SUPERSELECTION STRUCTURE OF MASSIVE QUANTUM FIELD THEORIES IN 1+1 DIMENSIONS

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We show that a large class of massive quantum field theories in $1 + 1$ dimensions, characterized by Haag duality and the split property for wedges, does not admit locally generated superselection sectors in the sense of Doplicher, Haag and Roberts. Thereby the extension of DHR theory to $1 + 1$ dimensions due to Fredenhagen, Rehren and Schroer is vacuous for such theories. Even charged representations which are localizable only in wedge regions are ruled out. Furthermore, Haag duality holds in all locally normal representations. These results are applied to the theory of soliton sectors. Furthermore, the extension of localized representations of a non-Haag dual net to the dual net is reconsidered. It must be emphasized that these statements do not apply to massless theories since they do not satisfy the above split property. In particular, it is known that positive energy representations of conformally invariant theories are DHR representations.

1. Introduction

It is well known that the superselection structure, i.e. the structure of physically relevant representations or "charges", of quantum field theories in low dimensional spacetimes gives rise to particle statistics governed by the braid group and is described by "quantum symmetries" which are still insufficiently understood. The meaning of "low dimensional" in this context depends on the localization properties of the charges under consideration. In the framework of algebraic quantum field theory [**?**, **?**] several selection criteria for physical representations of the observable algebra have been investigated. During their study of physical observables obtained from a field theory by retaining only the operators invariant under the action of a gauge group (of the first kind), Doplicher, Haag and Roberts were led to singling out the class of locally generated superselection sectors. A representation is of this type if it becomes unitarily equivalent to the vacuum representation when restricted to the observables localized in the spacelike complement of an arbitrary double cone (intersection of future and past directed light cones):

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$$\pi \restriction \mathcal{A}(\mathcal{O}') \cong \pi_0 \restriction \mathcal{A}(\mathcal{O}') \quad \forall\, \mathcal{O} \in \mathcal{K}\,. \tag{1.1}$$

Denoting the set of all double cones by $\mathcal{K}$ we consider a quantum field theory to be defined by its net of observables $\mathcal{K} \ni \mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$. This is a map which assigns to each double cone a C^*-algebra $\mathcal{A}(\mathcal{O})$ satisfying isotony:

$$\mathcal{O}_1 \subset \mathcal{O}_2 \;\Rightarrow\; \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)\,. \tag{1.2}$$

This net property allows the quasilocal algebra to be defined by

$$\mathcal{A} = \overline{\bigcup_{\mathcal{O}\in\mathcal{K}} \mathcal{A}(\mathcal{O})}^{\|\cdot\|}\,. \tag{1.3}$$

The net is local in the sense that

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = \{0\} \tag{1.4}$$

if $\mathcal{O}_1$, $\mathcal{O}_2$ are spacelike to each other. The algebra $\mathcal{A}(G)$ associated with an arbitrary subset of Minkowski space is understood to be the subalgebra of $\mathcal{A}$ generated (as a C^*-algebra) by all $\mathcal{A}(\mathcal{O})$ where $G \supset \mathcal{O} \in \mathcal{K}$. Furthermore, the Poincaré group acts on $\mathcal{A}$ by automorphisms $\alpha_{\Lambda,x}$ such that

$$\alpha_{\Lambda,x}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O} + x) \quad \forall\, \mathcal{O}\,. \tag{1.5}$$

This abstract approach is particularly useful if there is more than one vacuum.

One requires of a physically reasonable representation that at least the translations (Lorentz invariance might be broken) are unitarily implemented:

$$\pi \circ \alpha_x(A) = U_\pi(x)\pi(A)U_\pi(x)^*\,, \tag{1.6}$$

the generators of the representation $x \mapsto U(x)$, i.e. the energy-momentum operators, satisfying the spectrum condition (positivity of the energy). Vacuum representations are characterized by the existence of a unique (up to a phase) Poincaré invariant vector. Furthermore we assume them to be irreducible and to satisfy the Reeh–Schlieder property, the latter following from the other assumptions if weak additivity is assumed. In the analysis of superselection sectors satisfying (**??**) relative to a fixed vacuum representation one usually assumes the latter to satisfy Haag duality[a]

$$\pi_0(\mathcal{A}(\mathcal{O}))' = \pi_0(\mathcal{A}(\mathcal{O}'))'' \quad \forall\, \mathcal{O} \in \mathcal{K}\,, \tag{1.7}$$

which may be interpreted as a condition of maximality for the local algebras. In [**?**, **?**], based on (**??**), a thorough analysis of the structure of representations satisfying (**??**) was given, showing that the category of these representations

[a] $\mathcal{M}' = \{X \in \mathcal{B}(\mathcal{H}) | XY = YX\, \forall\, Y \in \mathcal{M}\}$ denotes the algebra of all bounded operators commuting with all operators in $\mathcal{M}$. If $\mathcal{M}$ is a unital $*$-algebra then $\mathcal{M}''$ is known to be the weak closure of a $\mathcal{M}$.

together with their intertwiners is monoidal (i.e. there is a product or, according to current fashion, fusion structure), rigid (i.e. there are conjugates) and permutation symmetric. In particular, the Bose–Fermi alternative, possibly with parastatistics, came out automatically although the analysis started from observable, i.e. strictly local, quantities. A lot more is known in this situation (cf. [**?**]) but we will not need that. A substantial part of this analysis, in particular concerning permutation statistics and the Bose–Fermi alternative, is true only in at least $2+1$ spacetime dimensions. The generalization to $1+1$ dimensions, where in general only braid group statistics obtains, was given in [**?**] and applied to conformally invariant theories in [**?**]. Whereas for the latter theories all positive energy representations are of the DHR type [**?**], it has been clear from the beginning that the criterion (**??**) cannot hold for charged sectors in gauge theories due to Gauss' law.

Implementing a programme initiated by Borchers, Buchholz and Fredenhagen proved [**?**] for every massive one-particle representation (where there is a mass gap in the spectrum followed by an isolated one-particle hyperboloid) the existence of a vacuum representation π_0 such that

$$\pi \upharpoonright \mathcal{A}(\mathcal{C}') \cong \pi_0 \upharpoonright \mathcal{A}(\mathcal{C}') \quad \forall\, \mathcal{C}\,. \tag{1.8}$$

Here the $\mathcal{C}$'s are spacelike cones which we do not need to define precisely. In $\geq 3+1$ dimensional spacetime the subsequent analysis leads to essentially the same structural results as the original DHR theory. Due to the weaker localization properties, however, the transition to braid group statistics and the loss of group symmetry occur already in $2+1$ dimensions, see [**?**]. In the $1+1$ dimensional situation with which we are concerned here, spacelike cones reduce to wedges (i.e. translates of $W_R = \{x \in \mathbb{R}^2 \mid x^1 \geq |x^0|\}$ and the spacelike complement $W_L = W_R'$). Furthermore, the arguments in [**?**] allow us only to conclude the existence of two *a priori* different vacuum representations π_0^L, π_0^R such that the restriction of π to left handed wedges (translates of W_L) is equivalent to π_0^L and similarly for the right handed ones. As for such representations, of course long well-known as soliton sectors, an operation of composition can only be defined if the "vacua fit together" [**?**], there is in general no such thing as permutation or braid group statistics. For lack of a better name soliton representations with coinciding left and right vacuum, i.e. representations which are localizable in wedges, will be called "wedge representations (or sectors)".

There have long been indications that the DHR criterion might not be applicable to massive $2d$-theories as it stands. The first of these was the fact, known for some time, that the fixpoint nets of Haag-dual field nets with respect to the action of a global gauge group do *not* satisfy duality even in simple sectors, whereas this is true in $\geq 2+1$ dimensions. This phenomenon has been analyzed thoroughly in [**?**] under the additional assumption that the fields satisfy the split property for wedges. This property, which is expected to be satisfied in all massive quantum field theories, plays an important role also in the present work which we summarize briefly.

In the next section we will prove some elementary consequences of Haag duality and the split property for wedges (SPW), in particular strong additivity and the time-slice property. The significance of our assumptions for superselection theory derives mainly from the fact that they preclude the existence of locally generated superselection sectors. More precisely, if the vacuum representation satisfies Haag duality and the SPW then every irreducible DHR representation is unitarily equivalent to the vacuum representation. This important and perhaps surprising result, to be proved in Sec. 3, indicates that the innocent-looking assumptions of the DHR framework are quite restrictive when they are combined with the split property for wedges. Although this may appear reasonable in view of the non-connectedness of $\mathcal{O}'$, our result also applies to the wedge representations which are only localizable in wedges provided left and right handed wedges are admitted. In Sec. 4 we will prove the minimality of the relative commutant for an inclusion of double cone algebras which, via a result of Driessler, implies Haag duality in all locally normal irreducible representations. In Sec. 5 the facts gathered in the preceding sections will be applied to the theory of quantum solitons thereby concluding our discussion of the representation theory of Haag-dual nets. Summing up the results obtained so far, the representation theory of such nets is essentially trivial. On the other hand, dispensing completely with a general theory of superselection sectors including composition of charges, braid statistics and quantum symmetry for massive theories is certainly not warranted in view of the host of more or less explicitly analyzed models exhibiting these phenomena. The only way to accommodate these models seems to be to relax the duality requirement by postulating only wedge duality. In Sec. 6 Roberts' extension of localized representations to the dual net will be reconsidered and applied to the theories considered already in [**?**], namely fixpoint nets under an unbroken inner symmetry group. In this work we will not attempt to say anything concerning the quantum symmetry question.

2. Strong Additivity and the Time-Slice Axiom

Until further notice we fix a vacuum representation π_0 (which is always faithful) on a separable Hilbert space $\mathcal{H}_0$ and omit the symbol $\pi_0(\cdot)$, identifying $\mathcal{A}(\mathcal{O}) \equiv \pi_0(\mathcal{A}(\mathcal{O}))$. Whereas we may assume the algebras $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ to be weakly closed, for more complicated regions X, in particular infinite ones like $\mathcal{O}'$, we carefully distinguish between the C^*-subalgebra

$$\mathcal{A}(X) \equiv \overline{\bigcup_{\mathcal{O}\in\mathcal{K},\mathcal{O}\subset X} \mathcal{A}(\mathcal{O})}^{\|\cdot\|} \tag{2.1}$$

of $\mathcal{A} \equiv \pi_0(\mathcal{A})$ and its ultraweak closure $\mathcal{R}(X) = \mathcal{A}(X)''$.

