

# Corrections to D. J. H. Garling's 'A course in Mathematical Analysis', Vol. 1

Michael Mürger  
Institute for Mathematics, Astrophysics and Particle Physics  
Radboud University Nijmegen, The Netherlands

February 27, 2019

I thank Fons van der Plas for contributing about 25 of the following corrections!

- p.5, l.-14: I strongly dislike the choice of having  $\subset$  denote **proper** inclusion, however “logical” that may seem.  
Very few authors do this. (Right now I can only think of Gaal’s ‘Point set topology’.) Most authors let  $\subset$  denote not-necessarily-proper inclusion, and deviating from this majority practice can only cause confusion.  
Anyway, a natural workaround is to avoid  $\subset$  altogether and only use  $\subseteq$  and  $\subsetneq$  (as also T. Tao does in his two-volume analysis course).
- p.7, l.-5: I don’t like the notation  $C(B)$  for the complement  $\mathbb{R}\setminus B$  and think one should stick to  $B^c$ .
- p.15, Exercise 1.4.1: In (c) and (d): Replace  $C, D$  by  $E, F$ , respectively.
- p.65, l.13: Garbled formula  $\psi(-m)l = \psi(-m)\psi(l)$ . Probably there just is a bracket missing:  $\psi((-m)l) = \psi(-m)\psi(l)$
- p.65, l.-10: Replace  $\psi(jn')$  by  $\psi(j'n)$ .
- p.67, l.-3: ‘upper bound for  $U$ ’ should be ‘upper bound for  $L$ ’.
- p.70, l.7: Here is the definition  $D(x) = \{r \in \mathbb{Q} \mid r < x\}$ . One may question that this makes sense.  $\mathbb{R}$ , as introduced in Theorem 2.9.5, is the set of Dedekind cuts. By definition, a cut just is a set of rationals satisfying some axioms. Thus for  $x \in \mathbb{R}$ ,  $x$  and  $D(x)$  are the same thing, and the notational distinction is hard to sell. (Of course one may say that  $D(x)$  is a set of rationals, whereas writing  $x \in \mathbb{R}$  one forgets about this fact and considers  $x$  as a structureless ‘point’ in  $\mathbb{R}$ . But still. . .)
- p.71, l.7: The middle term ‘ $\{r \in \mathbb{Q} \mid j(r) < -x\}$ ’ doesn’t really make sense since  $-x$  is not yet defined. (In fact, this line *is* the definition of  $-x$ .) At best, it serves as motivation for the third term.
- p.72, l.16: Replace ‘ $D^+x.D^+y$ ’ by ‘ $D^+(x) \cdot D^+(y)$ ’.
- p.73, l.9: Replace ‘ $x \in \mathbb{Q}$ ’ by ‘ $r \in \mathbb{Q}$ ’.
- p.74, l.-12: Replace  $(y - s^n)/(2^n - 1)y$  by  $\frac{y-s^n}{(2^n-1)s^n}$ .

- p.74: The proof of Theorem 2.10.11 uses Lemma 2.10.12, where the hypothesis  $0 < \varepsilon < 1$  is made. This must be taken into account when choosing  $\theta, \eta$ :

$$0 < \eta < \min\left(\frac{y - s^n}{(2^n - 1)s^n}, 1\right), \quad 0 < \theta < \min\left(\frac{s^n - y}{ns^n}, 1\right).$$

- p.83, Exercise 3.1.6: Under the first  $\sum$ -sign, replace  $1 = 1$  by  $i = 1$ . Under the second  $\sum$ , it would be better to write  $1 \leq i < j \leq n$ .
- p.84, l.10: Replace ' $l > 0$ ' by ' $\varepsilon > 0$ '.
- p.84, l.11: Replace ' $1/n < 1/n_0$ ' by ' $1/n \leq 1/n_0$ '.
- p.87, Item (v): It is simpler to just write  $a_n b_n - ab = a_n(b_n - b) + b(a_n - a)$ . The terms on the r.h.s. converge to zero by (ii) and (iv), respectively. Thus the sum goes to zero by (iii).
- p.87, Item (vi): This proof would be simpler if one noted earlier, preferably in the context of Prop. 3.2.3, that boundedness of  $\{a_n\}_{n \geq n_0}$  for some  $n_0$  implies boundedness of  $\{a_n\}$ .
- p.90, problem 3.2.4: add 'as  $n \rightarrow \infty$ '. (Omitting the variable is sloppy.)
- p.91 l.21: Replace  $\{x\} = x - [x]$  with  $\{x\} = x - \lfloor x \rfloor$ .
- p.92, l.-3: Closing bracket missing in  $k(j(x))$
- p.93, l.-12:  $C/ \sim$  is very ugly (too much white space). Inserting one negative space ( $\backslash!$ ) gives  $C/\sim$ , which is a bit better. I actually prefer  $C/\sim$  (two negative spaces). There likely are other instances of this.
- p.94, Theorem 3.4.1: I've come to the conclusion that the first proof (using a monotonous subsequence) is so much simpler that it is pointless to give the second. If one does, one should do that in two parts as follows, since the first has independent uses:

