GALOIS THEORY 2015/2016 EXERCISE SHEET 7

Questions marked with a star are optional.

(1) Let $L|K$ be a finite extension of fields, and let L_1 and L_2 be subextensions. Suppose $L_1|K$ is Galois, with Galois group G. Show that $L_1L_2|L_2$ is Galois, and that its Galois group is isomorphic to a subgroup of G.

(2) Let K be a finite extension of $\mathbb{Q}$. Let $f(X)$ be an irreducible polynomial in $K[X]$. Suppose that, for some n, $f(X)$ has a root in $K(\zeta_n)$. Show that $f(X)$ splits completely in $K(\zeta_n)$.

(3) (a) Let $L|K$ be an extension of fields, and for all $n \in \mathbb{Z}_{>0}$, let K_n be a subextension, such that $K_n|K$ is finite and Galois, and for all $n < m$, $K_n \subset K_m$. Show that, for all $n < m$, $\mathrm{Hom}_K(K_n, K_m) = \mathrm{Gal}(K_n|K)$.

(b) Show that for all $n < m$, there is a surjection $\mathrm{Gal}(K_m|K) \to \mathrm{Gal}(K_n|K)$.

(c) Deduce that every element of $\mathrm{Gal}(K_n|K)$ extends to an automorphism of the field K_∞, where $K_\infty := \cup_n K_n$.

(d) Let p be a prime, and m a positive integer. Let α_m be an element of $(\mathbb{Z}/p^m\mathbb{Z})^\times$. Show that for all $N > 0$, there is an $n > m$, and a lift of α_m to $\alpha_n$in $(\mathbb{Z}/p^n\mathbb{Z})^\times$, such that the order of α_n in $(\mathbb{Z}/p^n\mathbb{Z})^\times$ is greater than N.

(e) Use this to show that there is an automorphism of the field $\mathbb{Q}(\zeta_{p^\infty}) := \cup_n \mathbb{Q}(\zeta_{p^n})$ of infinite order.

(f) * Show that there is an automorphism of $\mathbb{Q}(\zeta_{p^\infty})$ of finite order.

Comments, corrections, questions etc to netandogra@gmail.com.