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Quantum Double Actions on Operator Algebras and Orbifold Quantum Field Theories

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Abstract: Starting from a local quantum field theory with an unbroken compact symmetry group G in 1+1-dimensional spacetime we construct disorder fields implementing gauge transformations on the fields (order variables) localized in a wedge region. Enlarging the local algebras by these disorder fields we obtain a nonlocal field theory, the fixpoint algebras of which under the appropriately extended action of the group G are shown to satisfy Haag duality in every simple sector. The specifically 1+1 dimensional phenomenon of violation of Haag duality of fixpoint nets is thereby clarified. In the case of a finite group G the extended theory is acted upon in a completely canonical way by the quantum double $D(G)$ and satisfies R-matrix commutation relations as well as a Verlinde algebra. Furthermore, our methods are suitable for a concise and transparent approach to bosonization. The main technical ingredient is a strengthened version of the split property which is expected to hold in all reasonable massive theories. In the appendices (part of) the results are extended to arbitrary locally compact groups and our methods are adapted to chiral theories on the circle.

1. Introduction

Since the notion of the "quantum double" was coined by Drinfel'd in his famous ICM lecture [30] there have been several attempts aimed at a clarification of its relevance to two dimensional quantum field theory. The quantum double appears implicitly in the work [19] on orbifold constructions in conformal field theory, where conformal quantum field theories (CQFTs) are considered whose operators are fixpoints under the of a symmetry group on another CQFT. Whereas the authors emphasize that "the fusion algebra of the holomorphic G-orbifold theory naturally combines both the representation and class algebra of the group G" the relevance of the double is fully recognized only in [20]. There the construction is also generalized by allowing for an arbitrary 3-cocycle in

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$H^3(G, U(1))$ leading only, however, to a quasi quantum group in the sense of [31]. The quantum double also appears in the context of integrable quantum field theories, e.g. [7], as well as in certain lattice models (e.g. [67]). Common to these works is the role of disorder operators or "twist fields" which are "local with respect to $\mathcal{A}$ up to the action of an element $g \in G$" [19]. Finally, it should be mentioned that the quantum double and its twisted generalization also play a role in spontaneously broken gauge theories in 2 + 1 dimensions (for a review and further references see [6]).

Regrettably most of these works (with the exception of [67]) are not very precise in stating the premises and the results in mathematically unambiguous terms. For example it is usually unclear whether the "twist fields" have to be constructed or are already present in some sense in the theory one starts with. As a means to improve on this state of affairs we propose to take seriously the generally accepted conviction that the physical content of a quantum field theory can be recovered by studying the inequivalent representations (superselection sectors) of the algebra $\mathcal{A}$ of observables (which in the framework of conformal field theory is known as the chiral algebra). This point of view, put forward as early as 1964 [45] but unfortunately widely ignored, has proved fruitful for the model independent study of (not necessarily conformally covariant) quantum field theories, for reviews see [46, 49]. Using the methods of algebraic quantum field theory we will exhibit the mechanisms which cause the quantum double to appear in *every* quantum field theory with group symmetry in 1 + 1 dimensions fulfilling (besides the usual assumptions like locality) only two technical assumptions (Haag duality and split property, see below) but independent of conformal covariance or exact integrability.

As in [21] we will consider a quantum field theory to be specified by a net of von Neumann algebras, i.e. a map

$$\mathcal{O} \mapsto \mathcal{F}(\mathcal{O}), \tag{1.1}$$

which assigns to any bounded region in 1 + 1 dimensional Minkowski space a von Neumann algebra (i.e. an algebra of bounded operators closed under hermitian conjugation and weak limits) on the common Hilbert space $\mathcal{H}$ such that isotony holds:

$$\mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow\; \mathcal{F}(\mathcal{O}_1) \subset \mathcal{F}(\mathcal{O}_2). \tag{1.2}$$

The quasilocal algebra $\mathcal{F}$ defined by the union

$$\mathcal{F} = \overline{\bigcup_{\mathcal{O} \in \mathcal{K}} \mathcal{F}(\mathcal{O})}^{\|\cdot\|} \tag{1.3}$$

over the set $\mathcal{K}$ of all double cones (diamonds) is assumed to be irreducible, i.e. $\mathcal{F}' = \mathbb{C}\mathbf{1}$.[1]

The net is supposed to fulfill Bose-Fermi commutation relations, i.e. any local operator decomposes into a bosonic and a fermionic part $F = F_+ + F_-$ such that for spacelike separated F and G we have

$$[F_+, G_+] = [F_+, G_-] = [F_-, G_+] = \{F_-, G_-\} = 0. \tag{1.4}$$

The above decomposition is achieved by

$$F_\pm = \frac{1}{2}(F \pm \alpha_-(F)), \tag{1.5}$$

[1] In general $\mathcal{M}' = \{X \in \mathcal{B}(\mathcal{H}) | XY = YX \, \forall Y \in \mathcal{M}\}$ denotes the algebra of all bounded operators commuting with all operators in $\mathcal{M}$.

where $\alpha_-(F) = VFV$ and $V = V^* = V^{-1}$ is the unitary operator which acts trivially on the space of bosonic vectors and like $-\mathbf{1}$ on the fermionic ones. To formulate this locality requirement in a way more convenient for later purposes we introduce the twist operation $F^t = ZFZ^*$, where

$$Z = \frac{1 + iV}{1 + i}, \quad (\Rightarrow Z^2 = V), \tag{1.6}$$

which leads to $ZF_+Z^* = F_+,\ ZF_-Z^* = iVF_-$ implying $[F, G^t] = 0$. The (twisted) locality postulate (1.4) can now be stated simply as

$$\mathcal{F}(\mathcal{O})^t \subset \mathcal{F}(\mathcal{O}')'. \tag{1.7}$$

Poincaré covariance is implemented by assuming the existence of a (strongly continuous) unitary representation on $\mathcal{H}$ of the Poincaré group $\mathcal{P}$ such that

$$\alpha_{(\Lambda,a)}(\mathcal{F}(\mathcal{O})) = Ad\, U(\Lambda, a)(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\Lambda\mathcal{O} + a). \tag{1.8}$$

The spectrum of the generators of the translations (momenta) is required to be contained in the closed forward lightcone and the existence of a unique vacuum vector Ω invariant under $\mathcal{P}$ is assumed. Covariance under the conformal group, however, is *not* required.

Our last postulate (for the moment) concerns the inner symmetries of the theory. There shall be a compact group G, represented in a strongly continuous fashion by unitary operators on $\mathcal{H}$ leaving invariant the vacuum such that the automorphisms $\alpha_g(F) = Ad\, U(g)(F)$ of $\mathcal{B}(\mathcal{H})$ respect the local structure:

$$\alpha_g(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\mathcal{O}). \tag{1.9}$$

The action shall be faithful, i.e. $\alpha_g \not\equiv \mathrm{id}\, \forall g \in G$. This is no real restriction, for the kernel of the homomorphism $G \to \mathrm{Aut}(\mathcal{F})$ can be divided out. (Compactness of G need not be postulated, as it follows [27, Thm. 3.1] from the split property which will be introduced later.) In particular, there is an element $k \in Z(G)$ of order 2 in the center of the group G such that $V = U(k)$. This implies that the observables which are now defined as the fixpoints under the action of G

$$\mathcal{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^G = \mathcal{F}(\mathcal{O}) \cap U(G)' \tag{1.10}$$

fulfill locality in the conventional untwisted sense. In 1+1 dimensions the representations of the Poincaré group and of the inner symmetries do not necessarily commute. In the appendix of [57] it is, however, proved that in theories satisfying the distal split property the translations commute with the inner symmetries whereas the boosts act by automorphisms on the group $G_{\max}$ of all inner symmetries. As we will postulate a stronger version of the split property in the next section the cited result applies to the situation at hand. What we still have to assume is that the one parameter group of Lorentz boosts maps the group G of inner symmetries, which in general will be a subgroup of $G_{\max}$, into itself and commutes with $V = U(k)$. This assumption is indispensable for the covariance of the fixpoint net $\mathcal{A}$ as well as of another net to be constructed later.

This framework was the starting point for the investigations in [21] where in particular properties of the observable net (1.10) and its representations on the sectors in $\mathcal{H}$, i.e. the G-invariant subspaces, were studied, implicitly assuming the spacetime to be of dimension $\geq 2 + 1$. While it is impossible to do any justice to the deep analysis which derives from this early work (e.g. [22–25, 61, 28] and the books [46, 49]) we have to sketch some of the main ideas in order to prepare the ground for our own work in the subsequent sections. One important notion examined in [21] was that of *duality* designating a certain maximality property in the sense that the local algebras cannot be enlarged (on the same Hilbert space) without violating spacelike commutativity. The postulate of twisted duality for the fields consists in strengthening the twisted locality (1.7) to

$$\mathcal{F}(\mathcal{O})^t = \mathcal{F}(\mathcal{O}')', \tag{1.11}$$

which means that $\mathcal{F}(\mathcal{O}')$, the von Neumann algebra generated by all $\mathcal{F}(\mathcal{O}_1)$, $\mathcal{O}' \supset \mathcal{O}_1 \in \mathcal{K}$ contains all operators commuting with $\mathcal{F}(\mathcal{O})$ after twisting. From this it has been derived [21, Thm. 4.1] that duality holds for the observables when restricted to a simple sector:

$$(\mathcal{A}(\mathcal{O}) \restriction \mathcal{H}_1)' = \mathcal{A}(\mathcal{O}') \restriction \mathcal{H}_1. \tag{1.12}$$

A sector $\mathcal{H}_1$ is called simple if the group G acts on it via multiplication with a character:

$$U(g) \restriction \mathcal{H}_1 = \chi(g) \cdot \mathbf{1} \restriction \mathcal{H}_1. \tag{1.13}$$

Clearly the vacuum sector is simple. Furthermore it has been shown [21, Thm. 6.1] that the irreducible representations of the observables on the charge sectors in $\mathcal{H}$ are strongly locally equivalent in the sense that for any representation $\pi(A) = A \restriction \mathcal{H}_\pi$ and any $\mathcal{O} \in \mathcal{K}$ there is a unitary operator $X_\mathcal{O} : \mathcal{H}_0 \to \mathcal{H}_\pi$ such that

$$X_\mathcal{O}\, \pi_0(A) = \pi(A)\, X_\mathcal{O} \ \ \forall A \in \mathcal{A}(\mathcal{O}'). \tag{1.14}$$

The fundamental facts (1.12) and (1.14), which have come to be called Haag duality and the DHR criterion, respectively, were taken as starting points in [23, 24] where a more ambitious approach to the theory of superselection sectors was advocated and developed to a large extent. The basic idea was that the physical content of any quantum field theory should reside in the observables and their vacuum representation. All other physically relevant representations as well as unobservable charged fields interpolating between those and the vacuum sector should be constructed from the observable data. The vacuum representation and the other representations of interest were postulated to satisfy

$$\pi_0(\mathcal{A}(\mathcal{O})) \ = \ \pi_0(\mathcal{A}(\mathcal{O}'))', \tag{1.15}$$

$$\pi \restriction \mathcal{A}(\mathcal{O}') \cong \pi_0 \restriction \mathcal{A}(\mathcal{O}') \ \ \forall \mathcal{O} \in \mathcal{K}, \tag{1.16}$$

respectively. It may be considered as one of the triumphs of the algebraic approach that it has finally been possible to prove [28, and references given there] the existence of an essentially unique net of field algebras with a unique compact group G of inner symmetries such that there is an isomorphism between the monoidal (strict, symmetric) category of the superselection sectors satisfying (1.16) with the product structure established in [23] and the category of finite dimensional representations of G. Before turning now to the two dimensional situation we should remark that the duality property (1.11) upon which the whole theory hinges has been proved to hold for free massive and massless fields (scalar [3] and Dirac [21]) in $\geq 1 + 1$ dimensions (apart from the massless scalar field in two dimensions) as well as for several interacting theories ($P(\phi)_2, Y_2$). Furthermore, there is a remarkable link [61] between Haag duality and spontaneous symmetry breakdown. For the rest of this paragraph we assume that only a subgroup G_0 of G is unbroken, i.e. unitarily implemented on $\mathcal{H}$. Then the net $\mathcal{B}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^{G_0}$ satisfies Haag duality in restriction to $\mathcal{H}_0 = \mathcal{H}^{G_0}$ whereas $\mathcal{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^G$, being a true subnet of $\mathcal{B}$, does not. Yet, defining the *dual net* (on $\mathcal{H}_0$) by

$$\mathcal{A}^d(\mathcal{O}) = \mathcal{A}(\mathcal{O}')', \tag{1.17}$$

the fixpoint net $\mathcal{A}$ still satisfies *essential duality*:

$$\mathcal{A}^d(\mathcal{O}) = \mathcal{A}^{dd}(\mathcal{O}). \tag{1.18}$$

(Haag duality, by contrast, is simply $\mathcal{A}(\mathcal{O}) = \mathcal{A}^d(\mathcal{O})$.) Furthermore, one finds $\mathcal{A}^d(\mathcal{O}) = \mathcal{B}(\mathcal{O})$. These matters have been developed further in [28, 15] in the context of reconstruction of the fields from the observables.

In 1 + 1 dimensions a large part of the analysis sketched above breaks down due to the following topological peculiarities of 1 + 1 dimensional Minkowski space. Firstly, there is a Poincaré invariant distinction between left and right, i.e. for a spacelike vector x the sign of x^1 is invariant under the unit component of $\mathcal{P}$. This fact accounts for the existence of *soliton sectors* which have been studied rigorously in the frameworks of constructive and general quantum field theory, see [39] and [37, 38, 63], respectively. We intend to make use of the latter in a sequel to this work.

In the present paper we focus on the other well known feature of the topology of $1+1$ dimensional Minkowski space, viz. the fact that the spacelike complement of a bounded connected region (in particular, a double cone) consists of two connected components. The implications of this fact are twofold. On one hand, in the adaption of the DHR analysis [23, 24] based on (1.15, 1.16) to 1 + 1 dimensions [35, 58, 36] the permutation group $\mathcal{S}_\infty$ governing the statistics is replaced by the *braid group* $\mathcal{B}_\infty$, as anticipated, e.g., in [40]. It is still not known by which structure the compact group appearing in the higher dimensional situation has to be replaced if a completely general solution to this question exists at all.

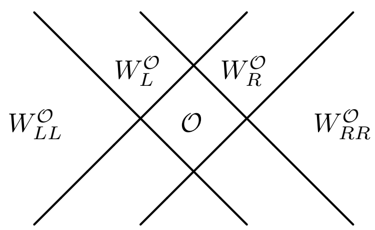
Besides the appearance of braid group statistics the disconnectedness of $\mathcal{O}'$ manifests itself also if one starts from a field net $\mathcal{F}$ with unbroken symmetry group G. It was mentioned above that in $\geq 2 + 1$ dimensions the restriction of the fixpoint net $\mathcal{A}$ to the simple sectors in $\mathcal{H}$ satisfies Haag duality provided the field net $\mathcal{F}$ fulfills (twisted) Haag duality. Since questions of Haag duality have been studied only in the framework of the algebraic approach the third peculiarity of quantum field theories in 1 + 1 dimensions (besides solitons and braid group statistics/quantum symmetry) is less widely known. We refer to the fact that the step from (1.11) to (1.15) may fail in 1+1 dimensions. This means that one cannot conclude from twisted duality of the fields that duality holds for the observables in simple sectors, which in fact is possible only in conformal theories. The origin of this phenomenon is easily understood. Let $\mathcal{O} \in \mathcal{K}$ be a double cone. One can then construct gauge invariant operators in $\mathcal{F}(\mathcal{O}')$ which are obviously contained in $\mathcal{A}(\mathcal{O})'$ but not in $\mathcal{A}(\mathcal{O}')$. This is seen remarking that the latter algebra, belonging to a disconnected region, is defined to be generated by the observable algebras associated with the left and right spacelike complements of $\mathcal{O}$, respectively. This algebra does not contain gauge invariant operators constructed using fields localized in both components.

We now come to the plan of this paper. Our aim will be to explore the relation between a quantum field theory with symmetry group G in 1+1 dimensions and the fixpoint theory. In addition to the general properties of such a theory stated above, twisted duality (1.11) is assumed to hold for the large theory. As explained above, in this situation duality of the fixpoint theory fails even in the case of unbroken group symmetry. Yet there is a local extension which satisfies Haag duality and one would like to obtain a complete understanding of this dual net. To this end we will need one additional postulate concerning the causal independence of one-sided infinite regions (wedges) which are separated from each other by a finite spacelike distance. This property rules out conformal theories and singles out a (presumably large) class of well-behaved massive theories. In Sect. 2 we prove the existence of unitary disorder operators which implement a global symmetry transformation on one wedge and act trivially on the spacelike complement of a slightly larger wedge. Using these operators we will in Sect. 3 consider a non-local extension $\hat{\mathcal{F}}(\mathcal{O})$ of the field net $\mathcal{F}(\mathcal{O})$. The fixpoint net $\hat{\mathcal{A}}(\mathcal{O})$ of the enlarged net $\hat{\mathcal{F}}(\mathcal{O})$ under the action of G is shown to coincide with the dual net $\mathcal{A}^d(\mathcal{O})$ (1.17) in restriction to the simple sectors. In conjunction with several technical results on actions of G this leads to an explicit characterization of the dual net. In Sect. 4 we will show that there is an action of the quantum double $D(G)$ on the extended net $\hat{\mathcal{F}}$ and that the spacelike commutation relations are governed by Drinfel'd's R-matrix. Since massive free scalar fields satisfy all assumptions we made on $\mathcal{F}$ this construction provides the first mathematically rigorous construction of quantum field theories with $D(G)$-symmetry for any finite group G. The quantum double may be considered a "hidden symmetry" of the original theory since it is uncovered only upon extending the latter. The $D(G)$-symmetry is spontaneously broken in that only the action of the subalgebra $\mathbb{C}G \subset D(G)$ is implemented in the Hilbert space $\mathcal{H}$. In analogy to Roberts' analysis this might be interpreted as the actual reason for the failure of Haag duality for the fixpoint net $\mathcal{A}$. The aim of the final Sect. 5 is to show that the methods introduced in the preceding sections are well suited for a discussion of Jordan-Wigner transformations and bosonization in the framework of algebraic quantum field theory.

Three appendices are devoted in turn to a summary of the needed facts on quantum groups and quantum doubles, a partial generalization of our results to infinite compact groups and an indication how an analysis similar to Sects. 2 to 4 can be done for chiral conformal theories on the circle.

2. Disorder Variables and the Split Property

2.1. Preliminaries. For any double cone $\mathcal{O} \in \mathcal{K}$ we designate the left and right spacelike complement by $W_{LL}^{\mathcal{O}}$ and $W_{RR}^{\mathcal{O}}$, respectively. Furthermore we write $W_L^{\mathcal{O}}$ and $W_R^{\mathcal{O}}$ for $W_{RR}^{\mathcal{O}}{}'$ and $W_{LL}^{\mathcal{O}}{}'$. These regions are wedge shaped, i.e. translates of the standard wedges $W_L = \{x \in \mathbb{R}^2 \mid x^1 < -|x^0|\}$ and $W_R = \{x \in \mathbb{R}^2 \mid x^1 > |x^0|\}$. We will not distinguish between open and closed regions, for definiteness one may consider $\mathcal{O}$ and all W-regions as open. With these definitions we have $\mathcal{O} = W_L^{\mathcal{O}} \cap W_R^{\mathcal{O}}$ and $\mathcal{O}' = W_{LL}^{\mathcal{O}} \cup W_{RR}^{\mathcal{O}}$ which graphically looks as follows:



(2.1)

Whereas, as we have shown in the introduction, Haag duality for double cones is violated in the fixpoint theory $\mathcal{A}$, one obtains the following weaker form of duality.

Proposition 2.1. *The representation of the fixpoint net $\mathcal{A}$ fulfills duality for wedges*

$$\mathcal{A}(W)' = \mathcal{A}(W') \tag{2.2}$$

and essential duality (1.18) in all simple sectors.

Proof. The spacelike complement of a wedge region is itself a wedge, thus connected, whereby the proof of [21, Thm. 4.1] applies, yielding the first statement. The second follows from wedge duality via

$$\mathcal{A}^d(\mathcal{O}) = \mathcal{A}(\mathcal{O}')' = (\mathcal{A}(W^{\mathcal{O}}_{LL}) \vee \mathcal{A}(W^{\mathcal{O}}_{RR}))' = \mathcal{A}(W^{\mathcal{O}}_R) \wedge \mathcal{A}(W^{\mathcal{O}}_L), \tag{2.3}$$

as locality of the dual net is equivalent to essential duality of $\mathcal{A}$. □

We will now introduce the central notion for this paper.