0.1 PROPOSITION *Let  $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$  be a decreasing sequence of closed intervals such that the lengths tend to zero:  $b_n - a_n \rightarrow 0$ . Then the intersection  $\bigcap_{n=1}^{\infty} [a_n, b_n]$  consists of precisely one point.*

*Proof.* The sequence  $\{a_n\}$  is increasing (weakly) and bounded above by  $b_1$ , thus convergent to some  $a \in \mathbb{R}$ . Similarly,  $\{b_n\}$  is decreasing and bounded below by  $a_1$ , thus  $b_n \rightarrow b$ . Now the assumption  $b_n - a_n \rightarrow 0$  implies  $b = a$ . Now  $a \geq a_n \forall n$  and  $a = b \leq b_n \forall n$  implies  $a \in \bigcap_{n=1}^{\infty} [a_n, b_n]$ . And if  $x$  is a point in the intersection then  $|x - a| \leq b_n - a_n$  (since both points are in  $[a_n, b_n]$ ) for all  $n$  implies  $x = a$ . ■

Now one uses this to prove BW:  $\{a_n\}$  is bounded, thus there is an interval  $[b_0, c_0]$  containing all  $a_n$ . Define  $d_0 = (b_0 + c_0)/2$ . Then at least one of the intervals  $[b_0, d_0], [d_0, c_0]$  contains  $a_n$  for infinitely many  $n$ . Choose such an interval and iterate the procedure. This gives a sequence  $\{[b_n, c_n]\}$  of intervals of lengths  $(c_0 - b_0)/2^n$ , thus the intersection consists of one point  $x$ . Now choose  $n_k$  such that  $a_{n_k} \in [b_k, c_k]$  and  $n_k > n_{k-1}$  for each  $k$ . This can be done since each of these intervals was chosen to contain  $a_n$  for infinitely many, thus arbitrarily large,  $n$ . Now  $b_k \leq a_{n_k} \leq c_k$  together with  $a_n \rightarrow x, b_n \rightarrow x$  implies  $a_{n_k} \rightarrow x$ . ■

- p.96 l.18: Replace  $a_M > r$  with  $a_m > r$ .

- p.108, l.7: Replace  $w_j$  by  $z_j$ .
- p.108, l.8 and l.6:  $\mathbb{N}^+$  has never been defined. The  $\sum_{j=0}^{\infty}$  in l.7 suggests that the author means  $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ , but then the statements  $s_{2n} = 0$  and  $s_{2n+1} = 1$  are false and should be replaced by  $s_{2n} = 1$  and  $s_{2n+1} = 0$ .
- p.109, l.13: Replace  $\sum_{j=1}^n$  by  $\sum_{j=0}^n$ .
- p.109, Coro. 4.2.2: It is true that in order to conclude convergence of  $\sum_{j=0}^{\infty} c_j$  from convergence of  $\sum_{j=0}^{\infty} a_j$  it suffices to have  $0 \leq c_j \leq a_j$  for  $j \geq j_0$ . But in order to conclude that  $\sum_{j=0}^{\infty} c_j \leq \sum_{j=0}^{\infty} a_j$ , we need  $0 \leq c_j \leq a_j$  to hold for **all**  $j \geq 0$ , i.e.  $j_0 = 0$  !!
- p.111, l.6: at the end of the line, there must be  $(a_{j_0} r^{-j_0}) r^j$  (i.e. minus sign is missing).
- p.113, l.2: See Garling's erratum, which erroneously lists this under p.123.
- p.114, l.1: 'Exercise 3.1.9' does not exist. Presumably this should be Exercise 3.2.11.
- p.115, l.15: Replace  $\sum_{j=0}^{\infty} |a_j|$  with  $\sum_{j=0}^{\infty} |z_j|$ .
- p.115: To prove of the first sentence in the proof of Prop. 4.3.1, we need not only  $|x_j| \leq |z_j|$  and  $|y_j| \leq |z_j|$ , but also  $|z_j| \leq |x_j| + |y_j|$ .
- p.116, l.13: Here  $a_{2n}$  must be  $a_{2n+2}$ , thus:

