# Properties of morphisms

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## 1 Introduction

This talk will be about some properties of morphisms of schemes. One of such properties was already discussed by Johan:

**Definition 1.1.** *Let $f\colon X \to Y$ be a morphism of schemes. $f$ is called* separated *if the image of the diagonal morphism $\Delta\colon X \to X \times_Y X$ is closed.*

This property is stable under base change and composition; this will be the case for many other properties we will discuss.

## 2 Finiteness properties

Let $f\colon A \to B$ be a morphism of commutative rings. $f$ is said to be *of finite type* if $B$ is finitely generated as an $A$-algebra. If this is the case, then $B \cong A[X_1, X_2, \ldots, X_n]/I$ for some integer $n$ and some ideal $I$. This motivates the following definition:

**Definition 2.1.** *Let $f\colon X \to Y$ be a morphism of schemes. $f$ is called* locally of finite type *if $Y$ can be covered by a collections of affine opens $\{U_i\}_{i\in\mathcal{I}}$ such that every $f^{-1}(U_i)$ can be covered by affine opens $\{V_{ij}\}_{j\in\mathcal{J}_i}$ such that every $f^\sharp\colon \mathcal{O}_Y(U_i) \to \mathcal{O}_X(V_{ij})$ is of finite type. If all $\mathcal{J}_i$ can be chosen to be finite, then $f$ is said to be* of finite type.

The 'geometric' intuition of 'of finite type' is that it is the algebraic equivalent of finite-dimensionality. For example, smooth reduced schemes of finite type over $\mathbf{C}$ correspond to (finite-dimensional) smooth complex manifolds.

Over non-Noetherian schemes, this is usually not a very useful finiteness property, since the ideal $I$ above may not be finitely generated; in this case, we still need infinitely many relations to describe $B$. Therefore, we say that $f\colon A \to B$ is *of finite presentation* if $B \cong A[X_1, X_2, \ldots, X_n]/I$ for some integer $n$ and some *finitely generated* ideal $I$. The property *locally of finite presentation* of morphisms of schemes is now easily defined (the notion *of finite presentation* is somewhat more involved). Over noetherian schemes, the properties *locally of finite type* and *locally of finite presentation* are equivalent. *Of finite presentation* has the following nice functorial property:

**Proposition 2.2.** *Let $f\colon X \to Y$ be a morphism of schemes. Then $f$ is locally of finite presentation if and only if for every directed system $I$ and any inverse system $\{T_i\}_{i\in I}$ of affine $Y$-schemes one has:*

$$\operatorname{Hom}_Y(\lim_i T_i, X) = \operatorname{colim}_i \operatorname{Hom}(T_i, X).$$

The second important finiteness property is defined as follows:

**Definition 2.3.** *Let $f\colon X \to Y$ be a morphism of schemes. $f$ is called* finite *if for any affine open $U \subset Y$, $f^{-1}(V)$ is affine and $\mathcal{O}_X(V)$ is a finitely generated $\mathcal{O}_Y(U)$-module. Equivalently, this has to hold for some affine cover.*

The terminology suggests that this has something to do with the topological definition of finite maps, i.e. $f^{-1}(y)$ is finite for every $y \in Y$. Such maps are called *quasi-finite* in algebraic geometry. As is to be expected, finite maps are always quasi-finite, but the reverse need not be true. For example $\operatorname{Spec}\bar{\mathbf{Q}} \to \operatorname{Spec}\mathbf{Q}$ is quasi-finite, but $\bar{\mathbf{Q}}$ is not a finite-dimensional $\mathbf{Q}$-vector space.

# 3 Proper morphisms

As in topology, a map between schemes is called *closed* if the image of every closed subset is closed. A map is called *universally closed* if all of its base changes are closed. For example, the projection map $\mathbf{A}^1_k \to \operatorname{Spec} k$ is closed, but not universally closed: basechanging over (another) affine line gives us the projection from the affine plane to an affine line, and the image of the hyperbola $xy = 1$ is not closed.

A morphism is called proper if it is of finite type, separated and universally closed. Note that the book does not require the first property. Since all of these properties are stable under composition and base change, the same holds for properness. There is an equivalent definition which requires some notation. For a discrete valuation ring $R$, let $K$ be the quotient field of $R$. $\operatorname{Spec} R$ has two points, the open point $\operatorname{Spec} K$ corresponding to the ideal 0 and and a closed point corresponding to the maximal ideal of $R$. As such, $U_R$ should be considered as an infinitesimal neighbourhood of the closed point, without the point itself. To see this, consider $R = \mathbf{C}[[t]]$. If we want to consider this as the ring of regular functions over a subset of $\mathbf{C}$, it would correspond to an infinitesimal neighbourhood of the origin. As such, $\operatorname{Spec} K$ is this infinitesimal neighbourhood with the origin removed.

**Proposition 3.1.** *Let $f\colon X \to Y$ be a morphism of schemes and suppose $Y$ is noetherian.*

1. *$f$ is separated if and only if for every DVR $R$ and for every diagram of the form*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

   *there is at most one map $h\colon SpecR \to X$ making the diagram commute.*

2. *$f$ is proper if and only if it is of finite type and the map $h$ as above exists and is unique.*

This proposition clarifies some of the topological intuition. If $Y$ is a point (the spectrum of a field), then intuitively, given a map $U_R \to X$, the closed point of $X_R$ has to be sent to a 'limit' of the image of $U_R$. So in this case, $X$ is separated if such a limit is unique if it exists, which corresponds to the Hausdorff property topologically. $X$ is proper if these 'limits' also exist, and this corresponds to being compact.

There are two more important statements regarding these properties:

**Proposition 3.2.** *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes.*

1. *$f$ is finite if and only if it is quasi-finite and proper.*

2. *Let $\mathcal{P} \in \{$quasi-finite, finite, proper$\}$. If $gf$ has property $\mathcal{P}$ then so does $f$.*