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Lectures on Operator Theory

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CHAPTER 23

AF-Algebras and Bratteli Diagrams

Brief History

- Glimm (1959): Introduced UHF algebras and techniques.
- Dixmier (1967): Matroid Algebras.
- Bratteli (1971): Introduced and classified AF algebras.
- Elliott (1973): Classified AF algebras in terms of groups.

23.1 AF-Algebras

Approximately finite dimensional (AF) C*-algebras are inductive limits of finite dimensional C*-algebras

$$A = \varinjlim A_i.$$

It is well known that every finite dimensional C*-algebra is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$

$$A_j \cong \bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C}) \quad \forall j \in \mathbb{N}.$$

Using this fact, it is easy to establish what kind of homomorphisms can exist between any two finite dimensional C*-algebras by just looking at what kind of homomorphisms exist between matrix algebras. Any homomorphism $M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is determined, up to an inner automorphism, by an integer $k \geq 0$, namely

$$A \mapsto \begin{bmatrix} A & & & & \\ & A & & & \\ & & \ddots & & \\ & & & A & \\ & & & & 0 \end{bmatrix}$$

where A appears k times on the diagonal (referred to as the *multiplicity* of the morphism), and 0 is the appropriate size zero matrix. We are making the assumption that $nk \leq m$.

23.2 Bratteli Diagrams

Given a morphism $A_i \to A_{i+1}$ of the directed system, we obtain for each summand $M_n(\mathbb{C})$ of A and each summand $M_m(\mathbb{C})$ of A_{i+1} an integer $k \geq 0$, namely the multiplicity (as above) of the composition $M_n(\mathbb{C}) \to A_i \to A_{i+1} \to M_m(\mathbb{C})$. In the Bratteli diagram of the direct system, these summands are represented by the integers n and m, and the multiplicity k by k lines connecting n to m.

Example 23.2.1 *Here are some examples "illustrating"* AF-algebras.

1.

$$2 \text{ --- } 4 \text{ --- } 6 \text{ --- } 8 \text{ --- } \cdots$$

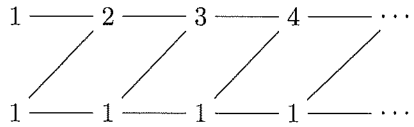
represents $M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_6(\mathbb{C}) \to \cdots$ with multiplicity 1 embeddings, which have direct limit $\mathcal{K}$

2.

$$2 = 4 = 8 = 16 = \cdots$$

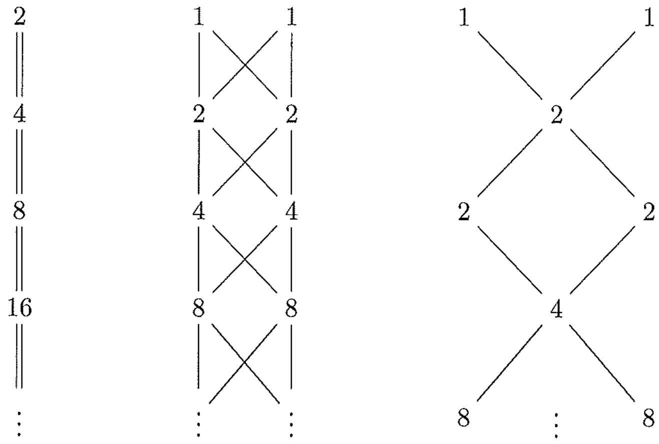
represents $M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \to \cdots$ with multiplicity 2 embeddings and direct limit the UHF algebra $M_{2^\infty}(\mathbb{C})$.

3. One can represent the unitized $\mathcal{K}$ as



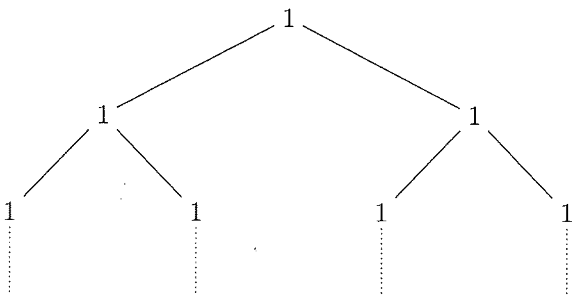
(the 1's on the bottom replace the 0 in the embedding of $M_n(\mathbb{C})$ in $M_{n+1}(\mathbb{C})$).

