

A SHORT COURSE ON ∞ -CATEGORIES

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ABSTRACT. These are notes on the theory of ∞ -categories building on a series of talks given by the author in Warsaw in January, 2010. The aim is to give a non-technical introduction to some of the main ideas of the theory in order to facilitate the digestion of the far more voluminous tomes due to Andre Joyal [Joy08b] and Jacob Lurie [Lur09e] where the theory is developed in full detail. Besides the basic ∞ -categorical notions, we mention the Joyal and Bergner model structures which are two approaches to the theory of $(\infty, 1)$ -categories. We then treat the theory of (symmetric) monoidal ∞ -categories as developed in [Lur09b, Lur09c] and introduce the notion of (commutative) algebra objects. We finish with a summary of Lurie's treatment of spectra and the smash product from the perspective of ∞ -categories [Lur09a] which allows us to give the definition of \mathcal{A}_∞ - and \mathcal{E}_∞ -ring spectra.

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0. INTRODUCTION AND PLAN

The aim of this short course is to give a non-technical account of some ideas in the theory of ∞ -categories (aka. quasi-categories or weak Kan complexes) as originally introduced by Boardman and Vogt [BV73, p.102] in their study of homotopy-invariant algebraic structures. Recently, these were intensively studied in particular by Joyal [Joy08b] and Lurie [Lur09e, Lur09a, Lur09b, Lur09c]. Since all proofs can be found in loc. cit. and since I would never claim to be able to improve on them, I will hardly include any proofs. Instead I put emphasis on some of the ideas and sketch the lines along which the theory has developed. In particular, there is no claim of originality. Category theory is one of the most important ‘modern’ mathematical disciplines in that almost all classes of mathematical objects like groups, vector spaces over a fixed field or manifolds can be organized into a category and from many typical constructions one frequently abstracts the categorical character behind it. Recall that a category \mathcal{C} consists of a class $\text{ob}(\mathcal{C})$ of objects and for any two objects $X, Y \in \text{ob}(\mathcal{C})$ a set $\text{hom}_{\mathcal{C}}(X, Y)$ of morphisms from X to Y together with a rule of how to compose arrows in an associative and unital way. More precisely, for each object $X \in \mathcal{C}$ there is an identity morphism $\text{id}_X \in \text{hom}_{\mathcal{C}}(X, X)$ and for three objects $X, Y, Z \in \mathcal{C}$ there is a composition map

$$\circ: \text{hom}_{\mathcal{C}}(Y, Z) \times \text{hom}_{\mathcal{C}}(X, Y) \longrightarrow \text{hom}_{\mathcal{C}}(X, Z), (g, f) \mapsto g \circ f$$

satisfying the expected associativity and unitality conditions. Small categories themselves can be organized into a category $\mathcal{C}at$ with the small categories as objects and the functors as morphisms. But somehow this is not the right thing to consider; in most cases, one is not asking whether two categories are *isomorphic* (which is frequently too strong) but instead one asks for the weaker situation that two categories are *equivalent*. Recall that two categories \mathcal{C}, \mathcal{D} are called equivalent if there are functors $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms

$$G \circ F \cong \text{id}_{\mathcal{C}}, \quad F \circ G \cong \text{id}_{\mathcal{D}}.$$

Since the concept of equivalence uses the notion of natural transformations i.e., morphisms between morphisms, and is hence a *2-categorical* one, one could say: instead of considering the *category* of small categories one should consider the *2-category* of categories.

One way to define the notion of a 2-category [Kel05b] is to say that it is a category enriched over categories. More vaguely, a 2-category is a mathematical species consisting of a class of objects, morphisms between two objects and 2-morphisms between parallel morphisms together with suitably compatible composition rules. There are other instances where one feels that in fact one considers a category in which there are morphisms at least up to dimension two. This is for example the case in the category of chain complexes in an abelian category with chain maps and chain homotopies and in the category of topological spaces with continuous maps and usual homotopies. One aim of the theory of $(\infty, 1)$ -categories is to find a good notion of ‘categories with morphisms of arbitrary dimension’ in which all morphisms of dimension at least 2 are invertible. There is the more general abstract concept of (∞, n) -categories, $n \geq 0$, by which one means ‘categories with morphisms of arbitrary dimension’ in which all morphisms of dimension greater than n are invertible. The translation of the standard concepts from classical category theory to the world of (∞, n) -categories for $n \geq 2$ is more complicated and in these notes we will only address the case $n = 1$. Nevertheless, we will not follow the convention of using the term ∞ -category for the *abstract concept* of an $(\infty, 1)$ -category because this abbreviation is in conflict with the notion of ∞ -categories as used by Lurie for a *specific model* for a theory of $(\infty, 1)$ -categories. This *specific*

model will be the main object of interest in these notes.

We now describe the contents of these notes. In Section 1, we give the central definitions and indicate how this definition of an ∞ -category gives a specific model for the concept of an $(\infty, 1)$ -category, in which arrows can be composed ‘uniquely up to a contractible choice’. Then the Joyal model structure on the category Set_Δ of simplicial sets and the Bergner model structure on the category Cat_Δ of simplicially enriched categories are introduced. These model structures organize two different approaches to the theory of $(\infty, 1)$ -categories. The coherent nerve construction as introduced by Cordier is described and a comparison result due to Lurie stating that these two model categories are Quillen equivalent is mentioned.

In Section 2, we sketch how to carry many familiar categorical concepts to the world of ∞ -categories, for example (co-)cones, slice constructions and (co-)limits. The notion of a presentable ∞ -category is mentioned and the strategy employed by Lurie to show that ‘nice model categories and nice ∞ -categories do the same job’ is sketched.

In Section 3, we introduce monoidal ∞ -categories. Motivated by the Grothendieck construction of classical category theory, one finds a reformulation of the axioms of a monoidal category. This reformulation is more easily translated to the world of ∞ -categories and encodes in both settings the coherence axioms in a very convenient way. After having briefly talked about algebra objects in a monoidal ∞ -category, we mention the relation between the algebra objects in a suitable monoidal model category and the algebra objects in the ∞ -category underlying the monoidal model category.

In Section 4, we indicate which modifications are needed in order to talk about *symmetric* monoidal ∞ -categories. Having briefly talked about commutative algebra objects in a symmetric monoidal ∞ -category, we mention how to establish the forgetful functors from this symmetric setting to the non-symmetric one. The last aim of the section is to quickly introduce the notion of (fully) dualizable objects in a symmetric monoidal ∞ -category. These and the corresponding objects in the more general situation of symmetric monoidal (∞, n) -categories are of central importance in Lurie’s classification of topological field theories [Lur09d].

In Section 5, we finally address the theory of spectra from the perspective of ∞ -categories. We introduce stable ∞ -categories and describe the stabilization process in some detail. As a byproduct, we can mention the ∞ -category of spectra as one of the most prominent examples for a stable ∞ -category. We close with a short sketch of Lurie’s construction of the smash product for spectra and show that it corresponds to the familiar smash product on highly structured (ring) spectra.

Remark: Before we can proceed, two more remarks are in order. The first concerns set-theoretical issues. In the theory of ∞ -categories, one frequently forms certain ‘categories of categories’ and considers the nerves of categories as simplicial *sets*, both of which are, strictly speaking, only allowed under certain smallness assumptions. In particular, we want to consider the nerves of model categories which are almost never small since small bicomplete categories can be shown to be posets [ML98, p.114]. In order to simplify the exposition, we will ignore all these issues throughout these notes and focus instead on the underlying concepts. Anyhow, all these things can be fixed as discussed in [Lur09e, pp.50-51 and subsection 5.4.1]. There is one exception to this remark: in the subsection on presentable ∞ -categories we will be more precise about these issues since they play a central role in that theory (as seen e.g. by the special form of the adjoint functor theorem for presentable categories).

The second remark is concerned with the terminology employed here. For many of the mathematical concepts to be introduced below, there are at least two different terminologies (most

frequently, one due to Joyal and one due to Lurie). Since we do not want to cause further confusion, we have to stick to one of these possible choices. As the last three sections of these notes are entirely concerned with topics discussed only in the expanded version of Lurie’s thesis [Lur09e, Lur09a, Lur09b, Lur09c], we will usually stick to Lurie’s terminology, but we will frequently mention the corresponding terminology employed by Joyal [Joy08b].

1. BASIC NOTIONS AND THE JOYAL RESP. BERGNER MODEL STRUCTURES

1.1. **Basic notions.** Before giving the central definition of an ∞ -category, we consider two classes of examples, which one definitely wants to be covered by the definition. The definition can then be seen as a common generalization of these two classes of examples.

Example 1.1. Given a topological space X , recall that associated to X there is the *fundamental groupoid* $\pi_1(X)$ of X . It is the groupoid where the objects are given by the points of X and, for two points $x, y \in X$, the morphisms from x to y are given by the homotopy classes of paths (relative to the boundary points) from x to y . This fundamental groupoid depends only on the 1-type of X . In order to encode more information, one can construct the *fundamental ∞ -groupoid* $\pi_{\leq \infty}(X)$ roughly in the following way: the objects are given by the points of X , the morphisms are the paths in X , the 2-morphisms are the homotopies between paths, and higher morphisms are given by higher homotopies. Note that the infinity groupoid $\pi_{\leq \infty}(X)$ seems to be an $(\infty, 0)$ -category since all morphisms are equivalences, i.e. invertible up to homotopy. It is a generally accepted principle of higher category theory, that all ∞ -groupoids should be given by topological spaces. Instead of working with topological spaces, one could also consider ‘simplicial models for spaces’, namely Kan complexes. Then the cited principle can be read: all ∞ -groupoids should come from Kan complexes. Using the approach to $(\infty, 1)$ -categories to be introduced below, it will be relatively easy to turn this into a theorem.

Example 1.2. Obviously, we want ordinary categories to give examples of ∞ -categories (we imagine all higher morphisms to be identities). Given a category \mathcal{C} , one can form the simplicial set $N(\mathcal{C})$, called the *nerve of \mathcal{C}* . By definition, we have $N(\mathcal{C})_n = \text{Fun}([n], \mathcal{C})$, where $[n]$ denotes the ordinal number $0 < \dots < n$, considered as a category. Since every category \mathcal{C} is equipped with a composition rule, the nerve $N(\mathcal{C})$ is not an arbitrary simplicial set, but instead satisfies certain extension properties: Two composable arrows

$$f: X \longrightarrow Y, \quad g: Y \longrightarrow Z$$

in \mathcal{C} define a simplicial map

$$(g, \bullet, f): \Lambda_1^2 \longrightarrow N(\mathcal{C}),$$

where Λ_i^n denotes the i -th n -horn, i.e. the simplicial set obtained from the standard n -simplex Δ^n by ‘removing the interior and the face opposite to the i -th vertex’. Stated differently, Λ_i^n is the simplicial subset of Δ^n obtained by forming the union of all faces of Δ^n with the exception of the i -th face, i.e. we have the following coequalizer diagram (see [GJ99, p.9] which is also a good reference for simplicial homotopy theory in general):

$$\bigsqcup_{0 \leq j < k \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{j \neq i} \Delta^{n-1} \rightarrow \Lambda_i^n.$$

Moreover, using the composition $g \circ f: X \longrightarrow Z$, we obtain an extension of the horn to a map $\Delta^2 \longrightarrow N(\mathcal{C})$ and this extension is unique.

If we would instead start with two morphisms

$$h: X \longrightarrow Z, \quad g: Y \longrightarrow Z$$

in \mathcal{C} with the same target, these give us a map

$$(g, h, \bullet): \Lambda_2^2 \longrightarrow N(\mathcal{C}).$$

This map though can not always be extended to a 2-simplex of $N(\mathcal{C})$. If for example $h = \text{id}_X$, then the existence of an extension is equivalent to the existence of a right inverse to $g: Y \longrightarrow X$. Similar observations are made, if we start with two morphisms having a common domain, so that in general a map $\Lambda_0^2 \longrightarrow N(\mathcal{C})$ cannot be extended to a 2-simplex of $N(\mathcal{C})$.

In the example we saw that the nerve $N(\mathcal{C})$ of a category \mathcal{C} has the horn extension property for Λ_1^2 , but not for the horns Λ_0^2 and Λ_2^2 . More generally, given a horn $\Lambda_i^n \longrightarrow N(\mathcal{C})$ in $N(\mathcal{C})$, this horn can always be extended to an n -simplex of $N(\mathcal{C})$ as long as we are not in one of the extremal cases $i = 0$ or $i = n$, i.e. it can be extended if we are given an *inner* horn as opposed to the case of an *outer* horn. In fact, there is the following characterization of nerves of categories among the simplicial sets [Lur09e, p.9].

Proposition 1.3. *Let $X \in \text{Set}_\Delta$ be a simplicial set. Then the following are equivalent:*

- i) *There is a category \mathcal{C} and an isomorphism $X \cong N(\mathcal{C})$.*
- ii) *Every inner horn $\Lambda_i^n \longrightarrow X$ of X can be uniquely extended to an n -simplex of X , i.e. all solid arrow diagrams as below can be uniquely completed to a commutative one by the indicated dashed arrow:*

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & X \\ \downarrow & \nearrow & \uparrow \\ \Delta^n & \xrightarrow{\exists!} & \end{array}$$

Since the Kan complexes have an important role to play, we recall the definition.

Definition 1.4. Let X be a simplicial set. Then X is called a *Kan complex* if it has the extension property for *all* horn inclusions. That is, for all horns $\lambda: \Lambda_i^n \longrightarrow X, 0 \leq i \leq n, n > 0$, there exists an n -simplex $\sigma: \Delta^n \longrightarrow X$ such that $\sigma|_{\Lambda_i^n} = \lambda$.

As a summary, in both cases, namely in the case of fundamental ∞ -groupoids and in the case of nerves of categories, we obtained simplicial sets with a certain horn extension properties. But these horn extension properties differ in two important aspects. In the case of Kan complexes, *all* horns can be extended to simplices, while in the case of nerves of categories, in general only the *inner* horns can be extended. The extension property for outer horns will not be fulfilled as soon as there are non-invertible morphisms in the category. The second important difference is that for nerves of categories the extension is *unique*. This is a property one wants to drop in the study of ∞ -categories: it is not important that there is a *unique* composition. Instead it suffices that the composition of morphisms can be performed and that the actual choice of composition is ‘homotopically unimportant’. This is similar to the situation of concatenation of paths in a given topological space: there is no associative composition law for composable paths parametrized by the unit interval but the actual choice of a composition (i.e. the parametrization of the glued path) is not important as all candidates are homotopic.

Definition 1.5. An ∞ -category \mathcal{C} is a simplicial set \mathcal{C} such that every inner horn $\Lambda_i^n \longrightarrow \mathcal{C}, 0 < i < n$, in \mathcal{C} can be extended to a simplex $\Delta^n \longrightarrow \mathcal{C}$.

We begin by introducing some language. Given an ∞ -category \mathcal{C} , the *objects of \mathcal{C}* are the 0-simplices \mathcal{C}_0 , while the *morphisms* are the 1-simplices \mathcal{C}_1 . Moreover, the *source map s* , the *target map t* and the *identity map id* are given by the simplicial structure maps of \mathcal{C} between degree zero and one:

$$s = d_1: \mathcal{C}_1 \longrightarrow \mathcal{C}_0, \quad t = d_0: \mathcal{C}_1 \longrightarrow \mathcal{C}_0, \quad \text{id} = s_0: \mathcal{C}_0 \longrightarrow \mathcal{C}_1.$$

For $f \in \mathcal{C}_1$, we write $f: x \longrightarrow y$ if $s(f) = x$ and $t(f) = y$.

Remark 1.6. Using the simplicial identities $d_0 s_0 = d_1 s_0 = \text{id}_{\mathcal{C}_0}$, we obtain that

$$s_0 x = \text{id}_x: x \longrightarrow x,$$

so id_x is in fact an endomorphism in \mathcal{C} . We will soon see that every ∞ -category \mathcal{C} has an associated homotopy category $\text{Ho}(\mathcal{C})$ (which is an ordinary category!) with the same objects and that in the homotopy category the morphism represented by $s_0 x$ is the identity of x .

Until now we did not use the fact that \mathcal{C} is an ∞ -category: what we did so far can be done with an arbitrary simplicial set. Now we turn towards the composition of morphisms where the horn extension property comes in. Let \mathcal{C} be an ∞ -category and let $f, g \in \mathcal{C}_1$ be two morphisms in \mathcal{C} , such that $t(f)$ equals $s(g)$. These morphisms give us an inner horn $(g, \bullet, f): \Lambda_1^2 \longrightarrow \mathcal{C}$ in \mathcal{C} which can be extended (not necessary uniquely!) to a simplex $\sigma: \Delta^2 \longrightarrow \mathcal{C}$. The face $d_1(\sigma)$ of σ opposite to vertex 1 is then *a candidate for a composition of the arrows g and f* .

This is one of the central points in which ∞ -category theory differs from ordinary category theory: one does not ask for uniquely determined compositions. Instead one demands only that there is a way to compose arrows and that any choice of such a composition is equally good: the space of all such choices is to be a contractible simplicial set. A more precise statement of this will be given in the short discussion of Theorem 1.11.

Now we are heading for the homotopy category associated to an ∞ -category \mathcal{C} .

Definition 1.7. Let \mathcal{C} be an ∞ -category and let $f, g: x \longrightarrow y$ be two morphisms in \mathcal{C} . Then f and g are called *homotopic* (notation: $f \simeq g$) if there is a 2-simplex $\sigma: \Delta^2 \longrightarrow \mathcal{C}$ with boundary $\partial\sigma = (g, f, \text{id}_x)$, i.e.

$$\begin{array}{ccc} & x & \\ \text{id}_x \nearrow & & \searrow g \\ x & \xrightarrow{f} & y \end{array}$$

For two objects $x, y \in \mathcal{C}$, we denote the set of morphisms from x to y by $\mathcal{C}_1(x, y)$. To establish the following elementary but important result one uses again the inner horn extension property of \mathcal{C} .

Proposition 1.8. *Let \mathcal{C} be an ∞ -category. Then the homotopy relation \simeq is an equivalence relation on $\mathcal{C}_1(x, y)$. We denote the homotopy class of a morphism $f \in \mathcal{C}_1(x, y)$ by $[f]$.*

We will give a partial proof to give an idea of how this works. For $f \in \mathcal{C}_1(x, y)$ consider $\kappa_f := s_0 f \in \mathcal{C}_2$. Using the simplicial identities, we obtain:

$$d_0 \kappa_f = d_1 \kappa_f = f \quad \text{and} \quad d_2 \kappa_f = d_2 s_0 f = s_0 d_1 f = \text{id}_x,$$

thus the boundary of κ_f is given by $\partial\kappa_f = (f, f, \text{id}_x)$, i.e. κ_f gives us a homotopy from f to f , called the *constant homotopy of f* . This shows the reflexivity of the homotopy relation. For the symmetry

one proceeds as follows: given a homotopy $\sigma: f \simeq g$ for $f, g \in \mathcal{C}_1(x, y)$, form the following inner horn in \mathcal{C} :

$$(\sigma, \kappa_g, \bullet, \kappa_{\text{id}_x}): \Lambda_2^3 \longrightarrow \mathcal{C}.$$

By the definition of an ∞ -category, this horn can be extended to a 3-simplex $\tau \in \mathcal{C}_3$. The face $\tilde{\sigma} := d_2\tau \in \mathcal{C}_2$ then gives the desired homotopy $\tilde{\sigma}: g \simeq f$.

We want to define the homotopy category $\text{Ho}(\mathcal{C})$ of an ∞ -category \mathcal{C} such that the objects are given by the 0-simplices \mathcal{C}_0 and that the morphisms are given by homotopy classes of morphisms in \mathcal{C} . The composition law on $\text{Ho}(\mathcal{C})$ is then obtained by representing the homotopy classes by morphisms in \mathcal{C} , choosing a *candidate composition* of the representatives and then taking the homotopy class of this candidate composition. Of course, in order to get an honest category there are a lot of things to be checked, but we instead content ourselves in just showing that all candidate compositions are homotopic: Given $f \in \mathcal{C}_1(x, y)$, $g \in \mathcal{C}_1(y, z)$ and let $\sigma_1 \in \mathcal{C}_2$ resp. $\sigma_2 \in \mathcal{C}_2$ determine h_1 resp. h_2 as candidate compositions of g with f . Then we can form the following horn in \mathcal{C} :

$$(\sigma_1, \sigma_2, \bullet, \kappa_f): \Lambda_2^3 \longrightarrow \mathcal{C}.$$

Again, we can find an extension to a 3-simplex $\tau \in \mathcal{C}_3$ and $d_2\tau \in \mathcal{C}_2$ gives us the desired homotopy $d_2\tau: h_2 \simeq h_1$. Using similar arguments, one can establish the following result.

Proposition 1.9. *Let \mathcal{C} be an ∞ -category. Then there is a category $\text{Ho}(\mathcal{C})$, called the homotopy category of \mathcal{C} , with the objects given by \mathcal{C}_0 and morphisms the homotopy classes of morphisms in \mathcal{C} . Moreover, we have:*

$$\text{composition: } [g] \circ [f] := [g \circ f], \quad \text{identities: } \text{id}_x := [\text{id}_x] = [s_0x],$$

where $g \circ f$ is an arbitrary candidate composition of g and f in \mathcal{C} .

Remark 1.10. • One guiding principle for the theory of ∞ -categories is that there should be a way to compose arrows and that the space of all such choices is a contractible space. Using the extension property for inner 2-horns, we saw that the space is non-empty. Using the extension property for inner horns up to dimension 3, we just checked that two candidate compositions are homotopic, i.e. that the space of all choices is connected. The extension property with respect to higher-dimensional inner horns can be thought of as guaranteeing the higher connectivity of the space of all such choices, giving finally that it is contractible (cf. the discussion of Theorem 1.11).

• A second guiding principle for the theory of ∞ -categories is that there should be morphisms of arbitrary dimension. Let \mathcal{C} be an ∞ -category and let x, y be two objects in \mathcal{C} . Then a *morphism* $f: x \longrightarrow y$ is given by

$$f: \Delta^1 \longrightarrow \mathcal{C} \text{ such that } f|_{\Delta^{\{0\}}} = x \text{ and } f|_{\Delta^{\{1\}}} = y.$$

Here and in the sequel, the notation is as follows: for vertices i_0, \dots, i_k in Δ^n , we denote by $\Delta^{\{i_0, \dots, i_k\}} \subseteq \Delta^n$ the k -simplex of Δ^n spanned by the given vertices. A homotopy between two parallel morphisms $x \longrightarrow y$ in \mathcal{C} can be interpreted as a *2-morphism* from x to y . Recall that a homotopy is given by

$$\sigma: \Delta^2 \longrightarrow \mathcal{C} \text{ such that } \sigma|_{\Delta^{\{0,1\}}} = x \text{ and } \sigma|_{\Delta^{\{2\}}} = y.$$

This can be generalized to higher dimensions: an *n -morphism* from x to y is a map of simplicial sets

$$\tau: \Delta^{n+1} \longrightarrow \mathcal{C} \text{ such that } \tau|_{\Delta^{\{0, \dots, n\}}} = x \text{ and } \tau|_{\Delta^{\{n+1\}}} = y.$$

For varying n , the n -morphisms in fact define a simplicial set $\text{Map}_{\mathcal{C}}(x, y) \in \text{Set}_{\Delta}$ which can be shown to be a Kan complex. There is an obvious dual way to define a *space* of maps from x to y

which turns out to be a weakly equivalent Kan complex. Thus the homotopy type of the mapping space $\text{Map}_{\mathcal{C}}(x, y)$ is well-defined.

• A third guiding principle for the theory of ∞ -categories is that they should give a model for $(\infty, 1)$ -categories, i.e. all higher morphisms should be invertible. To indicate that we succeeded in establishing such a framework, let us consider an ∞ -category and an arbitrary homotopy σ in \mathcal{C} :

$$\sigma: f \simeq g: x \longrightarrow y.$$

In order to prove the symmetry of the homotopy relation, we considered the inner horn

$$(\sigma, \kappa_g, \bullet, \kappa_{\text{id}_x}): \Lambda_2^3 \longrightarrow \mathcal{C},$$

which can be extended to a 3-simplex $\tau: \Delta^3 \longrightarrow \mathcal{C}$ and $\tilde{\sigma} = d_2\tau$ gives a homotopy $\tilde{\sigma}: g \simeq f$. Since we have $\tau|_{\Delta^{\{0,1,2\}}} = x$, τ is, by the above definition, a 3-morphism and can be interpreted as a 2-homotopy

$$\tau: \kappa_g \simeq \tilde{\sigma} \circ \sigma.$$

Thus every homotopy σ has (up to a 2-homotopy) a left inverse and a similar observation can be made for right inverses. Taking for granted that the horn extension property for higher dimensional horns allows us to deduce similar observations for higher homotopies, we see that the ∞ -categories really provide us with a model for $(\infty, 1)$ -categories.