Definition 2.2. *A family of disorder operators consists, for any $\mathcal{O} \in \mathcal{K}$ and any $g \in G$, of two unitary operators $U^{\mathcal{O}}_L(g)$ and $U^{\mathcal{O}}_R(g)$ verifying*

$$\begin{aligned} Ad\, U^{\mathcal{O}}_L(g) \restriction \mathcal{F}(W^{\mathcal{O}}_{LL}) &= Ad\, U^{\mathcal{O}}_R(g) \restriction \mathcal{F}(W^{\mathcal{O}}_{RR}) = \alpha_g, \\ Ad\, U^{\mathcal{O}}_L(g) \restriction \mathcal{F}(W^{\mathcal{O}}_{RR}) &= Ad\, U^{\mathcal{O}}_R(g) \restriction \mathcal{F}(W^{\mathcal{O}}_{LL}) = \ id. \end{aligned} \tag{2.4}$$

In words: the adjoint action of $U^{\mathcal{O}}_{L/R}(g)$ on fields located in the left and right spacelike complements of $\mathcal{O}$, respectively, equals the global group action on one side and is trivial on the other. As a consequence of (twisted) wedge duality we have at once

$$U^{\mathcal{O}}_L(g) \in \mathcal{F}(W^{\mathcal{O}}_L)^t,\ U^{\mathcal{O}}_R(g) \in \mathcal{F}(W^{\mathcal{O}}_R)^t. \tag{2.5}$$

On the other hand it is clear that disorder operators cannot be contained in the local algebras $\mathcal{F}(\mathcal{O}), \mathcal{F}(\mathcal{O})^t$ nor in the quasilocal algebra $\mathcal{F}$, for in this case locality would not allow their adjoint action to be as stated on operators located arbitrarily far to the left or right. Heuristically, assuming $U(g)$ arises from a conserved local current via $U(g) = e^{i\int j^0(t=0,x)dx}$, one may think of $U^{\mathcal{O}}_L(g)$ as given by

$$U^{\mathcal{O}}_L(g) = e^{i\int_{-\infty}^{x_0} j^0(x)dx}, \tag{2.6}$$

where integration takes place over a spacelike curve from left spacelike infinity to a point x_0 in $\mathcal{O}$. The need for a finitely extended interpolation region $\mathcal{O}$ arises from the distributional character of the current which necessitates a smooth cutoff. We refrain from discussing these matters further as they play no role in the sequel. In massive free field theories disorder operators can be constructed rigorously (e.g. [43, 1]) using the CCR/CAR structure and the criteria due to Shale.

Lemma 2.3. *Let $U^{\mathcal{O}}_{L,1}(g), U^{\mathcal{O}}_{L,2}(g)$ be disorder operators associated with the same double cone and the same group element. Then $U^{\mathcal{O}}_{L,1}(g) = F\, U^{\mathcal{O}}_{L,2}(g)$ with $F \in \mathcal{F}(\mathcal{O})^t$ unitary. An analogous statement holds for the right-handed disorder operators.*

Proof. Consider $F = U^{\mathcal{O}}_{L,1}(g)\, U^{\mathcal{O}*}_{L,2}(g)$. By construction $F \in \mathcal{F}(W^{\mathcal{O}}_L)^t$. On the other hand $Ad\, F \restriction \mathcal{F}(W^{\mathcal{O}}_{LL}) = id$ holds as $U^{\mathcal{O}}_{L,1}(g)$ and $U^{\mathcal{O}}_{L,2}(g)$ implement the same automorphism of $\mathcal{F}(W^{\mathcal{O}}_{LL})$. By (twisted) wedge duality we have $F \in \mathcal{F}(W^{\mathcal{O}}_R)^t$ and (twisted) duality for double cones implies $F \in \mathcal{F}(\mathcal{O})^t$. □

Remarks. 1. This result shows that disorder operators are unique up to unitary elements of $\mathcal{F}(\mathcal{O})^t$, the twisted algebra of the interpolation region. The obvious fact that $U_L^{\mathcal{O}}(g)\, U_L^{\mathcal{O}}(h)$ and $U(g)\, U_L^{\mathcal{O}}(h)\, U(g)^*$ are disorder operators for the group elements gh and ghg^{-1}, respectively, implies that a family of disorder operators constitutes a projective representation of G with the cocycle taking values in $\mathcal{F}(\mathcal{O})^t$.
2. Later on we will consider only bosonic disorder operators, which leads to the stronger result $F \in \mathcal{F}(\mathcal{O})_+$.

For the purposes of the present investigation the mere existence of disorder operators is not enough, for we need them to obey certain further restrictions. Our first aim will be to obtain such operators by a construction which is model independent to the largest possible extent, making use only of properties valid in any reasonable model. To this effect we reconsider an idea due to Doplicher [26] and developed further in, e.g., [27, 14]. It consists of using the split property [12] to obtain, for any $g \in G$ and any pair of double cones $\Lambda = (\mathcal{O}_1, \mathcal{O}_2)$ such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$, an operator $U_\Lambda(g) \in \mathcal{F}(\mathcal{O}_2)$ such that

$$U_\Lambda(g) F U_\Lambda(g)^* = U(g) F U(g)^* \quad \forall F \in \mathcal{F}(\mathcal{O}_1). \tag{2.7}$$

In order to be able to do the same thing with wedges we introduce our last postulate.

Definition 2.4. *An inclusion $A \subset B$ of von Neumann algebras is split [27], if there exists a type-I factor N such that $A \subset N \subset B$. A net of field algebras satisfies the "split property for wedges" if the inclusions $\mathcal{F}(W_{LL}^{\mathcal{O}}) \subset \mathcal{F}(W_L^{\mathcal{O}})$ and $\mathcal{F}(W_{RR}^{\mathcal{O}}) \subset \mathcal{F}(W_R^{\mathcal{O}})$ are split for every double cone $\mathcal{O}$. (In our case, where wedge duality holds, the split property for one of the above inclusions entails the same for the other as is seen by passing to commutants and twisting.)*

This property is discussed at some length in [57] and shown to be fulfilled for the free massive scalar and Dirac fields. In quantum field theories where there are lots of cyclic and separating vectors for the local algebras by the Reeh-Schlieder theorem, the split property is equivalent [27] to the existence, for any double cone $\mathcal{O}$, of a unitary operator $Y^{\mathcal{O}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ implementing an isomorphism between $\mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}})^t$ and the tensor product $\mathcal{F}(W_{LL}^{\mathcal{O}}) \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})^t$ (in the sense of von Neumann algebras)

$$Y^{\mathcal{O}}\, F_1 F_2^t\, Y^{\mathcal{O}*} = F_1 \otimes F_2^t \quad \forall F_1 \in \mathcal{F}(W_{LL}^{\mathcal{O}}), F_2 \in \mathcal{F}(W_{RR}^{\mathcal{O}}). \tag{2.8}$$

That one of the algebras $\mathcal{F}(W_{LL}^{\mathcal{O}})$ and $\mathcal{F}(W_{RR}^{\mathcal{O}})$, which are associated with spacelike separated regions, has to be twisted in order for an isomorphism as above to exist is clear as in general these algebras do not commute while the factors of a tensor product do commute. Analogously, there is a spatial isomorphism between $\mathcal{F}(W_{LL}^{\mathcal{O}})^t \vee \mathcal{F}(W_{RR}^{\mathcal{O}})$ and $\mathcal{F}(W_{LL}^{\mathcal{O}})^t \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})$ implemented by $\tilde{Y}^{\mathcal{O}}$. We will stick to the use of $Y^{\mathcal{O}}$ throughout. In order not to obscure the basic simplicity of the argument we assume for a moment that the theory $\mathcal{F}$ is purely bosonic, i.e. fulfills locality and duality without twisting. Using the isomorphism implemented by $Y^{\mathcal{O}}$ we then have the following correspondences:

$$\begin{array}{rcccc} \mathcal{F}(W_{LL}^{\mathcal{O}}) & \cong & \mathcal{F}(W_{LL}^{\mathcal{O}}) & \otimes & \mathbf{1}, \\ \mathcal{F}(W_{RR}^{\mathcal{O}}) & \cong & \mathbf{1} & \otimes & \mathcal{F}(W_{RR}^{\mathcal{O}}), \\ \mathcal{F}(W_L^{\mathcal{O}}) & \cong & \mathcal{B}(\mathcal{H}) & \otimes & \mathcal{F}(W_L^{\mathcal{O}}), \\ \mathcal{F}(W_R^{\mathcal{O}}) & \cong & \mathcal{F}(W_R^{\mathcal{O}}) & \otimes & \mathcal{B}(\mathcal{H}), \end{array} \tag{2.9}$$

whereas Haag duality for double cones yields

$$\mathcal{F}(\mathcal{O}) = \mathcal{F}(W_L^{\mathcal{O}}) \wedge \mathcal{F}(W_R^{\mathcal{O}}) \cong \mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}}). \tag{2.10}$$

(Taking the intersection separately for both factors of the tensor product is valid in this situation as can easily be proved using the lattice property of von Neumann algebras $\mathcal{M} \wedge \mathcal{N} = (\mathcal{M}' \vee \mathcal{N}')'$ and the commutation theorem for tensor products $(\mathcal{M} \otimes \mathcal{N})' = \mathcal{M}' \otimes \mathcal{N}'$.) We thus see that in conjunction with the well known fact [29] that the algebras associated with wedge regions are factors of type III_1 the split property for wedges implies that the algebras of double cones are type III_1 factors, too.

The following property of the maps $Y^{\mathcal{O}}$ will be pivotal for the considerations below. Given any unitary U implementing a local symmetry (i.e. $U\mathcal{F}(\mathcal{O})U^* = \mathcal{F}(\mathcal{O})\ \forall \mathcal{O}$) and leaving invariant the vacuum ($U\Omega = \Omega$) the following identity holds:

$$Y^{\mathcal{O}}\, U = (U \otimes U)\, Y^{\mathcal{O}}. \tag{2.11}$$

For the construction of $Y^{\mathcal{O}}$ as well as for the proof of (2.11) we refer to [27, 14], the difference that those authors work with double cones being unimportant.

2.2. Construction of disorder operators. The operators $Y^{\mathcal{O}}$ will now be used to obtain disorder operators. To this purpose we give the following

Definition 2.5. *For any double cone $\mathcal{O} \in \mathcal{K}$ and any $g \in G$ we set*

$$\begin{aligned} U_L^{\mathcal{O}}(g) &= Y^{\mathcal{O}*}\, (U(g) \otimes \mathbf{1})\, Y^{\mathcal{O}}, \\ U_R^{\mathcal{O}}(g) &= Y^{\mathcal{O}*}\, (\mathbf{1} \otimes U(g))\, Y^{\mathcal{O}}. \end{aligned} \tag{2.12}$$

As an immediate consequence of this definition we have the following

Proposition 2.6. *The disorder operators defined above satisfy*

$$\left[U_L^{\mathcal{O}}(g), U_R^{\mathcal{O}}(h)\right] = 0, \tag{2.13}$$

$$U_L^{\mathcal{O}}(g)\, U_R^{\mathcal{O}}(g) = U(g), \tag{2.14}$$

$$U(g)\, U_{L/R}^{\mathcal{O}}(h)\, U(g)^* = U_{L/R}^{\mathcal{O}}(ghg^{-1}). \tag{2.15}$$

Proof. The first statement is trivial and the second follows from (2.11). The covariance property (2.15) is another consequence of (2.11). □

Remark. We have thus obtained some kind of factorization of the global action of the group G into two commuting *true* (i.e. no cocycles) representations of G such that the original action is recovered as the diagonal. Furthermore, these operators transform covariantly under global gauge transformations. In particular they are bosonic since $k \in Z(G)$.

It remains to be shown that the $U_{L/R}^{\mathcal{O}}$ indeed fulfill the requirements of Definition 2.2. The second requirement follows from Definition 2.5, which with (2.9) obviously yields

$$U_L^{\mathcal{O}}(g) \in \mathcal{F}(W_L^{\mathcal{O}}),\ U_R^{\mathcal{O}}(g) \in \mathcal{F}(W_R^{\mathcal{O}}). \tag{2.16}$$

The first one is seen by the following computation valid for $F \in \mathcal{F}(W_{LL}^{\mathcal{O}})$:

$$\begin{aligned} U_L^{\mathcal{O}}(g) F U_L^{\mathcal{O}*}(g) &\cong (U(g) \otimes \mathbf{1})(F \otimes \mathbf{1})(U(g) \otimes \mathbf{1})^* \\ &= (U(g) F U(g)^* \otimes \mathbf{1}) \cong U(g) F U(g)^*, \end{aligned} \tag{2.17}$$

appealing to the isomorphism $\cong$ implemented by $Y^{\mathcal{O}}$.

Returning now to the more general case including fermions we have to consider the apparent problem that there are now two ways to define the operators $U_L^{\mathcal{O}}(g)$ and $U_R^{\mathcal{O}}(g)$, depending upon whether we choose $Y^{\mathcal{O}}$ or $\tilde{Y}^{\mathcal{O}}$. (By contrast, the tensor product factorization (2.10) of the local algebras is of a purely technical nature, rendering it irrelevant whether we use $Y^{\mathcal{O}}$ or $\tilde{Y}^{\mathcal{O}}$.) This ambiguity is resolved by remarking that the element $k \in G$ giving rise to V by $V = U(k)$ is central, implying that the operators $U(g),\ g \in G$, are bosonic (even). For even operators $F_1 \in \mathcal{F}(W_{LL}^{\mathcal{O}}), F_2 \in \mathcal{F}(W_{RR}^{\mathcal{O}})$ we have $F_1 = F_1^t, F_2 = F_2^t$ and thus

$$Y^{\mathcal{O}}\, F_1 F_2\, Y^{\mathcal{O}*} = \tilde{Y}^{\mathcal{O}}\, F_1 F_2\, \tilde{Y}^{\mathcal{O}*} = F_1 \otimes F_2, \tag{2.18}$$

so that the disorder variables are uniquely defined even operators.

The first two equations of (2.9) are replaced by

$$\begin{array}{rcl} \mathcal{F}(W_{LL}^{\mathcal{O}}) & \cong & \mathcal{F}(W_{LL}^{\mathcal{O}}) \otimes \mathbf{1}, \\ \mathcal{F}(W_{RR}^{\mathcal{O}})^t & \cong & \mathbf{1} \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})^t. \end{array} \tag{2.19}$$

By taking commutants we obtain

$$\begin{array}{rcl} \mathcal{F}(W_{L}^{\mathcal{O}}) & \cong & \mathcal{B}(\mathcal{H}) \otimes \mathcal{F}(W_{L}^{\mathcal{O}}), \\ \mathcal{F}(W_{R}^{\mathcal{O}})^t & \cong & \mathcal{F}(W_{R}^{\mathcal{O}})^t \otimes \mathcal{B}(\mathcal{H}), \end{array} \tag{2.20}$$

and an application of the twist operation to the second equations of (2.19) and (2.20) yields

$$\begin{array}{rcl} \mathcal{F}(W_{RR}^{\mathcal{O}}) & \cong & \mathbf{1} \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})_+ + V \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})_-, \\ \mathcal{F}(W_{R}^{\mathcal{O}}) & \cong & \mathcal{F}(W_{R}^{\mathcal{O}}) \otimes \mathcal{B}(\mathcal{H})_+ + \mathcal{F}(W_{R}^{\mathcal{O}})\, V \otimes \mathcal{B}(\mathcal{H})_-. \end{array} \tag{2.21}$$

The identity $\mathcal{F}(\mathcal{O}) = \mathcal{F}(W_L^{\mathcal{O}}) \wedge \mathcal{F}(W_R^{\mathcal{O}})$, which is valid in the fermionic case, too, finally leads to

$$\mathcal{F}(\mathcal{O}) \cong \mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}})_+ + \mathcal{F}(W_R^{\mathcal{O}})\, V \otimes \mathcal{F}(W_L^{\mathcal{O}})_-. \tag{2.22}$$

While this is not as nice as (2.10) it is still sufficient for the considerations in the sequel. That $\mathcal{F}(\mathcal{O}),\ \mathcal{O} \in \mathcal{K}$ is a factor is, however, less obvious than in the pure Bose case and will be proved only in Subsect. 3.3.

The following easy result will be of considerable importance later on.

Lemma 2.7. *The disorder operators $U_L^{\mathcal{O}}(g)$ and $U_R^{\mathcal{O}}(g)$ associated with the double cone $\mathcal{O}$ implement automorphisms of the local algebra $\mathcal{F}(\mathcal{O})$.*

Proof. In the pure Bose case this is obvious from Definition 2.5, (2.10) and the fact that $Ad\, U(g)$ acts as an automorphism on all wedge algebras. In the Bose-Fermi case (2.22) the same is true since $U(g)$ commutes with $V = U(k)$. □

Definition 2.8. $\alpha_g^{\mathcal{O}} = Ad\, U_L^{\mathcal{O}}(g),\ \ g \in G, \mathcal{O} \in \mathcal{K}$.

We close this section with one remark. We have seen that the split property for wedges implies the existence of disorder operators which constitute true representations of the symmetry group and which transform covariantly under the global symmetry. Conversely, one can show that the existence of disorder operators, possibly with group cocycle, in conjunction with the split property for wedges for the fixpoint net $\mathcal{A}$ implies the split property for the field net $\mathcal{F}$. This in turn allows to remove the cocyle using the above construction. We refrain from giving the argument which is similar to those in [26, pp. 79, 85].

3. Field Extensions and Haag Duality

3.1. The extended field net. Having defined the disorder variables we now take the next step, which at first sight may seem unmotivated. Its relevance will become clear in the sequel. We define a new net of algebras $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$ by adding the disorder variables associated with the double cone $\mathcal{O}$ to the fields localized in this region.

Definition 3.1.

$$\hat{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)''. \tag{3.1}$$

Remarks. 1. In accordance with the common terminology in statistical mechanics and conformal field theory, the operators which are composed of fields (order variables) and disorder variables might be called *parafermion operators*.
2. We could as well have chosen the disorder operators acting on the right-hand side. As there is a complete symmetry between left and right there would be no fundamental difference. We will therefore stick to the above choice throughout this paper. Including both the left and right-handed disorder operators would, however, have the unpleasant consequence that there would be translation invariant operators (namely the $U(g)$'s) in the local algebras.
3. The local algebra $\hat{\mathcal{F}}(\mathcal{O})$ of the above definition resembles the crossed product of $\mathcal{F}(\mathcal{O})$ by the automorphism group $\alpha_g^{\mathcal{O}}$, the interesting aspect being that the automorphism group depends on the region $\mathcal{O}$. These two constructions differ, however, with respect to the Hilbert space on which they are defined. Whereas the crossed product $\mathcal{F}(\mathcal{O}) \rtimes_{\alpha^{\mathcal{O}}} G$ lives on the Hilbert space $L^2(G, \mathcal{H})$, our algebras $\hat{\mathcal{F}}(\mathcal{O})$ are defined on the original space $\mathcal{H}$. For later purposes it will be necessary to know whether these algebras are isomorphic, but we prefer first to discuss those aspects which are independent of this question.

The first thing to check is, of course, that Definition 3.1 specifies a net of von Neumann algebras.

Proposition 3.2. *The assignment $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$ satisfies isotony.*

Proof. Let $\mathcal{O} \subset \hat{\mathcal{O}}$ be an inclusion of double cones. Obviously we have $\mathcal{F}(\mathcal{O}) \subset \hat{\mathcal{F}}(\hat{\mathcal{O}})$. In order to prove $U_L^{\mathcal{O}}(g) \in \hat{\mathcal{F}}(\hat{\mathcal{O}})$ we observe that $U_L^{\mathcal{O}}(g)$ is a disorder operator for the larger region $\hat{\mathcal{O}}$, too. Thus, by Lemma 2.3 we have $U_L^{\mathcal{O}}(g) = F\, U_L^{\hat{\mathcal{O}}}(g)$ with $F \in \mathcal{F}(\hat{\mathcal{O}})$. Now it is clear that $U_L^{\mathcal{O}}(g) \in \hat{\mathcal{F}}(\hat{\mathcal{O}})$. □

Remark. From this we can conclude that the net $\hat{\mathcal{F}}(\mathcal{O})$ is uniquely defined in the sense that any family of bosonic disorder operators gives rise to the same net $\hat{\mathcal{F}}(\mathcal{O})$ provided such operators exist at all. For most of the arguments in this paper we will, however, need the detailed properties proved above which follow from the construction via the split property.

It is obvious that the net $\hat{\mathcal{F}}$ is nonlocal. While the spacelike commutation relations of fields and disorder operators are known by construction we will have more to say on this subject later. On the other hand it should be clear that the nets $\hat{\mathcal{F}}$ and $\mathcal{A}$ are local relative to each other. This is simply the fact that the disorder operators commute with the fixpoints of α_g in both spacelike complements.

Proposition 3.3. *The net $\hat{\mathcal{F}}$ is Poincaré covariant with the original representation of $\mathcal{P}$. In particular $\alpha_a(U_L^{\mathcal{O}}(g)) = U_L^{\mathcal{O}+a}(g)$ whereas for the boosts we have*

$$\alpha_\Lambda(U_L^{\mathcal{O}}(g)) = U_L^{\Lambda\mathcal{O}}(h), \tag{3.2}$$

if $U(\Lambda)\,U(g)\,U(\Lambda)^* = U(h)$.

Proof. The family $Y^{\mathcal{O}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ of unitaries provided by the split property fulfills the identity

$$Y^{\Lambda\mathcal{O}+a} = (U(\Lambda, a) \otimes U(\Lambda, a))\, Y^{\mathcal{O}}\, U(\Lambda, a)^*, \tag{3.3}$$

as is easily seen to follow from the construction in [27, 14]. This implies

$$\begin{aligned} \alpha_{\Lambda,a}(U_L^{\mathcal{O}}(g)) &= U(\Lambda, a)\, Y^{\mathcal{O}^*}\, (U(g) \otimes \mathbf{1})\, Y^{\mathcal{O}}\, U(\Lambda, a)^* \\ &= Y^{\Lambda\mathcal{O}+a^*}\, (U(\Lambda, a)\, U(g)\, U(\Lambda, a)^* \otimes \mathbf{1})\, Y^{\Lambda\mathcal{O}+a} \\ &= U_L^{\Lambda\mathcal{O}+a}(h), \end{aligned} \tag{3.4}$$

where $U(\Lambda)\, U(g)\, U(\Lambda)^* = U(h)$. □

Proposition 3.4. *The vacuum vector Ω is cyclic and separating for $\hat{\mathcal{F}}(\mathcal{O})$.*

Proof. Follows from

$$\mathcal{F}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O}) \subset \mathcal{F}(W_L^{\mathcal{O}}) \tag{3.5}$$

since Ω is cyclic and separating for $\mathcal{F}(\mathcal{O})$ and $\mathcal{F}(W_L^{\mathcal{O}})$. □

Proposition 3.5. *The wedge algebras for the net $\hat{\mathcal{F}}$ take the form*

$$\hat{\mathcal{F}}(W_L^{\mathcal{O}}) = \mathcal{F}(W_L^{\mathcal{O}}), \;\; \hat{\mathcal{F}}(W_R^{\mathcal{O}}) = \mathcal{F}(W_R^{\mathcal{O}}) \vee U(G)'' = \mathcal{A}(W_{LL}^{\mathcal{O}})'. \tag{3.6}$$

As a consequence Ω is not separating for $\hat{\mathcal{F}}(W_R^{\mathcal{O}})$!