4. Bratteli diagrams are not unique. Here are 3 different representations of $M_{2^\infty}(\mathbb{C})$



Notice that the first two of these diagrams are subsequences of the third.

5. Our final example is the infinite "binary tree"



which represents the continuous functions on the Cantor set, C. This can be seen by recalling the "middle thirds" construction. The top 1 represents 1_C, the 1's on the second row represent projections onto the left and right "thirds" respectively, 1's on the third row are the projections onto the "ninths", etc.

Proposition 23.2.2 *If A and B are AF-algebras that have the same Bratteli diagram, then $A \cong B$.*

Proof Assume that $A = \varinjlim A_n$ and $B = \varinjlim B_n$ have the same diagram. Then there exist isomorphisms $\phi_i : A_i \to B_i$ and we have a (not necessarily commutative) diagram

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \cdots & \longrightarrow & A \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & & & \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \cdots & \longrightarrow & B. \end{array}$$

However, since the compositions in each square have the same diagram, the squares do commute up to an inner automorphism. Thus, one can inductively replace each ϕ_i with $\psi_i = \mathrm{Ad(u_i)} \circ \phi_\mathrm{i}$ with unitary u_i, and make the above diagram commute. The result now follows. □

23.3 Characterizing AF-Algebras

In this section, we obtain an internal characterization of AF-algebras, and an isomorphism theorem. Both theorems require a lemma by Glimm, which shows that if a finite dimensional sub-algebra, $A \subset D$ is "approximately contained" in another subalgebra, B, it can be twisted by a unitary so that it is contained in B. One must make the idea of "approximate containment" precise, and there is a natural way of doing this. How "close" the containment of A in B is can be defined by

$$\mathrm{dist(A,B)} = \sup_{\substack{\mathrm{a}\in\mathrm{A} \\ \|\mathrm{a}\|\le 1}} \mathrm{dist(a,B)} = \sup_{\substack{\mathrm{a}\in\mathrm{A} \\ \|\mathrm{a}\|\le 1}} \inf_{\mathrm{b}\in\mathrm{B}} \|\mathrm{a-b}\|.$$

Lemma 23.3.1 *Suppose $\epsilon > 0$ and $n \in \mathbb{N}$. Then there is a real number $\delta(\epsilon, n) > 0$ so that if $p_1, \cdots, p_n$ are pairwise orthogonal projections in a C^*-algebra B, and D a sub-algebra of B with $\mathrm{dist(p_i, D)} < \delta$, $1 \le \mathrm{i} \le \mathrm{n}$, there are pairwise orthogonal projections $q_i \in D$ such that $\|p_i - q_i\| < \epsilon$.*

Proof Let $n = 1$ and $\delta = \epsilon/2$ By assumption, we can find an element $x \in D$ with $\|x - p_1\| < \delta$. By replacing x with $(x + x^*)/2$ we can assume $x = x^*$ (this will not affect the norm estimate). Since $\sigma(p_1) = \{0, 1\}$ we have $\sigma(x) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta]$. Using the Borel functional calculus for normal operators, define a projection $q_1 = \chi_{[1-\delta,1+\delta]}(x) \in D$ where χ denotes the characteristic function. This will satisfy the norm estimate $\|p_1 - q_1\| < \epsilon$.

For arbitrary $n \in \mathbb{N}$, one uses a similar argument and induction, minding your ϵ's and δ's. (See **[2]**, Lemma II.3.1). □

Remark 23.3.2 If $\sum p_i = 1$ in the above lemma, one can choose the q_i's so that $\sum q_i = 1$.