The following theorem due to Joyal makes precise the idea that we succeeded in finding an axiomatic framework for categories with compositions determined up to contractible choices.

Theorem 1.11. *Let $X \in \text{Set}_{\Delta}$ be a simplicial set. Then X is an ∞ -category if and only if the restriction map $\text{Map}(\Delta^2, X) \longrightarrow \text{Map}(\Lambda_1^2, X)$ is an acyclic fibration of simplicial sets.*

Here Map denotes the usual simplicial mapping space between simplicial sets [GJ99, p.20],

$$\text{Map}(X, Y)_{\bullet} = \text{hom}_{\text{Set}_{\Delta}}(X \times \Delta^{\bullet}, Y).$$

Given a pair of composable arrows $f: x \longrightarrow y$ and $g: y \longrightarrow z$ in an ∞ -category \mathcal{C} , we obtain a map $\lambda: \Lambda_1^2 \longrightarrow \mathcal{C}$, i.e. a vertex $\lambda: \Delta^0 \longrightarrow \text{Map}(\Lambda_1^2, \mathcal{C})$. The fiber F of the above restriction map at this vertex, i.e. the following pullback

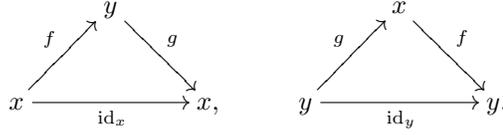
$$\begin{array}{ccc} F & \longrightarrow & \text{Map}(\Delta^2, X) \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & \text{Map}(\Lambda_1^2, X), \end{array}$$

can be regarded as the *space of all possible compositions of g with f* and is by the theorem a *contractible Kan complex*. This observation motivates us to henceforth suppress the ‘candidate’ in ‘candidate composition’. Now we come to the notion of equivalences in an ∞ -category.

Definition 1.12. Let \mathcal{C} be an ∞ -category and let $f: x \longrightarrow y$ be a morphism in \mathcal{C} . Then f is called an *equivalence* if $[f]: x \longrightarrow y$ is an isomorphism in the homotopy category $\text{Ho}(\mathcal{C})$.

It is immediate from this definition that the identities $\text{id}_x = s_0(x): x \longrightarrow x$ are equivalences and that a morphism $f: x \longrightarrow y$ in \mathcal{C} is an equivalence if and only if there is a morphism $g: y \longrightarrow x$ in

\mathcal{C} such that there are 2-simplices with boundaries



Moreover, for an equivalence $f: x \rightarrow y$ in \mathcal{C} every morphism homotopic to f is also an equivalence. We mentioned already the accepted principle that all ∞ -groupoids should come from spaces. Here is finally the precise definition of an ∞ -groupoid.

Definition 1.13. Let \mathcal{C} be an ∞ -category. Then \mathcal{C} is called an ∞ -groupoid if the homotopy category $\text{Ho}(\mathcal{C})$ is a groupoid.

Thus an ∞ -category \mathcal{C} is an ∞ -groupoid if and only if all morphisms in \mathcal{C} are equivalences. In the motivation of the definition of an ∞ -category, we saw that, in general, one should only demand the horn extension property for inner horns in order to obtain a good generalization of arbitrary categories (and not just of the groupoids!). Joyal established the following result, saying that the equivalences in an ∞ -category \mathcal{C} can in fact be characterized by the outer horns.

Proposition 1.14. Let \mathcal{C} be an ∞ -category and let f be a morphism in \mathcal{C} . Then f is an equivalence if and only if all horns $\lambda: \Lambda_0^n \rightarrow \mathcal{C}$, $n \geq 2$, with $\lambda|_{\Delta_{\{0,1\}}} = f$ can be extended to an n -simplex of \mathcal{C} .

Similarly, equivalences in an ∞ -category can be characterized using the outer horns Λ_n^n . As promised, the principle that all ∞ -groupoids should be given by spaces can now be turned into the following precise statement which is one of the main results of [Joy02] (see also [Lur09e, p.35]).

Corollary 1.15. Let \mathcal{C} be an ∞ -category. Then \mathcal{C} is an ∞ -groupoid if and only if \mathcal{C} is a Kan complex.

1.2. Simplicial categories and the comparison of the Joyal and the Bergner model structures. Now we come to an alternative approach to a theory of $(\infty, 1)$ -categories, namely the theory of *simplicial categories* (more precisely, *simplicially enriched categories*). There are further approaches which we will not discuss here but which are, for example, described in [Sim10]. We denote by Cat_Δ the category of simplicial categories together with the simplicial functors. Given two objects x, y of a simplicial category \mathcal{C} , we denote the simplicial mapping space from x to y by $\text{Map}_{\mathcal{C}}(x, y)$. This more rigid approach gives us, by definition, a notion of a category with morphisms of arbitrary dimensions. Building on work of Joyal and Bergner, Lurie has shown that these two approaches are equivalent in a precise sense (cf. Theorem 1.27, Remark 1.28).

Remark 1.16. Informally, one could say that $(\infty, 1)$ -categories should be categories enriched over $(\infty, 0)$ -categories, i.e. Kan complexes by Corollary 1.15. As we will see, the simplicial categories with the property that all mapping spaces are Kan complexes will play a special role in what follows.

Before giving the relation between ∞ -categories and simplicial categories, we consider again the nerve construction $N(\mathcal{C})$ of a category \mathcal{C} . By definition the nerve $N(\mathcal{C})$ is the simplicial set

$$N(\mathcal{C})_\bullet = \text{Fun}([\bullet], \mathcal{C}),$$

where $[\bullet]$ is the cosimplicial object in the category Cat of categories obtained by considering the finite ordinal numbers $[n]$ as categories. Thus the nerve functor is completely determined by this cosimplicial category. Given a *simplicial* category \mathcal{C} , we could forget the simplicial enrichment

and form the nerve of the underlying ordinary category. But this approach obviously forgets a lot of structure and instead one should proceed differently. This is done by replacing $[n] \in \mathcal{C}at$ by *simplicially thickened versions* $C[\Delta^n] \in \mathcal{C}at_\Delta$ and then building the simplicial set

$$N_\Delta(\mathcal{C})_\bullet = \text{Fun}_\Delta(C[\Delta^\bullet], \mathcal{C}),$$

where Fun_Δ denotes the set of *simplicial* functors. This will be done after the following remark.

Remark 1.17. The nerve construction $N: \mathcal{C}at \rightarrow \text{Set}_\Delta$ has a left adjoint called the *categorical realization functor*. Given a simplicial set $X \in \text{Set}_\Delta$, its categorical realization is defined to be the colimit $\text{colim}_{\Delta \downarrow X} [\bullet] \circ \text{pr}$, where the functor $[\bullet] \circ \text{pr}$ associates to each n -simplex $([n], x: \Delta^n \rightarrow X)$ the category $[n]$. The fact that we obtain an adjunction from Set_Δ to $\mathcal{C}at$ can be checked directly using only formal properties of colimits. This construction is a special case of the following more general observation, giving a conceptual explanation why the construction gives adjoint functors. Let \mathcal{C} be a cocomplete category, then there is an equivalence of categories $c\mathcal{C} \simeq \text{Adj}(\text{Set}_\Delta, \mathcal{C})$. Here $c\mathcal{C}$ denotes the category of cosimplicial objects in \mathcal{C} and, given two categories \mathcal{D}_1 and \mathcal{D}_2 , $\text{Adj}(\mathcal{D}_1, \mathcal{D}_2)$ denotes the category with objects the pairs of adjoint functors together with a unit transformation and morphisms the natural transformation of the left adjoints. Given a cosimplicial object Q^\bullet in \mathcal{C} , the right adjoint associated to Q^\bullet is obtained by sending $c \in \mathcal{C}$ to $\text{hom}_{\mathcal{C}}(Q^\bullet, c) \in \text{Set}_\Delta$ (compare to the nerve construction!). The left adjoint is given by the left Kan extension of the cosimplicial object Q^\bullet along the Yoneda embedding $Y = \Delta^\bullet: \Delta \rightarrow \text{Set}_\Delta$. This Kan extension exists by the cocompleteness of \mathcal{C} and can be calculated pointwise using colimits as in the above case. Thus the adjunction associated to $Q^\bullet \in c\mathcal{C}$ is given by

$$(\text{LKan}_{\Delta^\bullet} Q^\bullet, \text{hom}_{\mathcal{C}}(Q^\bullet, -)): \text{Set}_\Delta \rightarrow \mathcal{C}.$$

Note that this construction did not use any special property of the category Δ of the finite ordinals besides its smallness. Similar remarks can be made for arbitrary small categories and the construction of the left adjoint runs in that general situation under the name *Yoneda extension* [KS06, pp.62-64]. That this construction indeed gives us adjunctions is a special case of the adjoint functor theorem for presentable categories (cf. Proposition 2.19).

We now give the simplicial thickening $C[\Delta^n] \in \mathcal{C}at_\Delta$ of $[n]$. The idea is that $C[\Delta^n]$ encodes as objects the vertices of the standard simplex Δ^n together with all paths in increasing direction (morphisms) and all homotopies (2-morphisms) and higher homotopies (higher morphisms).

Example 1.18. In dimensions 0 and 1 nothing new happens. The categories $C[\Delta^0]$ and $C[\Delta^1]$ are just the ordinary categories $[0]$ and resp. $[1]$ considered as simplicial categories with discrete mapping spaces. Thus the pictures we have in mind are

$$C[\Delta^0]: 0 \quad \text{and} \quad C[\Delta^1]: 0 \rightarrow 1.$$

But from dimension 2 on, the simplicial picture is richer. In Δ^2 , there are two ways to pass from 0 to 2, namely the straight path and the path passing through 1. These paths should be encoded in $C[\Delta^2]$ together with a homotopy between them. The picture of $C[\Delta^2]$ is hence the following:

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & 2 \\ & \searrow & \nearrow \\ & 1 & \end{array}$$

Now we come to the precise definition of $C[\Delta^n]$. The objects of $C[\Delta^n]$ are again given by the numbers 0 to n . The strategy behind the definition of the simplicial mapping spaces is: Let

$i \leq j$ be two objects, then encode the possible paths from i to j by specifying the vertices of the corresponding path. Let $P_{i,j}$ be the following poset:

$$P_{i,j} = \{I \subseteq [i, j] \mid i, j \in I\}$$

ordered by inclusion. The simplicial mapping spaces can now be defined as

$$\mathrm{Map}_{\mathcal{C}[\Delta^n]}(i, j) = \begin{cases} N P_{i,j}, & i \leq j \\ \emptyset, & i > j \end{cases}.$$

The composition is induced by the union of subsets, which fits fine with the strategy to encode a path by specifying the vertices one passes along. Now one easily checks, that this definition gives the picture we had in mind in low dimensions. For example for dimension $n=2$, there is the following table of non-degenerate k -simplices in the mapping spaces $\mathrm{Map}_{\mathcal{C}[\Delta^2]}(i, j)$:

k	$i = j = 0$	$i = 0, j = 1$	$i = 0, j = 2$
0	{0}	{0, 1}	{0, 2}, {0, 1, 2}
1			{0, 2} \subseteq {0, 1, 2}

The association $[n] \mapsto \mathcal{C}[\Delta^n]$ can be seen to define a cosimplicial object in \mathcal{Cat}_Δ and we can thus give the following definition which appears to be due to Cordier [Cor82].

Definition 1.19. Let $\mathcal{C} \in \mathcal{Cat}_\Delta$ be a simplicial category, then the *coherent nerve* $N_\Delta \mathcal{C}$ of \mathcal{C} is the simplicial set

$$N_\Delta(\mathcal{C})_\bullet = \mathrm{Fun}_\Delta(\mathcal{C}[\Delta^\bullet], \mathcal{C}) \in \mathrm{Set}_\Delta.$$

Remark 1.20. Since the category \mathcal{Cat}_Δ of simplicial categories is cocomplete (in fact bicomplete), the above remark gives us a left adjoint to the coherent nerve construction functor. Namely, the left adjoint is given by the following left Kan extension: $\mathrm{LKan}_{\Delta^\bullet} \mathcal{C}[\Delta^\bullet]$. Denote this extension by $\mathcal{C}[-]: \mathrm{Set}_\Delta \rightarrow \mathcal{Cat}_\Delta: X \mapsto \mathcal{C}[X]$, so that the resulting adjunction can be written as

$$(\mathcal{C}[-], N_\Delta): \mathrm{Set}_\Delta \rightarrow \mathcal{Cat}_\Delta.$$

Since the left Kan extension along a fully-faithful functor is in fact an extension, this notation is not in conflict with the notation $\mathcal{C}[\Delta^n]$ for the simplicial thickening of $[n]$.

It is a result due to Lurie that this last adjunction is in fact a Quillen equivalence with respect to the *Joyal model structure* on Set_Δ and the *Bergner model structure* on \mathcal{Cat}_Δ . Since we will not make an intensive use of the Bergner model structure, we will only specify its weak equivalences.

Definition 1.21. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a simplicial functor. We call F a *weak equivalence* if the induced functor $\pi_0 F: \pi_0 \mathcal{C} \rightarrow \pi_0 \mathcal{D}$ is essentially surjective and if for all objects $x, y \in \mathcal{C}$ the map of simplicial sets $\mathrm{Map}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$ is a weak equivalence (in the usual Kan model structure, i.e. induces a weak equivalence on geometric realizations).

Recall that a functor between ordinary categories is an equivalence if and only if it is essentially surjective and fully-faithful. The definition of a weak equivalence F between simplicial categories can be interpreted as an immediate higher categorical generalization of the classical equivalences and can be read as demanding the simplicial functor F to be *homotopically essentially surjective* and *homotopically fully-faithful*. Such a functor is also called a *Dwyer-Kan equivalence*.

Remark 1.22. Recall from enriched category theory ([Bor94b, p.313-316] and as a general reference [Kel05a]), that given a monoidal functor $G: \mathcal{M} \rightarrow \mathcal{N}$, one obtains a *change of base functor*

$$G_* = \text{Cat}_G: \text{Cat}_{\mathcal{M}} \rightarrow \text{Cat}_{\mathcal{N}},$$

where $\text{Cat}_{\mathcal{M}}$ denotes the category of \mathcal{M} -enriched categories together with the \mathcal{M} -enriched functors as morphisms. For $\mathcal{C} \in \text{Cat}_{\mathcal{M}}$, $G_*\mathcal{C} \in \text{Cat}_{\mathcal{N}}$ is the \mathcal{N} -enriched category with the same objects as \mathcal{C} and with the morphisms spaces

$$\text{Map}_{G_*\mathcal{C}}(\bullet, \bullet) = G \text{Map}_{\mathcal{C}}(\bullet, \bullet).$$

On \mathcal{M} -enriched functors, G_* is defined similarly. As an example: $\pi_0: \text{Set}_{\Delta} \rightarrow \text{Set}$ is monoidal with respect to the Cartesian monoidal structures on both categories and the induced functor $\pi_0 F$ used in the above definition is just $(\pi_0)_*F$.

Building on work by Dwyer and Kan, Bergner [Ber07a] established the following theorem.

Theorem 1.23. *Cat_{Δ} carries a left proper combinatorial model structure with the weak equivalences introduced above. With respect to this model structure, a simplicial category \mathcal{C} is fibrant if and only if it is locally fibrant, i.e. if for all $x, y \in \mathcal{C}$ the simplicial mapping space $\text{Map}_{\mathcal{C}}(x, y)$ is a Kan complex.*

With a view towards the Joyal model structure on Set_{Δ} , we make the following definition.

Definition 1.24. A map $f: X \rightarrow Y$ in Set_{Δ} is a *categorical equivalence* if the induced simplicial functor $C[f]: C[X] \rightarrow C[Y]$ is a Dwyer-Kan equivalence.

Remark 1.25. This is not Joyal's original terminology. The maps in this definition are called *weak categorical equivalences* by Joyal [Joy08b], while he has a stronger notion of categorical equivalence. However, his notions of categorical equivalence and weak categorical equivalence coincide when only maps between ∞ -categories are considered.

With this terminology we can cite the following result due to Andre Joyal [Joy08b].

Theorem 1.26. *Set_{Δ} carries a left proper model structure with the monomorphisms as cofibrations and the categorical equivalences as weak equivalences. Moreover, a simplicial set X is fibrant with respect to this model structure if and only if X is an ∞ -category.*

As already remarked, the original definition of categorical equivalences due to Joyal is different [Joy08b]. He gives a definition without reference to simplicial categories and his proof of the existence of the Joyal model structure is purely combinatorial. Lurie gives this alternative definition because he is heading for the following comparison theorem [Lur09e, p.89].

Theorem 1.27. *The adjunction $(C[-], N_{\Delta})$ is a Quillen equivalence*

$$(C[-], N_{\Delta}): \text{Set}_{\Delta} \xrightarrow{\simeq} \text{Cat}_{\Delta},$$

where Set_{Δ} is endowed with the Joyal model structure and Cat_{Δ} with the Bergner model structure.

Remark 1.28. A similar result was also obtained by a combination of results due to Rezk, Joyal, Tierney, and Bergner. In [Rez01], Rezk introduces the notion of Segal spaces as an alternative model for a theory of $(\infty, 1)$ -categories. These are certain 'nice' bisimplicial sets and Rezk shows that there is an adapted model structure on the category of bisimplicial sets. This is combined with the result due to Joyal and Tierney [JT] that the model category for Segal spaces and the Joyal model structure on simplicial sets are Quillen equivalent. Finally, Julia Bergner [Ber07b] shows

that in turn the model category for Segal spaces and the Bergner structure on simplicial categories are Quillen equivalent through a zig-zag of Quillen equivalences. Thus, these results taken together also give a proof of the Quillen equivalence of the Joyal and the Bergner model structures through a zig-zag of Quillen equivalences.

As a corollary, we have the following result. Note that the corollary can also be proved directly, without any mention of model structures, but this way the result is put into perspective.

Corollary 1.29. Let \mathcal{C} be a locally fibrant simplicial category, then the coherent nerve $N_\Delta(\mathcal{C})$ of \mathcal{C} is an ∞ -category.

Now we come to the main source of examples for ∞ -categories. In fact, in Section 2 we will see that all sufficiently nice ∞ -categories are covered by this example.

Example 1.30. • Let \mathcal{M} be a simplicial model category [Qui67, section II.2] and let \mathcal{M}_{cf} be the full subcategory of \mathcal{M} spanned by the bifibrant, i.e. cofibrant and fibrant, objects. Then it is an immediate consequence of Quillen's axiom (SM7), that \mathcal{M}_{cf} is locally fibrant. Thus, via the coherent nerve construction, we obtain the ∞ -category $N_\Delta(\mathcal{M}_{cf})$, the ∞ -category associated to the simplicial model category \mathcal{M} .

• As a more concrete example, we can now take Set_Δ with the usual Kan model structure. This is a simplicial model category and with respect to this model structure, we have $(\text{Set}_\Delta)_{cf} = \text{Kan}$, the full subcategory spanned by the Kan complexes. This gives us a model for the ∞ -category of spaces: $S = N_\Delta(\text{Kan})$.

2. CATEGORICAL CONSTRUCTIONS AND THE RELATION OF ∞ -CATEGORIES TO MODEL CATEGORIES

The first main aim of this section is to extend some of the most important constructions from classical category theory to the world of ∞ -category in an *invariant manner*, i.e., given a categorical equivalence $\mathcal{C} \rightarrow \mathcal{D}$ of ∞ -categories, the result of the construction using \mathcal{C} should be categorically equivalent to that one obtained from the construction using \mathcal{D} . In the end, we want in particular be able to talk about limits and colimits in the ∞ -categorical setting. The reader who is less inclined towards abstract categorical constructions is asked to consider this as a justification for the discussion of the constructions in 2.2–2.3.

2.1. Functors. Since ∞ -categories are in particular simplicial sets, given two ∞ -categories \mathcal{C}, \mathcal{D} , we can say that a *functor from \mathcal{C} to \mathcal{D}* is just a map of simplicial sets. In fact, using the usual simplicial enrichment of Set_Δ , we obtain a *space* of functors from \mathcal{C} to \mathcal{D} : $\text{Fun}(\mathcal{C}, \mathcal{D}) := \text{Map}_{\text{Set}_\Delta}(\mathcal{C}, \mathcal{D})$. The vertices then give us the *functors*, while the 1-simplices give the *natural transformations*. To motivate that this definition is not only an easy but also a good one, we consider the following example.

Example 2.1. Let $I \in \text{Cat}$ be an ordinary category and \mathcal{M} be a locally fibrant simplicial category. By Corollary 1.29, we know that $\mathcal{C} = N_\Delta(\mathcal{M})$ is an ∞ -category. Consider a vertex

$$F \in \text{Fun}(NI, \mathcal{C})_0 = \text{Map}_{\text{Set}_\Delta}(NI, \mathcal{C})_0 = \text{hom}_{\text{Set}_\Delta}(NI, \mathcal{C}).$$

F thus associates, in particular, to each arrow $i_0 \rightarrow i_1$ in I a morphism $F i_0 \rightarrow F i_1$ in \mathcal{C} . Given two composable arrows $\sigma: i_0 \xrightarrow{f} i_1 \xrightarrow{g} i_2$ in I , we obtain a 2-simplex $F(\sigma) \in \mathcal{C}_2$, which shows, that there is a homotopy

$$F(\sigma): F(g \circ f) \simeq F(g) \circ F(f).$$

But there is still by far more information encoded by F , namely all the higher simplices obtained from longer sequences of composable arrows in I . These encode the idea that F is not only a ‘functor up to homotopy’ but gives us a ‘functor up to *coherent* homotopy’, i.e., a *homotopy coherent diagram*. For a precise statement on this see [Cor82].

In classical category theory, there is the straightforward observation that given two categories \mathcal{C}, \mathcal{D} the functors from \mathcal{C} to \mathcal{D} together with the natural transformations as morphism form again a category $\text{Fun}(\mathcal{C}, \mathcal{D})$. Moreover, given a category I and two equivalent categories $\mathcal{C} \xrightarrow{\simeq} \mathcal{D}$, the functor categories $\text{Fun}(I, \mathcal{C})$ and $\text{Fun}(I, \mathcal{D})$ are also equivalent. Similar results also hold in the world of ∞ -categories, but here they require a proof. In fact, one has to establish certain stability properties of the class of categorical equivalences and the so-called *inner anodyne maps*, which can e.g. be found in [Joy08a]. Using these properties, one is able to deduce the following result:

Proposition 2.2. *Let \mathcal{C}, \mathcal{D} be ∞ -categories and let K, M be simplicial sets. Then:*

- i) *The simplicial set $\text{Fun}(K, \mathcal{C})$ is an ∞ -category.*
- ii) *If $\mathcal{C} \rightarrow \mathcal{D}$ is a categorical equivalence, then the induced map $\text{Fun}(K, \mathcal{C}) \rightarrow \text{Fun}(K, \mathcal{D})$ is also a categorical equivalence.*
- iii) *If $K \rightarrow M$ is a categorical equivalence, then the induced map $\text{Fun}(M, \mathcal{C}) \rightarrow \text{Fun}(K, \mathcal{C})$ is also a categorical equivalence.*

Thus the formation of functor categories is an invariant notion (which should be the case for all categorical constructions in the world of ∞ -categories!).

Remark 2.3. • This proposition reveals one of the technical advantages of ∞ -categories over model categories: ∞ -categories are stable under the formation of functor categories without any further assumptions. In this respect, model categories are less well-behaved, since one has to impose certain conditions on the model categories involved to obtain this stability property: for cofibrantly-generated model categories, associated diagram categories always admit the *projective* model structure [Hir03, p.224], whereas in the case of combinatorial model categories the *projective* and the *injective* structure both always exist on the diagram categories [Lur09e, p.824].

• A further technical advantage of ∞ -categories over model categories is the following one. The ‘correct’ notion of equivalence for model categories is the notion of Quillen equivalence. Since in general, a Quillen equivalence can not be inverted, the equivalence relation generated by this notion is quite complicated: frequently model categories are only Quillen equivalent through a zig-zag of Quillen equivalences which may point in different directions. The appropriate notion of equivalence for ∞ -categories is the notion of categorical equivalence. Since the ∞ -categories are precisely the bifibrant objects with respect to the Joyal model structure, in which the weak equivalences are the categorical equivalences, a zig-zag of categorical equivalences can always be replaced by a single categorical equivalence.

• A third technical advantage of ∞ -categories was already mentioned, but will be repeated here for the sake of completeness. The notion of homotopy coherent diagrams is quite easily established in the world of ∞ -categories since it is simply a map of simplicial sets with the domain given by the nerve of an ordinary category. There will be further advantages of this flavor, i.e. where ‘higher coherences’ are easily encoded in the setting of ∞ -categories. For example the notions of \mathcal{A}_∞ - and \mathcal{E}_∞ -algebras are easily introduced in this setting as sections of certain Grothendieck opfibrations (as we will see in Section 3 and 4).

