A COMPARISON OF HOCHSCHILD HOMOLOGY IN ALGEBRAIC AND SMOOTH SETTINGS

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Abstract. Consider a complex affine variety $\tilde V$ and a real analytic Zariski-dense submanifold V of $\tilde V$. We compare modules over the ring $\mathcal O(\tilde V)$ of regular functions on $\tilde V$ with modules over the ring $C^\infty(V)$ of smooth complex valued functions on V.

Under a mild condition on the tangent spaces, we prove that $C^\infty(V)$ is flat as a module over $\mathcal O(\tilde V)$. From this we deduce a comparison theorem for the Hochschild homology of finite type algebras over $\mathcal O(\tilde V)$ and the Hochschild homology of similar algebras over $C^\infty(V)$.

We also establish versions of these results for functions on $\tilde V$ (resp. V) that are invariant under the action of a finite group G. As an auxiliary result, we show that $C^\infty(V)$ has finite rank as module over $C^\infty(V)^G$.

Contents

Introduction

Let $\tilde V$ be a complex affine variety and let $V \subset \tilde V$ be a smooth submanifold. The general goal of this paper is to compare modules over the algebra of regular functions $\mathcal O(\tilde V)$ with modules over the algebra of (complex-valued) smooth functions $C^\infty(V)$. One may pass from the algebraic setting to the smooth setting by tensoring with $C^\infty(V)$ over $\mathcal O(\tilde V)$, and we study that functor in detail. It may enable one to transfer various problems from one setting to the other.

The standard case is $\tilde V = V$ as sets, a non-singular complex affine variety considered both with its Zariski topology and with its analytic topology. In this case it follows from [Mal] (although we have not found an explicit account) that $C^\infty(V)$ is flat as a module over $\mathcal O(\tilde V)$.

In the more general situations which we consider, the affine variety $\tilde V$ may be singular. On the smooth side we allow minor singularities via an action of a finite

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group G, so that we actually consider an orbifold V/G. We assume that the action extends to $\tilde V$ and we want to compare $\mathcal{O}(\tilde V)^G = \mathcal{O}(\tilde V/G)$ with $C^\infty(V)^G$.

By well-known results of Noether [Eis, §13] $\tilde V/G$ is always an affine variety and $\mathcal{O}(\tilde V)$ is finitely generated as a module over $\mathcal{O}(\tilde V/G)$. While V/G need not be a smooth manifold and $C^\infty(V)^G$ can be substantially more complicated than $C^\infty(V)$, one part of Noether's algebraic results remains valid in this smooth setting:

Theorem A. (see Theorem 3.1)
Let V be a smooth manifold with a smooth action of a finite group G. Then $C^\infty(V)$ is finitely generated as a $C^\infty(V)^G$-module.

The precise conditions needed for our main results are:

Conditions B. (i) V is a real analytic Zariski-dense submanifold of $\tilde V$,
(ii) a finite group G acts algebraically on $\tilde V$ and stabilizes V,
(iii) for all $v \in V$, $T_v(\tilde V) = T_v(V) \otimes_{\mathbb{R}} \mathbb{C}$.

Typical examples come from real forms of $\tilde V$ (but maybe not all real forms qualify if $\tilde V$ is singular). Sometimes (iii) can be replaced by

(iii') G acts freely on V (e.g. $G = 1$) and for each $v \in V$, the real vector space $T_v(V)$ spans the complex vector space $T_v(\tilde V)$.

The assumptions (i) and (ii) guarantee that $\mathcal{O}(\tilde V)$ embeds G-equivariantly in $C^\infty(V)$. Either of (iii) and (iii') entails that at every point of V the formal completion of $\mathcal{O}(\tilde V)$ is a subalgebra of the formal completion of $C^\infty(V)$. Under condition (iii'), V/G can be endowed with the structure of a smooth manifold.

Theorem C. (see Theorem 1.5)
Assume that (i), (ii) and (iii) or (iii') hold. Then $C^\infty(V)^G$ is flat over $\mathcal{O}(\tilde V)^G$.

The proof runs mainly via formal completions of $C^\infty(V)^G$-modules. Theorem C tells us that the functor $C^\infty(V)^G \otimes_{\mathcal{O}(\tilde V)^G}$ is exact on finitely generated modules. Roughly speaking, that means that passing from $\tilde V/G$ to V/G is a reasonable operation, which does not lose information beyond shrinking the space.

Theorem C enables us to compare homological algebra with $\mathcal{O}(\tilde V)^G$-modules to homological algebra with $C^\infty(V)^G$-modules. Our main application is to the Hochschild homology of finite type algebras and their bimodules, as studied in [KNS]. Recall that a unital algebra A is a finite type $\mathcal{O}(\tilde V)^G$-algebra if an algebra homomorphism from $\mathcal{O}(\tilde V)^G$ to the centre of A is given, and makes A into a finitely generated $\mathcal{O}(\tilde V)^G$-module. Under the above conditions $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$ is a Fréchet algebra (this is why we need V to be real-analytic). Furthermore it is finitely generated as $C^\infty(V)^G$-module, so it is reasonable to regard it as a smooth finite type algebra. We stress that A and $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$ need not be commutative.

Theorem D. (see Theorem 2.3)
Let A be a unital finite type $\mathcal{O}(\tilde V)^G$-algebra and let M be a finitely generated A-bimodule. Assume that (i), (ii) and (iii) from Conditions B hold. There is a natural

isomorphism of Fréchet $C^\infty(V)^G$-modules

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} H_n(A,M) \longrightarrow H_n\Big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A, \big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op}\big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} M\Big).$$

We note that on the right hand side the Hochschild homology involves the topology of the algebra, via the complete projective tensor product of Fréchet spaces. Theorem D is a smooth version of an earlier result with formal completions [KNS, Theorem 3]. The special case of Theorem D with $M = A$ is an isomorphism for Hochschild homology of the algebras:

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} HH_n(A) \;\cong\; HH_n\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A\big).$$

With this one can reduce the computation of the Hochschild homology of certain Fréchet algebras to the Hochschild homology of finite type algebras, about which a lot is known from [KNS]. To facilitate that, we make the left hand side of Theorem D explicit in some cases. Recall that by Hochschild–Kostant–Rosenberg Theorem

$$HH_n(\mathcal{O}(\tilde V)) = \Omega^n(\tilde V) \quad \text{for nonsingular } \tilde V,$$

where Ω^n stands for algebraic differential forms. Denote the $C^\infty(V)$-module of smooth n-forms on V by $\Omega^n_{sm}(V)$.

Theorem E. (a special case of Lemma 3.4)
Suppose that (i), (ii) and (iii) from Conditions B hold. There is a natural isomorphism of Fréchet $C^\infty(V)^G$-modules

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} \Omega^n(\tilde V) \;\cong\; \Omega^n_{sm}(V).$$

From Theorems D and E one can easily deduce a smooth version of the Hochschild–Kostant–Rosenberg Theorem, see Section 4. Obviously that would be an extremely roundabout proof. The advantage of our methods is rather that they apply to much wider classes of algebras, possibly noncommutative. In particular our results will be useful for the computation of the Hochschild homology of the Harish-Chandra–Schwartz algebra of a reductive p-adic group, for which we refer to [Sol].

