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Conformal Field Theory and Doplicher-Roberts Reconstruction

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This contribution is dedicated to Sergio Doplicher and John E. Roberts on the occasion of their 60th birthdays.

Abstract. After a brief review of recent rigorous results concerning the representation theory of rational chiral conformal field theories (RC-QFTs) we focus on pairs $(\mathcal{A}, \mathcal{F})$ of conformal field theories, where $\mathcal{F}$ has a finite group G of global symmetries and $\mathcal{A}$ is the fixpoint theory. The comparison of the representation categories of $\mathcal{A}$ and $\mathcal{F}$ is strongly intertwined with various issues related to braided tensor categories. We explain that, given the representation category of $\mathcal{A}$, the representation category of $\mathcal{F}$ can be computed (up to equivalence) by a purely categorical construction. The latter is of considerable independent interest since it amounts to a Galois theory for braided tensor categories. We emphasize the characterization of modular categories as braided tensor categories with trivial center and we state a double commutant theorem for subcategories of modular categories. The latter implies that a modular category $\mathcal{M}$ which has a replete full modular subcategory $\mathcal{M}_1$ factorizes as $\mathcal{M} \simeq \mathcal{M}_1 \otimes_{\mathbb{C}} \mathcal{M}_2$ where $\mathcal{M}_2 = \mathcal{M} \cap \mathcal{M}_1'$ is another modular subcategory. On the other hand, the representation category of $\mathcal{A}$ is not determined completely by that of $\mathcal{F}$ and we identify the needed additional data in terms of soliton representations. We comment on 'holomorphic orbifold' theories, i.e. the case where $\mathcal{F}$ has trivial representation theory, and close with some open problems.

We point out that our approach permits the proof of many conjectures and heuristic results on 'simple current extensions' and 'holomorphic orbifold models' in the physics literature on conformal field theory.

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1 Introduction

As is well known and will be reviewed briefly in the next section, quantum field theories in Minkowski space of not too low dimension give rise to representation categories which are symmetric C^*-tensor categories with duals and simple unit. (The minimum number of space dimensions for this to be true depends on the class of representations under consideration.) As Doplicher and Roberts have shown, such categories are representation categories of compact groups [17] and every QFT is the fixpoint theory under a compact group action [18] of a theory admitting only the vacuum representation [9]. Thus the theory of (localized) representations of QFTs in higher dimensional spacetimes is essentially closed.

Though this is still far from being the case for low dimensional theories there has been considerable recent progress, of which we will review two aspects. The first of these concerns the general representation theory of rational chiral conformal theories, which have been shown [33] to give rise to unitary modular categories in perfect concordance with the physical expectations. See also [46] for a more self-contained and (somewhat) more accessible review. In this contribution we restrict ourselves to stating the main results insofar as they serve to motivate the subsequent considerations which form the core of this paper.

We will then study pairs $(\mathcal{F}, \mathcal{A})$ of quantum field theories in low dimension, mostly rational conformal, where $\mathcal{A}$ is the fixpoint theory of $\mathcal{F}$ w.r.t. the action of a finite group G of global symmetries. This scenario may seem quite special, as in fact it is, but it is justified by several arguments. First of all, as already alluded to, the fixpoint situation is the generic one in high dimensions. Whereas this is definitely not true in the case at hand, every attempt at classifying rational conformal field theories (or at least modular categories) will most likely make use of constructions which produce new conformal field theories from given ones. (Besides those we focus on there are, of course, other such procedures like the 'coset construction'.) The converse of the passage to G-fixpoints is provided by the construction [18] of Doplicher and Roberts, which in the case of abelian groups has appeared in the CQFT literature as 'simple current extension'. The latter are of considerable relevance in the classification of 'modular invariants', i.e. the construction of two-dimensional CQFTs out of chiral ones. It is therefore very satisfactory that we are able to provide rigorous proofs for many results in this area.

Finally the analysis of quantum field theories related by finite groups leads to many mathematical results which can be phrased in a purely categorical manner. As such they have applications to other areas of mathematics like subfactor theory or low-dimensional topology.

2 'Many' Spacetime Dimensions: Symmetric Categories

2.1 Global Symmetry Groups in $\geq 2+1$ Spacetime Dimensions. In this section we consider quantum field theories in Minkowski space with $d = s + 1$ dimensions where the *number s of space dimensions is at least two.* (See [27, 32] for more details.) We denote by $\mathcal{K}$ the set of double cones. Let $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O}), \mathcal{O} \in \mathcal{K}$ be a net (inclusion preserving assignment) of von Neumann algebras on a Hilbert space $\mathcal{H}$ satisfying irreducibility, locality $(\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow [\mathcal{F}(\mathcal{O}_1), \mathcal{F}(\mathcal{O}_2)] = \{0\})$ and covariance w.r.t. a positive energy representation of the Poincaré group with invariant vacuum vector Ω. We sharpen the locality requirement by imposing Haag duality

$$\mathcal{F}(\mathcal{O})' = \mathcal{F}(\mathcal{O}')^- \quad \text{where} \quad \mathcal{F}(\mathcal{O}') = \overline{\cup_{\tilde{\mathcal{O}}\in\mathcal{K},\tilde{\mathcal{O}}\subset\mathcal{O}'}\mathcal{F}(\tilde{\mathcal{O}})}^{\|\cdot\|}.$$

We assume that there is a compact group G with a strongly continuous faithful unitary representation U on $\mathcal{H}$ commuting with the representation of the Poincaré group, leaving Ω invariant and implementing global symmetries of $\mathcal{F}$:

$$\alpha_g(\mathcal{F}(\mathcal{O})) = \mathcal{F}(\mathcal{O}) \;\; \forall g \in G, \mathcal{O} \in \mathcal{K} \quad \text{where } \alpha_g = Ad\, U(g).$$

Consider the subnet $\mathcal{A}(\mathcal{O}) = \mathcal{F}(\mathcal{O})^G$ together with its vacuum representation π_0 on the subspace $\mathcal{H}_0$ of G-invariant vectors. π_0 can be shown to satisfy Haag duality [15]. The Hilbert space $\mathcal{H}$ decomposes as

$$\mathcal{H} = \bigoplus_{\xi\in\hat{G}} \mathcal{H}_\xi \otimes \mathbb{C}^{d(\xi)},$$

where $\hat{G}$ is the set of isomorphism classes of irreducible representation of G and the group G and the C^*-algebra $\mathcal{A} = \overline{\cup_{\mathcal{O}\in\mathcal{K}}\mathcal{A}(\mathcal{O})}^{\|\cdot\|}$ (the 'quasi-local algebra') act on $\mathcal{H}$ as follows:

$$U(g) = \bigoplus_{\xi\in\hat{G}} \mathbf{1} \otimes U_\xi(g), \quad \pi(A) = \bigoplus_{\xi\in\hat{G}} \pi_\xi(A) \otimes \mathbf{1}, \quad g \in G, A \in \mathcal{A}.$$

The representations π_ξ of $\mathcal{A}$ on $\mathcal{H}_\xi$ are irreducible and satisfy [15]

$$\pi_\xi \restriction \mathcal{A}(\mathcal{O}') \cong \pi_0 \restriction \mathcal{A}(\mathcal{O}') \quad \forall \mathcal{O} \in \mathcal{K}, \tag{2.1}$$

where π_0 is the representation of $\mathcal{A}$ on $\mathcal{H}_0$.

These observations motivate the analysis of the positive energy representations satisfying the 'DHR criterion' (2.1) for any irreducible local net $\mathcal{A}$ of algebras satisfying Haag duality and Poincaré covariance.

Definition 2.1 Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0)$ be a Haag dual net of algebras which is covariant w.r.t. a positive energy vacuum representation of $\mathcal{P}$. Then $\widetilde{DHR}(\mathcal{A})$ is the category of Poincaré covariant representations satisfying (2.1) (π_0 is the defining representation) together with their (bounded) intertwining operators.

For the purposes of the development of the theory another category is much more convenient.

Definition 2.2 Let $\mathcal{A}$ be as above. Then $DHR(\mathcal{A})$ denotes the category of localized transportable morphisms, i.e. bounded unital $*$-algebra endomorphisms ρ of the quasi-local algebra $\mathcal{A}$ such that $\rho \restriction \mathcal{A}(\mathcal{O}') = \mathrm{id}$ for *some* $\mathcal{O} \in \mathcal{K}$ and such that for *every* $\tilde{\mathcal{O}} \in \mathcal{K}$ there is $\rho_{\tilde{\mathcal{O}}}$ localized in $\tilde{\mathcal{O}}$ such that ρ and $\rho_{\tilde{\mathcal{O}}}$ are inner equivalent. The morphisms are the intertwiners in $\mathcal{A}$.

Theorem 2.3 [16] *$DHR(\mathcal{A})$ is canonically isomorphic to a full subcategory of $\widetilde{DHR}(\mathcal{A})$ which is equivalent to $\widetilde{DHR}(\mathcal{A})$. $DHR(\mathcal{A})$ is a strict symmetric C^*-tensor category with simple tensor unit. There is an additive and multiplicative dimension function on the objects with values in $\mathbb{N} \cup \infty$. (If the local algebras $\mathcal{A}(\mathcal{O}), \mathcal{O} \in \mathcal{K}$ are assumed to be factors then $d(\rho)^2 = [\mathcal{A}(\mathcal{O}) : \rho(\mathcal{A}(\mathcal{O}))]$ if ρ is localized in $\mathcal{O}$.) The full monoidal subcategory $DHR_f(\mathcal{A})$ of the ρ with finite statistics (i.e. $d(\rho) < \infty$) has conjugates in the sense of* [35]. *Viz. for every*

$\rho \in DHR_f(\mathcal{A})$ there are $\overline{\rho} \in DHR_f(\mathcal{A})$ and $r_\rho \in Hom(\mathbf{1}, \overline{\rho}\rho)$, $\overline{r}_\rho \in Hom(\mathbf{1}, \rho\overline{\rho})$ satisfying $r_\rho^* \circ r_\rho = \overline{r}_\rho^* \circ \overline{r}_\rho = d(\rho) id_{\mathbf{1}}$ and

$$id_\rho \otimes r_\rho^* \circ \overline{r}_\rho \otimes id_\rho = id_\rho, \quad id_{\overline{\rho}} \otimes \overline{r}_\rho^* \circ r_\rho \otimes id_{\overline{\rho}} = id_{\overline{\rho}}.$$

$DHR_f(\mathcal{A})$ is a ribbon category, i.e. has a twist, which on simple objects takes the values ± 1 (Bose-Fermi alternative).

