The perverse t-structure

Milan Lopuhaä

March 15, 2017

1 The perverse t-structure

The goal of today is to define the perverse t-structure and perverse sheaves, and to show some properties of both. In his talk Ben already defined $D_c^b(X, \mathbb{Q}_l)$, where X is separated of finite type over a field k in which l is invertible. We also want to consider a similar category over the complex numbers.

Definition 1.1. Let X be a complex algebraic variety. The category $D_c^b(X^{an}, \mathbb{Q})$ is the full subcategory of $K \in D(X^{an}, \mathbb{Q})$ that are bounded and are locally constant with respect to some algebraic stratification of X, i.e. there is some finite decomposition $X = \bigsqcup_i X_i$ of X into locally closed subschemes of X such that for every i and every integer n the sheaf $j_i^* \mathcal{H}^n K$ is locally constant, where $j_i \colon X_i \hookrightarrow X$ is the inclusion morphism.

Throughout this talk, X is either of the type of Bens's talk, or a complex algebraic variety.

Definition 1.2. Let X be as before. Then a complex $K \in D_c^b$ is in ${}^p D_c^{b,\leq 0}$ if for every point $x \in X$ with inclusion $i_x \colon \mathrm{Spec}(\kappa(x)) \hookrightarrow X$ and every $j > -\dim(x)$ we have $\mathcal{H}^j(i_X^* K) = 0$. Similarly, a complex $K \in D_c^b$ is in ${}^p D_c^{b,\geq 0}$ if for every point $x \in X$ with inclusion $i_x \colon \mathrm{Spec}(\kappa(x)) \hookrightarrow X$ and every $j < -\dim(x)$ we have $\mathcal{H}^j(i_X^! K) = 0$.

Remark 1.3. Another way to formulate the perverse t-structure is:

$$B \in {}^p D_c^{b,\geq 0} \Leftrightarrow \dim \mathrm{supp} \mathcal{H}^{-i} B \leq i \, \forall i,$$

$$B \in {}^p D_c^{b,\geq 0} \Leftrightarrow \dim \mathrm{supp} \mathcal{H}^{-i} \mathcal{D} B \leq i \, \forall i,$$

where $\mathcal{D}$ is the Verdier duality.

Remark 1.4. Let $K \in \mathcal{D}_c^b$, and let $U \xhookrightarrow{j} X$ be an open subset of X. Let $F \xhookrightarrow{i} X$ be its complement. Then

$$K \in {}^p D_c^{b,\leq 0} \Leftrightarrow j^* K \in {}^p D_c^{b,\leq 0} \text{ and } i^* X \in {}^p D_c^{b,\leq 0};$$

$$K \in {}^p D_c^{b,\geq 0} \Leftrightarrow j^! K \in {}^p D_c^{b,\geq 0} \text{ and } i^! X \in {}^p D_c^{b,\geq 0}.$$

If U is dense and universally smooth (i.e., $(U_{\bar{k}})_{\mathrm{red}}$ is smooth over $\bar{k}$), then this simplifies to

$$K \in {}^p D_c^{b,\leq 0} \Leftrightarrow j^* K \in D_c^{b,\leq -\dim(U)} \text{ and } i^* X \in {}^p D_c^{b,\leq 0};$$

$$K \in {}^p D_c^{b,\geq 0} \Leftrightarrow j^! K \in D_c^{b,\geq -\dim(U)} \text{ and } i^! X \in {}^p D_c^{b,\geq 0}.$$

Theorem 1.5. *Let X be as above. Then $({}^pD_c^{b,\leq 0}, {}^pD_c^{b,\geq 0})$ is a t-structure on D_c^b.*

Remark 1.6. This definition depends on the so-called *middle perversity*, i.e. the function $X \to \mathbb{Z}$ given by $x \mapsto \dim(x)$. Another such a function satisfying some conditions is called a *perversity*, and different perversities give rise to different t-structures. In this seminar we will only discuss the middle perversity.