Definition 2.1. An inclusion $A \subset B$ of von Neumann algebras is standard [**?**] if there is a vector Ω which is cyclic and separating for A, B, $A' \wedge B$.

Due to the Reeh–Schlieder property, the inclusion $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ ($\mathcal{R}(W_1) \subset \mathcal{R}(W_2)$) is standard whenever $\mathcal{O}_1 \subset\subset \mathcal{O}_2$ ($W_1 \subset\subset W_2$), i.e. the closure of $\mathcal{O}_1$ is contained in the interior of $\mathcal{O}_2$. ($W_1 \subset\subset W_2$ is equivalent to the existence of a double cone $\mathcal{O}$ such that $W_1 \cup W_2' = \mathcal{O}'$.)

Definition 2.2. An inclusion $A \subset B$ of von Neumann algebras is split [**?**], if there exists a type-I factor N such that $A \subset N \subset B$. A net of algebras satisfies the split property (for double cones) [**?**] if the inclusion $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ is split whenever $\mathcal{O}_1 \subset\subset \mathcal{O}_2$.

The importance of these definitions derives from the following result [**?**, **?**]:

Lemma 2.3. *Let $A \subset B$ be a standard inclusion. Then the following are equivalent*:

(i) *The inclusion $A \subset B$ is split.*
(ii) *The is a unitary Y such that $Y ab' Y^* = a \otimes b'$, $a \in A, b' \in B'$.*

Remarks. 1. The implication (ii)⇒(i) is trivial, an interpolating type-I factor being given by $N = Y^*(\mathcal{B}(\mathcal{H}_0) \otimes \mathbf{1})Y$.

2. The natural spatial isomorphism $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)' \cong \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)'$ implied by the split property whenever $\mathcal{O}_1 \subset\subset \mathcal{O}_2$ clearly restricts to

$$\mathcal{A}(\mathcal{O}_1) \vee \mathcal{R}(\mathcal{O}_2') \cong \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{R}(\mathcal{O}_2') . \tag{2.2}$$

As an important consequence, every pair of normal states $\phi_1 \in \mathcal{A}(\mathcal{O}_1)_*$, $\phi_2 \in \mathcal{R}(\mathcal{O}_2')_*$ extends to a normal state $\phi \in (\mathcal{A}(\mathcal{O}_1) \vee \mathcal{R}(\mathcal{O}_2'))_*$. Physically this amounts to a form of statistical independence between the regions $\mathcal{O}_1$ and $\mathcal{O}_2'$.

3. We emphasize that in the case where Haag duality fails ($\mathcal{A}(\mathcal{O}) \subsetneq \mathcal{A}(\mathcal{O}')'$), requiring (**??**) whenever $\mathcal{O}_1 \subset\subset \mathcal{O}_2$ defines a weaker notion of split property since one can conclude only the existence of a type-I factor N such that $\mathcal{A}(\mathcal{O}_1) \subset N \subset \mathcal{A}(\mathcal{O}_2')' = \mathcal{A}^d(\mathcal{O}_2)$.

In $1+1$ dimensions (and only there, cf. [**?**, p. 292]) the split property may be strengthened by extending it to wedge regions. In this paper we will examine the implications of the split property for wedges (SPW). The power of this assumption in combination with Haag duality derives from the fact that one obtains strong results on the relation between the algebras of double cones and of wedges. Some of these have already been explored in [**?**], where, e.g., it has been shown that the local algebras associated with double cones are factors. We recall some terminology introduced in [**?**]: the left and right spacelike complements of $\mathcal{O}$ are denoted by $W_{LL}^{\mathcal{O}}$ and $W_{RR}^{\mathcal{O}}$, respectively. Furthermore, defining $W_L^{\mathcal{O}} = W_{RR}^{\mathcal{O}}{}'$ and $W_R^{\mathcal{O}} = W_{LL}^{\mathcal{O}}{}'$ we have $\mathcal{O} = W_L^{\mathcal{O}} \cap W_R^{\mathcal{O}}$.

Before we turn to the main subject of this section, we remark on the relation between the two notions of Haag duality which are of relevance for this paper. In [**?**], as apparently in a large part of the literature, it was implicitly assumed that Haag duality for double cones implies duality for wedges, i.e.

$$\mathcal{R}(W)' = \mathcal{R}(W') \quad \forall\, W \in \mathcal{W} , \tag{2.3}$$

where $\mathcal{W}$ is the set of all wedge regions. Whereas there seems to be no general proof of this claim, for theories in $1+1$ dimensions satisfying the SPW we can give a straightforward argument, thereby also closing the gap in [**?**]. In view of Remark 3 after Lemma **??** the following definition of the split property for wedges is slightly weaker than the obvious modification of Definition **??**, but seems more natural from a physical point of view (cf. Remark 2):

Definition 2.4. A net of algebras satisfies the split property for wedges if the map $x \otimes y \mapsto xy$, $x \in \mathcal{R}(W_1)$, $y \in \mathcal{R}(W_2)$ extends to an isomorphism between $\mathcal{R}(W_1) \otimes \mathcal{R}(W_2)$ and $\mathcal{R}(W_1) \vee \mathcal{R}(W_2)$ whenever $W_1 \subset\subset W_2'$. By standardness this isomorphism is automatically spatial in the sense of Lemma **??** (ii). In the case where $W_1 = W_{LL}^{\mathcal{O}}$, $W_2 = W_{RR}^{\mathcal{O}}$ the canonical implementer [**?**] will be denoted $Y^{\mathcal{O}}$.

Proposition 2.5. *Let* $\mathcal{A}(\mathcal{O})$ *be a net of local algebras in* $1+1$ *dimensions, satisfying Haag duality* (*for double cones*) *and the SPW. Then* $\mathcal{A}$ *satisfies wedge duality and the inclusion* $\mathcal{R}(W_1) \subset \mathcal{R}(W_2)$ *is split whenever* $W_1 \subset\subset W_2$.

Proof. Appealing to the definition (**??**), duality for double cones is clearly equivalent to

$$\mathcal{A}(\mathcal{O})' = \mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{R}(W_{RR}^{\mathcal{O}}) \quad \forall\, \mathcal{O} \in \mathcal{K}\,. \tag{2.4}$$

Given a right wedge W, let $\mathcal{O}_i$, $i \in \mathbb{N}$ be an increasing sequence of double cones all of which have the same left corner as W and satisfying $\cup_i \mathcal{O}_i = W$. Then we clearly have $\mathcal{R}(W) = \bigvee_i \mathcal{A}(\mathcal{O}_i)$ and

$$\mathcal{R}(W)' = \bigwedge_i \mathcal{A}(\mathcal{O}_i)' = \bigwedge_i \left(\mathcal{R}(W') \vee \mathcal{R}(W_{RR}^{\mathcal{O}_i}) \right). \tag{2.5}$$

Using the unitary equivalence $Y^{\mathcal{O}_1}\, \mathcal{R}(W') \vee \mathcal{R}(W_{RR}^{\mathcal{O}_1})\, Y^{\mathcal{O}_1 *} = \mathcal{R}(W') \otimes \mathcal{R}(W_{RR}^{\mathcal{O}_1})$, the right-hand side of (**??**) is equivalent to

$$\bigwedge_i \left(\mathcal{R}(W') \otimes \mathcal{R}(W_R^{\mathcal{O}_i}) \right) = \mathcal{R}(W') \bigotimes \bigwedge_i \mathcal{R}(W_R^{\mathcal{O}_i}) = \mathcal{R}(W') \otimes \mathbb{C}\mathbf{1}\,, \tag{2.6}$$

where we have used the consequence $\wedge_i \mathcal{R}(W_R^{\mathcal{O}_i}) = \mathbb{C}\mathbf{1}$ of irreducibility. This clearly proves $\mathcal{R}(W)' = \mathcal{R}(W')$. The final claim follows from Lemma **??**. □

Now we are prepared for the discussion of additivity properties, starting with the easy

Lemma 2.6.

$$\mathcal{R}(W_{LL}^{\mathcal{O}}) \vee \mathcal{A}(\mathcal{O}) = \mathcal{R}(W_L^{\mathcal{O}})\,, \tag{2.7}$$

$$\mathcal{R}(W_{RR}^{\mathcal{O}}) \vee \mathcal{A}(\mathcal{O}) = \mathcal{R}(W_R^{\mathcal{O}})\,. \tag{2.8}$$

Remark. Equivalently, the inclusions $\mathcal{R}(W_{LL}^{\mathcal{O}}) \subset \mathcal{R}(W_L^{\mathcal{O}})$, $\mathcal{R}(W_{RR}^{\mathcal{O}}) \subset \mathcal{R}(W_R^{\mathcal{O}})$ are normal.

Proof. Under the unitary equivalence $\mathcal{R}(W^{\mathcal{O}}_{LL})\vee\mathcal{R}(W^{\mathcal{O}}_{RR}) \cong \mathcal{R}(W^{\mathcal{O}}_{LL})\otimes\mathcal{R}(W^{\mathcal{O}}_{RR})$ we have $\mathcal{R}(W^{\mathcal{O}}_{LL}) \cong \mathcal{R}(W^{\mathcal{O}}_{LL}) \otimes \mathbf{1}$ and $\mathcal{A}(\mathcal{O}) = \mathcal{R}(W^{\mathcal{O}}_{L}) \cap \mathcal{R}(W^{\mathcal{O}}_{R}) \cong \mathcal{R}(W^{\mathcal{O}}_{R}) \otimes \mathcal{R}(W^{\mathcal{O}}_{L})$. Thus $\mathcal{R}(W^{\mathcal{O}}_{LL}) \vee \mathcal{A}(\mathcal{O}) \cong (\mathcal{R}(W^{\mathcal{O}}_{LL}) \vee \mathcal{R}(W^{\mathcal{O}}_{R})) \otimes \mathcal{R}(W^{\mathcal{O}}_{L})$. Due to wedge duality and factoriality of the wedge algebras this equals $\mathcal{B}(\mathcal{H}_0)\otimes\mathcal{R}(W^{\mathcal{O}}_{L}) \cong \mathcal{R}(W^{\mathcal{O}}_{L})$. We emphasize that all above equivalences are established by the same unitary transformation. The second equation is proved in the same way. □

Remark. The proof of factoriality of wedge algebras in [**?**] relies, besides the usual net properties, on the spectrum condition and on the Reeh–Schlieder theorem. This is the only place where positivity of the energy and weak additivity enter into our analysis.

