The p-kernel of abelian varieties in characteristic p

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Affine algebraic groups

Let k be a field. All k-algebras are commutative and unital.

Definition

A *affine algebraic group over k* is a quadruple $G = (A, \Delta, \eta, \iota)$ consisting of

- a k-algebra A;
- a morphism $\Delta\colon A \to A \otimes_k A$ 'comultiplication',
- a morphism $\eta\colon A \to k$ 'counit',
- a morphism $\iota\colon A \to A$ 'coinversion',

satisfying 'some axioms'.

Affine algebraic groups

If $G = (A, \Delta, \eta, \iota)$ is an affine algebraic group, and R is a k-algebra, set $G(R) := \mathrm{Hom}(A, R)$. Then we get:

- A binary operator 'multiplication'
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For a right choice of 'some axioms' this turns $G(R)$ into a group.

Definition

The *order* of G is $\dim_k(A)$.

Examples

Γ abstract group: take

- $A = \bigoplus_{\gamma \in \Gamma} k \cdot e_\gamma$;
- $\Delta(e_\gamma) = \sum_{\gamma'\gamma''=\gamma} e_{\gamma'} \otimes e_{\gamma''}$;
- $\eta =$ projection onto $k \cdot e_1 \cong k$;
- $\iota(e_\gamma) = e_{\gamma^{-1}}$.

Then $G(k') = \Gamma$ for every field ext. k'/k, and the order of G is $\#\Gamma$.

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- take $A = k[X, Y]/(XY - 1)$, then $G(R) = R^{\times}$: multiplicative group $\mathbb{G}_{\mathsf{m}}$.
- take $A = k[X]/(X^n - 1)$, then $G(R) = \{r \in R : r^n = 1\}$: nth roots of unity μ_n.

In the last example, $\mathrm{ord}(\mu_n) = n$, but in general $\#\mu_n(k) \leq n$.

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If A is an abelian variety of dimension g, then its p-kernel $A[p]$ is a commutative affine algebraic group of order p^{2g} and exponent p.

Question: Can we classify all possible $A[p]$?

Classifying $A[p]$

Suppose $k = \bar{k}$.

- If $\mathrm{char}(k) \neq p$, then only one group scheme of order p^{2g} and exponent p: the group scheme corresponding to $(\mathbb{Z}/p\mathbb{Z})^{2g}$.

Classifying $A[p]$

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- If $\mathrm{char}(k) \neq p$, then only one group scheme of order p^{2g} and exponent p: the group scheme corresponding to $(\mathbb{Z}/p\mathbb{Z})^{2g}$.
- If $\mathrm{char}(k) = p$ more possibilities: consider μ_p.
  - $\mu_p(k) = 1$
  - take $R = k[\varepsilon]/(\epsilon^2)$; then $\mu_p(R) = 1 + k\varepsilon$.

Dieudonné theory

A commutative group scheme of order p^n and exponent p over k corresponds to a *Dieudonné module of dimension* g, i.e. a triple (D, F, V) consisting of

- A k-vector space D of dimension g;
- Two additive maps $F, V\colon D \to D$ such that for all $\lambda \in k^\times$

$$FV = VF = 0, F\lambda = \lambda^p F, V\lambda^p = \lambda V.$$

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Proposition

If (D, F, V) corresponds to $A[p]$ for some abelian variety A, then $\mathrm{im}(F) = \ker(V)$ *(hence* $\mathrm{im}(V) = \ker(F)$*).*

Classifying Dieudonné modules

So next two questions:

1. Can we classify Dieudonné modules of dimension n over k with $\mathrm{im}(F) = \ker(V)$?
2. Which of these correspond to $A[p]$ for some A?

For the remainder of this talk we focus on the first question.

The stack D_n

For a $\mathbb{F}_p$-algebra R, set $\mathsf{D}_n(R) :=$ category of Dieudonné modules of dimension n over R. Then

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is a *stack* $\approx$ categorical construction with some 'geometric structure'. We can count the 'points' of D_g by means of the zeta function:

$$Z(\mathsf{D}_n, t) = \exp\left(\sum_{v \geq 1} \frac{t^v}{v} \sum_{D \in [\mathsf{D}_n(\mathbb{F}_{p^v})]} \frac{1}{\#\mathsf{Aut}(D)}\right) \in \mathbb{Q}[[t]]$$

with $[\mathsf{D}_n(\mathbb{F}_{p^v})] := \mathsf{Ob}(\mathsf{D}_n(\mathbb{F}_{p^v}))/\cong$.