Proof of Theorem 1.5. We check the three properties of t-structures:

1. *If $K \in {}^pD_c^{b,\leq 0}$ and $L \in {}^pD_c^{b,>0}$, then $Hom(K, L) = 0$:* We prove this by induction on $\dim(X)$. It is clear for X zerodimensional. Suppose X is of dimension n and that the statement has been proven for all dimensions smaller than n. From the distinguished triangle $j_!j^!K \to K \to i_*i^*K$ we get an exact sequence
$$\mathrm{Hom}(i_*i^*K, L) \to \mathrm{Hom}(K, L) \to \mathrm{Hom}(j_!j^!K, L).$$
By the adjunctions we know, the first term is isomorphic to $\mathrm{Hom}(i^*K, i^!L)$, which is zero because of the induction hypothesis. Also, the last term in the sequence is isomorphic to $\mathrm{Hom}(j^*B, j^!C)$, which again is zero, because $j^*B \in D_\mathrm{c}^{\mathrm{b},\leq -\dim(U)}$ and $j^!C \in D_\mathrm{c}^{\mathrm{b},>-\dim(U)}$.

2. ${}^pD_\mathrm{c}^{\mathrm{b},\leq 0} \subset {}^pD_\mathrm{c}^{\mathrm{b},\leq 1}$, ${}^pD_\mathrm{c}^{\mathrm{b},\geq 1} \subset {}^pD_\mathrm{c}^{\mathrm{b},\geq 0}$: this is clear.

3. *There exists a distinguished triangle $A \to K \to B$ with $A \in {}^pD_c^{b,\leq 0}$ and $B \in {}^pD_c^{b,>0}$:* again we proceed by induction on $\dim(X)$. If $\dim(X) = 0$, the proposition is clear. Now assume X has dimension d and that the statement has been proven for all schemes of dimension $< d$. Let K be a complex in D_c^b, and let U be an essentially smooth (i.e. $(U_{\bar{k}})_\mathrm{red}$ is smooth over $\bar{k}$) dense open subscheme of X on which the H^nK are locally constant. Then $\tau_{\leq -\dim(U)}K|_U \in {}^\mathrm{p}D_\mathrm{c}^{\mathrm{b},\leq 0}(U)$ and $\tau_{>-\dim(U)}K|_U \in {}^\mathrm{p}D_\mathrm{c}^{\mathrm{b},>0}(U)$, so
$$\tau_{\leq -\dim(U)}K|_U \to K|_U\tau_{>-\dim(U)}K|_U$$
is a distinguished triangle on U that defines the required distinquished triangle on U. Let F be the complement of U; then $\dim(F) < \dim(X)$, so by the induction hypothesis the perverse t-structure is indeed a t-structure on F, and there exists a distinguished triangle
$${}^\mathrm{p}\tau_{\leq 0}K|_F \to K|_F{}^\mathrm{p}\tau_{>0} \to K|_F$$
that is our required distinguished triangle on F. Now, by what we know of recollements, we can glue the standard t-structure on U and the perverse t-structure on F to find a t-structure on D_c^b, and in this t-structure a triangle
$$A \to K \to B$$
that lifts the two triangles mentioned before. Although the t-structure we get in this way is not the perverse t-structure, the triangle is still distinguished, and in fact $A \in {}^pD_\mathrm{c}^{\mathrm{b},\leq 0}$ and $B \in {}^pD_\mathrm{c}^{\mathrm{b},>0}$ by Remark 1.4

□

Definition 1.7. The abelian category of perverse sheaves $\mathrm{Perv}(X)$ is the heart of the perverse t-structure.

2 t-exact functors

Definition 2.1. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two t-structured categories with hearts $\mathcal{C}_1$ and $\mathcal{C}_2$, and let $f: \mathcal{C} \to \mathcal{D}$ be a morphism of categories; then we denote ${}^{\mathrm{p}}f = \mathrm{H}^0 \circ f \circ \epsilon$. Here $\mathrm{H}^0 = \tau_{\geq 0}\tau_{\leq 0}$.