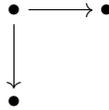
2.2. Join construction. Before giving the definitions, we recall the classical situation in category theory. Given two categories \mathcal{C}, \mathcal{D} , one can form a new category $\mathcal{C} \star \mathcal{D}$, called the *join construction*

of \mathcal{C} and \mathcal{D} in the following way. The class of objects is given by the disjoint union of the objects in \mathcal{C} and the objects in \mathcal{D} . For the morphisms, there are the following four different cases:

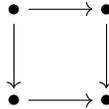
$$\mathrm{hom}_{\mathcal{C} \star \mathcal{D}}(X, Y) = \begin{cases} \mathrm{hom}_{\mathcal{C}}(X, Y), & X, Y \in \mathcal{C} \\ \mathrm{hom}_{\mathcal{D}}(X, Y), & X, Y \in \mathcal{D} \\ * & , \quad X \in \mathcal{C}, Y \in \mathcal{D} \\ \emptyset & , \quad X \in \mathcal{D}, Y \in \mathcal{C} \end{cases}$$

The composition is completely determined by requiring that \mathcal{C} and \mathcal{D} are full subcategories of $\mathcal{C} \star \mathcal{D}$. To get a feeling for this construction, some examples are in order.

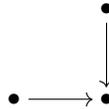
Example 2.4. • Let \mathcal{C} be arbitrary and let $\mathcal{D} = [0]$ be the terminal category, then $\mathcal{C}^\triangleright := \mathcal{C} \star [0]$ is called the *right cone on \mathcal{C}* . It is obtained from \mathcal{C} by adjoining a new object ∞ to \mathcal{C} and for each object $c \in \mathcal{C}$ a unique arrow $c \longrightarrow \infty$. This construction plays a central role in the study of colimits.
 • Dually, let $\mathcal{C} = [0]$ be the terminal category and let \mathcal{D} be arbitrary, then $\mathcal{D}^\triangleleft := [0] \star \mathcal{D}$ is called the *left cone on \mathcal{D}* . It is obtained from \mathcal{D} by adjoining a new object $-\infty$ to \mathcal{D} and for each object $d \in \mathcal{D}$ a unique arrow $-\infty \longrightarrow d$. This construction plays a central role in the study of limits.
 • To end with a more specific example, let \mathcal{C} be the category occurring in the study of pushout diagrams, i.e.



Then the right cone on \mathcal{C} is given by the square: $\mathcal{C}^\triangleright \cong [1]^2$, i.e. we have the following picture of $\mathcal{C}^\triangleright$:



Similarly, let \mathcal{D} be the diagram occurring in the study of pullback diagrams, i.e.



Then the left cone on \mathcal{D} is also given by the square: $\mathcal{D}^\triangleleft \cong [1]^2$.

The join construction can also be given for simplicial sets. There is a very conceptual approach to this construction as described by Joyal in [Joy08a], where also many ‘elementary relations’ are deduced. We give instead a more direct ‘definition’.

Definition 2.5. Let K, M be two simplicial sets. Then the *join of K and M* is defined to be the following simplicial set:

$$(K \star M)_n = K_n \cup M_n \cup \bigcup_{i+1+j=n} K_i \times M_j.$$

$K \star M$ is indeed a simplicial set. The join operation for simplicial sets is in fact characterized by the following two properties.

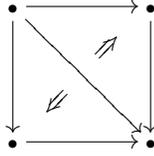
Proposition 2.6. i) The functors $K \star (-): \text{Set}_\Delta \rightarrow (\text{Set}_\Delta)_{K/}$ and $(-) \star M: \text{Set}_\Delta \rightarrow (\text{Set}_\Delta)_{M/}$ preserve colimits.

ii) For the standard simplices we find $\Delta^i \star \Delta^j \cong \Delta^{i+1+j}$.

To give some examples, we consider the left and right cone constructions and then again the pushout and pullback diagrams.

Example 2.7. • Let K be an arbitrary simplicial set and let $M = \Delta^0$. Then $K^\triangleright := K \star \Delta^0$ is called the *right cone on K* . Dually, let M be an arbitrary simplicial set and let $K = \Delta^0$. Then $M^\triangleleft := \Delta^0 \star M$ is called the *left cone on M* .

• Let $K = \Lambda_0^2$. Then the right cone on K is given by $(\Lambda_0^2)^\triangleright \cong (\Delta^1)^2$, i.e. we have the following picture:



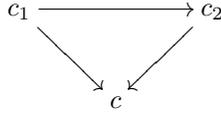
Similarly, the left cone on Λ_2^2 is also given by the square: $(\Lambda_2^2)^\triangleleft \cong (\Delta^1)^2$.

There is the following important proposition.

Proposition 2.8. Let \mathcal{C}, \mathcal{D} be ∞ -categories. Then the join $\mathcal{C} \star \mathcal{D}$ is again an ∞ -category.

Moreover, it can be shown that the join construction is an *invariant* notion.

2.3. Slice construction. We again begin by recalling the more classical situation of ordinary category theory. Given a category \mathcal{C} and an object $c \in \mathcal{C}$, one can form the *overcategory* $\mathcal{C}_{/c}$ where the objects are morphisms $c_1 \rightarrow c$ in \mathcal{C} and the morphisms are the following commutative triangles:



More generally, if we start with a functor $p: I \rightarrow \mathcal{C}$, we can form the *slice category* $\mathcal{C}_{/p}$ of objects over p . The objects are given by cones on p and the morphisms are the morphisms of cones. This slice construction satisfies the following universal property: for any category \mathcal{D} ,

$$\text{Fun}(\mathcal{D}, \mathcal{C}_{/p}) \cong \text{Fun}_p(\mathcal{D} \star I, \mathcal{C}),$$

where the right hand side denotes all functors from the join $\mathcal{D} \star I$ to \mathcal{C} whose restriction to I is given by p . These constructions can be carried over to the world of ∞ -categories as was done by Joyal [Joy02].

Proposition 2.9. Let M be a simplicial set, let \mathcal{C} be an ∞ -category and consider a map of simplicial sets $p: M \rightarrow \mathcal{C}$. Then there is an ∞ -category $\mathcal{C}_{/p}$ characterized by the following universal property: for every simplicial set K ,

$$\text{hom}_{\text{Set}_\Delta}(K, \mathcal{C}_{/p}) \cong \text{hom}_p(K \star M, \mathcal{C}),$$

where the right hand side denotes those simplicial maps $K \star M \rightarrow \mathcal{C}$ which restrict to p on M . We call the ∞ -category $\mathcal{C}_{/p}$ the ∞ -category of objects over p .

To check that there is such a *simplicial set* $\mathcal{C}_{/p}$, one can use the universal property as a definition: the special cases of the standard simplices $K = \Delta^n$ give us a description of the n -simplices of $\mathcal{C}_{/p}$. To show that one actually obtains an ∞ -category requires more work and will not be done here. Again, one can show that the slice construction is an *invariant* notion in a certain precise sense. Furthermore, it is obvious, in both the classical and the ∞ -categorical situation, that the constructions can be dualized. That way for example one obtains the ∞ -category $\mathcal{C}_{p/}$ of objects under p in the ∞ -categorical setting.

Example 2.10. Let \mathcal{C} be an ∞ -category and let $c \in \mathcal{C}_0$ be an object of \mathcal{C} , classified by the map $\Delta^0 \xrightarrow{\kappa_c} \mathcal{C}$. Then the ∞ -category $\mathcal{C}_{/\kappa_c}$ is called the *∞ -category of objects over c* and is simply denoted by $\mathcal{C}_{/c}$. Similarly, the ∞ -category $\mathcal{C}_{\kappa_c/}$ is called the *∞ -category of objects under c* and is denoted by $\mathcal{C}_{c/}$.

2.4. Initial and final objects. Now we come to the ∞ -categorical variant of the notion of initial and final objects. In classical category theory, final objects of a category are characterized by the property that for all objects there is a unique morphism to the final object. In these notes, we take a slightly different approach to the ∞ -categorical generalization (but see Remark 2.13). The following definition is slightly different to the original definition of Joyal [Joy02], but it is an equivalent one as shown in [Joy02].

Definition 2.11. Let \mathcal{C} be an ∞ -category and let $c \in \mathcal{C}_0$ be an object of \mathcal{C} . Then call c a *final object* of \mathcal{C} , if the canonical map $\mathcal{C}_{/c} \rightarrow \mathcal{C}$ is an acyclic fibration of simplicial sets.

There is the following proposition due to Joyal [Joy02].

Proposition 2.12. *Let \mathcal{C} be an ∞ -category and let $\mathcal{D} \subseteq \mathcal{C}$ be the full subcategory of \mathcal{C} spanned by the final objects of \mathcal{C} . Then \mathcal{D} is empty or a contractible Kan complex.*

Remark 2.13. • Since we haven't talked about subcategories yet, we will do it now. There is a general notion of subcategories of an ∞ -category \mathcal{C} associated to subcategories of its homotopy category $\mathrm{Ho}(\mathcal{C})$. Since we will not need this generality, we only consider the case of *full* subcategories. Given a subset $\mathcal{D}_0 \subseteq \mathcal{C}_0$ of the objects in \mathcal{C} , let $\mathcal{D} \subseteq \mathcal{C}$ be the simplicial subset consisting precisely of those simplices which have the property that all vertices belong to \mathcal{D}_0 . Then \mathcal{D} is an ∞ -category and is called the *full subcategory of \mathcal{C} spanned or determined by \mathcal{D}_0* .

• The conclusion of the last proposition, namely that a certain space 'parametrizing universal objects' is empty or contractible, is the typical form of an uniqueness statement in the theory of ∞ -categories. In classical category theory, if universal objects exist, they are unique up to unique isomorphism. In the world of ∞ -categories, if the space of universal objects is non-empty, then it is a *contractible* Kan complex. Already in the discussion of the composition of morphisms, we had the result, that the space of possible compositions is a *contractible* Kan complex. With this proposition, we have a further uniqueness statement of this sort and, since limits respectively colimits will be introduced below as certain final respectively initial objects, we have again by this proposition that if these exist, then the space of all such is a *contractible* Kan complex. Uniqueness results of this kind are also ubiquitous in the theory of model categories. Compare for example to [Hir03] where many categories of choices (for example, cofibrant replacements) are shown to have contractible nerves.

• To see that the notion of final objects in ∞ -categories is an expected analogue of the classical notion, we include this comment on mapping spaces into a final object of \mathcal{C} . In Section 1, we mentioned that for two objects c_1, c_2 in an ∞ -category \mathcal{C} , there is a well-defined homotopy type of a

space of maps $\text{Map}_{\mathcal{C}}(c_1, c_2) \in \text{Set}_{\Delta}$. One can deduce the following result: an object c_2 of \mathcal{C} is final if and only if for all $c_1 \in \mathcal{C}$ the mapping space $\text{Map}_{\mathcal{C}}(c_1, c_2)$ is contractible.

2.5. Colimits and limits. We will briefly reformulate the classical definition of the colimit of a functor in order to obtain a version which can be generalized directly to the setting of ∞ -categories. Let $F: I \rightarrow \mathcal{C}$ be an ordinary functor. A colimit of F is defined (if it exists) as an initial cocone on F , i.e. as an initial object of

$$\text{Cocone}(F) = \text{Fun}_F(I \star [0], \mathcal{C}) \cong \text{Fun}([0], \mathcal{C}_{F/}) \cong \mathcal{C}_{F/}.$$

This motivates the following definition which is due to Joyal [Joy02].

Definition 2.14. Let K be a simplicial set, let \mathcal{C} be an ∞ -category and consider a diagram $p: K \rightarrow \mathcal{C}$. A *colimit of p* is an initial object of $\mathcal{C}_{p/}$. Dually, a *limit of p* is a final object of $\mathcal{C}_{/p}$. An ∞ -category \mathcal{C} is called *cocomplete* if all diagrams have a colimit and *complete* if all diagrams have a limit. Finally, call an ∞ -category \mathcal{C} *bicomplete* if it is cocomplete and complete.

Remark 2.15. If (co-)limits exist, the space of all such (co-)limits forms by Proposition 2.12 a contractible Kan complex. More precisely, given an ∞ -category \mathcal{C} and a diagram $p: K \rightarrow \mathcal{C}$, the full subcategory of $\mathcal{C}_{p/}$ spanned by the colimits of p is empty or a contractible Kan complex.

Remark 2.16. • The definition of limits and colimits are the expected generalizations of the classical notions in category theory. But there is a much deeper justification for these definitions as discussed by Lurie in [Lur09e, Theorem 4.2.4.1]. Namely, there is a precise meaning in that this notion of (co-)limits coincides with the notion of homotopy (co-)limits in simplicial categories. In [Joy02], Joyal was already fully aware of this fact.

• Using this first remark, one can show that the ∞ -category associated to a simplicial model category is bicomplete. But to establish this result, one has to use quite a lot of theory. We only quickly mention the main steps. First, one has to introduce the notion of *cofinality* for morphisms of simplicial sets [Lur09e, p.224]. In the case that the target of the morphism is an ∞ -category, this is straightforward as one can give a definition which is the expected analogue of the one from classical category theory. For this, let $p: K \rightarrow \mathcal{C}$ be a diagram with \mathcal{C} an ∞ -category. Then for every object $c \in \mathcal{C}$, consider the following pullback diagram:

$$\begin{array}{ccc} \mathcal{C}_{c/} \times_{\mathcal{C}} K & \longrightarrow & \mathcal{C}_{c/} \\ \downarrow & & \downarrow \\ K & \longrightarrow & \mathcal{C} \end{array}$$

In this situation, p is called *cofinal* if the above pullback is *contractible* for all $c \in \mathcal{C}$. Recall from classical category theory, that the cofinality of a functor can be characterized by the *connectivity* of similar undercategories. The case of morphisms between arbitrary simplicial sets is more difficult and was first treated by Joyal [Joy08b]. Having this notion at hand, one then proves that for every simplicial set K , there is a category I and a cofinal functor $N(I) \rightarrow K$ [Lur09e, p.255]. Thus, concerning existence questions for colimits, one can restrict attention to diagrams defined on nerves of categories. A further deep ingredient is that homotopy coherent diagrams in the ∞ -category underlying a simplicial model category can be rigidified. Finally, all these results can then be combined to yield a proof of the bicompleteness of the ∞ -category $N_{\Delta}(\mathcal{M}_{cf})$ underlying a simplicial model category \mathcal{M} . Obviously, this lies outside the scope of these notes, but is treated in [Lur09e, section 4.1-4.2].

2.6. Presentable ∞ -categories and the relation to model categories. We begin this subsection with a short review of the theory of presentable categories without going into too much detail. As a motivation for their usefulness, we recall the adjoint functor theorem and the special form it takes when the categories under consideration are presentable. However, the main motivation for us to include a discussion of presentable ∞ -categories is the comparison of presentable ∞ -categories and model categories (cf. Theorem 2.28) and Lurie’s construction of the smash product on the ∞ -category of spectra (cf. Subsection 5.2).

Now, let \mathcal{C} and \mathcal{D} be cocomplete categories and consider a functor $F: \mathcal{C} \rightarrow \mathcal{D}$. If F has a right adjoint, then F preserves all colimits. Thus, this is obviously a *necessary* condition for the existence of a right adjoint. There is the celebrated *adjoint functor theorem* due to Freyd ([Fre03, pp.84-86], [ML98, p.121]) which gives *necessary and sufficient* conditions for the existence of a right adjoint functor: a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between cocomplete categories has a right adjoint if and only if it preserves all colimits and the so-called *solution set condition* is satisfied. The solution set condition states that a certain class of arrows in fact exists as a *set*, i.e. is small enough to form a set. Thus one can imagine that under certain conditions on the categories involved, ensuring that they are not too big (e.g. are ‘determined by something small’), then this solution set condition is automatically satisfied.

Definition 2.17. A category \mathcal{C} is called *presentable* if it is cocomplete and accessible.

The accessibility assumption is the smallness assumption alluded to above. Morally it says, that \mathcal{C} is formally determined by some small category. Slightly more precisely, a category \mathcal{C} is accessible if there is a small subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that every object of \mathcal{C} can be obtained canonically as a filtered colimit of objects in \mathcal{D} , i.e. a category \mathcal{C} is accessible if and only if \mathcal{C} is equivalent to the category $\text{Ind}(\mathcal{D})$ of ind-objects in some small category \mathcal{D} . For a precise definition and further properties of accessible and presentable categories see for example [AR94, GU71] or [Bor94a, Bor94b]. It can be shown that presentable categories are also complete, i.e. they are in fact bicomplete.

Remark 2.18. • Let \mathcal{C} be a small category. Then there is a universal way to construct a cocomplete category $\widehat{\mathcal{C}}$ together with a functor $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$, i.e. there is a *cocompletion* of \mathcal{C} . The fact that every contravariant set-valued functor on a small category is canonically a colimit of representable functors [ML98, p.76] can be reinterpreted as saying that the Yoneda-embedding $Y: \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Set}) =: \widehat{\mathcal{C}}$ is a model for the cocompletion. Moreover, one can establish the classification result that every presentable category \mathcal{E} is equivalent to a (‘nice’) localization $\mathbb{L}\widehat{\mathcal{C}}$ of a presheaf category $\widehat{\mathcal{C}}$ [AR94, pp.38-39]. If one interprets the localization process as a way to *impose relations* and if one considers the formation of the cocompletion as a *free generation* then it is reasonable to call for the moment a localization of a presheaf category a *presented category*. The classification result then reads: every *presentable* category is equivalent to a *presented* category.

• Recall that there is this nice class of abelian categories called *Grothendieck categories* [Gro57], [Fai73, chapter 14]. By definition an abelian category \mathcal{A} is Grothendieck if and only if \mathcal{A} has a generator and admits exact filtered colimits. It can be shown that an abelian category with exact filtered colimits is Grothendieck if and only if it is presentable. This gives a further important class of presentable categories.

We now come to the special form the adjoint functor theorem takes in the case of presentable categories.

Proposition 2.19. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between presentable categories. Then the following are equivalent:*

- i) *The functor F has a right adjoint.*
- ii) *The functor F preserves all colimits.*

Remark 2.20. There are similar results in slightly different settings as the Brown representability results for triangulated categories [Nee01, chapter 8] and of course the classical Brown representability theorem in stable homotopy theory [Bro62], Watt’s theorems in homological algebra [Rot79, section 5.3] and also representability theorems for Grothendieck categories [KS06, p.186].

Lurie [Lur09e, p.455] extends the notion of accessible and presentable categories to the world of ∞ -categories from where the following definition is taken.

Definition 2.21. An ∞ -category \mathcal{C} is called *presentable* if it is cocomplete and accessible.

It is possible to extend the classification result that all presentable categories are equivalent to localizations of presheaf categories to the world of ∞ -categories in a way we describe now. One reason why (*set-valued*) *presheaves* play such a central role in classical category theory is that all questions about the existence of universal constructions can be reformulated as representability questions for certain presheaves. In higher category theory, the representable functors take values in the category Set_Δ of simplicial sets. So it is not surprising that the central role played by the presheaf categories in classical category theory is taken by the *simplicial presheaf categories* in higher category theory. In the world of model categories, these were intensively studied by Jardine (cf. e.g. [Jar87]). Now, given an ∞ -category \mathcal{C} , there is an ∞ -category $\mathcal{P}(\mathcal{C})$ of (*simplicial*) *presheaves on \mathcal{C}* and a *Yoneda embedding* $Y: \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ which can be interpreted as a *cocompletion of \mathcal{C}* . Taking for granted a theory of adjoint functors in the ∞ -categorical setting [Lur09e, p.337], [Lur09b], we now describe the theory of localizations of ∞ -categories.

Definition 2.22. Let \mathcal{C}, \mathcal{D} be ∞ -categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a *localization* if F has a fully-faithful right adjoint.

Remark 2.23. In classical category theory, this type of situation is frequently called a *reflective localization* or *reflective subcategory* [Bor94a, sections 3.5 and 5.3]. In the setting of triangulated categories, this corresponds to the *Bousfield localizations* as opposed to the more general Verdier localizations [Kra].

With this definition at hand, the classification result for presentable ∞ -categories takes the following form as proved by Lurie in [Lur09e, p.456] in which he attributes the result to Simpson [Sim07].

Theorem 2.24. *For an ∞ -category \mathcal{C} the following are equivalent:*

- i) *The ∞ -category \mathcal{C} is presentable.*
- ii) *There is a small ∞ -category \mathcal{D} such that \mathcal{C} is an accessible localization of $\mathcal{P}(\mathcal{D})$.*

Remark 2.25. • Recall from classical category theory that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called *accessible* (or a *functor with rank* [Bor94b, p.272]) if there is a regular cardinal number κ such that F commutes with all κ -filtered colimits. For example all left adjoint functors are accessible. A further important class of accessible functors is given by the corepresented functors associated to κ -small objects. By an *accessible localization* one means a localization such that the fully-faithful right adjoint is accessible.

- We already recalled the special form of the adjoint functor theorem when the categories involved

are presentable: a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between presentable categories \mathcal{C}, \mathcal{D} is a left adjoint if and only if F preserves all colimits. One might now ask, if there is a similar result for right adjoint functors between presentable categories. Again, it is a formal consequence that a right adjoint functor between bicomplete categories necessarily preserves all limits. But even in the case of presentable categories, this is not sufficient. Instead, there is the following result: let \mathcal{C} and \mathcal{D} be presentable categories, then a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ is a right adjoint if and only if G preserves all limits and is accessible. Thus, in this situation a right adjoint functor is automatically accessible, i.e. commutes with κ -filtered colimits for sufficiently large regular cardinal numbers κ , which explains the occurrence of *accessible* localizations in the above classification result.

So, in order to understand presentable ∞ -categories it is important to understand the accessible localizations of ∞ -categories of presheaves or, more generally, of presentable ∞ -categories. This localization theory is very similar to the theory of Bousfield localizations of model categories [Bou75], [Hir03, part 1]. For the rest of this subsection, all ∞ -categories are assumed to be presentable unless otherwise stated. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be an accessible localization and let $L: \mathcal{C} \rightarrow \mathcal{C}$ be the composition of F with a fully-faithful right adjoint $\mathcal{D} \rightarrow \mathcal{C}$ of F . Moreover, denote the essential image of the localization functor L by $L\mathcal{C}$ and let \mathcal{S}_L be the class of morphisms in \mathcal{C} which are sent to equivalences in \mathcal{C} by L .

Definition 2.26. In this situation, call an object $c \in \mathcal{C}$ to be \mathcal{S}_L -local if for all $f \in \mathcal{S}_L$, $f: c_1 \rightarrow c_2$, the induced map $\text{Map}_{\mathcal{C}}(c_2, c) \rightarrow \text{Map}_{\mathcal{C}}(c_1, c)$ is a weak equivalence of simplicial sets.

Then it can be shown that the essential image of the localization functor consists precisely of the \mathcal{S}_L -local objects. Thus such an accessible localization is completely determined by the class \mathcal{S}_L of morphisms which are inverted by L . This class is not arbitrary, but instead has many closure properties: it is closed under the formation of colimits in \mathcal{S}_L as a subcategory of $\text{Fun}([1], \mathcal{C})$, is stable under the formation of retracts, contains the equivalences, satisfies the *2-out-of-3-axiom with respect to 2-simplices* and is stable under cobase change. We call a class of morphisms with these closure properties *strongly saturated*. Since for any family $\{\mathcal{S}_\alpha\}_\alpha$ of strongly saturated classes of morphisms the intersection $\bigcap_\alpha \mathcal{S}_\alpha$ is also strongly saturated, for each arbitrary class \mathcal{T} of morphisms in \mathcal{C} there is a smallest strongly saturated class of morphisms $\bar{\mathcal{T}}$ containing \mathcal{T} . We call a strongly saturated class of morphisms \mathcal{S} of *small generation* if there is a *subset* $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{S} = \bar{\mathcal{T}}$. The accessible localizations of a presentable ∞ -category \mathcal{C} are characterized, at least abstractly, by the following result [Lur09e, subsection 5.5.4].

Proposition 2.27. *Let \mathcal{C} be a presentable ∞ -category and let \mathcal{S} be a class of morphisms in \mathcal{C} . Then \mathcal{S} is strongly saturated of small generation if and only if there is an accessible localization $L: \mathcal{C} \rightarrow \mathcal{C}$ such that $\mathcal{S} = \mathcal{S}_L$.*

Having established all this theory, Lurie was able to build on Dugger's work on combinatorial model categories in order to deduce the comparison result which states informally that 'nice' ∞ -categories and 'nice' model categories do the same job. More precisely, Lurie proved the following result.