Proof. The first identity is obvious, while the second follows from $\mathcal{F}(W_R^{\mathcal{O}}) \ni U_R^{\hat{\mathcal{O}}}(g)\, \forall \hat{\mathcal{O}} \in W_R^{\mathcal{O}}$ and the factorization property (2.14). The last statement is equivalent to Ω not being cyclic for $\mathcal{A}(W_{LL}^{\mathcal{O}})$. □

Proposition 3.6. *Let $\hat{F} \in \mathcal{F}(\mathcal{O})U_L^{\mathcal{O}}(g)$. Then the following cluster properties hold.*

$$w - \lim_{x \to -\infty} \alpha_x(\hat{F}) = \langle \Omega, \hat{F}\Omega \rangle \cdot \mathbf{1}, \tag{3.7}$$

$$w - \lim_{x \to +\infty} \alpha_x(\hat{F}) = \langle \Omega, \hat{F}\Omega \rangle \cdot U(g). \tag{3.8}$$

Proof. The first identity follows from $\hat{F} \in \hat{\mathcal{F}}(W_L^{\mathcal{O}})$ and the usual cluster property. The second is seen by writing $\hat{F} = F\, U_R^{\mathcal{O}}(g^{-1})\, U(g)$ and applying the weak convergence of $U_R^{\mathcal{O}}$ as above, the translation invariance of $U(g)$ and the invariance of the vacuum under $U(g)$. □

3.2. Haag duality. Observing by (2.15) that the adjoint action of the global symmetry group leaves the 'localization' (in the sense of Definition 2.2) of the disorder operators invariant it is clear that the automorphisms $\alpha_g = Ad\, U(g)$ extend to local symmetries of the enlarged net $\hat{\mathcal{F}}$. We are thus in a position to define yet another net, the fixpoint net of $\hat{\mathcal{F}}$.

Definition 3.7.

$$\hat{\mathcal{A}}(\mathcal{O}) = \hat{\mathcal{F}}(\mathcal{O}) \wedge U(G)'. \tag{3.9}$$

Remark. We then have the following square of local inclusions:

$$\begin{array}{ccc} \hat{\mathcal{A}}(\mathcal{O}) & \subset & \hat{\mathcal{F}}(\mathcal{O}) \\ \cup & & \cup \\ \mathcal{A}(\mathcal{O}) & \subset & \mathcal{F}(\mathcal{O}). \end{array} \tag{3.10}$$

The conditional expectation $m(\cdot) = \int dg\, \alpha_g(\cdot)$ from $\hat{\mathcal{F}}(\mathcal{O})$ to $\hat{\mathcal{A}}(\mathcal{O})$ clearly restricts to a conditional expectation from $\mathcal{F}(\mathcal{O})$ to $\mathcal{A}(\mathcal{O})$. In Sect. 4 we will see that there is also a conditional expectation γ_e from $\hat{\mathcal{F}}(\mathcal{O})$ to $\mathcal{F}(\mathcal{O})$ which restricts to a conditional expectation from $\hat{\mathcal{A}}(\mathcal{O})$ to $\mathcal{A}(\mathcal{O})$, provided the group G is finite. Since γ_e commutes with m the square (3.10) then constitutes a commuting square in the sense of Popa.

Proposition 3.8. *The net $\mathcal{O} \mapsto \hat{\mathcal{A}}(\mathcal{O})$ is local.*

Proof. Let $\mathcal{O} < \tilde{\mathcal{O}}$ be two regions spacelike to each other, $\tilde{\mathcal{O}}$ being located to the right of $\mathcal{O}$. From $\hat{\mathcal{A}}(\mathcal{O}) \subset \mathcal{A}(W_L^{\mathcal{O}})$ and the relative locality of observables and fields we conclude that $\hat{\mathcal{A}}(\mathcal{O})$ commutes with $\mathcal{F}(\tilde{\mathcal{O}})$. On the other hand the operators $U_L^{\tilde{\mathcal{O}}}(g)$ commute with $\hat{\mathcal{A}}(\mathcal{O}) \subset \hat{\mathcal{F}}(W_L^{\mathcal{O}}) = \mathcal{F}(W_L^{\mathcal{O}})$ since $Ad\, U_L^{\tilde{\mathcal{O}}}(g) \restriction \mathcal{F}(W_L^{\mathcal{O}}) = \alpha_g$ and $\hat{\mathcal{A}}(\mathcal{O})$ is pointwise gauge invariant. □

We have just proved that the net $\hat{\mathcal{A}}$ constitutes a local extension of the observable net $\mathcal{A}$, thereby confirming our initial observation that $\mathcal{A}$ does not satisfy Haag duality. The elements of $\hat{\mathcal{A}}$ being gauge invariant they commute a fortiori with the central projections in the group algebra, thereby leaving invariant the sectors in $\mathcal{H}$. We will now prove a nice result which serves as our first justification for Definitions 3.1 and 3.7.

Lemma 3.9.

$$\mathcal{A}(\mathcal{O}')' = \mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'' \vee U_R^{\mathcal{O}}(G)''. \tag{3.11}$$

Proof. We already know that

$$\mathcal{A}(\mathcal{O}')' \supset \mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'' \vee U_R^{\mathcal{O}}(G)''. \tag{3.12}$$

In order to prove equality we consider the following string of identities, making use of the spatial isomorphisms due to the split property and omitting the superscript $\mathcal{O}$ on the wedge regions.

$$\begin{aligned} \mathcal{A}(\mathcal{O}')' &= (\mathcal{A}(W_{LL}) \vee \mathcal{A}(W_{RR}))' \\ &= ((\mathcal{F}(W_{LL}) \wedge U(G)') \vee (\mathcal{F}(W_{RR}) \wedge U(G)'))' \\ &\cong \big((\mathcal{F}(W_{LL}) \wedge U(G)') \otimes (\mathcal{F}(W_{RR}) \wedge U(G)')\big)' \\ &= (\mathcal{F}(W_R)^t \vee U(G)'') \otimes (\mathcal{F}(W_L)^t \vee U(G)''), \\ &= (\mathcal{F}(W_R) \otimes \mathcal{F}(W_L)) \vee (U(G)'' \otimes \mathbf{1}) \vee (\mathbf{1} \otimes U(G)'') \\ &\cong \mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'' \vee U_L^{\mathcal{O}}(G)''. \end{aligned} \tag{3.13}$$

In the third step we have used the identities $\mathcal{F}(W_{LL}) \wedge U(G)' \cong \mathcal{F}(W_{LL}) \wedge U(G)' \otimes \mathbf{1}$ and $\mathcal{F}(W_{RR}) \wedge U(G)' \cong \mathbf{1} \otimes \mathcal{F}(W_{RR}) \wedge U(G)'$ which are easily seen to follow from (2.20) and (2.21), respectively. The fourth step is justified by $\mathcal{F}(\cdot)^t \vee U(G)'' = \mathcal{F}(\cdot) \vee U(G)''$. □

Theorem 3.10. *In restriction to a simple sector $\mathcal{H}_1$ the net $\hat{\mathcal{A}}$ satisfies Haag duality, i.e. it coincides with the dual net $\mathcal{A}^d$ in this representation.*

Proof. Let P_1 be the projection on a simple sector, i.e. fulfilling

$$U(g)P_1 = P_1 U(g) = \chi(g) \cdot P_1, \tag{3.14}$$

where χ is a character of G. Making use of $U_L^{\mathcal{O}}(G)'' \vee U_R^{\mathcal{O}}(G)'' = U_L^{\mathcal{O}}(G)'' \vee U(G)''$ and (3.14) we have

$$\begin{aligned} &P_1\,(\mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'' \vee U_R^{\mathcal{O}}(G)'')\,P_1 = \\ &\quad P_1\,(\mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'' \vee U(G)'')\,P_1 = P_1\,\hat{\mathcal{F}}(\mathcal{O})\,P_1. \end{aligned} \tag{3.15}$$

With $m(F) = \int dg\, U(g) F U(g)^*$ and once again using (3.14) we obtain

$$P_1\,\hat{\mathcal{F}}(\mathcal{O})\,P_1 = P_1\,m(\hat{\mathcal{F}}(\mathcal{O}))\,P_1 = P_1\,\hat{\mathcal{A}}(\mathcal{O})\,P_1. \tag{3.16}$$

On the other hand

$$P_1\,\mathcal{A}(\mathcal{O}')'\,P_1 \restriction P_1\mathcal{H} = \left(P_1\,\mathcal{A}(\mathcal{O}')\,P_1 \restriction P_1\mathcal{H}\right)'. \tag{3.17}$$

The proof is now completed by applying the preceding lemma. □

Remark. The above arguments make it clear that Haag duality cannot hold for the net $\mathcal{A}(\mathcal{O})$ even in simple sectors. This is not necessarily so if the split property for wedges does not hold. In conformally invariant theories gauge invariant combinations of field operators in the left and the right spacelike complements of a double cone $\mathcal{O}$ may well be contained in $\mathcal{A}(\mathcal{O}')$ due to spacetime compactification. One would think, however, that this is impossible in massive theories, even those without the split property.

3.3. Outerness properties and computation of $\hat{\mathcal{A}}(\mathcal{O})$. While the above theorem allows us in principle to construct the dual net $\hat{\mathcal{A}}$ one would like to know more explicitly how the elements of $\hat{\mathcal{A}}$ look in terms of the fields in $\mathcal{F}$ and the disorder operators. In the case of an abelian group G this is easy to see. As a consequence of the covariance property (2.15) we then have

$$U(g)\,U_{L/R}^{\mathcal{O}}(h)\,U(g)^* = U_{L/R}^{\mathcal{O}}(ghg^{-1}) = U_{L/R}^{\mathcal{O}}(h), \tag{3.18}$$

that is the disorder operators are gauge invariant and thus contained in $\hat{\mathcal{A}}(\mathcal{O})$. It is then obvious that

$$\hat{\mathcal{A}}(\mathcal{O}) = \mathcal{A}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'', \quad (G \text{ abelian!}) \tag{3.19}$$

as $\hat{\mathcal{A}}(\mathcal{O})$ is spanned by operators of the form $FU_L^{\mathcal{O}}(g),\ F \in \mathcal{F}(\mathcal{O})$ which are invariant iff $F \in \mathcal{A}(\mathcal{O})$.

The case of the group G being non-abelian is more complicated and we limit ourselves to finite groups leading already to structures which are quite interesting. In order to proceed we would like to know that every operator $\hat{F} \in \hat{\mathcal{F}}(\mathcal{O})$ has a unique representation of the form

$$\hat{F} = \sum_{g \in G} F(g)\,U_L^{\mathcal{O}}(g), \quad F(g) \in \mathcal{F}(\mathcal{O}). \tag{3.20}$$

While this true for the crossed product $\mathcal{M} \rtimes G$ on $L^2(G, \mathcal{H})$ (only for finite groups!) it is not obvious for the algebra $\mathcal{M} \vee U(G)''$ on $\mathcal{H}$. The latter may be considered as the image of the former under a homomorphism which might have a nontrivial kernel. In this case there would be equations of the type

$$\sum_{g \in G} F(g)\, U_L^{\mathcal{O}}(g) = 0, \tag{3.21}$$

where not all $F(g)$ vanish. Fortunately at least for finite groups (infinite, thus noncompact, discrete groups are ruled out by the split property) this undesirable phenomenon can be excluded without imposing further assumptions using the following result due to Buchholz [16].

Proposition 3.11. *The automorphisms $\alpha_g = Ad\,U(g)$ act outerly on the wedge algebras.*

Proof. Let W be the standard wedge $W = \{x \in \mathbb{R}^2 \mid x^1 > |x^0|\}$ and assume there is a unitary $V_g \in \mathcal{F}(W)$ such that $Ad\,V_g \restriction \mathcal{F}(W) = \alpha_g$. Define $V_{g,x} = \alpha_x(V_g)$ for all $x \in W$. Obviously $V_{g,x} \in \mathcal{F}(W_x)$. By the commutativity $\alpha_x \circ \alpha_g = \alpha_g \circ \alpha_x$ of translations and gauge transformations we have $Ad\,V_{g,x} \restriction \mathcal{F}(W_x) = \alpha_g$. By the computation (for $x \in W$)

$$\begin{aligned} V_g\, V_{g,x}\, V_g^* = &\quad \alpha_g(V_{g,x}) \quad = \alpha_g \circ \alpha_x(V_g) = \alpha_x \circ \alpha_g(V_g) \\ = &\ \alpha_x(V_g\, V_g\, V_g^*) = \quad \alpha_x(V_g) \quad = \quad V_{g,x} \end{aligned} \tag{3.22}$$

we obtain

$$V_g\, V_{g,x} = V_{g,x}\, V_g \ \ \forall x \in W. \tag{3.23}$$

The von Neumann algebra

$$\mathcal{V} = \{V_{g,x},\ x \in W\}'' \tag{3.24}$$

is mapped into itself by translations α_x where $x \in W$ and the vacuum vector Ω it is separating for $\mathcal{V}$ as we have $\mathcal{V} \subset \mathcal{F}(W)$. This allows us to apply the arguments in [29] to conclude that $\mathcal{V}$ is either trivial (i.e. $\mathcal{V} = \mathbb{C}\mathbf{1}$) or a factor of type III_1. The assumed existence of V_g, which cannot be proportional to the identity due to the postulate $\alpha_g \not\equiv \mathrm{id}$, excludes the first alternative whereas the second is incompatible with (3.23) according to which V_g is central. Contradiction! □

Remark. This result may be interpreted as a manifestation of an ultraviolet problem. The automorphism α_g being inner on a wedge W, wedge duality would imply it to be inner on the complementary wedge W', too, giving rise to a factorization $U(g) = V_L(g)\, V_R(g),\ V_L(g) \in \mathcal{F}(W),\ V_R(g) \in \mathcal{F}(W')$. This would be incompatible with the distributional character of the local current from which $U(g)$ derives.

We cite the following well known result on automorphism groups of factors.

Proposition 3.12. *Let $\mathcal{M}$ be a factor and α an outer action of the finite group G. Then the inclusions $\mathcal{M}^G \subset \mathcal{M},\ \pi(\mathcal{M}) \subset \mathcal{M} \rtimes G$ are irreducible, i.e. $\mathcal{M} \rtimes G \cap \pi(\mathcal{M})' = \mathcal{M} \cap \mathcal{M}^{G'} = \mathbb{C}\mathbf{1}$. In particular $\mathcal{M} \rtimes G$ and $\mathcal{M}^G$ are factors. If the action α is unitarily implemented $\alpha_g = Ad\,U(g)$ then $\mathcal{M} \rtimes G$ and $\mathcal{M} \vee U(G)''$ are isomorphic.*

Proof. The irreducibility statements $\mathcal{M} \rtimes G \cap \pi(\mathcal{M})' = \mathcal{M} \cap \mathcal{M}^{G'} = \mathbb{C}\mathbf{1}$ are standard consequences of the relative commutant theorem [65, §22] for crossed products. Remarking that finite groups are discrete and compact the proof is completed by an application of [48, Corr. 2.3] which states that $\mathcal{M} \rtimes G$ and $\mathcal{M} \vee U(G)''$ are isomorphic if the former algebra is factorial and G is compact. □

We are now in a position to prove several important corollaries to Prop. 3.11.

Corollary 3.13. *The algebras $\mathcal{F}(\mathcal{O}),\ \mathcal{O} \in \mathcal{K}$ are factors also in the Bose-Fermi case.*

Proof. Since $Ad\,V$ acts outerly on the factor $\mathcal{F}(W_R^{\mathcal{O}})$ by Prop. 3.11, $M_1 = \mathcal{F}(W_R^{\mathcal{O}}) \vee \{V\}$ is a factor and there is an automorphism β of M_1 leaving $\mathcal{F}(W_R^{\mathcal{O}})$ pointwise invariant such that $\beta(V) = -V$. The automorphism $\beta \otimes \alpha_k$ of $M_1 \otimes \mathcal{F}(W_L^{\mathcal{O}})$ clearly has $Y^{\mathcal{O}}\,\mathcal{F}(\mathcal{O})\,Y^{\mathcal{O}*}$ as fixpoint algebra, cf. (2.22). Since α_k is outer the same holds [65, Prop. 17.6] for $\beta \otimes \alpha_k$. Thus the fixpoint algebra is factorial by another application of Prop. 3.12. □

Corollary 3.14. *Let $\mathcal{O} \in \mathcal{K}$. The automorphisms $\alpha_g = Ad\,U(g)$ and $\alpha_g^{\mathcal{O}} = Ad\,U_L^{\mathcal{O}}(g)$ act outerly on the algebra $\mathcal{F}(\mathcal{O})$.*

Proof. The pure Bose case is easy. $\mathcal{F}(\mathcal{O}),\ \alpha_g^{\mathcal{O}}$ and α_g are unitarily equivalent to $\mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}}),\quad \alpha_g \otimes id$, and $\alpha_g \otimes \alpha_g$, respectively. Since $\alpha_g = Ad\,U(g)$ is outer on $\mathcal{F}(W_R^{\mathcal{O}})$ the same holds by [65, Prop. 17.6] for the automorphisms $\alpha_g \otimes id$ and $\alpha_g \otimes \alpha_g$ of the above tensor product.

Turning to the Bose-Fermi case let $X_g \in \mathcal{F}(\mathcal{O})$ be an implementer of α_g or $\alpha_g^{\mathcal{O}}$ and define $\hat{X}_g = Y^{\mathcal{O}}\,X_g\,Y^{\mathcal{O}*}$. Then $(\mathbf{1} \otimes V)\hat{X}_g(\mathbf{1} \otimes V)$ also implements $\alpha_g \otimes id$ or $\alpha_g \otimes \alpha_g$, respectively, since k is central. $\mathcal{F}(\mathcal{O})$ being a factor this implies $(\mathbf{1} \otimes V)\hat{X}_g(\mathbf{1} \otimes V) = c_g\,\hat{X}_g$ with $c_g^2 = \pm 1$ due to $k^2 = e$. $\hat{X}_g$ is thus contained either in $\mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}})_+$ or in $\mathcal{F}(W_R^{\mathcal{O}})\,V \otimes \mathcal{F}(W_L^{\mathcal{O}})_-$. In the first case the restriction of $\alpha_g \otimes id$ or $\alpha_g \otimes \alpha_g$ to $\mathcal{F}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_L^{\mathcal{O}})_+$ is inner which can not be true by the same argument as for the Bose case. (Observe that $\mathcal{F}(W_L^{\mathcal{O}})_+$ is factorial.) On the other hand, no $\hat{X}_g \in \mathcal{F}(W_R^{\mathcal{O}})\,V \otimes \mathcal{F}(W_L^{\mathcal{O}})_-$ can implement $\alpha_g \otimes id$ or $\alpha_g \otimes \alpha_g$ since both automorphisms are trivial on the subalgebra $\mathbf{1} \otimes \mathcal{F}(W_L^{\mathcal{O}}) \cap U(G)'$ which requires $\hat{X}_g \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{F}(W_L^{\mathcal{O}})^{G'}$. This, however, is impossible: $\mathcal{F}(W_L^{\mathcal{O}})_- \cap \mathcal{F}(W_L^{\mathcal{O}})^{G'} = [\mathcal{F}(W_L^{\mathcal{O}}) \cap \mathcal{F}(W_L^{\mathcal{O}})^{G'}]_- = [\mathbb{C}\mathbf{1}]_- = \emptyset$, where we have used the irreducibility of $\mathcal{F}(W_L^{\mathcal{O}})^G \subset \mathcal{F}(W_L^{\mathcal{O}})$. □

Corollary 3.15. *Let the symmetry group G be finite. Then the enlarged algebra $\hat{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)''$ is isomorphic to the crossed product $\mathcal{F}(\mathcal{O}) \rtimes_{\alpha^{\mathcal{O}}} G$ and the inclusions $\mathcal{A}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O}),\ \mathcal{F}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$ are irreducible.*

Proof. Obvious from Prop. 3.12 and Cors. 3.13, 3.14. □

Remark. If G is a compact continuous group, outerness of the action does not allow us to draw these conclusions. In this case an additional postulate is needed. It would be sufficient to assume irreducibility of the inclusion $\mathcal{A}(W) \subset \mathcal{F}(W)$, for, as shown by Longo, this property in conjunction with proper infiniteness of $\mathcal{A}(W)$ implies dominance of the action and factoriality of the crossed product.

We are now able to give an explicit description of the dual net $\hat{\mathcal{A}}$.

