Algebraic spaces & algebraic stacks

Goal: arrive at/outline a definition of algebraic stack

The def of alg stack fits into a pattern of defining "spaces" by gluing local models.

Part 1: explain how this works for schemes, rather than stacks.
Part 2: explain how the stack constructions can be marginalized within stacks.

§1: Algebraic spaces

Let me start with a simple question:

\[ \text{Aff} = \text{Ring}^{\text{op}} \subseteq \text{Sch} \quad \text{with } \mathcal{X} \text{ a sheaf topology.} \]

Then \( \text{Sh}(\text{Sch}) \xrightarrow{\text{Sh}(\text{Aff})} \to \text{Sch} \subseteq \text{Sh}(\text{Aff}) \) fully faithful.

Q: when is a stack \( X: \text{Aff} \to \text{Sch} \) repre. by a scheme?

express this purely in terms of affines & schemes.

Let \( S := \text{class of Zariski open immersions } U: \text{Spec}(A) \to \text{Spec}(B) = V. \)

**Def.** A stack \( X \) is \((-1)-representable\) if \( X = \text{Spec}(A) \).

- A map \( f: X \to Y \) is \((-1)-representable\) if for all

  \[ U = \text{Spec}(A) \to Y, \text{ the pullback } X \times_Y U \text{ is } (-2)-representable. \]

- \( f \) is of class \((-1)-P\) if furthermore the map of affines \( X \times_Y U \to U \) is \( P. \)

Proceed inductively:

**Def.** For \( n \geq -2 \):

- \( X \) is \( n \)-representable if there exist a map \( p: \coprod U_i \to X \) (all \( U_i \) are affines) such that:
  1. \( p \) a surjective family of schemes.
  2. Each \( U_i = \text{Spec}(A_i) \) affine and \( U_i \to X \) is of class \((n-1)-P.\)

- \( f: X \to Y \) is \( n \)-representable if for all \( U = \text{Spec}(A) \to Y, \)

  \[ U \times_Y X \text{ is } n \text{-representable.} \]

- \( f \) is of class \( n-P \) if furthermore
  1. for an affine \( U: \text{Spec}(B) \to U \times_Y X \), each composite

        \[ \text{Spec}(B) \to U \times_Y X \to U = \text{Spec}(A) \text{ is } P. \]
  2. \( f \) is mono.

Formal properties: \( n \)-representable and \( n-P \) maps stable under base change

- any map between \( n \)-representable is \( n \)-representable
Formal properties:
- n-representable and n-transitive stable under base change
- Any map between n-representable is n-representable
- n-reps stable under pull.
- For m ≥ n, if n-reps + m-reps = n-reps, then m-reps.
- Def: say that g; X → Y is of class P if g; X is n-reps for some n.

Let g; X → Y and \( \bigcup Y_i \rightarrow Y \) surjective, each \( Y_i \rightarrow Y \) is P.

Then g is n-reps (n-P) iff each \( X \times X, Y_i \rightarrow Y \) is P.

(i.e., these properties are “tame locally” on the target).

Similarly “tame locally” on domain.

Lem: If X is n-representable, then X is o-representable

(\( \Rightarrow \) for \( n \geq o \): all notions coincide).

Proof: Let \( \bigcup U_i \rightarrow X \) an atlas, \( U_i = \text{Spec}(A_i) \rightarrow X \) class \( (n-1)\)-rep.

To show: \( U_i \rightarrow X \) is \( (n-2)\)-rep.

Take \( V = \text{Spec}(B) \rightarrow X \), then \( U_i \times X V \) is \( (n-1)\)-rep.

Let \( \bigcup V_i = \bigcup \text{Spec}(B_i) \rightarrow \bigcup U_i \times X V \) an atlas.

To show: \( V_i \rightarrow \bigcup U_i \times X V \) is \( (n-2)\)-rep.

Pick \( W = \text{Spec}(C) \rightarrow \bigcup U_i \times X V \). Then

\[ V_{ij} \times X W \cong V_{ij} \times X V \] is a pullback of objects, hence affine! \( \rightarrow \)

Example/Proposition: \( X \) \( (n-2)\)-rep \( \iff \) X represents by affine scheme.