1. Flatness of smooth functions as module over regular functions

Let V be a smooth manifold (without boundary) and let G be a finite group acting on V by diffeomorphisms. Consider the algebra $C^\infty(V)^G$ of G-invariant smooth complex-valued functions on V. For each $v \in V$ we have the closed maximal ideal $I_v \subset C^\infty(V)$ of functions vanishing at v and the closed ideal I_{Gv} of functions vanishing on Gv. The G-invariant elements in the latter form an ideal $I_{Gv}^G \subset C^\infty(V)^G$. Let FP_v be the Fréchet algebra of formal power series on an infinitesimal neighborhood of v in V and let $FP_v^{G_v}$ be the subalgebra of G_v-invariants. Then $FP_v \cong \varprojlim_n C^\infty(V)/I_v^n$ and

$$FP_v^{G_v} \cong \big(\textstyle\bigoplus_{v' \in Gv} FP_{v'}\big)^G \cong \big(\varprojlim_n C^\infty(V)/I_{Gv}^n\big)^G \cong \varprojlim_n C^\infty(V)^G/I_{Gv}^{n,G}. \tag{1.1}$$

By a theorem of Borel (see [Tou, Théorème IV.3.1 and Remarque IV.3.5] or [MeVo, Theorem 26.29]) the Taylor series map

$$\mathfrak{T}_v : C^\infty(V) \to FP_v$$

is surjective. Its kernel is the module I_v^∞ of functions that are flat at v. Similarly we have the ideal

$$I_{Gv}^\infty = \bigcap\nolimits_{v' \in Gv} I_{v'}^\infty \;\subset\; C^\infty(V)$$

of functions that are flat on Gv. In view of the surjectivity of $\mathfrak{T}_v$, (1.1) becomes an isomorphism

$$FP_v^{G_v} \cong C^\infty(V)^G / I_{Gv}^{\infty,G}. \tag{1.2}$$

For any Fréchet $C^\infty(V)^G$-module M we can form the "formal completion" at v:

$$\hat{M}_{Gv} := FP_v^{G_v} \underset{C^\infty(V)^G}{\hat{\otimes}} M \cong M / \overline{I_{Gv}^{\infty,G} M}. \tag{1.3}$$

In contrast with the algebraic setting, $\hat{M}_{Gv}$ is actually a quotient rather than a completion of M.

Lemma 1.1. *Let M be a finitely generated Fréchet $C^\infty(V)^G$-module. Let M_1 and M_2 be closed $C^\infty(V)^G$-submodules of M, such that $M_1 \supset M_2$ and $\widehat{M_1}_{Gv} = \widehat{M_2}_{Gv}$ for all $v \in V$. Then $M_1 = M_2$.*

Proof. By assumption there exists a finitely generated free $C^\infty(V)^G$-module N and a surjective homomorphism of Fréchet $C^\infty(V)^G$-modules $p : N \to M$. By the continuity of p, $N_i := p^{-1}(M_i)$ is a closed $C^\infty(V)^G$-submodule of N. For any $v \in V$ we have

$$\widehat{N_1/N_2}_{Gv} \cong \widehat{M_1/M_2}_{Gv} = 0.$$

From that and (1.3) we deduce

$$N_1/N_2 = \overline{I_{Gv}^{\infty,G}(N_1/N_2)}. \tag{1.4}$$

The inclusion $I_{Gv}^{\infty,G}(N_1/N_2) \subset \big(I_{Gv}^{\infty,G} N + N_2\big)/N_2$ induces an inclusion

$$\overline{I_{Gv}^{\infty,G}(N_1/N_2)} \;\subset\; \overline{I_{Gv}^{\infty,G} N + N_2/N_2} \;=\; \overline{I_{Gv}^{\infty,G} N + N_2}/N_2. \tag{1.5}$$

From (1.4) and (1.5) for all $v \in V$, we obtain

$$N_1 \;\subset\; \bigcap\nolimits_{v \in V} \overline{I_{Gv}^{\infty,G} N + N_2}.$$

Consider the finitely generated free $C^\infty(V)$-module $C^\infty(V) \underset{C^\infty(V)^G}{\hat{\otimes}} N$. In there we have $C^\infty(V)$-submodules

$$C^\infty(V) N_1 \;\subset\; C^\infty(V)\Big(\bigcap\nolimits_{v \in V} \overline{I_{Gv}^{\infty,G} N + N_2}\Big) \;\subset\; \bigcap\nolimits_{v \in V} \overline{I_{Gv}^{\infty} N + C^\infty(V) N_2}.$$

Applying the Taylor series map, we find

$$\mathfrak{T}_v(C^\infty(V)N_1) \;\subset\; \overline{\mathfrak{T}_v(C^\infty(V)N_2)}.$$

By [Tou, Corollaire V.1.6] for the variety $\{v\}$, the right hand side equals $\mathfrak{T}_v(C^\infty(V)N_2)$. As $N_1 \supset N_2$, we deduce that $C^\infty(V)N_1$ and $C^\infty(V)N_2$ have the same Taylor series at every $v \in V$. By Whitney's spectral theorem [Tou, Corollaire V.1.6], this implies

$$C^\infty(V)N_1 \subset \overline{C^\infty(V)N_2}. \tag{1.6}$$

Taking G-invariants inside $C^\infty(V) \hat{\otimes}_{C^\infty(V)^G} N$, we obtain

$$N_1 = (C^\infty(V) N_1)^G \subset \overline{C^\infty(V) N_2}^G = \overline{N_2} = N_2.$$

Hence $N_1 = N_2$ and $M_1 = M_2$. $\square$

In this context it is useful to mention the following slight generalization of a result of Malgrange [Tou, Corollaire VI.1.8].

Theorem 1.2. *Assume that V is real analytic and let $r \in \mathbb{N}$. Let M be a $C^\infty(V)^G$-submodule of $\big(C^\infty(V)^G\big)^r$ generated by finitely many real-analytic G-invariant functions from V to $\mathbb{C}^r$. Then M is closed in $\big(C^\infty(V)^G\big)^r$.*

Proof. Let $\{f_i\}$ be a finite set of analytic G-invariant functions from V to $\mathbb{C}^r$. By [Tou, Corollaire VI.1.8] they generate a closed $C^\infty(V)$-submodule M' of $C^\infty(V)^r$.

Assume that the f_i generate M as $C^\infty(V)^G$-module. Write $p_G = |G|^{-1} \sum_{g \in G} g$, an idempotent in $\mathbb{C}[G]$. Clearly $M \subset M' \cap (C^\infty(V)^G)^r$. On the other hand

$$\begin{aligned} M = \sum\nolimits_i C^\infty(V)^G f_i &= \sum\nolimits_i \big(p_G C^\infty(V)\big) f_i = p_G \sum\nolimits_i \big(p_G C^\infty(V) f_i\big) \\ &= p_G\big(\sum\nolimits_i C^\infty(V) f_i\big) = p_G M' \supset M' \cap (C^\infty(V)^G)^r. \end{aligned}$$

Hence $M = M' \cap (C^\infty(V)^G)^r$, which is closed in $(C^\infty(V)^G)^r$ because M' is closed in $C^\infty(V)^r$. $\square$

Let $\tilde{V}$ be a complex affine G-variety and recall the Conditions B.

Lemma 1.3. *Assume (i) and (ii) from Conditions B and let M be a finitely generated $\mathcal{O}(\tilde{V})^G$-module. The $C^\infty(V)^G$-modules*

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde{V})^G}{\otimes} M, \qquad FP_v^{G_v} \underset{\mathcal{O}(\tilde{V})^G}{\otimes} M \qquad \text{and} \qquad I_{Gv}^{\infty,G}\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde{V})^G}{\otimes} M\big)$$

are nuclear Fréchet. The first two are generated by a finite subset of M.

Proof. Any finite set of generators of M as $\mathcal{O}(\tilde{V})$-module also generates the first two $C^\infty(V)^G$-modules under consideration. By [OpSo, (30) and subsequent lines], every finitely generated $FP_v^{G_v}$-module is Fréchet, so in particular $FP_v^{G_v} \underset{\mathcal{O}(\tilde{V})^G}{\otimes} M$.

Pick $r \in \mathbb{Z}_{>0}$ and a $\mathcal{O}(\tilde{V})^G$-submodule N of $\big(\mathcal{O}(\tilde{V})^G\big)^r$ such that $M \cong \big(\mathcal{O}(\tilde{V})^G\big)^r / N$. The kernel of the surjective homomorphism of $C^\infty(V)^G$-modules

$$\begin{aligned} \big(C^\infty(V)^G\big)^r = C^\infty(V)^G \underset{\mathcal{O}(\tilde{V})^G}{\otimes} \big(\mathcal{O}(\tilde{V})^G\big)^r \longrightarrow \\ C^\infty(V)^G \underset{\mathcal{O}(\tilde{V})^G}{\otimes} \big(\mathcal{O}(\tilde{V})^G\big)^r / N = C^\infty(V)^G \underset{\mathcal{O}(\tilde{V})^G}{\otimes} M \end{aligned} \tag{1.7}$$

is generated by $1 \otimes N$. Since $\mathcal{O}(\tilde{V})^G$ is Noetherian, N is generated as $\mathcal{O}(\tilde{V})^G$-module by some finite subset S_N. Then the kernel of (1.7) is generated by $1 \otimes S_N$. The analyticity of V entails that S_N consists of analytic G-invariant functions from V to $\mathbb{C}^r$. Now Theorem 1.2 says that the kernel of (1.7) is closed in $\big(C^\infty(V)^G\big)^r$. Hence $C^\infty(V)^G \underset{\mathcal{O}(\tilde{V})^G}{\otimes} M$ is the quotient of $\big(C^\infty(V)^G\big)^r$ by a closed subspace, and in particular is a Fréchet space.