Theorem 3.16. *Every operator $\hat{A} \in \hat{\mathcal{A}}(\mathcal{O})$ can be uniquely written in the form*

$$\hat{A} = \sum_{g \in G} A(g)\,U_L^{\mathcal{O}}(g), \tag{3.25}$$

where the $A(g) \in \mathcal{F}(\mathcal{O})$ satisfy

$$A(kgk^{-1}) = \alpha_k(A(g))\ \ \forall g, k \in G. \tag{3.26}$$

Conversely, every choice of $A(g)$ complying with this constraint gives rise to an element of $\hat{\mathcal{A}}(\mathcal{O})$. An analogous representation for the algebras $\hat{\mathcal{A}}(W_R^{\mathcal{O}})$ is obtained by replacing $U_L^{\mathcal{O}}(g)$ by $U(g)$.

Remark. Condition (3.26) implies $A(g) \in \mathcal{F}(\mathcal{O}) \cap U(N_g)'$, where $N_g = \{h \in G \mid gh = hg\}$ is the normalizer of g in G.

Proof. By Prop. 3.11 any $\hat{A} \in \hat{\mathcal{A}}(\mathcal{O})$ can be represented uniquely according to (3.25). Using $\alpha_k(\hat{A}) = \sum_g \alpha_k(A(g))\, U_L^{\mathcal{O}}(kgk^{-1}) = \sum_g \alpha_k(A(k^{-1}gk))\, U_L^{\mathcal{O}}(g)$, Eq. (3.26) follows by comparing coefficients. It is obvious that the arguments can be reversed. The statement on the wedge algebras $\hat{\mathcal{A}}(W_R^{\mathcal{O}})$ follows from the fact that $\hat{\mathcal{F}}(W_R^{\mathcal{O}})$ is the crossed product of $\mathcal{F}(W_R^{\mathcal{O}})$ by the global automorphism group, cf. Prop. 3.5. □

3.4. The split property. The prominent role played by the split property in our investigations so far gives rise to the question whether it extends to the enlarged nets $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$. As to the net $\hat{\mathcal{F}}$ it is clear that a twist operation is needed in order to achieve commutativity of the algebras of two spacelike separated regions. Let $\mathcal{O}_1 < \mathcal{O}_2$ be double cones. Then one has $\hat{\mathcal{F}}(\mathcal{O}_2)^T \subset \hat{\mathcal{F}}(\mathcal{O}_1)'$, where

$$\left(\sum_g F(g)\, U_L^{\mathcal{O}}(g)\right)^T := \sum_g F(g)^t\, U_L^{\mathcal{O}}(g)\, U(g^{-1}) = \sum_g F(g)^t\, U_R^{\mathcal{O}}(g)^*, \qquad (3.27)$$

and the ^{t} on $F(g)$ denotes the Bose-Fermi twist of the introduction. (By the crossed product nature of the algebras $\hat{\mathcal{F}}(\mathcal{O})$ it is clear that this map is well defined and invertible.) That commutativity holds as claimed follows easily from $\hat{\mathcal{F}}(\mathcal{O}_1) \subset \mathcal{F}(W_L^{\mathcal{O}_1})$ and $\hat{\mathcal{F}}(\mathcal{O}_2)^T \subset \mathcal{F}(W_R^{\mathcal{O}_2})^t$. It is interesting to observe that the twist has to be applied to the algebra located to the right for this construction to work. This twist operation lacks, however, several indispensable features. Firstly, there is no unitary operator S implementing the twist as in the Bose-Fermi case. The second, more important objection refers to the fact that the map (3.27) becomes non-invertible when extended to right-handed wedge regions, for the operators $U_R^{\mathcal{O}}(g)$ are contained in $\mathcal{F}(W_R^{\mathcal{O}})$.

Concerning the net $\hat{\mathcal{A}}$ which, in contrast, is local there is no conceptual obstruction to proving the split property. We start by observing that $\hat{\mathcal{A}}(W_{LL}^{\mathcal{O}}) = \mathcal{A}(W_{LL}^{\mathcal{O}})$. Furthermore, in restriction to a simple sector $\mathcal{H}_1$ wedge duality (Prop. 2.1) implies $\hat{\mathcal{A}}(W_{RR}^{\mathcal{O}}) \restriction \mathcal{H}_1 = \mathcal{A}(W_{RR}^{\mathcal{O}}) \restriction \mathcal{H}_1$. As the split property for the fields carries over [26] to the observables in the vacuum sector there is nothing to do if we restrict ourselves to the latter. We intend to prove now that the net $\hat{\mathcal{A}}$ fulfills the split property on the big Hilbert space $\mathcal{H}$. To this purpose we draw upon the pioneering work [26] where it was shown that the split property (for double cones) of a field net with group symmetry and twisted locality follows from the corresponding property of the fixpoint net provided the group G is finite abelian. (The case of general groups constitutes an open problem, but given nuclearity for the observables and some restriction on the masses in the charged sectors nuclearity and thus the split property for the fields can be proved.)

Proposition 3.17. *The net $\mathcal{O} \mapsto \hat{\mathcal{A}}(\mathcal{O})$ satisfies the split property for wedge regions, provided the group G is finite.*

Proof. The split property for wedges is equivalent [11] to the existence, for every double cone $\mathcal{O}$, of a product state $\phi^{\mathcal{O}}$ satisfying $\phi^{\mathcal{O}}(AB) = \phi^{\mathcal{O}}(A)\cdot\phi^{\mathcal{O}}(B)\, \forall A \in \hat{\mathcal{A}}(W_{LL}^{\mathcal{O}}), B \in$

$\hat{\mathcal{A}}(W_{RR}^{\mathcal{O}})$. For the rest of the proof we fix one double cone $\mathcal{O}$ and omit it in the formulae. We have already remarked that for the net $\mathcal{A}$ product states ϕ_0 are known to exist. In order to construct a product state for $\hat{\mathcal{A}}$ we suppose γ_e is a conditional expectation from $\mathcal{A}(W_{LL}) \vee \hat{\mathcal{A}}(W_{RR})$ to $\mathcal{A}(W_{LL}) \vee \mathcal{A}(W_{RR})$ such that $\gamma_e(\hat{\mathcal{A}}(W_{RR})) = \mathcal{A}(W_{RR})$. Then $\gamma_e(AB) = \gamma_e(A)\,\gamma_e(B)$, where $A,\ B$ are as above, implying that $\phi = \phi_0 \circ \gamma_e$ is a product state. It remains to find the conditional expectation γ_e. To make plain the basic idea we consider abelian groups G first. In this case γ_e is given by

$$\gamma_e(\hat{A}) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \psi_\chi^* \hat{A} \psi_\chi, \tag{3.28}$$

where $\psi_\chi \in \mathcal{F}(\mathcal{O})$ is a unitary field operator transforming according to $\alpha_g(\psi_\chi) = \chi(g) \cdot \psi_\chi$ under the group G. This map has all the desired properties. The pointwise invariance of $\hat{\mathcal{A}}(W_{LL})$ follows from the fact that this algebra commutes with the unitaries ψ_χ. On the other hand

$$\psi_\chi^*\, U_L^{\tilde{\mathcal{O}}}(g)\, \psi_\chi = \chi(g) \cdot U_L^{\tilde{\mathcal{O}}}(g), \quad \tilde{\mathcal{O}} \subset W_{RR}^{\mathcal{O}} \tag{3.29}$$

in conjunction with the identity $\sum_{\chi \in \hat{G}} \chi(g) = |G|\, \delta_{g,e}$ (valid also for non-abelian groups) implies that the operators $U_L^{\tilde{\mathcal{O}}}(g) \in \hat{\mathcal{A}}(W_{RR}),\ g \neq e$ are annihilated by γ_e. Finally, the existence of $\psi_\chi \in \mathcal{F}(\mathcal{O})$ for all χ (i.e. the dominance of the group action α on $\mathcal{F}(\mathcal{O})$) is well known to follow from the outerness of the group action α. The generalization to non-abelian groups is straightforward. The unitaries ψ_χ are replaced by multiplets $\psi_{r,i}$ of isometries for all irreducible representations r of G. They fulfill the following relations of orthogonality and completeness:

$$\psi_{r,i}^*\, \psi_{r,j} = \delta_{i,j} \mathbf{1}, \tag{3.30}$$

$$\sum_{i=1}^{d_r} \psi_{r,i}\, \psi_{r,i}^* = \mathbf{1} \tag{3.31}$$

and transform according to

$$\alpha_g(\psi_{r,i}) = \sum_{i'} D^r_{i',i}(g)\, \psi_{r,i'} \tag{3.32}$$

under the group. That the conditional expectation γ_e given by

$$\gamma_e(\hat{A}) = \frac{1}{|G|} \sum_{r \in \hat{G}} \sum_{i=1}^{d_r} \psi_{r,i}^* \hat{A} \psi_{r,i}, \tag{3.33}$$

does the job follows from

$$\sum_{i=1}^{d_r} \psi_{r,i}^*\, U_L^{\tilde{\mathcal{O}}}(g)\, \psi_{r,i} = tr\, D^r(g) \cdot U_L^{\tilde{\mathcal{O}}}(g) = \chi_r(g) \cdot U_L^{\tilde{\mathcal{O}}}(g). \tag{3.34}$$

Again the existence of such multiplets is guaranteed by our assumptions. □

Remark. Tensor multiplets satisfying (3.30, 3.31) were first considered in [25] where the relation between the charged fields in a net of field algebras and the inequivalent representations of the observables was studied in the framework of [21]. Multiplets of this type will play a role in our subsequent investigations, too.

3.5. Irreducibility of $\mathcal{A}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$*.* The inclusions $\mathcal{A}(\mathcal{O}) \subset \mathcal{F}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$ are of the form

$$\mathcal{N} = \mathcal{P}^K \subset \mathcal{P} \subset \mathcal{P} \rtimes L = \mathcal{M}, \tag{3.35}$$

where K and L are finite subgroups of Aut $\mathcal{P}$, as studied in [8] (albeit for type II_1 factors). There $\mathcal{P}^K \subset \mathcal{P} \rtimes L$ was shown to be irreducible iff $K \cap L = \{e\}$ in Out $\mathcal{P}$ and to be of finite depth if and only if the subgroup Q of Out $\mathcal{P}$ generated by K and L is finite. Furthermore, the inclusion has depth two (i.e. $\mathcal{N}' \wedge \mathcal{M}_2$ is a factor where $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ is the Jones tower corresponding to the subfactor $\mathcal{N} \subset \mathcal{M}$) in the special case when $Q = K \cdot L$ (i.e. every $q \in Q$ can be written as $q = kl,\ k \in K, l \in L$).

In our situation, where $K = \mathrm{Diag}(G \times G)$ and $L = G \times \mathbf{1}$, all these conditions are fulfilled, as we have $Q = G \times G$ and $g \times h = (h \times h) \cdot (h^{-1}g \times e)$. The interest of this observation for our purposes derives from the following result, discovered by Ocneanu and proved, e.g., in [69, 55]. It states that an irreducible inclusion $\mathcal{N} \subset \mathcal{M}$ arises via $\mathcal{N} = \mathcal{M}^H = \{x \in \mathcal{M} \mid \gamma_a(x) = \varepsilon(x)\mathbf{1}\ \forall a \in H\}$ from the action of a Hopf algebra H on $\mathcal{M}$ iff the inclusion has depth two. In the next section this Hopf algebra will be identified and related to our quantum field theoretic setup.

For the irreducibility of $\mathcal{A}(\mathcal{O})$ in $\hat{\mathcal{F}}(\mathcal{O})$ we now give a proof independent of any sophisticated inclusion theoretic machinery.

Proposition 3.18. *For any* $\mathcal{O} \in \mathcal{K}$ *we have*

$$\hat{\mathcal{F}}(\mathcal{O}) \wedge \mathcal{A}(\mathcal{O})' = \mathbb{C}\mathbf{1}. \tag{3.36}$$

Proof. All unitary equivalences in this proof are implemented by $Y^{\mathcal{O}}$. With the abbreviations $\mathcal{M}_1 = \mathcal{F}(W_R^{\mathcal{O}})^t$ and $\mathcal{M}_2 = \mathcal{F}(W_L^{\mathcal{O}})$ we have $\mathcal{M}_1' \vee \mathcal{M}_2' \cong \mathcal{M}_1' \otimes \mathcal{M}_2'$. By (2.10) if $\mathcal{F}$ is bosonic or (2.22) in the Bose-Fermi case we have

$$\hat{\mathcal{F}}(\mathcal{O}) \cong \mathcal{F}(W_R^{\mathcal{O}}) \vee U(G)'' \otimes \mathcal{F}(W_L^{\mathcal{O}}) = \mathcal{M}_1 \vee U(G)'' \otimes \mathcal{M}_2, \tag{3.37}$$

where we have used $\mathcal{M}^t \vee U(G)'' = \mathcal{M} \vee U(G)''$ (which is true for every von Neumann algebra $\mathcal{M}$). Furthermore,

$$\begin{aligned} \mathcal{A}(\mathcal{O})' &= \mathcal{F}(\mathcal{O})' \vee U(G)'' = (\mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}}))^t \vee U(G)'' \\ &= \mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}}) \vee U(G)'' = \mathcal{F}(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}(W_{RR}^{\mathcal{O}})^t \vee U(G)'' \\ &= \mathcal{M}_1' \vee \mathcal{M}_2' \vee U(G)'' \cong (\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\ g \in G\}''. \end{aligned} \tag{3.38}$$

The relative commutant $\hat{\mathcal{F}}(\mathcal{O}) \wedge \mathcal{A}(\mathcal{O})'$ is thus equivalent to

$$(\mathcal{M}_1 \vee U(G)'' \otimes \mathcal{M}_2) \wedge [(\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\ g \in G\}'']. \tag{3.39}$$

The obvious inclusion $(\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\ g \in G\}'' \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_2' \vee U(G)''$ in conjunction with the irreducibility property $\mathcal{M}_2 \wedge (\mathcal{M}_2' \vee U(G)'') = \mathbb{C}\mathbf{1}$ (Cor. 3.15) yields

$$[(\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\ g \in G\}''] \wedge (\mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_2) \subset \mathcal{B}(\mathcal{H}) \otimes \mathbf{1}. \tag{3.40}$$

Now let X be an element of the algebra given by Eq. (3.39). By the same arguments as used earlier, every operator $X \in (\mathcal{M}_1' \otimes \mathcal{M}_2') \vee \{U(g) \otimes U(g),\ g \in G\}''$ has a unique representation of the form $X = \sum_g F_g\,(U(g) \otimes U(g))$, where $F_g \in \mathcal{M}_1' \otimes \mathcal{M}_2'$. The condition $X \in \mathcal{B}(\mathcal{H}) \otimes \mathbf{1}$ implies $F_g = 0$ for all $g \neq e$ and thereby $X \in \mathcal{M}_1' \otimes \mathbf{1}$. We thus have $X \in (\mathcal{M}_1' \wedge (\mathcal{M}_1 \vee U(G)'')) \otimes \mathbf{1}$ and, once again using the irreducibility of the group inclusions, $X \propto \mathbf{1} \otimes \mathbf{1}$. $\square$

4. Quantum Double Symmetry

4.1. Abelian groups. As we have shown above the algebras $\hat{\mathcal{F}}(\mathcal{O})$ may be considered as crossed products of $\mathcal{F}(\mathcal{O})$ with the actions of the respective automorphism groups $\alpha^{\mathcal{O}}$. In the case of abelian (locally compact) groups there is a canonical action [70] of the dual (character-) group $\hat{G}$ on $\mathcal{M} \rtimes G$ given by

$$\left.\begin{array}{l} \hat{\alpha}_\chi(\pi(x)) = \pi(x) \\ \hat{\alpha}_\chi(U_g) = \overline{\chi(g)} \cdot U_g \end{array}\right. ,\ \chi \in \hat{G}. \tag{4.1}$$

Making use of $U^{\mathcal{O}_1}(g)\,U^{\mathcal{O}_2}(g)^* \in \mathcal{F},\ \forall \mathcal{O}_i$ one can consistently define an action of $\hat{G}$ on the net $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$, respecting the local structure and thus extending to the quasilocal algebra $\hat{\mathcal{F}}$. The action of $\hat{G}$ commutes with the original action of G as extended to $\hat{\mathcal{F}}$, implying that the locally compact group $G \times \hat{G}$ is a group of local symmetries of the extended theory $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$. The square structure (3.10) can now easily be interpreted in terms of the larger symmetry:

$$\hat{\mathcal{A}} = \hat{\mathcal{F}}^G,\quad \mathcal{F} = \hat{\mathcal{F}}^{\hat{G}},\quad \mathcal{A} = \hat{\mathcal{F}}^{G \times \hat{G}}. \tag{4.2}$$

The symmetry between the subgroups G and $\hat{G}$ of $G \times \hat{G}$ is, however, not perfect, as only the automorphisms $\alpha_g,\ g \in G$ are unitarily implemented on the Hilbert space $\mathcal{H}$. That there can be no unitary implementer $U(\chi)$ for $\hat{\alpha}_\chi,\ \chi \in \hat{G}$ leaving invariant the vacuum Ω is shown by the following computation which would be valid for all $A \in \mathcal{A}(\mathcal{O})$:

$$\begin{aligned} \langle \Omega, AU_L^{\mathcal{O}}(g)\Omega \rangle &= \langle \Omega, U(\chi)\, AU_L^{\mathcal{O}}(g)\, U(\chi)^* \Omega \rangle \\ &= \langle \Omega, A\hat{\alpha}_\chi(U_L^{\mathcal{O}}(g))\Omega \rangle = \overline{\chi(g)} \cdot \langle \Omega, AU_L^{\mathcal{O}}(g)\Omega \rangle. \end{aligned} \tag{4.3}$$

This can only be true if $\chi(g) = 1$ or $\langle \Omega, AU_L^{\mathcal{O}}(g)\Omega \rangle = 0\, \forall A \in \mathcal{A}(\mathcal{O})$. The latter, however, can be ruled out, since the density of $\mathcal{A}(\mathcal{O})\Omega$ in $\mathcal{H}_0$ would imply $U_L^{\mathcal{O}}(g)\Omega \perp \mathcal{H}_0$ which is impossible, Ω being unitary and gauge invariant. This argument shows that the vacuum state $\omega = \langle \Omega, \cdot\, \Omega \rangle$ is not invariant under the automorphisms $\hat{\alpha}(\chi),\ \chi \in \hat{G}$, in other words, the symmetry under $\hat{G}$ is spontaneously broken.

The preceding argument is just a special case of the much more general analysis in [61], where non-abelian groups were considered, too. There, to be sure, the field net acted upon by the group was supposed to fulfill Bose-Fermi commutation relations, whereas in our case the field net is nonlocal. Furthermore, whereas the net $\mathcal{F}(\mathcal{O})$, the point of departure for our analysis, fulfills (twisted) duality, the extended net $\hat{\mathcal{F}}(\mathcal{O})$ enjoys no obvious duality properties. Nevertheless the analogy to [61] goes beyond the above argument. Indeed, as shown by Roberts, spontaneous breakdown of group symmetries is accompanied by a violation of Haag duality for the observables, restricted to the vacuum sector $\mathcal{H}_0$. Defining the net $\mathcal{B}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^{G_0}$, the fixpoint net under the action of the unbroken part $G_0 = \{g \in G \,|\, \omega_0 \circ \alpha_g = \omega_0\}$ of the symmetry group, a combination of the arguments in [21] and [61] leads to the conclusion that (in the vacuum sector $\mathcal{H}_0$) $\mathcal{B}(\mathcal{O})$ is just the dual net $\mathcal{A}^d(\mathcal{O})$ which verifies Haag duality. Our analysis in Sect. 2, leading to the identification of the dual net as $\mathcal{A}^d = \hat{\mathcal{A}} = \hat{\mathcal{F}}^G$, is obviously in accord with the general theory as we have shown above that G is the unbroken part, corresponding to G_0, of the full symmetry group $G \times \hat{G}$.

In the case of spontaneously broken group symmetries it is known that, irrespective of the nonexistence of global unitary implementers leaving invariant the vacuum, one can find local implementers for the whole symmetry group. This means that for each double cone $\mathcal{O}$ there exists a unitary representation $G \ni g \mapsto V_{\mathcal{O}}(g)$ satisfying $Ad\, V_{\mathcal{O}}(g) \restriction \mathcal{F}(\mathcal{O}) = \alpha_g$, the important point being the dependence on the region $\mathcal{O}$. (Due to the large commutant of $\mathcal{F}(\mathcal{O})$ such operators are far from unique.) A particularly nice construction, which applied to an unbroken symmetry g automatically yields the global implementer ($V_{\mathcal{O}}(g) = U(g)\, \forall \mathcal{O}$), was given in [15]. The construction given there applies without change to the situation at hand where the action of the dual group $\hat{G}$ on $\hat{\mathcal{F}}(\mathcal{O})$ is spontaneously broken.

An interesting example is provided by the free massive Dirac field which as already mentioned fulfills our postulates, including twisted duality and the split property. Its symmetry group $U(1)$ being compact and abelian, the extended net $\hat{\mathcal{F}}$ and the action of the dual group $\mathbb{Z}$ can be constructed as described above. By restriction of the net $\hat{\mathcal{A}}$ to the vacuum sector $\mathcal{H}_0$ one obtains a local net fulfilling Haag duality with symmetry group $\mathbb{Z}$. Wondering to which quantum field theory this net might correspond, it appears quite natural to think of the sine-Gordon theory at the free fermion point $\beta^2 = 4\pi$ as discussed, e.g., in [53].