\( X \) \( (n-1)\)-rep \( \iff \) X represents by a scheme with affine diagonal.

Proof: \( \Rightarrow \) Take \( \bigcup U_i \rightarrow X \) an affine open cover of the scheme X.

Then \( \bigcup U_i \rightarrow X \) is surj. and for all affine \( W: U_i \times X W \rightarrow U_i \times X W \) is affine, so \( U_i \times X W \rightarrow W \) is affine open subcover.

\( \Rightarrow \) Take \( \bigcup U_i \rightarrow X \) atlas, consider

\( \bigcup U_i \times X U_j \rightarrow \bigcup U_i \rightarrow X \) each \( U_i \rightarrow X \) each \( U_i \rightarrow X \).

Each \( U_i \times X U_j \rightarrow U_i \) affine open subcover

The scheme obtained by gluing the \( U_i \) along these open subcovers represents X.

\( X \) \( o\)-rep \( \iff \) X represents by a scheme.

A map between schemes \( X \rightarrow Y \) is of class P \( \iff \) open imm. in usual sense.
Observe: The above definition also make sense when 
\[ P = \text{etale maps} \]  
(\text{or } P = \text{small maps})

and without the condition that maps in \((n-P)\) are monic! 
(Otherwise, more etale \(= \text{open immersion} \).

\[ \Rightarrow \text{get notion of } n\text{-representable sheaf more general than a scheme.} \]

All of the above results remain true, with one (important) exception:

Lemma: Let \( \phi : X \to Y \) suppose \( \forall i \to Y \) any surjection of schemes.

If \( \phi \) and \( X \times_Y Y_i \to Y_i \) is \( n\text{-representable } (n-P) \), then \( \phi \) is too.

In other words, it is etale local on the target.

Proof: Representability only by sections on \( n \).

\( n \geq 2 \): Let \( U = \text{Spec}(A) \) affine, \( U \to Y \).

There is an etale cover \( U_i = \text{Spec}(A_i) \to U \) at \( \text{Spec}(A) \to Y \).

We get 
\[
\begin{array}{c}
\begin{array}{ccc}
U_i \times_Y X \to U_i \times_Y Y_i \\
\downarrow \\
U_{i\to Y}\end{array}
\end{array}
\]

\( \Rightarrow \text{etale } \to U \)

By assumption, the right two maps are affine maps.

Since affine maps surjective etale descent, the claim \( U \times_Y X \to U \) is also affine.

For \( n \geq 2 \): Exactly the same: \( \begin{array}{c}
\begin{array}{ccc}
U_i \times_Y X \to U \times_Y X \end{array}
\end{array} \) is \((1)\). The map 
\( \text{is } n\text{-representable } \text{descent} \).

Then an etale \( \phi \) is also an etale \( \phi \).

Case: \( \text{Let } R \to X \times X \) be an equivalence relation on \( X \) (internal to schemes)

and let \( R \times X \to X \times X \) be the quotient sheaf.

If \( R \times X \) is \( n\text{-representable} \) and the maps \( R \to X \) are etale, then \( X / R \) is \( 0\text{-representable} \).

Proof: It suffices to show that \( X / R \) is \( n\text{-representable } (\geq 0\text{-rep}) \).

Let \( U = \text{Spec}(A) \to X \) be an etale and \( \hat{R} = \text{Spec}(A \times_{X \times X} X \times X) \) the induced 
equivalence relation. Then the map \( U / \hat{R} \to X / R \) is an isomorphism of stalks.

If \( U = \text{Spec}(A) \) satisfies the conditions as well, one may assume \( X \) an isoprost of affine.

In that case, \( X \to X / R \) is etale \( (\geq 0\text{-rep}) \).

Indeed, pulling back along \( X / R \), we get \( X \times_X X \), which is etale \( \geq 0\text{-rep} \).
Ref.: In other works, the o-rup sheaves form the smallest class of sheaves containing affine $\mathbb{A}_k$-bundles under $\{\mid f\}$-sheaf equivalence relations.