In the short exact sequence of topological vector spaces

$$0 \to I_{Gv}^{\infty,G}\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M\big) \to C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \to FP_v^{G_v} \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \to 0$$

the middle term is Fréchet and the right hand side is Hausdorff. Hence the left hand side is a closed subspace of the middle term, and is itself Fréchet.

Next we address the nuclearity. Our arguments are based entirely on the inheritance properties for nuclearity, which can be found for instance in [MeVo, Satz 28.6–28.7] and [ScWo, Theorem 7.4]. The power series ring FP_v is a direct product of copies of $\mathbb{C}$, so it is nuclear. Then its subspace $FP_v^{G_v}$ and the finite direct sum $(FP_v^{G_v})^r$ with $r \in \mathbb{N}$ inherit nuclearity from FP_v. As $FP_v^{G_v} \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$ is a Hausdorff quotient of $(FP_v^{G_v})^r$ for a suitable r, it is nuclear as well.

The Fréchet space $C^\infty(V)$ is a standard example of a nuclear space [ScWo, p. 108]. Hence the subspace $C^\infty(V)^G$ and $(C^\infty(V)^G)^r$ are also nuclear. We showed that $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$ is a Hausdorff quotient of $(C^\infty(V)^G)^r$, and therefore nuclear. Finally, nuclearity is inherited by the subspace $I_{Gv}^{\infty,G}\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M\big)$. $\square$

From Lemma 1.3 we obtain a functor

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} : \mathrm{Mod}_{\mathrm{fg}}\big(\mathcal{O}(\tilde V)^G\big) \to \mathrm{Mod}_{\mathrm{Fr}}\big(C^\infty(V)^G\big), \tag{1.8}$$

where the subscripts fg and Fr stand for finitely generated and Fréchet, respectively.

Lemma 1.4. *Assume that (i), (ii) and either (iii) or (iii') from Conditions B hold and let $M \subset M'$ be finitely generated $\mathcal{O}(\tilde V)^G$-modules. Then the natural map*

$$\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M\big)^\wedge_{Gv} \longrightarrow \big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M'\big)^\wedge_{Gv} \quad \textit{is injective.}$$

Proof. Recall that the formal completion of the $\mathcal{O}(\tilde V)^G$-module M at $Gv \in V/G$ is defined as

$$\hat M_{Gv} = \varprojlim_n M/\big(I^n_{Gv} \cap \mathcal{O}(\tilde V)^G\big)M. \tag{1.9}$$

Let $\widetilde{FP}_v$ be the formal completion of $\mathcal{O}(\tilde V)$ at $v \in V$. Like in (1.1), $\widetilde{FP}_v^{G_v}$ is the formal completion of $\mathcal{O}(\tilde V)^G$ at Gv, and it can be considered as a subalgebra of $FP_v^{G_v}$. Since M is finitely generated, there is a natural isomorphism

$$\hat M_{Gv} \cong \widetilde{FP}_v^{G_v} \underset{\mathcal{O}(\tilde V)^G}{\otimes} M.$$

By definition

$$\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M\big)^\wedge_{Gv} = FP_v^{G_v} \underset{C^\infty(V)^G}{\hat\otimes} (C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M). \tag{1.10}$$

The right hand side is the completion of the algebraic tensor product

$$FP_v^{G_v} \underset{C^\infty(V)^G}{\otimes} C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M = FP_v^{G_v} \underset{\mathcal{O}(\tilde V)^G}{\otimes} M. \tag{1.11}$$

Since $\mathcal{O}(\tilde V)^G$ is Noetherian, M admits a finite presentation

$$(\mathcal{O}(\tilde V)^G)^k \to (\mathcal{O}(\tilde V)^G)^m \to M \to 0.$$

Tensoring with $FP_v^{G_v}$ over $\mathcal{O}(\tilde V)^G$ gives a finite presentation

$$(FP_v^{G_v})^k \to (FP_v^{G_v})^m \to FP_v^{G_v} \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \to 0. \tag{1.12}$$

The power series ring $FP_v^{G_v}$ is a Fréchet space of finite type, i.e. its topology is defined by an increasing sequence of seminorms all of whose cokernels have finite codimension. By [Kopp] all continuous linear maps between such Fréchet spaces have closed images. Now (1.12) shows that $FP_v^{G_v} \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$ is the quotient of $(FP_v^{G_v})^m$ by a closed linear subspace, so in particular is complete. Hence (1.11) and (1.10) are equal. Consequently there are isomorphisms of $FP_v^{G_v}$-modules

$$\begin{aligned}\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M\big)^\wedge_{Gv} &\cong FP_v^{G_v} \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \\ &\cong FP_v^{G_v} \underset{\widetilde{FP}_v^{G_v}}{\otimes} \widetilde{FP}_v^{G_v} \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \cong FP_v^{G_v} \underset{\widetilde{FP}_v^{G_v}}{\otimes} \hat{M}_{Gv}.\end{aligned} \tag{1.13}$$

By the exactness of the formal completion functor (1.9) for finitely generated $\mathcal{O}(\tilde V)^G$-modules,

$$\hat{M}_{Gv} \text{ is a } \widetilde{FP}_v^{G_v}\text{-submodule of } \hat{M}'_{Gv}. \tag{1.14}$$

Suppose that (iii) holds. Then $FP_v \cong \widetilde{FP}_v$ as G_v-representations, so $FP_v^{G_v} \cong \widetilde{FP}_v^{G_v}$. With the isomorphism (1.13) that immediately implies the statement.

Suppose that (iii') holds, so that $G_v = 1$. The canonical surjection

$$T_v(V) \otimes_{\mathbb{R}} \mathbb{C} \to T_v(\tilde V)$$

induces an injection

$$j : T_v(\tilde V)^* \to (T_v(V) \otimes_{\mathbb{R}} \mathbb{C})^*$$

Pick a basis $\{z_1, \ldots, z_d\}$ of $j(T_v(\tilde V)^*)$ and extend it with elements $\{w_1, \ldots, w_{\dim V - d}\}$ to a basis of $(T_v(V) \otimes_{\mathbb{R}} \mathbb{C})^*$. There are isomorphisms of Fréchet algebras

$$FP_v \cong \mathbb{C}[[z_1, \ldots, z_d, w_1, \ldots, w_{\dim V - d}]] \cong \widetilde{FP}_v[[w_1, \ldots, w_{\dim V - d}]]. \tag{1.15}$$

Thus the $\widetilde{FP}_v$-module FP_v is isomorphic to a product of copies of $\widetilde{FP}_v$, indexed by all the monomials built from $\{w_1, \ldots, w_{\dim V - d}\}$. Furthermore $\widetilde{FP}_v$ is Noetherian, and then [Cha, Theorem 2.1] says that FP_v is flat over $\widetilde{FP}_v$. Combine that with (1.13) and (1.14). $\square$

We note that Lemma 1.4 may become false if we assume only (i), (ii) and (iii') in a weaker version without the freedom of the G-action. For example, take $V = \tilde V = \mathbb{C}$, on which $G = \{1, -1\}$ acts by multiplication. For $v = 0$ we have

$$\widetilde{FP}_v^{G_v} = \mathbb{C}[[z^2]] \quad \text{and} \quad FP_v^{G_v} = \mathbb{C}[[z, \bar z]]^G = \mathbb{C}[[z^2, \bar z^2, z\bar z]].$$

Here $FP_v^{G_v}$ is not flat over $\widetilde{FP}_v^{G_v}$, and then we see from (1.13) that Lemma 1.4 fails.