4.2. Non-abelian groups. We refrain from further discussion of the abelian case and turn to the more interesting case of G being non-abelian and finite. (Infinite compact groups will be treated in Appendix B.) For non-abelian groups the dual object is not a group but either some Hopf algebraic structure or a category of representations. Correspondingly, the action of the dual group in [70] has to be replaced by a coaction of the group or the action of a group dual in the sense of [62]. For our present purposes these high-brow approaches will not be necessary. Instead we choose to generalize (4.1) in the following straightforward way. We observe that the characters of a compact abelian group constitute an orthogonal basis of the function space $L^2(G)$, whereas in the non-abelian case they span only the subspace of class functions. This motivates us to define an action of $\mathbb{C}(G)$, the $|G|$-dimensional space of *all* complex valued functions on G, on $\hat{\mathcal{F}}(\mathcal{O})$ in the following way:

$$\gamma_F \left(\sum_{g \in G} x(g)\, U_L^{\mathcal{O}}(g) \right) = \sum_{g \in G} F(g)\, x(g)\, U_L^{\mathcal{O}}(g), \quad x(g) \in \mathcal{F}(\mathcal{O}), F \in \mathbb{C}(G). \qquad (4.4)$$

Again this action of $\mathbb{C}(G)$ is consistent with the local structure of the net $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$ and extends to the quasilocal C^*-algebra $\hat{\mathcal{F}}$. In general, of course, γ_F is no homomorphism but only a linear map. (That the maps γ_F are well defined for every $F \in \mathbb{C}(G)$ should be obvious, see also the next section.) Introducing the "deltafunctions" $\delta_g(h) = \delta_{g,h}$ any function can be written as $F = \sum_g F(g)\, \delta_g$, and γ_{δ_g} will be abbreviated by γ_g. The latter are projections, i.e. they satisfy $\gamma_g^2 = \gamma_g$. The images of $\hat{\mathcal{F}}(\mathcal{O})$ and $\hat{\mathcal{F}}$ under these will be designated $\hat{\mathcal{F}}_g(\mathcal{O})$ and $\hat{\mathcal{F}}_g$, respectively. Obviously we have $\hat{\mathcal{F}}_g(\mathcal{O}) = \mathcal{F}(\mathcal{O})\, U_L^{\mathcal{O}}(g)$ and $\hat{\mathcal{F}}_g = \mathcal{F}\, U_L^{\mathcal{O}}(g)$ with $\mathcal{O} \in \mathcal{K}$ arbitrary. It should be clear that the decomposition

$$\hat{\mathcal{F}} = \bigoplus_{g\in G} \hat{\mathcal{F}}_g \tag{4.5}$$

represents a grading of $\hat{\mathcal{F}}$ by the group, i.e.

$$\hat{\mathcal{F}}_g\, \hat{\mathcal{F}}_h \subset \hat{\mathcal{F}}_{gh}\ \forall g, h \in G. \tag{4.6}$$

(In fact we have equality, but this will play no role in the sequel.) This group grading which is, of course, not surprising as it holds for every crossed product by a finite group allows us to state the behavior of γ_g under products:

$$\gamma_g(AB) = \sum_h \gamma_h(A)\, \gamma_{h^{-1}g}(B). \tag{4.7}$$

The novel aspect, however, is that $\hat{\mathcal{F}}$ is at the same time acted upon by the group G, these two structures being coupled by

$$\alpha_g(\hat{\mathcal{F}}_h) = \hat{\mathcal{F}}_{ghg^{-1}} \tag{4.8}$$

as a consequence of (2.15). This is equivalent to the relation

$$\alpha_g \circ \gamma_h = \gamma_{ghg^{-1}} \circ \alpha_g. \tag{4.9}$$

In this context it is of interest to remark that several years ago algebraists studied (see [18] and references given there) analogies between group graded algebras and algebras acted upon by a finite group. Similar studies have been undertaken in the context of inclusions of von Neumann algebras. As it turns out the situation at hand, which is rather more interesting, can be neatly described in terms of the action, as defined, e.g., in [68], of a Hopf algebra (in our case finite dimensional) on $\hat{\mathcal{F}}$. The relations fulfilled by the α_g and γ_h, in particular (4.9), motivate us to cite the following well known

Definition 4.1. *Let $\mathbb{C}(G)$ be the algebra of (complex valued) functions on the finite group G and consider the adjoint action of G on $\mathbb{C}(G)$ according to $\alpha_g:\ f \mapsto f \circ Ad(g^{-1})$. The quantum double $D(G)$ is defined as the crossed product $D(G) = \mathbb{C}(G) \rtimes_\alpha G$ of $\mathbb{C}(G)$ by this action. In terms of generators $D(G)$ is the algebra generated by elements U_g and $V_h,\ g, h \in G$ with the relations*

$$U_g\, U_h = U_{gh}, \tag{4.10}$$

$$V_g\, V_h = \delta_{g,h} V_g, \tag{4.11}$$

$$U_g\, V_h = V_{ghg^{-1}}\, U_g, \tag{4.12}$$

and the identification $U_e = \sum_g V_g = \mathbf{1}$.

It is easy to see that $D(G)$ is of the finite dimension $|G|^2$, where as a convenient basis one may choose $V(g)U(h),\ g, h \in G$, multiplying according to $V(g_1)U(h_1)\, V(g_2)U(h_2) = \delta_{g_1, h_1 g_2 h_1^{-1}} \cdot V(g_1)U(h_1 h_2)$. This is just a special case of a construction given by Drinfel'd [30] in greater generality which we do not bother to retain. For the purposes of this work it suffices to state the following well known properties of $D(G)$, referring to [30, 59, 20] for further discussion, see also Appendix A.

In order to define an action of a Hopf algebra on von Neumann algebras we further need a star structure on the former which in our case is provided by the following

Proposition 4.2. *With the definition* $U_g^* = U_{g^{-1}},\ V_h^* = V_h$ *and the appropriate extension,* $D(G)$ *is a *-algebra. D(G) is semisimple.*

Proof. Trivial calculation. Finite dimensional *-algebras are automatically semisimple. □

Before stating how the quantum double $D(G)$ acts on $\hat{\mathcal{F}}$ we define precisely the properties of a Hopf algebra action.

Definition 4.3. *A bilinear map* $\gamma : H \times \mathcal{M} \to \mathcal{M}$ *is an action of the Hopf *-algebra* H *on the *-algebra* $\mathcal{M}$ *iff the following hold for any* $a, b \in H,\ x, y \in \mathcal{M}$*:*

$$\gamma_{\mathbf{1}}(x) = x, \tag{4.13}$$

$$\gamma_a(\mathbf{1}) = \varepsilon(a)\mathbf{1}, \tag{4.14}$$

$$\gamma_{ab}(x) = \gamma_a \circ \gamma_b(x), \tag{4.15}$$

$$\gamma_a(xy) = \gamma_{a^{(1)}}(x)\gamma_{a^{(2)}}(y), \tag{4.16}$$

$$(\gamma_a(x))^* = \gamma_{S(a^*)}(x^*). \tag{4.17}$$

We have used the standard notation $\Delta(a) = a^{(1)} \otimes a^{(2)}$ *for the coproduct where on the right side there is an implicit summation. The map* γ *is assumed to be weakly continuous with respect to* $\mathcal{M}$ *and continuous with respect to some* C^**-norm on* H *(which is unique in the case of finite dimensionality).*

After these lengthy preparations it is clear how to define the action of $D(G)$ on $\hat{\mathcal{F}}$.

Theorem 4.4. *Defining* $\gamma_a(\hat{F}),\ \hat{F} \in \hat{\mathcal{F}}$ *for* $a \in \{U(g),\ V(h) | g, h \in G\}$ *by*

$$\gamma_{U_g}(\hat{F}) = \alpha_g(\hat{F}), \tag{4.18}$$

$$\gamma_{V_h}(\hat{F}) = \gamma_h(\hat{F}), \tag{4.19}$$

using (4.15) to define γ *on the basis* $V(g)U(h)$ *and extending linearly to* $D(G)$ *one obtains an action in the sense of Definition 4.3.*

Proof. Equation (4.13) follows from $\mathbf{1}_{D(G)} = \sum_g V_g$, (4.14) from $\mathbf{1}_{\hat{\mathcal{F}}} \in \hat{\mathcal{F}}_e$ and (A.4), whereas (4.15) is an obvious consequence of the definition. Furthermore, (4.16) is a consequence of α_g being a homomorphism, the coproduct property (4.7) and the definition (A.5). The statement (4.17) on the *-operation finally follows from $(\alpha_g(x))^* = \alpha_g(x^*)$ and $S(U_g^*) = U_g$ on the one hand and $(\hat{\mathcal{F}}_g)^* = \hat{\mathcal{F}}_{g^{-1}}$ and $S(V_g^*) = V_{g^{-1}}$ on the other. □

Remarks. 1. It should be obvious that the action of $D(G)$ on $\hat{\mathcal{F}}$ commutes with the translations and that it commutes with the boosts iff the group G does. Otherwise, $U(\Lambda)\,U(g)\,U(\Lambda)^* = U_h$ implies $\alpha_\Lambda \circ \gamma_g = \gamma_h \circ \alpha_\Lambda$.
2. In the case of G being abelian $U_\chi = \sum_{g \in G} \overline{\chi(g)} \cdot V_g,\ \chi \in \hat{G}$ constitutes an alternative basis for the subalgebra $\mathbb{C}(G) \subset D(G)$. The resulting formulae $U_\chi\, U_\rho = U_{\chi\rho},\ \Delta(U_\chi) = U_\chi \otimes U_\chi$ and $\gamma_{U_\chi}(\cdot) = \hat{\alpha}_\chi(\cdot)$ establish the equivalence of the quantum double with the group $G \times \hat{G}$. The abelian case is special insofar as $D(G)$ is spanned by its grouplike elements, which is not true for G non-abelian.

4.3. Spontaneously broken quantum symmetry. Having shown in the abelian case that the symmetry under the dual group $\hat{G}$ is spontaneously broken it should not come as a surprise that the same holds for non-abelian groups G where, of course, the notion of unitary implementation has to be generalized.

Definition 4.5. *An action γ of the Hopf algebra H on the *-algebra $\mathcal{M}$ is said to be implemented by the (homomorphic) representation $U : H \to \mathcal{B}(\mathcal{H})$ if for all $a \in H, x \in \mathcal{M}$*

$$U(a)\,x = \gamma_{a^{(1)}}(x)\,U(a^{(2)}) \tag{4.20}$$

or equivalently

$$\gamma_a(x) = U(a^{(1)})\,x\,U(S(a^{(2)})). \tag{4.21}$$

*The representation is said to be unitary if the map U is a *-homomorphism.*

In complete analogy to the abelian case we see that only a subalgebra of $D(G)$, namely the group algebra $\mathbb{C}G$ is implemented in the above sense. A similar phenomenon has already been observed to occur in the Coulomb gas representation of the minimal models [44] and in [7] where two dimensional theories without conformal covariance were considered. It would be interesting to know whether there exists, in some sense, a "quantum version" of Goldstone's theorem for spontaneously broken Hopf algebra symmetries.

In an earlier section we defined a twist operation (3.27) which bijectively maps $\hat{\mathcal{F}}(\mathcal{O})$ into an algebra $\hat{\mathcal{F}}(\mathcal{O})^T$ which commutes with all field operators localized in the left spacelike complement $W_{LL}^{\mathcal{O}}$ of $\mathcal{O}$. With the notation introduced in this chapter this operation can be written as $F^T = \sum_g \gamma_g(F)^t\,U(g^{-1})$. One might wonder whether there is a map $\bar{T}$ which achieves the same thing for the right spacelike complement $W_{RR}^{\mathcal{O}}$. If the quantum symmetry were not spontaneously broken, such a map would be given by

$$F^{\bar{T}} = \sum_g \alpha_g(F)^t\,V(g), \tag{4.22}$$

where the $V(g)$ are the projectors implementing the dual $\mathbb{C}(G)$ of the group G. Using the spacelike commutation relations and the property $U^{\mathcal{O}}(g)\,V(h) = V(gh)\,U^{\mathcal{O}}(g)$ this claim is easily verified.

In the discussion of the abelian case we have mentioned that one can construct, e.g. by the method given in [15], local implementers of the dual group $\hat{G}$. For the quantum double $D(G)$ of a non-abelian group G, however, which is not spanned by its grouplike elements, another approach is needed.

Proposition 4.6. *For every double cone $\mathcal{O} \in \mathcal{K}$ there is a family of orthogonal projections $V_{\mathcal{O}}(g)$ fulfilling*

$$V_{\mathcal{O}}(g)\,V_{\mathcal{O}}(h) = \delta_{g,h}\,V_{\mathcal{O}}(g)\,, \quad \sum_g V_{\mathcal{O}}(g) = \mathbf{1}, \tag{4.23}$$

$$\gamma_g \restriction \hat{\mathcal{F}}(\mathcal{O}) = \sum_h V_{\mathcal{O}}(gh)\,\cdot\,V_{\mathcal{O}}(h) \tag{4.24}$$

and transforming correctly under the (unbroken) group G,

$$U(g)\,V_{\mathcal{O}}(h)\,U(g)^* = V_{\mathcal{O}}(ghg^{-1}). \tag{4.25}$$

Proof. In order to obtain operators with these properties we make use of the isomorphism, for every wedge W, between $\mathcal{F}(W) \vee U(G)''$ and $\mathcal{F}(W) \rtimes_\alpha G$. We briefly recall the construction of the crossed product $\mathcal{M} \rtimes G$. It is represented on the Hilbert space $\underline{\mathcal{H}} = L^2(G, \mathcal{H})$ of square integrable functions from G to $\mathcal{H}$. The algebra $\mathcal{M}$ acts according to $(\pi(x)f)(g) = \alpha_{g^{-1}}(x)\,f(g)$ whereas the group G is unitarily represented by $(\underline{U}(k)f)(g) = f(k^{-1}g)$. With these definitions one can easily verify the equation

$\overline{U}(k)\,\pi(x)\,\overline{U}(k)^* = \pi\circ\alpha_k(x)$. If the group G is finite one can furthermore define the projections $(\overline{E}(k)f)(g) = \delta_{g,k}\, f(g)$ for which one obviously has $\overline{U}(g)\,\overline{E}(k) = \overline{E}(gk)\,\overline{U}(g)$. As already discussed above there is, as a consequence of the outerness of the action of the group, an isomorphism between the algebras $\mathcal{M}\vee U(G)''$ and $\mathcal{M}\rtimes_\alpha G$ sending $\sum_g x_g\,U(g)$ to $\sum_g \pi(x_g)\,\overline{U}(g)$. As both algebras are of type III and live on separable Hilbert spaces this isomorphism is unitarily implemented and can be used to pull back the projections $\overline{E}(k)$ to the Hilbert space $\mathcal{H}$, where we denote them by $E(k)$. ($E(e)$ is nothing but the Jones projection in the extension $\mathcal{M}_2$ of the inclusion $\mathcal{M}\subset\mathcal{M}\vee U(G)''$.) Applying these considerations to the algebras of the wedges $W_L^{\mathcal{O}}$ and $W_R^{\mathcal{O}}$ we obtain the families of projections $E_{L/R}^{\mathcal{O}}(k)$, satisfying

$$U(g)\,E_{L/R}^{\mathcal{O}}(k)\,U(g)^* = E_{L/R}^{\mathcal{O}}(gk), \tag{4.26}$$

which we use to define

$$V_{\mathcal{O}}(g) = Y^{\mathcal{O}*}\,(\sum_h E_R^{\mathcal{O}}(gh)\otimes E_L^{\mathcal{O}}(h))\,Y^{\mathcal{O}}. \tag{4.27}$$

The properties (4.23) of orthogonality and completeness are obvious whereas covariance (4.25) follows from (4.26) and $U(k) = Y^{\mathcal{O}*}\,U(k)\otimes U(k)\,Y^{\mathcal{O}}$ as follows:

$$\begin{aligned} Ad\,U(k)(V_{\mathcal{O}}(g)) &= Y^{\mathcal{O}*}\,(\sum_h E_R^{\mathcal{O}}(kgh)\otimes E_L^{\mathcal{O}}(kh))\,Y^{\mathcal{O}} \\ &= Y^{\mathcal{O}*}\,(\sum_h E_R^{\mathcal{O}}(kgk^{-1}h)\otimes E_L^{\mathcal{O}}(h))\,Y^{\mathcal{O}} \\ &= V_{\mathcal{O}}(kgk^{-1}). \end{aligned} \tag{4.28}$$

It remains to show the implementation property (4.24). Using the fact that $E_L^{\mathcal{O}}(g)\,\mathcal{F}(W_L^{\mathcal{O}})\,E_L^{\mathcal{O}}(h) = \{0\}$ if $g\neq h$ and $\hat{\mathcal{F}}(\mathcal{O})\cong\mathcal{F}(W_R^{\mathcal{O}})\vee U(G)''\otimes\mathcal{F}(W_L^{\mathcal{O}})$ we obtain

$$\begin{aligned} &Y^{\mathcal{O}}\,\sum_h V_{\mathcal{O}}(gh)\,\hat{F}\,V_{\mathcal{O}}(h)\,Y^{\mathcal{O}*} \\ &\quad= \sum_{h,k,l} E_R^{\mathcal{O}}(ghk)\otimes E_L^{\mathcal{O}}(k)\,F_1\otimes F_2\,E_R^{\mathcal{O}}(hl)\otimes E_L^{\mathcal{O}}(l) \\ &\quad= \sum_{h,k} E_R^{\mathcal{O}}(ghk)\otimes E_L^{\mathcal{O}}(k)\,F_1\otimes F_2\,E_R^{\mathcal{O}}(hk)\otimes E_L^{\mathcal{O}}(k) \\ &\quad= (\sum_h E_R^{\mathcal{O}}(gh)\,F_1\,E_R^{\mathcal{O}}(h))\otimes(\sum_k E_L^{\mathcal{O}}(k)\,F_2\,E_L^{\mathcal{O}}(k)) \\ &\quad= \sum_h E_R^{\mathcal{O}}(gh)\,F_1\,E_R^{\mathcal{O}}(h)\otimes F_2, \end{aligned} \tag{4.29}$$

where we have written (abusively) $F_1\otimes F_2$ for $Y^{\mathcal{O}}\,\hat{F}\,Y^{\mathcal{O}*}$. Since we have $\sum_h E_R^{\mathcal{O}}(gh)$ $U(k)\,E_R^{\mathcal{O}}(h) = \delta_{g,k}U(k)$ it is clear that the above map projects $\hat{\mathcal{F}}(\mathcal{O})$ onto $\mathcal{F}(\mathcal{O})U_L^{\mathcal{O}}(g)$, thus implementing the restriction of γ_g to $\hat{\mathcal{F}}(\mathcal{O})$. □

Remark. It should be remarked that the simpler definition $\tilde{V}_{\mathcal{O}}(g) = Y^{\mathcal{O}*}(E_R^{\mathcal{O}}(g)\otimes\mathbf{1})Y^{\mathcal{O}}$, which also satisfies (4.24), does not lead to a representation of $D(G)$ as these $V_{\mathcal{O}}$'s do not transform according to the adjoint representation (4.25).

4.4. Spectral properties. The above discussion was to a large extent independent of the quantum field theoretic application insofar as the action of the quantum double on a certain class of *-algebras was concerned. As we have seen, any *-algebra which is at the same time acted upon by a finite group G and graded by G supports an action of the double provided the relation (4.8) holds. The converse is also true. Let $\mathcal{M}$ be a *-algebra on which the double acts. Then $\mathcal{M}_g = \gamma_g(\mathcal{M})$ induces a G-grading satisfying (4.8). It may however happen that $\mathcal{M}_g = \{0\}$ for g in a normal subgroup. This possibility can be eliminated by demanding the existence of a unitary representation of G in $\mathcal{M}: \ G \ni g \mapsto U(g) \in \mathcal{M}_g$. In the situation at hand this condition is fulfilled by construction.

We now turn to the spectral properties of the action of the double. To this purpose we introduce the following notion [62], already encountered implicitly in the proof of Prop. 3.17.

Definition 4.7. *A normclosed linear subspace $\mathcal{T}$ of a von Neumann algebra $\mathcal{M}$ is called a Hilbert space in $\mathcal{M}$ if $x^*x \in \mathbb{C}\mathbf{1}$ for all $x \in \mathcal{T}$ and $x \in \mathcal{M}$ and $xa = 0 \ \forall a \in \mathcal{T}$ implies $x = 0$.*

The name is justified as $\langle x, y\rangle \mathbf{1} = x^*y$ defines a scalar product in $\mathcal{T}$. One can thus choose a basis $\psi_i, \ i = 1, \ldots, d_{\mathcal{T}}$ satisfying the requirements (3.30, 3.31). The interest of this definition stems from the following well known lemma, the easy proof of which we omit.

Lemma 4.8. *Let $\mathcal{T}$ be a finite dimensional Hilbert space in $\mathcal{M}$ globally invariant under the action γ_H of H on $\mathcal{M}$. A basis of the above type gives rise to a unitary representation of H according to*

$$\gamma_a(\psi_i) = \sum_{i'=1}^{d} D_{i'i}(a)\, \psi_{i'}. \tag{4.30}$$

Our aim will now be to show that the extended algebras $\hat{\mathcal{F}}(\mathcal{O}), \ \mathcal{O} \in \mathcal{K}$ in fact contain such tensor multiplets for every irreducible representation of $D(G)$. In order to do this we make use of the representation theory of the double developed in [20]. ($D(G)$ being semisimple, every finite dimensional representation decomposes into a direct sum of irreducible ones.) The (equivalence classes of) irreducible representations are labeled by pairs (c, π), where $c \in C(G)$ is a conjugacy class and π is an irreducible representation of the normalizer group N_c. Here N_c is the abstract group corresponding to the mutually isomorphic normalizers N_g for $g \in c$, already encountered in the remark following Thm. 3.16. The representation $\hat{\pi}$ labeled by (c, π) is obtained by choosing an arbitrary $g_0 \in c$ and inducing up from the representation

$$\hat{\pi}(V_g\, U_h) = \delta_{g,g_0}\, \pi(h) \tag{4.31}$$

of the subalgebra $\mathcal{B}_{g_0}$ of $D(G)$ generated by $V(g)\,, g \in G$ and $U(h)\,, h \in N_{g_0}$. The representation space of $\hat{\pi}_{(c,\pi)}$ is thus $V_{(c,\pi)} = D(G) \otimes_{\mathcal{B}_{g_0}} V_\pi$. For a more complete discussion we refer to [20] remarking only that $\hat{\pi}_{(c,\pi)}(V_g\, U_h) = 0$ if $g \notin c$.

