What are stacks and why should you care?

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Todays goal is twofold: I want to tell you why you would want to study stacks in the first place, and I want to define what a stack is. This is a purely category theoretical concept; the main objects of study for this seminar will be algebraic stacks, which will be defined by a volunteer next week.

1 Why should you care?

Algebraic stacks can be thought of as a suitable generalisation of schemes. As one may expect, they arise in situations where the object one would like to study turns out not be a scheme, but still has some 'geometric' properties. This section is dedicated to describing such a situation.

Definition 1.1. Let S be a scheme. An *elliptic curve* over S is a smooth projective connected group scheme over S of relative dimension 1.

You have probably already seen elliptic curves over fields at some point in your life. There are two properties of elliptic curves that will be relevant in the following discussion:

1. All elliptic curves are commutative group schemes.
2. Over a field of characteristic 0, an elliptic curve is a curve in the projective plane given by the equation $y^2 = x^3 + ax + b$ for some (not necessarily unique) constants a and b.

If $S \to S'$ is a morphism of schemes, and E' is an elliptic curve over S', then the pullback $E' \times_{S'} S$ is an elliptic curve over S. Hence we can consider the functor

$$\begin{aligned} F\colon \mathsf{Schm}^{\mathrm{opp}} &\to \mathsf{Set} \\ S &\mapsto \{\text{Ell. curves}/S\}/\cong . \end{aligned}$$

One may wonder if this functor is *representable*, i.e. whether there exists a scheme M such that $F \cong \mathrm{Hom}(-, M)$ as functors $\mathsf{Schm}^{\mathrm{opp}} \to \mathsf{Set}$; such a scheme is called a *moduli space*[1]. If this were true, this

[1] Often called *fine moduli spaces* to distinguish them from the coarse moduli spaces that will pop up soon.

would make studying elliptic curves a bit easier: The identity $\mathrm{id}_M \in \mathrm{Hom}(M, M)$ would correspond to a unique isomorphism class of elliptic curves $\mathcal{E}/M$. This

would then be called the *universal* elliptic curve, and it would have the property that for every scheme S and for every elliptic curve E/S, there would be a unique morphism $f\colon S \to M$ such that $E \cong \mathcal{E} \times_{M,f} S$ as elliptic curves over S.

Unfortunately, such an M cannot exist for this specific functor. There are two reasons for this (which are actually instances of the same reason):

1. Let E be an elliptic curve over $\mathbb{C}$. Since E is commutative, the inverse map $-1\colon E \to E$ is an automorphism of E as a group variety over $\mathbb{C}$. It turns out that this automorphism is nontrivial (in other words, E does not have exponent 2). Now we consider an elliptic curve $\mathcal{E}$ over $S = \mathrm{Spec}\mathbb{C}[X, X^{-1}]$, that we may describe analytically as follows. We know $S(\mathbb{C}) = \mathbb{C}^\times$, and $\pi_1(\mathbb{C}^\times, 1) \cong \mathbb{Z}$. Now let $\mathcal{E}$ be the elliptic curve over S such that the fibre over every point is equal to E, but such that walking along a cycle that winds once around the origin induces the automorphism -1 on a fibre. This is just an analytic construction, but by using the étale fundamental group one can do the same construction algebraically, to actually get a group scheme $\mathcal{E}$ over S.

   Now suppose F can be represented by a scheme M; then $\mathcal{E}$ corresponds to a unique morphism $S \to M$. Similarly, E corresponds to a unique $x \in M(\mathbb{C})$; hence the map $S \to M$ maps every $\mathbb{C}$-point of S to x. Hence this map is constant; but this corresponds to the constant elliptic curve $E \times_{\mathbb{C}} S$ over S, rather than $\mathcal{E}$.

2. Let k be a non-algebraically closed field, and E an elliptic curve over k given by the equation $y^2 = x^3 + ax + b$. Let $d \in k^2 \backslash k$, and let E_d be the *twist* of E given by the equation $y^2 = x^3 + d^2ax + d^3b$. Over k, these elliptic curves are nonisomorphic, but they become isomorphic over $k(\sqrt{d})$, via the transformation $(x, y) \mapsto (d^{-1}x, d^{-\frac{3}{2}}y)$. If F were represented by a scheme M, then E and E_d would correspond to two different points in $M(k)$, that map to the same point in $M(k(\sqrt{d}))$; but since M is a scheme, the map $M(k) \to M(k(\sqrt{d}))$ is injective, which is a contradiction.