The main result of this section generalizes the flatness of $C^\infty(\tilde V)$ over $\mathcal{O}(\tilde V)$. As pointed out in an answer to a question on MathOverflow[1], that case can be shown quickly with results of Malgrange [Mal] about complex analytic functions.

[1] mathoverflow.net/questions/226136/is-the-sheaf-of-smooth-functions-flat

Theorem 1.5. *Assume that (i), (ii) and either (iii) or (iii') from Conditions B hold. Then $C^\infty(V)^G$ is flat as an $\mathcal{O}(\tilde V)^G$-module. In particular the functor* (1.8) *is exact.*

Proof. According to [Eis, Proposition 6.1], flatness can be checked by testing it with finitely generated modules. Let $M \subset M'$ be finitely generated $\mathcal{O}(\tilde V)^G$-modules. We need to show that the natural map

$$\mu : C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \to C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M'$$

is injective. We want to apply Lemma 1.1 inside the domain of μ, which by Lemma 1.3 has the right properties. The submodules will be $M_2 = 0$ and $M_1 = \ker(\mu)$, which is a closed submodule of the domain because μ is continuous and $C^\infty(V)^G$-linear. Lemma 1.1 yields the desired conclusion $\ker(\mu) = 0$, provided we can check that all formal completions of the $C^\infty(V)^G$-module $\ker(\mu)$ are zero.

From Lemma 1.4 we know that $\hat\mu_{Gv}$ is injective. We would like to apply the exactness of the formal completion functor from [OpSo, Theorem 2.5] to

$$0 \to \ker(\mu) \to C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \to C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M',$$

but unfortunately $\ker(\mu)$ could be a topological vector space of a more general kind than allowed by [OpSo, Theorem 2.5]. It turns out that we can still use the proof of [OpSo, Theorem 2.5], which relies on technical constructions in [MeTo, Chapitre 1]. Consider an element of $\widehat{\ker(\mu)}_{Gv}$ represented by $m \in \ker(\mu) \subset C^\infty(V)^G \otimes M$. By the injectivity of $\hat\mu_{Gv}$ and (1.3), m belongs to

$$\overline{I^{\infty,G}_{Gv}\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M\big)}.$$

Here taking the closure is superfluous, for by Lemma 1.3 it is already a closed subspace of $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$. Hence there are finitely many $f_j \in I^{\infty,G}_{Gv}$ and $m_j \in M$ such that $m = \sum_j f_j \otimes m_j$. By [MeTo, p. 183] there exists a $\psi \in I^\infty_{Gv}$ such that

$$f_j/\psi \in I^\infty_{Gv} \subset C^\infty(V) \text{ for all } j.$$

The construction of ψ in [MeTo, p. 184] runs via a sequence of functions

$$\epsilon_i \in I^\infty_{Gv} \quad \text{such that} \quad \sum\nolimits_i \epsilon_i = \psi \quad \text{and} \quad \epsilon_i/\psi \in I^\infty_{Gv}.$$

Since the set Gv is G-stable, we may assume that the copy of $\mathbb{R}^n$ in [MeTo, p. 183–184], which is obtained from a neighborhood of Gv, inherits a G-action. Next we average the metric on $\mathbb{R}^n$ over G, so that G acts isometrically. Then the sets U_i, F_i, G_i in [MeTo, p. 183–184] are G-stable and we can average all the functions ϵ_i over G, that preserves their properties used in [MeTo]. Hence we may assume that all the ϵ_i are G-invariant and that $\psi \in I^{\infty,G}_{Gv} \subset C^\infty(V)^G$. Then

$$m/\psi = \sum\nolimits_j f_j/\psi \otimes m_j \in C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$$

is well-defined. By G-invariance $(\epsilon_i/\psi)m$ lies in $I^{\infty,G}_{Gv}\ker(\mu)$. Now we can write

$$m = \psi \cdot m/\psi = \sum\nolimits_i \epsilon_i \cdot m/\psi = \sum\nolimits_i (\epsilon_i/\psi) \cdot m \in \overline{I^{\infty,G}_{Gv}\ker(\mu)}.$$

The sums converge by the equalities (although $\sum_i (\epsilon_i/\psi)$ need not converge). Hence $m = 0$ in $\widehat{\ker(\mu)}_{Gv}$, and $\widehat{\ker(\mu)}_{Gv} = 0$. $\square$

2. Finite type algebras and their smooth versions

We will apply Theorem 1.5 to finite type algebras. By an $\mathcal{O}(\tilde V)^G$-algebra we mean a (not necessarily unital) algebra A together with a unital algebra homomorphism from $\mathcal{O}(\tilde V)^G$ to the centre of the multiplier algebra of A. Recall from [KNS] that A has finite type (as $\mathcal{O}(\tilde V)^G$-algebra) if it is finitely generated as module over $\mathcal{O}(\tilde V)^G$. The structure, homology and representation theory of such algebras were studied in [KNS]. In particular A is always a polynomial identity algebra. We want to compare A and $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$. By Lemma 1.3 the latter is a Fréchet algebra, and it is finitely generated as a module over $C^\infty(V)^G$. It is also a polynomial identity algebra, and we regard it as a smooth version of a finite type algebra.

Assume that A is unital. Let M be a finitely generated A-module. By [KNS, Lemma 3] it has a resolution $(A \otimes_{\mathbb{C}} F_*, d_*)$ consisting of finitely generated free A-modules.

Lemma 2.1. *Assume that (i), (ii) and (iii) or (iii') from Conditions B hold and put* $C_n = C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A \underset{\mathbb{C}}{\otimes} F_n$.

(a) $(C_n, \mathrm{id} \otimes d_n)$ *is a resolution of the*
$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A\text{-module} \quad C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$$
by finitely generated free modules.

(b) *Suppose in addition that* $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$ *and* $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$ *are isomorphic (as Fréchet spaces) to direct summands of the space of rapidly decreasing sequences* $\mathcal{S}(\mathbb{N})$*. Then the resolution from part (a) is split exact as a complex of Fréchet spaces.*

Proof. (a) The exactness of
$$C_* \longrightarrow C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \tag{2.1}$$
is a direct consequence of Theorem 1.5.

(b) Let $\mathcal{D}$ be the category of Fréchet spaces that are isomorphic to direct summands of $\mathcal{S}(\mathbb{N})$. Recall that $\mathcal{S}(\mathbb{N})^d \cong \mathcal{S}(\mathbb{N})$ for all $d \in \mathbb{N}$. Hence C_n belongs to $\mathcal{D}$ and (2.1) is an exact sequence in $\mathcal{D}$. By [Vog, Theorems 1.8 and 5.1], every exact sequence in $\mathcal{D}$ admits a continuous linear splitting. $\square$

We turn to a comparison of the Hochschild homologies of A and of $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$. For a unital finite type algebra A and an A-bimodule M, this can be defined as
$$H_n(A, M) = \mathrm{Tor}_n^{A \otimes A^{op}}(A, M),$$
see [Lod, Proposition 1.1.13]. The special case $M = A$ is by definition the Hochschild homology $HH_n(A)$.

For Fréchet algebras like $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$, the topology must be taken into account. This is done best by fixing a (completed) topological tensor product and building all differential complexes with respect to this tensor product, see for instance [Tay].

We do it slightly differently though, with bornologies and bornological modules [Mey1, §2]. This approach has the advantage that both A and $C^\infty(V)^G \underset{\mathcal{O}(\tilde{V})^G}{\otimes} A$ can be regarded as complete bornological algebras. For A it boils down to the standard purely algebraic setup, while for Fréchet algebras/modules the bornological structure is equivalent to the topological structure. The appropriate tensor product is the complete bornological tensor product $\hat{\otimes}$, which for Fréchet spaces agrees with the complete projective tensor product [Mey2, Theorem I.87]. By default we endow all finitely generated $\mathcal{O}(\tilde{V})^G$-modules with the fine bornology [Mey1, §2.1], so that complete bornological tensor products also make sense for them (and they agree with the algebraic tensor products).