Applying the general formalism to fixpoint nets as above one obtains:

Proposition 2.4 [15] Let $\mathcal{A}$ be a fixpoint net as above. Then $DHR_f(\mathcal{A})$ contains a full monoidal subcategory $\mathcal{S}$ which is equivalent (as a symmetric monoidal category) to the category $G-\text{mod}$ of finite dimensional continuous unitary representations of G. For a simple object $\rho_\xi \in \mathcal{S}$ the dimension $d(\rho_\xi)$ coincides with the dimension $d(\xi)$ of the associated representation U_ξ of G.

Now the question arises under which circumstances one obtains all DHR representations of $\mathcal{A}$ in this way.

Proposition 2.5 [51] Assume $\mathcal{F}$ has trivial representation category $DHR(\mathcal{F})$ (in the sense of quasi-trivial one-cohomology). Then $\mathcal{A}$ has no irreducible DHR representations of infinite statistics and $DHR(\mathcal{A}) \simeq G-\text{mod}$.

It is thus natural to conjecture that every net $\mathcal{A}$ satisfying the above axioms is the fixpoint net under the action of a compact group G of a net $\mathcal{F}$ with trivial representation structure.

2.2 The Reconstruction Theory of Doplicher and Roberts.

Theorem 2.6 [17] *Let $\mathcal{S}$ be a symmetric C^*-tensor category with conjugates and simple unit such that every simple object has twist $+1$. Then there is a compact group G, unique up to isomorphism, such that one has an equivalence $\mathcal{S} \simeq G-mod$ of symmetric tensor $*$-categories with conjugates.*

Remark 2.7 1. If there are objects with twist -1 then there is a compact group G together with a central element k of order two such that $\mathcal{S} \simeq G-\text{mod}$ as a tensor category and the twist of a simple object equals the value of k in the corresponding irreducible representation of G.

2. Most categories in this paper will be closed w.r.t. direct sums and subobjects (i.e. all idempotents split). Yet, in order not to have to require this everywhere, all equivalences of ((braided/symmetric) monoidal) categories in this paper will be understood as equivalences of the respective categories after completion w.r.t. direct sums and subobjects. See, e.g., [25] for these constructions and note that equivalence of the completed categories is equivalent to Morita equivalence [25]. We believe that this is the appropriate notion of equivalence for semisimple k-linear categories. By the coherence theorem for braided tensor categories [30] we may and do assume that all tensor categories are strict. (In fact, most of the categories under consideration here are so by construction.)

Theorem 2.8 [18] *Let $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}_0)$ as above. Then there is a net of algebras $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ where $\mathcal{H} \supset \mathcal{H}_0$ such that*

- *$\mathcal{F}$ is a graded local net (which is local iff all objects in $DHR_f(\mathcal{A})$ have twist $+1$),*

- *the group G corresponding to the symmetric tensor category $DHR_f(\mathcal{A})$ is unitarily and faithfully represented on $\mathcal{H}$, implementing global symmetries of $\mathcal{F}$,*
- $\mathcal{F}(\mathcal{O})^G \restriction \mathcal{H}_0 = \mathcal{A}(\mathcal{O}) \quad \forall \mathcal{O} \in \mathcal{K}$,
- *the reducible representation of $\mathcal{A}$ on $\mathcal{H}$ contains every irreducible DHR sector π_ξ of $\mathcal{A}$ (of finite dimension $d(\pi_\xi)$) with multiplicity $d(\pi_\xi)$,*
- *the charged (non-G-invariant) fields intertwine the vacuum and the DHR sectors.*

The net $\mathcal{F}$, which we denote $\mathcal{F} = \mathcal{A} \rtimes DHR_f(\mathcal{A})$, is unique up to unitary equivalence. (One may also consider the crossed product $\mathcal{A} \rtimes \mathcal{S}$ with a full monoidal subcategory $\mathcal{S}$ of $DHR_f(\mathcal{A})$.)

It is natural to ask whether there is a converse to Prop. 2.5 to the effect that $\mathcal{F} = \mathcal{A} \rtimes DHR_f(\mathcal{A})$ has trivial representation theory. A first result was proved independently in [8] and [39]:

Proposition 2.9 Assume that $\mathcal{A}$ has finitely many unitary equivalence classes of irreducible DHR representations of finite statistics, all with twist $+1$. Then the local net $\mathcal{F} = \mathcal{A} \rtimes DHR_f(\mathcal{A})$ has no non-trivial DHR representations of finite statistics.

This result has the obvious weakness of being restricted to theories with finite representation theory. On the positive side, we do not need to make assumptions on potential representations of $\mathcal{A}$ with infinite statistics. For most purposes of the present paper this result is sufficient, but we cite the following recent result.

Theorem 2.10 [9] *Assume $\mathcal{A}$ lives on a separable Hilbert space and all DHR representations are direct sums of irreducible DHR representations with finite statistics and twist $+1$. Then $\mathcal{F} = \mathcal{A} \rtimes DHR(\mathcal{A})$ has no non-trivial sectors of finite or infinite statistics.*

In $1+1$-dimensional Minkowski space or on $\mathbb{R}$ (i.e. no time: '$1+0$ dimensions') the DHR analysis must be modified [20] since there one can only prove that $DHR(\mathcal{A})$ is braided. We will therefore give a brief discussion of some pertinent results on braided tensor categories. (See [30] or [31] for the basic definitions.)

3 Few Spacetime Dimensions and Modular Categories

3.1 Categorical Interlude 1: Braided Tensor Categories and Their Center. Throughout we denote morphisms in a category by small Latin letters and objects by capital Latin or, in the quantum field context, by small Greek letters. We often write XY instead of $X \otimes Y$.

Definition 3.1 A TC^* is a C^*-tensor category [35] with simple unit and conjugates (and therefore finite dimensional hom-spaces). A BTC^* is a TC^* with unitary braiding. A STC^* is a symmetric BTC^*.

A TC^* (more generally, a semisimple spherical category) will be called finite dimensional if the set Γ of isomorphism classes of simple objects is finite. Then its dimension is defined by

$$\dim \mathcal{C} \equiv \sum_{i \in \Gamma} d(X_i)^2,$$

where the $X_i, i \in \Gamma$ are representers for these classes. If $\mathcal{C}$ is braided then there is another numerical invariant, which we call the Gauss sum, defined by

$$\Delta_{\mathcal{C}} \equiv \sum_{i\in\Gamma} d(X_i)^2 \omega(X_i)^{-1}.$$

The dimensions in a TC^* (not necessarily braided!) are quantized [35] in the same way as the square roots of indices in subfactor theory:

$$d(\rho) \in \left\{2\cos\frac{\pi}{n}, n\geq 3\right\} \cup [2,\infty).$$

The twist $\omega(\rho)$ of a simple object may a priori take any value in the circle group $\mathbb{T}$. In a finite dimensional TC^*, every $d(\rho)$ is a totally real algebraic integer and $\omega(\rho)$ is a root of unity. In the braided case there is no known replacement for Thm. 2.6.

The deviation of a braided category $\mathcal{C}$ from being symmetric is measured by the monodromies

$$c_M(X,Y) = c(Y,X)\circ c(X,Y) \in \mathrm{Hom}(XY,XY), \quad X,Y\in \mathrm{Obj}\,\mathcal{C}.$$

If $\mathcal{C}$ has conjugates (in the sense of Thm. 2.3) and the unit $\mathbf{1}$ is simple then

$$S'(X,Y)\mathrm{id}_{\mathbf{1}} = (r_X^* \otimes \overline{r}_Y^*)\circ (id_{\overline{X}} \otimes c_M(X,Y)\otimes \mathrm{id}_{\overline{Y}})\circ (r_X\otimes \overline{r}_Y)$$

defines a number which depends only on the isomorphism classes of X,Y. These numbers, for irreducible X,Y, were called statistics characters in [49]. (They also give the invariant for the Hopf link with the two components colored by X,Y.) Picking arbitrary representers $X_i, i\in\Gamma$ we define the matrix $S'_{i,j} = S'(X_i,X_j),\ i,j\in\Gamma$. The matrix of statistics characters is of particular interest if the category is finite dimensional.

Then, as proved independently be Rehren [49] and Turaev [54], if S' is invertible then

$$S = (\dim\mathcal{C})^{-1/2} S', \quad T = \left(\frac{\Delta_{\mathcal{C}}}{|\Delta_{\mathcal{C}}|}\right)^{1/3} Diag(\omega_i)$$

are unitary and satisfy the relations

$$S^2 = (ST)^3 = C, \quad TC = CT,$$

where $C_{ij} = \delta_{i,\bar{j}}$ is the charge conjugation matrix (which satisfies $C^2=1$). (Whereas the dimension of a TC^* is always non-zero, this is not true in general. Yet, when S' is invertible then $\dim\mathcal{C}\neq 0$, cf. [54].) Since these relations give a presentation of the modular group $SL(2,\mathbb{Z})$ we obtain a finite dimensional unitary representation of the latter, which motivated the terminology 'modular category' [54]. Furthermore, the 'fusion coefficients' $N_{ij}^k = \dim\mathrm{Hom}(X_iX_j,X_k)$ are given by the Verlinde relation [56]

$$N_{ij}^k = \sum_m \frac{S_{im}S_{jm}S_{km}^*}{S_{0m}}.$$

The assumption that S' is invertible is not very conceptual and therefore unsatisfactory. A better understanding of its significance is obtained from the following considerations.

Definition 3.2 Let $\mathcal{C}$ be a braided monoidal category and $\mathcal{K}$ a full subcategory. Then the relative commutant $\mathcal{C}\cap\mathcal{K}'$ of $\mathcal{K}$ in $\mathcal{C}$ is the full subcategory defined by

$$\mathrm{Obj}\ \mathcal{C}\cap\mathcal{K}' = \{X\in\mathcal{C}\mid c_M(X,Y)=\mathrm{id}_{XY}\quad \forall Y\in\mathcal{K}\}.$$

($\mathcal{C} \cap \mathcal{K}'$ is automatically monoidal and replete.) The center of a braided monoidal category $\mathcal{C}$ is $\mathcal{Z}(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'$.

Remark 3.3 1. If there is no danger of confusion about the ambient category $\mathcal{C}$ we will occasionally write $\mathcal{K}'$ instead of $\mathcal{C} \cap \mathcal{K}'$.

2. $\mathcal{Z}(\mathcal{C})$ is a symmetric tensor category for every $\mathcal{C}$. $\mathcal{C}$ is symmetric iff $\mathcal{Z}(\mathcal{C}) = \mathcal{C}$.