Main result (for now)

Theorem

There exist a finite set $\mathcal{I}$ and a function $d\colon \mathcal{I} \to \mathbb{Z}_{\geq 0}$ that can explicitely be described, such that

$$Z(D_n, t) = \prod_{i \in \mathcal{I}} \frac{1}{1 - p^{d(i)} t}.$$

Next: sketch of how to obtain this result.

Canonical filtration

Let $D \in \mathrm{D}_n(k)$. Then the *canonical filtration* on D is the coarsest filtration $0 = D_0 \subset D_1 \subset \ldots \subset D_r = D$ such that for every i, $F(D_i) = D_j$ and $V^{-1}(D_i) = D_{j'}$ for some $j, j' \leq n$.

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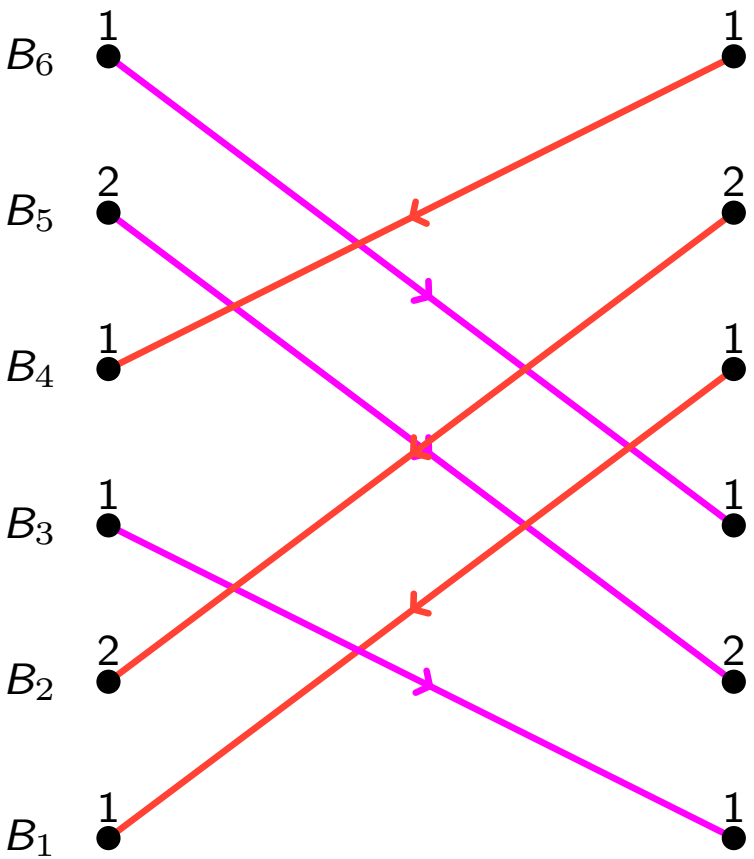
Set $B_i := D_i/D_{i-1}$; then for every i two options:

- $\exists \tau(i)$ such that F induces a Fr_p-semilinear $B_i \xrightarrow{\sim} B_{\tau(i)}$;
- $\exists \tau(i)$ such that V induces a Fr_p-semilinear $B_i \xleftarrow{\sim} B_{\tau(i)}$;

and $\tau \in S_r$.

Graph of a Dieudonné module



F, V, $\dim(B_i)$.