Definition 4.9. *The action γ of a group or Hopf algebra on a von Neumann algebra $\mathcal{M}$ is dominant iff the algebra of fixed points is properly infinite and the monoidal spectrum of γ is complete, i.e. for every finite dimensional unitary representation π of the group or Hopf algebra, respectively, there is a γ-invariant Hilbert space $\mathcal{T}$ in $\mathcal{M}$ such that $\gamma \upharpoonright \mathcal{T}$ is equivalent to π.*

Proposition 4.10. *Let $\hat{\mathcal{M}}$ be a von Neumann algebra supporting an action of the quantum double $D(G)$. Assume further that there is a unitary representation of G in $\hat{\mathcal{M}}$, where $\bar{U}(g) \in \hat{\mathcal{M}}_g$ and $\alpha_h(\bar{U}(g)) = \bar{U}(hgh^{-1})$. Then the action of $D(G)$ on $\hat{\mathcal{M}}$ is dominant if and only if the action of G on $\mathcal{M} = \gamma_e(\hat{\mathcal{M}})$ is dominant.*

Proof. As a consequence of $\mathcal{M}^G = \hat{\mathcal{M}}^{D(G)}$ the conditions of proper infiniteness of the fixpoint algebras coincide. The "only if" statement is easily seen by considering the representations of the double corresponding to the conjugacy class $c = \{e\}$. For these $N_c \cong G$ holds, implying that the representations of $D(G)$ with $c = \{e\}$ are in one-to-one correspondence to the representations of G. A multiplet in $\hat{\mathcal{M}}$ transforming according to $(\{e\}, \pi)$ is nothing but a π-multiplet in $\mathcal{M}$.

The "if" statement requires more work. We have to show that for every pair (c, π), where π is an irreducible representation of the normalizer N_c, there exists a multiplet of isometries transforming according to $\hat{\pi}_{(c,\pi)}$. To begin with, choose $g \in c$ arbitrarily and find in $\mathcal{M}$ a multiplet of isometries $\psi_i,\ i = 1, \ldots, d = \dim(\pi)$ transforming according to the representation π under the action of $N_g \subset G$. The existence of such a multiplet follows from the dominance of the group action on $\mathcal{M}$. Now, let $x_1, \ldots, x_n$ be representatives of the cosets G/N_g, where $n = [G : N_g] = |c|$. Furthermore, the proper infiniteness of the fixpoint algebra allows us to choose a family $V_1, \ldots, V_n$ of isometries in $\mathcal{M}^G = \hat{\mathcal{M}}^{D(G)}$ satisfying $V_i^* V_j = \delta_{i,j},\ \sum_i V_i V_i^* = \mathbf{1}$. Defining

$$\Psi_{ij} = V_i\, \alpha_{x_i}(\bar{U}(g)\, \psi_j), \quad i = 1, \ldots, n,\ j = 1, \ldots, d \tag{4.32}$$

one verifies that the Ψ_{ij} constitute a complete family of mutually orthogonal isometries spanning a vectorspace of dimension $nd = \dim(\hat{\pi}_{(c,\pi)})$. That this space is mapped into itself by the action of the double follows from the fact that, for every $k \in G$, $k\, x_i$ can uniquely be written as $x_j\, h,\ h \in N_g$. Finally, the multiplet transforms according to the representation (c, π) of $D(G)$, which is evident from the definition of the latter in [20, (2.2.2)]. □

Remark. Since in our field theoretic application the conditions of the proposition are satisfied thanks to Lemma 3.14 and the discussion in Subsect. 4.2 we can conclude that $\hat{\mathcal{F}}(\mathcal{O}),\ \mathcal{O} \in \mathcal{K}$ has full $D(G)$-spectrum.

4.5. Commutation relations and statistics. Up to this point our investigations in this section have focused on the local inclusion $\mathcal{A}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$ for any fixed double cone $\mathcal{O}$. Having clarified the relation between these algebras in terms of the action of the quantum double we can now complete our discussion of the latter. To this purpose we recall that the double construction has been introduced in [30] as a means of obtaining quasitriangular Hopf algebras (quantum groups) in the sense defined there, i.e. Hopf algebras possessing a "universal R-matrix," cf. Appendix A. As it turns out the latter appears very naturally in our approach when considering the spacelike commutation relations of irreducible $D(G)$-multiplets as defined in the preceding subsection.

Proposition 4.11. *Assume the net $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ is bosonic, i.e. fulfills untwisted locality. Let $\mathcal{O}_2 < \mathcal{O}_1$ and $\psi_i^1, i = 1, \ldots, d_1$ and $\psi_j^2, j = 1, \ldots, d_2$ be $D(G)$-tensors in $\hat{\mathcal{F}}(\mathcal{O}_1),\ \hat{\mathcal{F}}(\mathcal{O}_2)$, respectively. They then fulfill C-number commutation relations*

$$\psi_i^1\, \psi_j^2 = \sum_{i'j'} \psi_{j'}^2\, \psi_{i'}^1\, (D^1_{i'i} \otimes D^2_{j'j})(R), \tag{4.33}$$

where $D^1,\ D^2$ *are the matrices of the respective representations of* $D(G)$ *and*

$$R = \sum_{g\in G} V_g \otimes U_g \in D(G)\otimes D(G). \tag{4.34}$$

Proof. The equation $\sum_g V_g = \mathbf{1}$ in $D(G)$ implies $\sum_g \gamma_g = id$. We can thus compute

$$\begin{aligned}\psi_i^1\,\psi_j^2 &= \sum_{g\in G}\gamma_g(\psi_i^1)\,\psi_j^2 = \sum_{g\in G}\alpha_g(\psi_j^2)\,\gamma_g(\psi_i^1) \\ &= \sum_{g\in G}\sum_{i'j'}\psi_{j'}^2\,\psi_{i'}^1\,D^2_{j'j}(U_g)\,D^1_{i'i}(V_g),\end{aligned} \tag{4.35}$$

where the second identity follows from $\gamma_g(\psi_i^1) \in \mathcal{F}(\mathcal{O}_1)\,U_L^{\mathcal{O}_1}(g)$ and $Ad\,U_L^{\mathcal{O}_1}(g) \restriction \hat{\mathcal{F}}(\mathcal{O}_2) = \alpha_g$. The rest is clear. □

Remarks. 1. Commutation relations of the above general type have apparently first been considered in [40]. For the special case of $Z(N)$ order disorder duality they date back at least to [66].
2. By this result the field extension of Definition 3.1 in conjunction with Thm. 4.4 may be considered a local version of the construction of the double. (If we had used the $U_R^{\mathcal{O}}(g)$ we would have ended up with R^{-1} which would do just as well.)
3. If the net $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ is fermionic an additional sign $\pm$ appears on the right-hand side of (4.33) depending on the Bose/Fermi nature of the fields. Using the bosonization prescription of the next section this sign can be eliminated.

We now turn to a discussion of the localized endomorphisms of the observable algebra $\mathcal{A}$ which are implemented by the charged fields in $\hat{\mathcal{F}}$ as in [25]. Let $\psi_i,\ i = 1,\ldots,d_\psi$ be a multiplet of isometries in $\hat{\mathcal{F}}(\mathcal{O})$ transforming according to the irreducible representation r of $D(G)$. Then the map

$$\rho(\cdot) = \sum_{i=1}^{d_\psi}\psi_i\cdot\psi_i^* \tag{4.36}$$

defines a unital *-endomorphism of $\hat{\mathcal{F}}$. The relative locality of $\mathcal{A}$ and $\hat{\mathcal{F}}$ implies the restriction of ρ to $\mathcal{A}$ to be localized in $\mathcal{O}$ in the sense that $\rho(A) = A\ \forall A \in \mathcal{A}(\mathcal{O}')$. Furthermore, ρ maps $\mathcal{A}(\mathcal{O}_1)$ into itself if $\mathcal{O}_1 \supset \mathcal{O}$ as follows from the $D(G)$-invariance of $\rho(x)$ for $x \in \mathcal{A}$. (The conventional argument using duality would allow us only to conclude $\rho(\mathcal{A}(\mathcal{O}_1)) \subset \hat{\mathcal{A}}(\mathcal{O}_1)$.)

Proposition 4.12. *In restriction to* $\mathcal{A}(\mathcal{O}_1),\ \mathcal{O}_1 \supset \mathcal{O}$ *the endomorphism* ρ *is irreducible.*

Proof. The proof is omitted as it is identical to the proof of [54, Prop. 6.9], where compact groups are considered. □

Remarks. 1. In application to the net $\hat{\mathcal{A}}$ the endomorphisms ρ are localized only in wedge regions, i.e. they are of solitonic character.
2. Due to the spontaneous breakdown of the quantum symmetry the endomorphisms ρ which arise from non-group representations of $D(G)$ should not be considered as true superselection sectors of the net $\mathcal{A} \restriction \mathcal{H}_0$. This would be justified if the symmetry were unbroken. Nevertheless, one can analyze their statistics, as will be done in the rest of this subsection.

Whereas the endomorphisms ρ defined above need not be invertible one can always find left inverses [21] ϕ such that $\phi \circ \rho = id$. For ρ as defined by (4.36) the left inverse is easily verified to be given by

$$\phi_\rho = \frac{1}{d_\psi} \sum_{i=1}^{d_\rho} \psi_i^* \cdot \psi_i. \tag{4.37}$$

In order to study the statistics of endomorphisms one introduces [23, 35] the statistics operators

$$\varepsilon(\rho_1, \rho_2) = U_2^* \, \rho_1(U_2) \in (\rho_1\rho_2, \rho_2\rho_1), \tag{4.38}$$

where U_2 is a charge transporter intertwining ρ_2 and $\tilde{\rho}_2$, the latter being localized in the left spacelike complement of the localization region of ρ_1. Such an intertwiner is given by $U_2 = \sum_i \tilde{\psi}_i^2 \psi_i^{2*} \in \mathcal{F}^G = \mathcal{A}$, where $\tilde{\psi}_i$ is a multiplet in $\hat{\mathcal{F}}(\tilde{\mathcal{O}})$, $\tilde{\mathcal{O}} < \mathcal{O}_1$ transforming according the same representation of $D(G)$ as ψ_i, such that U_2 is $D(G)$-invariant and thus in $\mathcal{A}$.

Lemma 4.13. *Let $\psi_i^1 \in \hat{\mathcal{F}}(\mathcal{O}_1), i = 1, \ldots, d_1$ and $\psi_j^2, j = 1, \ldots, d_2$ be $D(G)$-multiplets corresponding to the representations D^1, D^2 and let ρ_1, ρ_2 be the associated endomorphisms. Then the statistics operator is given by*

$$\varepsilon(\rho_1, \rho_2) = \sum_{ijkl} \psi_i^2 \, \psi_l^1 \, \psi_j^{2*} \, \psi_k^{1*} \, (D^1_{lk} \otimes D^2_{ij})(R). \tag{4.39}$$

The statistics parameter [21] for the morphism ρ which is implemented by the irreducible $D(G)$-tensor $\psi_i, i = 1, \ldots, d_\psi$ is

$$\lambda_\rho = \frac{\omega_\rho}{d_\rho}, \tag{4.40}$$

with $d_\rho = d_\psi$ and $D_{lj}(X) = \delta_{lj} \, \omega_\rho$, where $X = \sum_g V_g \, U_g$ is a unitary element in the center of $D(G)$.

Proof. With $U_2 = \sum_i \tilde{\psi}_i^2 \psi_i^{2*}$ we have

$$\varepsilon(\rho_1, \rho_2) = \sum_{ijk} \psi_i^2 \tilde{\psi}_i^{2*} \, \psi_j^1 \, \tilde{\psi}_k^2 \psi_k^{2*} \, \psi_j^{1*}. \tag{4.41}$$

Then (4.39) follows by an application of (4.33) to ψ_j^1 and $\tilde{\psi}_k^2$ and appealing to the orthogonality relation $\tilde{\psi}_i^{2*} \tilde{\psi}_j^2 = \delta_{ij} \mathbf{1}$. With the identification $\psi^1 = \psi^2 = \psi$ in (4.39), (4.37) and using once more the orthogonality relation we compute the statistics parameter as follows:

$$\lambda_\rho \, \mathbf{1} = \phi_\rho(\varepsilon_{\rho,\rho}) = \frac{1}{d_\psi} \sum_{ijl} \psi_l \, \psi_j^* \, (D_{li} \otimes D_{ij})(R) = \frac{1}{d_\psi} \sum_{jl} \psi_l \, \psi_j^* \, M_{lj}, \tag{4.42}$$

where

$$M_{lj} = \sum_i (D_{li} \otimes D_{ij})(R) = D_{lj}(\sum_g V_g \, U_g). \tag{4.43}$$

An easy calculation shows that $X = \sum_g V_g \, U_g$ is a unitary element in the center of $D(G)$ such that it is represented by a phase times the unit matrix in the irreducible representation D: $M_{lj} = \delta_{lj} \, \omega, \; \omega \in S^1$. □

Remarks. 1. The statistical dimension of the sector ρ, defined as $d_\rho = |\lambda_\rho|^{-1}$ coincides with the dimension of the corresponding representation of the quantum double. This was to be expected and is in accord with the fact [55] that the action of finite dimensional Hopf algebras cannot give rise to non-integer dimensions.
2. Recalling Lemma 4.8 we see that in restriction to a field operator in a multiplet transforming according to the irreducible representation r the action of γ_X amounts to multiplication by ω_r. The unitary $X \in D(G)$ may thus be interpreted as the quantum double analogue of the group element k which distinguishes between bosons and fermions. This is reminiscent of the notion of ribbon elements in the framework of quantum groups, see Appendix A. In fact, the operator X defined above is just the inverse of Drinfel'd's $u = \sum_g V_g U_{g^{-1}}$ which itself is a ribbon element due to $S(u) = u$.
3. Appealing to the representation theory of $D(G)$ as expounded in [20] it is easy to compute the phase ω_r for the representation $r = (c, \pi)$. It is given by the scalar to which $g \in c$, obviously being contained in the center of the normalizer N_g, is mapped by the irreducible representation π of N_g. As an immediate consequence [19] ω_r is an nth root of unity where n is the order of g.

We now turn to the calculation of the monodromy operator

$$\varepsilon_M(\rho_1, \rho_2) = \varepsilon(\rho_1, \rho_2)\, \varepsilon(\rho_2, \rho_1), \tag{4.44}$$

which measures the deviation from permutation group statistics and of the *statistics characters* [58]

$$Y_{ij}\, \mathbf{1} = d_i d_j\, \phi_i(\varepsilon_M(\rho_i, \rho_j)^*). \tag{4.45}$$

In the latter expression ρ_i, ρ_j are irreducible morphisms such that the right-hand side is a C-number since $\phi_i(\varepsilon_M(\rho_i, \rho_j)^*)$ is a selfintertwiner of ρ_j and due to the irreducibility of the latter, cf. Prop. 4.12.) We thus obtain a square matrix of complex numbers indexed by the superselection sectors, i.e. in our case the irreducible representations of the quantum double $D(G)$.

Proposition 4.14. *In terms of the fields the monodromy operator is given by*

$$\varepsilon_M(\rho_1, \rho_2) = \sum_{ijkl} \psi_i^2\, \psi_j^1\, \psi_k^{1*}\, \psi_l^{2*}\, (D^1_{jk} \otimes D^2_{il})(I), \tag{4.46}$$

where

$$I = R\, \sigma(R). \tag{4.47}$$

The statistics characters are given by

$$Y_{ij} = (tr_i \otimes tr_j) \circ (D^i \otimes D^j)(I^*). \tag{4.48}$$

Proof. Inserting the statistics operators according to (4.39) and using twice the orthogonality relation we obtain

$$\varepsilon_M(\rho_1, \rho_2) = \sum_{\substack{ijkl \\ k'l'}} \psi_i^2\, \psi_j^1\, \psi_{k'}^{1*}\, \psi_{l'}^{2*}\, (D^1_{jl} \otimes D^2_{ik})(R)\, (D^2_{kl'} \otimes D^1_{lk'})(R). \tag{4.49}$$

The numerical factor to the right (including the summations over k, l) can be simplified to

$$\sum_{g,h\in G} D^1_{jk'}(V_g\,U_h)\,D^2_{il'}(U_g\,V_h) = (D^1_{jk'}\otimes D^2_{il'})(I). \tag{4.50}$$

Omitting the primes on k', l' we obtain (4.46). The formula (4.48) follows in analogy to the computation of λ_ρ from (4.46), (4.37) and $Y_{ij} \propto \mathbf{1}$. □

Remark. $I = R\,\sigma(R)$ can be considered as the quantum group version of the monodromy operator.

In [2] it was shown that Y is invertible, in fact $\frac{1}{|G|}Y$ is unitary. In conjunction with the known facts concerning the representation theory one concludes [2] that the quantum double $D(G)$ is a *modular Hopf algebra* in the sense of [60]. We are now in a position to complete our demonstration of the complete parallelism between quantum group theory and quantum field theory (which we claim only for the quantum double situation at hand!). What remains to be discussed is the Verlinde algebra structure [71] behind the fusion of representations of the double and the associated endomorphisms of $\hat{\mathcal{F}}$, respectively. The fusion rules are said to be diagonalized by a unitary matrix S if

$$N^k_{ij} = \sum_m \frac{S_{im}\,S_{jm}\,S^*_{km}}{S_{0m}}. \tag{4.51}$$

(For a comprehensive survey of fusion structures see [41].) One speaks of a Verlinde algebra if, in addition, S is symmetric, there is a diagonal matrix T of phases satisfying $TC = CT = T$ ($C_{ij} = \delta_{i\bar{j}}$ is the charge conjugation matrix) and S and T constitute a representation of $SL(2,\mathbb{Z})$ (in general not of $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$), i.e.

$$S^2 = (ST)^3 = C. \tag{4.52}$$

On the one hand the representation categories of modular Hopf algebras are known [60] to be modular, i.e. to satisfy (4.51) and (4.52), where the phases in T are given by the values of the ribbon element X in the irreducible representations.

On the other hand this structure has been shown [58] to arise from the superselection structure of *every* rational quantum field theory in 1 + 1 dimensions. In this framework the phases in T are given by the phases of the statistics parameters (4.40), whereas the matrix S arises from the statistics characters

$$T = \left(\frac{\sigma}{|\sigma|}\right)^{1/3} \mathrm{Diag}(\omega_i), \quad S = |\sigma|^{-1}\,Y. \tag{4.53}$$

For nondegenerate theories the number $\sigma = \sum_i \omega_i^{-1}\,d_i^2$ satisfies $|\sigma|^2 = \sum_i d_i^2$. Using the result [2] $\sigma = |G|$ this condition is seen to be fulfilled, for the semisimplicity of $D(G)$ gives $\sum_i d_i^2 = \dim(D(G)) = |G|^2$.

We thus observe, for the orbifold theories under study, a perfect parallelism between the general superselection theory [58] for quantum field theories in low dimensions and the representation theory of the quantum double [20]. This parallelism extends beyond the Verlinde structure. One observes, e.g., that Eqs. (2.4.2) of [20] and (2.30) of [58], both stating that the monodromy operator is diagonalized by certain intertwining operators, are identical although derived in apparently unrelated frameworks.

5. Bosonization

In this section we will show how the methods expounded in the preceding sections can be used to obtain an understanding of the Bose/Fermi correspondence in 1+1 dimensions in the framework of local quantum theory. This is so say, we will show how one can pass from a fermionic net of algebras with twisted duality to a bosonic net satisfying Haag duality *on the same Hilbert space*, and vice versa. Our method amounts to a continuum version of the Jordan-Wigner transformation and is reminiscent of Araki's approach to the XY-model [4].

