The perverse $t$-structure

Milan Lopuhaä

March 15, 2017

1 The perverse $t$-structure

The goal of today is to define the perverse $t$-structure and perverse sheaves, and to show some properties of both. In his talk Ben already defined $D^b_c(X, \mathbb{Q}_l)$, where $X$ is separated of finite type over a field $k$ in which $l$ is invertible. We also want to consider a similar category over the complex numbers.

**Definition 1.1.** Let $X$ be a complex algebraic variety. The category $D^b_c(X^{an}, \mathbb{Q})$ is the full subcategory of $K^2D(X^{an}, \mathbb{Q})$ that are bounded and are locally constant with respect to some algebraic stratification of $X$, i.e. there is some finite decomposition $X = \bigsqcup_i X_i$ of $X$ into locally closed subschemes of $X$ such that for every $i$ and every integer $n$ the sheaf $j_i^*H^nK$ is locally constant, where $j_i : X_i \hookrightarrow X$ is the inclusion morphism.

Throughout this talk, $X$ is either of the type of Bens’s talk, or a complex algebraic variety.

**Definition 1.2.** Let $X$ be as before. Then a complex $K \in D^b_c$ is in $pD^b_c; 0$ if for every point $x \in X$ with inclusion $i_x : \text{Spec}(\mathbb{C}(x)) \hookrightarrow X$ and every $j > -\dim(x)$ we have $H^j(i_x^*K) = 0$. Similarly, a complex $K \in D^b_c$ is in $pD^b_c; 0$ if for every point $x \in X$ with inclusion $i_x : \text{Spec}(\mathbb{C}(x)) \hookrightarrow X$ and every $j < -\dim(x)$ we have $H^j(i_x^*K) = 0$.

**Remark 1.3.** Another way to formulate the perverse $t$-structure is:

$$B \in pD^b_c; \leq 0 \iff \dim \text{supp} \mathcal{H}^{-1}B \leq i \forall i,$$

$$B \in pD^b_c; \geq 0 \iff \dim \text{supp} \mathcal{H}^{-1}DB \leq i \forall i,$$

where $D$ is the Verdier duality.

**Remark 1.4.** Let $K \in D^b_c$, and let $U \xrightarrow{j} X$ be an open subset of $X$. Let $F \xrightarrow{i} X$ be its complement. Then

$$K \in pD^b_c; \leq 0 \iff j^*K \in pD^b_c; \leq 0 \text{ and } i^*X \in pD^b_c; \leq 0;$$

$$K \in pD^b_c; \geq 0 \iff j^!K \in pD^b_c; \geq 0 \text{ and } i^!X \in pD^b_c; \geq 0.$$
\textbf{Theorem 1.5.} Let $X$ be as above. Then $(pD^b_{\leq 0}, pD^b_{> 0})$ is a $t$-structure on $D_c^b$.

\textbf{Remark 1.6.} This definition depends on the so-called middle perversity, i.e. the function $X \to \mathbb{Z}$ given by $x \mapsto \dim(x)$. Another such a function satisfying some conditions is called a perversity, and different perversities give rise to different $t$-structures. In this seminar we will only discuss the middle perversity.

\textbf{Proof of Theorem 1.5.} We check the three properties of $t$-structures:

1. If $K \in pD^b_{\leq 0}$ and $L \in pD^b_{> 0}$, then $\text{Hom}(K, L) = 0$: We prove this by induction on $\dim(X)$. It is clear for $X$ zerodimensional. Suppose $X$ is of dimension $n$ and that the statement has been proven for all dimensions smaller than $n$. From the distinguished triangle $j_! j^! K \to K \to i_* i^* K$ we get an exact sequence
   \[ \text{Hom}(i_* i^* K, L) \to \text{Hom}(K, L) \to \text{Hom}(j_! j^! K, L). \]

   By the adjunctions we know, the first term is isomorphic to $\text{Hom}(i^* K, i^! L)$, which is zero because of the induction hypothesis. Also, the last term in the sequence is isomorphic to $\text{Hom}(j^* B, j^! C)$, which again is zero, because $j^* B \in D_{\leq -\dim(U)}^b$ and $j^! C \in D_{> -\dim(U)}^b$.

2. $pD^b_{\leq 0} \subset pD^b_{\leq 1}, pD^b_{> 1} \subset pD^b_{> 0}$: this is clear.