Consider now the situation depicted in Fig. 1. In particular, $\mathcal{O}$, $\tilde{\mathcal{O}}$ are spacelike separated double cones the closures of which share one point. Such double cones will be called *adjacent*.

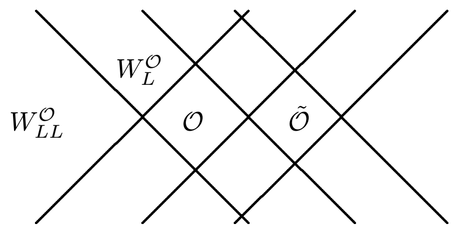

Fig. 1. Double cones sharing one point.

Lemma 2.7. *Let* $\hat{\mathcal{O}} = \sup(\mathcal{O}, \tilde{\mathcal{O}})$ *be the smallest double cone containing* $\mathcal{O}, \tilde{\mathcal{O}}$. *Then*

$$\mathcal{A}(\mathcal{O}) \vee \mathcal{A}(\tilde{\mathcal{O}}) = \mathcal{A}(\hat{\mathcal{O}})\,. \tag{2.9}$$

Proof. In the situation of Fig. 1 we have $\hat{\mathcal{O}} = W^{\mathcal{O}}_{R} \cap W^{\tilde{\mathcal{O}}}_{L}$. Under the unitary equivalence considered above we have $\mathcal{A}(\tilde{\mathcal{O}}) \cong \mathbf{1} \otimes \mathcal{A}(\tilde{\mathcal{O}})$ since $\tilde{\mathcal{O}} \subset W^{\mathcal{O}}_{RR}$. Thus $\mathcal{A}(\mathcal{O})\vee\mathcal{A}(\tilde{\mathcal{O}}) \cong \mathcal{R}(W^{\mathcal{O}}_{R})\otimes(\mathcal{R}(W^{\mathcal{O}}_{L})\vee\mathcal{A}(\tilde{\mathcal{O}}))$. But now $W^{\mathcal{O}}_{L} = W^{\tilde{\mathcal{O}}}_{LL}$ leads to $\mathcal{R}(W^{\mathcal{O}}_{L})\vee \mathcal{A}(\tilde{\mathcal{O}}) = \mathcal{R}(W^{\tilde{\mathcal{O}}}_{L})$ via the preceding lemma. Thus $\mathcal{A}(\mathcal{O})\vee\mathcal{A}(\tilde{\mathcal{O}}) \cong \mathcal{R}(W^{\mathcal{O}}_{R})\otimes\mathcal{R}(W^{\tilde{\mathcal{O}}}_{L})$ which in turn is unitarily equivalent to $\mathcal{R}(W^{\mathcal{O}}_{R}) \wedge \mathcal{R}(W^{\tilde{\mathcal{O}}}_{L}) = \mathcal{A}(\hat{\mathcal{O}})$. □

Remark. In analogy to chiral conformal field theory we denote this property *strong additivity*.

With these lemmas it is clear that the quantum field theories under consideration are *n-regular* in the sense of the following definition for all $n \geq 2$.

Definition 2.8. A quantum field theory is n-regular if

$$\mathcal{R}(W_1) \vee \mathcal{A}(\mathcal{O}_1) \vee \cdots \vee \mathcal{A}(\mathcal{O}_{n-2}) \vee \mathcal{R}(W_2) = \mathcal{B}(\mathcal{H}_0)\,, \tag{2.10}$$

whenever $\mathcal{O}_i,\ i = 1, \ldots, n-2$ are mutually spacelike double cones such that the sets $\overline{\mathcal{O}_i} \cap \overline{\mathcal{O}_{i+1}},\ i = 1, \ldots, n-3$ each contain one point and where the wedges W_1, W_2 are such that

$$W_1 \cup W_2 = \left(\bigcup_{i=1}^{n-2} \mathcal{O}_i \right)' . \tag{2.11}$$

Corollary 2.9. *A quantum field theory in* $1+1$ *dimensions satisfying Haag duality and the SPW fulfills the* (*von Neumann version of the*) *time-slice axiom*, *i.e.*

$$\mathcal{R}(S) = \mathcal{B}(\mathcal{H}_0)\,, \tag{2.12}$$

whenever $S = \{x \in \mathbb{R}^2 \,|\, x \cdot \eta \in (a, b)\}$ *where* $\eta \in \mathbb{R}^2$ *is timelike and* $a < b$.

Proof. The time-slice S contains an infinite string $\mathcal{O}_i,\ i \in \mathbb{Z}$ of mutually spacelike double cones as above. Thus the von Neumann algebra generated by all these double cones contains each $\mathcal{A}(\mathcal{O}),\ \mathcal{O} \in \mathcal{K}$ from which the claim follows by irreducibility. □

Remarks. 1. We wish to emphasize that this statement on von Neumann algebras is weaker than the C^*-version of the time-slice axiom, which postulates that the C^*-algebra $\mathcal{A}(S)$ generated by the algebras $\mathcal{A}(\mathcal{O}),\ \mathcal{O} \subset S$ equals the quasilocal algebra $\mathcal{A}$. We follow the arguments in [**?**, Sec. III.3] to the effect that this stronger assumption should be avoided.

2. It is interesting to confront the above result with the investigations concerning the time-slice property [**?**] and the split property [**?**, Theorem 10.2] in the context of generalized free fields (in $3+1$ dimensions). In the cited works it was proved that generalized free fields possess the time-slice property iff (roughly) the spectral measure vanishes sufficiently fast at infinity. On the other hand, the split property imposes strong restrictions on the spectral measure, in particular it must be atomic without an accumulation point at a finite mass. The split property (for double cones) is, however, neither necessary nor sufficient for the time-slice property.

3. Absence of Localized Charges

Whereas the results obtained so far are intuitively plausible, we will now prove a no-go theorem which shows that the combination of Haag duality and the SPW is extremely strong.

Theorem 3.1. *Let* $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ *be a net of observables satisfying Haag duality and the split property for wedges. Let* π *be a representation of the quasilocal algebra* $\mathcal{A}$ *which satisfies*

$$\pi \restriction \mathcal{A}(W) \cong \pi_0 \restriction \mathcal{A}(W) \quad \forall\ W \in \mathcal{W}\,, \tag{3.1}$$

where $\mathcal{W}$ *is the set of all wedges* (*left and right handed*). *Then* π *is equivalent to an at most countable direct sum of representations which are unitarily equivalent* *to* π_0 :

$$\pi = \bigoplus_{i\in I} \pi_i\,, \quad \pi_i \cong \pi_0\,. \tag{3.2}$$

In particular, if π *is irreducible it is unitarily equivalent to* π_0.

Remark. A fortiori, this applies to DHR representations (**??**).

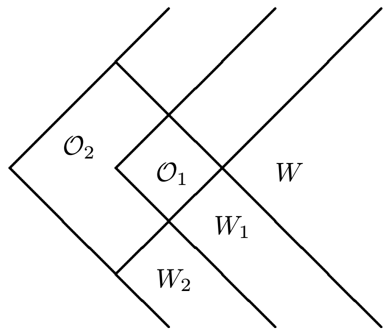

Fig. 2. A split inclusion of wedges.

Proof. Consider the geometry depicted in Fig. 2. If π is a representation satisfying (**??**) then there is a unitary $V : \mathcal{H}_\pi \to \mathcal{H}_0$ such that, setting $\rho = V\pi(\cdot)V^*$, we have $\rho(A) = A$ if $A \in \mathcal{A}(W')$. Due to normality on wedges and wedge duality, ρ continues to normal endomorphisms of $\mathcal{R}(W)$, $\mathcal{R}(W_1)$. By the split property there are type-I factors M_1, M_2 such that

$$\mathcal{R}(W) \subset M_1 \subset \mathcal{R}(W_1) \subset M_2 \subset \mathcal{R}(W_2)\,. \tag{3.3}$$

Let $x \in M_1 \subset \mathcal{R}(W_1)$. Then $\rho(x) \in \mathcal{R}(W_1) \subset M_2$. Furthermore, ρ acts trivially on $M_1' \cap \mathcal{R}(W_2) \subset \mathcal{R}(W)' \cap \mathcal{R}(W_2) = \mathcal{A}(\mathcal{O}_2)$, where we have used Haag duality. Thus ρ maps M_1 into $M_2 \cap (M_1' \cap \mathcal{R}(W_2))' \subset M_2 \cap (M_1' \cap M_2)' = M_1$, the last identity following from M_1, M_2 being type-I factors. By [**?**, Corollary 3.8] every endomorphism of a type-I factor is inner, i.e. there is a (possibly infinite) family of isometries $V_i \in M_1$, $i \in I$ with $V_i^* V_j = \delta_{i,j}$, $\sum_{i\in I} V_i V_i^* = \mathbf{1}$ such that

$$\rho(A) = \eta(A) \quad \forall\ A \in M_1\,, \tag{3.4}$$

where

$$\eta(A) \equiv \sum_{i\in I} V_i\, A\, V_i^*,\ \ A \in \mathcal{B}(\mathcal{H}_0)\,. \tag{3.5}$$

(The sum over I is understood in the strong sense.) Now, ρ and thus η act trivially on $M_1 \cap \mathcal{R}(W)' \subset \mathcal{R}(W_1) \cap \mathcal{R}(W)' = \mathcal{A}(\mathcal{O}_1)$, which implies

$$V_i \in M_1 \cap (M_1 \cap \mathcal{R}(W)')' = \mathcal{R}(W)\,. \tag{3.6}$$

Thanks to Lemma **??** we know that for every wedge $\hat{W} \supset\supset W$

$$\mathcal{R}(\hat{W}) = \mathcal{R}(W) \vee \mathcal{A}(\mathcal{O})\,, \tag{3.7}$$

where $\mathcal{O} = \hat{W} \cap W'$. From the fact that ρ acts trivially on $\mathcal{A}(W')$ it follows that (**??**) is true also for $A \in \mathcal{A}(\mathcal{O})$. By assumption, ρ is normal also on $\mathcal{A}(\hat{W})$ which leads to (**??**) on $\mathcal{A}(\hat{W})$. As this holds for every $\hat{W} \supset\supset W$, we conclude that

$$\pi(A) = \sum_{i \in I} V^* V_i \, A \, V_i^* V \quad \forall\, A \in \mathcal{A}\,. \tag{3.8}$$

□

Remarks. 1. The main idea of the proof is taken from [**?**, Proposition 2.3].