3. The objects of the center have previously been called *degenerate* (Rehren), *transparent* (Bruguières) and *pseudotrivial* (Sawin). Yet, calling them *central* seems the best motivated terminology since the above definition is the correct analogue for braided tensor categories of the center of a monoid, as can be seen appealing to the theory of n-categories.

4. We say a semisimple category (thus in particular a BTC^*) has trivial center, denoted symbolically $\mathcal{Z}(\mathcal{C}) = \mathbf{1}$, if every object of $\mathcal{Z}(\mathcal{C})$ is a direct sum of copies of the monoidal unit $\mathbf{1}$ of if, equivalently, every simple object in $\mathcal{Z}(\mathcal{C})$ is isomorphic to $\mathbf{1}$.

5. Note that the center of a braided tensor category as given in Defin. 3.2 must not be confused with another notion of center [29, 36] which is defined for all tensor categories (not necessarily braided) and which in a sense generalizes the quantum double of Hopf algebras. See also Subsect. 5.2.

Proposition 3.4 [49] Let $\mathcal{C}$ be a BTC^* with finitely many classes of simple objects. Then the following are equivalent:

(i) The S'-matrix is invertible, thus $\mathcal{C}$ is modular.
(ii) The center of $\mathcal{C}$ is trivial.
(iii) $|\Delta_{\mathcal{C}}|^2 = \dim \mathcal{C}$.

Remark 3.5 The direction $(i) \Rightarrow (ii)$ is obvious, and $(ii) \Rightarrow (i)$ has been generalized by Bruguières [6] to a class of categories without $*$-operation, in fact over arbitrary fields. He proves that a 'pre-modular' category [4] is modular iff its dimension is non-zero (which is automatic for $*$-categories) and its center is trivial. This provides a very satisfactory characterization of modular categories and we see that modular categories are related to symmetric categories like factors to commutative von Neumann algebras. Recalling that finite dimensional *symmetric* BTC^*s are representation categories of finite groups by the DR duality theorem, one might say that modular categories ($\mathcal{Z}(\mathcal{C}) = \mathbf{1}$) differ from finite groups ($\mathcal{Z}(\mathcal{C}) = \mathcal{C}$) by the change of a single symbol in the respective definitions!

3.2 General Low Dimensional Superselection Theory. As already mentioned, in low dimensions the category $DHR(\mathcal{A})$ is only braided. As a consequence the proofs [49, 26] of the existence of conjugate (dual) representations have to proceed in a fashion completely different from [16, II]. More importantly, Thm. 2.6 and, a fortiori, Thm. 2.8 are no more applicable. (There is a weak substitute for the DR field net, cf. [21, 26] for the reduced field bundle, which however is not very useful in practice.) The facts expounded in the Categorical Interlude imply that every low dimensional QFT whose DHR category has finitely many simple objects and trivial center gives rise to a unitary representation of $SL(2,\mathbb{Z})$. This is consistent with the physics literature on rational conformal models but at first sight rather surprising in non-conformal models. (Note, however, that Haag dual theories which are massive in a certain strong sense have trivial DHR representation theory [38], implying that for them the question concerning the rôle of $SL(2,\mathbb{Z})$ does not arise.)

What remains is the issue of triviality of the center of $DHR_f(\mathcal{A})$ which does not obviously follow from the axioms. A first result in this direction was the following which proves a conjecture in [49].

Theorem 3.6 [39] *Let $\mathcal{A}$ be a Haag dual theory on 1+1 dimensional Minkowski space or on $\mathbb{R}$. Assume that $DHR_f(\mathcal{A})$ is finite and that all objects in $\mathcal{Z}(DHR_f(\mathcal{A}))$ are even, i.e. bosonic. Then $\mathcal{F} = \mathcal{A} \rtimes \mathcal{Z}(DHR_f(\mathcal{A}))$ is local and Haag dual and $DHR_f(\mathcal{F})$ has trivial center, thus is modular.*

In other terms, every rational QFT whose representation category has non-trivial center is the fixpoint theory of a theory with modular representation category under the action of a finite group of global symmetries. In the next subsection we will cite results according to which a large class of models automatically has a modular representation category. For these models the above theorem is empty, but the analysis of [39] is still relevant for the study of $\mathcal{F} = \mathcal{A} \rtimes \mathcal{S}$ where $\mathcal{S} \subset DHR_f(\mathcal{A})$ is any full symmetric subcategory, not necessarily contained in $\mathcal{Z}(DHR_f(\mathcal{A}))$.

3.3 Completely Rational Chiral Conformal Field Theories. In this section we consider chiral conformal field theories, i.e. quantum field theories on the circle. We refer to [46] for a more complete and fairly self-contained account. Let $\mathcal{I}$ be the set of intervals on S^1, i.e. connected open non-dense subsets of S^1. For every $J \subset S^1$, J' is the interior of the complement of J, and for $M \subset \mathcal{B}(\mathcal{H}_0)$ we posit $M' = \{x \in \mathcal{B}(\mathcal{H}_0) \mid xy = yx \ \forall y \in M\}$.

Definition 3.7 A chiral conformal field theory consists of

1. A Hilbert space $\mathcal{H}_0$ with a distinguished non-zero vector Ω,
2. an assignment $\mathcal{I} \ni I \mapsto \mathcal{A}(I) \subset \mathcal{B}(\mathcal{H}_0)$ of von Neumann algebras to intervals,
3. a strongly continuous unitary representation U on $\mathcal{H}_0$ of the Möbius group $PSU(1,1) = SU(1,1)/\{\mathbf{1}, -\mathbf{1}\}$, i.e. the group of those fractional linear maps $\mathbb{C} \to \mathbb{C}$ which map the circle into itself.

These data must satisfy

- Isotony: $I \subset J \ \Rightarrow\ \mathcal{A}(I) \subset \mathcal{A}(J)$,
- Locality: $I \subset J' \ \Rightarrow\ \mathcal{A}(I) \subset \mathcal{A}(J)'$,
- Irreducibility: $\vee_{I \subset S^1} \mathcal{A}(I) = \mathcal{B}(\mathcal{H}_0)$ (equivalently, $\cap_{I \in \mathcal{I}} \mathcal{A}(I)' = \mathbb{C}\mathbf{1}$),
- Covariance: $U(a)\mathcal{A}(I)U(a)^* = \mathcal{A}(aI) \quad \forall a \in PSU(1,1),\ I \in \mathcal{I}$,
- Positive energy: $L_0 \geq 0$, where L_0 is the generator of the rotation subgroup of $PSU(1,1)$,
- Vacuum: every vector in $\mathcal{H}_0$ which is invariant under the action of $PSU(1,1)$ is a multiple of Ω.

For consequences of these axioms see, e.g., [24]. We limit ourselves to pointing out some facts:

- Reeh-Schlieder property: $\overline{\mathcal{A}(I)\Omega} = \overline{\mathcal{A}(I)'\Omega} = \mathcal{H}_0 \ \ \forall I \in \mathcal{I}$.
- Type: The von Neumann algebra $\mathcal{A}(I)$ is a factor of type III_1 for every $I \in \mathcal{I}$.
- Haag duality: $\mathcal{A}(I)' = \mathcal{A}(I') \ \ \forall I \in \mathcal{I}$.
- The modular groups and conjugations associated with $(\mathcal{A}(I), \Omega)$ have a geometric meaning, cf. [7, 24].

Now one studies coherent representations $\pi = \{\pi_I, I \in \mathcal{I}\}$ of $\mathcal{A}$ on Hilbert spaces $\mathcal{H}$, where π_I is a representation of $\mathcal{A}(I)$ on $\mathcal{H}$ such that

$$I \subset J \quad \Rightarrow \quad \pi_J \restriction \mathcal{A}(I) = \pi_I.$$

One can construct [21] a unital C^*-algebra $C^*(\mathcal{A})$, the global algebra of $\mathcal{A}$, such that the coherent representations of $\mathcal{A}$ are in one-to-one correspondence with the representations of $C^*(\mathcal{A})$. We therefore simply speak of representations. A representation is covariant if there is a positive energy representation U_π of the universal covering group $\widetilde{PSU(1,1)}$ of the Möbius group on $\mathcal{H}$ such that

$$U_\pi(a)\pi_I(x)U_\pi(a)^* = \pi_{aI}(U(a)xU(a)^*) \quad \forall a \in \widetilde{PSU(1,1)}, I \in \mathcal{I}.$$

A representation is locally normal iff each π_I is strongly continuous.

In order to obtain further results we introduce additional axioms.

Definition 3.8 Two disjoint intervals $I, J \in \mathcal{I}$ are called adjacent if they have exactly one common boundary point. A chiral CQFT satisfies strong additivity if

$$I, J \text{ adjacent} \quad \Rightarrow \quad \mathcal{A}(I) \vee \mathcal{A}(J) = \mathcal{A}((I \cup J)'').$$

A chiral CQFT satisfies the split property if $I, J \in \mathcal{I}$ such that $\overline{I} \cap \overline{J} = \emptyset$ implies the existence of an isomorphism

$$\eta : \ \mathcal{A}(I) \vee \mathcal{A}(J) \to \mathcal{A}(I) \otimes \mathcal{A}(J)$$

of von Neumann algebras satisfying $\eta(xy) = x \otimes y \ \forall x \in \mathcal{A}(I), y \in \mathcal{A}(J)$.

Remark 3.9 By Möbius covariance strong additivity holds in general if it holds for one pair I, J of adjacent intervals. Strong additivity has been verified in all known rational models. Furthermore, every CQFT can be extended canonically to one satisfying strong additivity. If the split property holds then $\mathcal{H}_0$ is separable, and thanks to the Reeh-Schlieder theorem $\mathcal{A}(I) \vee \mathcal{A}(J)$ and $\mathcal{A}(I) \otimes \mathcal{A}(J)$ are actually unitarily equivalent. The split property follows if $Tr e^{-\beta L_0} < \infty$ for all $\beta > 0$, which is satisfied in all reasonable models.

