1. Moduli of sheaves

1.1. **Families.** In this talk I will give an overview of moduli of sheaves mostly following Huybrechts-Lehn. Throughout $(X, \mathcal{O}_X(1))$ is a projective k-scheme with very ample line bundle.[1] The goal is to consider the collection M of isomorphism classes $[\mathcal{F}]$, where $\mathcal{F}$ is a coherent sheaf on X and make this set into a scheme "in a natural way". This means that the scheme structure on M should be well-behaved with respect to *families* of coherent sheaves.

Definition 1.1. Let B be any base scheme. A *flat family* of coherent sheaves over B is a coherent sheaf on $X \times B$ which is B-flat.

Since M is huge, we want to fix some "topological data" which does not change in flat families. For each coherent sheaf $\mathcal{E}$ on X

$$P_{\mathcal{E}}(t) := \chi(\mathcal{E}(t)) = \sum_{i=0}^{d} \frac{\alpha_i(\mathcal{E})}{i!} t^i \in \mathbb{Q}[t],\ \alpha_d(\mathcal{E}) \neq 0$$

is known as the *Hilbert polynomial* of $\mathcal{E}$. Moreover, $p_{\mathcal{E}}(t) := P_{\mathcal{E}}(t)/\alpha_d(\mathcal{E})$ is known as the *reduced Hilbert polynomial.* One way to see polynomiality is by Hirzebruch-Riemann-Roch (overkill). Facts:

- $d = \dim(\mathcal{E}) := \dim(\mathrm{Supp}(\mathcal{E}))$ and $\mathrm{rk}(\mathcal{E}) := \frac{\alpha_d(\mathcal{E})}{\alpha_d(\mathcal{O}_X)}$,
- for any family $\mathcal{F}$ over B: if $\mathcal{F}$ is B-flat then $P_{\mathcal{F}_b}(t)$ is locally constant on B.

Flat families form a contravariant functor under pull-back: for any $P(t) \in \mathbb{Q}[t]$, we define $\underline{M}_P(B)$ to be the collection of isomorphism classes of B-flat families $\mathcal{F}$ such that

$$P_{\mathcal{F}_b}(t) = P(t),\ \forall b \in B.$$

It is convenient to work with a slightly more general equivalence relation on families: we identify two B-flat families $\mathcal{F}, \mathcal{F}'$ whenever $\mathcal{F} \cong \mathcal{F}' \otimes p_B^* L$, where L is a line bundle on B and p_B denotes projection. Then

$$\underline{M}_P(-) : (\mathrm{Sch}/k)^o \longrightarrow \mathrm{Sets},$$

defines a contravariant functor known as the moduli functor. Note: $\underline{M}_P(\mathrm{Spec}\ k)$ is the set we are originally interested in.

[1] All schemes in this talk are k-schemes of finite type, where k is an algebraically closed field.

1.2. **Deformations.** Before considering arbitrary flat families, it is natural to study flat families over $B = \text{Spec } A$, where A is a "thickened point", or more formally "a local Artinian k-algebra with residue field k". For a fixed sheaf $\mathcal{E}$ on X, the restriction

$$(\text{Artin}/k) \longrightarrow \text{Sets}, A \mapsto \{[\mathcal{F}] \in \underline{M}_P(\text{Spec } A) : \mathcal{F}|_{X\times \text{Spec}\,(k)} \cong \mathcal{E}\}$$

is called the *deformation functor* of $\mathcal{E}$. Here Spec $k \hookrightarrow$ Spec A is induced by the map to the residue field.

The first non-trivial local Artinian ring is $D := k[\epsilon]/(\epsilon^2)$ known as the *ring of dual numbers*. Deformations over the dual numbers are known as *first order deformations*. I will now explain why $\text{Ext}^1_X(\mathcal{E},\mathcal{E})$ is the set of first order deformations. We have a natural short exact sequence

$$0 \to (\epsilon) \to D \to k \to 0. \tag{1}$$

Given a first order deformation $\mathcal{F}$ of $\mathcal{E}$. Then $\mathcal{F} \otimes_D k \cong \mathcal{E}$. Using flatness, we get an induced sequence of coherent $\mathcal{O}_X \otimes D$-modules

$$0 \to \mathcal{F} \otimes_D (\epsilon) \to \mathcal{F} \to \mathcal{E} \to 0.$$