2. The above result may seem inconvenient as it trivializes the DHR/FRS superselection theory [**?**, **?**, **?**] for a large class of massive quantum field theories in $1+1$ dimensions. It is not so clear what this means with respect to field theoretical models since little is known about Haag duality in nontrivial models.

3. Conformal quantum field theories possessing no representations besides the vacuum representation, or "holomorphic" theories, have been the starting point for an analysis of "orbifold" theories in [**?**]. In [**?**], which was motivated by the desire to obtain a rigorous understanding of orbifold theories in the framework of massive two-dimensional theories, the present author postulated the split property for wedges and claimed it to be weaker than the requirement of absence of nontrivial representations. Whereas this claim is disproved by Theorem **??**, as far as localized (DHR or wedge) representations of Haag dual theories are concerned, none of the results of [**?**] is invalidated or rendered obsolete.

4. Haag Duality in Locally Normal Representations

A further crucial consequence of the split property for wedges is observed in the following:

Proposition 4.1. *Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ be a net satisfying Haag duality (for double cones) and the split property for wedges. Then for every pair $\mathcal{O} \subset\subset \hat{\mathcal{O}}$ we have*

$$\mathcal{A}(\hat{\mathcal{O}}) \wedge \mathcal{A}(\mathcal{O})' = \mathcal{A}(\mathcal{O}_L) \vee \mathcal{A}(\mathcal{O}_R)\,, \tag{4.1}$$

where $\mathcal{O}_L, \mathcal{O}_R$ are as in Fig. 3.

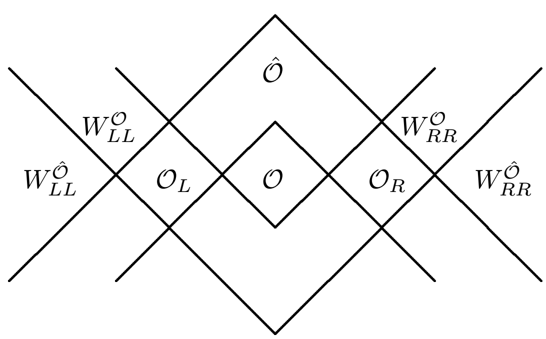

Fig. 3. Relative commutant of double cones.

Proof. By the split property for wedges there is a unitary operator $Y^{\mathcal{O}} : \mathcal{H}_0 \to \mathcal{H}_0 \otimes \mathcal{H}_0$ such that $\mathcal{R}(W^{\mathcal{O}}_{LL}) \vee \mathcal{R}(W^{\mathcal{O}}_{RR}) = Y^{\mathcal{O}*}(\mathcal{R}(W^{\mathcal{O}}_{LL}) \otimes \mathcal{R}(W^{\mathcal{O}}_{RR}))Y^{\mathcal{O}}$. More specifically,

$$Y^{\mathcal{O}}\, xy\, Y^{\mathcal{O}*} = x \otimes y \quad \forall\, x \in \mathcal{R}(W^{\mathcal{O}}_{LL}),\ y \in \mathcal{R}(W^{\mathcal{O}}_{RR})\,. \tag{4.2}$$

By Haag duality $\mathcal{A}(\mathcal{O})' = \mathcal{R}(W^{\mathcal{O}}_{LL}) \vee \mathcal{R}(W^{\mathcal{O}}_{RR}) \cong \mathcal{R}(W^{\mathcal{O}}_{LL}) \otimes \mathcal{R}(W^{\mathcal{O}}_{RR})$ and $\mathcal{A}(\hat{\mathcal{O}})' = \mathcal{R}(W^{\hat{\mathcal{O}}}_{LL}) \vee \mathcal{R}(W^{\hat{\mathcal{O}}}_{RR})$. Now $\mathcal{R}(W^{\hat{\mathcal{O}}}_{LL/RR}) \subset \mathcal{R}(W^{\mathcal{O}}_{LL/RR})$ implies $\mathcal{A}(\hat{\mathcal{O}})' \cong \mathcal{R}(W^{\hat{\mathcal{O}}}_{LL}) \otimes \mathcal{R}(W^{\hat{\mathcal{O}}}_{RR})$ under the same equivalence $\cong$ provided by $Y^{\mathcal{O}}$, and thus

$$\mathcal{A}(\hat{\mathcal{O}}) \cong (\mathcal{R}(W^{\hat{\mathcal{O}}}_{LL}) \otimes \mathcal{R}(W^{\hat{\mathcal{O}}}_{RR}))' = \mathcal{R}(W^{\hat{\mathcal{O}}}_{R}) \otimes \mathcal{R}(W^{\hat{\mathcal{O}}}_{L})\,, \tag{4.3}$$

where we have used wedge duality and the commutation theorem for tensor products. Now we can compute the relative commutant as follows:

$$\begin{aligned} \mathcal{A}(\hat{\mathcal{O}}) \wedge \mathcal{A}(\mathcal{O})' &\cong (\mathcal{R}(W^{\hat{\mathcal{O}}}_{R}) \otimes \mathcal{R}(W^{\hat{\mathcal{O}}}_{L})) \wedge (\mathcal{R}(W^{\mathcal{O}}_{LL}) \otimes \mathcal{R}(W^{\mathcal{O}}_{RR}))\\ &= (\mathcal{R}(W^{\hat{\mathcal{O}}}_{R}) \wedge \mathcal{R}(W^{\mathcal{O}}_{LL})) \otimes (\mathcal{R}(W^{\hat{\mathcal{O}}}_{L}) \wedge \mathcal{R}(W^{\mathcal{O}}_{RR}))\\ &= \mathcal{A}(\mathcal{O}_L) \otimes \mathcal{A}(\mathcal{O}_R)\\ &\cong \mathcal{A}(\mathcal{O}_L) \vee \mathcal{A}(\mathcal{O}_R)\,. \end{aligned} \tag{4.4}$$

We have used Haag duality in the form $\mathcal{R}(W^{\hat{\mathcal{O}}}_{R}) \wedge \mathcal{R}(W^{\mathcal{O}}_{LL}) = \mathcal{A}(\mathcal{O}_L)$ and similarly for $\mathcal{A}(\mathcal{O}_R)$. $\square$

Remarks. 1. Readers having qualms about the above computation of the intersection of tensor products are referred to [**?**, Corollary 5.10], which also provides the justification for the arguments in Sec. 2.

2. Recalling that $\mathcal{R}(\mathcal{O}) = \mathcal{A}(\mathcal{O})$ and that the algebras of regions other than double cones are defined by additivity, (**??**) can be restated as follows:

$$\mathcal{R}(\hat{\mathcal{O}}) \cap \mathcal{R}(\mathcal{O})' = \mathcal{R}(\hat{\mathcal{O}} \cap \mathcal{O}')\,. \tag{4.5}$$

In conjunction with the assumed properties of isotony, locality and Haag duality for double cones (**??**) entails that the map $\mathcal{O} \mapsto \mathcal{R}(\mathcal{O})$ is a homomorphism of orthocomplemented lattices as proposed in [**?**, Sec. III.4.2]. While the discussion in [**?**, Sec. III.4.2] can be criticized, the class of models considered in this paper provides examples where the above lattice homomorphism is in fact realized.

The proposition should contribute to the understanding of Theorem **??** as far as DHR representations are concerned. In fact, it already implies the absence of DHR sectors as can be shown by an application of the triviality criterion for local 1-cohomologies [**?**] given in [**?**], see also [**?**].

Sketch of proof. Let $z \in Z^1(\mathcal{A})$ be the local 1-cocycle associated according to [**?**, **?**] with a representation π satisfying the DHR criterion. Due to Proposition **??** it satisfies $z(b) \in \mathcal{A}(|\partial_0 b|) \vee \mathcal{A}(|\partial_1 b|)$ for every $b \in \Sigma_1$ such that $|\partial_0 b| \subset\subset |\partial_1 b|'$. Thus the arguments in the proof of [**?**, Theorem 3.5] are applicable despite the fact that we are working in 1+1 dimensions. We thereby see that there are unique Hilbert spaces $H(\mathcal{O}) \subset \mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \Sigma_0 \equiv \mathcal{K}$ of support **1** such that $z(b)H(\partial_1 b) = H(\partial_0 b)\ \forall\ b \in \Sigma_1$. Each of these Hilbert spaces implements an endomorphism $\rho_\mathcal{O}$ of $\mathcal{A}$ such that $\rho_\mathcal{O} \cong \pi$. This implies that ρ is either reducible or an inner automorphism. □

Remark. This argument needs the split property for double cones. It is not completely trivial that the latter follows from the split property for wedges. It is clear that the latter implies unitary equivalence of $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ and $\mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)$ if $\mathcal{O}_1$, $\mathcal{O}_2$ are double cones separated by a finite spacelike distance. The split property for double cones requires more, namely unitary equivalence of $\mathcal{A}(\mathcal{O})\vee\mathcal{A}(\hat{\mathcal{O}})'$ and $\mathcal{A}(\mathcal{O})\otimes\mathcal{A}(\hat{\mathcal{O}})'$ whenever $\mathcal{O} \subset\subset \hat{\mathcal{O}}$, which is equivalent to the existence of a type-I factor N such that $\mathcal{A}(\mathcal{O}) \subset N \subset \mathcal{A}(\hat{\mathcal{O}})$.