Lemma 23.3.3 (Glimm) *Let D be a unital C^*-algebra. Then for every $n \in \mathbb{N}$ and $\epsilon > 0$ there is a $\delta(\epsilon, n) > 0$ such that if $A, B \subset D$ are C^*-subalgebras satisfying $\dim(A) \le n$ and A has matrix units $\{\mathrm{e}_{ij}^{(s)}\}$ satisfying $\mathrm{dist}(\mathrm{e}_{\mathrm{ij}}^{(s)}, \mathrm{B}) < \delta$, there is a unitary $u \in D$ such that $\|u - 1\| < \epsilon$ and $uAu^* \subseteq B$.*

Proof If $e_{ij}^{(s)}$ are a system of matrix units for A, one first gets a desired unitary on the projections $e_{ii}^{(s)}$. To do so, let $\eta = \epsilon/n + 1$ and use Lemma 23.3.1

to find projections $f_{ii}^{(s)} \in B$ with $\|e_{ii}^{(s)} - f_{ii}^{(s)}\| < \delta(\eta, n)$. Let $x = \sum_{i=1}^{n} f_{ii}^{(s)} e_{ii}^{(s)}$. Using our norm estimate, one can show that both x^*x and xx^* are invertible, so x is invertible, and $f_{ii}^{(s)} x = x e_{ii}^{(s)}$ for each i.

Let u be the unitary in the polar decomposition of $x = u|x|$. One checks directly that $ue_{ii}^{(s)}u^* = f_{ii}^{(s)} \in B$, and $\|u - I\| < \epsilon$. The matrix units $f_{ij}^{(s)} = ue_{ij}^{(s)}u^*$ may not be in B whenever $i \neq j$. However, the $f_{ij}^{(s)}$ are 'close enough' to B so that one can do a similar (though more technical) trick as above to obtain a unitary w that "twists" the $f_{ij}^{(s)}$ into B. The unitary wu is then the candidate that satisfies the lemma's conclusion. Full details can be found in ([**2**], Lemma II.3.2). □

Remark 23.3.4 Erik Christensen showed that one can choose δ in the above lemma independent of n.

Remark 23.3.5 By construction, the unitary u in the above lemma is in the C*-algebra generated by A and B. Furthermore, one can take $u \in (A \cap B)'$. (See [**2**]).

Using Glimm's lemma, we can get a local characterization of AF-algebras.

Theorem 23.3.6 *A C*-algebra A is an AF-algebra iff it is separable and*

$$\begin{array}{l}\forall \epsilon > 0,\ \forall a_1, \cdots, a_n \in A,\ n \in \mathbb{N},\ \exists \text{ finite dimensional} \\ *\text{-subalgebra } B \text{ of } A \text{ such that } dist(a_i, B) < \epsilon,\ \ 1 \le i \le n.\end{array} \qquad (*)$$

Proof ($\Rightarrow$) is trivial.

($\Leftarrow$) Let $\{a_i\}_{i\in\mathbb{N}}$ be a dense subset of the unit ball of A with $a_1 = 0$. Let $\epsilon_i > 0$ be a sequence decreasing to 0. We use induction to find finite dimensional subalgebras A_i such that $\text{dist}(a_i, A_k) < \epsilon_k$ for $1 \le i \le k$, and $A_{k-1} \subseteq A_k$.

The result holds trivially for $k = 1$. So suppose we have the induction hypothesis for some $k > 1$. Let $n = \dim(A_k)$, and choose $\delta = \delta(\epsilon_{k+1}/3, n)$ as in Lemma 23.3.3. Fix a set of matrix units, $e_{ij}^{(s)}$ for A_k, and use $(*)$ to find a finite dimensional subalgebra B so that $\text{dist}(e_{ij}^{(s)}, B) < \delta$ for all the matrix units and $\text{dist}(A_i, B) < \epsilon_{k+1}/3$ for $1 \le i \le k+1$. By Lemma 23.3.3 there is a unitary u such that $uA_ku^* \subset B$ and $\|u - 1\| < \epsilon_{k+1}/3$. Let $A_{k+1} = u^*Bu$. Then A_{k+1} is a finite dimensional algebra containing A_k. We also have

$$\text{dist}(a_i, A_{k+1}) = \text{dist}(ua_iu^*, B) \le 2\|u - 1\| + \epsilon_{k+1}/3 < \epsilon_{k+1}.$$

Continuing in this manner gives us an increasing sequence of finite dimensional sub-algebras of A whose closure is all of A. So A is AF. □

Theorem 23.3.7 *Suppose* $A = \overline{\cup_n A_n}$ *and* $B = \overline{\cup_n B_n}$ *are AF algebras. Then* $A \cong B$ *iff* $\cup_n A_n \cong \cup_n B_n$.