Our starting point is as defined in the introduction, i.e. a net of field algebras with fermionic commutation relations (1.4) and twisted duality (1.11) augmented by the split property for wedge regions introduced in Sect. 2. As before there exists a selfadjoint unitary operator V distinguishing between even and odd operators. For the present investigations, however, the existence of further inner symmetries is ignored as they are irrelevant for the spacelike commutation relations. Therefore we now repeat the field extension of Sect. 3 replacing the group G by the subgroup $\mathbb{Z}_2 = \{e, k\}$. This amounts to simply extending the local algebras by the disorder operator associated with the only nontrivial group element k,

$$\hat{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O}) \vee \{V^{\mathcal{O}}\}, \tag{5.1}$$

where $V^{\mathcal{O}} = U_L^{\mathcal{O}}(k)$. Again, the assignment $\mathcal{O} \mapsto \hat{\mathcal{F}}(\mathcal{O})$ is isotonous, i.e. a net. This is of course the simplest instance of the situation discussed at the beginning of Sect. 4 where it was explained that there is an action of the dual group $\hat{G}$ on the extended net. We thus have an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the quasilocal algebra $\hat{\mathcal{F}}$ generated by $\alpha = Ad\, V$ and β,

$$\alpha(F + GV^{\mathcal{O}}) = F_+ - F_- + (G_+ - G_-)V^{\mathcal{O}}, \tag{5.2}$$
$$\beta(F + GV^{\mathcal{O}}) = F - GV^{\mathcal{O}}, \tag{5.3}$$

where $F, G \in \mathcal{F}$. We now define $\tilde{\mathcal{F}}(\mathcal{O})$ as the fixpoint algebra under the *diagonal* action $\alpha \circ \beta = \beta \circ \alpha$:

$$\tilde{\mathcal{F}}(\mathcal{O}) = \{x \in \hat{\mathcal{F}}(\mathcal{O}) \mid x = \alpha \circ \beta(x)\}. \tag{5.4}$$

Obviously $\tilde{\mathcal{F}}(\mathcal{O})$ can be represented as the following sum:

$$\tilde{\mathcal{F}}(\mathcal{O}) = \mathcal{F}(\mathcal{O})_+ + \mathcal{F}(\mathcal{O})_-\, V^{\mathcal{O}}. \tag{5.5}$$

It is instructive to compare $\tilde{\mathcal{F}}(\mathcal{O})$ with the twisted algebra

$$\mathcal{F}(\mathcal{O})^t = \mathcal{F}(\mathcal{O})_+ + \mathcal{F}(\mathcal{O})_-\, V, \tag{5.6}$$

the only difference being that in the former expression $V^{\mathcal{O}}$ appears instead of V. This reflects just the difference between Jordan-Wigner and Klein transformations. It is well known that the net $\mathcal{F}^t$ is local relative to $\mathcal{F}$. That the former cannot be local itself, however, follows clearly from the fact that it is unitarily equivalent to the latter by $\mathcal{F}(\mathcal{O})^t = Z\mathcal{F}(\mathcal{O})Z^*$.

Lemma 5.1. *Let W_L and W_R be left and right wedges, respectively. Then the wedge algebras of $\tilde{\mathcal{F}}$ are given by*

$$\tilde{\mathcal{F}}(W_L) = \mathcal{F}(W_L), \tag{5.7}$$
$$\tilde{\mathcal{F}}(W_R) = \mathcal{F}(W_R)^t. \tag{5.8}$$

Wedge duality holds for the net $\tilde{\mathcal{F}}$.

Proof. $V^{\mathcal{O}}$ is contained in $\mathcal{F}(W_L)_+$ for any $\mathcal{O} \subset W_L$. Thus, $\mathcal{F}(W_L)_- V^{\mathcal{O}} = \mathcal{F}(W_L)_-$, whence the first identity. Similarly we have $V_R^{\mathcal{O}} \in \mathcal{F}(W_R)_+$ for $\mathcal{O} \in W_R$, from which we obtain $\mathcal{F}(W_L)_- V^{\mathcal{O}} = \mathcal{F}(W_L)_- V$. Wedge duality for $\tilde{\mathcal{F}}$ now follows immediately from twisted duality for $\mathcal{F}$. □

Proposition 5.2. *The net $\mathcal{O} \mapsto \tilde{\mathcal{F}}(\mathcal{O})$ is local.*

Proof. Let $\mathcal{O}_1, \mathcal{O}_2$ be mutually spacelike double cones. We may assume $\mathcal{O}_1 < \mathcal{O}_2$ such that $W_L^{\mathcal{O}_1}$ and $W_R^{\mathcal{O}_2}$ are mutually spacelike. The commutativity of $\tilde{\mathcal{F}}(\mathcal{O}_1)$ and $\tilde{\mathcal{F}}(\mathcal{O}_1)$ follows from the preceding lemma and twisted locality for $\mathcal{F}$ since $\mathcal{O}_1 \subset W_L^{\mathcal{O}_1}$ and $\mathcal{O}_2 \subset W_R^{\mathcal{O}_2}$. □

Remark. A more intuitive proof goes as follows. Let $F_i \in \mathcal{F}(\mathcal{O}_i)_-, i = 1, 2$. Then commuting $F_1 V^{\mathcal{O}_1}$ through $F_2 V^{\mathcal{O}_2}$ gives exactly two factors of -1. The first arises from $F_1 F_2 = -F_2 F_1$ and the other from $V^{\mathcal{O}_2} F_1 = -F_1 V^{\mathcal{O}_2}$, whereas $V^{\mathcal{O}_1} F_2 = F_2 V^{\mathcal{O}_1}$.

Proposition 5.3. *The net $\tilde{\mathcal{F}}$ fulfills Haag duality for double cones.*

Proof. We have to prove $\tilde{\mathcal{F}}(\mathcal{O}) = \tilde{\mathcal{F}}(W_L^{\mathcal{O}}) \wedge \tilde{\mathcal{F}}(W_L^{\mathcal{O}})$. Using the lemma the right-hand side is seen to equal $\mathcal{F}(W_L^{\mathcal{O}}) \wedge \mathcal{F}(W_R^{\mathcal{O}})^t$ which by (2.20) is unitarily equivalent to $\mathcal{F}(W_R^{\mathcal{O}})^t \otimes \mathcal{F}(W_L^{\mathcal{O}})$. On the other hand (2.22) leads to

$$\begin{aligned}
\tilde{\mathcal{F}}(\mathcal{O}) &= \mathcal{F}(\mathcal{O})_+ + \mathcal{F}(\mathcal{O})_- V^{\mathcal{O}} \\
&\cong \mathcal{F}(W_R^{\mathcal{O}})_+ \otimes \mathcal{F}(W_L^{\mathcal{O}})_+ + \mathcal{F}(W_R^{\mathcal{O}})_- V \otimes \mathcal{F}(W_L^{\mathcal{O}})_- \\
&\quad + [\mathcal{F}(W_R^{\mathcal{O}})_- \otimes \mathcal{F}(W_L^{\mathcal{O}})_+ + \mathcal{F}(W_R^{\mathcal{O}})_+ V \otimes \mathcal{F}(W_L^{\mathcal{O}})_-]\, V \otimes \mathbf{1} \\
&= \mathcal{F}(W_R^{\mathcal{O}})^t \otimes \mathcal{F}(W_L^{\mathcal{O}}),
\end{aligned} \tag{5.9}$$

which completes the proof. □

It is obvious that the net $\tilde{\mathcal{F}}$ is Poincaré covariant with respect to the original representation of $\mathcal{P}$. Finally, the group G acts on $\tilde{\mathcal{F}}$ via the adjoint representation $g \mapsto Ad\, U(g)$. In particular $Ad\, U(k) = Ad\, V$ acts trivially on the first summand of the decomposition (5.5) and by multiplication with -1 on the second, i.e. the bosonized theory carries an action of $\mathbb{Z}_2$ in a natural way.

It should be clear that the same construction can be used to obtain a twisted dual fermionic net from a Haag dual bosonic net with a $\mathbb{Z}_2$ symmetry. It is not entirely trivial that these operations performed twice lead back to the net one started with, as the operators $V^{\mathcal{O}}$ constructed with the original and the bosonized net might differ. That this is not the case, however, can be derived from Lemma 5.1, the easy argument is left to the reader.

6. Conclusions and Outlook

In this final section we summarize our results and relate them to some of those in the literature. Starting from a local quantum field theory in 1 + 1 dimensions with an *unbroken* group symmetry we have discussed disorder operators which implement a global symmetry on some wedge region and commute with the operators localized in the spacelike complement of a somewhat larger wedge. Whereas disorder operators are only localized in wedge regions, they can in a natural way be associated to the bounded region where the interpolation between the global group action and the trivial action takes place. Extending the local algebras $\mathcal{F}(\mathcal{O})$ of the original theory by the disorder operators corresponding to the double cone $\mathcal{O}$ gives rise to a nonlocal net $\hat{\mathcal{F}}$ which is uniquely defined. We have shown that for every quantum field theory fulfilling a sufficiently strong version of the split property disorder operators exist and can be chosen such as to transform nicely under the global group action. As a consequence, the extended theory supports an action of the quantum double $D(G)$ which, however, is spontaneously broken in the sense that only the subalgebra $\mathbb{C}G$ is implemented by operators on the Hilbert space. Nevertheless, all other aspects of the quantum symmetry, like R-matrix commutation relations and the Verlinde algebra, show up and correspond nicely to the structures expected due to the general analysis [35, 58]. The spontaneous breakdown of the quantum symmetry is in accord with the findings of [50] where it was argued (in the case of a cyclic group $Z(N)$) that the vacuum expectation values of order and disorder variables can vanish jointly, as they must in the case of unbroken quantum symmetry, only if there is no mass gap. Massless theories are, however, ruled out by the postulate of the split property for wedges upon which our analysis hinges.

The fact that in the situation studied in this paper "one half" of the quantum double symmetry is spontaneously broken hints at an alternative construction which we describe briefly. Given a local net of C^*-algebras with group symmetry there may of course be vacuum states which are *not* gauge invariant. Let us assume that ω_e is such that $\omega_g = \omega_e \circ \alpha_g \neq \omega_e \, \forall g \neq e$, i.e. the symmetry is completely broken. One may now consider the reducible representation $\oplus_g \pi_g$ of $\mathcal{F}$ on the Hilbert space $\hat{\mathcal{H}} = L^2(G, \mathcal{H})$, where π_g is the GNS-representation corresponding to a soliton state which connects the vacua ω_e and ω_g. The existence of such states follows from the same set of assumptions as was used in the present investigation [63]. Again, one can construct operators $U^{\mathcal{O}}(g)$ enjoying similar algebraic properties as the disorder operators appearing in this paper. Their interpretation is different, however, in that they are true soliton operators intertwining the vacuum representation and the soliton sectors. Extending the local algebras according to (3.1) gives rise to a field net $\hat{\mathcal{F}}$ which acts irreducibly on $\hat{\mathcal{H}}$. The details of the construction outlined above, which is complementary in many ways to the one studied in the present work, will be given in a forthcoming publication. In the solitonic variant there is also an action of the quantum double $D(G)$, where the action of $C(G)$ is implemented in the obvious way, whereas the group symmetry is spontaneously broken.

Although the split property for wedges should be satisfied by reasonable massive quantum field theories it definitely excludes conformally invariant models, which via [19, 20] provided part of the motivation for the present investigation. Concerning this somewhat disturbing point we confine ourselves to the following remarks. It is well known that quantum field theories in 1 + 1 dimensions, like the $\mathcal{P}(\phi)_2$ models, possess a unique symmetric vacuum for some range of the parameters whereas spontaneous symmetry breakdown and vacuum degeneracy occur for other choices. The construction sketched above shows that the algebraic structure of order/disorder duality is the same in both massive regimes. It is furthermore known that the $\mathcal{P}(\phi)_2$ theory with interaction $\lambda\phi^4 - \delta\phi^2$ possesses a critical point at the interface between the symmetric and broken phases. Unfortunately, little is rigorously known about the possible conformal invariance of the theory at this point. The case of conformal invariance is quite different anyway, for Haag duality of the net $\mathcal{F}$ and of the fixpoint net $\mathcal{A}$ are compatible in contrast to the massive case.

In the framework of lattice models things are easier as the local degrees of freedom are amenable to more direct manipulation. The authors of [67] considered a class of models where the disorder as well as the order operators were explicitly defined by specifying their action on the Hilbert space associated to a finite region. They then had to assume the existence of a vacuum state which is invariant under the action of the quantum double. In his approach [4] to the XY-model Araki similarly defines an automorphism of the algebra of order variables which is localized in a halfspace and then constructs the crossed product. In the continuum solitonic automorphisms can be defined for some models [39], but for a model independent analysis there seems to be no alternative to our abstract approach.

As to the interpretation of the structures found in the present work and outlined above, we have already remarked that they may be considered as a local version of the construction of the quantum double. The quantum double was invented by Drinfel'd as a means to obtain quasitriangular Hopf algebras, and in [59] it was shown to be "factorizable," see Appendix A. Furthermore, every finite dimensional factorizable Hopf algebra can be obtained as a quotient of a quantum double by a two-sided ideal. One may therefore expect that quantum doubles will play an important role in an extension of the constructions in [28] to low dimensional theories.

A. Quantum Groups and Quantum Doubles

A Hopf algebra is an algebra H which at the same time is a coalgebra, i.e. there are homomorphisms $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbb{C}$ satisfying

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{A.1}$$

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}, \tag{A.2}$$

with the usual identification $H \otimes \mathbb{C} = \mathbb{C} \otimes H = H$. Furthermore, there is an antipode, i.e. an antihomomorphism $S : H \to H$ for which

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon(\cdot)\mathbf{1}, \tag{A.3}$$

where $m : H \otimes H \to H$ is the multiplication map of the algebra.

Remark. By (A.2) the counit, which is simply a one dimensional representation, is the "neutral element" with respect to the comultiplication.

For the quantum double $D(G)$ defined in Definition 4.1 these maps are given by

$$\varepsilon(V(g)U(h)) = \delta_{g,e}, \tag{A.4}$$

$$\Delta(V(g)U(h)) = \sum_k V(hk)U(h) \otimes V(k^{-1})U(h), \tag{A.5}$$

$$S(V(g)U(h)) = V(h^{-1}g^{-1}h)U(h^{-1}) \tag{A.6}$$

on the basis $\{V(g)U(h)\, g, h \in G\}$ and extended to $D(G)$ by linearity.

A Hopf algebra H is *quasitriangular*, or simply a quantum group, if there is an element $R \in H \otimes H$ satisfying

$$\Delta'(\cdot) = R\, \Delta(\cdot)\, R^{-1}, \tag{A.7}$$

where $\Delta' = \sigma \circ \Delta$ with $\sigma(a \otimes b) = b \otimes a$ and

$$(\Delta \otimes \mathrm{id})(R) = R_{13}\, R_{23}, \tag{A.8}$$

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}\, R_{12}. \tag{A.9}$$

Here $R_{12} = R \otimes \mathbf{1}, R_{23} = \mathbf{1} \otimes R$ and $R_{13} = (id \otimes \sigma)(R \otimes \mathbf{1})$. As a consequence, R satisfies the Yang-Baxter equation

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}. \tag{A.10}$$

It is easy to verify that the R-matrix (4.34) satisfies these requirements.

Remark. As shown by Drinfel'd, for quantum groups the square of the antipode is inner, i.e. $S^2(a) = uau^{-1}$, where u is given by $u = m \circ (S \otimes \mathrm{id}) \circ \sigma(R)$. The operator u satisfies $\varepsilon(u) = 1,\ \Delta(u) = (\sigma(R)R)^{-1}\,(u \otimes u) = (u \otimes u)(\sigma(R)R)^{-1}$. For quantum doubles of finite groups the antipode is even involutive ($S^2 = \mathrm{id}$, equivalently u is central). This holds for all finite dimensional Hopf-*****-algebras, whether quantum groups or not.

A quantum group is called *factorizable* [59] if the map $H^* \to H$ given by $H^* \ni x \mapsto \langle x \otimes id, I\rangle$ is nondegenerate, where I is as in (4.47). Quantum doubles are automatically factorizable.

A quasitriangular Hopf algebra possessing a (non-unique) central element v satisfying the conditions

$$v^2 = u\, S(u),\ \varepsilon(v) = 1,\ S(v) = v, \tag{A.11}$$

$$\Delta(v) = (\sigma(R)R)^{-1}\,(v \otimes v), \tag{A.12}$$

where u is the operator defined in the above remark, is called a *ribbon Hopf algebra* [60].

Finally, *modular Hopf algebras* are defined by some restrictions on their representation structure, the most important of which is the nondegeneracy of the matrix Y defined in (4.48). Obviously, the conditions of factorizability and modularity are strongly related.

B. Generalization to Continuous Groups

In this appendix we will generalize our considerations on quantum double actions to arbitrary locally compact groups (the quantum field theoretic framework gives rise only to compact groups.) In Sect. 4 we identified von Neumann algebras acted upon by the double $D(G)$ of a finite group with von Neumann algebras which are simultaneously graded by the group and automorphically acted upon by the latter, satisfying in addition the relation (4.8). The concept of group grading, however, loses its meaning for continuous groups. This problem is solved by appealing to the well known fact (see e.g. the introduction to [52]) that an algebra A (von Neumann or unital C^*) graded by a finite group G is the same as an algebra with a coaction of the group. A coaction is a homomorphism δ from A into $A \otimes \mathbb{C}G$ satisfying

$$(\delta \otimes id) \circ \delta = (id \otimes \delta_G) \circ \delta, \tag{B.1}$$

where $\delta_G : \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G$ is the coproduct given by $g \mapsto g \otimes g$. The correspondence between these notions is as follows. Given a G-graded algebra $A = \oplus_g A_g,\ A_g A_h \subset A_{gh}$ and defining $\delta(x) = x \otimes g$ for $x \in A_g$, one obtains a coaction. The converse is also true. The relation $\alpha_g(A_h) = A_{ghg^{-1}}$ between the group action and the grading obviously translates to

$$\delta \circ \alpha_g = (\alpha_g \otimes Ad\, g) \circ \delta. \tag{B.2}$$

The concept of coaction extends to continuous groups, where the group algebra $\mathbb{C}G$ is replaced by the von Neumann algebra $\mathcal{L}(G)$ (here we will treat only quantum double actions on von Neumann algebras) of the left regular representation which is generated by the operators $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$ on the Hilbert space $L^2(G)$.

In the next step we give a precise definition of the double of a continuous group. To this purpose we have to put a topology on the crossed product of some algebra of functions on the group by the adjoint action of the latter. There are many ways of doing this, as is generally the case with infinite dimensional vector spaces. For compact *Lie* groups two different constructions, one of which appears to generalize to arbitrary compact groups, have been given in [10]. The most important virtue of this work is that the topological Hopf algebras obtained there are reflexive as topological vector spaces, making the duality between $D(G)$ and $D(G)^*$ very explicit. From the technical point of view, however, the Fréchet topologies on which this approach relies are not very convenient.

In the following we will define the quantum double in the framework of Kac algebras [32, 33]. The latter has been invented as a generalization of locally compact groups which is closed under duality. As the C^* and von Neumann versions of Kac algebras have been proved [33] equivalent (generalizing the equivalence between locally compact groups and measurable groups) it is just a matter of convenience which formulation we use. We therefore consider first the von Neumann version which is technically easier.

We start with the von Neumann algebra $M = L^\infty(G)$ of essentially bounded measurable functions acting on the Hilbert space $H = L^2(G)$ by pointwise multiplication. With the coproduct $\Gamma(f)(g,h) = f(gh)$ and the involution $\kappa(f)(g) = f(g^{-1})$ it is a coinvolutive Hopf von Neumann algebra. This means Γ is a coassociative isomorphism of M into $M \otimes M$, κ is an anti-automorphism (complex linear, antimultiplicative and $\kappa(x^*) = \kappa(x)^*$) and $\Gamma \circ \kappa = \sigma \circ (\kappa \otimes \kappa) \circ \Gamma$ holds where σ is the flip. The weight φ, defined on M_+ by $\varphi(f) = \int_G dg\, f(g)$, is normal, faithful, semifinite (n.f.s.) and fulfills

1. For all $x \in M_+$ one has $(\imath \otimes \varphi)\Gamma(x) = \varphi(x)\mathbf{1}$.
2. For all $x, y \in \mathfrak{n}_\varphi$ one has $(\imath \otimes \varphi)((\mathbf{1} \otimes y^*)\Gamma(x)) = \kappa((\imath \otimes \varphi)(\Gamma(y^*)(\mathbf{1} \otimes x)))$.
3. $\kappa \circ \sigma_t^\varphi = \sigma_{-t}^\varphi \circ \kappa\ \forall t \in \mathbb{R}$.

This makes $(M, \Gamma, \kappa, \varphi)$ a Kac algebra in the sense of [32], well known as $KA(G)$. The dual Kac algebra [32] of $KA(G)$ is $KS(G) = (\mathcal{L}(G), \hat{\Gamma}, \hat{\kappa}, \hat{\varphi})$, the von Neumann algebra of the left regular representation equipped with the coproduct $\hat{\Gamma}(\lambda(g)) = \lambda(g) \otimes \lambda(g)$, the coinvolution $\hat{\kappa}(\lambda(g)) = \lambda(g^{-1})$ and the weight $\hat{\varphi}$ which we do not bother to state (see e.g. [47]).

Defining now an action of G on M by the automorphisms $\alpha_g(f)(h) = f(g^{-1}hg)$ it is trivial to check weak continuity with respect to g. Furthermore, α_g is unitarily implemented by $u_g = \lambda(g)\rho(g)$, where $(\rho(g)\xi)(h) = \Delta(g)^{1/2}\xi(hg)$ is the right regular representation. We can thus consider the crossed product (in the usual von Neumann algebraic sense [70]) $\tilde{M} = M \rtimes_\alpha G$ on $H \otimes L^2(G)$ $(= L^2(G) \otimes L^2(G))$, generated by $\pi(M)$ and $\lambda_1(g) = \mathbf{1}_M \otimes \lambda(g),\ g \in G$.