   This example is actually related to the previous one in the following sense. To give a form of E over k is to give the elliptic curve $E_{\bar{k}}$, together with an action of $\mathrm{Gal}(\bar{k}/k)$ on $E(\bar{k})$. If $E_{\bar{k}}$ has automorphisms, one can 'twist' the Galois action on $E(\bar{k})$ to get a different curve over k that is the same over $\bar{k}$. The examples are even more similar when you realise that $\pi^1_{\text{ét}}(\mathrm{Spec} k) = \mathrm{Gal}(\bar{k}/k)$.

From these two examples we see that the fact that elliptic curves have nontrivial automorphisms is an obstruction to the existence to a moduli space. There are three ways to solve this problem:

1. One can decorate the elliptic curves with additional data, so that they no longer have nontrivial automorphisms. For example, for a fixed scheme S one can look at isomorphism classes of pairs (E, P), where E is an elliptic curve over S and $P \in E(S)$ is a global section of a fixed order $N \geq 3$. It turns out if restrict ourselves to schemes over $\mathbb{Z}[N^{-1}]$, these pairs have no nontrivial automorphisms, and we get a moduli space over $\mathbb{Z}[N^{-1}]$, along with a universal curve over it.

2. If F is not representable, one can hope for the existence of a 'best possible approximation', i.e. a scheme M such that there is a natural transformation $F \to \mathrm{Hom}(-, M)$, and such that M is universal with respect to this property. Such a scheme M is called a *coarse moduli space*. In the example where $F(S)$ is the set of isomorphism classes of elliptic curves over S, such a coarse moduli space exists; it is the 'j-line' $M = \mathrm{Spec}\mathbb{Z}[j]$, and the map $F \to \mathrm{Hom}(-, M)$ is given by sending an elliptic curve over S to its j-invariant. Of course, this object does not have the nice properties that a moduli space would have: for example, there is no universal elliptic curve over M. We will discuss coarse moduli spaces in more detail in the talk about Deligne-Mumford stacks.

3. The third solution is to somehow incorporate these automorphisms into F. Instead of letting $F(S)$ be the set of isomorphism classes of elliptic curves, we can consider the groupoid $\mathsf{F}(S)$, whose objects are elliptic curves over S, and whose morphisms are isomorphisms. It turns out that the morphism
$$\mathsf{F}\colon \mathsf{Schm} \to \mathsf{Grpd}$$
is not 'represented by a scheme' in any sensible way, but it still has quite a few algebro-geometrical properties.

The F from the last point is an example of an *algebraic stack*, which will be the main objects of study for this seminar. Before we can define them as algebro-geometrical objects, we first need to develop some category theory.

2 What are stacks?

It is best to think of a stack as a "sheaf of categories over a category". We emphasize that this is *not* a definition! The next sections will be devoted to defining increasingly category-theoretic notions of presheaves and sheaves.

2.1 Sites —"sheaves of sets on a category"

A site is to be thought of as a generalisation of a topological space. To a topological space X one can associate the category X whose objects are opens $U \subset X$, and whose morphisms are inclusion maps. Then a presheaf F on X is a functor $F\colon \mathsf{X}^{\mathrm{op}} \to \mathsf{Set}$. For F to be a sheaf, the following property has to hold: for every $U \in \mathsf{X}$ and every covering $(U_i \hookrightarrow U)_{i\in I}$, the canonical map

$$F(U) \to \left\{ (f_i)_i \in \prod_{i\in I} F(U_i) : f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j} \right\}$$

is bijective. We can rewrite this slightly to make it more 'categorical'. First note that $U_i \cap U_j = U_i \times_U U_j$ in the category X. There are two maps $\prod_{i\in I} F(U_i) \to \prod_{i,j\in I} F(U_i \times_U U_j)$, induced by the projections to the first and second coordinate, respectively. To say that F is a sheaf is exactly to say that the sequence

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

is exact, i.e. every element of $\prod_{i\in I} F(U_i)$ that has the same image under both maps, comes from a unique element of $F(U)$.

If we want to define sheaves on a general category, we can make sense of the last statement, provided that we know what a covering is. Since there is no canonical candidate for this, the solution is to make the coverings part of the data.