Lemma 4.2. *Let $\mathcal{A}$ be a local net satisfying Haag duality and the split property for wedges. Then the split property for double cones holds.*

Proof. Using the notation of the preceding proof we have

$$\mathcal{A}(\mathcal{O}) \cong \mathcal{R}(W_R^{\mathcal{O}}) \otimes \mathcal{R}(W_L^{\mathcal{O}})\,, \tag{4.6}$$

$$\mathcal{A}(\hat{\mathcal{O}}) \cong \mathcal{R}(W_R^{\hat{\mathcal{O}}}) \otimes \mathcal{R}(W_L^{\hat{\mathcal{O}}})\,. \tag{4.7}$$

By the SPW there are type-I factors N_L, N_R such that $\mathcal{R}(W_L^{\mathcal{O}}) \subset N_L \subset \mathcal{R}(W_L^{\hat{\mathcal{O}}})$ and $\mathcal{R}(W_R^{\mathcal{O}}) \subset N_R \subset \mathcal{R}(W_R^{\hat{\mathcal{O}}})$. Thus $Y^{\mathcal{O}*}(N_R \otimes N_L)Y^{\mathcal{O}}$ is a type-I factor sitting between $\mathcal{A}(\mathcal{O})$ and $\mathcal{A}(\hat{\mathcal{O}})$. □

Having disproved the existence of nontrivial representations localized in double cones or wedges, we will now prove a result which concerns a considerably larger class of representations.

Theorem 4.3. *Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ be a net of observables satisfying Haag duality and the SPW. Then every irreducible, locally normal representation of the quasilocal algebra $\mathcal{A}$ fulfills Haag duality.*

Proof. We will show that our assumptions imply those of [**?**, Theorem 1]. $\mathcal{A}$ satisfies the split property for double cones (called "funnel property" in [**?**, **?**]) by Lemma **??**, whereas we also assume condition (1) of [**?**, Theorem 1] (Haag duality and irreducibility). Condition (3), which concerns relative commutants $\mathcal{A}(\mathcal{O}_2) \cap \mathcal{A}(\mathcal{O}_1)'$, $\mathcal{O}_2 \supset\supset \mathcal{O}_1$ in the vacuum representation, is an immediate consequence of Proposition **??** (we may even take $\mathcal{O} = \mathcal{O}_1$, $\mathcal{O}_2 = \mathcal{O}_3$). Finally, Lemma **??** implies

$$\mathcal{A}(\mathcal{O})' = \mathcal{A}(\hat{\mathcal{O}})' \vee \mathcal{A}(\mathcal{O}_L) \vee \mathcal{A}(\mathcal{O}_R)\,, \tag{4.8}$$

where we again use the notation of Fig. 3. This is more than required by Driessler's condition (2). Now [**?**, Theorem 1] applies and we are done. □

Remarks. 1. In [**?**] a slightly simplified version of [**?**, Theorem 1] is given which dispenses with condition (2) at the price of a stronger form of condition (3). This condition is still (more than) fulfilled by our class of theories.

2. Observing that soliton representations are locally normal with respect to both asymptotic vacua [**?**, **?**], we conclude at once that Haag duality holds for every irreducible soliton sector where at least one of the vacua satisfies Haag duality and the SPW. Consequences of this fact will be explored in the next section. We remark without going into details that our results are also of relevance for the construction of soliton sectors with prescribed asymptotic vacua in [**?**].

5. Applications to the Theory of Quantum Solitons

In [**?**] it has been shown that every factorial massive one-particle representation (massive one-particle representation) in $\geq 2+1$ dimensions is a multiple of an irreducible representation which is localizable in every spacelike cone. (Here, massive one-particle representation means that the lower bound of the energy-momentum spectrum consists of a hyperboloid of mass $m > 0$ which is separated from the rest of the spectrum by a mass gap.) In $1+1$ dimensions one is led to irreducible soliton sectors [**?**] which we will now reconsider in the light of Theorems **??** and **??**. In this section, where we are concerned with inequivalent vacuum representations, we will consider a QFT to be defined by a net of abstract C^*-algebras instead of the algebras in a concrete representation. Given two vacuum representations π_0^L, π_0^R, a representation π is said to be a soliton representation of type (π_0^L, π_0^R) if it is translation covariant and

$$\pi \restriction \mathcal{A}(W_{L/R}) \cong \pi_0^{L/R} \restriction \mathcal{A}(W_{L/R}) \,, \tag{5.1}$$

where W_L, W_R are arbitrary left and right handed wedges, respectively. An obvious consequence of (**??**) is local normality of π_0^L, π_0^R with respect to each other. In order to formulate a useful theory of soliton representations [**?**] one must assume $\pi_0^{L/R}$ to satisfy wedge duality. After giving a short review of the formalism in [**?**], we will show in this section that considerably more can be said under the stronger assumption that one of the vacuum representations satisfies duality for double cones and the SPW. (Then the other vacuum is automatically Haag dual, too.)

Let π_0 be a vacuum representation and $W \in \mathcal{W}$ a wedge. Then by $\mathcal{A}(W)_{\pi_0}$ we denote the W^*-completion of the C^*-algebra $\mathcal{A}(W)$ with respect to the family of seminorms given by

$$\|A\|_T = |\mathrm{tr}\, T\pi_0(A)| \,, \tag{5.2}$$

where T runs through the set of all trace class operators in $\mathcal{B}(\mathcal{H}_{\pi_0})$. Furthermore, we define extensions $\mathcal{A}^L_{\pi_0}$, $\mathcal{A}^R_{\pi_0}$ of the quasilocal algebra $\mathcal{A}$ by

$$\mathcal{A}^{L/R}_{\pi_0} = \overline{\bigcup_{W \in \mathcal{W}_{L/R}} \mathcal{A}(W)_{\pi_0}}^{\|\cdot\|} \,, \tag{5.3}$$

where $\mathcal{W}_L$, $\mathcal{W}_R$ are the sets of left and right wedges, respectively. Now, it has been demonstrated in [**?**] that, given a (π_0^L, π_0^R)-soliton representation π, there are homomorphisms ρ from $\mathcal{A}^R_{\pi_0^R}$ to $\mathcal{A}^R_{\pi_0^L}$ such that

$$\pi \cong \pi_0^L \circ \rho\,. \tag{5.4}$$

(Strictly speaking, π_0^L must be extended to $\mathcal{A}^R_{\pi_0^L}$, which is trivial since $\mathcal{A}(W)_{\pi_0}$ is isomorphic to $\pi_0(\mathcal{A}(W))''$.) The morphism ρ is localized in some right wedge W in the sense that

$$\rho \restriction \mathcal{A}(W') = \mathrm{id} \restriction \mathcal{A}(W')\,. \tag{5.5}$$

Provided that the vacua of two soliton representations π, π' "fit together" $\pi_0^R \cong \pi_0'^L$ one can define a soliton representation $\pi \times \pi'$ of type π_0^L, $\pi_0'^R$ via composition of the corresponding morphisms:

$$\pi \times \pi' \cong \pi_0^L \circ \rho\rho' \restriction \mathcal{A}\,. \tag{5.6}$$

Alternatively, the entire analysis may be done in terms of left localized morphisms η from $\mathcal{A}^L_{\pi_0^L}$ to $\mathcal{A}^L_{\pi_0^R}$. As proved in [**?**], the unitary equivalence class of the composed representation depends neither on the use of left or right localization nor on the concrete choice of the morphisms.

Whereas for soliton representations there is no analog to the theory of statistics [**?**, **?**, **?**], there is still a "dimension" $\mathrm{ind}(\rho)$ defined by

$$\mathrm{ind}(\rho) \equiv [\mathcal{A}(W)_{\pi_0^L} : \rho(\mathcal{A}(W)_{\pi_0^R})]\,, \tag{5.7}$$

where ρ is localized in the right wedge W and $[M : N]$ is the Jones index of the inclusion $N \subset M$.

Proposition 5.1. *Let π be an irreducible soliton representation such that at least one of the asymptotic vacua π_0^L, π_0^R satisfies Haag duality and the SPW. Then π and both vacua satisfy the SPW and duality for double cones and wedges. The associated soliton-morphism satisfies* $\mathrm{ind}(\rho) = 1$.

Proof. By symmetry it suffices to consider the case where π_0^L satisfies HD + SPW. By Theorem **??** also the representations π and π_0^R satisfy Haag duality since they are locally normal w.r.t. to π_0^L. Let now $W_1 \subset\subset W_2$ be left wedges. By Proposition **??**, wedge-duality holds for π_0^L and $\pi_0^L(\mathcal{A}(W_1))'' \subset \pi_0^L(\mathcal{A}(W_2))''$ is split. Since $\pi_0^L(\mathcal{A}(W_2))''$ is unitarily equivalent to $\pi(\mathcal{A}(W_2))''$, also $\pi(\mathcal{A}(W_1))'' \subset \pi(\mathcal{A}(W_2))''$ splits. A fortiori, π satisfies the SPW in the sense of Definition **??** and thus wedge duality by Proposition **??**. By a similar argument the SPW is carried over to π_0^R. Now, for a right wedge W we have

$$\pi_0^L \circ \rho(\mathcal{A}(W))^- = \pi_0^L \circ \rho(\mathcal{A}(W'))' = \pi_0^L(\mathcal{A}(W'))' = \pi_0^L(\mathcal{A}(W))^-\,. \tag{5.8}$$

By ultraweak continuity on $\mathcal{A}(W)$ of π_0^L and of ρ this implies

$$\rho(\mathcal{A}(W)_{\pi_0^R}) = \mathcal{A}(W)_{\pi_0^L}\,, \tag{5.9}$$

whence the claim. □

This result rules out soliton sectors with infinite index so that [**?**, Theorem 3.2] applies and yields equivalence of the various possibilities of constructing antisoliton sectors considered in [**?**]. In particular the antisoliton sector is uniquely defined up to unitary equivalence. Now we can formulate our main result concerning soliton representations.