Lemma 3.10 [33] *Let $\mathcal{A}$ be a CQFT satisfying strong additivity and the split property. Let $I_k \in \mathcal{I}, k = 1, \dots, n$ be intervals with mutually disjoint closures and denote $E = \cup_k I_k$. Then $\mathcal{A}(E) \subset \mathcal{A}(E')'$ is an irreducible inclusion (of type III_1 factors) and the index $[\mathcal{A}(E')' : \mathcal{A}(E)]$ depends only on the number n but not on the choice of the intervals. Let μ_n be the index for the n-interval inclusion. These numbers are related by*

$$\mu_n = \mu_2^{n-1} \quad \forall n \in \mathbb{N}.$$

(In particular $\mu_1 = 1$, which is just Haag duality.) Thus every CQFT satisfying strong additivity and the split property comes along with a numerical invariant $\mu_2 \in [1, \infty]$ whose meaning is elucidated by the main result of [33] stated below.

Definition 3.11 A chiral CQFT is completely rational if it satisfies (a) strong additivity, (b) the split property and (c) $\mu_2 < \infty$.

All known classes of rational CQFTs are completely rational in the above sense, see [57, 58] for the WZW models connected to loop groups and [59, 43] for orbifold models. Very strong results on both the structure and representation theory of completely rational theories can be proved. (All representations are understood to be non-degenerate.)

Theorem 3.12 [33, 46] *Let $\mathcal{A}$ be a completely rational CQFT. Then*

- *Every representation of $C^*(\mathcal{A})$ on a separable Hilbert space is locally normal and completely reducible, i.e. the direct sum of irreducible representations.*
- *Every irreducible separable representation has finite statistical dimension $d_\pi \equiv [\pi(\mathcal{A}(I'))' : \pi(\mathcal{A}(I))]^{1/2}$ (independent of $I \in \mathcal{I}$). It therefore* [26] *has a conjugate representation $\overline{\pi}$ and is automatically Möbius covariant with positive energy.*
- *For a representation π the following are equivalent: (a) π is Möbius covariant with positive energy, (b) π is locally normal, (c) π is a direct sum of separable representations.*
- *The category $Rep(\mathcal{A})$ of finite direct sums of separable irreducible representations has a monoidal structure with simple unit and duals (conjugates). The number of unitary equivalence classes of separable irreducible representations is finite and*

  $$\dim Rep(\mathcal{A}) = \mu_2(\mathcal{A}).$$

  Furthermore, there is a non-degenerate braiding. $Rep(\mathcal{A})$ thus is a unitary modular category in the sense of Turaev [54].

Remark 3.13 1. In the way of structure theoretical results we mention that for completely rational theories the subfactors $\mathcal{A}(E) \subset \mathcal{A}(E')'$, $E = \cup_{i=1}^n I_i$ can be analyzed quite explicitly, generalizing some of the results of [58]. Yet [33] by no means supersedes the ingenious computation in [58] of the indices $[\mathcal{A}(E')' : \mathcal{A}(E)]$ in the case of loop group models.

2. In view of the above results we do not need to worry about representations with infinite statistics when dealing with completely rational CQFTs. From now on we will write $Rep(\mathcal{A})$ instead of $DHR(\mathcal{A})$ since the (separable) representation theory can be developed without any selection criterion [46]. Some of our results hold for low-dimensional theories without the assumption of complete rationality. For this we refer to [43].

4 From $Rep(\mathcal{A})$ to $Rep(\mathcal{F})$

4.1 Pairs of Quantum Field Theories Related by a Symmetry Group. In the rest of this paper we will be concerned with pairs $(\mathcal{F}, \mathcal{A})$ of quantum field theories in one or two dimensions where $\mathcal{F}$ has a compact group G of global symmetries (acting non-trivially for $g \neq e$) and $\mathcal{A} = \mathcal{F}^G \restriction \mathcal{H}_0$. We assume that both $\mathcal{A}$ and $\mathcal{F}$ satisfy Haag duality. Then there is a full symmetric subcategory $\mathcal{S} \subset Rep(\mathcal{A})$ such that $\mathcal{S} \simeq G-\mathrm{mod}$ and $\mathcal{F} \cong \mathcal{A} \rtimes \mathcal{S}$. This situation is summarized in the quadruple $(\mathcal{F}, G; \mathcal{A}, \mathcal{S})$. Our aim will be to compute the representation category of $\mathcal{F}$ from that of $\mathcal{A}$ and vice versa. The nicest case clearly is the one where both $\mathcal{A}$ and $\mathcal{F}$ are completely rational CQFTs (then G must be finite), but some of our results hold in larger generality.

Theorem 4.1 [59, 33, 43] *Consider a pair $(\mathcal{F}, G; \mathcal{A}, \mathcal{S})$ of chiral theories. If $\mathcal{A}$ is completely rational then $\mathcal{F}$ is completely rational. With $\mathcal{F}$ completely rational, $\mathcal{A}$ is completely rational iff G is finite. In this case $\mu_2(\mathcal{A}) = |G|^2 \mu_2(\mathcal{F})$, thus*

$$\dim Rep(\mathcal{A}) = |G|^2 \dim Rep(\mathcal{F}). \tag{4.1}$$

Remark 4.2 That fixpoint nets inherit the split property from field nets is classical [14], and that $\mathcal{F}$ satisfies strong additivity if $\mathcal{A}$ does is almost trivial. The converses of these two implications are non-trivial and require the full force of complete rationality. The implication $\mathcal{F}$ completely rational $\Rightarrow$ $\mathcal{A}$ satisfies strong additivity is proved in [59], and $\mathcal{A}$ completely rational $\Rightarrow$ $\mathcal{F}$ satisfies split will be proved below. The computation of the invariant $\mu_2(\mathcal{A})$ is done already in [33].

Remark 4.3 The completely different structure (symmetric instead of modular) of the representation categories in $\geq 2+1$ dimensions is reflected in a replacement of $|G|^2$ in (4.1) by $|G|$.

In Subsect. 4.3 we will show that the representation category of $\mathcal{F}$ depends only on $Rep(\mathcal{A})$ and the symmetric subcategory $\mathcal{S}$. More precisely, let $\mathcal{A}_1, \mathcal{A}_2$ be QFTs such that $Rep(\mathcal{A}_1) \simeq Rep(\mathcal{A}_2)$ and let $\mathcal{S}_i \subset Rep(\mathcal{A}_i), i = 1, 2$ be replete full symmetric subcategories which correspond to each other under the above equivalence. Then we claim $Rep(\mathcal{F}_1) \simeq Rep(\mathcal{F}_2)$ where $\mathcal{F}_i = \mathcal{A}_i \rtimes \mathcal{S}_i$. The most natural way to prove such a result clearly is to construct a braided tensor category from $Rep(\mathcal{A})$ and $\mathcal{S}$ and to prove that it is equivalent to $Rep(\mathcal{A} \rtimes \mathcal{S})$ independently of the fine structure of $\mathcal{A}$. The next categorical interlude will provide such a construction.

4.2 Categorical Interlude 2: Galois Extensions of Braided Tensor Categories. The following result realizes a conjecture in [39].

Theorem 4.4 [4, 41] *Let $\mathcal{C}$ be a BTC^*. Let $\mathcal{S} \subset \mathcal{C}$ be a replete full monoidal subcategory which is symmetric (with the braiding of $\mathcal{C}$). Then there exists a TC^* $\mathcal{C} \rtimes \mathcal{S}$ together with a tensor functor $F : \mathcal{C} \to \mathcal{C} \rtimes \mathcal{S}$ such that*

- *F is faithful and injective on the objects, thus an embedding.*
- *F is dominant, i.e. for every simple object $X \in \mathcal{C} \rtimes \mathcal{S}$ there is $Y \in \mathcal{C}$ such that X is a subobject of $F(Y)$.*
- *F trivializes $\mathcal{S}$, i.e. $X \in \mathcal{S} \;\Rightarrow\; F(X) \cong \mathbf{1} \oplus \ldots \oplus \mathbf{1}$, where $\mathbf{1}$ appears with multiplicity $d(X)$ (which is in $\mathbb{N}$ by [17]).*
- *The pair $(\mathcal{C} \rtimes \mathcal{S}, F)$ is the universal solution for the above problem, i.e. if $F' : \mathcal{C} \to \mathcal{E}$ has the same properties then F' factorizes through F.*

Remark 4.5 This result was arrived at independently by the author [41] and (somewhat earlier) by Bruguières [4]. The above statement incorporates some results of [4]. The construction in [4] relying on Deligne's duality theorem [10] instead of the one of [18] it is slightly more general, but one must assume that the objects in $\mathcal{S}$ have integer dimension since this is no more automatic if there is no positivity. On the other hand, in [4] $\mathcal{S}$ is assumed finite dimensional (thus G is finite) and to be contained in $\mathcal{Z}(\mathcal{C})$, restrictions which are absent in [41]. Applications of the above construction to quantum groups and invariants of 3-manifolds are found in [4] and [52], the latter reference considering also relations with products of braided categories and of TQFTs.

Remark 4.6 By the universal property $\mathcal{C} \rtimes \mathcal{S}$ is unique up to equivalence. The existence is proved by explicit construction. Essentially, one adds morphisms to $\mathcal{C}$ which trivialize the objects in $\mathcal{S}$. (Then one completes such that all idempotents split, but this is of minor importance.) Here essential use is made of the fact that there is a compact (respective finite) group G such that $\mathcal{S} \simeq G-\text{mod}$.

Many facts are known about the category $\mathcal{C} \rtimes \mathcal{S}$:

Proposition 4.7 [4, 42] If $\mathcal{C}$ is finite dimensional then

$$\dim \mathcal{C} \rtimes \mathcal{S} = \frac{\dim \mathcal{C}}{\dim \mathcal{S}} = \frac{\dim \mathcal{C}}{|G|}. \tag{4.2}$$

Remark 4.8 Heuristically, the passage from $\mathcal{C}$ to $\mathcal{C} \rtimes \mathcal{S}$ amounts to dividing out the subcategory $\mathcal{S}$, an idea which is further supported by (4.2). Yet, this is not done by killing the objects of $\mathcal{S}$ in a quotient operation but rather by adding morphisms which trivialize them. Therefore the notation $\mathcal{C} \rtimes \mathcal{S}$, which is also in line with [18], seems more appropriate. We consider $\mathcal{C} \rtimes \mathcal{S}$ as a Galois extension of $\mathcal{C}$ as is amply justified by the following result.

Proposition 4.9 [41, 5] We have $G \cong \mathrm{Aut}_{\mathcal{C}}(\mathcal{C} \rtimes \mathcal{S})$ and there is a Galois correspondence between closed subgroups H of G and TC*s $\mathcal{E}$ satisfying $\mathcal{C} \subset \mathcal{E} \subset \mathcal{C} \rtimes \mathcal{S}$. (The correspondence is given by $\mathcal{E} = (\mathcal{C} \rtimes \mathcal{S})^H$ and $H = \mathrm{Aut}_{\mathcal{E}}(\mathcal{C} \rtimes \mathcal{S})$.) Here H is normal iff $\mathcal{E} = \mathcal{C} \rtimes \mathcal{T}$ with $\mathcal{T}$ a replete full subcategory of $\mathcal{S}$, in which case $\mathrm{Aut}_{\mathcal{C}}(\mathcal{E}) \cong G/H$.