$$s_{2n+2} = s_{2n} - (a_{2n+1} - a_{2n+2}) \leq s_{2n}.$$

- p.116, l.9: The  $a_{2n+1}$  must be  $a_{2n+2}$ .
- p.117, l.8-9:  $s_n = \sum_{j=0}^n a_j z_j$  is used before it is defined.
- p.117, l.16: Abel's formula and its uses are better known as 'partial summation'.
- p.117, l.10 and also l.9: superfluous left bracket in  $|(a_j - a_{j+1}|$
- p.117, l.2: Since  $\{a_j\}$  is decreasing, thus  $a_j - a_{j-1} \leq 0$ , the correct formula is:

$$\sum_{j=1}^{\infty} |a_j - a_{j-1}| = \sum_{j=1}^{\infty} (a_{j-1} - a_j) = a_0.$$

- p.120, l.4: There must be  $k = \sup\{\sigma(j) : \mathbf{0} \leq j \leq n\}$ .
- p.121, l.10: The end of the third displayed formula must be

$$+ \dots + \left( \frac{1}{\sqrt{4j+1}} + \frac{1}{\sqrt{4j+3}} - \frac{1}{\sqrt{2j+2}} \right)$$

in order for this general term to be consistent with the first two (corresponding to  $j = 0$  and  $j = 1$ ).

- p.128, item 5 of the list: uniform convergence has not yet been defined!
- p.128, item 6 of the list: Replace  $e^z = e(z)$  with  $e^z = \exp(z)$ .

- p.131, l-10: “It is an easy exercise to show that every interval is of one of these forms.” Add: ‘or equal to  $\mathbb{R}$ ’.
- p.133, l.11: Replace ‘ $a \in \bar{A} = A$ ’ by ‘ $b \in \bar{A} = A$ ’.
- p.138, Exercise 5.3.1: The claimed statement is **not** equivalent to connectedness of  $A$ ! If  $U_1, U_2$  are as in the definition of connectedness (p.137) and one puts  $F_i = U_i^c$  then connectedness of  $A$  is equivalent to the statement that  $F_1, F_2$  closed,  $F_1 \cup F_2 = \mathbb{R}$ ,  $A \cap F_1 \cap F_2 = \emptyset \Rightarrow A \subseteq F_1 \vee A \subseteq F_2$ . The condition  $A \cap F_1 \cap F_2 = \emptyset$ , without which there is no connection between  $A$  and  $F_1, F_2$ , is missing in Garling.
- p.138, Exercise 5.3.3: In the fourth line, replace ‘ $b \in c_0, d_0$ ’ by ‘ $b \in [c_0, d_0]$ ’.
- p.139, l.-6: Replace ‘First,  $c > a$ ’ by ‘First,  $s > a$ ’.
- p.140, l.11: After ‘Suppose... bounded’, there should be ‘and let  $\mathcal{U}$  be an open cover of  $B$ .’.
- p.140, l.19: The first closing bracket should be a brace.
- p.142, l.-12: Delete the assumption ‘ $a \in \mathbb{R}$ ’ that is not referred to in the rest of the statement of the Proposition.
- p.149, l.10: Delete ‘and that  $l \in \mathbb{R}$ ’. (No  $l$  appears in (ii) and (iii), whereas in (i) we have ‘there exists  $l$ ’.)
- p.149, l.15: Replace  $N_\delta^*(b)$  by  $N_\delta^*(b) \cap A$ .
- p.150, Corollary 6.1.5: Of course this also requires  $f$  to be increasing!
- p.157, l.2: Typo of “... *there exist* ...”.
- p.158, l.15: Replace ‘Theorem 6.3.4’ by ‘Proposition 6.3.4’.
- p.159, l.12: ‘= =’.
- p.159, l.15: Replace ‘ $e$  is not continuous’ by ‘ $f$  is not continuous’.
- p.161, l.3: Replace  $a \in A$  with  $x \in A$ .
- p.163, in Corollary 6.4.6: Replace  $k \in \mathbb{N}$  with  $n \in \mathbb{N}$ .
- p.166, l.5: Replace  $\sum_{j=m+1}^n |f(s)| < \epsilon$  with  $\sum_{j=m+1}^n |f_j(s)| < \epsilon$ .
- p.166, l.-2: Replace ‘ $H$ ’ by ‘ $h$ ’.
- p.167, l.11: Replace  $\sum_{j=0}^{\infty} a_j z_j$  with  $\sum_{j=0}^{\infty} f_j z_j$ .
- p.167: Theorem 6.6.1 is literally identical to Theorem 6.5.1. At least ‘real-valued’ should be changed to ‘complex-valued’, but probably also the functions are defined on subsets of  $\mathbb{C}$ ?
- p.168, l.6: The ‘ $r - s$ ’ in the denominator should be ‘ $s - r$ ’.
- p.172, l.7: Replace  $\sum_{j=0}^{\infty} a_j z_j$  with  $\sum_{j=0}^{\infty} f_j z_j$ .
- p.173, l.-6:  $\eta$  is not defined. “for some  $\eta > 0$ ” should be added after  $(a - \eta, a + \eta) \subseteq I$ .
- p.177, l.12: Superfluous bracket in  $f(a + h) = b + (s(h))$ .

