture will be described in a forthcoming paper of the second author.

2 Preliminaries

2.1 Notations

We assume for the rest of the article that R is a commutative $\mathbb{Q}$-algebra. Let A be an R-algebra containing R. Let $Spec(R)$ be the collection of all prime ideals of R. So $\cap_{\mathfrak{p} \in Spec(R)} \mathfrak{p}$ equals the collection of nilpotent elements of R, which we denote by η. Throughout this paper D denotes an R-derivation on A. We say that an element

$s \in A$ is a *slice* of a derivation D if $D(s) = 1$. If $A = R[X] = R[X_1, \ldots, X_n]$ and $D = a_1\partial_{X_1} + \ldots + a_n\partial_{X_n}$ then the *divergence* of D, denoted by $div(D)$, equals $\sum_{i=1}^n \partial_{X_i} a_i$.

2.2 Tools

Now follows a score of lemmas which prove themselves useful in the proofs of the next section.

Lemma 2.1. *If D is a locally nilpotent R-derivation on A then D has a slice if and only if D surjective.*

Proof. If D is surjective then among others 1 is in the image, and hence some $s \in A$ is mapped onto 1. So let us assume we have a locally nilpotent derivation having some slice s. Let $F \in A$. Define $G = \sum_{i=0}^\infty (-1)^i \frac{s^{i+1}}{(i+1)!} D^i(F)$. $G \in A$ because the sum is finite: $D^i(F) = 0$ for $i \geq N$ for some N, since D is locally nilpotent. Now

$$\begin{aligned} D(G) = & \sum_{i=0}^\infty (-1)^i D(\tfrac{s^{i+1}}{(i+1)!} D^i(F)) \\ = & \sum_{i=0}^\infty (-1)^i (\tfrac{s^i}{i!} D^i(F) + \tfrac{s^{i+1}}{(i+1)!} D^{i+1}(F)) \\ = & \sum_{i=0}^\infty (-1)^i \tfrac{s^i}{i!} D^i(F) + \sum_{i=0}^\infty (-1)^i \tfrac{s^{i+1}}{(i+1)!} D^{i+1}(F) \\ = & F. \end{aligned}$$

So D is surjective. □

Definition 2.2. If I is any ideal of R then we write $D_I := D \bmod(I)$, the induced derivation on A/AI. Also if $F \in A$ then write $F_I := F \bmod(IA)$.

Lemma 2.3. *Let D be an R-derivation on A. Let $I, J \subset R$ be ideals of R and suppose D_I has a slice and D_J is surjective. Then D_{IJ} has a slice.*

Proof. There exists $s \in A$ such that $D_I(s_I) = 1$ and hence $D(s) = 1 + f$ for some $f \in IA$. Write $f = \sum f_\alpha a_\alpha$ where $f_\alpha \in I$ and $a_\alpha \in A$. Since D_J is surjective there exists $F_\alpha \in A$ such that $D(F_\alpha) = a_\alpha + h_\alpha$ where $h_\alpha \in JA$. Denote $S := s - \sum f_\alpha F_\alpha$. Then

$$\begin{aligned} D(S) = & D(s - \textstyle\sum f_\alpha F_\alpha) \\ = & D(s) - \textstyle\sum f_\alpha D(F_\alpha) \\ = & 1 + f - \textstyle\sum (f_\alpha a_\alpha + f_\alpha h_\alpha) \\ = & 1 - \textstyle\sum f_\alpha h_\alpha \end{aligned}$$

and since $f_\alpha h_\alpha \in IJ$ we have $D_{IJ}(S_{IJ}) = 1$. □

Lemma 2.4. *Let D_{I_i} be surjective for the ideals $I_1, \ldots, I_r \subset R$. Then $D_{I_1 \cdot \ldots \cdot I_r}$ is also surjective.*

Proof. It is enough to show that if D_I, D_J are surjective that D_{IJ} is too. Let $a \in A$ be arbitrary. There exists $b \in A$ such that $D_I(b_I) = a_I$ hence $D(b) = a + i$ where $i \in IA$. Write $i = \sum_{k=0}^t i_k c_k$ where $i_k \in I$, $c_k \in A$. Then for every c_k there exists some d_k such that $D(d_k) = c_k + j_k$ some $j_k \in JA$ since D_J surjective. Now $D(b - \sum_{k=0}^t i_k d_k) = a - \sum_{k=0}^t i_k j_k$. Since $\sum_{k=0}^t i_k j_k \in IJA$ we're done. □