Theorem 4.10 [41] *The braiding of $\mathcal{C}$ lifts to a braiding of $\mathcal{C} \rtimes \mathcal{S}$ iff $\mathcal{S} \subset \mathcal{Z}(\mathcal{C})$. In this case $\mathcal{C} \rtimes \mathcal{S}$ has trivial center iff $\mathcal{S} = \mathcal{Z}(\mathcal{C})$. $\mathcal{C} \rtimes \mathcal{Z}(\mathcal{C})$ is called the modular closure $\overline{\mathcal{C}}^m$ of $\mathcal{C}$ since it is modular if $\mathcal{C}$ is finite dimensional.*

Remark 4.11 This result has obvious applications to the topology of 3-manifolds since it provides a means of constructing a modular category out of every finite dimensional braided tensor category (which must not be symmetric). In fact, ad hoc versions of the above constructions in simple special cases motivated by topology had appeared before.

If $\mathcal{S} \not\subset \mathcal{Z}(\mathcal{C})$ then $\mathcal{C} \rtimes \mathcal{S}$ fails to have a braiding in the usual sense. Yet, there is a braiding in the following generalized sense.

Definition 4.12 Let $\mathcal{C}$ be a semisimple k-linear category over a field k. If G is a group then $\mathcal{C}$ is G-graded if

1. With every simple object X is associated an element $gr(X) \in G$.
2. If X, Y are simple and isomorphic then $gr(X) = gr(Y)$.
3. Let $\mathcal{C}_g$ be the full subcategory of $\mathcal{C}$ whose objects are finite direct sums of objects with grade g. Then $X \in \mathcal{C}_g, Y \in \mathcal{C}_h$ implies $X \otimes Y \in \mathcal{C}_{gh}$.

If $\mathcal{C}$ is G-graded and carries a G-action such that

$$\alpha_g(\mathcal{C}_h) = \mathcal{C}_{ghg^{-1}} \quad \forall g, h \in G,$$

then $\mathcal{C}$ is a crossed G-category. A crossed G-category is braided if there are isomorphisms $c(X,Y) : X \otimes Y \to \alpha_{gr(X)}(Y) \otimes X$ for all Y and all homogeneous X, i.e. $X \in \mathcal{C}_g$ for some $g \in G$. For the relations which c must satisfy cf. [55].

Remark 4.13 Our definitions are slightly more general than those given by Turaev in [55] in that we allow for direct sums of objects of different grade. This complicates the definition of the braiding, but the gained generality is needed in our applications.

Theorem 4.14 [42] *Let $\mathcal{C}$ be a BTC* and $\mathcal{S}$ a replete full symmetric subcategory. Then $\mathcal{C} \rtimes \mathcal{S}$ is a crossed G-category. The zero-grade part of $\mathcal{C} \rtimes \mathcal{S}$ is given by*

$$(\mathcal{C} \rtimes \mathcal{S})_e = (\mathcal{C} \cap \mathcal{S}') \rtimes \mathcal{S},$$

which is always a BTC and has trivial center iff $\mathcal{Z}(\mathcal{C}) \subset \mathcal{S}$. The set H of $g \in G$ for which $\mathcal{C}_g$ is non-empty is a closed normal subgroup which corresponds to $\mathcal{S} \cap \mathcal{Z}(\mathcal{C})$ under the bijection between closed normal subgroups of G and replete full monoidal subcategories of $\mathcal{S}$. Thus the grading is full ($(\mathcal{C} \rtimes \mathcal{S})_g \neq \emptyset \ \forall g \in G$) iff $\mathcal{S} \cap \mathcal{Z}(\mathcal{C}) = \mathbf{1}$*

and trivial ($\mathcal{C} \rtimes \mathcal{S} = (\mathcal{C} \rtimes \mathcal{S})_e$) *iff* $\mathcal{S} \subset \mathcal{Z}(\mathcal{C})$. *If* $\mathcal{C}$ *is modular then* $\mathcal{C} \rtimes \mathcal{S}$ *is modular in the sense of* [55] *with full grading for every* $\mathcal{S}$.

This result will be relevant when we compute $Rep(\mathcal{A})$ in Sect. 5.

Proposition 4.15 [41] Let $X \in \mathcal{C}$ be simple. Then all simple subobjects X_i of $F(X) \in \mathcal{C} \rtimes \mathcal{S}$ occur with the same multiplicity and have the same dimension. If $\mathcal{S} \subset \mathcal{Z}(\mathcal{C})$, thus $\mathcal{C} \rtimes \mathcal{S}$ is braided, then all X_i have the same twist as X, and they are either all central or all non-central according to whether X is central or non-central.

Given irreducible objects $X \ncong Y$ in $\mathcal{C}$ we should also understand whether they can have equivalent subobjects in $\mathcal{C} \rtimes \mathcal{S}$. We have

Proposition 4.16 [42] Let X, Y be simple objects in $\mathcal{C}$. We write $X \sim Y$ iff there is $Z \in \mathcal{S}$ such that $\mathrm{Hom}(ZX, Y) \neq \{0\}$. This defines an equivalence relation which is weaker than isomorphism $X \cong Y$. If $X \sim Y$ then $F(X)$ and $F(Y)$ contain the same (isomorphism classes of) simple objects of $\mathcal{C} \rtimes \mathcal{S}$ (whose multiplicity in $F(X)$ and $F(Y)$ need not be the same), otherwise $\mathrm{Hom}(F(X), F(Y)) = \{0\}$.

In the case of abelian extensions one can give a more complete analysis. Recall that G is abelian iff every simple object in $\mathcal{S} \cong G-\mathrm{mod}$ has dimension one (equivalently, is invertible up to isomorphism). In this case the set of isomorphism classes of simple objects in $\mathcal{S}$ is an abelian group $K \cong \hat{G}$ (as opposed to an abelian semigroup in the general case). Since the tensor product of a simple and an invertible object is simple, K acts on the set Γ of isomorphism classes of simple objects of $\mathcal{C}$ (by tensoring of representers). For every simple $X \in \mathcal{C}$ we define $K_X = \{[Z] \in K \mid [Z][X] = [X]\}$, the stabilizer of X, which is a finite subgroup of K. (Non-trivial stabilizers can exist since generically there is no cancellation in Γ!)

Proposition 4.17 [41] For every simple $X \in \mathcal{C}$ there is a subgroup $L_X \subset K_X$ such that $N_X = [K_X : L_X]^{1/2} \in \mathbb{N}$ and

$$F(X) \cong N_X \bigoplus_{\chi \in \widehat{L_X}} X_\chi,$$

where the X_χ are mutually inequivalent simple objects in $\mathcal{C} \rtimes \mathcal{S}$. K_X and L_X depend only on the image $\underline{X}$ of X in Γ/K (i.e. the K-orbit in Γ which contains $[X]$). The isomorphism classes of simple objects in $\mathcal{C} \rtimes \mathcal{S}$ are labeled by pairs $(\underline{X}, \chi)$, where $\underline{X} \in \Gamma/K$ and $\chi \in \widehat{L_{\underline{X}}}$.

Proposition 4.18 [42] Assume $\mathcal{S} \subset \mathcal{Z}(\mathcal{C})$. (If necessary enforce this by replacing $\mathcal{C} \to \mathcal{C} \cap \mathcal{S}'$.) There are unitary matrices $(S^{[Z]}(\underline{X}, \underline{Y}))_{\underline{X},\underline{Y}}$ where $\underline{X}, \underline{Y} \in \Gamma/K$ are Z-fixpoints (i.e. $[Z] \in K_{\underline{X}} \cap K_{\underline{Y}}$), such that

$$S_{\mathcal{C} \rtimes \mathcal{S}}((\underline{X}, \chi), (\underline{Y}, \nu)) = \frac{|G|}{|K_{\underline{X}}||L_{\underline{X}}||K_{\underline{Y}}||L_{\underline{Y}}|} \sum_{Z \in L_{\underline{X}} \cap L_{\underline{Y}}} \chi(Z) \overline{\nu(Z)}\, S^{[Z]}(\underline{X}, \underline{Y}). \quad (4.3)$$

Together with the (appropriately restricted) T-matrix of $\mathcal{C}$ the matrices $S^{[Z]}$ satisfy the relations of the mapping class group of the torus with a puncture [1].

The two preceding propositions are abstract versions of heuristically derived results in [23] which provided decisive motivation.

4.3 Computation of $Rep(\mathcal{F})$. Given a $\rho \in Rep(\mathcal{A})$ of $\mathcal{A}$ which is localized in some double cone $\mathcal{O}$ there exists [50, 39] an extension $\hat{\rho}$ to an endomorphism of $\mathcal{F} = \mathcal{A} \rtimes \mathcal{S}$ which commutes with the action of G. It is determined by

$$\hat{\rho}(\psi) = c(\gamma, \rho)\psi, \quad \text{where} \quad \psi \in \mathcal{H}_\gamma,$$

where $\gamma \in \mathcal{S}$ and the spaces

$$\mathcal{H}_\gamma = \{\psi \in \mathcal{F} \mid \psi A = \gamma(A)\psi \ \ \forall A \in \mathcal{A}\}$$

generate $\mathcal{F}$ linearly. In $\geq 2+1$ dimensions this extension is unique and again localized in $\mathcal{O}$ since $c(\gamma, \rho) = 1$ whenever ρ, γ have spacelike localization regions. For theories in $1+1$ dimensional Minkowski space or on $\mathbb{R}$, however, there is an a priori different extension obtained by replacing $c(\gamma, \rho)$ by $c(\rho, \gamma)^*$, and for spacelike localized ρ, γ a priori only one of these equals $\mathbf{1}$. Thus the two extensions are solitonic, i.e. localized in left and right, respectively, wedges or half-lines. They coincide and are localized in $\mathcal{O}$ iff the two braidings are the same for all $\gamma \in \mathcal{S}$, thus precisely if $\rho \in Rep(\mathcal{A}) \cap \mathcal{S}'$. For theories on S^1 an extension $\hat{\rho}$ does not even exist as a representation of $C^*(\mathcal{A})$ if $\rho \notin \mathcal{S}'$. The map $Rep(\mathcal{A}) \cap \mathcal{S}' \to Rep(\mathcal{F}), \ \rho \mapsto \hat{\rho}$ being functorial, it follows easily from the definition of $\mathcal{F} = \mathcal{A} \rtimes \mathcal{S}$ and $\mathcal{C} \rtimes \mathcal{S}$ that $(Rep(\mathcal{A}) \cap \mathcal{S}') \rtimes \mathcal{S}$ is (equivalent as a braided monoidal category to) a replete full monoidal subcategory of $Rep(\mathcal{F})$. In fact, by an argument similar to the one used in Sect. 2.2 one can prove that this exhausts the sectors of $\mathcal{F}$ and one obtains:

Theorem 4.19 [43] *The representation category of $\mathcal{F}$ is given (up to equivalence of braided tensor $*$-categories) by*

$$Rep(\mathcal{F}) \simeq (Rep(\mathcal{A}) \cap \mathcal{S}') \rtimes \mathcal{S}. \tag{4.4}$$

Thus $Rep(\mathcal{F})$ depends only on $Rep(\mathcal{A})$ and the symmetric subcategory $\mathcal{S}$.