The category of bornological modules (always tacitly assumed to be complete) of a complete bornological algebra B is made into an exact category by allowing only extensions of B-modules that are split as extensions of bornological vector spaces. For extensions of Fréchet B-modules, this just means that they must be split as extensions of Fréchet spaces. It was checked in [Mey1, §3] that this is an excellent setting for homological algebra.

Assume that B is unital, and let N be a bornological B-bimodule. Equivalently, N is a bornological module over $B \hat{\otimes} B^{op}$, where B^{op} denotes the opposite algebra of B. A good definition of the Hochschild homology of B with coefficients in N is

$$H_n(B,N) = \mathrm{Tor}_n^{B \hat{\otimes} B^{op}}(B,N), \tag{2.2}$$

where Tor is computed in the exact category of bornological B-bimodules. For $N = B$ this yields the Hochschild homology $HH_n(B)$. For Fréchet algebras and modules, (2.2) agrees with the definition in terms of the completed projective tensor product [Tay].

From (2.2) we see that we will have to consider some modules over

$$C^\infty(V)^G \hat{\otimes} C^\infty(V)^G \cong C^\infty(V \times V)^{G \times G}. \tag{2.3}$$

Lemma 2.2. *Suppose that (i), (ii) and (iii) from Conditions B hold.*

(*a*) *$\mathcal{O}(\tilde{V})^G$ is dense in $C^\infty(V)^G$.*

(*b*) *For a finitely generated $\mathcal{O}(\tilde{V})^G$-module M, considered also as bimodule with the same action from the right, there is a natural isomorphism of $C^\infty(V)^G$-modules*

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde{V})^G}{\otimes} M \to C^\infty(V)^G \hat{\otimes} C^\infty(V)^{G,op} \underset{\mathcal{O}(\tilde{V})^G \otimes \mathcal{O}(\tilde{V})^{G,op}}{\otimes} M : f \otimes m \mapsto f \otimes 1 \otimes m.$$

When M is an $\mathcal{O}(\tilde{V})^G$-algebra, this map is an algebra isomorphism.

Remark. Of course $K^{op} = K$ for any commutative algebra K. The above superscripts *op* merely indicate which tensor factor acts from the right on the bimodule.

Proof. (a) It suffices to show that $\mathcal{O}(\tilde{V})$ is dense in $C^\infty(V)$, because from that we can obtain the statement by applying the idempotent p_G. For any $v \in V$, (iii) yields a natural isomorphism

$$\widehat{\mathcal{O}(\tilde{V})}_v = \widetilde{FP}_v \cong FP_v = \widehat{C^\infty(V)}_v.$$

According to [Tou, Corollaire V.1.6] this implies that the closure of $\mathcal{O}(\tilde V)$ in $C^\infty(V)$ is $C^\infty(V)$.
(b) By (2.3) and Lemma 1.3 (applied to $\tilde V \times \tilde V$ with the $G \times G$-action),
(2.4)

$$\big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op}\big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} M \cong \big(C^\infty(V)^{G,op} \hat\otimes C^\infty(V)^G\big) \underset{\mathcal{O}(\tilde V)^{G,op} \otimes \mathcal{O}(\tilde V)^G}{\otimes} M$$

is a Fréchet space. Let $x \in C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$ and $f \in C^\infty(V)^{G,op}$. By part (a) there exists a sequence $(f_n)_{n=1}^\infty$ in $\mathcal{O}(\tilde V)^{G,op}$ converging to f. The space (2.4) is Hausdorff, so limits are unique in there and we can compute

$$f \otimes x = \lim_{n\to\infty} f_n \otimes x = \lim_{n\to\infty} 1 \otimes f_n x = 1 \otimes f x.$$

Consequently (2.4) equals

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\hat\otimes} M. \tag{2.5}$$

Since $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$ is already Fréchet (by Lemma 1.3), it equals (2.5). It is easy to see that this isomorphism of Fréchet $C^\infty(V)^G$-modules is given by the map in the statement.

When M is in addition an $\mathcal{O}(\tilde V)^G$-algebra, the map in the statement is also an algebra homomorphism, so in fact an algebra isomorphism. □

Lemmas 2.1 and 2.2 together say that, under the topological condition from Lemma 2.1.b, the embedding of bornological algebras

$$A \to C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A \cong \big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op}\big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} A \tag{2.6}$$

is a homological epimorphism. That implies several comparison results for homological properties of the derived module categories of the two involved algebras, see [Mey1, Theorem 35] (where this is called an isocohomological embedding).

Since the Fréchet space $C^\infty(V)^G$ is isomorphic to a direct summand of $\mathcal{S}(\mathbb{N})$ when V is compact [MeVo, Satz 31.16], it seems likely that in many cases $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$ has the same property. Proving that is another matter though. Fortunately, we can work around the existence of continuous linear splittings of our resolutions by involving properties of nuclear Fréchet spaces.

One can compute $H_n(B, N)$ (at least when B is unital) with a completed version of the standard bar-resolution of B [Lod, §1], but the definition as a derived functor is more flexible. The inclusion $A \to C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$ induces a chain map between the respective bar-resolutions, and hence induces a natural map
(2.7)

$$H_n(A, M) \to H_n\Big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A, \big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op}\big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} M\Big).$$

Notice that by Lemma 1.3

$$\big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op}\big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} M$$

is a Fréchet $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A$-bimodule, so the right hand side of (2.7) is defined.

Theorem 2.3. *Let A be unital and let M be a finitely generated A-bimodule. Assume that (i), (ii) and (iii) from Conditions B are fulfilled. Then* (2.7) *induces a natural isomorphism of Fréchet $C^\infty(V)^G$-modules*

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} H_n(A,M) \longrightarrow$$
$$H_n\Big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A, \big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op} \big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} M \Big).$$

In the special case $M = A$ this gives a natural isomorphism

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} HH_n(A) \;\cong\; HH_n\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A\big).$$

Proof. The algebra $A \otimes A^{op}$ is of finite type over $\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^G$. Hence [KNS, Lemma 3] applies to it, and yields a resolution $(A \otimes A^{op} \otimes F_*, d_*)$ of A by finitely generated free $A \otimes A^{op}$-modules. (Here each F_n is just a finite dimensional $\mathbb{C}$-vector space, so later we may use $\otimes F_n$ and $\hat\otimes F_n$ interchangeably.) By definition

$$H_n(A,M) = H_n\big((A \otimes A^{op}) \otimes F_* \underset{A \otimes A^{op}}{\otimes} M, d_* \otimes \mathrm{id} \big) = H_n(F_* \otimes M, d_*). \tag{2.8}$$

We note that by [KNS, Proposition 2 and Corollary 1] $H_n(A,M)$ is a finitely generated $\mathcal{O}(\tilde V)^G$-module, so applying $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes}$ to it yields a Fréchet $C^\infty(V)^G$-module (Lemma 1.3). We abbreviate

$$B = C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A \qquad \text{and} \qquad N = \big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op} \big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} M.$$

By the associativity of completed bornological tensor products [Mey1, §2.1] there is a natural algebra isomorphism

$$\big(C^\infty(V)^G \hat\otimes C^\infty(V)^G \big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^G}{\otimes} (A \otimes A^{op}) \cong B \hat\otimes B^{op}.$$

Using that we put

$$C_n = \big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op} \big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} (A \otimes A^{op}) \otimes F_n \cong B \hat\otimes B^{op} \otimes F_n.$$

Then Lemma 2.1 says that (C_*, d_*) is a finitely generated free $B \hat\otimes B^{op}$-resolution of

$$\big(C^\infty(V)^G \hat\otimes C^\infty(V)^{G,op} \big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} A. \tag{2.9}$$

We warn that this resolution need not be split in the category of Fréchet spaces. By Lemma 2.2 the algebra (2.9) is just B.