Theorem 2.28. *Let \mathcal{C} be an ∞ -category. Then \mathcal{C} is presentable if and only if there is a combinatorial, simplicial model category \mathcal{M} such that \mathcal{C} is equivalent to the ∞ -category $N_\Delta(\mathcal{M}_{cf})$ associated to \mathcal{M} .*

The class of combinatorial model categories was introduced by J.Smith. Recall that a model category \mathcal{M} is called *combinatorial* if \mathcal{M} is cofibrantly generated and if the underlying category is

presentable. One important property of the class of combinatorial model categories is that it admits a good theory of Bousfield localizations. In [Dug01a], Dugger showed that any combinatorial model category \mathcal{M} ‘has a presentation’. By this he means that up to Quillen equivalence, \mathcal{M} can be obtained as a left Bousfield localization of a model structure on a category of simplicial presheaves. Having these similar classification results for presentable ∞ -categories on the one side and for combinatorial model categories on the other side, gives some evidence for this theorem. The proof can be found in [Lur09e, pp.905-906].

3. MONOIDAL ∞ -CATEGORIES AND ALGEBRA OBJECTS

The aim of this section is to introduce the notion of monoidal ∞ -categories as given and intensively studied by Lurie [Lur09b]. We follow him in motivating the definition by recalling a reformulation of the classical concept of a monoidal category. In order to do so we start by introducing the notion of *Grothendieck opfibrations* [Bor94b, chapter 8], [Vis05].

3.1. Grothendieck opfibrations and monoidal categories. Consider the following classical situation. Let $p: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between ordinary categories and let $d \in \mathcal{D}$ be an object. Define \mathcal{C}_d , the *fiber of p over d* , by the following pullback diagram

$$\begin{array}{ccc} \mathcal{C}_d & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow p \\ [0] & \xrightarrow{d} & \mathcal{D} \end{array}$$

i.e. $\mathcal{C}_d \subseteq \mathcal{C}$ is the subcategory with objects those objects of \mathcal{C} which are mapped to d by p and morphisms all morphisms in \mathcal{C} which are mapped to id_d by p . The aim is to find conditions which ensure that the fiber \mathcal{C}_d ‘depends covariantly on the object $d \in \mathcal{D}$ ’.

Definition 3.1. Let $p: \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $f: c_1 \rightarrow c_2$ be a morphism of \mathcal{C} over the morphism $\alpha: d_1 \rightarrow d_2$ in \mathcal{D} , i.e. $p(f) = \alpha$. Then call f *p -coCartesian* if for all objects $c_3 \in \mathcal{C}$ the following diagram is a pullback diagram:

$$\begin{array}{ccc} \text{hom}_{\mathcal{C}}(c_2, c_3) & \longrightarrow & \text{hom}_{\mathcal{C}}(c_1, c_3) \\ \downarrow & & \downarrow \\ \text{hom}_{\mathcal{D}}(p(c_2), p(c_3)) & \longrightarrow & \text{hom}_{\mathcal{D}}(p(c_1), p(c_3)) \end{array}$$

Unwinding the definition, we obtain that a morphism f in \mathcal{C} , as in the definition, is p -coCartesian if and only if the following holds: Let $h: c_1 \rightarrow c_3$ be an arrow in \mathcal{C} and let γ be the image of h , i.e. $\gamma = p(h): d_1 \rightarrow d_3 = p(c_3)$. Then for every $\beta: d_2 \rightarrow d_3$ such that $\gamma = \beta \circ \alpha$ there is a unique $g: c_2 \rightarrow c_3$ in \mathcal{C} such that $\beta = p(g)$ and $h = g \circ f$. We have given the diagrammatic definition of p -coCartesian arrows as this translates most easily to the setting of ∞ -categories (cf. Definition 3.6). To get used to the notion of p -coCartesian arrows, one can easily establish the following lemma.

Lemma 3.2. *Let $p: \mathcal{C} \rightarrow \mathcal{D}$ be a functor and let $f_1: c_1 \rightarrow c_2$ and $f_2: c_1 \rightarrow c_3$ be two p -coCartesian arrows above $\alpha = p(f_1) = p(f_2): d_1 \rightarrow d_2$. Then there is a unique isomorphism*

$\phi: c_2 \rightarrow c_3$ such that $p(\phi) = \text{id}_{d_2}$ and such that the following diagram commutes

$$\begin{array}{ccc} & c_1 & \\ f_1 \swarrow & & \searrow f_2 \\ c_2 & \xrightarrow{\phi} & c_3 \end{array}$$

Thus, as soon as one has fixed an object c_1 in the fibre \mathcal{C}_{d_1} the target of a p -coCartesian lift of a morphism $\alpha: d_1 \rightarrow d_2$ is –if it exists at all– unique up to unique isomorphism in the fibre \mathcal{C}_{d_2} . To obtain a ‘covariant dependence’ of the fibres \mathcal{C}_d on $d \in \mathcal{D}$, one should thus ask for a sufficient supply of p -coCartesian arrows; this is the idea behind Grothendieck opfibrations.

Definition 3.3. Let $p: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then p is a *Grothendieck opfibration* if for all $c_1 \in \mathcal{C}$ and for all $\alpha: p(c_1) = d_1 \rightarrow d_2$ there is a p -coCartesian arrow $f: c_1 \rightarrow c_2$ in \mathcal{C} such that $p(f) = \alpha$.

Let $p: \mathcal{C} \rightarrow \mathcal{D}$ be a Grothendieck opfibration and choose for each $c \in \mathcal{C}$ and for each morphism $\alpha: p(c) \rightarrow d$ a p -coCartesian lift as given in the definition. Now fix a morphism $\alpha: d_1 \rightarrow d_2$ in \mathcal{D} and define

$$\alpha_1: \mathcal{C}_{d_1} \rightarrow \mathcal{C}_{d_2}: c_1 \mapsto c_2,$$

where c_2 is the codomain of the p -coCartesian lift $f: c_1 \rightarrow c_2$ chosen above. This defines α_1 on objects and it is easy to define α_1 on morphisms using the universal property of p -coCartesian arrows. Consider now a pair of composable arrows $d_1 \xrightarrow{\alpha} d_2 \xrightarrow{\beta} d_3$ in \mathcal{D} . This way, we obtain the following associated functors:

$$\mathcal{C}_{d_1} \xrightarrow{\alpha_1} \mathcal{C}_{d_2} \xrightarrow{\beta_1} \mathcal{C}_{d_3} \quad \text{and} \quad \mathcal{C}_{d_1} \xrightarrow{(\beta \circ \alpha)_!} \mathcal{C}_{d_3}.$$

In general, the two functors $(\beta \circ \alpha)_!$ and $\beta_1 \circ \alpha_1$ are not equal. But since both functors are defined by choosing p -coCartesian lifts and since these p -coCartesian lifts are unique up to unique isomorphism it follows that the two functors are naturally isomorphic:

$$(\beta \circ \alpha)_! \cong \beta_1 \circ \alpha_1.$$

This is an important observation since it allows us to encode a lot of structure in a Grothendieck opfibration as the following example shows.

Example 3.4. Let \mathcal{M} be a monoidal category with $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ the monoidal product and with $I \in \mathcal{M}$ the monoidal unit. Then we form a new category \mathcal{M}^\otimes in the following way. The objects of \mathcal{M}^\otimes are given by (possibly empty) finite sequences of objects in \mathcal{M} :

$$(M_1, \dots, M_n), \quad n \geq 0, \quad M_i \in \mathcal{M}.$$

Given two such sequences (M_1, \dots, M_n) and (L_1, \dots, L_k) , a morphism

$$(\alpha, \{f_i\}): (M_1, \dots, M_n) \rightarrow (L_1, \dots, L_k)$$

consists of a morphism $\alpha: [k] \rightarrow [n]$ in Δ together with morphisms

$$f_i: M_{\alpha(i-1)+1} \otimes \dots \otimes M_{\alpha(i)} \rightarrow L_i, \quad i = 1, \dots, k.$$

Thus in such a morphism, α encodes the domains of the f_i . In particular, if there is an $i \in [k]$ such that $\alpha(i-1) = \alpha(i)$, then the corresponding map f_i is a map $f_i: I \rightarrow L_i$. The composition of morphisms in \mathcal{M}^\otimes is defined using the composition in Δ and the associativity constraints of the monoidal structure on \mathcal{M} , while the identity of (M_1, \dots, M_n) is given by $(\text{id}_{[n]}, \{\text{id}_{M_i}\})$.

There is an obvious projection functor $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$ which sends an object (M_1, \dots, M_n) to

$[n]$ and a morphism in \mathcal{M}^\otimes to its first component. This functor p is a Grothendieck opfibration. Indeed, let (M_1, \dots, M_n) be an object of the fiber $\mathcal{M}_{[n]}^\otimes$ and $\alpha: [k] \rightarrow [n]$ be a morphism in Δ . Then a p -coCartesian lift of α is given by a family of isomorphisms

$$f_i: M_{\alpha(i-1)+1} \otimes \cdots \otimes M_{\alpha(i)} \xrightarrow{\cong} L_i, \quad i = 1, \dots, k.$$

More precisely, these L_i specify an object $(L_1, \dots, L_k) \in \mathcal{M}_{[k]}^\otimes$ and the morphism

$$(\alpha, \{f_i\}): (M_1, \dots, M_n) \rightarrow (L_1, \dots, L_k)$$

is p -coCartesian. Let $\iota_{\{i-1, i\}}: [1] \rightarrow [n]$ be the unique monomorphism in Δ with image $\{i-1, i\}$ and let $\iota_{\{i-1, i\}}^{op}: [n] \rightarrow [1]$ be the same map considered as a morphism in Δ^{op} . Using the fact, that $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$ is a Grothendieck opfibration, we thus obtain induced functors

$$(\iota_{\{i-1, i\}}^{op})!: \mathcal{M}_{[n]}^\otimes \rightarrow \mathcal{M}_{[1]}^\otimes = \mathcal{M}.$$

These functors together induce an equivalence

$$\mathcal{M}_{[n]}^\otimes \xrightarrow{\cong} (\mathcal{M}_{[1]}^\otimes)^{\times n} = \mathcal{M}^{\times n}.$$

The monoidal product $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ is encoded by the Grothendieck opfibration p as follows. Let $d^{1op}: [2] \rightarrow [1]$ denote the morphism in Δ^{op} which is opposite to the morphism $d^1: [1] \rightarrow [2]$. Then the induced map

$$\mu = (d^{1op})!: \mathcal{M}_{[2]}^\otimes \rightarrow \mathcal{M}_{[1]}^\otimes = \mathcal{M}$$

may be identified up to natural isomorphism under the equivalence $\mathcal{M}_{[2]}^\otimes \simeq \mathcal{M}^{\times 2}$ with the monoidal structure \otimes on \mathcal{M} . The associativity constraints of the monoidal structure on \mathcal{M} are also encoded by p . The cosimplicial identity

$$d^2 \circ d^1 = d^1 \circ d^1: [1] \rightarrow [3]$$

implies that we obtain a natural isomorphism of the following induced functors between the fibers:

$$(d^{1op})! \circ (d^{2op})! \cong (d^{1op})! \circ (d^{1op})!: \mathcal{M}_{[3]}^\otimes \rightarrow \mathcal{M}_{[1]}^\otimes = \mathcal{M}.$$

Under the identification $\mathcal{M}_{[3]}^\otimes \cong \mathcal{M}^{\times 3}$, this natural isomorphism gives us the associativity constraint

$$\alpha: M_1 \otimes (M_2 \otimes M_3) \cong (M_1 \otimes M_2) \otimes M_3, \quad M_1, M_2, M_3 \in \mathcal{M}.$$

It is now straightforward to see how the different factorizations of the map $\iota_{\{0, 4\}}: [1] \rightarrow [4]$ will give the pentagon axiom which is one of the coherence axioms imposed on a monoidal structure. The monoidal unit and the remaining coherence axioms are encoded in a similar way by p .

The point of this example was to show that there is an equivalent way to encode the structure of a monoidal category \mathcal{M} . Given a monoidal category \mathcal{M} , the Grothendieck opfibration $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$ can be constructed which has the further property that the different inclusions of the interval $[1]$ in $[n]$ induce an equivalence $\mathcal{M}_{[n]}^\otimes \rightarrow (\mathcal{M}_{[1]}^\otimes)^{\times n}$. Conversely, let us assume that we are given a Grothendieck opfibration $\mathcal{M}^\otimes \rightarrow \Delta^{op}$, such that the inclusions $\iota_{\{i-1, i\}}$ induce $\mathcal{M}_{[n]}^\otimes \xrightarrow{\cong} (\mathcal{M}_{[1]}^\otimes)^{\times n}$. Then one obtains a monoidal structure on $\mathcal{M} := \mathcal{M}_{[1]}$ mutatis mutandis as in the example. Thus the datum of such a Grothendieck opfibration is equivalent to the datum of a monoidal category.

Remark 3.5. The advantage of the approach via Grothendieck opfibrations is that the coherence axioms are hidden by the opfibration. In the ∞ -categorical setting, one expects a monoidal ∞ -category to be given by an ∞ -category \mathcal{M} together with a family of products $m_n: \mathcal{M}^{\times n} \rightarrow \mathcal{M}$ which satisfy many coherence axioms. These coherence axioms should be similar to the axioms for

an A_∞ -algebra and which can be made explicit using the combinatorics of the Stasheff associahedra. The point is that these *by far more complicated* coherence axioms can also be hidden if one is willing to use the associated Grothendieck opfibration.

3.2. ∞ -categorical generalizations: coCartesian fibrations and monoidal ∞ -categories.

Now we turn to the ∞ -categorical variants of the above concepts. Recall that given an ∞ -category \mathcal{C} and an object $c \in \mathcal{C}$ then there is the ∞ -category of objects under c denoted by $\mathcal{C}_{c/}$. Similarly, given a morphism $f: c_1 \rightarrow c_2$ in an ∞ -category \mathcal{C} , as a special case the slice construction applied to the map $\Delta^1 \rightarrow \mathcal{C}$ classifying f gives the ∞ -category of objects under f , which we denote by $\mathcal{C}_{f/}$. With this notation at hand, there is the following more or less straightforward generalization of p -coCartesian arrows to the ∞ -categorical setting [Lur09e, p.115].

Definition 3.6. Let \mathcal{C}, \mathcal{D} be ∞ -categories, let $p: \mathcal{C} \rightarrow \mathcal{D}$ be a functor and let $f: c_1 \rightarrow c_2$ be a morphism in \mathcal{C} . Then f is called to be *p -coCartesian* (above $\alpha = p(f)$) if the map

$$\mathcal{C}_{f/} \rightarrow \mathcal{C}_{c_1/} \times_{\mathcal{D}_{p(c_1)/}} \mathcal{D}_{p(f)/}$$

is an acyclic fibration of simplicial sets.

The ∞ -categorical concept corresponding to Grothendieck opfibrations is that of a coCartesian fibration. The main idea is again to axiomatically demand a sufficient supply of coCartesian morphisms.

Definition 3.7. Let \mathcal{C}, \mathcal{D} be ∞ -categories and $p: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then p is called a *coCartesian fibration* if p satisfies the following two properties:

- i) p is an inner fibration.
- ii) For every object $c_1 \in \mathcal{C}$ and every morphism $\alpha: p(c_1) = d_1 \rightarrow d_2$ in \mathcal{D} , there is a p -coCartesian arrow $f: c_1 \rightarrow c_2$ in \mathcal{C} such that $p(f) = \alpha$.

Remark 3.8. • We did not talk about inner fibrations yet. Recall, that an ∞ -category is a simplicial set with the horn extension property for all inner horns. The notion of an inner fibration, due to Joyal [Joy08b], is the relative version of an ∞ -category. More precisely, if $p: \mathcal{C} \rightarrow \mathcal{D}$ is a map of simplicial sets, call p an *inner fibration* if it has the right lifting property with respect to all inner horn inclusions, i.e. if for all diagrams with $0 < i < n$

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow p \\ \Delta^n & \longrightarrow & \mathcal{D} \end{array}$$

there is a map $\Delta^n \rightarrow \mathcal{C}$ making the diagram commutative. One can think of inner fibrations as families of ∞ -categories parametrized by \mathcal{D} , but which are functorial in $d \in \mathcal{D}$ only in a very weak sense (in the sense of *correspondences* [Lur09e, p.97], which are also known as *distributors, profunctors or bimodules* in the classical setting [Bor94a, section 7.8]). Given a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of ordinary categories, the induced map $N(F): N(\mathcal{C}) \rightarrow N(\mathcal{D})$ of simplicial sets is automatically an inner fibration, so that the notion of inner fibrations does not have a classical analogue.

- The terminology introduced by Lurie differs from the one due to Joyal. Joyal [Joy08b] uses the term *mid-fibrations* instead of inner fibration and *Grothendieck opfibration* instead of coCartesian fibration.
- Let $p: \mathcal{C} \rightarrow \mathcal{D}$ be a coCartesian fibration. The existence of sufficiently many coCartesian arrows ensures that p gives us a family of ∞ -categories parametrized by \mathcal{D} which depends functorially in a

stronger sense on $d \in \mathcal{D}$. In fact, Lurie established an ∞ -categorical analogue of the Grothendieck construction which makes this idea precise [Lur09e, Theorem 3.2.0.1].

In the last subsection we saw that monoidal categories can be alternatively encoded by certain Grothendieck opfibrations. Having introduced the corresponding notion of coCartesian fibrations, it is now straightforward to talk about monoidal ∞ -categories [Lur09b].

Definition 3.9. A *monoidal ∞ -category* is a coCartesian fibration $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\Delta^{op})$ such that for all $n \geq 0$ the inclusions $\iota_{\{i-1, i\}}: [1] \rightarrow [n]$ together induce a categorical equivalence of ∞ -categories

$$\mathcal{M}_{[n]}^\otimes \xrightarrow{\sim} (\mathcal{M}_{[1]}^\otimes)^{\times n}.$$

For simplicity, we refer to the ∞ -category $\mathcal{M} := \mathcal{M}_{[1]}^\otimes$ as a *monoidal ∞ -category*.

The interpretation of such a coCartesian fibration $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\Delta^{op})$ is now similar to the situation in classical category theory. To give an example we just make the following remark. One immediate consequence of the axioms is that the fiber $\mathcal{M}_{[0]}^\otimes$ over $[0]$ gives a contractible space. The unique map $n: [1] \rightarrow [0]$ induces a functor $\eta = (n^{op})_!: \mathcal{M}_{[0]}^\otimes \rightarrow \mathcal{M}_{[1]}^\otimes$. Call any object of $\mathcal{M} = \mathcal{M}_{[1]}^\otimes$ which lies in the image of η a *unit of the monoidal structure on \mathcal{M}* .

Remark 3.10. This definition of a monoidal ∞ -category $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\Delta^{op})$ really encodes quite a lot of structure on the underlying ∞ -category \mathcal{M} , namely that of a monoidal product which is associative and unital up to coherent homotopy (cf. also to Remark 3.15).

The last notion we want to introduce in this subsection is the notion of a morphism of monoidal ∞ -categories. It is immediate that given two monoidal ∞ -categories \mathcal{M}^\otimes and \mathcal{N}^\otimes a morphism between them should be a functor $F: \mathcal{M}^\otimes \rightarrow \mathcal{N}^\otimes$ which is compatible with the projections to $\mathbf{N}(\Delta^{op})$. Moreover, in the classical situation we saw that the monoidal structure is encoded by the coCartesian arrows which holds true in a similar way in the ∞ -categorical setting. Thus the following definition is reasonable [Lur09b].

Definition 3.11. Let $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\Delta^{op})$ and $q: \mathcal{N}^\otimes \rightarrow \mathbf{N}(\Delta^{op})$ be monoidal ∞ -categories. A *monoidal functor* from \mathcal{M}^\otimes to \mathcal{N}^\otimes is a functor $F: \mathcal{M}^\otimes \rightarrow \mathcal{N}^\otimes$ over $\mathbf{N}(\Delta^{op})$, i.e. such that the diagram

$$\begin{array}{ccc} \mathcal{M}^\otimes & \xrightarrow{F} & \mathcal{N}^\otimes \\ & \searrow p & \swarrow q \\ & \mathbf{N}(\Delta^{op}) & \end{array}$$

commutes, which carries p -coCartesian arrows in \mathcal{M}^\otimes to q -coCartesian arrows in \mathcal{N}^\otimes .

Let $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\Delta^{op})$, $q: \mathcal{N}^\otimes \rightarrow \mathbf{N}(\Delta^{op})$ be monoidal ∞ -categories, then the monoidal functors from \mathcal{M}^\otimes to \mathcal{N}^\otimes are themselves organized in an ∞ -category the following way. Denote by

$$\mathrm{Map}_{\mathbf{N}(\Delta^{op})}(\mathcal{M}^\otimes, \mathcal{N}^\otimes) \subseteq \mathrm{Fun}(\mathcal{M}^\otimes, \mathcal{N}^\otimes)$$

the full subcategory of the functor category spanned by the functors over $\mathbf{N}(\Delta^{op})$. Then the ∞ -category of *monoidal functors from \mathcal{M}^\otimes to \mathcal{N}^\otimes* is given by the full subcategory

$$\mathrm{Fun}^{\mathrm{Mon}}(\mathcal{M}, \mathcal{N}) \subseteq \mathrm{Map}_{\mathbf{N}(\Delta^{op})}(\mathcal{M}^\otimes, \mathcal{N}^\otimes)$$

spanned by the monoidal functors $F: \mathcal{M} \rightarrow \mathcal{N}$.

3.3. Algebra objects in monoidal ∞ -categories. Let \mathcal{M} be an ordinary monoidal category with monoidal pairing \otimes and unit object $I \in \mathcal{M}$. Then an algebra object in \mathcal{M} is an object $A \in \mathcal{M}$ together with a multiplication map $\mu: A \otimes A \rightarrow A$ and a unit map $\eta: I \rightarrow A$ which satisfy the expected associativity and unitality conditions. At the beginning of this section, we saw how to associate to a given monoidal category \mathcal{M} a Grothendieck opfibration $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$ which encodes the monoidal structure. If we now try to find a definition for an algebra object in the Grothendieck opfibration picture, a first guess would be that an algebra object A is given by a certain section of p . Thus consider an arbitrary section

$$A: \Delta^{op} \rightarrow \mathcal{M}^\otimes.$$

Using the isomorphism $\mathcal{M}_{[n]}^\otimes \cong \mathcal{M}^{\times n}$ and the fact that A is a section, we deduce that the value of A at $[n] \in \Delta^{op}$ is determined by n objects of \mathcal{M} :

$$A_{[n]} \longleftrightarrow A_1^n, \dots, A_n^n \in \mathcal{M}.$$

Let us consider now the map $d^1: [1] \rightarrow [2]$. The section evaluated on that map gives us, under the above identification, a map

$$A(d^{1op}): (A_1^2, A_2^2) \rightarrow A_1^1.$$

By the discussion in subsection 3.1, we have a description of the p -coCartesian arrows for the Grothendieck opfibration $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$. Applied to our situation, a p -coCartesian lift of d^{1op} starting at $A_{[2]}$ corresponds under the usual identification to the morphism

$$(A_1^2, A_2^2) \rightarrow A_1^2 \otimes A_2^2.$$

By the universal property of this p -coCartesian lift, we obtain an induced map

$$A_1^2 \otimes A_2^2 \rightarrow A_1^1.$$

If we now want to obtain a classical algebra object, we would like this induced map to be a map of the form $M \otimes M \rightarrow M$ for some $M \in \mathcal{M}$. Thus we should ensure that, among other things, the objects A_1^2, A_2^2 and A_1^1 are isomorphic.

Definition 3.12. Let $\alpha: [n] \rightarrow [k]$ be a morphism in Δ . Then call α *convex* if α is injective such that the image $\text{Im}(\alpha)$ is convex, i.e. the image is given by the interval $[\alpha(0), \alpha(n)]$.

From the structure of the p -coCartesian arrows of the Grothendieck opfibration $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$, we see that the p -coCartesian lifts of convex maps $\alpha: [n] \rightarrow [k]$ in Δ induce the projection functors $\mathcal{M}^{\times k} \rightarrow \mathcal{M}^{\times n}$. In particular, a p -coCartesian lift associated to $\iota_{\{0,1\}}$ resp. $\iota_{\{1,2\}}: [1] \rightarrow [2]$ starting at $A_{[2]}$ can be identified with

$$(A_1^2, A_2^2) \rightarrow A_1^2, \text{ resp. } (A_1^2, A_2^2) \rightarrow A_2^2.$$

If the images of $\iota_{\{0,1\}}, \iota_{\{1,2\}}: [2] \rightarrow [1]$ under A are p -coCartesian arrows, then by Lemma 3.2 we obtain the desired sequence of isomorphisms

$$A_1^2 \cong A_1^1 \cong A_2^2.$$

With this preparation it is now straightforward to establish the following result. The last part of the statement is a reformulation of MacLane's coherence result.