(This result together with Thm. 4.10 provides another proof of Thm. 3.6.)

The operation $\mathcal{A} \to \mathcal{F} = \mathcal{A} \rtimes \mathcal{S}$ where G is abelian is called a *simple current extension*. Bringing to bear our results on abelian Galois extensions we obtain the following theorem which proves most of the observations of [23] many of which were based on consistency checks rather than proofs.

Theorem 4.20 [43] *Consider $(\mathcal{F}, G; \mathcal{A}, \mathcal{S})$ with G abelian. Then the equivalence classes of irreducible localized representations of $\mathcal{F}$ are labeled by pairs $(\underline{X}, \chi)$, where $\underline{X} \in Obj(Rep(\mathcal{A}) \cap \mathcal{S}')/\cong/K$ and $\chi \in \widehat{L_{\underline{X}}}$. Here $K \cong \hat{G}$ is the set of isomorphism classes of simple objects in $\mathcal{S}$ and $L_{\underline{X}}$ is as in Prop. 4.17. The modular S-matrix of $Rep(\mathcal{F})$ is given by formula (4.3).*

Contrary to the fixpoint problem $\mathcal{F} \to \mathcal{A} = \mathcal{F}^G$, non-abelian *extensions* $\mathcal{A} \to \mathcal{F} = \mathcal{A} \rtimes \mathcal{S}$ seem not to have been considered in the physics literature. (This is perhaps not surprising since they require the duality theorems either of Doplicher/Roberts or Deligne.)

Assuming that $\mathcal{A}$ is completely rational we know by Thm. 4.1 that $\mathcal{F}$ is completely rational, thus $Rep(\mathcal{F})$ is modular. In view of the 'explicit' formula (4.4) for $Rep(\mathcal{F})$ and of Thm. 4.10 we can conclude that

$$\mathcal{Z}(Rep(\mathcal{A}) \cap \mathcal{S}') = \mathcal{S}. \tag{4.5}$$

Furthermore, the dimension of $Rep(\mathcal{F})$ is given by $\dim Rep(\mathcal{A})/|G|^2$. Comparing this with (4.2) we infer

$$\dim(Rep(\mathcal{A}) \cap \mathcal{S}') = \frac{\dim Rep(\mathcal{A})}{\dim \mathcal{S}}. \tag{4.6}$$

There should clearly be a purely categorical proof of these two observations. In fact, the result holds in considerably larger generality and is the subject of our next categorical interlude.

4.4 Categorical Interlude 3: Double Commutants in Modular Categories. For obvious reasons the following result will be called the (double) commutant theorem for modular categories.

Theorem 4.21 [42] *Let $\mathcal{C}$ be a modular BTC^* and let $\mathcal{K} \subset \mathcal{C}$ be a replete full sub TC^* closed w.r.t. direct sums and subobjects (as far as they exist in $\mathcal{C}$). Then we have*

(a) $\mathcal{K}'' = \mathcal{K}$,
(b) $\dim \mathcal{K} \cdot \dim \mathcal{K}' = \dim \mathcal{C}$.

Remark 4.22 The double commutant property (a) appears first (without published proof) in the notes [48] in connection with Ocneanu's asymptotic subfactor [47]. In the subfactor setting (a) and (b) are proved in [28]. A simple argument proving (a) and (b) in one stroke in the more general setting of C^*-categories appears in [42]. Finally, the theorem was then extended [6] to categories $\mathcal{C}$ which are semisimple spherical with non-zero dimension. It seems likely that this is the most general setting where it holds.

Thm. 4.21 has many applications, the first of which is the desired purely categorical proof of (4.5). Let thus $\mathcal{C}$ be modular and $\mathcal{S}$ a replete full monoidal subcategory. Then

$$\mathcal{Z}(\mathcal{C} \cap \mathcal{S}') = \mathcal{C} \cap \mathcal{S}' \cap (\mathcal{C} \cap (\mathcal{C} \cap \mathcal{S}')') = \mathcal{S}' \cap \mathcal{S}'' = \mathcal{S}' \cap \mathcal{S} = \mathcal{Z}(\mathcal{S}). \tag{4.7}$$

If $\mathcal{S}$ is symmetric, thus $\mathcal{Z}(\mathcal{S}) = \mathcal{S}$, (4.5) follows at once, and (4.6) is just a special case of (b).

Consider now a modular category $\mathcal{C}$ with a replete full *modular* subcategory $\mathcal{K}$. Modularity being equivalent to triviality of the center by Prop. 3.4, (4.7) implies the following.

Corollary 4.23 Let $\mathcal{C}$ be a modular BTC^* and let $\mathcal{K} \subset \mathcal{C}$ be a replete full modular sub TC^*. Then $\mathcal{L} = \mathcal{C} \cap \mathcal{K}'$ is modular, too.

A surprisingly easy argument now proves:

Theorem 4.24 [42] *Let $\mathcal{K} \subset \mathcal{C}$ be a replete full inclusion of modular BTC^*s and let $\mathcal{L} = \mathcal{C} \cap \mathcal{K}'$. Then there is an equivalence of braided monoidal categories:*

$$\mathcal{C} \simeq \mathcal{K} \otimes_{\mathbb{C}} \mathcal{L},$$

where $\otimes_{\mathbb{C}}$ is the product in the sense of enriched category theory.

Remark 4.25 This result implies that every modular category is a direct product of prime ones, the latter being defined by the absence of proper replete full modular subcategories. Again this holds beyond the setting of $*$-categories [6]. The question in which sense this factorization might be unique is quite non-trivial.

It is also interesting to note the analogy with the well-known result from the theory of von Neumann algebras where an inclusion $A \subset B$ of type I factors gives rise to an isomorphism $B \cong A \otimes (B \cap A')$.

5 From $Rep(\mathcal{F})$ to $Rep(\mathcal{A})$

5.1 Does $Rep(\mathcal{F})$ Determine $Rep(\mathcal{A})$? By the results of Subsect. 4.3 we have

$$Rep(\mathcal{F}) \simeq (Rep(\mathcal{A}) \cap \mathcal{S}') \rtimes \mathcal{S}.$$

The Galois group G (which is determined up to isomorphism by $G - \text{mod} \simeq \mathcal{S}$) acts on $Rep(\mathcal{F})$ and the fixpoints are given by

$$Rep(\mathcal{F})^G \simeq Rep(\mathcal{A}) \cap \mathcal{S}' \subset Rep(\mathcal{A}).$$

Thus the category $Rep(\mathcal{F})^G$, which consists just of those localized transportable endomorphisms of $\mathcal{F}$ which commute with all α_g, is only a full subcategory of $Rep(\mathcal{A})$, viz. precisely $Rep(\mathcal{A}) \cap \mathcal{S}'$. The latter cannot coincide with $Rep(\mathcal{A})$ since this would mean $\mathcal{S} \subset \mathcal{Z}(Rep(\mathcal{A}))$, whereas we know that $Rep(\mathcal{A})$ has trivial center.

Abstractly the situation is the following. We have a non-modular category $\mathcal{C}_0 = Rep(\mathcal{A}) \cap \mathcal{S}'$ and its modular closure $\overline{\mathcal{C}_0}^m = \mathcal{C}_0 \rtimes \mathcal{Z}(\mathcal{C}_0) \cong Rep(\mathcal{F})$ familiar from Sect. 4.2. But we also have a modular category $\mathcal{C} = Rep(\mathcal{A})$ which contains $\mathcal{C}_0$ as a full subcategory. The dimensions of the categories in question are:

$$\begin{aligned} \dim \overline{\mathcal{C}_0}^m &= \frac{\dim \mathcal{C}_0}{\dim \mathcal{Z}(\mathcal{C}_0)}, \\ \dim \mathcal{C} &= \dim \mathcal{C}_0 \cdot \dim \mathcal{Z}(\mathcal{C}_0). \end{aligned} \tag{5.1}$$

This suggests the conjecture that every non-modular category $\mathcal{C}_0$ embeds as a full subcategory into a modular category $\mathcal{C}$ such that (5.1) holds. We will look into this problem in the next Categorical Interlude, without however giving a proof.

5.2 Categorical Interlude 4: Constructing Modular Categories. In Categorical Interlude 2 we have constructed modular categories out of braided categories by adding morphisms, which heuristically amounts to dividing out the center. In Subsection 4.3 we have seen that this categorical construction reflects what happens in the passage from $Rep(\mathcal{A})$ to $Rep(\mathcal{F})$.

Given a braided tensor category $\mathcal{C}$ with non-trivial center one might wish to construct a modular category $\mathcal{M}$ into which $\mathcal{C}$ is embedded as a full subcategory, i.e. without tampering with $\mathcal{C}$ as done in Subsect. 4.2.