The reverse implication is obvious. The idea behind the proof of the forward implication is as follows. Suppose $\phi : A \to B$ is the isomorphism in question. By restricting ϕ to A_1, one has a finite dimensional subalgebra of B which will be "close" to some B_k, $k \in \mathbb{N}$. One uses lemma 23.3.3 to find a unitary u so that $u\phi(A_1)u^* \subseteq B_k$. Continuing along in this manner yields the result needed. □

Theorem 3.7 can be used to construct an isomorphism invariant for AF-algebras. Since the algebras A_n, B_n are all finite dimensional, it follows that the isomorphism $\cup A_n \to \cup B_n$ will carry each A_n into some B_m and conversely. Thus two

AF-algebras $A = \overline{\cup A_n}$ and $B = \overline{\cup B_m}$ are isomorphic if and only if there is a commutative diagram

$$\begin{array}{ccccccccc} A_{n_1} & \longrightarrow & A_{n_2} & \longrightarrow & A_{n_3} & \longrightarrow & \cdots & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & & & \\ B_{m_1} & \longrightarrow & B_{m_2} & \longrightarrow & B_{m_3} & \longrightarrow & \cdots & \longrightarrow & B \end{array} \qquad (*)$$

where $1 = n_1 < n_2 < n_3 < \cdots$ and $1 \le m_1 < m_2 < m_3 < \cdots$. Thus two diagrams define the same AF-algebra if and only if they are equivalent in the equivalence relation generated by the following two processes.

1. From a given diagram, remove the vertices in certain levels and insert the appropriate edges between the remaining vertices. This process is for example used in going from the right-hand diagram in Example 2.1 (4) to the left-hand diagram: Remove level 1, 3, 5,$\cdots$.
2. The converse of (1): Introduce new levels of vertices, like in going form the left-hand diagram to the right-hand diagram in Example 2.1 (4).

By using (1), one goes from

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$$

to

$$A_{n_1} \longrightarrow A_{n_2} \longrightarrow A_{n_3} \longrightarrow \cdots.$$

Then one uses (2) to go to

$$A_{n_1} \longrightarrow B_{m_1} \longrightarrow A_{n_2} \longrightarrow B_{m_2} \longrightarrow A_{n_3} \longrightarrow \cdots$$

and then (1) to go to

$$B_{m_1} \longrightarrow B_{m_2} \longrightarrow B_{m_3} \longrightarrow \cdots$$

and finally (2) to obtain

$$B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow \cdots.$$

Thus, one obtains any equivalent diagram by using (1), then (2), then (1), and finally (2).

Another way of formulating Theorem 3.7 is by means of K-theory. Replace each of the vertices in a level of the Bratteli diagram by a copy of $\mathbb{Z}$, so that the level is represented by $\mathbb{Z}^n$, where n is the number of vertices at that level, and map the groups into each other by the rule defined by the diagram. Define an order on $\mathbb{Z}^n$ by $(m_1, \cdots, m_n) \ge 0$ iff $m_i \ge 0$ for every i, and equip the inductive limit group with the limit order. Using that each $\mathbb{Z}^n$ is finitely generated, it is not hard to see that two such ordered groups are isomorphic if and only if one has a commutative diagram of the type (*), only with A_{n_i} and B_{n_j} replaced by ordered groups of the type $\mathbb{Z}^n$. This group, together with the range of the canonical map from projections in the algebra into the group (which is a hereditary subset) is the Elliott invariant of AF-algebras. A deep result by Effros, Handelman and Shen states that an ordered group is the ordered group of an AF-algebra if and only if it is countable, abelian, has the Riesz interpolation property, and is unperforated.

An ordered group has the Riesz interpolation property if whenever g_1, g_2, h_1, h_2 are group elements and $g_i \le h_j$ for $i, j = 1, 2$ then there is a group element k such that $g_i \le k \le h_j$ for $i, j = 1, 2$. It is unperforated if whenever $n \in \mathbb{N}$ and ng is positive then g is positive.

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