Proposition 3.13. *Let \mathcal{M} be a monoidal category and $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$ be the associated Grothendieck opfibration. Then a section $A: \Delta^{op} \rightarrow \mathcal{M}^\otimes$ of $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$ which has the property that the image of convex arrows under A are p -coCartesian arrows in \mathcal{M}^\otimes encodes an algebra object*

in \mathcal{M} . The underlying object of this algebra object can be chosen to be $A_{[1]} \in \mathcal{M}$. Conversely, any algebra object $A \in \mathcal{M}$ determines such a section of $p: \mathcal{M}^{\otimes} \rightarrow \Delta^{op}$.

Thus an alternative way to encode algebra objects in a monoidal category is to specify nice sections of the associated Grothendieck opfibration. In the world of ∞ -categories, we turn this observation into a definition [Lur09b].

Definition 3.14. Let $p: \mathcal{M}^{\otimes} \rightarrow \mathbf{N}(\Delta^{op})$ be a monoidal ∞ -category. A section A of p is called an *algebra object in \mathcal{M}^{\otimes}* if A sends convex morphisms to p -coCartesian arrows in \mathcal{M}^{\otimes} .

Remark 3.15. • Let $\mathcal{M}^{\otimes} \rightarrow \mathbf{N}(\Delta^{op})$ be a monoidal ∞ -category and let $\mathcal{M} = \mathcal{M}_{[1]}^{\otimes}$ be the underlying ∞ -category. Then it is already a certain abuse of language to speak of algebra objects of \mathcal{M}^{\otimes} since the notion of algebra object obviously also depends on the coCartesian fibration. A further comfortable abuse of language is to simply speak of algebra objects in \mathcal{M} .

• The notion of algebra objects in monoidal ∞ -categories can be seen to be a special case of lax monoidal functors between monoidal ∞ -categories. For this purpose we include the following definition. Let

$$p: \mathcal{M}^{\otimes} \rightarrow \mathbf{N}(\Delta^{op}), \quad q: \mathcal{N}^{\otimes} \rightarrow \mathbf{N}(\Delta^{op})$$

be monoidal ∞ -categories and let $F: \mathcal{M}^{\otimes} \rightarrow \mathcal{N}^{\otimes}$ be a functor over $\mathbf{N}(\Delta^{op})$. Then call F a *lax monoidal functor* if F sends all p -coCartesian arrows above convex morphisms in $\mathbf{N}(\Delta^{op})$ to q -coCartesian arrows. If one considers now the special case of the monoidal ∞ -category given by $\text{id}: \mathbf{N}(\Delta^{op}) \rightarrow \mathbf{N}(\Delta^{op})$, then algebra objects in \mathcal{M} can be identified with lax monoidal functors $\mathbf{N}(\Delta^{op}) \rightarrow \mathcal{M}$.

• Similar to the case of monoidal functors, the lax monoidal functors between two monoidal ∞ -categories \mathcal{M}^{\otimes} and \mathcal{N}^{\otimes} are organized into an ∞ -category $\text{Fun}^{\text{Lax}}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes})$, namely the full subcategory

$$\text{Fun}^{\text{Lax}}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}) \subseteq \text{Map}_{\mathbf{N}(\Delta^{op})}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes})$$

spanned by the lax monoidal functors. As a special case, given a single monoidal ∞ -category \mathcal{M}^{\otimes} , we see that the algebra objects in \mathcal{M} are themselves organized into an ∞ -category $\text{Alg}(\mathcal{M})$, the *∞ -category of algebra objects in \mathcal{M}* :

$$\text{Alg}(\mathcal{M}) = \text{Fun}^{\text{Lax}}(\mathbf{N}(\Delta^{op}), \mathcal{M}^{\otimes}).$$

• Similar to the case of monoidal structures on ∞ -categories themselves, an algebra object encodes quite a lot of structure: namely, given an algebra object $A \in \text{Alg}(\mathcal{M})$, the *underlying object* $A_{[1]}$ is endowed with a unit and a multiplication map which is associative and unital up to *coherent homotopy*. This similarity can be turned into a precise result in the following way. Given an ∞ -category \mathcal{C} with finite products, then there is an essentially unique way to endow \mathcal{C} with the *Cartesian monoidal structure* $\mathcal{C}^{\times} \rightarrow \mathbf{N}(\Delta^{op})$. On the other hand, in an arbitrary ∞ -category \mathcal{C} , one can talk about monoid objects which are certain simplicial objects M_{\bullet} in \mathcal{C} . More precisely, given a simplicial object $M_{\bullet}: \mathbf{N}(\Delta^{op}) \rightarrow \mathcal{C}$ in \mathcal{C} , call M_{\bullet} a *monoid object in \mathcal{C}* if for all $n \geq 0$ the inclusions of the interval $\iota_{\{i-1, i\}}: [1] \rightarrow [n]$ identify $M_{[n]}$ with an n -fold product of $M_{[1]}$ in \mathcal{C} :

$$M_{[n]} \xrightarrow{\sim} M_{[1]} \times \cdots \times M_{[1]}.$$

Now given an ∞ -category with finite products, the notions of *monoid object in \mathcal{C}* and *algebra object in \mathcal{C}^{\times}* are essentially the same, i.e. we have an equivalence of ∞ -categories

$$\text{Alg}(\mathcal{C}^{\times}) \simeq \text{Mon}(\mathcal{C}).$$

We want to apply this to the ∞ -category $\widehat{\mathcal{C}at}_\infty$ of ∞ -categories. Note that this ∞ -category admits finite products. It can then be shown, that monoidal ∞ -categories are essentially the same as monoid objects in $\widehat{\mathcal{C}at}_\infty$. Thus, by the above equivalence of monoids with algebra objects, monoidal ∞ -categories themselves are special cases of algebra objects, namely algebra objects in $\widehat{\mathcal{C}at}_\infty^\times$, which explains why in both cases we have the same type of coherence data.

Given a monoidal ∞ -category \mathcal{M}^\otimes such that the underlying ∞ -category \mathcal{M} has certain completeness or cocompleteness properties, one might wonder which of these properties are (under probably further assumptions on the monoidal structure) inherited by the ∞ -category $\text{Alg}(\mathcal{M})$ of algebra objects in \mathcal{M} . This question is answered by Lurie in [Lur09b]. It turns out that the case of limits is easier and the case of colimits needs a very careful analysis. Already in the classical situation, there is this distinction. The limits in, say, groups can be calculated in the category of sets whereas this is in general not the case for the colimits. However, arbitrary colimits can be obtained from the so-called *sifted colimits* and the finite coproducts. From these two cases, the case of sifted colimits is the easier one in that these can again be calculated in the category of sets. On the contrary, the coproduct of two groups G and H is given by the *free product* $G \star H$ whose underlying set is *very* different from the disjoint union of the underlying sets of G and H . Also in the ∞ -categorical setting, the difficulty is in guaranteeing the existence of finite coproducts in the ∞ -category $\text{Alg}(\mathcal{M})$ of algebra objects in \mathcal{M} . Nevertheless, the case of the empty coproduct is relatively simple and can be obtained without any further assumptions.

Proposition 3.16. *Let $\mathcal{M}^\otimes \rightarrow \mathbf{N}(\Delta^{op})$ be a monoidal ∞ -category, then the ∞ -category $\text{Alg}(\mathcal{M})$ of algebra objects in \mathcal{M} has an initial object. Moreover, an algebra object $A \in \text{Alg}(\mathcal{M})$ is initial if and only if the unit map $I_{\mathcal{M}} \rightarrow A_{[1]}$ is an equivalence in \mathcal{M} .*

3.4. Monoidal model categories and monoidal ∞ -categories. In this subsection we briefly describe the relation between monoidal model categories and monoidal ∞ -categories. We begin with the following elementary observation.

Proposition 3.17. *Let \mathcal{M} be a monoidal category and $p: \mathcal{M}^\otimes \rightarrow \Delta^{op}$ be the associated Grothendieck opfibration. Then the induced map of simplicial sets*

$$\mathbf{N}(p): \mathbf{N}(\mathcal{M}^\otimes) \rightarrow \mathbf{N}(\Delta^{op})$$

endows $\mathbf{N}(\mathcal{M})$ with the structure of a monoidal ∞ -category.

We would like to obtain a similar result for model categories which carry a monoidal and a simplicial structure such that these three structures are compatible in a precise sense.

Definition 3.18. Let \mathcal{M} be a closed monoidal, simplicial category. Then the monoidal structure on \mathcal{M} is *compatible with the simplicial structure* if the monoidal product $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ is endowed with the structure of a simplicial functor compatible with the unitality and associativity constraints and if the adjunctions expressing the fact that our given monoidal structure is closed are in fact enriched adjunctions (i.e. we have natural isomorphism of the relevant mapping spaces).

This definition thus describes a reasonable compatibility assumption between a *monoidal* and a *simplicial* structure. The third structure, namely the *model* structure, now enters the game.

Definition 3.19. Let \mathcal{M} be a simplicial model category endowed with a closed monoidal structure $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$. Then the monoidal structure is *compatible with the simplicial model structure* if the following are satisfied:

- i) The monoidal structure is compatible with the simplicial structure.
 ii) The monoidal pairing $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ is a *left Quillen bifunctor*, i.e. for every pair of cofibrations $i: M_1 \rightarrow M_2$, $j: N_1 \rightarrow N_2$ in \mathcal{C} the induced map

$$i \odot j: (M_1 \otimes N_2) \coprod_{M_1 \otimes N_1} (M_2 \otimes N_1) \rightarrow M_2 \otimes N_2$$

is a cofibration which is acyclic if i or j is.

- iii) The unit object $I_{\mathcal{M}} \in \mathcal{M}$ is cofibrant.

Let now \mathcal{M} be a simplicial model category with a compatible monoidal structure. Associated to the monoidal structure, we obtain as usual our Grothendieck opfibration $\mathcal{M}^{\otimes} \rightarrow \Delta^{op}$. Let

$$\mathcal{M}_{cf}^{\otimes} \subseteq \mathcal{M}^{\otimes}$$

be the full subcategory spanned by those objects $(M_1, \dots, M_n) \in \mathcal{M}_{[n]}^{\otimes}$, $n \geq 0$, with the property that all M_i are bifibrant. This category $\mathcal{M}_{cf}^{\otimes}$ can be endowed with a simplicial structure. Let $(M_1, \dots, M_n), (L_1, \dots, L_k)$ be two objects of $\mathcal{M}_{cf}^{\otimes}$, then the corresponding simplicial mapping space is given by

$$\mathrm{Map}_{\mathcal{M}_{cf}^{\otimes}}((M_1, \dots, M_n), (L_1, \dots, L_k)) = \prod_{\alpha: [k] \rightarrow [n]} \prod_{i=1}^k \mathrm{Map}_{\mathcal{M}}(M_{\alpha(i-1)+1} \otimes \dots \otimes M_{\alpha(i)}, L_i).$$

With this notation there is the following result.

Proposition 3.20. *Let \mathcal{M} be a simplicial model category with a compatible monoidal structure. Then the underlying ∞ -category $N_{\Delta}(\mathcal{M}_{cf}^{\otimes})$ is a monoidal ∞ -category with the monoidal structure specified by*

$$N_{\Delta}(\mathcal{M}_{cf}^{\otimes}) \rightarrow N(\Delta^{op}).$$

We now want to describe in which sense the formation of algebra objects is under certain assumptions compatible with the formation of coherent nerves. Thus for the remainder of this subsection, let \mathcal{M} be a *combinatorial* simplicial model category with a compatible monoidal structure. Consider the following three possible assumptions on \mathcal{M} :

- (A): All objects of \mathcal{M} are cofibrant.
 (B): \mathcal{M} satisfies the monoid axiom of Schwede-Shipley [SS00] and the monoidal product \otimes is symmetric.
 (C): The model structure is left proper and the class of cofibrations is generated by cofibrations between cofibrant objects.

In their paper [SS00], Schwede and Shipley addressed the question under which assumptions on a given monoidal model category, the associated categories of algebra objects or the associated category of modules over a fixed algebra object are again naturally endowed with a model structure. In fact, they discussed more general situations, but for our purpose we only need the following special case.

Theorem 3.21. *Either of the conditions (A) or (B) implies that the category $\mathrm{Alg}(\mathcal{M})$ of algebra objects in \mathcal{M} admits a combinatorial simplicial model structure with the property that the forgetful functor $\mathrm{Alg}(\mathcal{M}) \rightarrow \mathcal{M}$ creates the fibrations and the weak equivalences.*

Under the assumptions of the theorem, we can thus consider the combinatorial simplicial model category $\text{Alg}(\mathcal{M})$ of algebra objects in \mathcal{M} and its associated ∞ -category $\text{N}_\Delta(\text{Alg}(\mathcal{M})_{cf})$. Moreover, by Proposition 3.20, we can consider the monoidal ∞ -category $\text{N}_\Delta(\mathcal{M}_{cf})$ underlying \mathcal{M} and the ∞ -category $\text{Alg}(\text{N}_\Delta(\mathcal{M}_{cf}))$ of algebra objects therein. There is a canonical functor

$$\text{N}_\Delta(\text{Alg}(\mathcal{M})_{cf}) \longrightarrow \text{Alg}(\text{N}_\Delta(\mathcal{M}_{cf}))$$

and Lurie [Lur09b] has established conditions when this canonical functor is an equivalence.

Theorem 3.22. *Either of the conditions (A) or ((B) and (C)) implies that the canonical functor*

$$\text{N}_\Delta(\text{Alg}(\mathcal{M})_{cf}) \xrightarrow{\sim} \text{Alg}(\text{N}_\Delta(\mathcal{M}_{cf}))$$

is an equivalence of ∞ -categories.

Remark 3.23. • The ∞ -category $\text{N}_\Delta(\text{Alg}(\mathcal{M})_{cf})$ on the left hand side can be seen as an ∞ -category of *strict algebra objects*, whereas the ∞ -category $\text{Alg}(\text{N}_\Delta(\mathcal{M}_{cf}))$ can be seen as an ∞ -category of *coherent algebra objects*, and the canonical functor sends a strict algebra object to the canonical coherent one determined by it. Thus the statement of the theorem can be interpreted by saying that every coherent algebra object can be rigidified to a strict algebra object, i.e. is equivalent to a strict one. The proof of this theorem as given by Lurie in [Lur09b] uses a lot of theory and the rough structure is as follows. The first main step is to establish an ∞ -categorical Barr-Beck theorem which characterizes the monadic adjunctions among the adjunctions. Building on this ∞ -categorical Barr-Beck theorem, Lurie is able to deduce a result of the following form. Given two adjunctions

$$(F_1, G_1): \mathcal{C} \rightleftarrows \mathcal{D}_1, \quad (F_2, G_2): \mathcal{C} \rightleftarrows \mathcal{D}_2$$

of ∞ -categories together with a functor $H: \mathcal{D}_1 \rightarrow \mathcal{D}_2$, then under certain conditions the functor $H: \mathcal{D}_1 \rightarrow \mathcal{D}_2$ is an equivalence of ∞ -categories. This general result, can finally be applied to the following situation: the two adjunctions are given by the free and forgetful functors

$$\text{N}_\Delta(\mathcal{M}_{cf}) \rightleftarrows \text{N}_\Delta(\text{Alg}(\mathcal{M})_{cf}), \quad \text{N}_\Delta(\mathcal{M}_{cf}) \rightleftarrows \text{Alg}(\text{N}_\Delta(\mathcal{M}_{cf}))$$

and the role of the functor H is played by the canonical functor $\text{N}_\Delta(\text{Alg}(\mathcal{M})_{cf}) \rightarrow \text{Alg}(\text{N}_\Delta(\mathcal{M}_{cf}))$.

• With Theorem 2.28, we mentioned the comparison result stating that an ∞ -category \mathcal{C} is presentable if and only if \mathcal{C} is equivalent to the ∞ -category underlying a combinatorial simplicial model category. This can be extended to the situation of monoidal ∞ -categories: Given a presentable ∞ -category \mathcal{M} with a monoidal structure such that the monoidal functor $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ commutes with colimits separately in each variable, then \mathcal{M} is equivalent as a monoidal ∞ -category to the ∞ -category $\text{N}_\Delta(\mathcal{N}_{cf})$ underlying a combinatorial simplicial model category \mathcal{N} with a compatible monoidal structure. For a sketch of a proof see [Lur09b].

4. SYMMETRIC MONOIDAL ∞ -CATEGORIES AND (FULLY) DUALIZABLE OBJECTS

4.1. Symmetric monoidal ∞ -categories. In this subsection, we want to consider the modifications which are in order to obtain a theory of *symmetric* monoidal ∞ -categories. This theory is inspired by the theory of Γ -spaces [Seg74, Sch99] and this inspiration can be put into perspective using the ∞ -categorical Grothendieck construction.

We start again with the classical situation of ordinary category theory. In the last section, we obtained an alternative description of monoidal categories \mathcal{M} by certain Grothendieck opfibrations $\mathcal{M}^\otimes \rightarrow \Delta^{op}$. In that picture, the monoidal product was encoded by the induced functor

$$\mu = (d^{1op})_!: \mathcal{M}_{[2]}^\otimes \longrightarrow \mathcal{M}_{[1]}^\otimes.$$

If we want Grothendieck opfibrations to describe *symmetric* monoidal categories, we must be able to encode the symmetry isomorphism, thus we must, in particular, be able to encode the flip map

$$t: \mathcal{M}^{\times 2} \longrightarrow \mathcal{M}^{\times 2}: (X, Y) \mapsto (Y, X)$$

and, more generally, any permutation of n objects in \mathcal{M} . Thus it is plausible, that the role of the category Δ is taken by ‘a category of finite sets with all maps between them’. The details are as follows. Given a finite set I , denote by I_* the set obtained by adding a disjoint base-point:

$$I_* = I \sqcup \{*\}.$$

Moreover, for an arbitrary integer $n \geq 0$, let $\langle n \rangle$ be the (possibly empty) finite set

$$\langle n \rangle = \{1 < \dots < n\}.$$

In the next definition, we consider $\langle n \rangle$ only as a set but the natural ordering on $\langle n \rangle$ will be used later in the formation of (higher) monoidal products.

Definition 4.1. Let $\mathcal{F}in_*$ denote the following category. The objects are given by the finite pointed sets $\langle n \rangle_*$, $n \geq 0$, and the morphisms between two such finite pointed sets $\langle n \rangle_*$, $\langle m \rangle_*$ are given by the pointed maps $\alpha: \langle n \rangle_* \longrightarrow \langle m \rangle_*$. For each $n \geq 1$ and each $j = 1, \dots, n$, let $\alpha^{j, \langle n \rangle}$ be the following map in $\mathcal{F}in_*$:

$$\alpha^{j, \langle n \rangle}: \langle n \rangle_* \longrightarrow \langle 1 \rangle_*: i \mapsto \begin{cases} 1 & \text{for } i = j, \\ * & \text{else.} \end{cases}$$

Let now \mathcal{M} be a symmetric monoidal category with monoidal product $\otimes: \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$ and monoidal unit $I \in \mathcal{M}$. Following a pattern similar to last section, we construct a new category \mathcal{M}^{\otimes} as follows. The objects are again given by the (possibly empty) finite sequences of objects in \mathcal{M} :

$$(M_1, \dots, M_n), \mathcal{M}_i \in \mathcal{M}, n \geq 0.$$

Given two such sequences (M_1, \dots, M_n) , (L_1, \dots, L_k) , a morphism

$$(\alpha, \{f_i\}): (M_1, \dots, M_n) \longrightarrow (L_1, \dots, L_k)$$

between them in \mathcal{M}^{\otimes} is given by a morphism $\alpha: \langle n \rangle_* \longrightarrow \langle k \rangle_*$ in $\mathcal{F}in_*$ together with morphisms

$$f_i: \bigotimes_{j \in \alpha^{-1}(i)} M_j \longrightarrow L_i, \quad i = 1, \dots, k,$$

where the product is formed according to the ordering on $\alpha^{-1}(i)$. Again, if the set $\alpha^{-1}(i)$ is empty, the map f_i is simply a map $f_i: I \longrightarrow L_i$. This category is equipped with a projection functor $p: \mathcal{M}^{\otimes} \longrightarrow \mathcal{F}in_*$ which sends an object (M_1, \dots, M_n) , $n \geq 0$, to $\langle n \rangle_* \in \mathcal{F}in_*$ and a morphism $(\alpha, \{f_i\}): (M_1, \dots, M_n) \longrightarrow (L_1, \dots, L_k)$ to its first component $\alpha: \langle n \rangle_* \longrightarrow \langle k \rangle_*$ in $\mathcal{F}in_*$.

Lemma 4.2. *Given a symmetric monoidal category \mathcal{M} , the projection functor $p: \mathcal{M}^{\otimes} \longrightarrow \mathcal{F}in_*$ is a Grothendieck opfibration.*

In fact, given a morphism $\alpha: \langle n \rangle_* \longrightarrow \langle k \rangle_*$ in $\mathcal{F}in_*$ and an object (M_1, \dots, M_n) in \mathcal{M}^{\otimes} over $\langle n \rangle_*$, an associated p -coCartesian lift can be given by specifying isomorphisms

$$f_i: \bigotimes_{j \in \alpha^{-1}(i)} M_j \longrightarrow L_i, \quad i = 1, \dots, k.$$

More precisely, the L_i define an object $(L_1, \dots, L_k) \in \mathcal{M}_{\langle k \rangle_*}^{\otimes}$ and the morphism

$$(\alpha, \{f_i\}): (M_1, \dots, M_n) \longrightarrow (L_1, \dots, L_k)$$

is easily checked to be a p -coCartesian lift of α . Having a description of the p -coCartesian arrows, we can in particular describe the p -coCartesian lifts of the morphisms

$$\alpha^{j,\langle n \rangle}: \langle n \rangle_* \longrightarrow \langle 1 \rangle_*.$$

Given an object $(M_1, \dots, M_n) \in \mathcal{M}_{\langle n \rangle_*}^\otimes$, an associated p -coCartesian lift can be chosen to be

$$(\alpha^{j,\langle n \rangle}, \text{id}_{M_j}): (M_1, \dots, M_n) \longrightarrow M_j.$$

The associated functors $\alpha_1^{j,\langle n \rangle}: \mathcal{M}_{\langle n \rangle_*}^\otimes \longrightarrow \mathcal{M}_{\langle 1 \rangle_*}^\otimes$ are, up to natural isomorphism, projection functors and thus play a similar role to the inclusions of the intervals $\iota_{\{i-1, i\}}: [1] \longrightarrow [n]$ in the non-symmetric situation. In fact, there is the following proposition.