Definition 2.2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two t-structured categories with hearts $\mathcal{C}_1$ and $\mathcal{C}_2$, and let $f: \mathcal{C} \to \mathcal{D}$ be amorphism of triangulated abelian categories. We say that f is *left t-exact* if $f(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$, *right t-exact* if $f(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$, and *$t$-exact* if both are true.

Lemma 2.3. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two t-structured categories with hearts $\mathcal{C}_1$ and $\mathcal{C}_2$, and let $f: \mathcal{C} \to \mathcal{D}$ be a morphism of categories.*

1. *If f is (left, right) t-exact, then ${}^{p}f$ is (left, right) exact.*
2. *If f is left (right) t-exact and K is an element of $\mathcal{D}_1^{\geq 0}$, then ${}^{p}fH^0K \xrightarrow{\sim} H^0fK$.*
3. *Suppose f has a left adjoint $g: \mathcal{D}_2 \to \mathcal{D}_1$. Then g is right t-exact if and only if f is left t-exact, and in this case $({}^{p}g, {}^{p}f)$ form an adjoint pair.*
4. *If both f and some $h: \mathcal{D}_2 \to \mathcal{D}_3$ are (left, right) t-exact, then $h \circ f$ is as well, and ${}^{p}(h \circ f) = {}^{p}h \circ f$.*

Proof.

1. Let $0 \to X \to Y \to Z$ be a short exact sequence in $\mathcal{C}_1$; then $fX, fY, fZ \in \mathcal{D}_2 \geq 0$, so the long exact cohomology sequence gives

$$0 \to \mathrm{H}^0 fX \to \mathrm{H}^0 fY \to \mathrm{H}^0 fZ.$$

The statement on right exactness is dual to this one.

2. If $K \in \mathcal{D}_1^{\geq 0}$, then $\mathrm{H}^0 K \to K \to \tau_{>0} K$ is a distinguished triangle, hence so is $f\mathrm{H}^0 K \to fK \to f\tau_{>0}K$. Since $f\tau_{>0}K$ is an element of $\mathcal{D}_2^{>0}$, the long exact sequence gives an isomorphism $\mathrm{H}^0 f\mathrm{H}^0 K \xrightarrow{\sim} \mathrm{H}^0 fK$.

3. Suppose f is left t-exact, and let $U \in \mathcal{D}_1^{>0}$ and $V \in D_2^{\leq 0}$. Then $\mathrm{Hom}(gV, U) = \mathrm{Hom}(V, fU) = 0$. Since this is true for all U, one has $\tau_{>0} gV = 0$, hence $gV \in \mathcal{D}_1^{\leq 0}$, so g is right t-exact. For $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$, we find $\mathrm{H}^0 gB = \tau_{\leq 0} gB$ and $\mathrm{H}^0 fA = \tau_{\geq 0} fA$; this gives a functorial isomorphism

$$\mathrm{Hom}(\mathrm{H}^0 gB, A) = \mathrm{Hom}(gB, A) = \mathrm{Hom}(B, fA) = \mathrm{Hom}(B, \mathrm{H}^0 fA).$$

4. The first point is trivial; furthermore for every $A \in \mathcal{C}_1$ one has

$${}^{\mathrm{p}}(h \circ f)A = \mathrm{H}^0 hfA = \mathrm{H}^0 h\mathrm{H}^0 fA$$

by point 2.

□

3 t-exactness in the geometric setting

Proposition 3.1. *Let X be as before, let $U \overset{j}{\hookrightarrow} X$ be a Zariski open on X, and let $F \overset{i}{\hookrightarrow} X$ be its closed complement. Consider the perverse t-structure on all schemes.*