Next we check all the conditions for [Tay, Proposition 4.5]. The exactness of (C_*, d_*) entails that $\mathrm{im}(d_{n+1}) = \ker(d_n)$ is a closed subspace of C_n, and in particular it is also Fréchet. The open mapping theorem for Fréchet spaces says that $d_n : C_n \to \mathrm{im}(d_n)$ is open, which in the terminology of [Tay, §4] means that it is a topological homomorphism. By Lemma 1.3 for $\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^G$, N is a Fréchet space and B and all the C_n are nuclear Fréchet spaces. Now we can apply [Tay, Proposition 4.5], which says that $H_n(B,N)$ can be computed as

$$H_n\big(B \hat\otimes B^{op} \otimes F_* \underset{B \hat\otimes B^{op}}{\hat\otimes} N, d_* \otimes \mathrm{id} \big) = H_n(F_* \otimes N, d_*). \tag{2.10}$$

By the exactness of $\big(C^\infty(V)^G \hat{\otimes} C^\infty(V)^{G,op}\big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes}$ from Theorem 1.5, there are natural isomorphisms of $C^\infty(V)^G$-bimodules

$$
\begin{aligned}
H_n(F_* \otimes N, d_*) &\cong H_n\big(F_* \otimes \big(C^\infty(V)^G \hat{\otimes} C^\infty(V)^{G,op}\big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} M, d_*\big) \\
&\cong \big(C^\infty(V)^G \hat{\otimes} C^\infty(V)^{G,op}\big) \underset{\mathcal{O}(\tilde V)^G \otimes \mathcal{O}(\tilde V)^{G,op}}{\otimes} H_n(F_* \otimes M, d_*).
\end{aligned} \tag{2.11}
$$

From (2.8) and Lemma 2.2 we see that (2.11) is isomorphic to

$$
C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} H_n(A, M). \tag{2.12}
$$

By (2.10), (2.12) is also isomorphic to $H_n(B, N)$. This shows that $H_n(B, N)$ is Hausdorff. In its construction as

$$
H_n(F_* \otimes N, d_*) = \ker(d_n)/\mathrm{im}(d_{n+1}),
$$

$\ker(d_n)$ is closed by the continuity of d_n. By Hausdorffness, the image of d_{n+1} must be closed as well, which implies that the quotient $H_n(F_* \otimes N, d_*)$ is Fréchet.

The statement about the special case $M = A$ follows from the general case and Lemma 2.2.b. $\square$

3. Modules consisting of differential forms

We preserve the setting of the previous paragraph. To make good use of Theorem 2.3, we will make both its sides more explicit in some relevant classes of examples. As we are dealing with algebraic tensor products, this involves checking that some modules are finitely generated. There have been ample investigations of the structure of $C^\infty(V)^G$, starting with [Sch]. On the other hand, $C^\infty(V)$ has hardly been studied as $C^\infty(V)^G$-module.

Let π be a representation of G on a finite dimensional real vector space W. By classical results of Noether, see for instance [Eis, §13.3], the ring of real valued polynomial functions $S(W^*)$ on W is finitely generated as module over $S(W^*)^G$.

Theorem 3.1. *Let G be a finite group.*

(a) $C^\infty(W)$ is generated as $C^\infty(W)^G$-module by a finite subset of $S(W^*)$.

(b) Let V be a smooth manifold with a smooth G-action. Then $C^\infty(V)$ is finitely generated as $C^\infty(V)^G$-module.

Proof. (a) This is contained in [Poe, Lemme III.1.4.1], but in disguise. Namely, it is stated there that, for any finite dimensional real G-representation (π', W'),

$$
C^\infty_G(W, W') = \{f \in C^\infty(W, W') : f(\pi(g)w) = \pi'(g)f(w)\, \forall g \in G, w \in W\}
$$

is a finitely generated $C^\infty(W)^G$-module. We claim that, for $W' = \mathbb{C}[G]$ the left regular representation, there is an isomorphism of $C^\infty(W)^G$-modules

$$
\begin{array}{ccc}
C^\infty(W) & \longleftrightarrow & C^\infty_G(W, \mathbb{C}[G]), \\
f & \mapsto & [w \mapsto \sum_{g \in G} f(\pi(g^{-1})w)g] \\
\phi_1 & \leftarrow\!\shortmid & \phi = \sum_{g \in G} \phi_g g
\end{array} . \tag{3.1}
$$

Indeed, the equivariance condition $\phi(\pi(g)w) = g\phi(w)$ means precisely that

$$
\phi_g(w) = \phi_1(\pi(g^{-1})w) \text{ for all } w \in W.
$$

Hence the two maps in (3.1) are mutually inverse. The proof of [Poe, Lemme III.1.4.1] uses only polynomial functions on $W \otimes W'^*$ as generators, so via the isomorphism (3.1) we can conclude that $C^\infty(W)$ is generated by a finite subset of $S(W^*)$. In fact any set that generates $S(W^*)$ as $S(W^*)^G$-module will do.
(b) By [Mos, Theorem 6.1], V can be embedded G-equivariantly as a closed submanifold in a space W as in part (a). Thus we may and do regard V as a subspace of W. With part (a) we choose a finite set of generators $\{f_i\}_i$ for $C^\infty(W)$ as $C^\infty(W)^G$-module. According to [Tou, Théorème IX.4.3], the restriction map

$$C^\infty(W) \to C^\infty(V) : f \mapsto f|_V$$

is surjective. Hence the functions $f_i|_V$ generate $C^\infty(V)$ as $C^\infty(V)^G$-module. $\Box$

In the algebraic setting, a theorem of Serre says that $\Omega^n(\tilde V)$ is finitely generated as $\mathcal{O}(\tilde V)$-module, and hence also as $\mathcal{O}(\tilde V)^G$-module. Similarly, the smooth Serre–Swan theorem says that $\Omega^n_{sm}(V)$ is finitely generated as $C^\infty(V)$-module, for any $n \in \mathbb{Z}_{\geq 0}$. This holds for any smooth manifold V, compact or not [Mor]. By Theorem 3.1 $\Omega^n_{sm}(V)$ also finitely generated as $C^\infty(V)^G$-module.

In view of the Hochschild–Kostant–Rosenberg Theorem [Lod, Theorem 3.4.4], the Hochschild homology of finite type algebras will involve differential forms on varieties related to $\tilde V$. We will study this in a setting that starts with (i) and (ii) from Conditions B. We assume that an embedding $\imath : \tilde Y_1 \to \tilde V$ is given, such that

- the image of $\imath$ is closed in $\tilde V$ and $\imath : \tilde Y_1 \to \imath(\tilde Y_1)$ is an isomorphism of affine algebraic varieties,
- $Y_1 := \imath^{-1}(V)$ is a real analytic Zariski-dense submanifold of $\tilde Y_1$ and $\imath|_{Y_1} : Y_1 \to \imath(Y_1)$ is a diffeomorphism.

Thus $\imath$ induces algebra homomorphisms

$$\imath^* : C^\infty(V) \to C^\infty(Y_1) \quad \text{and} \quad \imath^* : \mathcal{O}(\tilde V) \to \mathcal{O}(\tilde Y_1).$$

Let $\tilde Y$ be a finite disjoint union of complex affine varieties $\tilde Y_j$ ($j \in J$), not necessarily of the same dimension, each of which has the same properties as those of $\tilde Y_1$ just listed. Let Y be the disjoint union of the Y_j.

The above setup is used to study Schwartz algebras of reductive p-adic groups [Sol, §3.1]. However, let us point out that the standard and most instructive case of the upcoming results is simply $\tilde Y = \tilde V, Y = V$.