3. There exists a distinguished triangle $A \to K \to B$ with $A \in pD^b_{\leq 0}$ and $B \in pD^b_{> 0}$: again we proceed by induction on $\dim(X)$. If $\dim(X) = 0$, the proposition is clear. Now assume $X$ has dimension $d$ and that the statement has been proven for all schemes of dimension $< d$. Let $K$ be a complex in $D^b_c$, and let $U$ be an essentially smooth (i.e. $(U_k)_{\text{red}}$ is smooth over $k$) dense open subscheme of $X$ on which the $H^n K$ are locally constant. Then $\tau_{\leq -\dim(U)} K|_U \in pD^b_{\leq 0}(U)$ and $\tau_{> -\dim(U)} K|_U \in pD^b_{> 0}(U)$, so
   \[ \tau_{\leq -\dim(U)} K|_U \to K|_U \to \tau_{> -\dim(U)} K|_U \]
   is a distinguished triangle on $U$ that defines the required distinguished triangle on $U$. Let $F$ be the complement of $U$; then $\dim(F) < \dim(X)$, so by the induction hypothesis the perverse $t$-structure is indeed a $t$-structure on $F$, and there exists a distinguished triangle
   \[ p\tau_{\leq 0}K|_F \to K|_F \to p\tau_{> 0}K|_F \]
   that is our required distinguished triangle on $F$. Now, by what we know of recollements, we can glue the standard $t$-structure on $U$ and the perverse $t$-structure on $F$ to find a $t$-structure on $D^b_c$, and in this $t$-structure a triangle
   \[ A \to K \to B \]
   that lifts the two triangles mentioned before. Although the $t$-structure we get in this way is not the perverse $t$-structure, the triangle is still distinguished, and in fact $A \in pD^b_{\leq 0}$ and $B \in pD^b_{> 0}$ by Remark 1.4.

\textbf{Definition 1.7.} The abelian category of perverse sheaves $\text{Perv}(X)$ is the heart of the perverse $t$-structure.
2 \( t \)-exact functors

**Definition 2.1.** Let \( D_1 \) and \( D_2 \) be two \( t \)-structured categories with hearts \( C_1 \) and \( C_2 \), and let \( f : C \to D \) be a morphism of categories; then we denote \( f^* = H^0 \circ f \circ \epsilon \). Here \( H^0 = \tau_{\geq 0} \tau_{\leq 0} \).

**Definition 2.2.** Let \( D_1 \) and \( D_2 \) be two \( t \)-structured categories with hearts \( C_1 \) and \( C_2 \), and let \( f : C \to D \) be a morphism of triangulated abelian categories. We say that \( f \) is left \( t \)-exact if \( f(D_1^{>0}) \subset D_2^{>0} \), right \( t \)-exact if \( f(D_1^{<0}) \subset D_2^{<0} \), and \( t \)-exact if both are true.

**Lemma 2.3.** Let \( D_1 \) and \( D_2 \) be two \( t \)-structured categories with hearts \( C_1 \) and \( C_2 \), and let \( f : C \to D \) be a morphism of categories.

1. If \( f \) is (left, right) \( t \)-exact, then \( f^* \) is (left, right) exact.
2. If \( f \) is left (right) \( t \)-exact and \( K \) is an element of \( D_1^{\geq 0} \), then \( f^* H^0 K \cong H^0 f K \).
3. Suppose \( f \) has a left adjoint \( g : D_2 \to D_1 \). Then \( g \) is right \( t \)-exact if and only if \( f \) is left \( t \)-exact, and in this case \((f^*, g^*)\) form an adjoint pair.
4. If both \( f \) and some \( h : D_2 \to D_3 \) are (left, right) \( t \)-exact, then \( h \circ f \) is as well, and \( f^* (h \circ f) = h^* \circ f \).

**Proof.**

1. Let \( 0 \to X \to Y \to Z \) be a short exact sequence in \( C_1 \); then \( fX, fY, fZ \in D_2^{\geq 0} \), so the long exact cohomology sequence gives
\[
0 \to H^0 fX \to H^0 fY \to H^0 fZ.
\]

The statement on right exactness is dual to this one.

2. If \( K \in D_1^{\leq 0} \), then \( H^0 K \to K \to \tau_{>0} K \) is a distinguished triangle, hence so is \( fH^0 K \to fK \to f\tau_{>0} K \). Since \( f\tau_{>0} K \) is an element of \( D_2^{\geq 0} \), the long exact sequence gives an isomorphism \( H^0 fH^0 K \cong H^0 fK \).

3. Suppose \( f \) is left \( t \)-exact, and let \( U \in D_1^{>0} \) and \( V \in D_2^{<0} \). Then \( \text{Hom}(gV, U) = \text{Hom}(V, fU) = 0 \).

