$\mathbb{Z}$ is difficult, polynomials are easy.

Stefan Maubach

Saginaw, October 2008

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Are there any other sets with something like "prime numbers"?

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Same way:

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In $\mathbb{C}[X]$ one may describe "$\gcd(f, g) = 1$" by saying: "f and g have different zeroes".

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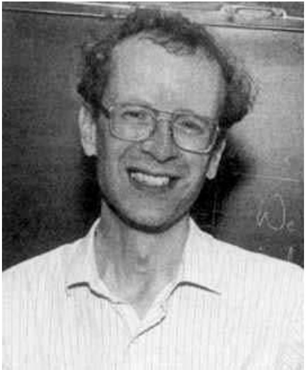

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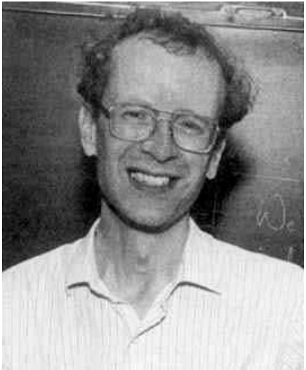

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Proof of Wiles is very difficult! My guess is: no one present in this room has read and understood the proof. . . !

Fermat's Last Theorem for $\mathbb{C}[X]$

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$$f^n \qquad + g^n \qquad = h^n$$

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so $g^{n-1}(f'g - fg') = h^{n-1}(f'h - fh')$.

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g divides g', and f divides f' - That is only possible if f, g are constant and then h is automatically constant! So this case is done. So we can assume that $f'g - fg', f'h - f'h$, and $g'h - gh'$ are unequal to 0.

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$(n-3)(deg(f) + deg(g) + deg(h)) \leq -3$, contradiction!!

What's the story about $l, m, n \in \mathbb{N}$ large enough and $x^l + y^m = z^n$? If $x, y, z \in \mathbb{Z}$ then Wiles only gave a proof for $l = m = n$!

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Version for $\mathbb{C}[X]$:
Let $f, g, h \in \mathbb{C}[X]$ satisfy $f + g = h$, $\gcd(f, g, h) = 1$, then

$$deg(f) < N(fgh)$$

where $N(fgh)$ is the number of zeroes of fgh.

If ABC conjecture true, then Fermat is an immediate consequence. And more stuff ($x^l + y^m = z^n$). I'll not prove this today, but - I'll prove the ABC conjecture for polynomials!!

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$$deg(\gcd(f, f')) + deg(\gcd(g, g')) + deg(\gcd(h, h')) \leq deg(f) + deg(g) - 1.$$

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. . . (krijtbord?) Using the lemma we get Mason's!

Theorem:

Let $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$. If $F, G, H \in \mathbb{C}[X]$ satisfying $\gcd(F, G, H) = 1$ and $F^p + G^q = H^r$ then $F, G, H \in \mathbb{C}$.

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Divide by $p deg(F)$:
$1 < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$. Contradiction!

Notice: $p = q = r$ gives $\frac{1}{n} + \frac{1}{n} + \frac{1}{n} \leq 1$ so $n \geq 3$.

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What is wrong in these lines if $f, g, h \in \mathbb{Z}$? Exactly! In $\mathbb{C}[X]$ one can take derivatives!

$\mathbb{C}[X]$ has a *derivation*: a map δ satisfying

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Well, let's make one on $\mathbb{Z}$, so we can prove stuff!

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$D(2^2 a) = 2^2 D(a) + 2 \cdot 2a = 2^2(D(a) + a)$ so that one increases and increases if $a > 1$!

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Comes down to finding a locally nilpotent derivation D on the ring $\mathbb{C}[X, Y, Z]/(X^2 + Y^3 + Z^7)$.

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****** THANK YOU ******