Lemma 3.2. *With the above assumptions, let $C^\infty(V)^G$ act on $\Omega^n_{sm}(Y)$ via $\imath^*$.*
(*a*) *$\Omega^n(\tilde Y)$ is finitely generated as an $\mathcal{O}(\tilde V)^G$-module.*
(*b*) *$\Omega^n_{sm}(Y)$ is generated, as a $C^\infty(V)^G$-module, by a finite subset of $\Omega^n(\tilde Y)$.*

Proof. (a) By assumption $\imath(\tilde Y)$ is closed in $\tilde V$, so the restriction map $\mathcal{O}(\tilde V) \to \mathcal{O}(\imath(\tilde Y))$ is surjective. As $\imath\big|_{\tilde Y}$ is an isomorphism $\imath^* : \mathcal{O}(\tilde V) \to \mathcal{O}(\tilde Y)$ is surjective. In particular $\Omega^n(\tilde Y)$ is a finitely generated module, over $\mathcal{O}(\tilde V)$ as well as over $\mathcal{O}(\tilde Y)$. Since $\mathcal{O}(\tilde V)$ is the integral closure of $\mathcal{O}(\tilde V)^G$ in the quotient field of $\mathcal{O}(\tilde V)$, it has finite rank over $\mathcal{O}(\tilde V)^G$ [Eis, Proposition 13.14]. Hence $\Omega^n(\tilde Y)$ is also finitely generated as $\mathcal{O}(\tilde V)^G$-module.
(b) By the smooth Serre–Swan theorem, $\Omega^n_{sm}(Y_j)$ is finitely generated over $C^\infty(Y_j)$. As $\imath(Y_j)$ is a closed submanifold of V, the restriction map $C^\infty(V) \to C^\infty(\imath(Y_j))$ is

surjective [Tou, Théorème IX.4.3]. Since $\imath\big|_{Y_j}$ is a diffeomorphism, also

$$\imath^* : C^\infty(V) \to C^\infty(Y_j) \text{ is surjective.} \tag{3.2}$$

In particular $\Omega^n_{sm}(Y_j)$ is a finitely generated $C^\infty(V)$-module, and so is $\Omega^n_{sm}(Y) = \bigoplus_{j\in J} \Omega^n(Y_j)$. From the definition of the module structures we see that the tensor products

$$C^\infty(V)^G \otimes \Omega^n(\tilde Y), \quad C^\infty(V)^G \otimes \mathcal{O}(\tilde Y) \otimes \Omega^n(\tilde Y), \quad C^\infty(V)^G \otimes \mathcal{O}(\tilde V) \otimes \Omega^n(\tilde Y).$$

have the same image in $\Omega^n_{sm}(Y)$, under the natural action maps. By Theorem 3.1.b and (3.2) the last one has the same image as

$$C^\infty(V) \otimes \Omega^n(\tilde Y) \quad \text{and} \quad C^\infty(Y) \otimes \Omega^n(\tilde Y).$$

The latter equals $\Omega^n_{sm}(Y)$, so $\Omega^n(\tilde Y)$ generates $\Omega^n_{sm}(Y)$ as $C^\infty(V)^G$-module. By part (a) that can be achieved with a finite subset of $\Omega^n(\tilde Y)$. □

Consider a $\mathcal{O}(\tilde V)^G$-submodule M of $\Omega^n(\tilde Y)$, where the action goes via $\imath^*$. Although it might seem obvious that $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M$ embeds in $\Omega^n_{sm}(Y)$, that is actually about as difficult as Theorem 1.5.

Proposition 3.3. *Assume that (i), (ii) and (iii) from Conditions B hold and let M, Y and $\tilde Y$ be as above. The natural homomorphism of Fréchet $C^\infty(V)^G$-modules $C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \to \Omega^n_{sm}(Y)$ is injective.*

Proof. By Theorem 1.5 the natural map

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} M \to C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} \Omega^n(\tilde Y)$$

is injective. Therefore we may assume that $M = \Omega^n(\tilde Y)$. Then the statement factors naturally as a direct sum indexed by $j \in J$. It suffices to consider one such direct summand, say

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} \Omega^n(\tilde Y_1) \to \Omega^n_{sm}(Y_1). \tag{3.3}$$

The formal completion of $\Omega^n_{sm}(Y_1)$ as $C^\infty(V)^G$-module at $Gv \in V/G$ is

$$\bigoplus_{y\in \imath^{-1}(Gv)} FP_v^{G_v} \underset{C^\infty(V)^G}{\otimes} \widehat{C^\infty(Y_1)}_y \underset{\mathbb{R}}{\otimes} \bigwedge\nolimits^n (T_y(Y_1)^*) = \bigoplus_{y\in \imath^{-1}(Gv)} \widehat{C^\infty(Y_1)}_y \underset{\mathbb{R}}{\otimes} \bigwedge\nolimits^n (T_y(Y_1)^*).$$

Using assumption (iii) we can also compute the formal completion of the left hand side of (3.3):

$$\begin{aligned}\Big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} \Omega^n(\tilde Y_1)\Big)^\wedge_{Gv} &= \bigoplus_{y\in \imath^{-1}(Gv)} FP_v^{G_v} \underset{\widehat{FP}_v^{G_v}}{\otimes} \widehat{\mathcal{O}(\tilde Y_1)}_y \underset{\mathbb{C}}{\otimes} \bigwedge\nolimits^n \big(T_y(\tilde Y_1)^*\big) \\ &= \bigoplus_{y\in \imath^{-1}(Gv)} \widehat{\mathcal{O}(\tilde Y_1)}_y \underset{\mathbb{C}}{\otimes} \bigwedge\nolimits^n \big(T_y(\tilde Y_1)^*\big).\end{aligned}$$

Assumption (iii) and the construction of Y_1 imply that $T_y(\tilde Y_1) = T_y(Y_1) \otimes_{\mathbb{R}} \mathbb{C}$. From that and the above we see that the map

$$\big(C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} \Omega^n(\tilde Y_1)\big)^\wedge_{Gv} \longrightarrow \widehat{\Omega^n_{sm}(Y_1)}_{Gv}$$

induced by (3.3) is injective. Now the same argument as for μ in the proof of Theorem 1.5 shows that (3.3) is injective. $\square$

Describing the image of the map from Proposition 3.3 is another issue. One would like to think of it as some closure of M in $\Omega^n_{sm}(Y)$, but in general it is not clear whether the image is closed. To overcome that, we specialize to submodules of $\Omega^n(\tilde Y)$ that are direct summands. Let p be an idempotent in the ring of continuous $C^\infty(V)^G$-linear endomorphisms of $\Omega^n_{sm}(Y)$, such that p stabilizes $\Omega^n(\tilde Y)$. Then

$$\Omega^n_{sm}(Y) = p\Omega^n_{sm}(Y) \oplus (1-p)\Omega^n_{sm}(Y), \tag{3.4}$$

so $p\Omega^n_{sm}(Y)$ is a closed $C^\infty(V)^G$-submodule of $\Omega^n_{sm}(Y)$. Similarly $p\Omega^n(\tilde Y)$ is a $\mathcal O(\tilde V)^G$-submodule and a direct summand of $\Omega^n(\tilde Y)$.

Lemma 3.4. *Assume (i), (ii) and (iii) from Conditions B. The natural map*

$$\mu : C^\infty(V)^G \underset{\mathcal O(\tilde V)^G}{\otimes} p\Omega^n(\tilde Y) \;\to\; p\Omega^n_{sm}(Y)$$

is an isomorphism of Fréchet $C^\infty(V)^G$-modules.

Proof. By construction the image of μ is contained in $p\Omega^n_{sm}(Y)$ and we know from Proposition 3.3 that μ is injective. By Lemma 3.2.b any $m \in p\Omega^n_{sm}(Y)$ can be written as a finite sum $m = \sum_i f_i\omega_i$ with $f_i \in C^\infty(V)^G$ and $\omega_i \in \Omega^n(\tilde Y)$. We compute

$$m = p(m) = p\big(\sum\nolimits_i f_i\omega_i\big) = \sum\nolimits_i f_i p(\omega_i) \in \mu\big(C^\infty(V)^G \underset{\mathcal O(\tilde V)^G}{\otimes} p\Omega^n(\tilde Y)\big).$$

In other words, μ is surjective. In view of Proposition 3.3, μ is a continuous bijection between Fréchet spaces. Now the open mapping theorem says that it is a homeomorphism. $\square$

4. Special cases

Consider the algebra $C^\infty(V)$ with V and $\tilde V$ as in Conditions B, for the moment without any group action. By Theorem 2.3

$$HH_n(C^\infty(V)) = HH_n\Big(C^\infty(V) \underset{\mathcal O(\tilde V)}{\otimes} \mathcal O(\tilde V)\Big) \cong C^\infty(V) \underset{\mathcal O(\tilde V)}{\otimes} HH_n\big(\mathcal O(\tilde V)\big). \tag{4.1}$$

Here we may remove the singular locus of $\tilde V$, because it does not meet V. Then $\tilde V$ is nonsingular, so we can invoke the Hochschild–Kostant–Rosenberg Theorem [Lod, Theorem 3.4.4]. Next we apply Lemma 3.4 to the right hand side of (4.1) and we find natural isomorphisms

$$HH_n(C^\infty(V)) \cong C^\infty(V) \underset{\mathcal O(\tilde V)}{\otimes} \Omega^n(\tilde V) \cong \Omega^n_{sm}(V).$$

In this way we recover Connes' version of the Hochschild–Kostant–Rosenberg Theorem [Con], for the Hochschild homology of the Fréchet algebra of smooth functions on a real analytic manifold V. Because of the techniques that we used, our proof only applies when V can be embedded in a complex affine variety $\tilde V$ such that $T_v(\tilde V) = T_v(V) \otimes_{\mathbb R} \mathbb C$ for all $v \in V$.