Thm.: For a sheaf $X$, TFAE:

1. $X$ is o-rup for $P$-sheaf (or $P$-small)
2. There is a map $\eta: U \rightarrow X$ such that:
   a. $P$-ary of sheaves
   b. $U$ is a sheaf and for any $V \in \pi(\mathcal{B}) \rightarrow X$,
      $UX \rightarrow V$ is an o-rup map of sheaves.
   c. The diagonal $\Delta: X \rightarrow XX$ is representable by sheaves.

Def.: An algebraic space is a sheaf $X \in \mathcal{X}(\mathbb{A}_k(\text{Aff}) \cap \mathbf{Sht}(\text{Sch}))$ satisfying (1) in the above.

Def.: Variety: $\mathcal{X}(\mathbb{A}_k(\text{Aff}) \cap \mathbf{Sht}(\text{Sch}))$ also requires $UX \eta \rightarrow UX \eta$ quasi-finite.

[Sketch of proof:

1. $X$ is an o-rup sheaf
   (like a quotient by $\mathcal{E}(\text{Sch} \cap \mathbf{07FL})$)

"Ideal proof": (1) $\Rightarrow$ (i) shows: sheaves are definitely o-rup, o-rup closed under o-rup equivalence.

(i) $\Rightarrow$ (ii) Suffices to show that algebraic spaces are also closed
under quotient of sheaf equivalence. $P = X \rightarrow X/R$
This is quite complicated [Springer 0455, L-MB Prop. 1.6].

Step 1 (final) can reduce to $X$ a sheaf.

Step 2: $R = X \times X$ and $R \rightarrow X$ sheaf $\Rightarrow R$ is also a sheaf

Step 3: Quotient of sheaves by sheaf equivalence are algebraic spaces

[Sketch of proof 0455]

Examples:

$S$ a sheaf, $G \times S$ free action by a finite group

Then the quotient $S/G$ is an algebraic space.

- $G = \text{GL}_r$: $\mathbb{A}^k$, $G \times S = S$ as sheaf.

- $G = \text{GL}_k$, $U = \mathbb{A}^k$, $U \times S = S$ as sheaf.

Then $UX \eta = \text{Spec} \mathcal{O}(\mathbb{A}^k(\text{Aff}) \cap \mathbf{Sht}(\text{Sch})) = \Delta(U) + \Delta^*(U)$

(see $\Delta^*$)
Then $U \times U = \text{spec}(\mathcal{O}_R(U \times U)) = \text{spec}(\mathcal{O}_R(U) \otimes \mathcal{O}_R(U))$.

where $\Delta^R = (\Delta^R_U) : U \to U \times U$.

Let $R = \Delta^R(U) \otimes \mathcal{O}_R(U) \otimes \mathcal{O}_R(U)$.

Then $R \subset U \times U$ defines an equivalence relation $\sim$ on $X = U/R$ on algebraic space over $S$.

Proposition:

1. $X \to S$ is over $S$-isom.
2. $X \to S$ is smooth $U \to S$ is.
3. $U$ is open over $S$.
4. $\Delta^R_U = \text{spec} \mathcal{O}_R(U)$.
5. $X$ is not a scheme: otherwise $G_{X/P}$ would be a local ring with fraction field $\text{field}(x)$, residue field $\text{field}(x)$.
6. $X \times U \cong U$, $\Delta^R_U$ is a scheme (the origin, double origin).

What can you do with them? Any sort of map which is local on source and target induces a lift of map between $\mathcal{S}$.

$X \to x \times x$,

the open-aff, sheaves.

§ 2: Algebraic stacks.

Now: interpret Definition 1 not in schemes over Aff, but stacks over Aff.

Def: An algebraic stack is a stack $X$ over Aff (or $S$) such that

is 1-representable in the sense of Definition 1, where $P = \text{small}$ maps.

Equivalently: $X$ is an algebraic stack if

(i) $X \to x \times x$ representable by algebraic spaces

(ii) there is a scheme $U$, a surjection of stacks $p : U \to X$ which is representable by algebraic spaces and smooth.

Category theory of stacks

Restricting then $\text{fun}(\text{stacks } C) \to \text{Psh}(C)$.