Definition 2.1. Let C be a category. A *Grothendieck topology* on C consists of a set $\mathrm{Cov}(X)$ of collections of morphisms $\{X_i \to X\}_{i\in I}$ for every object $X \in C$ such that the following hold:

1. If $X' \xrightarrow{\sim} X$ is an isomorphism, then $\{X' \xrightarrow{\sim} X\} \in \mathrm{Cov}(X)$.
2. If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$, and $Y \to X$ is any morphism, then the products $Y_i := X_i \times_X Y$ exist and $\{Y_i \to Y\}_{i\in I} \in \mathrm{Cov}(Y)$.
3. If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $\{V_{i,j} \to X_i\}_{j\in J_i} \in \mathrm{Cov}(X_i)$ for each $i \in I$, then

$$\{V_{i,i} \to X_i \to X\}_{i\in I, j\in J_i} \in \mathrm{Cov}(X).$$

A category C together with a Grothendieck topology on C is called a *site*.

Definition 2.2. Let C be a site.

1. A *presheaf on* C is a functor $\mathsf{C} \to \mathsf{Set}$.

2. A presheaf on C is called *separated* if for every $X \in \mathsf{C}$ and every $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ the map $F(X) \to \prod_i F(X_i)$ is injective.
3. A presheaf on C is called a *sheaf* if for every $X \in \mathsf{C}$ and every $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ the following sequence is exact:
$$F(X) \to \prod_{i \in I} F(X_i) \rightrightarrows \prod_{i,j \in I} F(X_i \times_X X_j).$$

Example 2.3.

1. Consider the category X associated with a topological space X. For every open $U \subset X$, we let $\mathrm{Cov}(U)$ be the set of open coverings of U (in the subspace topology), which we can consider as a set of collections of morphisms in X. This defines a Grothendieck topology on X, and sheaves on the site X coincide with sheaves on the topological space X.
2. Let C be any category, and for any $X \in \mathsf{C}$, let $\mathrm{Cov}(X) = \{\{X' \xrightarrow{\sim} X\} : X' \cong X\}$, i.e. the only coverings are isomorphisms. This is a Grothendieck topology on C. On this site all presheaves are sheaves.
3. Let X be a scheme, and consider the category Schm_X. We can put a Grothendieck topology on this category as follows: for every X-scheme U, the set $\mathrm{Cov}(U)$ is the set of collections $\{U_i \to U\}_{i \in I}$ of morphisms of X-schemes such that each $U_i \to U$ is étale and the map $\coprod_{i \in I} U_i \to U$ is surjective. Since étale morphisms are stable under base change and composition, this is indeed a Grothendieck topology. The resulting site is called the *big étale site* over X. Of course, one can replace the adjective *étale* with several others to get different sites from the same category. In particular, when we use the adjective *étale*, we mean that each $U_i \to U$ is an open immersion.
4. Let C be a site, and let $X \in \mathsf{C}$. Let C/X be the category whose objects are morphisms $Y \to X$ in C, and whose morphisms are commutative triangles
$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$
For an $Y \in \mathsf{C}/X$, define $\mathrm{Cov}(Y \to X)$ to be the set of collections of X-morphisms $\{Y_i \to Y\}_{i \in I}$ such that the same collection is a covering of Y in C. This turns C/X into a site.

2.2 Filtered categories —"presheaves of categories on a category"

Definition 2.4.

1. Let C be a category. A *category over* C is a pair (F, p) of a category F and a functor $p\colon \mathsf{F} \to \mathsf{C}$. For any object $U \in \mathsf{C}$, we write $\mathsf{F}(U)$ for the subcategory of F whose objects are $u \in \mathsf{F}$ such that $p(u) = U$, and whose morphisms are $\varphi\colon u \to u'$ in F such that $p(\varphi) = \mathrm{id}_U$.

2. A morphism $\varphi\colon u \to v$ in F is called *cartesian* if for every $\psi\colon w \to v$ in F and a factorisation

$$p(w) \xrightarrow{h} p(u) \xrightarrow{p(\varphi)} p(v)$$

of $p(\psi)$, there exists a unique $\lambda\colon w \to u$ such that $\varphi \circ \lambda = \psi$ and $p(\lambda) = h$.

3. A *fibred category over* C is a category $p\colon \mathsf{F} \to \mathsf{C}$ over C such that for every $f\colon U \to V$ in C and $v \in \mathsf{F}(V)$, there is a cartesian morphism $\varphi\colon u \to v$ such that $p(\varphi) = f$.