As D-modules, $(\epsilon) \cong k$ and $\mathcal{F} \otimes_D (\epsilon) \cong \mathcal{F} \otimes_D k \cong \mathcal{E}$. We obtain an element of $\text{Ext}^1_X(\mathcal{E},\mathcal{E})$. Conversely, for an extension $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{E} \to 0$ of coherent $\mathcal{O}_X$-modules, we define multiplication by ϵ on $\mathcal{F}$ by

$$\epsilon\cdot : \mathcal{F} \to \mathcal{E} \to \mathcal{F}.$$

Inductively, having a deformation $\mathcal{F}$ of $\mathcal{E}$ over $k[\epsilon]/(\epsilon^{k-1})$, one can consider

$$0 \to (\epsilon^{k-1}) \to k[\epsilon]/(\epsilon^k) \to k[\epsilon]/(\epsilon^{k-1}) \to 0$$

and ask to classify deformations of $\mathcal{E}$ over $k[\epsilon]/(\epsilon^k)$, which pull-back to $\mathcal{F}$. More generally, given any short exact sequence $0 \to \mathfrak{a} \to A' \to A \to 0$ in Artin/k with $\mathfrak{a}$ a principal ideal satisfying $\mathfrak{a} \cdot \mathfrak{m}' = 0$ and a flat family $\mathcal{F}$ over A, one can ask to classify the lifts. The answer is more complicated. *Suppose $\mathcal{E}$ is simple, i.e.* $\text{End}(\mathcal{E}) = \mathbb{C}$. Then it turns out that the set of lifts forms an $\text{Ext}^1_X(\mathcal{E},\mathcal{E}) \otimes_k \mathfrak{a}$-torsor. This means: the set of lifts may be empty, but if not then there is a fully faithful transitive action of $\text{Ext}^1(\mathcal{E},\mathcal{E}) \otimes_k \mathfrak{a}$ on the set of lifts. I will mention at the end how this fact follows for $\mathcal{E}$ stable using the construction of the moduli space, Luna's Étale Slice Theorem, and Schlessinger's Criterion (which I will not discuss). A direct proof can be found in Hartshorne's book on deformation theory.

1.3. **Obstructions.** The following approach is taken from Artamkin. Given a setup $0 \to \mathfrak{a} \to A' \to A \to 0$ as above, a flat family $\mathcal{F}$ over A with $\mathcal{E} \cong \mathcal{F} \otimes_A k$ simple. We claim there exists a natural class $ob \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \otimes_k \mathfrak{a}$ such that $ob = 0$ if and only if lifts exist. Consider the short exact sequences

$$0 \to \mathfrak{m} \to A \to k \to 0$$
$$0 \to \mathfrak{a} \to \mathfrak{m}' \to \mathfrak{m} \to 0.$$

The key observation is that the latter can be seen as a short exact sequence of A-modules using the fact that $\mathfrak{a} \cdot \mathfrak{m}' = 0$. Applying $\mathcal{F} \otimes_A -$ and using flatness gives short exact sequences

$$0 \to \mathcal{F} \otimes_A \mathfrak{m} \to \mathcal{F} \to \mathcal{E} \to 0$$
$$0 \to \mathcal{E} \otimes_k \mathfrak{a} \to \mathcal{F} \otimes_A \mathfrak{m}' \to \mathcal{F} \otimes_A \mathfrak{m} \to 0,$$

where we use $\mathcal{F} \otimes_A \mathfrak{a} \cong \mathcal{E} \otimes_k \mathfrak{a}$. We call the second extension ξ and use the first extension to obtain a long exact sequence:

$$\cdots \to \mathrm{Ext}^1_X(\mathcal{F}, \mathcal{E} \otimes_k \mathfrak{a}) \to \mathrm{Ext}^1_X(\mathcal{F} \otimes_A \mathfrak{m}, \mathcal{E} \otimes_k \mathfrak{a}) \to \mathrm{Ext}^2_X(\mathcal{E}, \mathcal{E} \otimes_k \mathfrak{a}) \to \cdots$$

Denoting the image of ξ in $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes_k \mathfrak{a})$ by ob, we see that $ob = 0$ if and only if ξ comes from an extension

$$0 \to \mathcal{E} \otimes_k \mathfrak{a} \to \mathcal{F}' \to \mathcal{F} \to 0.$$

Working out the details you will find that $\mathcal{F}'$ exactly produces the lift (this requires work: you need to construct an A'-module structure on $\mathcal{F}'$). See Artamkin's paper for details.