Lemma 2.5. *Let D be a locally nilpotent R-derivation on A. If $I_1, \ldots, I_r \subset R$ are ideals of R and D_{I_i} has a slice for all i then $D_{I_1 \cdot \ldots \cdot I_r}$ has a slice too.*

Proof. It is enough to show that if D_I, D_J both have a slice then D_{IJ} has one too. By lemma 2.1 D_I and D_J are surjective. By lemma 2.4 D_{IJ} is surjective. In particular, D_{IJ} has a slice. □

Lemma 2.6. *If $I_1, \ldots, I_r \subset R$ are ideals of R and D_{I_i} is locally nilpotent for all i then $D_{I_1 \cdot \ldots \cdot I_r}$ is locally nilpotent too.*

Proof. It is enough to show that if D_I, D_J are locally nilpotent then D_{IJ} is locally nilpotent. Let $a \in A$. One knows there exists $N \in \mathbb{N}$ such that $D_I^N(a_I) = 0$ hence $D^N(a) = \sum_{k=0}^t i_k b_k$ where $i_k \in I, b_k \in A$. Now there exists $M_k \in \mathbb{N}$ such that $D^{M_k}(b_k) \in JA$. Let $M = max_k(M_k)$. Then $D^{N+M}(a) = D^M(\sum_{k=0}^t i_k b_k) = \sum_{k=0}^t i_k D^M(b_k) \in IJA$. □

3 Divergence zero derivations

Throughout this section let $A = R[X, Y]$ and D a non-zero R-derivation on A with divergence zero. Then it is well-known that $D = P_Y \partial_X - P_X \partial_Y$ for some $P \in A$ (where $P_X = \partial_X(P), P_Y = \partial_Y(P)$ are the derivatives of P) which is unique if one assumes $P(0,0) = 0$. We denote this element by $P(D)$. We say that R has property $B(R)$ if and only if the following holds:

$B(R)$ Any locally nilpotent derivation D on A with $div(D) = 0$ and $1 \in (D(X), D(Y))$ has a slice and satisfies $A^D = R[P(D)]$.

The main aim of this section is to show that $B(R)$ holds for any $\mathbb{Q}$-algebra R (Theorem 3.7). We first reduce to the case that R is Noetherian. Therefore let R' be the $\mathbb{Q}$-subalgebra of R generated by the coefficients of the polynomials P, a and b where a, b are such that $1 = aP_X + bP_Y$. Notice that R' is noetherian, regardless of R. Write $A' = R'[X, Y]$, D' the restriction of D to A'.

Lemma 3.1. *If D' has a slice and ${A'}^{D'} = R'[P]$ then D has a slice and $A^D = R[P]$.*

Proof. Let $S \in A'$ such that $D'(S) = 1$. Then since $A' \subseteq A$ we have $S \in A$ and $D(S) = D'(S) = 1$. So let ${A'}^{D'} = R'[P]$. In general for any locally nilpotent derivation having a slice S one has $R[X] = R[X]^D[S]$. Hence $R'[X, Y] = A' = {A'}^{D'}[S] = R'[P, S]$. So there exist $F, G \in R'[X, Y]$ such that $F(P, S) = X$ and $G(P, S) = Y$. But since all is contained in $R[X, Y]$ we have

$$R[X, Y] = R[F(P, S), G(P, S)] \subseteq R[P, S] \subseteq R[X, Y].$$

Hence $A^D = R[P, S]^D = R[P]$. □

To prove $B(R)$ for Noetherian domains containing $\mathbb{Q}$, we first need a lemma from [1]

Lemma 3.2. *Let R be a domain containing $\mathbb{Q}$ and $P \in R[X,Y]$ such that $1 \in (P_X, P_Y)$. Then $K[P] \cap R[X,Y] = R[P]$, where $K = Q(R)$, its field of fractions.*

Proof. If $K[P] \cap R[X,Y] \not\subseteq R[P]$, then there exists an $F \in K[T] \setminus R[T]$ with $F(P) \in R[X,Y]$. Choose one of minimal degree. Observe that $F(P) \in R[X,Y]$ implies that $F'(P)F_X$ and $F'(P)F_Y$ belong to $R[X,Y]$.
Since there are $g, h \in R[X,Y]$ with $P_X g + P_Y h = 1$, we deduce $F'(P) = F'(P)P_X g + F'(P)P_Y h \in R[X,Y]$. So $F'(T) \in K[T]$ and $F'(P) \in R[X,Y]$, thus by minimality of the degree of F we must conclude, that $F' \in R[T]$. Now write $F = \sum_{i=0}^{d} f_i T^i$, then $F' \in R[T]$ implies (since R is a $\mathbb{Q}$-algebra) that $f_i \in R$ for all $i \geq 1$, thus yielding $f_0 = F(P) - \sum_{i=1}^{d} f_i P^i \in R[X,Y] \cap K = R$, contradicting the assumption, that $F \notin R[T]$. $\square$