4. Let $p\colon \mathsf{F} \to \mathsf{C}$ and $q\colon \mathsf{G} \to \mathsf{C}$ be two fibred categories over C. A *morphism* of fibred categories $g\colon \mathsf{F} \to \mathsf{G}$ is a functor $g\colon \mathsf{F} \to \mathsf{G}$ such that

   (a) $qg = p$;

   (b) g sends cartesian morphisms in F to cartesian morphisms in G.

5. Let $g, g'\colon \mathsf{F} \to \mathsf{G}$ be two morphisms of fibred categories over C. A *base preserving natural transformation* $\alpha\colon g \to g'$ is a natural transformation such that for every $u \in \mathsf{F}$ the morphism $\alpha_u\colon g(u) \to g'(u)$ projects to the identity morphism in C.

6. We denote by $\mathsf{HOM}_{\mathsf{C}}(\mathsf{F}, \mathsf{G})$ the category whose objects are morphisms of fibred categories and whose morphisms are base preserving natural transformations.

Remark 2.5. Let $p\colon \mathsf{F} \to \mathsf{C}$ be a category over C, and let $f\colon U \to V$ be a morphism in C, let $v \in \mathsf{F}$ be a lift of V, and let $\varphi\colon u \to v$ be a cartesian lift of f. Then u is called a *pullback of v along f*, and lives in $\mathsf{F}(U)$. The cartesian property tells us that if $\varphi\colon u \to v$ and $\varphi'\colon u' \to v$, then there is a unique $\lambda\colon u \xrightarrow{\sim} u'$ in $\mathsf{F}(U)$ such that $\varphi'\lambda = \varphi$. If we choose, for every object $v \in \mathsf{F}(V)$, a pullback f^*v in $\mathsf{F}(U)$, this gives us a functor

$$f^*\colon \mathsf{F}(U) \to \mathsf{F}(V).$$

The functor itself depends on the choice of the pullbacks, but different choices give a unique transformation from one pullback functor to the other.

Example 2.6. Let C be a category, and let $F\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a presheaf. We can turn F into a fibred category F as follows. We let the objects of F be pairs (X, x), where $X \in \mathsf{C}$ and $x \in F(X)$. A morphism $(X, x) \to (Y, y)$ is a morphism $f\colon X \to Y$ in C such that $f^*y = x$. Then the natural forgetful functor $\mathsf{F} \to \mathsf{C}$ turns F into a fibred category over C. For an object X the category $\mathsf{F}(X)$ has as objects the elements of the set $F(X)$, and only identities as morphisms.

Example 2.7. Let $f\colon X \to Y$ be a morphism of schemes. For any Y-scheme T one can consider the fibre product $T \times_Y X$. However, this is only defined up to unique isomorphism. Usually this is not a problem, but it presents a technical obstacle when developing the theory of stacks. The solution lies in fibred categories. Fix $f\colon X \to Y$ as above, and let C be the category of Y-schemes. Let F be the category of cartesian diagrams

$$\begin{array}{ccc} P & \xrightarrow{b} & X \\ \downarrow a & & \downarrow f \\ T & \xrightarrow{t} & Y \end{array}$$

with the obvious notion of morphism. Then we get a forgetful functor $p\colon \mathsf{F} \to \mathsf{C}$ by sending the diagram above to the Y-scheme T, and this makes F a fibred category over C (the axiom of a fibred category comes down to the existence of fibre products). Furthermore, for any Y-scheme T, the category $\mathsf{F}(T)$ consists of all 'fibre products' of T and X over Y. Any two objects are uniquely isomorphic.

Definition 2.8. Let $p\colon \mathsf{F} \to \mathsf{C}$ be a fibred category over C. We say that F is *fibred in groupoids* if for every $U \in \mathsf{C}$ the category $\mathsf{F}(U)$ is a groupoid.