Coarse/fine moduli spaces. Suppose $\underline{M}$ is a moduli functor (such as $\underline{M}_P$ above). The best you may wish for is:

Definition 1.2. We say $\underline{M}$ is *representable* by a scheme M if there exists a natural isomorphism $\Phi : \underline{M} \Rightarrow \mathrm{Hom}(-, M)$.

In the above setting, M is called a *fine moduli space*. Moreover, the identity map $M \to M$ in $\mathrm{Hom}(M, M)$ produces an element $\mathcal{U} \in \underline{M}(M)$ known as the *universal family*. Exercise: Show that for any family $\mathcal{F} \in \underline{M}(B)$ we have $\Phi_B(\mathcal{F})^* \mathcal{U} = \mathcal{F}$. Here are some famous examples of fine moduli spaces:

- For V an n-dimensional vector space and $r > 0$, the *Grassmannian* $\mathrm{Gr}(r, V)$. Here a family over B is a quotient $V \otimes_k \mathcal{O}_B \twoheadrightarrow \mathcal{Q}$, where $\mathcal{Q}$ is locally free with fibres of dimension $n - r$.
- For $\mathcal{E}$ a coherent sheaf on X and $P(t) \in \mathbb{Q}[t]$, Grothendieck's *Quot scheme* $\mathrm{Quot}(\mathcal{E}, P)$. Here a family over B is a quotient $\mathcal{E} \otimes_k \mathcal{O}_B \twoheadrightarrow \mathcal{Q}$, where $\mathcal{Q}$ is a B-flat coherent sheaf on $X \times B$ and $P_{\mathcal{Q}_b}(t) = P(t)$ for

all $b \in B$. If X is projective (which we assumed), then $\mathrm{Quot}(\mathcal{E}, P)$ is a projective k-scheme, which may have singular and non-reduced components. In the case $\mathcal{E} = \mathcal{O}_X$ this reduces to the Hilbert scheme $\mathrm{Hilb}(X, P)$ of closed subschemes $Z \hookrightarrow X$ with Hilbert polynomial $P(t)$.

The construction of Quot schemes uses the construction of Grassmannians. In turn, the construction of moduli spaces of stable sheaves uses Quot schemes.

The bad news is that we cannot expect a fine moduli space for our functor of sheaves $\underline{M}_P$. The reason is simple. Take $X = \mathrm{Spec}\, k$ and $P = r > 1$. Suppose $\underline{M}_P$ is representable, i.e. we have a natural isomorphism $\Phi : \underline{M}_P \Rightarrow \mathrm{Hom}(-, M)$. Clearly M has a unique closed point corresponding to the vector space $k^{\oplus r}$. Now just take any variety B with locally free sheaf $\mathcal{F}$ of rank r on B, which does *not* decompose as a sum of line bundles. Then $\Phi_{\mathcal{O}_B^{\oplus r}}$ factors through the closed point, whereas $\Phi_{\mathcal{F}}$ does not ($\mathcal{O}_B^{\oplus r}$, $\mathcal{F}$ are different families). However, since $\mathcal{F}$ trivializes over an open cover U_α of B, the restriction of $\Phi_{\mathcal{F}}$ to each U_α factors through the closed point; contradiction.

Therefore, we content ourselves with a weaker notion of moduli space:

Definition 1.3. We say $\underline{M}$ is *co-representable* by a scheme M if there exists a smallest natural transformation $\Phi : \underline{M} \Rightarrow \mathrm{Hom}(-, M)$ (i.e. any other such natural transformation factors through it).

In the above setting we call M a *coarse moduli space*. Exercise: Show M is unique up to isomorphism.

Typically in applications, we also want Φ_k to be a bijection so that we are indeed classifying the objects of interest. However, even this is problematic. Indeed, suppose we have a non-trivial extension of a coherent sheaf $\mathcal{E}$ on X

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.$$

The affine line $\mathbb{A}^1 \subset \mathrm{Ext}^1(\mathcal{E}'', \mathcal{E}')$ through 0 and the extension produces a flat family $\mathcal{F}$ over $\mathbb{A}^1$ such that

$$\begin{aligned} &\mathcal{F}_b \cong \mathcal{E}, \ \forall b \in \mathbb{A}^1 \setminus \{0\}, \\ &\mathcal{F}_0 \cong \mathcal{E}' \oplus \mathcal{E}''. \end{aligned}$$

So if $\underline{M}_P$ were representable by a fine moduli space M_P (with $P := P_{\mathcal{E}}$), we get a morphism $\mathbb{A}^1 \to M_P$ which is constant on $\mathbb{A}^1 \setminus \{0\}$. But then it is constant on $\mathbb{A}^1$ and $\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{E}''$, contradiction. This example also shows that, at our current level of generality, we cannot expect to have a natural transformation $\underline{M}_P \Rightarrow \mathrm{Hom}(-, M_P)$, which is a bijection on $B = \mathrm{Spec}\, k$!