Lemma 5.1 *Let $\mathcal{M}$ be a modular BTC* and $\mathcal{C} \subset \mathcal{M}$ a replete full sub TC*. Then*

$$\dim \mathcal{M} \geq \dim \mathcal{C} \cdot \dim \mathcal{Z}(\mathcal{C}). \tag{5.2}$$

Proof. The obvious inclusion

$$\mathcal{M} \cap \mathcal{C}' \supset \mathcal{C} \cap \mathcal{C}' = \mathcal{Z}(\mathcal{C})$$

in conjunction with Thm. 4.21 implies for any modular extension $\mathcal{M}$

$$\dim(\mathcal{M} \cap \mathcal{C}') = \frac{\dim \mathcal{M}}{\dim \mathcal{C}} \geq \dim \mathcal{Z}(\mathcal{C})$$

and thus the bound (5.2). ∎

Conjecture 5.2 For every BTC^* $\mathcal{C}$ of finite dimension there exists a modular extension $\mathcal{M}$ of dimension

$$\dim \mathcal{M} = \dim \mathcal{C} \cdot \dim \mathcal{Z}(\mathcal{C}). \tag{5.3}$$

An equivalent conjecture was formulated, in fact claimed to be true without proof, by Ocneanu [48]. We do not have any doubt concerning its correctness but, unfortunately, we are not aware of a proof. The considerations of the preceding subsection show that the conjecture is in fact true for all categories of the form $Rep(\mathcal{A}) \cap \mathcal{S}'$, where $\mathcal{A}$ is a CQFT and $\mathcal{S}$ is a full symmetric subcategory of $Rep(\mathcal{A})$. Since we do not know that all BTC^*s actually appear as representation categories of some CQFT this provides evidence for the conjecture, but no proof.

If $\mathcal{C}$ is already modular, i.e. $\dim \mathcal{Z}(\mathcal{C}) = 1$, then $\mathcal{M} = \mathcal{C}$ clearly is a minimal modular extension. On the other hand, if $\mathcal{Z}(\mathcal{C}) = \mathcal{C}$ then $\mathcal{C} \simeq G - \text{mod}$ for a finite group G and a modular extension of dimension $|G|^2 = (\dim \mathcal{C})^2$, thus minimal, is given by $D^\omega(G) - \text{mod}$, where $\omega \in Z^3(G, \mathbb{T})$ and $D^\omega(G)$ is the twisted quantum double [12]. Since $D^{\omega_1}(G) - \text{mod} \not\simeq D^{\omega_2}(G) - \text{mod}$ if $[\omega_1] \neq [\omega_1]$ this example shows already that the minimal extension need not be unique. But apart from these easy cases it is a priori not obvious that it is at all possible to fully embed braided tensor categories into modular ones, even in a non-minimal way. This is proven by the 'center construction' for tensor categories [29, 36], a construction which produces a braided tensor category $\mathcal{D}(\mathcal{C})$ out of any (not necessarily braided!) tensor category $\mathcal{C}$. If $\mathcal{C}$ happens to be braided then it imbeds into $\mathcal{D}(\mathcal{C})$ as a replete full subcategory. The category $\mathcal{D}(\mathcal{C})$ generalizes the quantum double $D(H)$ of a finite dimensional Hopf algebra H in the sense that there is an equivalence of braided tensor categories

$$\mathcal{D}(H - \text{mod}) \simeq D(H) - \text{mod}.$$

(See [31, Sect. XIII.4] for a nice presentation of all this.) For this reason – and also in order to avoid confusion with the center $\mathcal{Z}(\mathcal{C})$ of a braided tensor category as defined in Sect. 3.1 – we refer to $\mathcal{D}(\mathcal{C})$ as the quantum double of $\mathcal{C}$. In [19] it has been shown $\mathcal{D}(\mathcal{C})$ is a modular category if $\mathcal{C} \simeq H - \text{mod}$ where H is a finite dimensional semisimple Hopf algebra over a field of characteristic zero. This can been generalized to the much wider setting of tensor categories:

Theorem 5.3 [45] *Let $\mathcal{C}$ be a semisimple spherical tensor category $\mathcal{C}$ with non-zero dimension over an algebraically closed field (of arbitrary characteristic). (Finite dimensional BTC^* belong to this class). Then $\mathcal{D}(\mathcal{C})$ is semisimple, spherical and modular with*

$$\dim \mathcal{D}(\mathcal{C}) = (\dim \mathcal{C})^2. \tag{5.4}$$

Remark 5.4 Eq. (5.4) clearly is the only identity which is compatible with the special case $\mathcal{C} \simeq H - \text{mod}$. Concerning the proofs we must limit ourselves to the remark that they are based on the adaption [44] of results from subfactor theory to category theory. These works owe much to [28] which provides not only the crucial motivation but also bits of the proof.

By the above, $\mathcal{D}(\mathcal{C})$ provides a modular extension of $\mathcal{C}$, which is minimal iff $\mathcal{C} = \mathcal{Z}(\mathcal{C})$, i.e. if $\mathcal{C}$ is symmetric, as one sees comparing (5.4) with (5.3). One might hope that a *minimal* modular extension can be constructed by a modification of the quantum double.

As another application of the double commutant theorem we exhibit a construction which provides many examples of BTC^*s which admit a minimal modular extension.

Proposition 5.5 [42] Let $\mathcal{C}$ be a finite dimensional BTC^* and let $\mathcal{E}$ be the full monoidal subcategory of the quantum double $\mathcal{D}(\mathcal{C})$ which is generated by $\mathcal{C}$ and $\mathcal{D}(\mathcal{C}) \cap \mathcal{C}'$. Then $\mathcal{Z}(\mathcal{E}) = \mathcal{Z}(\mathcal{C})$ and $\dim \mathcal{E} = (\dim \mathcal{C})^2 / \dim \mathcal{Z}(\mathcal{C})$, thus $\dim \mathcal{D}(\mathcal{C}) = \dim \mathcal{E} \cdot \dim \mathcal{Z}(\mathcal{E})$ and the quantum double $\mathcal{D}(\mathcal{C})$ is a minimal modular extension of $\mathcal{E}$.

5.3 Computing $Rep(\mathcal{A})$: Soliton Endomorphisms. We have seen that it is not possible to compute $Rep(\mathcal{A})$ knowing just $Rep(\mathcal{F})$. Thus we must use properties of $\mathcal{F}$ which go beyond the localized representations. The aim of this subsection is to identify the additional information we need. We have already used the fact that every localized endomorphism ρ of $\mathcal{A}$ extends to an endomorphism $\hat{\rho}$ of $\mathcal{F}$ which is localized iff $\rho \in \mathcal{S}'$. Thus, trivially, every $\rho \in Rep(\mathcal{A})$ is obtained as restriction of $\hat{\rho}$ to $\mathcal{A}$. This makes clear that we should understand the nature of $\hat{\rho}$ for $\rho \notin \mathcal{S}'$.

In the following discussion we consider theories on $\mathbb{R}$ or $1+1$ dimensional Minkowski space. (In the case of theories living on S^1 one must remove an arbitrary point 'at infinity' in order for $\hat{\rho}$ to be well defined.) For any double cone $\mathcal{O}$ (or interval I) we denote by $\mathcal{O}_L$ (resp. $\mathcal{O}_R$) its left (resp. right) spacelike complement. A endomorphism of $\mathcal{F}$ which acts on $\mathcal{F}(\mathcal{O}_R)$ like α_g for some $g \in G$ and as the identity on $\mathcal{F}(\mathcal{O}_L)$ is called a right handed g-soliton endomorphism associated with in $\mathcal{O}$. (Left handed soliton endomorphisms are of course defined analogously, but it is sufficient to consider one species.) A G-soliton endomorphism is a g-soliton associated with some $\mathcal{O} \in \mathcal{K}$ for some $g \in G$. We emphasize that for $\rho \in \mathcal{S}'$, $\hat{\rho}$ is a bona fide superselection sector (possibly reducible) of $\mathcal{F}$, but the soliton endomorphisms $\hat{\rho}$ of the quasilocal algebra $\mathcal{F}$ arising if $\rho \notin \mathcal{S}'$ provably do not admit extension to locally normal representations on S^1. Heuristically this is clear since they 'act discontinuously at infinity'.

Lemma 5.6 [43] *Consider $(\mathcal{F}, G; \mathcal{A}, \mathcal{S})$ with G compact but not necessarily finite. Let ρ be an irreducible transportable endomorphism of $\mathcal{A}$ localized in a double cone $\mathcal{O}$. Let $\hat{\rho}$ be its (right hand localized) extension to $\mathcal{F}$ and let $\hat{\rho}_1$ be an irreducible submorphism of $\hat{\rho}$. (If $E \in \mathcal{F} \cap \hat{\rho}(\mathcal{F})' \subset \mathcal{F}(\mathcal{O})$ is the corresponding minimal projection we pick an isometry in $\mathcal{F}(\mathcal{O})$ in order to define $\hat{\rho}_1$.) Then there is $g \in G$ such that*

$$\hat{\rho}_1 \restriction \mathcal{F}(\mathcal{O}_R) = \alpha_g.$$

In particular, if $\hat{\rho}$ is irreducible then there is $g \in Z(G)$ such that

$$\hat{\rho} \restriction \mathcal{F}(\mathcal{O}_R) = \alpha_g.$$

The latter is, a fortiori, the case if ρ is a localized automorphism of $\mathcal{A}$.

Lemma 5.6 shows that every irreducible localized endomorphism of $\mathcal{A}$ is the restriction to $\mathcal{A}$ of a direct sum of G-soliton endomorphisms of $\mathcal{F}$. The following is more precise.

Proposition 5.7 [43] Let $\mathcal{A}, \mathcal{F}, \rho, \hat{\rho}$ be as in Lemma 5.6. Then there is a conjugacy class c of G such that $\hat{\rho}$ contains an irreducible g-soliton endomorphism iff $g \in c$. The adjoint action of the group G on the (equivalence classes of) irreducible submorphisms of $\hat{\rho}$ is transitive. Thus all irreducible soliton endomorphisms contained in $\hat{\rho}$ have the same dimension and appear with the same multiplicity.

Now we make the connection between Thm. 4.14 and the case of QFT at hand.

Definition 5.8 Let $\mathcal{F}$ be a CQFT with compact global symmetry group G. The category $G - \mathrm{Sol}(\mathcal{F})$ is the category whose objects are transportable G-soliton endomorphisms with finite index and all finite direct sums of them (not necessarily corresponding to the same $g \in G$). The morphisms are the intertwiners in $\mathcal{F}$.

Proposition 5.9 [43] The category $G - \mathrm{Sol}(\mathcal{F})$ is a crossed G-catgory in the sense of Defin. 4.12. (The action of G is by $\rho \mapsto \alpha_g \circ \rho \circ \alpha_g^{-1}$ on the objects and by $s \mapsto \alpha_g(s)$ on the morphisms $s \in \mathcal{F}$.)