Theorem 5.2. *Let* π_0^L, π_0^R *be vacuum representations, at least one of which satisfies Haag duality and the SPW. Then all soliton representations of type* (π_0^L, π_0^R) *are unitarily equivalent.*

Remark. Equivalently, up to unitary equivalence, a soliton representation is completely characterized by the pair of asymptotic vacua.

Proof. Let π, π' be irreducible soliton representations of types (π_0, π_0') and (π_0', π_0), respectively. They may be composed, giving rise to a soliton representation of type (π_0, π_0) (or (π_0', π_0')). This representation is irreducible since the morphisms ρ, ρ' must be isomorphisms by the proposition. Now, $\pi \times \pi'$ is unitarily equivalent to π_0 on left *and* right handed wedges, which by Theorem **??** and irreducibility implies $\pi \times \pi' \cong \pi_0$. We conclude that every (π_0', π_0)-soliton is an antisoliton of every (π_0, π_0')-soliton. This implies the statement of the theorem since for every soliton representation with finite index there is a corresponding antisoliton which is unique up to unitary equivalence. □

Remark. The above proof relies on the absence of nontrivial representations which are localizable in wedges. Knowing just that DHR sectors do not exist, as follows already from Proposition **??**, is not enough.

6. Solitons and DHR Representations of Non-Haag Dual Nets

6.1. *Introduction and an instructive example*

We have observed that the theory of localized representations of Haag-dual nets of observables which satisfy the SPW is trivial. There are, however, quantum field theories in 1 + 1 dimensions where the net of algebras which is most naturally considered as the net of observables does not fulfill Haag duality in the strong form (**??**). As mentioned in the introduction, this is the case if the observables are defined as the fixpoints under a global symmetry group of a field net which satisfies (twisted) duality and the SPW. The weaker property of wedge duality (**??**) remains, however. This property is also known to hold automatically whenever the local algebras arise from a Wightman field theory [**?**]. However, for the analysis in [**?**, **?**, **?**] as well as Sec. 4 above one needs full Haag duality. Therefore it is of relevance that, starting from a net of observables satisfying only (**??**), one can define a larger but still local net

$$\mathcal{A}^d(\mathcal{O}) \equiv \mathcal{R}(W_L^{\mathcal{O}}) \wedge \mathcal{R}(W_R^{\mathcal{O}}) \tag{6.1}$$

which satisfies Haag duality, whence the name *dual net*. Here $W_L^{\mathcal{O}}$, $W_R^{\mathcal{O}}$ are wedges such that $W_L^{\mathcal{O}} \cap W_R^{\mathcal{O}} = \mathcal{O}$ and duality is seen to follow from the fact that the wedge algebras $\mathcal{R}(W)$, $W \in \mathcal{W}$ are the same for the nets $\mathcal{A}$, $\mathcal{A}^d$. (For observables arising as group fixpoints the dual net has been computed explicitly in [**?**].) It is known [**?**, **?**] that *in* $\geq 2+1$ *dimensions* representations π satisfying the DHR criterion (**??**) extend uniquely to DHR representations $\hat{\pi}$ of the (appropriately defined) dual net. Furthermore, the categories of DHR representations of $\mathcal{A}$ and $\mathcal{A}^d$, respectively, and their intertwiners are isomorphic. Thus, instead of $\mathcal{A}$ one may as well study $\mathcal{A}^d$ to which the usual methods are applicable. (The original net is needed only to satisfy essential duality, which is implied by wedge duality.) In $1+1$ dimensions things are more complicated. As shown in [**?**] there are in general two different extensions $\hat{\pi}^L$, $\hat{\pi}^R$. They coincide iff one (thus both) of them is a DHR representation. Even before defining precisely these extensions we can state the following consequence of Theorem **??**.

Proposition 6.1. *Let $\mathcal{A}$ be a net of observables satisfying wedge duality and the SPW. Let π be an irreducible DHR or wedge representation of $\mathcal{A}$ which is not unitarily equivalent to the defining* (*vacuum*) *representation. Then there is no extension $\hat{\pi}$ to the dual net $\mathcal{A}^d$ which is still localized in the DHR or wedge sense.*

Proof. Assume π to be the restriction to $\mathcal{A}$ of a wedge-localized representation $\hat{\pi}$ of $\mathcal{A}^d$. As the latter is known to be either reducible or unitarily equivalent to π_0, the same holds for π. This is a contradiction. □

The fact that the extension of a localized representation of $\mathcal{A}$ to the dual net $\mathcal{A}^d$ cannot be localized, too, partially undermines the original motivation for considering these extensions. Nevertheless, one may entertain the hope that there is something to be learnt which is useful for a model-independent analysis of the phenomena observed in models. Before we turn to the general examination of the extensions $\hat{\pi}^L$, $\hat{\pi}^R$ we consider the most instructive example.

It is provided by the fixpoint net under an unbroken global symmetry group of a field net as studied in [**?**]. We briefly recall the framework. Let $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ be a (for simplicity) bosonic, i.e. local, net of von Neumann algebras acting on the Hilbert space $\mathcal{H}$ and satisfying Haag duality and the SPW. On $\mathcal{H}$ there are commuting strongly continuous representations of the Poincaré group and of a group G of inner symmetries. Both groups leave the vacuum Ω invariant. Defining the fixpoint net

$$\mathcal{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^G = \mathcal{F}(\mathcal{O}) \cap U(G)' \tag{6.2}$$

and its restriction

$$\mathfrak{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}) \upharpoonright \mathcal{H}_0 \tag{6.3}$$

to the vacuum sector (= subspace of G-invariant vectors) we consider $\mathfrak{A}(\mathcal{O})$ as the observables. It is well-known that the net $\mathfrak{A}$ satisfies only wedge duality. Nevertheless, one very important result of [**?**] remains true, namely that the restrictions of

$\mathcal{A}$ to the charged sectors $\mathcal{H}_\chi$ which are labeled by the characters $\chi \in \hat{G}$, interpreted as representations of the abstract C^*-algebra $\mathcal{A}$, satisfy the DHR criterion and are connected to the vacuum by charged fields, i.e. the representation of $\mathcal{A}$ in $\mathcal{H}_\chi$ is of the form

$$\pi_\chi(A) = A \restriction \mathcal{H}_\chi \cong \pi_\chi^{\mathcal{O}}(A) = \psi\, A\, \psi^* \restriction \mathcal{H}_0\,, \tag{6.4}$$

where $\psi \in \mathcal{F}(\mathcal{O})$ and $\alpha_g(\psi) = \chi(g)\psi$.

It was shown in [**?**, Theorem 3.10] that the dual net in the vacuum sector is given by

$$\mathfrak{A}^d(\mathcal{O}) = \hat{\mathcal{A}}_L(\mathcal{O}) \restriction \mathcal{H}_0 = \hat{\mathcal{A}}_R(\mathcal{O}) \restriction \mathcal{H}_0\,, \tag{6.5}$$

where

$$\hat{\mathcal{A}}_{L/R}(\mathcal{O}) = \hat{\mathcal{F}}_{L/R}(\mathcal{O})^G = \hat{\mathcal{F}}_{L/R}(\mathcal{O}) \cap U(G)'\,. \tag{6.6}$$

Here the nonlocal nets $\hat{\mathcal{F}}_{L/R}(\mathcal{O})$ are obtained by adjoining to $\mathcal{F}(\mathcal{O})$ the *disorder operators* [**?**] $U_L^{\mathcal{O}}(G)$ or $U_R^{\mathcal{O}}(G)$, respectively, which satisfy

$$\begin{aligned} \operatorname{Ad} U_L^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) &= \alpha_g = \operatorname{Ad} U_R^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{RR}^{\mathcal{O}})\,, \\ \operatorname{Ad} U_L^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{RR}^{\mathcal{O}}) &= \mathrm{id} = \operatorname{Ad} U_R^{\mathcal{O}}(g) \restriction \mathcal{F}(W_{LL}^{\mathcal{O}}) \end{aligned} \tag{6.7}$$

and transform covariantly under the global symmetry:

$$U(g)\, U_{L/R}^{\mathcal{O}}(h)\, U(g)^* = U_{L/R}^{\mathcal{O}}(ghg^{-1})\,. \tag{6.8}$$

For the moment we restrict to the case of abelian groups G. The disorder operators commuting with G, $\hat{\mathcal{A}}_{L/R}(\mathcal{O})$ is simply $\mathcal{A}(\mathcal{O}) \vee U_{L/R}^{\mathcal{O}}(G)''$. On the C^*-algebras $\hat{\mathcal{A}}_L$ and $\hat{\mathcal{A}}_R$ there is an action of the dual group $\hat{G}$ which acts trivially on $\mathcal{A}$ and via

$$\hat{\alpha}_\chi(U_{L/R}^{\mathcal{O}}(g)) = \chi(g)\, U_{L/R}^{\mathcal{O}}(g) \quad \forall\ \mathcal{O} \in \mathcal{K} \tag{6.9}$$

on the disorder operators. Since this action commutes with the Poincaré group and since it is spontaneously broken ($\omega_0 \circ \hat{\alpha}_\chi \neq \omega_0\ \forall\ \chi \neq e_{\hat{G}}$) it gives rise to inequivalent vacuum states on $\hat{\mathcal{A}}$ via

$$\omega_\chi = \omega_0 \circ \hat{\alpha}_\chi\,. \tag{6.10}$$

The extensions $\hat{\pi}_{\chi,L}$, $\hat{\pi}_{\chi,R}$ of π_χ to the dual net $\mathcal{A}^d$ can now defined using the right-hand side of (**??**) by allowing A to be in $\hat{\mathcal{A}}_L$ or $\hat{\mathcal{A}}_R$. As is obvious from the commutation relation (**??**) between fields and disorder operators, the extension $\hat{\pi}_{\chi,L}$ ($\hat{\pi}_{\chi,R}$) is nothing but a soliton sector interpolating between the vacua ω_0 and $\omega_{\chi^{-1}}$ (ω_χ and ω_0). The moral is that the net $\mathcal{A}^d$, while not having nontrivial localized representations by Theorem **??**, admits soliton representations. Furthermore, with respect to $\mathcal{A}^d$, the charged fields ψ_χ are creation operators for solitons since they intertwine the representations of $\mathcal{A}^d$ on $\mathcal{H}_0$ and $\mathcal{H}_\chi$. Due to $U_L^{\mathcal{O}}(g)\,U_R^{\mathcal{O}}(g) = U(g)$ and $U(g) \upharpoonright \mathcal{H}_\chi = \chi(g)\mathbf{1}$ we have