Conclusion: we need to use coarse instead of fine moduli spaces and we need to restrict the class of objects that we consider.

Geometric Invariant Theory. In moduli theory we typically want to build a moduli space of *isomorphism classes* of objects on X. The idea is to define a big scheme $\widetilde{M}$ who's elements are the objects and define a group action G on $\widetilde{M}$ such that the orbits $\widetilde{M}/G$ are the isomorphism classes.

Quotients in algebraic geometry are subtle and form the subject of GIT. The weakest notion of quotient is the following:

Definition 1.4. Let G be an affine algebraic group acting on a scheme X. A scheme Y is called a *categorical quotient* if it co-represents the functor

$$(\mathrm{Sch}/k)^o \to \mathrm{Sets},\ \ B \mapsto \mathrm{Hom}(B,X)/\,\mathrm{Hom}(B,G),$$

where the quotient is set theoretic.

In the above definition, the identity morphism $X \to X$ provides an induced morphism $\pi : X \to Y$.

From now on we restrict attention to reductive groups (e.g. $\mathrm{GL}(n,k)$, $\mathrm{SL}(n,k)$, $\mathrm{PGL}(n,k)$, $\mathbb{G}_m^n$). The reason is that if G is reductive and $X = \mathrm{Spec}\ R$ is affine, then the ring of invariants R^G is finitely generated (Nagata). This gives an induced morphism $\pi : X \to Y$, where $Y = \mathrm{Spec}\ R^G$ is of finite type. One can show that Y is a categorical quotient. In practise one often wants more:

Definition 1.5. Let G be an affine algebraic group acting on a scheme X. A morphism of schemes $\pi : X \to Y$ is called a *geometric quotient* if it is categorical (+ some extra properties I do not want you to know) and the fibres of closed points of Y are exactly the G-orbits of closed points of X.

The local affine construction can be globalized as follows. Let G be a reductive group acting on a scheme X with G-equivariant line bundle L. Then $x \in X$ is called *semistable* w.r.t. L if there exists a G-invariant section $\sigma \in \Gamma(X, L^{\otimes n})$ for some n, such that $X_\sigma := \{\sigma \neq 0\}$ is affine and contains x. Moreover $x \in X$ is called *stable* w.r.t. L if in addition the the stabilizer G_x is finite and the orbits of G in X_σ are closed. The collections of such points are denoted by

$$X^s(L) \subset X^{ss}(L) \subset X$$

and are Zariski open but potentially empty.

Theorem 1.6 (Mumford-Fogarty-Kirwan)**.** *There exists a categorical quotient $\pi : X^{ss}(L) \to Y$, where Y is a quasi-projective scheme. Moreover, there exists an open subset $V \subset Y$ such that $\pi^{-1}(V) = X^s(L)$ and $\pi : X^s(L) \to V$ is a geometric quotient.*

In this context we write $X^{ss}(L)/\!\!/G := Y$ and $X^s(L)/G := V$. The scheme $X^{ss}(L)/\!\!/G$ in this theorem is patched together from the local pieces $X_\sigma/\!\!/G$.

Also if X is projective, then $X^{ss}(L)/\!\!/G$ is projective. It is a corollary of *Luna's Étale Slice Theorem*, that if $x \in X^{ss}(L)$ has trivial stabilizer, then $\pi : X^{ss}(L) \to X^{ss}(L)/\!\!/G$ is a principal G-bundle in a neighborhood of $\pi(x)$ (in the étale topology).

1.4. **The construction.**
Step 1: Introduce stability of sheaves. A coherent sheaf $\mathcal{E}$ on X is *(Gieseker) (semi)-stable* w.r.t. $\mathcal{O}_X(1)$ if $p_{\mathcal{G}}(t)(\leq)p_{\mathcal{E}}(t)$ for all $0 \neq \mathcal{G} \subsetneq \mathcal{E}$ and $t \gg 0$.

Step 2: By Serre vanishing: for each semistable sheaf $\mathcal{E}$ on X with Hilbert polynomial P, there exists an $m \gg 0$ such that $\mathcal{E}(m)$ is globally generated and $H^{>0}(\mathcal{E}(m)) = 0$. The *Boundedness Theorem* for semistable sheaves states that the first two quantors can be reversed.