1. *$j_!$ and i^* are right t-exact, j_* and $i^!$ are left t-exact, and $j^*(= j^!)$ and $i_*(= i_!)$ are t-exact.*
2. *There are adjunctions $({}^pi^*, {}^pi_*)$, $({}^pi_!, {}^pi^!)$, $({}^pj_!, {}^pj^!)$, $({}^pj^*, {}^pj_*)$.*
3. *The compositions ${}^pj^* \circ {}^pi_*, {}^pi^*{}^pj_!, {}^pi^!{}^pj_*$ are zero.*
4. *For $A \in Perv(F)$ and $B \in Perv(U)$ one has*
$$Hom({}^pj_!B, {}^pi_*A) = Hom({}^pi_*A, {}^pj_*B) = 0.$$
5. *For $A \in Perv(X)$ the sequences*
$${}^pj_!{}^pj^*A \to A \to {}^pi_*{}^pi^*A \to 0$$
and
$$0 \to {}^pi_*{}^pi^!A \to A \to {}^pj_*{}^pj^*A$$
are exact.
6. *${}^pi_*$, ${}^pj_!$ and ${}^pj_*$ are fully faithful, i.e. the natural transformations ${}^pi^*{}^pi_* \to id \to {}^pi^!{}^pi_*$ and ${}^pj^*{}^pj_* \to id \to {}^pj^*{}^pj_!$ are isomorphisms.*

Proof.
1. By definition of the perverse t-structure, $j^* = j^!$ is t-exact, i^* is right t-exact, and $i^!$ is left t-exact. By Lemma 2.3.3 $i_* = i_!$ is t-exact, $j_!$ is right t-exact, and $i^!$ is left t-exact. The result now follows from Lemma 2.3.1.
2. This now follows directly from Lemma 2.3.3.
3. This follows from $j^*i_* = 0$ (etc).
4. This is a direct consequence of the former statement.
5. This follows from the fact that
$$j_!j^*A \to A \to i_*i^*A$$
and
$$i_*i^!A \to A \to j_*j^*A$$
are distinguished.
6. This follows from the fact that these are isomorphisms without the ^{p}.

□

Proposition 3.2. *Let $f\colon X \to Y$ be a quasifinite morphism. Then $f_!$ and f^* are right t-exact, and $f^!$ are f_* are left t-exact.*

Proof. Let $K \in D^{\mathrm{b}}_{\mathrm{c}}(X)$. One has that $K \in D^{\mathrm{b},\leq 0}_{\mathrm{c}}(X)$ if and only if $\dim\mathrm{Supp}\mathrm{H}^i K \geq i$. $f_!$ is exact on sheaves, so $\mathrm{H}^i\mathrm{R}f_! K = f_!\mathrm{H}^i K$, so $\mathrm{Supp}\mathrm{H}^i f_! K = \overline{f(\mathrm{Supp}\,\mathrm{H}^i K)}$ and $\dim \mathrm{Supp}\,\mathrm{H}^i f_! K = \dim \mathrm{Supp}\,\mathrm{H}^i K$, which proves that $f_!$ is right t-exact. The proof for f^* is similar, and the result for $f^!$ and f_* follow by adjunction. □

Theorem 3.3. *Let X and Y be separated schemes of finite type over k, and let l be a prime invertible in k. If $f\colon X \to Y$ is an affine morphism, the functor $f_*\colon D^b_c(X, \bar{\mathbb{Q}}_l) \to D^b_c(Y, \bar{\mathbb{Q}}_l)$ is right t-exact.*

Proof. Let F be a constructible sheaf (on X or Y), and let $d(F)$ be the smallest integer such that $F \in {}^pD_c^{\leq d(F)}$. If K is an object in the derived category, then we define $d(K) = \sup(i + d(\mathrm{H}^i K))$; then again $d(K)$ is the smallest d such that $K \in {}^pD_c^{\leq d(K)}$; hence $K \in {}^pD_c^{\leq 0}$ if and only if all the $\mathrm{H}^i K[-i]$ are. Hence we need to show that if F is a sheaf such that $d(F) \leq d$, then $d(\mathrm{R}^i f_* F) \leq d - i$; but this is proven in (SGA4, XIV 3.1). □

Corollary 3.4. *With the notation as above, $f_!$ is left t-exact.*

Proof. This follows from Verdier duality. □

Corollary 3.5. *If $f\colon X \to Y$ is quasi-finite and affine, the functors $f_!$ and f_* are t-exact.*