Proposition B.1. *There are mappings* $\tilde{\Gamma}, \tilde{\kappa}, \tilde{\varphi}$ *on* $\tilde{M}$ *such that the quadruple* $(\tilde{M}, \tilde{\Gamma}, \tilde{\kappa}, \tilde{\varphi})$ *is a Kac algebra, which we call the quantum double* $\mathcal{D}(G)$*. On the subalgebras* $\pi(M)$ *and* $\lambda_1(G)'' = \mathbf{1}_M \otimes \mathcal{L}(G)$ *the coproduct and the coinvolution act according to*

$$\tilde{\Gamma}(\pi(x)) = (\pi \otimes \pi)(\Gamma(x)),\ \ \tilde{\kappa}(\pi(x)) = \pi(\kappa(x)),\ \ x \in M, \tag{B.3}$$

$$\tilde{\Gamma}(\lambda_1(g)) = \lambda_1(g) \otimes \lambda_1(g),\ \tilde{\kappa}(\lambda_1(g)) = \lambda_1(g^{-1}),\ g \in G. \tag{B.4}$$

The Haar weight $\tilde{\varphi}$ is given by the dual weight [47]

$$\tilde{\varphi} = \varphi \circ \pi^{-1} \circ (\imath_{\tilde{M}} \otimes \hat{\varphi})(\tilde{\delta}(x)), \tag{B.5}$$

where $\tilde{\delta}$ is the dual coaction from $\tilde{M}$ to $\tilde{M} \otimes \mathcal{L}(G)$ which acts according to

$$\tilde{\delta}(\pi(x)) = \pi(x) \otimes \mathbf{1}_{\mathcal{L}(G)}, \quad x \in M, \tag{B.6}$$
$$\tilde{\delta}(\lambda_1(g)) = \lambda_1(g) \otimes \lambda(g), \quad g \in G. \tag{B.7}$$

Proof. The automorphisms α_g of M are easily shown to satisfy $\Gamma \circ \alpha_g = (\alpha_g \otimes \alpha_g) \circ \Gamma$ and $\kappa \circ \alpha_g = \alpha_g \circ \kappa$. (The first identity is just $g^{-1}(hk)g = (g^{-1}hg)(g^{-1}kg)$, the second $(g^{-1}hg)^{-1} = g^{-1}h^{-1}g$.) Thus $\alpha : G \to Aut\, M$ constitutes an action of G on the Kac algebra $(M, \Gamma, \kappa, \varphi)$ in the sense of [17]. We can now apply [17, Thm. 1] to conclude that there exist a coproduct, a coinvolution and a Haar weight on $\tilde{M}$ such that the axioms of a Kac algebra are satisfied. Equations (B.3, B.4) are restatements of [17, Props. 3.1, 3.3] whereas the Haar weight is as in [17, Déf. 1.9]. □

Proposition B.2. *The dual Kac algebra of the quantum double is $\widehat{\mathcal{D}(G)} = (\mathcal{L}(G) \otimes L^\infty(G), \hat{\tilde{\Gamma}}, \hat{\tilde{\kappa}}, \hat{\varphi} \otimes \varphi)$. The coproduct and the counit are*

$$\hat{\tilde{\Gamma}}(x) = R\,(\mathbf{1} \otimes \sigma \otimes \mathbf{1})\,(\hat{\Gamma} \otimes \Gamma)(x)\,(\mathbf{1} \otimes \sigma \otimes \mathbf{1})\,R^*, \tag{B.8}$$
$$\hat{\tilde{\kappa}}(x) = V^*\,(\hat{\kappa} \otimes \kappa)(x)\,V, \tag{B.9}$$

where R and V are given by

$$(R\xi)(g,h) = (u_h \otimes \mathbf{1})\,\xi(g,h), \tag{B.10}$$
$$(V\xi)(g) = u_g\,\xi(g). \tag{B.11}$$

Proof. This is just the specialization of [17, Thm. 2] to the situation at hand. According to this theorem the von Neumann algebra underlying the dual of the crossed product Kac algebra $K \rtimes_\alpha G$ is $\hat{M} \otimes L^\infty(G)$, where $\hat{M}$ is the von Neumann algebra of $\hat{K}$. In our case $M = L^\infty(G)$ such that $\hat{M} = \mathcal{L}(G)$. The formulae for $\hat{\tilde{\Gamma}}$ and $\hat{\tilde{\kappa}}$ are stated in [17, Prop. 4.10]. □

Remark. If the group G is not finite the quantum double is neither compact nor discrete, for the weights $\tilde{\varphi}, \hat{\tilde{\varphi}} = \hat{\varphi} \otimes \varphi$ are both infinite.

We are now in a position to define a coaction of the dual double $\widehat{\mathcal{D}(G)}$ on an algebra A, provided A supports an action α and a coaction δ satisfying (B.2) (with g replaced by $\lambda(g)$). In order to remove the apparent asymmetry between $\alpha : A \times G \to A$ and $\delta : A \to A \otimes \mathcal{L}(G)$ we write the former as the homomorphism $\alpha : A \to A \otimes L^\infty(G)$ which maps $x \in A$ into $g \mapsto \alpha_g(x) \in L^\infty(G, A)$. We now show that the maps α and δ can be put together to yield a coaction.

Definition B.3. *The map $\Delta : A \to A \otimes \mathcal{L}(G) \otimes L^\infty(G) = A \otimes \widehat{\mathcal{D}(G)}$ is defined by*

$$\Delta = (\imath_A \otimes \sigma) \circ (\alpha \otimes \imath_{\mathcal{L}(G)}) \circ \delta, \tag{B.12}$$

where $\sigma : x \otimes y \mapsto y \otimes x$ is the flip map from $L^\infty(G) \otimes \mathcal{L}(G)$ to $\mathcal{L}(G) \otimes L^\infty(G)$.

Theorem B.4. *The map* Δ *is a coaction of* $\widehat{\mathcal{D}(G)}$ *on* A*, i.e. it satisfies*

$$(\Delta \otimes \imath_{\hat{\mathcal{D}}}) \circ \Delta = (\imath_A \otimes \hat{\hat{\Gamma}}) \circ \Delta. \tag{B.13}$$

Proof. Appealing to the isomorphism $A \otimes L^\infty(G) \cong L^\infty(G, A)$ we identify $A \otimes \mathcal{L}(G) \otimes L^\infty(G) \otimes \mathcal{L}(G) \otimes L^\infty(G)$ with $L^\infty(G \times G, A \otimes \mathcal{L}(G) \otimes \mathcal{L}(G))$. We compute $(\Delta \otimes \imath) \circ \Delta(x)$ as follows (abbreviating $\imath_{\mathcal{L}(G)}$ by $\imath_{\mathcal{L}}$)

$$\begin{aligned}
((\Delta \otimes \imath) \circ \Delta(x))(g, h) &= (\alpha_g \otimes \imath_{\mathcal{L}} \otimes \imath_{\mathcal{L}}) \circ (\delta \otimes \imath_{\mathcal{L}}) \circ (\alpha_h \otimes \imath_{\mathcal{L}}) \circ \delta(x) \\
&= (\alpha_g \otimes \imath_{\mathcal{L}} \otimes \imath_{\mathcal{L}}) \circ (\alpha_h \otimes Ad\, \lambda_h \otimes \imath_{\mathcal{L}}) \circ (\delta \otimes \imath_{\mathcal{L}}) \circ \delta(x) \qquad \text{(B.14)} \\
&= (\imath_A \otimes Ad\, \lambda_h \otimes \imath_{\mathcal{L}}) \circ (\alpha_{gh} \otimes \hat{\Gamma}) \circ \delta(x).
\end{aligned}$$

The second equality follows from the connection (B.2) between the action α and the coaction δ whereas the third derives from the defining property (B.1) of the coaction $\hat{\Gamma}$. Now $(\alpha_{gh} \otimes \hat{\Gamma}) \circ \delta(x)$ is seen to be nothing but $[(\mathbf{1} \otimes \sigma \otimes \mathbf{1})\, (\hat{\Gamma} \otimes \Gamma)(x)\, (\mathbf{1} \otimes \sigma \otimes \mathbf{1})](g, h)$, and the adjoint action of R in (B.8) is seen to have the same effect as $Ad\, (\imath_A \otimes Ad\, \lambda_h \otimes \imath_{\mathcal{L}})$ due to $\rho(g) \in \mathcal{L}(G)'$. □

Proposition B.5. *The fixpoint algebra under the coaction* Δ*, defined as* $A^\Delta = \{x \in A \mid \Delta(x) = x \otimes \mathbf{1}_{\hat{\mathcal{D}}}\}$*, is given by*

$$A^\Delta = A^\alpha \cap A^\delta, \tag{B.15}$$

where $A^\alpha,\ A^\delta$ *are defined analogously.*

Proof. Obvious consequence of Definition B.3. □

The coaction of the dual double $\widehat{\mathcal{D}(G)}$ on A constructed above is exactly the kind of output the theory of depth-2 inclusions [55, 34] would give when applied to the inclusion $A^{\mathcal{D}(G)} \subset A$, which in the quantum field theoretical application corresponds to $\mathcal{A}(\mathcal{O}) \subset \hat{\mathcal{F}}(\mathcal{O})$. Nevertheless it is perhaps not exactly what one might have desired from a generalization of the results of Sect. 4 to compact groups. At least to a physicist, some kind of bilinear map $\gamma : A \times D(G) \to A$, as it was defined above for finite G, would seem more intuitive. This map should be well defined on the whole algebra A. Such a map can be constructed, provided the von Neumann double $\mathcal{D}(G)$ is replaced by its C^*-variant, which is uniquely defined by the above mentioned results [5, 33]. The details will be given in a subsequent publication.

The representation theory of the quantum double in the (locally) compact case was studied in a recent preprint [51] of which I became aware after completion of the present work. An application of the results expounded there in analogy to Sect. 4 should be possible but is deferred for reasons of space.

C. Chiral Theories on the Circle

For the foregoing analysis in this chapter the split property for wedges was absolutely crucial. While this property has been proved only for free massive fields it is expected to be true for all reasonable theories with a mass gap. For conformally invariant theories in 1 + 1 dimensions, however, it has no chance to hold. This is a consequence of the fact that two wedges $W_1 \subset W_2$ "touch at infinity." More precisely, there is an element of the conformal group transforming W_1, W_2 into double cones having a corner in common. For such regions there can be no interpolating type I factor, see e.g. [11]. On the other hand, for chiral theories on a circle, into which a 1+1 dimensional conformal theory should factorize, an appropriate kind of split property makes sense. For a general review of the framework, including a proof of the split property from the finiteness of the trace of $e^{-\tau L_0}$, we refer to [42]. We restrict ourselves to a concise statement of the axioms.

For every interval I on the circle such that $\overline{I} \neq S^1$, there is a von Neumann algebra $\mathfrak{A}(I)$ on the common Hilbert space $\mathcal{H}$. The assignment $I \mapsto \mathfrak{A}(I)$ fulfills isotony and locality:

$$I_1 \subset I_2 \Rightarrow \mathfrak{A}(I_1) \subset \mathfrak{A}(I_2), \tag{C.1}$$

$$I_1 \cap I_2 = \emptyset \Rightarrow \mathfrak{A}(I_1) \subset \mathfrak{A}(I_2)'. \tag{C.2}$$

Furthermore, there is a strongly continuous unitary representation of the Möbius group $SU(1,1)$ such that $\alpha_g(\mathfrak{A}(I)) = Ad\, U(g)(\mathfrak{A}(I)) = \mathfrak{A}(gI)$. Finally, the generator L_0 of the rotations is supposed to be positive and the existence of a unique invariant vector Ω is assumed.

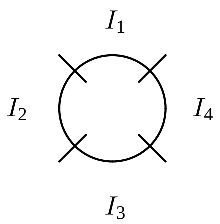
Starting from these assumptions one can prove, among other important results, that the local algebras $\mathfrak{A}(I)$ are factors of type III_1 for which the vacuum is cyclic and separating. Furthermore, Haag duality [42] is fulfilled automatically:

$$\mathfrak{A}(I)' = \mathfrak{A}(I'). \tag{C.3}$$

Given a chiral theory in its defining (vacuum) representation π_0 one may consider inequivalent representations. An important first result [13] states that all positive energy representations are locally equivalent to the vacuum representation, i.e. $\pi \restriction \mathfrak{A}(I) \cong \pi_0 \restriction \mathfrak{A}(I)\ \forall I$. This implies that all superselection sectors are of the DHR type and can be analyzed accordingly [37, 36]. As a means of studying the superselection theory of a model it has been proposed [64] to examine the inclusion

$$\mathfrak{A}(I_1) \vee \mathfrak{A}(I_3) \subset (\mathfrak{A}(I_2) \vee \mathfrak{A}(I_4))' = \mathfrak{A}(I_{341}) \wedge \mathfrak{A}(I_{123}), \tag{C.4}$$

where $I_{1,\ldots,4}$ are quadrants of the circle and $I_{ijk} = \overline{I_i \cup I_j \cup I_k}$:



(C.5)

At least for strongly additive theories, where $\mathfrak{A}(I_1) \vee \mathfrak{A}(I_2) = \mathfrak{A}(I)$ if $\overline{I_1 \cup I_2} = I$, the inclusion (C.4) is easily seen to be irreducible. In the presence of nontrivial superselection sectors this inclusion is strict as the intertwiners between endomorphisms localized in I_1, I_3, respectively, are contained in the larger algebra of (C.4) by Haag duality but not in the smaller one. Furthermore, for rational theories the inclusion (C.4) is expected to have finite index.

While we have nothing to add in the way of model independent analysis the techniques developed in the preceding sections can be applied to a large class of interesting models. These are chiral nets obtained as fixpoints of a larger one under the action of a group. I. e. we start with a net $I \mapsto \mathcal{F}(I)$ on the Hilbert space $\mathcal{H}$ fulfilling isotony and locality, the latter possibly twisted. The Möbius group $SU(1,1)$ and the group G of inner symmetries are unitarily represented with common invariant vector Ω. Again, the net $\mathcal{F}$ is supposed to fulfill the split property (with the obvious modifications due to the different geometry). The net $I \mapsto \mathfrak{A}(I)$ is now defined by $\mathcal{A}(I) = \mathcal{F}(I) \wedge U(G)'$ and $\mathfrak{A}(I) = \mathcal{A}(I) \upharpoonright \mathcal{H}_0$, where $\mathcal{H}_0$ is the space of G-invariant vectors. The proof of Haag duality for chiral theories referred to above applies also to the net $\mathfrak{A}$, implying that there is no analogue of the violation of duality for the fixpoint net as occurs in 1+1 dimensions. This is easily understood as a consequence of the fact that the spacelike complement of an interval is again an interval, thus connected. However, our methods can be used to study the inclusion (C.4).

It is clear that due to the split property

$$\mathfrak{A}(I_1) \vee \mathfrak{A}(I_3) \cong \mathcal{F}(I_1) \otimes \mathcal{F}(I_3)^{G\times G} \upharpoonright \mathcal{H}_0 \otimes \mathcal{H}_0. \tag{C.6}$$

Our aim will now be to compute $(\mathfrak{A}(I_2) \vee \mathfrak{A}(I_4))'$. In analogy to the 1+1 dimensional case we use the split property to construct unitaries $Y_1, \ldots, Y_4 : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ implementing the following isomorphisms:

$$Y_i \, F_i F_{i+2}^t \, Y_i^* = F_i \otimes F_{i+2}^t \;\; \forall F_i \in \mathcal{F}(I_i). \tag{C.7}$$

(One easily checks that $Y_{i+2} = T\, Y_i$, where $T\, x \otimes y = y \otimes x$.) These unitaries can in turn be used to define local implementers of the gauge transformations

$$U_i(g) = Y_i^* \, (U(g) \otimes \mathbf{1}) \, Y_i \tag{C.8}$$

with the localization $U_i(g) \in \mathcal{F}(I_{i+2})'$. (The index arithmetic takes place modulo 4.) These operators satisfy

$$Ad\, U_i(g) \upharpoonright \mathcal{F}(I_i) = \alpha_g, \tag{C.9}$$

$$[U_i(g), U_{i+2}(h)] = 0, \tag{C.10}$$

$$U_i(g)\, U_{i+2}(g) = U(g). \tag{C.11}$$

In a manner analogous to the proof of Lemma 3.9 one shows ($\mathcal{F}_i \equiv \mathcal{F}(I_i)$ etc.)

$$(\mathcal{A}_2 \vee \mathcal{A}_4)' = (\mathcal{F}_2 \vee \mathcal{F}_4)' \vee U_2(G)'' \vee U_4(G)''. \tag{C.12}$$

At this point we strengthen the property of Haag duality for the net $\mathcal{F}$ by requiring

$$(\mathcal{F}_1 \vee \mathcal{F}_3)' = (\mathcal{F}_2 \vee \mathcal{F}_4)^t, \tag{C.13}$$

which by the above considerations amounts to $\mathcal{F}$ having no nontrivial superselection sectors. This condition is fulfilled, e.g., by the CAR algebra on the circle which also possesses the split property. The chiral Ising model as discussed in [9] is covered by our general framework (with the group $\mathbb{Z}_2$).

While (C.13) is a strong restriction it is the same as in [19] where the larger theory was supposed to be "holomorphic." At this place it might be appropriate to emphasize that the requirement of (twisted) Haag duality (1.11) made above when considering 1+1 dimensional theories by no means excludes nontrivial superselection sectors.

Making use of (C.13) we can now state quite explicitly how $(\mathfrak{A}_2 \vee \mathfrak{A}_4)'$ looks. In analogy to Thm. 3.10 we obtain

$$(\mathfrak{A}_2 \vee \mathfrak{A}_4)' = m\left(\mathcal{F}_1 \vee \mathcal{F}_3 \vee U_2(G)''\right) \upharpoonright \mathcal{H}_0. \tag{C.14}$$

Again, using (C.13) one can check that $\alpha_2(g) = Ad\, U_2(g)$ restrict to automorphisms of $\mathcal{F}_1 \vee \mathcal{F}_3$ rendering the algebra $\mathcal{F}_1 \vee \mathcal{F}_3 \vee U_2(G)''$ a crossed product. Recalling

$$\mathfrak{A}_1 \vee \mathfrak{A}_3 = m(\mathcal{F}_1) \vee m(\mathcal{F}_3) \restriction \mathcal{H}_0, \tag{C.15}$$

we have the following natural sequence of inclusions:

$$\mathfrak{A}_1 \vee \mathfrak{A}_3 \subset m(\mathcal{F}_1 \vee \mathcal{F}_3) \restriction \mathcal{H}_0 \subset (\mathfrak{A}_2 \vee \mathfrak{A}_4)', \tag{C.16}$$

both of which have index $|G|$. It is interesting to remark that the intermediate algebra $m(\mathcal{F}_1 \vee \mathcal{F}_3) \restriction \mathcal{H}_0$ equals $(m(\mathcal{F}_2 \vee \mathcal{F}_4) \restriction \mathcal{H}_0)'$. For general chiral theories the existence of such an intermediate subfactor between $\mathfrak{A}_1 \vee \mathfrak{A}_3$ and $(\mathfrak{A}_2 \vee \mathfrak{A}_4)'$ is not known. In the case of G being abelian where the $U_i(g)$ are invariant under global gauge transformations we obtain a square structure similar to the one encountered in Sect. 3.:

$$\begin{array}{ccc} \mathfrak{A}_1 \vee \mathfrak{A}_3 \vee U_2(G)'' & \subset & (\mathfrak{A}_2 \vee \mathfrak{A}_4)' \\ \cup & & \cup \\ \mathfrak{A}_1 \vee \mathfrak{A}_3 & \subset & m(\mathcal{F}_1 \vee \mathcal{F}_3) \restriction \mathcal{H}_0. \end{array} \tag{C.17}$$

It may be instructive to compare the above result with the situation prevailing in $2+1$ or more dimensions. There, as already mentioned in the introduction, the superselection theory for localized charges is isomorphic to the representation theory of a (unique) compact group. Furthermore, there is a net of field algebras acted upon by this group, such that the observables arise as the fixpoints. The analogue of the inclusion (C.4) then is

$$\mathfrak{A}(\mathcal{O}_1) \vee \mathfrak{A}(\mathcal{O}_2) \subset \mathfrak{A}(\mathcal{O}_1' \cap \mathcal{O}_2')', \tag{C.18}$$

where $\mathcal{O}_1, \mathcal{O}_2$ are spacelike separated double cones. Under natural assumptions it can be shown that the larger algebra equals $m(\mathcal{F}(\mathcal{O}_1) \vee \mathcal{F}(\mathcal{O}_2)) \restriction \mathcal{H}_0$, implying that the inclusion (C.18) is of the type $(\mathcal{F}_1 \otimes \mathcal{F}_2)^{G\times G} \subset (\mathcal{F}_1 \otimes \mathcal{F}_2)^{\mathrm{Diag}(G)}$ just as the first one in (C.16). That the index of the inclusion (C.4) is $|G|^2$ instead of $|G|$ as for (C.18) is a consequence of the low dimensional topology comparable to the phenomena occurring in 1 + 1 dimensions.

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Note added in proof. In Appendix C we claimed that the combination of Haag duality and the split property for wedges is weaker than the requirement of absence of charged sectors which was made in [19] where conformal orbifold theories were considered. After submission of this paper we discovered that this claim is wrong! While this does not affect any result of the present work it shows that the analysis of massive models based on the former assumptions is even stronger related to the one in [19] than expected. Furthermore, if the vacuum sector satisfies HD+SPW then Haag duality holds in all irreducible locally normal representations. In particular, on can replace "simple sector" by "irreducible $\mathcal{A}$-stable subspace of $\mathcal{H}$" in Thm. 3.10. The proofs as well as applications to the theory of quantum solitons will be found in [56].

I thank Prof. B. Schroer for drawing my attention to [72], where massive quantum field theories in 1 + 1 have been considered. In particular it has been shown that the statistics of charged fields is arbitrary in the sense that the same particle states can be created by Bose and by Fermi fields. Even though scattering theory aspects have not been discussed in the present work, the cited result fits well with that of our Sect. 5.

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