Proposition 4.3. *Let \mathcal{M} be a symmetric monoidal category, then $p: \mathcal{M}^\otimes \longrightarrow \mathcal{F}in_*$ is a Grothendieck opfibration and the functors $\alpha_1^{j,\langle n \rangle}: \mathcal{M}_{\langle n \rangle_*}^\otimes \longrightarrow \mathcal{M}_{\langle 1 \rangle_*}^\otimes \cong \mathcal{M}$ induce an isomorphism of categories*

$$\mathcal{M}_{\langle n \rangle_*}^\otimes \longrightarrow \mathcal{M}^{\times n}.$$

Conversely, given such a Grothendieck opfibration $p: \mathcal{M}^\otimes \longrightarrow \mathcal{F}in_$, there is a symmetric monoidal structure on $\mathcal{M} := \mathcal{M}_{\langle 1 \rangle_*}^\otimes$ encoded by p .*

We already saw the construction of the Grothendieck opfibration p associated to a symmetric monoidal category \mathcal{M} . Now we sketch a construction in the converse direction. Given a Grothendieck opfibration $p: \mathcal{M}^\otimes \longrightarrow \mathcal{F}in_*$ such that the $\alpha^{j,\langle n \rangle}$ induce isomorphisms

$$\mathcal{M}_{\langle n \rangle_*}^\otimes \xrightarrow{\cong} \mathcal{M}^{\times n},$$

we describe the main steps how to establish a symmetric monoidal structure on $\mathcal{M} = \mathcal{M}_{\langle 1 \rangle_*}^\otimes$. Consider the map m in $\mathcal{F}in_*$ determined by $m(1) = m(2) = 1$. Choices of p -coCartesian lifts of m induce a functor

$$\otimes := m_!: \mathcal{M}^{\times 2} \cong \mathcal{M}_{\langle 2 \rangle_*}^\otimes \xrightarrow{m_!} \mathcal{M}_{\langle 1 \rangle_*}^\otimes = \mathcal{M},$$

which is our candidate for the symmetric monoidal product. The symmetry isomorphism is obtained as follows. The *twist map* t in $\mathcal{F}in_*$ is the automorphism of $\langle 2 \rangle_*$ that interchanges 1 and 2. Since we have the equality $m = m \circ t: \langle 2 \rangle_* \longrightarrow \langle 1 \rangle_*$, we obtain the following natural isomorphism

$$\sigma: m_! \cong m_! \circ t_!: \mathcal{M}^{\times 2} \cong \mathcal{M}_{\langle 2 \rangle_*}^\otimes \longrightarrow \mathcal{M}_{\langle 1 \rangle_*}^\otimes = \mathcal{M}.$$

It is straightforward to verify that $t_!$ can be identified with the twist map on $\mathcal{M} \times \mathcal{M}$. Thus this natural isomorphism σ gives the intended symmetry constraint of the monoidal product \otimes :

$$\sigma_{M_1, M_2}: M_1 \otimes M_2 \xrightarrow{\cong} M_2 \otimes M_1.$$

Moreover, it is immediate from the axioms, that $\mathcal{M}_{\langle 0 \rangle_*}^\otimes$ is isomorphic to the terminal category $[0]$. For the unique map $u: \langle 0 \rangle_* \longrightarrow \langle 1 \rangle_*$ in $\mathcal{F}in_*$, the induced functor

$$\eta := u_!: \mathcal{M}_{\langle 0 \rangle_*}^\otimes \longrightarrow \mathcal{M}_{\langle 1 \rangle_*}^\otimes = \mathcal{M}$$

can thus be identified with an object $I \in \mathcal{M}$. In order to verify that this object behaves like a monoidal unit with respect to \otimes , we introduce the following notation. For $i = 1, 2$, let

$$\iota_i: \langle 1 \rangle_* \longrightarrow \langle 2 \rangle_*$$

be the unique morphism in $\mathcal{F}in_*$ sending $1 \in \langle 1 \rangle_*$ to $i \in \langle 2 \rangle_*$. Then it is straightforward to deduce that the induced functors $(\iota_i)_!: \mathcal{M} \longrightarrow \mathcal{M} \times \mathcal{M}$ can be identified with

$$(\iota_1)_!: M \mapsto (M, I) \quad \text{and} \quad (\iota_2)_!: M \mapsto (I, M).$$

The equalities $m \circ \iota_1 = \text{id}_{\langle 1 \rangle_*} = m \circ \iota_2$ give rise to natural isomorphisms $m_! \circ (\iota_1)_! \cong \text{id} \cong m_! \circ (\iota_2)_!$, i.e. we have the following isomorphisms which are natural in $M \in \mathcal{M}$:

$$\rho_M: M \otimes I \xrightarrow{\cong} M \quad \text{and} \quad \lambda_M: I \otimes M \xrightarrow{\cong} M.$$

By similar arguments, one can obtain the associativity constraints and establish the remaining coherence axioms. As in the non-symmetric case, we now turn this observation into a definition [Lur09c].

Definition 4.4. A *symmetric monoidal ∞ -category* is a coCartesian fibration $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\mathcal{F}in_*)$ such that, for all $n \geq 0$, the morphisms $\alpha^{j, \langle n \rangle}: \langle n \rangle_* \rightarrow \langle 1 \rangle_*$, $j = 1, \dots, n$, fit together to induce an equivalence of ∞ -categories

$$\mathcal{M}_{\langle n \rangle_*}^\otimes \xrightarrow{\sim} (\mathcal{M}_{\langle 1 \rangle_*}^\otimes)^{\times n}.$$

Remark 4.5. • Let $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}_\Delta(\mathcal{F}in_*)$ be as in the definition. Then one frequently abuses language in also calling the underlying ∞ -category $\mathcal{M} := \mathcal{M}_{\langle 1 \rangle_*}$ a symmetric monoidal ∞ -category.

• This definition of a symmetric monoidal ∞ -category $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\mathcal{F}in_*)$ really encodes a lot of structure on the underlying ∞ -category \mathcal{M} , namely that of a pairing which is *associative and commutative up to coherent homotopies*.

• In classical category theory, there is the so-called Grothendieck construction: Given a small category I together with a $\mathcal{C}at$ -valued functor

$$F: I \rightarrow \mathcal{C}at,$$

one can form a new category $\int F$ which ‘glues the categories $F(i)$ along the functors $F(\alpha)$ ’, for α a morphism in I . This Grothendieck construction $\int F$ is naturally endowed with a Grothendieck opfibration $p: \int F \rightarrow I$. Conversely, given such a Grothendieck opfibration $p: \mathcal{C} \rightarrow I$, one obtains a bifunctor $F: I \rightarrow \mathcal{C}at$ which associates to each object $i \in I$ the fiber \mathcal{C}_i of the Grothendieck opfibration over i . These two constructions can be seen to be inverse to each other in a certain precise sense. In [Lur09e, chapter 3], Lurie has generalized these constructions to the ∞ -categorical setting. Roughly speaking, he has established a result saying that a coCartesian fibration $p: \mathcal{C} \rightarrow \mathcal{D}$ is, in a certain sense, equivalent to giving a functor $\mathcal{D} \rightarrow \mathcal{C}at_\infty$. Applying this to our situation of symmetric monoidal ∞ -categories $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\mathcal{F}in_*)$, we see that the datum of such a p is equivalent to a functor $\mathbf{N}(\mathcal{F}in_*) \rightarrow \mathcal{C}at_\infty$. Informally, this can be read as saying that a symmetric monoidal ∞ -category can be interpreted as a *commutative monoid object* in $\mathcal{C}at_\infty$. For further evidence of this fact, observe that since the Grothendieck opfibration $p: \mathcal{M}^\otimes \rightarrow \mathbf{N}(\mathcal{F}in_*)$ has the property that the induced functors $\alpha_!^{j, \langle n \rangle}$ define an equivalence of ∞ -categories $\mathcal{M}_{\langle n \rangle_*}^\otimes \xrightarrow{\sim} (\mathcal{M}_{\langle 1 \rangle_*}^\otimes)^{\times n}$, a similar property is also shared by the associated functor $\mathbf{N}(\mathcal{F}in_*) \rightarrow \mathcal{C}at_\infty$ as expected from a commutative monoid object. Similar observations can also be made in the non-symmetric case where we end up with $\mathcal{C}at_\infty$ -valued monoids. In the classical situation of topological monoids such an approach is also described by Adams in [Ada78, section 2.5].

Before we give the definition of morphisms between symmetric monoidal ∞ -categories, we mention two expected results.

Proposition 4.6. • Let \mathcal{M} be an ordinary symmetric monoidal category and let $p: \mathcal{M}^\otimes \rightarrow \mathcal{F}in_*$ be the associated Grothendieck opfibration as discussed above. Then under the nerve construction, this gives us the underlying symmetric monoidal ∞ -category $\mathbf{N}(\mathcal{M})$:

$$\mathbf{N}(p): \mathbf{N}(\mathcal{M}^\otimes) \rightarrow \mathbf{N}(\mathcal{F}in_*).$$

- Given a symmetric monoidal ∞ -category $p: \mathcal{M}^\otimes \rightarrow \mathcal{N}(\mathcal{F}in_*)$, then the homotopy category $\mathrm{Ho}(\mathcal{M})$ of \mathcal{M} naturally carries the structure of an ordinary symmetric monoidal category.

Remark 4.7. Similarly to the situation of monoidal ∞ -categories, the first example can be generalized to certain results concerning the coherent nerves of simplicial model categories with suitably compatible symmetric monoidal structures. But this will not be treated here. Nevertheless, we want to mention, that, in contrast to the situation of monoidal ∞ -categories, the converse to this result is not known to be true. Thus given a presentable ∞ -category \mathcal{M} and a coCartesian fibration $p: \mathcal{M}^\otimes \rightarrow \mathcal{N}(\mathcal{F}in_*)$ which endows \mathcal{M} with the structure of a symmetric monoidal ∞ -category such that the monoidal product $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ commutes with colimits separately in each variable, it is not known whether, in general, \mathcal{M} is equivalent as a symmetric monoidal ∞ -category to the symmetric monoidal ∞ -category underlying a suitable simplicial and symmetric monoidal model category.

In the previous remark we used morphisms between symmetric monoidal ∞ -categories, so we should actually define them. Again, we saw in the classical situation, that given a symmetric monoidal category \mathcal{M} in the form of a Grothendieck opfibration

$$p: \mathcal{M}^\otimes \rightarrow \mathcal{F}in_*,$$

the whole monoidal structure is encoded by the p -coCartesian arrows [Lur09c].

Definition 4.8. Let $p: \mathcal{M}^\otimes \rightarrow \mathcal{N}(\mathcal{F}in_*)$ and $q: \mathcal{N}^\otimes \rightarrow \mathcal{N}(\mathcal{F}in_*)$ be symmetric monoidal ∞ -categories. Then a functor $F: \mathcal{M}^\otimes \rightarrow \mathcal{N}^\otimes$ over $\mathcal{N}(\mathcal{F}in_*)$ is called *symmetric monoidal* if F carries p -coCartesian arrows in \mathcal{M}^\otimes to q -coCartesian arrows in \mathcal{N}^\otimes . The full subcategory

$$\mathrm{Fun}^{s\mathrm{Mon}}(\mathcal{M}^\otimes, \mathcal{N}^\otimes) \subseteq \mathrm{Fun}(\mathcal{M}^\otimes, \mathcal{N}^\otimes)$$

is called the ∞ -category of symmetric monoidal functors from \mathcal{M}^\otimes to \mathcal{N}^\otimes .

Remark 4.9. Alternatively, these symmetric monoidal functors could be called *strongly* symmetric monoidal functors in order to emphasize that there also is a notion of *lax symmetric monoidal functors* [Lur09c].

4.2. Commutative algebra objects in symmetric monoidal ∞ -categories. Let \mathcal{M} be a symmetric monoidal category with monoidal unit $I \in \mathcal{M}$ and monoidal product \otimes . Let $p: \mathcal{M}^\otimes \rightarrow \mathcal{F}in_*$ be the associated Grothendieck opfibration. Then one might ask how to describe commutative algebra objects in terms of p . Again, it is a reasonable starting point to take a section

$$E: \mathcal{F}in_* \rightarrow \mathcal{M}^\otimes$$

of p . Under the identifications $\mathcal{M}_{\langle n \rangle_*}^\otimes \xrightarrow{\cong} (\mathcal{M}_{\langle 1 \rangle_*}^\otimes)^{\times n}$, we see that the value of such a section A at $\langle n \rangle_* \in \mathcal{F}in_*$ is determined by n objects of \mathcal{M} :

$$E_{\langle n \rangle_*} \leftrightarrow (E_1^n, \dots, E_n^n), \quad E_i^n \in \mathcal{M}.$$

Since a commutative algebra object E is, in particular, an object $E \in \mathcal{M}$ together with a multiplication map $E \otimes E \rightarrow E$, one wants to ensure that all components E_i^n of such a section $E: \mathcal{F}in_* \rightarrow \mathcal{M}^\otimes$ are isomorphic to the underlying object $E_1^1 \in \mathcal{M}$. Hence, we start by axiomatizing those morphisms $\alpha: \langle n \rangle_* \rightarrow \langle k \rangle_*$ in $\mathcal{F}in_*$, such that the induced functors $\alpha_!: \mathcal{M}_{\langle n \rangle_*}^\otimes \rightarrow \mathcal{M}_{\langle k \rangle_*}^\otimes$ are under the standard identifications just projection and permutation functors.

Definition 4.10. Let $\alpha: \langle n \rangle_* \rightarrow \langle k \rangle_*$ be a morphism in $\mathcal{F}in_*$. Then call α *collapsing* if for each $i \in \langle k \rangle_*$, $i \neq *$, there is a unique $j \in \langle n \rangle_*$ such that $\alpha(j) = i$.

As an example, let us consider the morphisms $\alpha^{j,\langle n \rangle}: \langle n \rangle_* \rightarrow \langle 1 \rangle_*$ introduced above. Since $j \in \langle n \rangle_*$ is by definition the only element of $\langle n \rangle_*$ which is not mapped to $*$, these morphisms are collapsing. In the last subsection, we described the structure of the p -coCartesian arrows of the Grothendieck opfibration $p: \mathcal{M}^\otimes \rightarrow \mathcal{F}in_*$. Recall that given an object (M_1, \dots, M_n) in $\mathcal{M}_{\langle n \rangle_*}^\otimes$, a p -coCartesian lift of $\alpha^{j,\langle n \rangle}$ starting at (M_1, \dots, M_n) can be chosen to be the morphism

$$(\alpha^{j,\langle n \rangle}, \text{id}_{M_j}): (M_1, \dots, M_n) \rightarrow M_j.$$

Thus the induced functor $\alpha_1^{j,\langle n \rangle}$ gives, under the standard identification, up to natural isomorphism, the projection on the j -th factor. Similarly, one checks that a general collapsing morphism induces a corresponding projection and permutation functor.

Let us now come back to the section $E: \mathcal{F}in_* \rightarrow \mathcal{M}^\otimes$ of the Grothendieck opfibration p which we want to encode the algebra structure. By the above discussion, it is now straightforward to check that in this picture the commutative algebra objects of \mathcal{M} are precisely encoded by those sections which send collapsing morphisms in $\mathcal{F}in_*$ to p -coCartesian arrows in \mathcal{M}^\otimes . Thus the following definition is reasonable [Lur09c].

Definition 4.11. Let $p: \mathcal{M}^\otimes \rightarrow \mathcal{N}(\mathcal{F}in_*)$ be a symmetric monoidal ∞ -category. Then a *commutative algebra object* E of \mathcal{M}^\otimes is a section $E: \mathcal{N}(\mathcal{F}in_*) \rightarrow \mathcal{M}^\otimes$ of p which sends collapsing morphisms in $\mathcal{N}(\mathcal{F}in_*)$ to p -coCartesian arrows in \mathcal{M}^\otimes . Given such a commutative algebra object E , one calls $E_{\langle 1 \rangle_*} \in \mathcal{M} := \mathcal{M}_{\langle 1 \rangle_*}^\otimes$ the *underlying object of the algebra object*.

Remark 4.12. • If the symmetric monoidal structure on an ∞ -category \mathcal{M} is understood, one simply speaks of commutative algebra objects of \mathcal{M} .

• Similarly to the non-symmetric situation, it is easy to see that given a symmetric monoidal ∞ -category $p: \mathcal{M}^\otimes \rightarrow \mathcal{N}(\mathcal{F}in_*)$, the commutative algebra objects of \mathcal{M} are naturally organized into an ∞ -category $\text{CAlg}(\mathcal{M})$, called the *∞ -category of commutative algebra objects in \mathcal{M}* . Namely, one takes the full subcategory of $\text{Map}_{\mathcal{N}(\mathcal{F}in_*)}(\mathcal{N}(\mathcal{F}in_*), \mathcal{M}^\otimes)$ spanned by the commutative algebra objects.

• As in the case of symmetric monoidal ∞ -categories themselves, the following remark is in order. Given a commutative algebra object $E \in \text{CAlg}(\mathcal{M})$, then this section E encodes quite a lot of structure on the underlying object, namely that of a multiplication which is *associative and commutative up to coherent homotopy*.

Obviously, given the notion of commutative algebra objects, one would now like to talk about module objects over a fixed commutative algebra object and similarly in the non-symmetric monoidal case. More generally, one would like to talk about ∞ -categories, which are *tensoring* over a given (symmetric) monoidal ∞ -category. Such a theory exists and is apparently due to Lurie, who developed this with an amazing amount of detail in [Lur09b, Lur09c]. In particular, he also describes which completeness and cocompleteness properties of a symmetric monoidal ∞ -category \mathcal{M} are inherited (under possibly further assumptions on \mathcal{M}) by $\text{CAlg}(\mathcal{M})$ and by the ∞ -category of E -modules for a fixed commutative algebra object E . Moreover, he also gives sufficient conditions ensuring that the coherent nerve \mathcal{N}_Δ commutes with the formation of commutative algebra objects for nice simplicial and symmetric monoidal model categories. But instead of giving an introduction to these results, we will content ourselves with a description of how to forget structure in a symmetric monoidal situation in order to obtain a non-symmetric monoidal situation. For this purpose, we construct the following functor

$$\phi: \Delta^{op} \rightarrow \mathcal{F}in_*.$$

On objects, ϕ is simply defined to send $[n] \in \Delta^{op}$ to $\langle n \rangle_* \in \mathcal{F}in_*$. Let now $\alpha: [k] \rightarrow [n]$ be a morphism in Δ . Then the induced map $\phi(\alpha): \langle n \rangle_* \rightarrow \langle k \rangle_*$ is defined by

$$\phi(\alpha)(j) = \begin{cases} i & \text{if there is an } i \text{ such that } j \in [\alpha(i-1) + 1, \alpha(i)], \\ * & \text{else.} \end{cases}$$

Since α is monotone, such an i is unique if it exists, and considering easy examples one can convince oneself that this definition is the one we were heading for.

Example 4.13. Let again $\iota_{\{i-1, i\}}: [1] \rightarrow [n]$ denote the inclusion of an interval, then it is straightforward to check that we have $\phi(\iota_{\{i-1, i\}}) = \alpha^{i, \langle n \rangle}: \langle n \rangle_* \rightarrow \langle 1 \rangle_*$ with $\alpha^{i, \langle n \rangle}$ as above.

This example gives an explanation why the maps $\alpha^{i, \langle n \rangle}$ play a role in the theory of symmetric monoidal ∞ -category similar to that of the inclusions of intervals $\iota_{\{i-1, i\}}$ in the theory of monoidal ∞ -categories. With a view towards algebra objects, we already record the following easily established result.

Lemma 4.14. *Let $\alpha: [k] \rightarrow [n]$ be a morphism in Δ . Then the morphism $\phi(\alpha): \langle n \rangle_* \rightarrow \langle k \rangle_*$ in $\mathcal{F}in_*$ is collapsing if and only if the morphism $\alpha: [k] \rightarrow [n]$ in Δ is convex.*

Applying the nerve construction to ϕ , we obtain an induced map $N(\phi): N(\Delta^{op}) \rightarrow N(\mathcal{F}in_*)$ of simplicial sets and make the following definition.

Definition 4.15. Let $p: \mathcal{M}^\otimes \rightarrow N(\mathcal{F}in_*)$ be a symmetric monoidal ∞ -category, then the *monoidal ∞ -category $U(\mathcal{M})$ underlying \mathcal{M}* is defined to be monoidal ∞ -category $q: U(\mathcal{M}) \rightarrow N(\Delta^{op})$ which occurs in the following pullback diagram:

$$\begin{array}{ccc} \mathcal{M}^\otimes & \times_{N(\mathcal{F}in_*)} & N(\Delta^{op}) & \longrightarrow & \mathcal{M}^\otimes \\ & q \downarrow & & & \downarrow p \\ & N(\Delta^{op}) & \xrightarrow{N(\phi)} & & N(\mathcal{F}in_*) \end{array}$$

Similarly, given a commutative algebra object $E \in \text{CAlg}(\mathcal{M})$ for a symmetric monoidal ∞ -category \mathcal{M} , using the universal property of the pullback, the map $E \circ \phi: N(\Delta^{op}) \rightarrow \mathcal{M}^\otimes$ induces a section $U(E)$ of $q: U(\mathcal{M}) \rightarrow N(\Delta^{op})$. By Lemma 4.14, this section is obviously an algebra object of the underlying monoidal structure $U(\mathcal{M})$. Thus given a symmetric monoidal ∞ -category, we obtain a canonical forgetful functor

$$U: \text{CAlg}(\mathcal{M}) \rightarrow \text{Alg}(U(\mathcal{M}))$$

from commutative algebra objects to algebra objects.

4.3. Dualizable objects in symmetric monoidal ∞ -categories. In this subsection we discuss the notion of (left, right) dualizable objects in (symmetric) monoidal categories and extend these notions in a straightforward way to the setting of (symmetric) monoidal ∞ -categories. The notions are motivated by the following example.

Example 4.16. Let $\text{Vect}(k)$ denote the symmetric monoidal category of vector spaces over a field k with the tensor product $\otimes = \otimes_k$ as monoidal product and the ground field k as monoidal unit. For an arbitrary vector space $V \in \text{Vect}(k)$ denote by V^* the dual vector space: $V^* = \text{hom}_k(V, k)$. Then there is the *evaluation map*

$$\text{ev}_V: V \otimes V^* \rightarrow k: v \otimes \alpha \mapsto \alpha(v).$$

Dually, given a *finite-dimensional* vector space V , one can define the so-called *coevaluation map* $\text{coev}_V: k \rightarrow V^* \otimes V$. This map can for example be obtained by choosing a (finite!) basis (v_1, \dots, v_n) of V and by setting

$$\text{coev}_V: k \rightarrow V^* \otimes V: 1 \mapsto \sum_{i=1}^n v^i \otimes v_i,$$

where (v^1, \dots, v^n) is the dual basis of (v_1, \dots, v_n) . This map coev_V is independent of the choice of basis of V and it is straightforward to check, that coev_V and ev_V satisfy the following two relations:

$$\begin{aligned} \text{id}_V: V \cong V \otimes k &\xrightarrow{V \otimes \text{coev}_V} V \otimes V^* \otimes V \xrightarrow{\text{ev}_V \otimes V} k \otimes V \cong V, \\ \text{id}_{V^*}: V^* \cong k \otimes V^* &\xrightarrow{\text{coev}_V \otimes V^*} V^* \otimes V \otimes V^* \xrightarrow{V^* \otimes \text{ev}_V} V^* \otimes k \cong V^*. \end{aligned}$$

This observation motivates the following definition.

Definition 4.17. Let \mathcal{M} be a monoidal category with monoidal product \otimes and monoidal unit $I \in \mathcal{M}$. Then a pair (X, Y) of objects $X, Y \in \mathcal{M}$ is called a *dual pair* if there are morphisms

$$\eta: I \rightarrow Y \otimes X \quad \text{and} \quad \epsilon: X \otimes Y \rightarrow I,$$

such that the following relations are satisfied:

$$\begin{aligned} \text{id}_X: X \cong X \otimes I &\xrightarrow{X \otimes \eta} X \otimes Y \otimes X \xrightarrow{\epsilon \otimes X} I \otimes X \cong X, \\ \text{id}_Y: Y \cong I \otimes Y &\xrightarrow{\eta \otimes Y} Y \otimes X \otimes Y \xrightarrow{Y \otimes \epsilon} Y \otimes I \cong Y. \end{aligned}$$

In such a situation, call X a *left dual to Y* and call Y a *right dual to X* .

Remark 4.18. • The two relations coming up in the definition are very similar to the *triangular identities* [ML98, p.82] one is used to from adjunctions. And in fact, this similarity can be made precise. Given a monoidal category \mathcal{M} , one can consider the associated bicategory $\text{B}\mathcal{M}$ with one object $*$ and the endomorphism category $\text{Hom}_{\text{B}\mathcal{M}}(*, *)$ given by \mathcal{M} . Recall that a *bicategory* is similar to a 2-category with the difference that the composition is only unital and associative up to coherent isomorphisms (thus the construction B associates a bicategory to a monoidal category, while it associates a 2-category to a strict monoidal category). In any bicategory, one can introduce the notion of adjoint 1-morphisms by turning the obvious triangular identities into a definition. With that notion of adjoints we have: two objects $X, Y \in \mathcal{M}$ define a *dual pair* (X, Y) in \mathcal{M} if and only if considered as 1-morphisms $X, Y \in \text{Hom}_{\text{B}\mathcal{M}}(*, *)$ they define an *adjoint pair* (X, Y) in $\text{B}\mathcal{M}$. • With this last remark in mind, the coevaluation map coev of the above example thus plays the role of the unit η , while the evaluation map ev plays the role of the counit ϵ .

In light of the last remark, it follows immediately that left and right duals are unique up to unique isomorphism since this is the case for adjoint morphisms in bicategories. But this can of course also be proved directly by establishing the following lemma.

Lemma 4.19. *Let \mathcal{M} be a monoidal category and let (X, Y) be a pair of dual objects in \mathcal{M} . Then we have the following two adjunctions:*

$$(X \otimes (-), Y \otimes (-)): \mathcal{M} \rightarrow \mathcal{M}, \quad ((-) \otimes Y, (-) \otimes X): \mathcal{M} \rightarrow \mathcal{M}.$$

Corollary 4.20. i) Let \mathcal{M} be a monoidal category and let $Y \in \mathcal{M}$. If X_1, X_2 are left dual objects to Y , then X_1 and X_2 are canonically isomorphic.

ii) Let \mathcal{M} be a monoidal category and let $X \in \mathcal{M}$. If Y_1, Y_2 are right dual objects to X , then Y_1 and Y_2 are canonically isomorphic.

In fact, left dual objects to Y are corepresenting objects for the functor $\mathrm{hom}_{\mathcal{M}}(I, Y \otimes (-))$, while right dual objects to X corepresent the functor $\mathrm{hom}_{\mathcal{M}}(I, (-) \otimes X)$.

Remark 4.21. In the case of a *symmetric* monoidal category \mathcal{M} , a pair (X, Y) of objects in \mathcal{M} is a dual pair if and only if the pair (Y, X) is one. Thus in the symmetric context one simply speaks of *dual* and *dualizable objects*.