Theorem 5.10 [43] *Consider a pair $(\mathcal{F}, G; \mathcal{A}, \mathcal{S})$ with G finite. Then we have the equivalences*

$$\begin{aligned} G - Sol(\mathcal{F}) &\simeq Rep(\mathcal{A}) \rtimes \mathcal{S}, \\ Rep(\mathcal{A}) &\simeq [G - Sol(\mathcal{F})]^G \end{aligned}$$

of braided G-crossed tensor categories and braided tensor categories, respectively.

The situation can be neatly summarized in the following diagram, where the horizontal inclusions of categories are full. (A very similar diagram appeared in [37] in a massive context where, however, one has to do with partially broken quantum symmetries.)

$$\begin{array}{ccccccc} & & \xleftarrow{\quad 0-\text{grade} \quad} & & & & \\ & & Rep(\mathcal{F}) & \subset & \text{G}-\text{Sol}(\mathcal{F}) & & \\ \cdot \rtimes \mathcal{S} & \Big\uparrow & \cup & & \cup & \Big\downarrow & G - \text{fixpoints} \\ & & Rep(\mathcal{A}) \cap \mathcal{S}' & \subset & Rep(\mathcal{A}) & & \end{array}$$

In view of these results it is clearly desirable to know which soliton endomorphisms a theory $\mathcal{F}$ with global symmetry G admits. This is partially answered by the following result. We say that $\mathcal{F}$ admits g-soliton endomorphisms if for every $\mathcal{O} \in \mathcal{K}$ there is an irreducible g-soliton endomorphism associated with $\mathcal{O}$.

Corollary 5.11 Let $\mathcal{F}$ be completely rational and let α_g be a global symmetry of finite order, i.e. $\alpha_g^N = \mathrm{id}$ for some $N \in \mathbb{N}$. Then $\mathcal{F}$ admits g-soliton endomorphisms.

Proof. Let G be the finite cyclic group generated by α_g and $\mathcal{A} = \mathcal{F}^G$. By Thm. 4.1 and Thm. 3.12 we have $\mathcal{Z}(Rep(\mathcal{A})) = \mathbf{1}$. Then Thm. 4.14 implies that the grading of $G - \mathrm{Sol}(\mathcal{F}) \simeq Rep(\mathcal{A}) \rtimes \mathcal{S}$ is full. ■

Remark 5.12 1. Now we can complete the proof of the implication $\mathcal{A}$ completely rational $\Rightarrow$ $\mathcal{F}$ satisfies split (thus complete rationality). By Thm. 3.12, $Rep(\mathcal{A})$ is modular, thus the grading of $Rep(\mathcal{A}) \rtimes \mathcal{S}$ is full by Thm. 4.14. Let $I, J \in \mathcal{I}$ satisfy $\overline{I} \cap \overline{J} = \emptyset$ and $x \notin \overline{I \cup J}$. By Thm. 5.10 (whose proof does not assume complete rationality of $\mathcal{F}$!) $\mathcal{F} \restriction S^1 - \{x\}$ admits g-soliton endomorphisms for all $g \in G$. Using the latter one can construct a normal conditional expectation

$$m_2 : \mathcal{F}(I) \vee \mathcal{F}(J) \longrightarrow \mathcal{F}(I) \vee \mathcal{A}(J).$$

The rest of the proof works as in [14, Sect. 5].
2. If $\mu_2(\mathcal{F}) = 1$ one can prove that $\mathcal{F}$ admits soliton *auto*morphisms, see the next subsection. We expect that there are direct proofs of Coro. 5.11 and the above fact which avoid the detour through the fixpoint theory and its modularity and thus might work without the periodicity restriction.

Putting everything together we have the following generalization of Thm. 3.12:

Theorem 5.13 *Let $\mathcal{F}$ be a completely rational CQFT with finite group G of global symmetries. Then $G - Sol(\mathcal{F})$ is a modular crossed G-category in the sense of* [55] *with full grading.*

We end this section with the computation of $Rep(\mathcal{A})$ in a relatively simple albeit non-trivial and instructive example.

5.4 An Example: Holomorphic Orbifold Models. In order to illustrate the computation of the representation category of a fixpoint theory we consider the simplest possible example, namely the case where the net $\mathcal{F}$ is completely rational with $\mu_2(\mathcal{F}) = 1$, i.e. without non-trivial sectors. Even though the analysis of this particular case reduces essentially to an exercise in low dimensional group cohomology it is quite instructive and allows us to clarify, prove and extend the results of the heuristic discussion in [11] and to expose the link with [12] and – to a lesser extent – with [13].

By the analysis in Subsect. 5.3 we know that the grading of $G - Sol(\mathcal{F})$ is full, thus $\dim[G - Sol(\mathcal{F})]_g \geq 1 \ \forall g \in G$. Together with $\dim G - Sol(\mathcal{F}) = |G| \cdot \dim Rep(\mathcal{F}) = |G|$ this clearly implies that each of the categories $[G - Sol(\mathcal{F})]_g$ has exactly one isomorphism class of simple objects, all of dimension one and thus invertible. Therefore $G - Sol(\mathcal{F})$ is (equivalent to) the monoidal category $\mathcal{C}(G, \Phi)$ determined (up to equivalence) by G and $\Phi \in H^3(G, \mathbb{T})$, which is considered in [22, Chap. 7.5] and [55, Ex. 1.3]. (In our field theoretic language this means that $\mathcal{F}$ admits G-soliton *auto*morphisms which are unique up to inner unitary equivalence. In an operator algebraic setting it is long known [53] that 'G-kernels', i.e. homomorphisms $G \to \mathrm{Out}\mathcal{M} = \mathrm{Aut}\mathcal{M}/\mathrm{Inn}\mathcal{M}$ are classified by $H^3(G, \mathbb{T})$ if $\mathcal{M}$ is a factor. This analysis is immediately applicable to the present approach to CQFTs.) Now, in [12] starting from a finite group G and a 3-cocycle $\phi \in Z^3(G, \mathbb{T})$ a quasi-Hopf algebra $D^\phi(G)$ was defined, the 'twisted quantum double'. For the trivial cocycle this is just the ordinary quantum double. For cocycles in the same cohomology class the corresponding twisted quantum doubles are related by a twist of the coproduct which induces an equivalence as rigid braided tensor categories of the representation categories.

To make a long story short we state the following result:

Theorem 5.14 [43] *Let $\mathcal{F}$ be completely rational with $\mu_2(\mathcal{F}) = 1$, let G be a finite group of symmetries and let $\mathcal{A} = \mathcal{F}^G$. Then there is $\Phi \in H^3(G, \mathbb{T})$ such that the following equivalences of braided crossed G-categories and braided categories,* *respectively, hold:*

$$\begin{aligned} G - Sol(\mathcal{F}) &\simeq \mathcal{C}(G,\Phi), \\ Rep(\mathcal{A}) &\simeq D^\phi(G) - mod, \end{aligned}$$

where $[\phi] = \Phi$*. In particular,* $Rep(\mathcal{A})$ *and* $D^\phi(G) - mod$ *give rise to the same representation of* $SL(2,\mathbb{Z})$*.*

The proof proceeds by explicitly constructing (typically reducible if G is non-abelian) endomorphisms of $\mathcal{F}$ as direct sums of soliton automorphisms and considering their restriction to $\mathcal{A}$. One finds enough inequivalent irreducible sectors of $\mathcal{A}$ to saturate the bound $\dim Rep(\mathcal{A}) \leq |G|^2$ and concludes that there are no others. Then modularity of $Rep(\mathcal{A})$ follows by an easy argument based on [39, Coro. 4.3] even without invoking the main theorem of [33].

We conclude this discussion by emphasizing that the above analysis, satisfactory as it is, holds only if $\mathcal{F}$ is a local, i.e. purely bosonic, theory. If $\mathcal{F}$ is fermionic (graded local) then new phenomena may appear, as is illustrated by the following well known example [3]. Let $\mathcal{F}$ be a theory of N free real fermions on S^1 and let $\mathcal{A} = \mathcal{F}^G$ where $G = \mathbb{Z}/2$ acts by $\psi \mapsto -\psi$. $\mathcal{F}$ satisfies twisted duality for all disconnected intervals E, thus $\mu_2(\mathcal{F}) = 1$ (in a generalized sense) and $\mu_2(\mathcal{A}) = 4$. One finds $|H^3(G,\mathbb{T})| = 2$, corresponding to the fusion rules $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$ for $D^\phi(G)$, which in fact govern the cases $N = 4M$ and $N = 4M - 2$, respectively. For odd N, however, $Rep(\mathcal{A})$ has only three simple objects (of dimensions $1, 1, \sqrt{2}$) with Ising fusion rules [2]. The latter case clearly is not covered by [11] and our elaboration [43] of it. While the appearance of the object with non-integer dimension can be traced back to the fact that $\mathcal{F}$ does not admit soliton automorphisms for $g \neq e$ but rather soliton endomorphisms, a general model independent analysis of the fermionic case is still lacking and seems desirable.

6 Summary and Open Problems

At least on an abstract level the relation between the representation categories of rational CQFTs $\mathcal{F}$ and $\mathcal{A} = \mathcal{F}^G$ for finite G has been elucidated in quite a satisfactory way by Thm. 4.19 and Thm. 5.10. We have seen that this leads to fairly interesting structures, results and conjectures of an essentially categorical nature. When considering concrete QFT models the computations can, of course, still be quite tedious as is amply demonstrated by [59] and [43].

We close with a list of important open problems.

1. Extend the results from Props. 4.17, 4.18 to extensions $\mathcal{C} \rtimes \mathcal{S}$ where G is non-abelian. Thus, (i) given a simple object $X \in \mathcal{C}$, understand how $F(X) \in \mathcal{C} \rtimes \mathcal{S}$ decomposes into simple objects. (ii) Clarify the structure of the set of isomorphism classes of simple objects in $\mathcal{C} \rtimes \mathcal{S}$. (iii) Compute the fusion rules of $\mathcal{C} \rtimes \mathcal{S}$ and the S-matrix of $(\mathcal{C} \cap \mathcal{S}') \rtimes \mathcal{S}$.
2. Prove a form of unique factorization for modular categories into prime ones.
3. Prove Conjecture 5.2 on the existence of minimal modular extensions.
4. Give a more direct proof of Coro. 5.11 on the existence of soliton endomorphisms.

We cannot help remarking that the results of our Categorical Interludes strongly resemble well-known facts in Galois theory and algebraic number theory. (Note, e.g., the striking similarity between our Prop. 4.15 and Coro. 2-3 of [34, §I.7] on the decomposition of prime ideals in Galois extensions of quotient fields of Dedekind rings, thus in particular algebraic number fields.) The same remark applies to questions 1-3 above.

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