Commuting derivations on UFDs

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(ii). $D_i = \frac{\partial}{\partial s_i}$.

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Weak Abhyankar-Sataye Conjecture: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $k(f)[X_1, \ldots, X_n] \cong_{k(f)} k(f)[Y_1, \ldots, Y_{n-1}]$. Then f is a coordinate in A.

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$\mathbb{C}^{[2]} \cong A/(x) = \mathbb{C}[Z, T, Y]/(Z^2 + T^3)$.
Contradiction, so $A \ncong \mathbb{C}^{[3]}$!

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2. . $D_1 \mod (f-\alpha), \ldots, D_n \mod (f-\alpha)$ independent over $A/(f-\alpha)$ for all $\alpha \in k$
   $\Rightarrow A \cong \mathbb{C}^{[n+1]}$, f coordinate.

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$\mathcal{P}_i$ can be seen as the set of "preslices of D_i on $\mathcal{A}_i$".

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$D_i(\tilde{p}_i - h_i(f)p_i) = r_i(f)$ so $r_i = 0$. So $D_i(\tilde{p}_i) \in q_i(f)\mathbb{C}[f]$. □

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(2) $D_i(\mathcal{P}_i) = q_i(f)k[f]$ for some nonzero polynomial q_i.
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Proof.

(3) Elegant, but too long for a presentation. □

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Can we construct such E_i, given D_i, which are optimal in some way?

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Or, if this equality does not hold always, what type of rings A do have equality?

Final remark:

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THANK YOU

(for watching at 94 slides!)