Interesting examples arise from imposing conditions in terms of an affine subvariety $\tilde W \subset \tilde V$. For instance, let $k \in \mathbb N$ and consider the unital finite type $\mathcal O(\tilde V)$-algebra

$$A = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb C) \otimes \mathcal O(\tilde V) : c \text{ vanishes to the order } k \text{ on } \tilde W\}.$$

From Lemma 3.4 one can deduce that

$$C^\infty(V) \underset{\mathcal{O}(\tilde V)}{\otimes} A = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{C}) \otimes C^\infty(V) : c \text{ vanishes to the order } k \text{ on } \tilde W \cap V \}.$$

In principle $HH_*(A)$ can be computed with the techniques from [KNS]. Thus Theorem 2.3 provides an approach to analyse $HH_*\big(C^\infty(V) \underset{\mathcal{O}(\tilde V)}{\otimes} A\big)$.

Consider the crossed product algebra $\mathcal{O}(\tilde V) \rtimes G$, where G is a finite group acting on V and on $\tilde V$. Its Hochschild homology has been determined in [Nis, Theorem 2.11]:

$$HH_n(\mathcal{O}(\tilde V) \rtimes G) \cong \bigoplus\nolimits_{g \in \langle G \rangle} \Omega^n(\tilde V^g)^{Z_G(g)}, \tag{4.2}$$

where $\langle G \rangle$ is a set of representatives for the conjugacy classes in G. Similary, it is known from [Bry, Proposition 6] that

$$HH_n(C^\infty(V) \rtimes G) \cong \bigoplus\nolimits_{g \in \langle G \rangle} \Omega^n_{sm}(V^g)^{Z_G(g)}. \tag{4.3}$$

As $\mathcal{O}(\tilde V)$ has finite rank over $\mathcal{O}(\tilde V)^G$, $\mathcal{O}(\tilde V) \rtimes G$ is a finite type $\mathcal{O}(\tilde V)^G$-algebra. By Lemma 3.4

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} \mathcal{O}(\tilde V) \rtimes G \cong C^\infty(V) \rtimes G.$$

Now Theorem 2.3 says that

$$\begin{aligned} HH_n(C^\infty(V) \rtimes G) &\cong C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} HH_n(\mathcal{O}(\tilde V) \rtimes G) \\ &\cong C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} \bigoplus\nolimits_{g \in \langle G \rangle} \Omega^n(\tilde V^g)^{Z_G(g)}. \end{aligned} \tag{4.4}$$

By Lemma 3.4, with $p|_{\Omega^n_{sm}(V^g)}$ the projection to $Z_G(g)$-invariants, the right hand side of (4.4) is isomorphic to

$$\bigoplus\nolimits_{g \in \langle G \rangle} \Omega^n_{sm}(V^g)^{Z_G(g)}.$$

Thus our results agree with the earlier findings from [Bry, Nis].

A more challenging class of examples arises as follows. Suppose that G acts on $M_n(\mathbb{C}) \otimes \mathcal{O}(\tilde V) = M_n(\mathcal{O}(\tilde V))$ by

$$g \cdot f = u_g (f \circ g^{-1}) u_g^{-1},$$

where $u_g \in M_n(\mathcal{O}(\tilde V))^\times$ and f is regarded as a map from $\tilde V$ to $M_n(\mathbb{C})$. Then

$$A = M_n(\mathcal{O}(\tilde V))^G \tag{4.5}$$

is a finite type $\mathcal{O}(\tilde V)^G$-algebra. Special cases of this construction are $\mathcal{O}(\tilde V)^G$ (for $n = 1$) and $\mathcal{O}(\tilde V) \rtimes G$, for $M_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}[G])$. As far as we are aware, there is no general formula for the Hochschild homology of such algebras. By Lemma 3.4

$$C^\infty(V)^G \underset{\mathcal{O}(\tilde V)^G}{\otimes} A \;\cong\; M_n(C^\infty(V))^G. \tag{4.6}$$

Algebras of the form (4.5) and (4.6) are relevant because they arise in abundance from reductive p-adic groups, see for instance [Sol].

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References

[Bry] J.L. Brylinski, "Cyclic homology and equivariant theories", Ann. Inst. Fourier **37.4** (1987), 15–28

[Cha] S.U. Chase, "Direct products of modules", Trans. Amer. Math. Soc. **97** (1960), 457–473

[Con] A. Connes, "Noncommutative differential geometry", Publ. Math. Inst. Hautes Études Sci. **62** (1985), 41–144

[Eis] D. Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York NJ, 1995

[KNS] D. Kazhdan, V. Nistor, P. Schneider, "Hochschild and cyclic homology of finite type algebras", Sel. Math. New Ser. **4.2** (1998), 321–359

[Kopp] M.K. Kopp, "Fréchet algebras of finite type", Arch. Math. **83.3** (2004), 217–228

[Lod] J.-L. Loday, *Cyclic homology 2nd edition*, Mathematischen Wissenschaften **301**, Springer-Verlag, Berlin, 1997

[Mal] B. Malgrange, *Ideals of differentiable functions*, Tata Institute of Fundamental Research Studies in Mathematics **3**, Oxford University Press, London, 1967

[Mey1] R. Meyer, "Embeddings of derived categories of bornological modules", arXiv:math.FA/0410596, 2004

[Mey2] R. Meyer, *Local and analytic cyclic homology*, European Mathematical Society Publishing House, 2007

[MeVo] R. Meise, D. Vogt, *Einführung in die Funktionalanalysis*, Vieweg Studium: Aufbaukurs Mathematik **62**, Friedr. Vieweg & Sohn, 1992

[MeTo] J. Merrien, J.-C. Tougeron, "Ideaux de fonctions différentiables II", Ann. Inst. Fourier **20.1** (1970), 179–233

[Mor] A.S. Morye, "Note on the Serre-Swan theorem", Math. Nachrichten **286.2-3** (2013), 272–278

[Mos] G.D. Mostow, "Equivariant embeddings in euclidean space", Ann. Math. **65** (1957), 432–446

[Nis] V. Nistor, "A non-commutative geometry approach to the representation theory of reductive p-adic groups: Homology of Hecke algebras, a survey and some new results", pp. 301–323 in: *Noncommutative geometry and number theory*, Aspects of Mathematics **E37** Vieweg Verlag, Wiesbaden, 2006

[OpSo] E. Opdam, M. Solleveld, "Extensions of tempered representations", Geom. And Funct. Anal. **23** (2013), 664–714

[Poe] V. Poénaru, *Singularités C^∞ en présence de symétrie*, Lecture Notes in Mathematics **510**, Springer Verlag, Berlin, 1976

[ScWo] H.H. Schaefer, M.P. Wolff, *Topological vector spaces. Second edition*, Graduate Texts in Mathematics **3**, Springer-Verlag, New York, 1999

[Sch] G.W. Schwarz, "Smooth functions invariant under the action of a compact Lie group", Topology **14** (1975), 63–68

[Sol] M. Solleveld, "Hochschild homology of reductive p-adic groups", J. Noncommut. Geom. **18** (2024), 1–65

[Tay] J.L. Taylor, "Homology and cohomology for topological algebras", Adv. Math. **9** (1972), 137–182

[Tou] J.C. Tougeron, *Idéaux de fonctions différentiables*, Ergebnisse der Mathematik und ihrer Grenzgebiete **71**, Springer-Verlag, Berlin, 1972

[Vog] D. Vogt, "On the functors $Ext^1(E,F)$ for Fréchet spaces", Studia Math. **85.2** (1987), 163–197

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