- p.178, l.2: The word “*but*” seems misplaced, and a new sentence should start after  $f'(a) \geq 0$ .
- p.179, l.9-14: The enumeration here uses Arabic numbering (1., 2., 3., 4.), but in the remainder of the proof, these items are referred to using (i), (ii), (iii), (iv).
- p.180, Exercise 7.1.3: Replace ‘ $f$  is differentiable at 0’ by ‘ $g$  is differentiable at 0’.

- p.186, Theorem 7.3.2 (Rolle): I dislike this proof, which obscures the matter by unnecessarily bringing in monotonicity. A cleaner argument is as follows:

Let  $A := f(a) = f(b)$ . If  $f$  is constant, i.e.  $f(x) = A$  for all  $x \in (a, b)$ , then  $f'(x) = 0$  for all  $x \in (a, b)$ , and we are done. Assume  $f$  is non-constant. Then either  $\sup_{x \in [a, b]} f(x) > A$  or  $\inf_{x \in [a, b]} f(x) < A$  (or both). Assume the first holds. By Theorem 6.3.6,  $S = \sup_{x \in [a, b]} f(x) < \infty$ , and there is an  $x \in [a, b]$  such that  $f(x) = S$ . Since  $S > A$ , we have  $a \neq x \neq b$ , thus  $x \in (a, b)$ . Thus the global maximum at  $x$  is a local maximum, thus  $f'(x) = 0$  by Proposition 7.3.1. QED

- p.187, Theorem 7.3.5 (Mean value theorem): Garling omits the proof of  $h_\lambda(a) = h_\lambda(b)$ , which actually is quite tedious for his choice of  $h_\lambda$ . If instead we define  $h_\lambda(x) = f(x) - \lambda(x - a)$  the computations are quite trivial: We trivially have  $h_\lambda(a) = f(a)$ . And

$$h_\lambda(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - (f(b) - f(a)) = f(a) = h_\lambda(a).$$

- p.192, last line: There must be  $x \searrow 0$ .
- p.193, l.-3: Replace  $\sin' x = -\cos x$  with  $\sin' x = \cos x$ .
- p.194, l.6: A global factor  $w^2$  has gone missing.
- p.194, lines 8 and 9: The factors  $|w|^2$  are missing.
- p.194, l.-7: There must be  $+\sin x$  instead of  $-\sin x$  (as in the correct preceding equation).
- p.196, l.-1: “. . . , *respectively*.” should be added.
- p.205, -9: Replace ‘If  $0 \leq x < 1$ ’ by ‘If  $-\frac{1}{2} < x < 1$ ’. (If  $-\frac{1}{2} < x < 0$  then  $|x| < \frac{1}{2}$  and  $|1 + \theta x| > \frac{1}{2}$ , thus  $\left| \frac{x}{1 + \theta x} \right| < 1$  still holds.)
- p.212, l.-3: Replace  $\sum_{j=1}^k M_j \chi_j$  with  $\sum_{j=1}^k M(I_j) \chi_j$ .
- p.213, l.6: Similarly, replace  $\sum_{j=1}^k m_j \chi_j$  with  $\sum_{j=1}^k m(I_j) \chi_j$ .
- p.214, l.14: Replace  $M_p l(J_p) l(J_p)$  with  $M(J_p) l(J_p)$ .
- p.216, l.-3: Add “ $\in \mathbb{N}$ ” after “*Choose N*”.
- p.216, l.-1: Replace  $M_j$  and  $m_j$  with  $M(I_j)$  and  $m(I_j)$ , respectively.
- p.216, l.-1: Superfluous closing brace.
- p.217, l.7: Missing closing bracket after  $(f(b) - f(a))$ .
- p.219, l.12:  $G \cup B$  should be a partition of the interval *indices* (in  $\mathbb{N}$ ), but  $G$  and  $B$  are incorrectly defined as the interval *boundaries* (in  $\mathbb{R}$ ).
- p.232 in Theorem 8.7.3: Replace “ $k$  times differentiable” with “ $n$  times differentiable”.