Let $p\colon \mathsf{F} \to \mathsf{C}$ be a category fibred in groupoids, and let $X \in \mathsf{C}$. Let $x, x' \in \mathsf{F}(X)$. We can then define a presheaf $\underline{\mathrm{Isom}}(x, x')\colon (\mathsf{C}/X)^{\mathrm{opp}} \to \mathsf{Set}$ as follows. For any morphism $f\colon Y \to X$, choose pullbacks $f^*x, f^*x' \in \mathsf{F}(Y)$; we set

$$\underline{\mathrm{Isom}}(x, x')(f\colon Y \to X) := \mathrm{Isom}_{\mathsf{F}(Y)}(f^*x, f^*x').$$

If we have a composition

$$Z \xrightarrow{g} Y \xrightarrow{f} X,$$

then $(fg)^*x$ and $(fg)^*x'$ are pullbacks of f^*x and f^*x' along g, and as such we get a canonical map

$$g^*\colon \underline{\mathrm{Isom}}(x, x')(f\colon Y \to X) \to \underline{\mathrm{Isom}}(x, x')(fg\colon Z \to X).$$

The presheaf $\underline{\mathrm{Isom}}(x, x')$ is independent of the choices of pullbacks, up to canonical isomorphism.

2.3 Descent —"the sheaf property for presheaves of categories"

Let $p\colon \mathsf{F} \to \mathsf{C}$ be a fibred category. Let $\{X_i \to Y\}_{i\in I}$ be a collection of morphisms in C. We define a category $\mathsf{F}(\{X_i \to Y\}_{i\in I})$ as follows:

- The objects are collections of data $(\{E_i\}_{i\in I}, \{\sigma_{i,j}\}_{i,j\in I})$ where $E_i \in \mathsf{F}(X_i)$, and $\sigma_{i,j}\colon \mathrm{pr}_1^* E_i \to \mathrm{pr}_2^* E_j$ is an isomorphism in $\mathsf{F}(X_i \times_Y X_j)$ such that the following diagram in $\mathsf{F}(X_i \times_Y X_j \times_Y X_k)$ commutes for all $i, j, k \in I$:

$$\begin{array}{ccccc} \mathrm{pr}_{12}^*\mathrm{pr}_1^* E_i & \xrightarrow{\mathrm{pr}_{12}^*\sigma_{ij}} & \mathrm{pr}_{12}^*\mathrm{pr}_2^* E_j & = & \mathrm{pr}_{23}^*\mathrm{pr}_1^* E_j \\ \| & & & & \downarrow \mathrm{pr}_{23}^*\sigma_{jk} \\ \mathrm{pr}_{13}^*\mathrm{pr}_1^* E_i & \xrightarrow{\mathrm{pr}_{13}^*\sigma_{ik}} & \mathrm{pr}_{13}^*\mathrm{pr}_2^* E_k & = & \mathrm{pr}_{23}^*\mathrm{pr}_2^* E_k \end{array}$$

- The morphisms are collections of morphisms $\{g_i \colon E_i \to E'_i\}_{i\in I}$ such that $\sigma'_{ij}\mathrm{pr}_1^* g_i = \mathrm{pr}_2^* g_j \sigma_{ij}$.

The commutativity condition on the σ_{ij} is to be interpreted as follows: there are two ways to compare the pullbacks of E_i and E_k in $\mathsf{F}(X_i \times_Y X_j \times_Y X_k)$, namely via σ_{ik}, or via σ_{ij} and σ_{jk}, and these two ways should give the same result. The collection $\{\sigma_{ij}\}_{i,j\in I}$ is sometimes referred to as *descent data* on the $\{E_i\}_{i\in I}$. There is a natural functor $\mathsf{F}(Y) \to \mathsf{F}(\{X_i \xrightarrow{f_i} Y\}_{i\in I})$ given by sending an object $E \in \mathsf{F}(Y)$ to $(\{f_i^*E\}_{i\in I}, \{\sigma_{ij}\}_{i,j\ inI})$ where the $\sigma_{ij} \colon \mathrm{pr}_1^* f_i^* E \to \mathrm{pr}_2^* f_j^* E$ are the unique morphisms induced by the fact that both $\mathrm{pr}_1^* f_i^* E$ and $\mathrm{pr}_2^* f_j^* E$ are pullbacks of E along the morphism $X_i \times_Y X_j \to Y$.

Definition 2.9. The collection of morphisms $\{X_i \to Y\}_{i\in I}$ is *of effective descent for* F if the functor $\mathsf{F}(Y) \to \mathsf{F}(\{X_i \to Y\}_{i\in I})$ is an equivalence of categories.