Now we can prove the next theorem :

Theorem 3.3. *Let R be a Noetherian domain containing $\mathbb{Q}$, $K = Q(R)$, and let D be a locally nilpotent derivation on $R[X,Y]$ with $1 \in (D(X), D(Y))$.*
Then $R[X,Y]^D = R[P]$ for some $P \in R[X,Y]$ and D has a slice $t \in R[X,Y]$.

Proof. Extend D to $K[X,Y]$ the natural way. We know by [3] (Th.1.2.25) or [5] that there is a $Q \in K[X,Y]$ with $K[X,Y]^D = K[Q]$. Because D is locally nilpotent, we know that $div(D) = 0$, so there is a $P \in R[X,Y]$ with $D(X) = P_Y$ and $D(Y) = -P_X$. This means that $D(P) = 0$, and, as a consequence, $P \in K[X,Y]^D = K[Q]$. So write $P = g(Q)$ with $g \in K[T]$. We now have $P_X = g'(Q)Q_X$ and $P_Y = g'(Q)Q_Y$. Notice that $(P_Y, P_X) = (D(X), D(Y)) = (1)$ (also in $K[X,Y]$), which means that $g'(Q) \in K^*$. Then there are $\lambda, \mu \in K, \lambda \neq 0$ satisfying $P = g(Q) = \lambda Q + \mu$, yielding $K[P] = K[Q]$. By the previous lemma, $R[X,Y]^D = K[X,Y]^D \cap R[X,Y] = K[P] \cap R[X,Y] = R[P]$.
Hence we proved our first claim. Now we can use Theorem 4.7 in [4] to conclude that

$$R[X,Y] = R[P][s] \text{ for some } s \in R[X,Y] \tag{1}$$

This means that $f : R[X,Y] \longrightarrow R[X,Y]$ defined by $f(X) = P(X,Y)$ and $f(Y) = s(X,Y)$ satisfies $f \in Aut_R R[X,Y]$. A well-known consequence is that

$$\det JF(X) \in R[X,Y]^* = R^* \tag{2}$$

But this determinant is equal to $-P_Y s_X + P_X s_Y = -D(s)$. So $D(s) \in R^*$, whence $t := s/D(s)$ satisfies $D(t) = 1$ and we are done. $\square$

Combining lemma 3.1 and theorem 3.3 we have

Theorem 3.4. *Let R be any domain containing $\mathbb{Q}$. Then $B(R)$ holds.*

Lemma 3.5. *Let D be an R-derivation on A and $I_1, \ldots, I_r \subseteq R$ ideals of R. Suppose there exists $P \in A$ such that $R/I_i[X,Y]^{D_{I_i}} = R/I_i[P_{I_i}]$ for all i. Then $A^D \subseteq R[P] + I_1 \cdot \ldots \cdot I_r A^D$.*

Proof. It is enough to prove the lemma for $r = 2$. So let I, J be ideals in R. We know $R/I[X,Y]^{D_I} = R/I[P_I]$. Hence $A^D \subseteq R[P] + IA^D$. In the same way $A^D \subseteq R[P] + JA^D$. Substituting the latter in the first we get

$$\begin{aligned} A^D \subseteq\ & R[P] + IA^D \\ & \subseteq R[P] + I(R[P] + JA^D) \\ & \subseteq R[P] + IJA^D \end{aligned}$$

□

Now we assume R to be a reduced ring, that is, its nilradical η equals (0). We will prove $B(R)$ for these rings.