Definition 4.22. Let \mathcal{M} be a monoidal category. Then one says that \mathcal{M} *has dual objects* if for every object $X \in \mathcal{M}$ there is a left and a right dual object to X .

Example 4.23. With this terminology, a vector space $V \in \mathrm{Vect}(k)$ has a dual if and only if V is finite-dimensional. Thus the monoidal category $\mathrm{Vect}(k)$ does not have dual objects, while the monoidal category $\mathrm{Vect}^{\mathrm{fd}}(k)$ of finite-dimensional vector spaces has dual objects.

Since the homotopy category $\mathrm{Ho}(\mathcal{M})$ of a (symmetric) monoidal ∞ -category \mathcal{M} carries canonically the structure of a (symmetric) monoidal category, the above concepts can be extended to the ∞ -categorical setting. We only treat the case of symmetric monoidal ∞ -categories.

Definition 4.24. Let $p: \mathcal{M}^{\otimes} \rightarrow \mathcal{N}(\mathcal{F}in_*)$ be a symmetric monoidal ∞ -category. Then an object X of \mathcal{M} is called *dualizable* if it is a dualizable object when considered as an object of the homotopy category $\mathrm{Ho}(\mathcal{M})$. Say that the symmetric monoidal ∞ -category \mathcal{M} *has duals* if the homotopy category $\mathrm{Ho}(\mathcal{M})$ with the induced symmetric monoidal structure has duals.

Since the concept of dualizable objects is a natural one, making use only of categorical terms and the symmetric monoidal structure, the following proposition is immediate.

Proposition 4.25. *Let $p: \mathcal{M}^{\otimes} \rightarrow \mathcal{N}(\mathcal{F}in_*)$ be a symmetric monoidal ∞ -category. Then there is another symmetric monoidal category $\mathcal{M}^{\mathrm{fd}}$ and a symmetric monoidal functor $\iota: \mathcal{M}^{\mathrm{fd}} \rightarrow \mathcal{M}$ with the following properties:*

- i) *The symmetric monoidal ∞ -category $\mathcal{M}^{\mathrm{fd}}$ has duals.*
- ii) *For every symmetric monoidal ∞ -category \mathcal{N} which has duals and for every symmetric monoidal functor $F: \mathcal{N} \rightarrow \mathcal{M}$ there exists a functor $f: \mathcal{N} \rightarrow \mathcal{M}^{\mathrm{fd}}$ and an equivalence $F \cong \iota \circ f$.*

$$\begin{array}{ccc}
 \mathcal{N} & \xrightarrow{F} & \mathcal{M} \\
 & \searrow f & \downarrow \Downarrow \\
 & & \mathcal{M}^{\mathrm{fd}} \\
 & & \nearrow \iota
 \end{array}$$

Moreover, such a functor f is unique up to equivalence.

In fact, this can be achieved if one takes for ι the full subcategory $\iota: \mathcal{M}^{\mathrm{fd}} \subseteq \mathcal{M}$ of \mathcal{M} spanned by the dualizable objects.

Remark 4.26. • In these notes, we are only concerned with a specific model for a theory of $(\infty, 1)$ -categories, namely the ∞ -categories, in contrast to the more general theories of (∞, n) -categories for $n \geq 2$. For these theories, there is apparently also an adapted theory of symmetric monoidal (∞, n) -categories. Given a symmetric monoidal (∞, n) -category \mathcal{M} , Lurie [Lur09d] introduces the stronger notion of *fully dualizable objects* in \mathcal{M} and, in this more general situation, he also establishes an analogue to the above proposition. Now, for a symmetric monoidal ∞ -category \mathcal{M} , an object $X \in \mathcal{M}$ is dualizable if and only if it is fully dualizable, which motivates the notation $\mathcal{M}^{\mathrm{fd}}$ chosen for the symmetric monoidal ∞ -category occurring in the above proposition.

- The fully dualizable objects play a central role in Lurie’s classification of topological field theories

[Lur09d]. Roughly speaking, given a fully dualizable object X in a symmetric monoidal (∞, n) -category \mathcal{M} , there is a \mathcal{M} -valued topological field Z such that $Z(*) \simeq X$. Conversely, given a \mathcal{M} -valued topological field theory Z , then the value $Z(*)$ of Z at the point $*$ is a fully dualizable object of \mathcal{M} . For a precise statement and a (quite long) sketch of a proof see [Lur09d].

5. SPECTRA AND THE SMASH PRODUCT FROM THE PERSPECTIVE OF ∞ -CATEGORIES

In this section, we want to sketch Lurie's approach to the theory of spectra and, more generally, the stabilization process in the world of ∞ -categories. The central notion is that of a stable ∞ -category [Lur09a] which is the ∞ -categorical analogue of a stable model category [Hov99, chapter 7] or a triangulated derivator [Mal01, Fra96, Kel91]. We then turn to the important class of presentable stable ∞ -categories and see from a very conceptual perspective how the smash product enters the scene [Lur09b].

5.1. Stable ∞ -categories, the process of stabilization and the ∞ -category of spectra. In order to speak about exact and coexact triangles in ∞ -categories, we first need the notion of pointed ∞ -categories.

Definition 5.1. Let \mathcal{C} be an ∞ -category. Then call \mathcal{C} *pointed*, if there is a *zero object* in \mathcal{C} , i.e. an object $0 \in \mathcal{C}$, which is initial and final.

Thus, an ∞ -category is pointed if there is an object $0 \in \mathcal{C}$ such that for all $X \in \mathcal{C}$ the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, 0)$ and $\mathrm{Map}_{\mathcal{C}}(0, X)$ are contractible. It follows, that for any two objects X, Y there is a zero map

$$0 = 0_{X,Y}: X \longrightarrow Y,$$

well-defined up to a contractible choice. Again by Proposition 2.12, if an ∞ -category \mathcal{C} is pointed, then the full subcategory spanned by the zero objects is a contractible Kan complex. Let \mathcal{C} now be a pointed ∞ -category. A *triangle* τ in \mathcal{C} is a diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z, \end{array}$$

more precisely the triangles are given by this full subcategory of the category of squares in \mathcal{C} , i.e. of $\mathrm{Fun}((\Delta^1)^2, \mathcal{C})$. Thus a triangle in \mathcal{C} encodes two composable arrows g and f , a further arrow h together with a homotopy $h \simeq g \circ f$ and a *null-homotopy* $h \simeq 0$. Recall from Subsection 2.2 that the square $(\Delta^1)^2$ can be interpreted as a left resp. right cone as follows:

$$(\Lambda_0^2)^{\triangleright} \cong (\Delta^1)^2 \cong (\Lambda_2^2)^{\triangleleft}.$$

Definition 5.2. Let \mathcal{C} be a pointed ∞ -category. Then a triangle τ in \mathcal{C} is called *exact*, if the diagram $\tau: (\Lambda_2^2)^{\triangleleft} \longrightarrow \mathcal{C}$ is a limit diagram in \mathcal{C} . Dually, call a triangle τ in \mathcal{C} *coexact*, if the diagram $\tau: (\Lambda_0^2)^{\triangleright} \longrightarrow \mathcal{C}$ is a colimit diagram in \mathcal{C} .

For simplicity, we assume for the rest of this subsection, that, unless otherwise stated, all ∞ -categories \mathcal{C} are finitely bicomplete. Further, we leave it to the reader to find the minimal assumptions needed to perform the constructions. Given such a pointed \mathcal{C} , denote by

$$\mathcal{M}^{\Sigma} \subseteq \mathrm{Fun}((\Delta^1)^2, \mathcal{C})$$

the full subcategory spanned by the coexact triangles of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y, \end{array}$$

where 0 and $0'$ are zero objects in \mathcal{C} . Dually, denote by

$$\mathcal{M}^\Omega \subseteq \text{Fun}((\Delta^1)^2, \mathcal{C})$$

the full subcategory spanned by the exact triangles of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y, \end{array}$$

with $0, 0'$ again zeros. Morally, such diagrams should be determined by the object in the upper left corner in the first and by the object in the lower right corner in the second case. More precisely, there is the following result.

Proposition 5.3. *Let \mathcal{C} be a finitely bicomplete, pointed ∞ -category, then the evaluation maps*

$$\text{ev}_{(0,0)}: \mathcal{M}^\Sigma \longrightarrow \mathcal{C} \quad \text{and} \quad \text{ev}_{(1,1)}: \mathcal{M}^\Omega \longrightarrow \mathcal{C}$$

are acyclic Kan fibrations.

This proposition follows from a general criterion established in [Lur09e] on when certain maps of simplicial sets are acyclic fibrations. Since we will apply this criterion only to cases where one feels that the map under consideration must be an acyclic fibration, we will not give any details here but instead refer to [Lur09e, Proposition 4.3.2.15] where the precise statement and a proof can be found. Thus under the assumption of the proposition, we can choose sections

$$s_\Sigma: \mathcal{C} \longrightarrow \mathcal{M}^\Sigma, \quad \text{resp.} \quad s_\Omega: \mathcal{C} \longrightarrow \mathcal{M}^\Omega$$

of $\text{ev}_{(0,0)}$, resp. $\text{ev}_{(1,1)}$ and can use these to obtain suspension resp. loop functors.

Definition 5.4. Let \mathcal{C} be a finitely bicomplete, pointed ∞ -category. Then the *suspension functor* $\Sigma = \Sigma_{\mathcal{C}}$ is defined to be the composition

$$\Sigma: \mathcal{C} \xrightarrow{s_\Sigma} \mathcal{M}^\Sigma \xrightarrow{\text{ev}_{(1,1)}} \mathcal{C}.$$

Similarly, the *loop functor* $\Omega = \Omega_{\mathcal{C}}$ is obtained as the composition

$$\Omega: \mathcal{C} \xrightarrow{s_\Omega} \mathcal{M}^\Omega \xrightarrow{\text{ev}_{(0,0)}} \mathcal{C}.$$

Remark 5.5. Since we chose sections of the acyclic fibrations $\text{ev}_{(0,0)}$, resp. $\text{ev}_{(1,1)}$ in the definition of the suspension resp. loop functor, these are not well-defined on the nose but only well-defined up to a contractible choice which is fine for all ∞ -categorical purposes.

Although we have not talked about adjunctions in the world of ∞ -categories, we nevertheless want to mention the following proposition.

Proposition 5.6. *Let \mathcal{C} be a finitely bicomplete, pointed ∞ -category. Then the suspension functor and the loop functor are an adjoint pair:*

$$(\Sigma, \Omega): \mathcal{C} \dashv \mathcal{C}.$$

The central notion of stable ∞ -categories captures axiomatically the situation that this adjunction is in fact an equivalence.

Definition 5.7. Let \mathcal{C} be an ∞ -category. Then call \mathcal{C} *stable*, if \mathcal{C} is finitely bicomplete, pointed and if a triangle in \mathcal{C} is exact if and only if it is coexact.

Thus, by the very definition of stable ∞ -categories, we have for \mathcal{C} stable that $\mathcal{M}^\Sigma = \mathcal{M}^\Omega$. From the diagram

$$\begin{array}{ccccc} X & & \mathcal{C} & \xrightarrow{s_\Sigma} & \mathcal{M}^\Sigma & \xrightarrow{\text{ev}(1,1)} & \mathcal{C} & & \Sigma\Omega(Y) \\ \downarrow & & \parallel & & \parallel & & \parallel & & \uparrow \\ \Omega\Sigma(X) & & \mathcal{C} & \xleftarrow{\text{ev}(0,0)} & \mathcal{M}^\Omega & \xleftarrow{s_\Omega} & \mathcal{C} & & Y \end{array}$$

one deduces the following result.

Proposition 5.8. *Let \mathcal{C} be a stable ∞ -category, then the suspension functor and the loop functor are a pair of inverse equivalences $(\Sigma, \Omega): \mathcal{C} \xrightarrow{\cong} \mathcal{C}$.*

Remark 5.9. Recall the following fact from classical category theory. Let \mathcal{C} be a category and consider a diagram in \mathcal{C} which has the following shape

$$\begin{array}{ccccc} X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 \\ \downarrow & & \downarrow & & \downarrow \\ X_2 & \longrightarrow & Y_2 & \longrightarrow & Z_2. \end{array}$$

Then if the left and the right squares are pullback squares so is the compound diagram. Conversely, if the right square and the compound square are pullbacks, then this is also the case for the left square. Dual results hold for pushout diagrams. In the setting of ∞ -categories, the corresponding results are also true. It can be seen that in a stable ∞ -category a square is a pullback diagram if and only if it is a pushout square. Thus in stable ∞ -categories, we have the 2-out-of-3-property for pullbacks and pushouts.

As one is used to from stable model categories, one also expects the homotopy category $\text{Ho}(\mathcal{C})$ of a stable ∞ -category \mathcal{C} to be a triangulated category. For this purpose, consider diagrams in $\text{Ho}(\mathcal{C})$ of the following shape

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X.$$

Call such a diagram a *distinguished triangle* if in \mathcal{C} there is a diagram

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow \tilde{g} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\tilde{h}} & U \end{array}$$

such that the squares are pushout diagrams, $0, 0'$ are zero objects, \tilde{f} , resp. \tilde{g} represent f , resp. g and h can be identified under the induced isomorphism $U \cong \Sigma X$ with the homotopy class of \tilde{h} . We remark that the induced isomorphism is obtained from the fact that the compound diagram is also a pushout diagram (cf. Remark 5.9). In [Lur09a], Lurie establishes the following result.

Theorem 5.10. *Let \mathcal{C} be a stable ∞ -category. Then with the above class of distinguished triangles, the homotopy category $\mathrm{Ho}(\mathcal{C})$ is a triangulated category.*

The natural class of functors between stable ∞ -categories is the class of exact functors.

Definition 5.11. Let \mathcal{C} and \mathcal{D} be stable ∞ -categories and let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor mapping zero objects in \mathcal{C} to zero objects in \mathcal{D} . Then call F an *exact functor* if F carries exact triangles in \mathcal{C} to exact triangles in \mathcal{D} .

Remark 5.12. • Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between stable ∞ -categories. Then we have the following equivalent statements:

$$F \text{ is exact} \iff F \text{ is left exact} \iff F \text{ is right exact}$$

- Obviously, the identity functors and compositions of exact functors are exact.

Now we come to the stabilization process which is given by considering spectrum objects in a given nice ∞ -category. Similar work was done in the world of model categories for example by Schwede [Sch97] and Hovey [Hov01]. In the world of ∞ -categories, this was carried out by Lurie [Lur09a] and all results from this subsection are taken from this reference.

Definition 5.13. Let \mathcal{C} be a finitely bicomplete, pointed ∞ -category. Then a *prespectrum object* in \mathcal{C} is a functor

$$X: \mathbb{N}(\mathbb{Z} \times \mathbb{Z}) \rightarrow \mathcal{C}$$

such that for all $i \neq j$ the value $X(i, j)$ is a zero object in \mathcal{C} . As a full subcategory of a functor category, the prespectrum objects form an ∞ -category which is denoted by $\mathrm{PSp}(\mathcal{C})$.

Here, \mathbb{Z} is considered as a category using the standard ordering on \mathbb{Z} . Since only the diagonal entries may differ from zero objects, we use from now on the shorthand notation X_n for $X(n, n)$. Graphically, a part of such a prespectrum object X can be depicted as in the next diagram.

$$\begin{array}{ccccc}
 & & 0 & \longrightarrow & X_{n+1} \\
 & & \uparrow & & \uparrow \\
 & & 0'' & \longrightarrow & X_n & \longrightarrow & 0' \\
 & & \uparrow & & \uparrow & & \uparrow \\
 X_{n-1} & \longrightarrow & 0''' & & & &
 \end{array}$$

By definition of the suspension functor Σ and the loop functor Ω , given a prespectrum object X we obtain the following induced morphisms:

$$\Sigma X_{n-1} \longrightarrow X_n \quad \text{and} \quad X_n \longrightarrow \Omega X_{n+1}.$$

Definition 5.14. Let \mathcal{C} be a finitely bicomplete, pointed ∞ -category and let $X \in \mathrm{PSp}(\mathcal{C})$, then:

- X is called a *spectrum object* if all induced maps $X_n \xrightarrow{\sim} \Omega X_{n+1}$ are equivalences in \mathcal{C} . The full subcategory of $\mathrm{PSp}(\mathcal{C})$ spanned by the spectrum objects is denoted by $\mathrm{Sp}(\mathcal{C})$.
- X is called a *spectrum below n* if the induced maps $X_{m-1} \xrightarrow{\sim} \Omega X_m$ are equivalences in \mathcal{C} for all $m \leq n$. The full subcategory of $\mathrm{PSp}(\mathcal{C})$ spanned by the spectra below n is denoted by $\mathrm{Sp}_n(\mathcal{C})$.

Remark 5.15. Let \mathcal{C} be a finitely bicomplete, pointed ∞ -category, then it can be shown that the ∞ -category $\mathrm{Sp}(\mathcal{C})$ of spectrum objects in \mathcal{C} can be identified with the homotopy inverse limit of

$$\dots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}.$$

Thus $\mathrm{Sp}(\mathcal{C})$ is in fact given by an ∞ -category of *infinite loop objects* in \mathcal{C} [Lur09a].

Let \mathcal{C} be a finitely bicomplete ∞ -category and let $*$ $\in \mathcal{C}$ be a final object. Then denote by \mathcal{C}_* the ∞ -category of *pointed objects* in \mathcal{C} , i.e. $\mathcal{C}_* = \mathcal{C}_{*/}$.

Definition 5.16. Let \mathcal{C} be a finitely bicomplete ∞ -category, then the *stabilization* $\mathrm{Stab}(\mathcal{C})$ of \mathcal{C} is defined as

$$\mathrm{Stab}(\mathcal{C}) = \mathrm{Sp}(\mathcal{C}_*).$$

We now come to one of the most important examples.

Example 5.17. Recall that one model for an ∞ -category \mathcal{S} of spaces can be obtained in the following way. Let $\mathcal{Kan} \subseteq \mathrm{Set}_\Delta$ be the full subcategory spanned by the Kan complexes, then $\mathcal{S} = \mathcal{N}_\Delta(\mathcal{Kan})$. The *∞ -category of spectra* Sp is defined by $\mathrm{Sp} = \mathrm{Stab}(\mathcal{S}) = \mathrm{Sp}(\mathcal{S}_*)$.

Let us come back to the general situation. We denote by $\iota: \mathrm{Sp}(\mathcal{C}) \hookrightarrow \mathrm{P}\mathrm{Sp}(\mathcal{C})$ the inclusion. The next aim is to obtain a left adjoint to this inclusion, i.e. to find a *spectrification functor* $\mathrm{P}\mathrm{Sp}(\mathcal{C}) \rightarrow \mathrm{Sp}(\mathcal{C})$.

Proposition 5.18. *Let \mathcal{C} be a finitely bicomplete, pointed ∞ -category, then we have an adjunction*

$$(L_n, \iota): \mathrm{P}\mathrm{Sp}(\mathcal{C}) \rightarrow \mathrm{Sp}_n(\mathcal{C}).$$

The idea is of course that spectra below a certain level are somehow determined by the higher levels. And in fact, the left adjoint L_n is defined as follows: first restrict a prespectrum X to

$$Q_n = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid i \neq j \text{ or } i = j \geq n\}$$

and then set

$$L_n(X) = \mathrm{R}\mathrm{Kan}_{Q_n \hookrightarrow \mathbb{N}(\mathbb{Z} \times \mathbb{Z})}(X|_{Q_n}).$$

Here $\mathrm{R}\mathrm{Kan}$ stands for an ∞ -categorical variant of the usual right Kan extension whose existence we take for granted [Lur09e, section 4.3]. Under suitable completeness assumptions on the ∞ -categories involved, right Kan extensions can again be calculated pointwise, i.e. are given by limits over certain slice categories. In our situation, the corresponding slice categories are cofinally finite, thus our assumption on the ∞ -category \mathcal{C} to be *finitely* bicomplete allows for this construction. By definition, the essential image of L_n consists of the spectra below n . With a bit more care, one can now observe that there is a sequence of functors

$$\mathrm{id} \rightarrow L_0 \rightarrow L_1 \rightarrow L_2 \rightarrow \dots : \mathrm{P}\mathrm{Sp}(\mathcal{C}) \rightarrow \mathrm{P}\mathrm{Sp}(\mathcal{C})$$

and it is tempting to set $L := \mathrm{colim}_n L_n$. This in fact works if one imposes some mild conditions on \mathcal{C} .

Proposition 5.19. *Let \mathcal{C} be a pointed ∞ -category with finite limits and countable colimits. If $\Omega_{\mathcal{C}}$ commutes with sequential colimits, then*

$$L := \mathrm{colim}_n L_n : \mathrm{P}\mathrm{Sp}(\mathcal{C}) \rightarrow \mathrm{P}\mathrm{Sp}(\mathcal{C})$$

is a localization with essential image $\mathrm{Sp}(\mathcal{C})$. Under these assumptions, call L the spectrification functor.

One of the very important examples to which this proposition can be applied is the ∞ -category of pointed spaces: $\mathcal{C} = \mathbf{S}_*$. In this case, let $\mathcal{D}_n \subseteq \mathbf{Sp}_n$ be the full subcategory of the spectra below n spanned by those X which satisfy

$$\Sigma X_m \xrightarrow{\simeq} X_{m+1}, \quad m \geq n.$$

Thus, morally, such a prespectrum $X \in \mathcal{D}_n$ is formally determined by its n -th value. And in fact, the evaluation map $\text{ev}_n: \mathcal{D}_n \rightarrow \mathbf{S}_*$ can be shown to be an acyclic fibration. A proof of this makes again use of the already mentioned criterion on when certain maps of simplicial sets are acyclic fibrations and can be found in [Lur09a]. Choose a section $s_{\tilde{\Sigma}^{\infty-n}}: \mathbf{S}_* \rightarrow \mathcal{D}_n$ of ev_n and set

$$\tilde{\Sigma}^{\infty-n}: \mathbf{S}_* \xrightarrow{s_{\tilde{\Sigma}^{\infty-n}}} \mathcal{D}_n \rightarrow \mathbf{PSp}.$$

If we denote the n -th evaluation functor $\mathbf{PSp} \rightarrow \mathbf{S}_*$ by $\tilde{\Omega}^{\infty-n}$, then there is the following result.

Proposition 5.20. *The above construction gives an adjunction*

$$(\tilde{\Sigma}^{\infty-n}, \tilde{\Omega}^{\infty-n}): \mathbf{S}_* \rightarrow \mathbf{PSp}.$$

Furthermore, the forgetful functor from pointed spaces to spaces has a left adjoint by adding a disjoint basepoint which gives us the adjunction

$$(+, -): \mathbf{S} \rightarrow \mathbf{S}_*.$$

Put together with the spectrification adjunction, we thus obtain the composite adjunction

$$(\Sigma_+^{\infty-n}, \Omega_-^{\infty-n}): \mathbf{S} \rightarrow \mathbf{S}_* \rightarrow \mathbf{PSp} \rightarrow \mathbf{Sp}.$$

Since we will need it later, we introduce one more piece of notation. Forgetting about the adjunction $(+, -)$, by the above proposition we obtain a further composite adjunction:

$$(\Sigma^{\infty-n}, \Omega^{\infty-n}): \mathbf{S}_* \rightarrow \mathbf{Sp}.$$

This adjunction will have the universal property of a stabilization as we will state further down. But before that, we quickly mention the expected and important result [Lur09a].

Theorem 5.21. *The ∞ -category \mathbf{Sp} of spectra is a presentable, stable ∞ -category.*

Since we now have a model of an ∞ -category of spectra, we would like to see how this model is related to the different models of spectra in the theory of model categories [Sch01, MMSS01]. A comparison result will be mentioned in Subsection 5.2 when we have the monoidal structure on \mathbf{Sp} at our disposal.

An adjunction similar to the adjunction $(\Sigma^{\infty-n}, \Omega^{\infty-n}): \mathbf{S}_* \rightarrow \mathbf{Sp}$ can also be obtained in more general situations. Obviously, the evaluation functor $\Omega^{\infty-n}: \mathbf{Sp}(\mathcal{C}) \rightarrow \mathcal{C}$ makes sense for every finitely bicomplete, pointed ∞ -category. In the case that the ∞ -category \mathcal{C} is in addition presentable, it can be seen that Ω satisfies the conditions of the special adjoint functor theorem as described in Remark 2.25. Thus, the existence of a left adjoint $\Sigma^{\infty-n}: \mathcal{C} \rightarrow \mathbf{Sp}(\mathcal{C})$ can be ensured in this context, i.e. we have an adjunction

$$(\Sigma^{\infty-n}, \Omega^{\infty-n}): \mathcal{C} \rightarrow \mathbf{Sp}(\mathcal{C}).$$

Call the left adjoint a *suspension spectrum functor*. Although there is not such a nice description of $\Sigma^{\infty-n}$ as in the case of spaces, the following important result can be established [Lur09a]. For this purpose, given two presentable ∞ -categories \mathcal{C} and \mathcal{D} , let $\mathcal{P}r^L(\mathcal{C}, \mathcal{D})$ denote the ∞ -category of colimit-preserving functors from \mathcal{C} to \mathcal{D} .