Since this is true for all \( U \), one has \( \tau_{>0} gV = 0 \), hence \( gV \in D_1^{\leq 0} \), so \( g \) is right \( t \)-exact. For \( A \in C_1 \) and \( B \in C_2 \), we find \( H^0 gB = \tau_{<0} gB \) and \( H^0 fA = \tau_{\geq 0} fA \); this gives a functorial isomorphism
\[
\text{Hom}(H^0 gB, A) = \text{Hom}(gB, A) = \text{Hom}(B, fA) = \text{Hom}(B, H^0 fA).
\]

4. The first point is trivial; furthermore for every \( A \in C_1 \) one has
\[
\tau(h \circ f)A = H^0 h f A = H^0 h H^0 f A
\]

by point 2. 

\[\square\]
3 $t$-exactness in the geometric setting

**Proposition 3.1.** Let $X$ be as before, let $U \hookrightarrow X$ be a Zariski open on $X$, and let $F \hookrightarrow X$ be its closed complement. Consider the perverse $t$-structure on all schemes.

1. $j_!$ and $i^*$ are right $t$-exact, $j_*$ and $i^!$ are left $t$-exact, and $j^* (= j^!)$ and $i_* (= i_!)$ are $t$-exact.
2. There are adjunctions $(p_! p_*, p_!)$, $(p_! p_!)$, $(p_! p_!)$, $(p_! p_!)$.  
3. The compositions $p_j^* \circ p_i^* p_i^* j_!$, $p_i^! p_j^! j_!$ are zero.
4. For $A \in \text{Perv}(F)$ and $B \in \text{Perv}(U)$ one has
   $$\text{Hom}(p_j^! B, p_i^* A) = \text{Hom}(p_i^* A, p_j^! B) = 0.$$  
5. For $A \in \text{Perv}(X)$ the sequences
   $$p_j^* p_j^* A \to A \to p_i^* p_i^* A \to 0$$  
   and
   $$0 \to p_i^* p_i^! A \to A \to p_j^* p_j^! A$$  
   are exact.
6. $p_!$, $p_! j_!$ and $p_! j_!$ are fully faithful, i.e. the natural transformations $p_! p_! A \to \text{id} \to p_! p_! A$ and $p_! p_! j_! \to \text{id} \to p_! p_! j_!$ are isomorphisms.

**Proof.**
1. By definition of the perverse $t$-structure, $j^* = j^!$ is $t$-exact, $i^*$ is right $t$-exact, and $i^!$ is left $t$-exact. By Lemma 2.3.3 $i_! = i_*$ is $t$-exact, $j_!$ is right $t$-exact, and $i^!$ is left $t$-exact. The result now follows from Lemma 2.3.1.
2. This now follows directly from Lemma 2.3.3.
3. This follows from $j^* i_* = 0$ (etc).
4. This is a direct consequence of the former statement.
5. This follows from the fact that $j_! j^* A \to A \to i_* i^* A$  
   and
   $$i_* i^! A \to A \to j_* j^* A$$  
   are distinguished.
6. This follows from the fact that these are isomorphisms without the $p$.

**Proposition 3.2.** Let $f : X \to Y$ be a quasifinite morphism. Then $f_!$ and $f^*$ are right $t$-exact, and $f^!$ are $f_*$ are left $t$-exact.
Proof. Let $K \in D^b_c(X)$. One has that $K \in D^{p\leq 0}_c(X)$ if and only if $\dim \text{Supp} H^i K \geq i$. $f_!$ is exact on sheaves, so $H^i Rf_! K = f_* H^i K$, so $\text{Supp} H^i f_! K = f(\text{Supp} H^i K)$ and $\dim \text{Supp} H^i f_! K = \dim \text{Supp} H^i K$, which proves that $f_!$ is right $t$-exact. The proof for $f^*$ is similar, and the result for $f^!$ and $f_*$ follow by adjunction.

\textbf{Theorem 3.3.} Let $X$ and $Y$ be separated schemes of finite type over $k$, and let $l$ be a prime invertible in $k$. If $f : X \to Y$ is an affine morphism, the functor $f_* : D^b_c(X, \mathbb{Q}_l) \to D^b_c(Y, \mathbb{Q}_l)$ is right $t$-exact.

\textbf{Proof.} Let $F$ be a constructible sheaf (on $X$ or $Y$), and let $d(F)$ be the smallest integer such that $F \in p \mathcal{D}^{d(F)}_c$. If $K$ is an object in the derived category, then we define $d(K) = \sup (i + d(H^i K))$; then again $d(K)$ is the smallest $d$ such that $K \in p \mathcal{D}^{d(K)}_c$, hence $K \in p \mathcal{D}^{\leq 0}_c$ if and only if all the $H^i K[-i]$ are. Hence we need to show that if $F$ is a sheaf such that $d(F) \leq d$, then $d(R^i f_* F) \leq d - i$; but this is proven in (SGA4, XIV 3.1).

\textbf{Corollary 3.4.} With the notation as above, $f_!$ is left $t$-exact.

\textbf{Proof.} This follows from Verdier duality.

\textbf{Corollary 3.5.} If $f : X \to Y$ is quasi-finite and affine, the functors $f_!$ and $f_*$ are $t$-exact.