There is an alternate of $U$-construction:

Stacked construction: Suppose $F : C \to \text{Set}$ to a (stack) diagram of groupoids: $F(a) \to F(c)$ for $a \to c$.

Define a stack $\mathcal{S}_F$ on $F$ objects $(c, x)$ along $C \in C$, $x \in F(c)$.

maps $\Delta^C : (c, x) \to (x, \Delta^C(x), (\Delta^C(x))_d) : a \to c \to d \in C$

$\Delta^C : (c, x) \to (x, \Delta^C(x), (\Delta^C(x))_d) : a \to c \to d \in C$

Then the structures $\mathcal{S}_F \to C$ is fibrad in groupoids, fibers are $F(c)'s$. 
Inverse:  \( p: \mathcal{D} \to \mathcal{C} \) filtered in groupoids.

\[
\text{St}(p): \mathcal{C} \to \text{Grpd}: \mathcal{C} \to \text{groupoid of functors } \mathcal{C}/_{/x} \rightarrow \mathcal{D}
\]

proving criterion maps.

For \( x: c \to d \), \( x^*: \text{St}(p)(x) \to \text{St}(p)(c) \) restricts along \( \mathcal{C}/c \to \mathcal{C}/d \)

\((x \to c) \mapsto (x \to c \cong d) \).

Then \( \text{St}(p)(c) \xrightarrow{a_0(c,x)} p^*(c) \) an iso of cats.

For: These two constructions are mutually inverse, \( p \to (\text{natural}) \) equivalence.

So a stack is really just a certain presheaf of groupoids. In particular, it determines a sheaf \( \tilde{\mathcal{F}}_o(X) \): \( \mathcal{C} \to \text{Set} \), the associated sheaf of the presheaf \( c \mapsto \mathcal{O}_c(X(c))/_{/i_0} \).

**Def.** A map of stacks \( X \to Y \) is a surjection iff the map

\( \tilde{\mathcal{F}}_o(X) \to \tilde{\mathcal{F}}_o(Y) \) is a surjection.

**Weak limit of stacks**

let \( C: I \to \text{Cat} \) a diagram of categories: \( C \xrightarrow{\alpha: c \mapsto C_0} \) for \( x \in I \).

The pseudo-limit \( \lim C \) \( \mathcal{C} \) is the following category:

- objects: tuples \( (x, c, h, h_0) \) for \( x \in I \)

The composition of \( (x, c, h, h_0) \) \( \xrightarrow{\alpha} \tilde{\mathcal{C}}_1 \) \( (y, d, h', h_1) \) is given by

\[
(\alpha^{-1}(x,y), c \to d, h_0 \circ \alpha_0, h' \
\]

is universal among cones of the form

\[
\begin{array}{ccc}
C_i & \xrightarrow{h_i} & C_j \\
\downarrow \alpha_i & & \downarrow \alpha_j \\
C_k & \xrightarrow{h_k} & C_l
\end{array}
\]

**Ex:** \( G \) a group, \( B(G) = \{ d \in \text{Aut}(x) \mid G \} \). Then \( \times x H \times \cong G \).

For: \( X: C^o \to \text{Grpd} \), \( \{ U_i \to X \} \) a cover in \( C \).

\[
\text{diagram } X(u_i) \xrightarrow{\Delta} \times X(u_i) \xrightarrow{\Delta} \times X(u_i, u_j, u_k) \xrightarrow{\Delta} \times X(u_i, u_j, u_k, u_
l)
\]

Then \( \text{Desc}(U, X) \equiv \text{holim of the simplicial diagram, } i.e. \)

\( X \) is a stack if \( \Delta \) is a holim diagram.

**Homotopy limit of stacks:** In terms of factors \( C^o \to \text{Grpd} \): compute holim of gods.

**In terms of fibered categories:** compute the holim over \( \mathcal{C} \).

(For pullbacks, it does not matter.)

**Ex:** \( \text{Vec}^o: C^o \to \text{Grpd} \) is an algebraic stack.
Ex: $\text{Vect}^P : C^p \to \text{Grp}$ is an adjoint stack.

An also is given by $x \mapsto \text{Vec}^x$ taking the trivial vector bundle.