Theorem 5.22. *Let \mathcal{C}, \mathcal{D} be presentable ∞ -categories and let \mathcal{D} in addition be stable. Then the suspension spectrum functor $\Sigma^\infty: \mathcal{C} \rightarrow \mathrm{Sp}(\mathcal{C})$ induces an equivalence of ∞ -categories*

$$\mathcal{P}r^L(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathcal{P}r^L(\mathcal{C}, \mathcal{D}).$$

As an important special case, consider once again the ∞ -category S_* of pointed spaces and its stabilization $\mathrm{Sp} = \mathrm{Stab}(S_*)$. Call the image of the zero sphere under Σ^∞ the *sphere spectrum*.

Corollary 5.23. *Let \mathcal{D} be a stable, presentable ∞ -category, then the evaluation at the sphere spectrum induces the following equivalence of ∞ -categories*

$$\mathcal{P}r^L(\mathrm{Sp}, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Thus, the ∞ -category Sp of spectra is the *free stable ∞ -category on one generator*, namely on the sphere spectrum. In fact, the equivalence of ∞ -categories is obtained as the following sequence of equivalences, where the first one is a special case of Theorem 5.22:

$$\mathcal{P}r^L(\mathrm{Sp}, \mathcal{D}) \xrightarrow{\sim} \mathcal{P}r^L(S, \mathcal{D}) \xrightarrow{\sim} \mathcal{D}.$$

Remark 5.24. The second equivalence can be read as saying that the ∞ -category S of spaces is the free cocomplete ∞ -category on one generator. Recall, from the discussion in Subsection 2.6, that in classical category theory presheaf categories can be considered as free cocompletions. The argument was that every such presheaf is canonically a colimit of representable functors. In the world of higher categories, a corresponding result would state that simplicial presheaf categories should be the free cocompletions. Applied to the special case of one generator, this is precisely the statement that we have the second equivalence. A proof of this result can be found in [Lur09e, p.462]. A corresponding result in the world of model categories was established by Dugger in [Dug01b]. Given a small category \mathcal{C} , Dugger shows that the model category $U(\mathcal{C})$ of simplicial presheaves on \mathcal{C} is the universal model category constructed out of \mathcal{C} . One central argument is to show that every simplicial presheaf is canonically a *homotopy colimit* of representable functors.

5.2. Presentable stable ∞ -categories and the smash product. The aim of this subsection is to sketch Lurie's construction of the smash product on the ∞ -category Sp of spectra. The construction is given indirectly by introducing the tensor product of stable, presentable ∞ -categories and remarking that Sp is a monoidal unit. We will give some details for the construction of the tensor product of ordinary presentable categories and then take it for granted that similar constructions can be carried out in the world of ∞ -categories [Lur09b, Lur09c].

Since we will need to distinguish between different functor categories, so we begin by fixing some notation. For ordinary categories $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ and \mathcal{D} we will consider the following functor categories: the colimit resp. limit preserving functors from \mathcal{C} to \mathcal{D} are denoted by $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$, resp. $\mathrm{Fun}^R(\mathcal{C}, \mathcal{D})$, while the left adjoint resp. right adjoint functors from \mathcal{C} to \mathcal{D} are denoted by $\mathrm{LAdj}(\mathcal{C}, \mathcal{D})$ resp. $\mathrm{RA}dj(\mathcal{C}, \mathcal{D})$. Moreover, we denote by $\mathrm{Fun}^{L,L}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$ the functors $\mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{D}$ which commute separately with colimits in each variable and by $\mathrm{Fun}^{\mathrm{acc}}(\mathcal{C}, \mathcal{D})$ the accessible functors from \mathcal{C} to \mathcal{D} . Using the obvious combination of these notations, recall that the adjoint functor theorems for presentable categories can be written in the following way. Let \mathcal{C}, \mathcal{D} be presentable categories, then:

$$\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) = \mathrm{LAdj}(\mathcal{C}, \mathcal{D}) \quad \text{and} \quad \mathrm{Fun}^{R,\mathrm{acc}}(\mathcal{C}, \mathcal{D}) = \mathrm{RA}dj(\mathcal{C}, \mathcal{D}).$$

The aim is to show that the presentable categories together with the colimit preserving functors form a symmetric monoidal closed category (or, more precisely, a 2-categorical variant of this since some structure maps will only be equivalences of categories but we will not care about this detail).

The monoidal pairing will be the tensor product and the idea is that given two presentable categories $\mathcal{C}_1, \mathcal{C}_2$ the tensor product $\mathcal{C}_1 \otimes \mathcal{C}_2$ is to be the universal recipient of a *bilinear* map $\mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{C}_1 \otimes \mathcal{C}_2$, i.e. of a functor which commutes with colimits separately in each variable.

Lemma 5.25. *Let \mathcal{C}_1 and \mathcal{C}_2 be presentable categories. Then the functor category $\text{Fun}^{\text{R}}(\mathcal{C}_1^{\text{op}}, \mathcal{C}_2)$ is again a presentable category.*

We give a proof only for the case that \mathcal{C}_1 is a presheaf category. Thus let $\mathcal{C}_1 = \widehat{\mathcal{E}}_1 = \text{Fun}(\mathcal{E}_1^{\text{op}}, \text{Set})$. Then we have the following sequence of equivalences of categories

$$\text{Fun}^{\text{R}}(\mathcal{C}_1^{\text{op}}, \mathcal{C}_2) \simeq \text{Fun}^{\text{L}}(\widehat{\mathcal{E}}_1, \mathcal{C}_2^{\text{op}})^{\text{op}} \simeq \text{Fun}(\mathcal{E}_1, \mathcal{C}_2^{\text{op}})^{\text{op}} \simeq \text{Fun}(\mathcal{E}_1^{\text{op}}, \mathcal{C}_2),$$

where the last category is again presentable since the class of presentable categories is closed under the formation of functor categories. With some more effort the general case can also be established.

Definition 5.26. Let Cat^{Pr} be the category whose objects are presentable categories and whose morphisms are given by colimit preserving functors.

In view of Lemma 5.25, we make the following definition.

Definition 5.27. The *tensor product of two presentable categories* $\mathcal{C}_1, \mathcal{C}_2$ is defined to be the presentable category $\mathcal{C}_1 \otimes \mathcal{C}_2 := \text{Fun}^{\text{R}}(\mathcal{C}_1^{\text{op}}, \mathcal{C}_2)$. This construction is obviously functorial and gives us the *tensor product functor*

$$\otimes: \text{Cat}^{\text{Pr}} \times \text{Cat}^{\text{Pr}} \longrightarrow \text{Cat}^{\text{Pr}}.$$

Remark 5.28. A better definition of a tensor product of presentable categories $\mathcal{C}_1, \mathcal{C}_2$ would of course be a third presentable category $\mathcal{C}_1 \otimes \mathcal{C}_2$ together with a universal bilinear morphism $\mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathcal{C}_1 \otimes \mathcal{C}_2$. We will obtain such a morphism once the universal property of this construction is established. Using the definition we have given, the universal property of the tensor product takes its usual form, i.e. bilinear maps out of a product are the same as linear maps out of a tensor product.

The next proposition justifies that this definition gives us the universal recipient of a bilinear map.

Proposition 5.29. *For three presentable categories $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D} \in \text{Cat}^{\text{Pr}}$ we have an equivalence of categories*

$$\text{Fun}^{\text{L,L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \xrightarrow{\sim} \text{Fun}^{\text{L}}(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D}).$$

The proof goes as follows. We have the following equivalences of categories:

$$\begin{aligned} \text{Fun}^{\text{L,L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) &\cong \text{Fun}^{\text{L}}(\mathcal{C}_1, \text{Fun}^{\text{L}}(\mathcal{C}_2, \mathcal{D})) \\ &= \text{Fun}^{\text{L}}(\mathcal{C}_1, \text{LAdj}(\mathcal{C}_2, \mathcal{D})) \\ &\simeq \text{Fun}^{\text{L}}(\mathcal{C}_1, \text{RAdj}(\mathcal{D}, \mathcal{C}_2)^{\text{op}}) \\ &= \text{Fun}^{\text{L}}(\mathcal{C}_1, \text{Fun}^{\text{R,acc}}(\mathcal{D}, \mathcal{C}_2)^{\text{op}}). \end{aligned}$$

By forgetting the accessibility assumption, the last category embeds into $\text{Fun}^{\text{L}}(\mathcal{C}_1, \text{Fun}^{\text{R}}(\mathcal{D}, \mathcal{C}_2)^{\text{op}})$ and we thus have

$$\begin{aligned}
\mathrm{Fun}^L(\mathcal{C}_1, \mathrm{Fun}^{\mathrm{R}, \mathrm{acc}}(\mathcal{D}, \mathcal{C}_2)^{\mathrm{op}}) &\hookrightarrow \mathrm{Fun}^L(\mathcal{C}_1, \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C}_2)^{\mathrm{op}}) \\
&\cong \mathrm{Fun}^L(\mathcal{C}_1, \mathrm{Fun}^L(\mathcal{D}^{\mathrm{op}}, \mathcal{C}_2^{\mathrm{op}})) \\
&\cong \mathrm{Fun}^{\mathrm{L}, \mathrm{L}}(\mathcal{C}_1 \times \mathcal{D}^{\mathrm{op}}, \mathcal{C}_2^{\mathrm{op}}) \\
&\cong \mathrm{Fun}^L(\mathcal{D}^{\mathrm{op}}, \mathrm{Fun}^L(\mathcal{C}_1, \mathcal{C}_2^{\mathrm{op}})) \\
&\cong \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathrm{Fun}^L(\mathcal{C}_1, \mathcal{C}_2^{\mathrm{op}})^{\mathrm{op}})^{\mathrm{op}}
\end{aligned}$$

One checks that the image of this inclusion under the sequence of isomorphisms consists of the functors which are limit preserving and accessible, thus putting these two chains of equivalences together we obtain an equivalence

$$\mathrm{Fun}^{\mathrm{L}, \mathrm{L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{R}, \mathrm{acc}}(\mathcal{D}, \mathrm{Fun}^L(\mathcal{C}_1, \mathcal{C}_2^{\mathrm{op}})^{\mathrm{op}})^{\mathrm{op}}.$$

Proceeding similarly, we obtain the following equivalences

$$\begin{aligned}
\mathrm{Fun}^{\mathrm{L}, \mathrm{L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) &\simeq \mathrm{Fun}^{\mathrm{R}, \mathrm{acc}}(\mathcal{D}, \mathrm{Fun}^L(\mathcal{C}_1, \mathcal{C}_2^{\mathrm{op}})^{\mathrm{op}})^{\mathrm{op}} \\
&= \mathrm{RAdj}(\mathcal{D}, \mathrm{Fun}^L(\mathcal{C}_1, \mathcal{C}_2^{\mathrm{op}})^{\mathrm{op}})^{\mathrm{op}} \\
&\simeq \mathrm{LAdj}(\mathrm{Fun}^L(\mathcal{C}_1, \mathcal{C}_2^{\mathrm{op}})^{\mathrm{op}}, \mathcal{D}) \\
&\cong \mathrm{Fun}^L(\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}_1^{\mathrm{op}}, \mathcal{C}_2), \mathcal{D}) \\
&= \mathrm{Fun}^L(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D}).
\end{aligned}$$

This gives us the desired equivalence of categories. In particular, setting $\mathcal{D} = \mathcal{C}_1 \otimes \mathcal{C}_2$ and following the identity $\mathrm{id}: \mathcal{C}_1 \otimes \mathcal{C}_2 \rightarrow \mathcal{C}_1 \otimes \mathcal{C}_2$ through the sequence of equivalences, we obtain a universal bilinear map

$$\mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}_1 \otimes \mathcal{C}_2.$$

Since $\mathrm{Fun}^{\mathrm{L}, \mathrm{L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$ is symmetric in \mathcal{C}_1 and \mathcal{C}_2 , the tensor product $\mathcal{C}_1 \otimes \mathcal{C}_2$ is a symmetric pairing on $\mathrm{Cat}^{\mathrm{Pr}}$. A simple example of such a tensor product is the following. Let \mathcal{C} be an arbitrary presentable category, then we have

$$\mathrm{Set} \otimes \mathcal{C} = \mathrm{Fun}^{\mathrm{R}}(\mathrm{Set}^{\mathrm{op}}, \mathcal{C}) \cong \mathrm{Fun}^L(\mathrm{Set}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \cong \mathrm{Fun}(*, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \cong \mathcal{C},$$

thus the category of sets behaves as a monoidal unit. Recall that when \mathcal{C}_1 and \mathcal{C}_2 are presentable categories, then $\mathrm{Fun}^L(\mathcal{C}_1, \mathcal{C}_2)$ is also presentable. Thus the equivalences

$$\mathrm{Fun}^L(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}, \mathrm{L}}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{C}_1, \mathrm{Fun}^L(\mathcal{C}_2, \mathcal{D}))$$

show that we have a closed symmetric monoidal structure on $\mathrm{Cat}^{\mathrm{Pr}}$. We summarize the preceding discussion in the following proposition.

Proposition 5.30. *The tensor product functor $\otimes: \mathrm{Cat}^{\mathrm{Pr}} \times \mathrm{Cat}^{\mathrm{Pr}} \rightarrow \mathrm{Cat}^{\mathrm{Pr}}$ endows $\mathrm{Cat}^{\mathrm{Pr}}$ with the structure of a closed symmetric monoidal category. The unit is given by the category Set of sets and the inner hom is given by the category of colimit preserving functors $\mathrm{Fun}^L(\mathcal{C}_1, \mathcal{C}_2)$.*

Let us return to the world of ∞ -categories. Denote by $\widehat{\mathrm{Cat}}_\infty$ the ∞ -category of ∞ -categories and let $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L}, \mathrm{Pr}}$ denote the subcategory of presentable ∞ -categories with the colimit-preserving functors. The tensor product can also be defined for presentable ∞ -categories and endows $\widehat{\mathrm{Cat}}_\infty^{\mathrm{L}, \mathrm{Pr}}$ with the structure of a monoidal ∞ -category. Let us record two important examples.

Example 5.31. • Let \mathcal{C} be an arbitrary presentable ∞ -category, then we have by Remark 5.24:

$$\mathcal{S} \otimes \mathcal{C} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{S}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{S}, \mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \simeq \mathcal{C}.$$

Furthermore, since the representability criterion for presentable ∞ -categories states that a (simplicial) presheaf is representable if and only if it commutes with limits, we have

$$\mathcal{C} \otimes \mathcal{S} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{C}.$$

Thus the ∞ -category \mathcal{S} of spaces behaves like a monoidal unit for the tensor product.

• Let \mathcal{C} be an arbitrary presentable ∞ -category, then, using Remark 5.15 and again the representability criterion for presentable ∞ -categories, one deduces

$$\begin{aligned} \mathcal{C} \otimes \mathrm{Sp} &\simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \\ &\simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathrm{holim}(\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)) \\ &\simeq \mathrm{holim}(\dots \xrightarrow{\Omega} \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_*) \xrightarrow{\Omega} \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_*)) \\ &\simeq \mathrm{holim}(\dots \xrightarrow{\Omega} \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_* \xrightarrow{\Omega} \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_*) \\ &\simeq \mathrm{holim}(\dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*) \\ &\simeq \mathrm{Stab}(\mathcal{C}). \end{aligned}$$

Thus the tensor product with the ∞ -category Sp of spectra is stabilization.

Let $\widehat{\mathcal{C}at}_{\infty}^{\mathrm{L}, \mathrm{Pr}, \sigma} \subseteq \widehat{\mathcal{C}at}_{\infty}^{\mathrm{L}, \mathrm{Pr}}$ be the full subcategory spanned by the *stable* presentable ∞ -categories. Then the tensor product of stable presentable ∞ -categories is again a stable presentable ∞ -category. Taking for granted that the stabilization process is idempotent, the above example shows that Sp behaves like a monoidal unit. In fact, there is the following result which is proved in [Lur09b].

Theorem 5.32. *The tensor product of presentable ∞ -categories induces a monoidal structure on $\widehat{\mathcal{C}at}_{\infty}^{\mathrm{L}, \mathrm{Pr}, \sigma}$ with the ∞ -category Sp of spectra as monoidal unit.*

Recall from Proposition 3.16 that, given a monoidal ∞ -category $\mathcal{M}^{\otimes} \rightarrow \mathbf{N}(\Delta^{\mathrm{op}})$, the monoidal unit $I_{\mathcal{M}}$ gives us an initial object of $\mathrm{Alg}(\mathcal{M})$. We want to apply this to $\widehat{\mathcal{C}at}_{\infty}^{\mathrm{L}, \mathrm{Pr}, \sigma}$. For this purpose, denote the algebra objects in $\widehat{\mathcal{C}at}_{\infty}^{\mathrm{L}, \mathrm{Pr}, \sigma}$ by

$$\widehat{\mathcal{C}at}_{\infty}^{\sigma, \mathrm{Mon}} := \mathrm{Alg}(\widehat{\mathcal{C}at}_{\infty}^{\mathrm{L}, \mathrm{Pr}, \sigma}).$$

In this context, Proposition 3.16 reads as follows.

Theorem 5.33. *The ∞ -category Sp of spectra gives an initial object Sp^{\otimes} of $\widehat{\mathcal{C}at}_{\infty}^{\sigma, \mathrm{Mon}}$.*

Remark 5.34. Recall from Remark 3.15 that monoidal ∞ -categories can be regarded as certain algebra objects, namely as algebra objects in $\widehat{\mathcal{C}at}_{\infty}^{\times}$, where $\widehat{\mathcal{C}at}_{\infty}^{\times}$ denotes the Cartesian monoidal ∞ -category of ∞ -categories. Similarly, algebra objects in $\widehat{\mathcal{C}at}_{\infty}^{\mathrm{L}, \mathrm{Pr}}$ can be identified with presentable monoidal ∞ -categories \mathcal{M} such that the monoidal product $\otimes: \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ is bilinear. Finally, the same holds for $\widehat{\mathcal{C}at}_{\infty}^{\sigma, \mathrm{Mon}}$, where in addition the stability condition is imposed.

Thus the ∞ -category Sp carries a bilinear monoidal structure. There is the following uniqueness property of this monoidal structure. Recall that by Corollary 5.23 the stable presentable ∞ -category

Sp is freely generated on one generator. Thus given an object in a further stable presentable ∞ -category \mathcal{D} , we obtain an essentially unique map $\mathrm{Sp} \rightarrow \mathcal{D}$ *classifying the object*. This can, in particular, be applied to the case of a monoidal unit.

Corollary 5.35. Let $\mathcal{D}^{\otimes} \rightarrow \mathbf{N}(\Delta^{\mathrm{op}})$ be a monoidal ∞ -category, such that the underlying ∞ -category \mathcal{D} is stable and presentable and such that the monoidal product \otimes on \mathcal{D} is bilinear. If the map $\mathrm{Sp} \rightarrow \mathcal{D}$ classifying the monoidal unit $I_{\mathcal{D}} \in \mathcal{D}$ is an equivalence of ∞ -categories, then there is a monoidal equivalence $\mathrm{Sp}^{\otimes} \rightarrow \mathcal{D}^{\otimes}$. Moreover, the space of such equivalences is contractible.

Remark 5.36. In the setting of model categories, Shipley [Shi01] obtained a similar uniqueness result for certain stable, monoidal model categories \mathcal{M} having homotopy categories which are equivalent in a strong sense (namely as triangulated, monoidal, π_*^s -linear categories) to the stable homotopy category \mathcal{SHC} . Under certain assumptions on \mathcal{M} , she obtains a strong monoidal left Quillen functor from the category of symmetric spectra endowed with the positive model structure to \mathcal{M} . If the monoidal unit I of \mathcal{M} satisfies some natural conditions (I is to be a small weak generator of $\mathrm{Ho}(\mathcal{M})$ and the graded endomorphism ring $[I, I]_*$ is to be free of rank one over π_*^s with the identity as generator), then the obtained Quillen adjunction is, in fact, a Quillen equivalence.

Thus the monoidal structure on Sp is essentially uniquely characterized by the properties that the monoidal product is bilinear and that the sphere spectrum acts as monoidal unit.

Definition 5.37. Let $\otimes: \mathrm{Sp} \times \mathrm{Sp} \rightarrow \mathrm{Sp}$ be any bilinear monoidal product with the sphere spectrum as monoidal unit. Then call \otimes the *smash product on spectra*.

Remark 5.38. • Once one has established the fact that the ∞ -category Sp of spectra is equivalent to the coherent nerve of –say– symmetric spectra, the uniqueness property of the smash product can be used to show that this definition is compatible with the usual smash product on symmetric spectra [Hov00]. Using the tensor product of stable presentable ∞ -categories, Lurie has thus obtained an *intrinsic* description of the smash product, i.e. a description given completely in the world of ∞ -categories without making reference to a model category of spectra with a strict monoidal smash product.

• In the world of model categories, Lenhardt [Len] has given a similar approach to the construction of the smash product on spectra. First, he extends the theory of frames of Dwyer-Kan [DK80] to a theory of *stable* frames on stable model categories. Then, he shows that the category Sp of Bousfield-Friedlander spectra of simplicial sets is initial in the following sense. Given a stable model category \mathcal{M} and a bifibrant object $X \in \mathcal{M}$, there is an essentially unique left Quillen functor $L: \mathrm{Sp} \rightarrow \mathcal{M}$ sending the sphere spectrum \mathbb{S} to $L(\mathbb{S}) \cong X$. This induces a pairing $\mathcal{SHC} \times \mathrm{Ho}(\mathcal{M}) \rightarrow \mathrm{Ho}(\mathcal{M})$ and, in particular, a pairing $\mathcal{SHC} \times \mathcal{SHC} \rightarrow \mathcal{SHC}$. He then shows that the ladder pairing can be identified with the smash product on \mathcal{SHC} and that the first pairing endows the homotopy category of a stable model category with the structure of a \mathcal{SHC} -module.

Having the ∞ -categorical version of the smash product at our disposal, one can consider the associated algebra objects.

Definition 5.39. An \mathcal{A}_{∞} -ring is an algebra object of the monoidal ∞ -category $\mathrm{Sp}^{\otimes} \rightarrow \mathbf{N}(\Delta^{\mathrm{op}})$. The ∞ -category of \mathcal{A}_{∞} -rings is denoted by $\mathcal{A}_{\infty} := \mathrm{Alg}(\mathrm{Sp})$.

Having this notion at hand, one could now develop a theory of modules over \mathcal{A}_{∞} -rings and redo an amazing amount of classical noncommutative ring theory in this setting. Since this is done in detail in [Lur09b], we will instead finish this course by the following comments on the commutative variants. In the discussion of the tensor product for ordinary presentable categories, we remarked

that the tensor product is in fact a symmetric pairing. With some work, a similar result can be shown to hold true in the world of ∞ -categories [Lur09c].

Theorem 5.40. • *The monoidal structure on $\widehat{\mathcal{C}at}_\infty^{\text{L,Pr}}$ given by the tensor product of presentable ∞ -categories can be promoted to a symmetric monoidal structure.*

• *The ∞ -category $\widehat{\mathcal{C}at}_\infty^{\text{L,Pr},\sigma}$ inherits a symmetric monoidal structure from $\widehat{\mathcal{C}at}_\infty^{\text{L,Pr}}$ with Sp as monoidal unit.*

Using a variant of Proposition 3.16 for symmetric monoidal ∞ -categories, one obtains the existence of a symmetric monoidal pairing on Sp . In fact, there is the following characterization of this pairing.

Corollary 5.41. The ∞ -category Sp of spectra has an essentially unique symmetric monoidal structure $\text{Sp}^\otimes \rightarrow \mathbf{N}(\mathcal{F}in_*)$ characterized by the properties that the sphere spectrum is a monoidal unit and that the monoidal product is bilinear. This monoidal product is again called the *smash product pairing*.

Definition 5.42. An \mathcal{E}_∞ -ring is a commutative algebra object of the symmetric monoidal ∞ -category $\text{Sp}^\otimes \rightarrow \mathbf{N}(\mathcal{F}in_*)$. Denote the ∞ -category of all \mathcal{E}_∞ -rings by $\mathcal{E}_\infty := \text{CAlg}(\text{Sp})$.

Again, we will not pursue this any further, since the theory of these rings and of their modules is developed in detail in [Lur09c]. There, Lurie also discusses how this definition of \mathcal{E}_∞ -rings is related to the more classical approach using the theory of model categories.

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