Theorem 3.6. *Let R be any reduced $\mathbb{Q}$-algebra. Then $B(R)$ holds.*

Proof. Let $D = P_Y \partial_X - P_X \partial_Y$ be an arbitrary locally nilpotent derivation having $div(D) = 0$ and $1 \in (P_X, P_Y)$. We have to prove that D has a slice and that $A^D = R[P]$. By lemma 3.1 we may assume R to be Noetherian. We know that for any prime ideal $\mathfrak{p}$ we have $R/\mathfrak{p}$ is a domain. Hence by theorem 3.4 $D_\mathfrak{p}$ has a slice and $A/\mathfrak{p}A^{D_\mathfrak{p}} = R/\mathfrak{p}[X,Y]^{D_\mathfrak{p}} = R/\mathfrak{p}[P_\mathfrak{p}]$. Since R is assumed to be Noetherian there are finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Write $\mathfrak{q} := \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_n$. Now using lemma 2.5 we see that $D_\mathfrak{q}$ has a slice too and by lemma 3.5 we have $A/\mathfrak{q}^{D_\mathfrak{q}} = A/\mathfrak{q}[P_\mathfrak{q}]$. But since

$$\mathfrak{q} = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_n \subseteq \cap_{i=1}^n \mathfrak{p}_i = \eta = (0)$$

we are done. □

Now we do the main theorem:

Theorem 3.7. *Let R be any $\mathbb{Q}$-algebra. Then $B(R)$ holds.*

Proof. Let $D = P_Y \partial_X - P_X \partial_Y$ be an arbitrary locally nilpotent derivation having $div(D) = 0$ and $1 \in (P_X, P_Y)$. We have to prove that D has a slice and that $A^D = R[P]$. By lemma 3.1 we may assume R to be noetherian. Hence $\eta^N = 0$ for some $N \in \mathbb{N}$. By theorem 3.6 we know $D_\eta(s_\eta) = 1$ for some $s \in A$ and $A/\eta^{D_\eta} = R/\eta[P_\eta]$. Now using lemma 2.5 we see that D_{η^N} has a slice too and by lemma 3.5 we have $A/\eta^{N^{D_{\eta^N}}} = A/\eta^N[P_{\eta^N}]$. But since $\eta^N = 0$ we are done. □

Finally we consider surjective R-derivations on $R[X,Y]$ having divergence zero. We say that a $\mathbb{Q}$-algebra R satisfies property $S(R)$ if and only if the following holds:

$S(R)$ Any surjective R-derivation of $R[X,Y]$ having divergence zero is locally nilpotent.

Theorem 3.8. *$S(R)$ holds for any Noetherian $\mathbb{Q}$-algebra.*

Proof. i) If R is a field the result was proved by Stein in [2]. One easily deduces that $S(R)$ holds for any domain R.
ii) Now assume that R is a reduced ring. So $(0) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$ for some prime

ideals $\mathfrak{p}_i$. Let D be a surjective derivation on $R[X,Y]$ satisfying $div(D) = 0$. Then each induced derivation $D_{\mathfrak{p}_i} : R/\mathfrak{p}_i[X,Y] \longrightarrow R/\mathfrak{p}_i[X,Y]$ is surjective and satisfies $div(D_{\mathfrak{p}_i}) = 0$. So by i) each $D_{\mathfrak{p}_i}$ is locally nilpotent, hence by lemma 2.6 D is locally nilpotent.
iii) Finally let R be any Noetherian $\mathbb{Q}$-algebra. Let η be the nilradical. Since R is Noetherian there exists some $N \in \mathbb{N}$ such that $\eta^N = 0$. $D_\eta : R/\eta[X,Y] \longrightarrow R/\eta[X,Y]$ is surjective and $div(D_\eta) = 0$. So by ii) D_η is locally nilpotent. Then it follows by lemma 2.6 that D locally nilpotent. □

Comment: Theorem 3.8 above is a special case of the Jacobian Conjecture, namely the surjectivity of D certainly implies that $1 \in Im(D)$ i.e. $D(s) = 1$ for some $s \in R[X,Y]$ or equivalently, writing $D = P_Y\partial_X - P_X\partial_Y$ that $detJ(s,P) = 1$. So if the two-dimensional Jacobian Conjecture is true then apparently the condition $1 \in Im(D)$ is equivalent to the surjectivity of D. So in order to try to make the gap between theorem 3.8 and the Jacobian Conjecture smaller one can pose the following questions:

Question 1: Can one give a finite number of elements $a_1, \ldots, a_m$ in $R[X,Y]$ such that $a_i \in Im(D)$ for all i implies that D is surjective (of course assuming $div(D) = 0$)?

Or more concretely:

Question 2: Does $\{1, X, Y\} \subset Im(D)$ imply that D is surjective?

If the answer to the first question is affirmative one can improve theorem 3.8 to arbitrary $\mathbb{Q}$-algebras (instead of Noetherian $\mathbb{Q}$-algebras) using an argument similar to the one used in the proof of lemma 3.1.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

Email : berson@sci.kun.nl, stefanm@sci.kun.nl, essen@sci.kun.nl