Definition 2.10. For an object $(\{E_i\}_{i\in I}, \{\sigma_{i,j}\}_{i,j\in I}) \in \mathsf{F}(\{X_i \to Y\}_{i\in I})$ we say that the descent data $\{\sigma_{ij}\}_{i,j\in I}$ is *effective* if $(\{E_i\}_{i\in I}, \{\sigma_{i,j}\}_{i,j\in I})$ is in the essential image of the functor

$$\mathsf{F}(Y) \to \mathsf{F}(\{X_i \to Y\}_{i\in I}).$$

Example 2.11. Let C be a site, and let F be a presheaf on C. Let $\mathsf{F} \to \mathsf{C}$ be the associated fibred category over C. Then F is a sheaf if and only if for every $Y \in \mathsf{C}$, every covering of Y is of effective descent for F. Note that in this example the commutation relation on the σ_{ij} is vacuous, since the only isomorphisms are identities.

2.4 Stacks

Definition 2.12. Let C be a site. A category F fibred in groupoids over C is a *stack* if for every $X \in \mathsf{C}$ and every covering $\{X_i \to X\}_{i\in I}$ of X the functor

$$\mathsf{F}(X) \to \mathsf{F}(\{X_i \to X\}_{i\in I}).$$

is an equivalence of categories.

The following result is straightforward:

Proposition 2.13. *Let F be a category fibred in groupoids over C. Then F is a stack if and only if the following two conditions hold:*

1. *For any $X \in \mathsf{C}$ and $x, y \in \mathsf{F}(X)$, the presheaf $\underline{Isom}(x, y)$ on C/X is a sheaf.*
2. *For any covering $\{X_i \to X\}$ of an object $X \in \mathsf{C}$, any descent data with respect to this covering is effective.*

Example 2.14. Let C be the category of schemes, regarded as the big Zariski site over $\mathrm{Spec}\mathbb{Z}$ as before; hence a covering of a scheme X is simply a collection of open immersions $\{X_i \hookrightarrow X\}_{i\in I}$ whose images cover X. Let F be the category whose objects are pairs (X, E) of a scheme X and an elliptic curve E over X, and whose morphisms are cartesian diagrams

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

Then the forgetful functor $\mathsf{F} \to \mathsf{C}$ sending (X, E) to X turns F into a fibred category over C (the existence of pullbacks is again just the existence of pullbacks of elliptic curves). Also, any morphism in F between (X, E) and (X, E') that is the identity on X is an isomorphism, so F is fibred in groupoids. Finally F is a stack: to show this with the proposition above we have to show the following two things:

1. For any scheme X, any two elliptic curves E, E' over X, any covering $\{X_i \hookrightarrow X\}$ of X, and any 'compatible' collection of isomorphisms

$$E \times_X X_i \xrightarrow{\sim} E' \times_X X_i$$

   there is a unique isomorphism $E \tilde{E}'$ of elliptic curves over X gluing these data.

2. For any covering $\{X_i \hookrightarrow X\}_{i \in I}$ of a scheme X, any collection of elliptic curves E_i/X_i, and any compatible collection of isomorphisms

$$E_i \times_{X_i} (X_i \times_X X_j) \xrightarrow{\sim} E_j \times_{X_j} (X_i \times_X X_j)$$

   there is a E/X gluing these data.

This is algebraic geometry however, and since this talk was just about category theory we will leave this problem be for now.

To finish this talk we present the following theorem without proof.

Theorem 2.15. *Let* F *be a category fibred in groupoids over* C*. Then there exists a 'unique' stack over* F^a *over* C*, called the* stackification of F*, and a morphism of fibred categories* $q\colon \mathsf{F} \to \mathsf{F}^a$ *such that for any stack* G *over* C *the induced functor*

$$\mathsf{HOM}_{\mathsf{C}}(\mathsf{F}^a, \mathsf{G}) \to \mathsf{HOM}_{\mathsf{C}}(\mathsf{F}, \mathsf{G})$$

is an equivalence of categories.

Remark 2.16. The word *unique* in the theorem is to be interpreted as follows. If $(\mathsf{F}^{\mathrm{a}\prime}, q')$ also satisfies the conditions of the theorem, there is a unique pair (h, η), where $h\colon \mathsf{F}^{\mathrm{a}\prime} \to \mathsf{F}^{\mathrm{a}}$ is an equivalence of fibred categories, and $\eta\colon hq' \cong q$ is an isomorphism in $\mathsf{HOM}_{\mathsf{C}}(\mathsf{F}, \mathsf{F}^{\mathrm{a}})$.