Step 3: Define $V := k^{\oplus P(m)}$ and $\mathcal{H} = \mathcal{O}_X(-m) \otimes_k V$. Then for any $\mathcal{E}$ as before, there exists a surjection

$$[\mathcal{H} \twoheadrightarrow \mathcal{E}] \in \mathrm{Quot}(\mathcal{H}, P) =: Q.$$

Semi-stability is an open condition, so we get open subsets $R^s \subset R^{ss} \subset Q$ of quotients for which $\mathcal{F}$ is (semi-)stable. The natural action of $G = \mathrm{PGL}(V)$ on V induces actions on Q (and R^s, R^{ss}). Observation: set theoretically R^{ss}/G and R^s/G are the collections of isomorphism classes of (semi)stable sheaves on X with Hilbert polynomial P!

Step 4: Let $\mathbb{F}$ be the universal quotient on $Q \times X$ and consider

$$L := \det(p_{Q*}(\mathbb{F}(\ell))).$$

For $\ell \gg 0$ this is a very ample line bundle. Fact: there exists a natural G-equivariant structure on L such that $Q^{ss}(L) = R^{ss}$ and $Q^s(L) = R^s$.[2] From GIT we get a categorical quotient $R^{ss}/\!\!/G =: M_P^{ss}$ and a geometric quotient $R^s/G =: M_P^s$.

Theorem 1.7 (Gieseker, Maruyama, Simpson,...). *We have a projective k-scheme M_P^{ss} corepresenting $\underline{M}_P^{ss}$ and an open subset M_P^s corepresenting $\underline{M}_P^s$.*

The hard part is Steps 2 and 4. The rest is fairly standard. Remarks:

- In this notation I am suppressing the dependence on $\mathcal{O}_X(1)$. Studying this dependence is very interesting and leads to so-called *wall-crossing* phenomena.

[2] Maybe Q must be replaced by $\overline{R}$?

- The closed points of M_P^s are exactly the isomorphism classes of stable sheaves on X with Hilbert polynomial P. However, the closed points of M_P^{ss} are more difficult to interpret. They are so-called *S-equivalence classes* of semistable sheaves on X with Hilbert polynomial P.
- By Luna's Étale Slice Theorem, $R^s \to M_P^s$ is a principal G-bundle (in the étale topology). Although M_P^s in general does not have a universal family, étale locally it does. Moreover, the moduli functor induces isomorphisms $\mathcal{M}_P^s(\mathrm{Spec}\ A) \cong \mathrm{Hom}(\mathrm{Spec}\ A, M_P^s)$ for $A \in \mathrm{Artin}/k$.
- Combining the previous observation with Schlessinger's Criterion gives the earlier description of deformations. We have seen that first order deformations of $[\mathcal{E}] \in \mathcal{M}_P^s$ are given by $\mathrm{Ext}_X^1(\mathcal{E},\mathcal{E})$. By the previous observation, the first order deformations of $\mathcal{E}$ are also given by morphisms
$$\mathrm{Hom}(\mathrm{Spec}\ k[\epsilon]/(\epsilon^2), M_P^s),$$
for which the point maps to $[\mathcal{E}]$. The latter is nothing but the Zariski tangent space at $[\mathcal{E}]$ (Exercise). Hence $\dim M_P^s \leq \dim \mathrm{Ext}_X^1(\mathcal{E},\mathcal{E})$. Fact from deformation theory:
$$\dim M_P^s \geq \dim \mathrm{Ext}_X^1(\mathcal{E},\mathcal{E}) - \dim \mathrm{Ext}_X^2(\mathcal{E},\mathcal{E}).$$
We deduce that M_P^s is smooth at $[\mathcal{E}]$ if $\dim \mathrm{Ext}_X^2(\mathcal{E},\mathcal{E}) = 0$.

It is often easier to work in situations, where there are no strictly semistable sheaves. Given $P(t) = \sum_{i=0}^d \frac{\alpha_i}{i!} t^i$, recall that $\mathrm{rk} := \alpha_d/\alpha_d(\mathcal{O}_X)$ and $\deg := \alpha_{d-1} - \mathrm{rk} \cdot \alpha_{d-1}(\mathcal{O}_X)$. Exercise: prove that if $\gcd(\mathrm{rk}, \deg) = 1$, then $M_P^{ss} = M_P^s$. In this case one may also replace Gieseker stability by the easier notion of slope (a.k.a. Mumford-Takemoto) stability.