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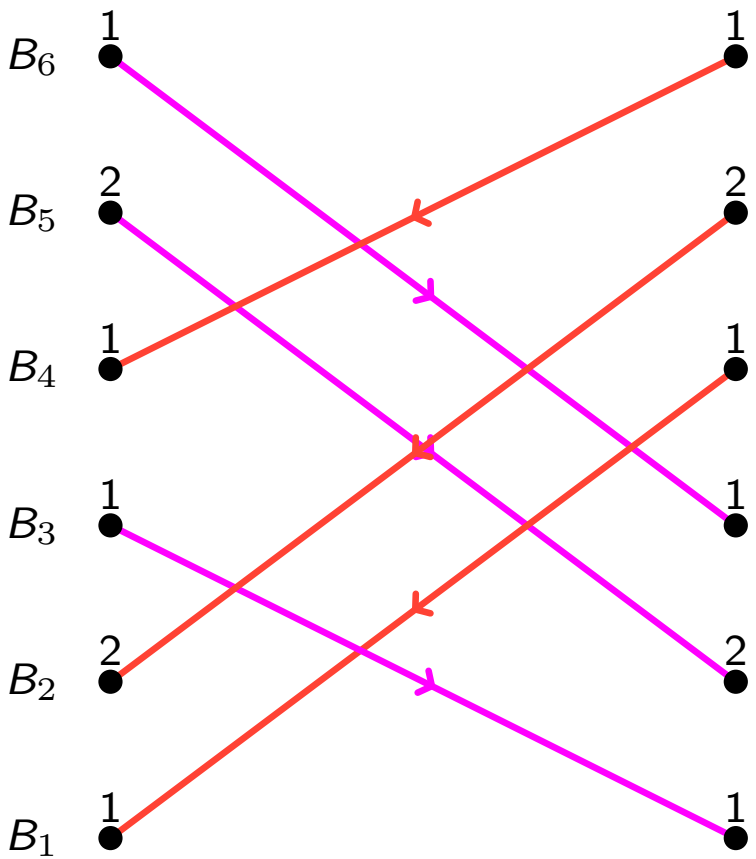

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Theorem

(Kraft 1975) Over $k = \bar{k}$, two Dieudonné modules with the same graph are isomorphic.

Hence there are only finitely many Dieudonné modules of dimension n over $\bar{\mathbb{F}}_p$, and we can describe them explicitely.

Automorphism group

Over $\bar{\mathbb{F}}_p$, we may assume that
Any automorphism has to fix the canonical filtration, so $\mathrm{Aut}(D)$ is a subgroup of

$$\begin{pmatrix} \mathrm{GL}(B_n) & 0 & \cdots & 0 \\ * & \mathrm{GL}(B_{n-1}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \mathrm{GL}(B_1) \end{pmatrix}$$

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Lemma

The image H of $\mathrm{Aut}(D)$ in $\prod_i \mathrm{GL}(B_i)$ is finite.

Over $\mathbb{F}_q$

Which Dieudonné modules over $\mathbb{F}_q$ correspond to a given graph?

- Take one (D, F, V); then D defines a semilinear isomorphism $\mathrm{Fr}_q \colon D_{\bar{\mathbb{F}}_p} \to D_{\bar{\mathbb{F}}_p}$.

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- Any other Dieudonné module with the same graph is obtained by 'twisting' the Frobenius action: $\tilde{\mathrm{Fr}}_q = \gamma \circ \mathrm{Fr}_q$ for some $\gamma \in \mathrm{Aut}(D_{\bar{\mathbb{F}}_p})$.

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- $\gamma \sim \gamma'$ give isomorphic $D \Leftrightarrow \exists \delta \in \mathrm{Aut}(D_{\bar{\mathbb{F}}_p}) : \gamma' = \delta\gamma'\mathrm{Fr}_q(\delta)^{-1}$.
- Set $\mathrm{H}^1(\mathbb{F}_q, \mathrm{Aut}(D_{\bar{\mathbb{F}}_p})) = \mathrm{Aut}(D_{\bar{\mathbb{F}}_p})/\sim$;
- Then $\mathrm{H}^1(\mathbb{F}_q, \mathrm{Aut}(D_{\bar{\mathbb{F}}_p})) \cong \mathrm{H}^1(\mathbb{F}_q, H)$, and this can be explicitely calculated.

We can also calculate the automorphism group of the Dieudonné module corresponding to γ, and this allows us to calculate the Zeta function.

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Research in progress:

- Zeta function for arbitrary G-zips;
- Look at $A[p^2]$ rather than $A[p]$.