$$U_L^{\mathcal{O}}(g) \upharpoonright \mathcal{H}_\chi = \chi(g)\, U_R^{\mathcal{O}}(g^{-1}) \upharpoonright \mathcal{H}_\chi\,, \tag{6.11}$$

so that the algebras $\hat{\mathcal{A}}_{L/R}(\mathcal{O}) \upharpoonright \mathcal{H}_\chi$ are independent of whether we use the left or right localized disorder operators. In particular, in the vacuum sector $U_L^{\mathcal{O}}(g)$ and $U_R^{\mathcal{O}}(g^{-1})$ coincide, but due to the different localization properties it is relevant whether $U_L^{\mathcal{O}}(g)$, considered as an element of $\mathcal{A}^d$, is represented on $\mathcal{H}_\chi$ by $U_L^{\mathcal{O}}(g)$ or by $\chi(g)\,U_R^{\mathcal{O}}(g^{-1})$. This reasoning shows that the two possibilities for extending a localized representation of a general non-dual net to a representation of the dual net correspond in the fixpoint situation at hand to the choice between the nets $\hat{\mathcal{A}}_L$ and $\hat{\mathcal{A}}_R$ arising from the field extensions $\hat{\mathcal{F}}_L$ and $\hat{\mathcal{F}}_R$.

6.2. *General Analysis*

We begin by first assuming only that π is localizable in wedges. Let $\mathcal{O}$ be a double cone and let W_L, W_R be left and right handed wedges, respectively, containing $\mathcal{O}$. By assumption the restriction of π to $\mathcal{A}(W_L)$, $\mathcal{A}(W_R)$ is unitarily equivalent to π_0. Choose unitary implementers U_L, U_R such that

$$\begin{aligned} Ad\,U_L \upharpoonright \mathcal{A}(W_L) &= \pi \upharpoonright \mathcal{A}(W_L)\,, \\ Ad\,U_R \upharpoonright \mathcal{A}(W_R) &= \pi \upharpoonright \mathcal{A}(W_R)\,. \end{aligned} \tag{6.12}$$

Then $\hat{\pi}^L$, $\hat{\pi}^R$ are defined for $A \in \mathcal{A}^d(\mathcal{O})$ by

$$\begin{aligned} \hat{\pi}^L(A) &= U_L\, A\, U_L^*\,, \\ \hat{\pi}^R(A) &= U_R\, A\, U_R^*\,. \end{aligned} \tag{6.13}$$

Independence of these definitions of the choice of W_L, W_R and the implementers U_L, U_R follows straightforwardly from wedge duality. We state some immediate consequences of this definition.

Proposition 6.2. *$\hat{\pi}^L, \hat{\pi}^R$ are irreducible, locally normal representations of $\mathcal{A}^d$ and satisfy Haag duality. $\hat{\pi}^L, \hat{\pi}^R$ are normal on left and right handed wedges, respectively.*

Proof. Irreducibility is a trivial consequence of the assumed irreducibility of π whereas local normality is obvious from the definition (**??**). Thus, Theorem **??** applies and yields Haag duality in both representations. Normality of, say, $\hat{\pi}^L$ on left handed wedges W follows from the fact that we may use the same auxiliary wedge $W_L \supset W$ and implementer U_L for all double cones $\mathcal{O} \subset W$. □

Clearly, the extensions $\hat{\pi}^L$, $\hat{\pi}^R$ cannot be normal w.r.t. π_0 on right and left wedges, respectively, for otherwise Theorem **??** would imply unitary equivalence to π_0. In general, we can only conclude localizability in the following weak sense. Given an arbitrary left handed wedge W, $\hat{\pi}^L$ is equivalent to a representation ρ on $\mathcal{H}_0$ such that $\rho(A) = A \; \forall \; A \in \mathcal{A}(W)$. Furthermore, by duality ρ is an isomorphism of $\mathcal{A}(W')$ onto a weakly dense subalgebra of $\mathcal{R}(W')$ which is only continuous in the norm. In favorable cases like the one considered above this is a local symmetry, acting as an automorphism of $\mathcal{A}(W')$. But we will see shortly that there are perfectly non-pathological situations where the extensions are not of this particularly nice type. In complete generality, the best one can hope for is normality with respect to another vacuum representation π_0'. In particular, this is automatically the case if π is a massive one-particle representation [?] which we did not assume so far.

If the representation π satisfies the DHR criterion, i.e. is localizable in double cones, we can obtain stronger results concerning the localization properties of the extended representations $\hat{\pi}_L$, $\hat{\pi}_R$. By the criterion, there are unitary operators $X^{\mathcal{O}} : \mathcal{H}_\pi \to \mathcal{H}_0$ such that

$$\pi^{\mathcal{O}}(A) \equiv X^{\mathcal{O}}\, \pi(A)\, X^{\mathcal{O}*} = A \quad \forall \; A \in \mathcal{A}(\mathcal{O}')\,. \tag{6.14}$$

(By wedge duality, $X^{\mathcal{O}}$ is unique up to left multiplication by operators in $\mathcal{A}^d(\mathcal{O})$.) Considering the representations

$$\hat{\pi}^{\mathcal{O}}_{L/R} = X^{\mathcal{O}} \hat{\pi}_{L/R} X^{\mathcal{O}*} \tag{6.15}$$

on the vacuum Hilbert space $\mathcal{H}_0$, it is easy to verify that

$$\hat{\pi}^{\mathcal{O}}_L \restriction \mathcal{A}^d(W^{\mathcal{O}}_{LL}) = \mathrm{id} \restriction \mathcal{A}^d(W^{\mathcal{O}}_{LL})\,, \tag{6.16}$$

$$\hat{\pi}^{\mathcal{O}}_R \restriction \mathcal{A}^d(W^{\mathcal{O}}_{RR}) = \mathrm{id} \restriction \mathcal{A}^d(W^{\mathcal{O}}_{RR})\,. \tag{6.17}$$

We restrict our attention to $\hat{\pi}^{\mathcal{O}}_L$, the other extension behaving similarly. If $A \in \mathcal{A}(\tilde{\mathcal{O}})$ then $\hat{\pi}_L(A) = X^{\mathcal{O}_r *}\, A\, X^{\mathcal{O}_r}$ whenever $\mathcal{O}_r > \tilde{\mathcal{O}}$. Therefore

$$\hat{\pi}^{\mathcal{O}}_L(A) = X^{\mathcal{O}} X^{\mathcal{O}_r *}\, A\, X^{\mathcal{O}_r} X^{\mathcal{O}*}\,, \tag{6.18}$$

where the unitary $X^{\mathcal{O}} X^{\mathcal{O}_r *}$ intertwines $\pi^{\mathcal{O}}$ and $\pi^{\mathcal{O}_r}$. Associating with every pair $(\mathcal{O}_1, \mathcal{O}_2)$ two other double cones by

$$\hat{\mathcal{O}} = \sup(\mathcal{O}_1, \mathcal{O}_2)\,, \tag{6.19}$$

$$\mathcal{O}_0 = \hat{\mathcal{O}} \cap \mathcal{O}_1' \cap \mathcal{O}_2' \tag{6.20}$$

($\mathcal{O}_0$ may be empty) and defining

$$\mathcal{C}(\mathcal{O}_1, \mathcal{O}_2) = \mathcal{A}^d(\hat{\mathcal{O}}) \cap \mathcal{A}(\mathcal{O}_0)'\,, \tag{6.21}$$

we can conclude by wedge duality that

$$X^{\mathcal{O}} X^{\mathcal{O}_r *} \in \mathcal{C}(\mathcal{O}, \mathcal{O}_r)\,. \tag{6.22}$$

Thus $\hat{\pi}^{\mathcal{O}}_L(A)$ as given by (**??**) is contained in $\mathcal{A}^d(\sup(\mathcal{O}, \tilde{\mathcal{O}}, \mathcal{O}_r))$ which already shows that $\hat{\pi}^{\mathcal{O}}_L$ maps the quasilocal algebra $\mathcal{A}^d$ into itself (this does not follow if

π is only localizable in wedges). Since the double cone $\mathcal{O}_r > \mathcal{O}$ may be chosen arbitrarily small and appealing to outer regularity of the dual net $\mathcal{A}^d$ we even have $\hat{\pi}_L^{\mathcal{O}}(A) \in \mathcal{A}^d(\sup(\mathcal{O}, \tilde{\mathcal{O}}))$ and thus finally

$$\hat{\pi}_L^{\mathcal{O}}(\mathcal{A}^d(\tilde{\mathcal{O}})) \subset \mathcal{C}(\mathcal{O}, \tilde{\mathcal{O}})\,. \tag{6.23}$$

This result has two important consequences. Firstly, it implies that the representation $\hat{\pi}_L^{\mathcal{O}}$ maps the quasilocal algebra into itself:

$$\hat{\pi}_L^{\mathcal{O}}(\mathcal{A}^d) \subset \mathcal{A}^d\,. \tag{6.24}$$

This fact is of relevance since it allows the extensions $\hat{\pi}_{1,L}^{\mathcal{O}}$, $\hat{\pi}_{2,L}^{\mathcal{O}}$ of two DHR representations π_1, π_2 to be composed in much the same way as the endomorphisms of $\mathcal{A}$ derived from DHR representations in the Haag dual case. In this respect, the extensions $\hat{\pi}_{L/R}$ are better behaved than completely general soliton representations as studied in [**?**].

The second consequence of (**??**) is that the representations $\hat{\pi}_L^{\mathcal{O}}$ (and $\hat{\pi}_R^{\mathcal{O}}$), while still mapping local algebras into local algebras, may deteriorate the localization. We will see below that this phenomenon is not just a theoretical possibility but really occurs. Whereas one might hope that one could build a DHR theory for non-dual nets upon the endomorphism property of the extended representations, their weak localization properties and the inequivalence of $\hat{\pi}_L$ and $\hat{\pi}_R$ seem to constitute serious obstacles. It should be emphasized that the above considerations owe a lot to Roberts' local 1-cohomology [**?**, **?**, **?**], but (**??**) seems to be new.

