A three dimensional UFD cancellation counterexample

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Central question:

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Or: how to distinguish two varieties?

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2006 (Finston/Maubach)	$dim(A) = 3$, A $\mathbb{C}$-algebra UFD

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Examples: Let $a_1, \ldots, a_n \in \mathbb{C}[X_1, \ldots, X_n]$. Then

$$a_1 \frac{\partial}{\partial X_1} + a_2 \frac{\partial}{\partial X_2} + \ldots + a_n \frac{\partial}{\partial X_n}$$

is a derivation.

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Not locally nilpotent: $X\frac{\partial}{\partial X}$ on $\mathbb{C}[X]$.

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Known fact:

$\varphi \in Aut_{\mathbb{C}}(A)$, then

$\varphi(\mathrm{ML}(A)) = \mathrm{ML}(A)$.

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So let us focus on getting such a rigid ring R!

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Mason's Theorem: *(ABC-conjecture)*

Let $f, g, h \in k[X]$ where k is an algebraically closed field.

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Then $max\{deg(f), deg(g), deg(h)\} < \#Z(fgh)$.

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Example: extend $\frac{\partial}{\partial X}$ on $\mathbb{Z}[X]$, to $\mathbb{Q}[X]$, and $\mathbb{C}[X]$.

Using Mason's in rings

Definition: $R := \mathbb{C}[X, Y, Z]/(X^a + Y^b + Z^c) = \mathbb{C}[x, y, z]$ where $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1$.

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K fraction field of $\tilde{R}^D$

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p preslice, $D(p) = q$, $D(q) = 0$, $s := pq^{-1}$

K fraction field of $\tilde{R}^D$

$\bar{K}$ algebraic closure of K

Using Mason's in rings

Definition: $R := \mathbb{C}[X, Y, Z]/(X^a + Y^b + Z^c) = \mathbb{C}[x, y, z]$
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Corollary implies: $f, g, h \in \overline{K}$.

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Corollary implies: $f, g, h \in \overline{K} \cap R$.

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Corollary implies: $f, g, h \in \overline{K} \cap R = R^D$. So $LND(R) = \{0\}$!!

Just one more thing to say:

THANK YOU