6.3. *Fixpoint nets*: *non-abelian case*

We now generalize our analysis of fixpoint nets to non-abelian (finite) groups G, where the outcome is less obvious *a priori*. Let $\hat{A} = \sum_{g\in G} F_g U_L^{\tilde{\mathcal{O}}}(g) \in \hat{\mathcal{A}}_L(\tilde{\mathcal{O}}_1)$ (F_g must satisfy the condition given in [**?**, Theorem 3.16]) and let $\psi_i \in \mathcal{F}(\mathcal{O}_2)$, where $\mathcal{O}_2 < \mathcal{O}_1$ (i.e. $\mathcal{O}_2 \subset W_{LL}^{\mathcal{O}_1}$) be a multiplet of field operators transforming according to a finite dimensional representation of G. Then

$$\sum_i \psi_i \left(\sum_{g\in G} F_g U_L^{\mathcal{O}_1}(g)\right) \psi_i^* = \sum_{g\in G}\left(\sum_i \psi_i \alpha_g(\psi_i^*)\right) F_g U_L^{\mathcal{O}_1}(g)\,. \tag{6.25}$$

In contrast to the abelian case where $\psi\alpha_g(\psi^*)$ is just a phase, $O_g \equiv \sum_i \psi_i\alpha_g(\psi_i^*)$ is a nontrivial unitary operator

$$O_g^{-1} = O_g^* = \sum_i \alpha_g(\psi_i)\psi_i^* \tag{6.26}$$

satisfying

$$\alpha_k(O_g) = O_{kgk^{-1}}\,. \tag{6.27}$$

In particular (**??**) is not contained in $\mathcal{A}^d(\mathcal{O}_1)$ which implies that the map $\hat{A} \mapsto \sum_i \psi_i\,\hat{A}\,\psi_i^*$ does not reduce to a local symmetry on $\hat{\mathcal{A}}_L(W_{RR}^{\mathcal{O}_2})$. Rather, we obtain a monomorphism into $\hat{\mathcal{A}}_L(W_R^{\mathcal{O}_2})$. Defining $\hat{\mathcal{O}}$ and $\mathcal{O}_0$ as above we clearly see that (**??**) is contained in $\mathcal{A}^d(\hat{\mathcal{O}})$. Furthermore, due to the relative locality of the net $\mathcal{A}$ with respect to $\mathcal{A}^d$ and $\mathcal{F}$, (**??**) commutes with $\mathcal{A}(\mathcal{O}_0)$. Thus we obtain precisely the localization properties which were predicted by our general analysis above.

We close this section with a discussion of the duality properties in the extended representations $\hat{\pi}$. In the case of abelian groups G Haag duality holds in all charged sectors since these are all simple. Our abstract result in Theorem **??** to the effect that duality obtains in *all* locally normal irreducible representations of the dual net applies, of course, to the situation at hand. We conclude that Haag duality also holds for the non-simple sectors which by necessity occur for non-abelian groups G. Since this result is somewhat counterintuitive (which explains why it was overlooked in [**?**]) we verify it by the following direct calculation.

Lemma 6.3. *The commutants of the algebras* $\hat{\mathcal{A}}_L(\mathcal{O})$ *are given by*

$$\hat{\mathcal{A}}_L(\mathcal{O})' = \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{F}}_L(W_{RR}^{\mathcal{O}}) \quad \forall\, \mathcal{O} \in \mathcal{K}\,. \tag{6.28}$$

Proof. For simplicity we assume $\mathcal{F}$ to be a local net for a moment. Then

$$\begin{aligned}\hat{\mathcal{A}}_L(\mathcal{O})' &= (\hat{\mathcal{F}}_L(\mathcal{O}) \wedge U(G)')' = \hat{\mathcal{F}}_L(\mathcal{O})' \vee U(G)'' \\ &= (\mathcal{F}_L(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'')' \vee U(G)'' = (\mathcal{F}_L(\mathcal{O})' \wedge U_L^{\mathcal{O}}(G)') \vee U(G)'' \\ &= ((\mathcal{F}_L(W_{LL}^{\mathcal{O}}) \vee \mathcal{F}_L(W_{RR}^{\mathcal{O}})) \wedge U_L^{\mathcal{O}}(G)') \vee U(G)'' \\ &= (\mathcal{F}_L(W_{LL}^{\mathcal{O}}) \wedge U_L^{\mathcal{O}}(G)') \vee \mathcal{F}_L(W_{RR}^{\mathcal{O}}) \vee U(G)'' \\ &= \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{F}}_L(W_{RR}^{\mathcal{O}})\,.\end{aligned} \tag{6.29}$$

The fourth line follows from the third using the split property. In the last step we have used the identities $\hat{\mathcal{A}}_L(W_L) = \mathcal{A}_L(W_L)$ and $\mathcal{F}_L(W_R) \vee U(G)'' = \hat{\mathcal{F}}_L(W_R)$ which hold for all left (right) handed wedges W_L (W_R), cf. [**?**, Proposition 3.5]. Now, if $\mathcal{F}$ satisfies twisted duality, (2.23) of [**?**] leads to $\mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'' \cong \mathcal{F}(W_R^{\mathcal{O}}) \vee U(G)'' \otimes \mathcal{F}(W_L^{\mathcal{O}})$ and $(\mathcal{F}(\mathcal{O}) \vee U_L^{\mathcal{O}}(G)'')' \cong \mathcal{A}(W_R^{\mathcal{O}}) \otimes \mathcal{F}(W_{RR}^{\mathcal{O}})^t$. Using this it is easy to verify that (**??**) is still true. □

Proposition 6.4. *The net* $\hat{\mathcal{A}}_L$ *satisfies Haag duality in restriction to every invariant subspace of* $\mathcal{H}$ *on which* $\hat{\mathcal{A}}_L$ *acts irreducibly.*

Proof. We recall that the representation π of $\hat{\mathcal{A}}_{L/R}$ on $\mathcal{H}$ is of the form $\pi = \oplus_{\xi \in \hat{G}}\, d_\xi \pi_\xi$. Let thus P be an orthogonal projection onto a subspace $\mathcal{H}_\xi \subset \mathcal{H}$ on which $\hat{\mathcal{A}}_L$ acts as the irreducible representation π_ξ. Since P commutes with $\mathcal{A}_L(\mathcal{O})$ and $\mathcal{A}_L(W_{LL}^{\mathcal{O}})$ we have

$$\begin{aligned}P\, \hat{\mathcal{A}}_L(\mathcal{O})'\, P &= P\, \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{F}}_L(W_{RR}^{\mathcal{O}})\, P \\ &= \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee (P\, \hat{\mathcal{F}}_L(W_{RR}^{\mathcal{O}})\, P) \\ &= P\, \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{A}}_L(W_{RR}^{\mathcal{O}})\, P\,,\end{aligned} \tag{6.30}$$

which implies

$$(\hat{\mathcal{A}}_L(\mathcal{O}) \upharpoonright \mathcal{H}_\xi)' = \hat{\mathcal{A}}_L(W_{LL}^{\mathcal{O}}) \vee \hat{\mathcal{A}}_L(W_{RR}^{\mathcal{O}}) \upharpoonright \mathcal{H}_\xi\,. \tag{6.31}$$

□

This provides a concrete verification of Theorem **??** in a special, albeit important situation.

7. Conclusions and Outlook

We have seen that the combination of Haag duality with the split property for wedges has remarkable unifying power. It implies factoriality of the double cone algebras, n-regularity for all n and irreducibility of time-slice algebras. As a consequence of the minimality of relative commutants of double cone algebras we obtain Haag duality in all irreducible, locally normal representations. The strongest result concerns the absence not only of locally generated superselection (DHR) sectors but also of charges localized in wedges. This in turn implies the uniqueness up to unitary equivalence of soliton sectors with prescribed asymptotic vacua. In the following we briefly relate these results to what is known in concrete models in $1+1$ dimensions.

(a) *The free massive scalar field.* Since this model is known to satisfy Haag duality and the SPW, Theorem **??** constitutes a high-brow proof of the well-known absence of local charges. Furthermore, there are no non-trivial soliton sectors, since the vacuum representation is unique [**?**]. Thus, the irreducible representations constructed in [**?**], which are inequivalent to the vacuum, must be rather pathological. In fact, they are equivalent to the (unique) vacuum only on left wedges.

(b) $\mathcal{P}(\phi)_2$-*models.* These models have been shown [**?**] to satisfy Haag duality in all pure phases, but there is no proof of the SPW. Yet, the split property for double cones, the minimality of relative commutants and strong additivity, thus also the time slice property, follow immediately from the corresponding properties for the free field via the local Fock property. These facts already imply the non-existence of DHR sectors and Haag duality in all irreducible locally normal sectors. All these consequences are compatible with the conjecture that the SPW holds. There seems, however, not to be a proof of the absence of wedge sectors.

(c) *The sine-Gordon/Thirring model.* For this model neither Haag duality nor the SPW are known. In the case $\beta^2 = 4\pi$, however, for which the SG model corresponds to the free massive Dirac field, there seems to be no doubt that the net $\hat{\mathcal{A}}$ constructed like in Sec. 6 from the free Dirac field is exactly the local net of the SG model. As shown in [**?**], also $\hat{\mathcal{A}}$ satisfies Haag duality and the SPW. Since from the point of view of constructive QFT there is nothing special about $\beta^2 = 4\pi$ one may hope that both properties hold for all $\beta \in [0, 8\pi)$.

In view of the results of this paper as well as of [**?**] it is highly desirable to clarify the status of the SPW in interacting massive models like (b) and (c) as well as that of Haag duality in case (c). (Also the Gross–Neveu model might be expected to satisfy both assumptions.) The most promising approach to this problem should be identifying conditions on a set of Wightman (or Schwinger) distributions which imply Haag duality and the SPW, respectively, for the net of algebras generated by the fields. For a first step in this direction see [**?**, Sec. IIIB].

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