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From subfactors to categories and topology II: The quantum double of tensor categories and subfactors

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Abstract

For every tensor category $\mathscr{C}$ there is a braided tensor category $\mathscr{Z}(\mathscr{C})$, the 'center' of $\mathscr{C}$. It is well known to be related to Drinfel'd's notion of the quantum double of a finite dimensional Hopf algebra H by an equivalence $\mathscr{Z}(H\text{-mod}) \overset{\otimes}{\simeq}_{\rm br} D(H)\text{-mod}$ of braided tensor categories. In the Hopf algebra situation, whenever $D(H)$-mod is semisimple (which is the case iff $D(H)$ is semisimple iff H is semisimple and cosemisimple iff $S^2 = \mathrm{id}$ and char $\mathbb{F} \nmid \dim H$) it is modular in the sense of Turaev, i.e. its S-matrix is invertible. (This was proven by Etingof and Gelaki in characteristic zero. We give a fairly general proof in the appendix.) The present paper is concerned with a generalization of this and other results to the quantum double (center) of more general tensor categories.

We consider $\mathbb{F}$-linear tensor categories $\mathscr{C}$ with simple unit and finitely many isomorphism classes of simple objects. We assume that $\mathscr{C}$ is either a $*$-category (i.e. $\mathbb{F} = \mathbb{C}$ and there is a positive $*$-operation on the morphisms) or semisimple and spherical over an algebraically closed field $\mathbb{F}$. In the latter case we assume $\dim \mathscr{C} \equiv \sum_i d(X_i)^2 \neq 0$, where the summation runs over the isomorphism classes of simple objects. We prove that $\mathscr{Z}(\mathscr{C})$ (i) is a semisimple spherical (or $*$-) category and (ii) is weakly monoidally Morita equivalent (in the sense of Müger (J. Pure Appl. Algebra 180 (2003) 81–157)) to $\mathscr{C} \otimes_{\mathbb{F}} \mathscr{C}^{\rm op}$. This implies $\dim \mathscr{Z}(\mathscr{C}) = (\dim \mathscr{C})^2$. (iii) We analyze the simple objects of $\mathscr{Z}(\mathscr{C})$ in terms of certain finite dimensional algebras, of which Ocneanu's tube algebra is the smallest. We prove the conjecture of Gelfand and Kazhdan according to which the number of simple objects of $\mathscr{Z}(\mathscr{C})$ coincides with the dimension of the state space $\mathscr{H}_{S^1 \times S^1}$ of the torus in the triangulation TQFT built from $\mathscr{C}$. (iv) We prove that $\mathscr{Z}(\mathscr{C})$ is modular and

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we compute $\Delta_\pm(\mathscr{Z}(\mathscr{C})) \equiv \sum_i \theta(X_i)^{\pm 1} d(X_i)^2 = \dim \mathscr{C}$. (v) Finally, if $\mathscr{C}$ is already modular then $\mathscr{Z}(\mathscr{C}) \overset{\otimes}{\simeq}_{\rm br} \mathscr{C} \otimes_{\mathbb{F}} \tilde{\mathscr{C}} \overset{\otimes}{\simeq} \mathscr{C} \otimes_{\mathbb{F}} \mathscr{C}^{\rm op}$, where $\tilde{\mathscr{C}}$ is the tensor category $\mathscr{C}$ with the braiding $\tilde{c}_{X,Y} = c_{Y,X}^{-1}$.

MSC: 18D10; 18D05; 46L37

1. Introduction

Define the 'center' $Z_0(X)$ of a set X to be the monoid of all functions $f: X \to X$ (with composition as product and the identity map as unit). Then the usual center $Z_1 \equiv Z$ of the monoid $Z_0(X)$ is trivial: $Z_1(Z_0(X)) = \{\mathrm{id}_X\}$. The cardinality of $Z_0(X)$ is given by $\#Z_0(X) = \#X^{\#X}$. The aim of the present work is to prove 1-categorical analogues of these trivial set theoretic (= 0-categorical) observations. (I owe the above definition of $Z_0(X)$ to J. Baez.)

Given an arbitrary monoidal category (or tensor category) $\mathscr{C}$ its center $\mathscr{Z}(\mathscr{C})$ is a braided monoidal category which was defined independently by Drinfel'd (unpublished), Majid [35] and Joyal and Street [19]. (See Section 3 for the definition.) In order to avoid confusion with another notion of center, we will write $\mathscr{Z}_1(\mathscr{C})$ throughout. In the present work, as in [35,19], we will assume $\mathscr{C}$ to be strict, but this is exclusively for notational convenience. The definition of the center $\mathscr{Z}_1$ and all results in this paper extend immediately to the non-strict case. The other assumptions which we must make on $\mathscr{C}$ are more restrictive, but we are still left with a class of categories which appears in contexts like low dimensional topology and subfactor theory. We assume $\mathscr{C}$ to be linear over a ground field which is algebraically closed. Furthermore, $\mathscr{C}$ is semisimple with simple tensor unit and spherical [6]. (A semisimple category is spherical iff it is pivotal [6] (= sovereign) and every simple object has the same dimension as its dual, cf. [37, Lemma 2.8].) See [6] or [37, Section 2] for the precise definitions.

Definition 1.1. Let $\mathscr{C}$ be a semisimple spherical tensor category with simple unit and let Γ be the set of isomorphism classes of simple objects. If Γ is finite we define

$$\dim \mathscr{C} = \sum_{i \in \Gamma} d(X_i)^2,$$

otherwise we write $\dim \mathscr{C} = \infty$.

If $\mathscr{C}$ is finite dimensional and braided then the Gauss sums of $\mathscr{C}$ are given by

$$\Delta_\pm(\mathscr{C}) = \sum_{i \in \Gamma} \omega(X_i)^{\pm 1} d(X_i)^2,$$

where $\theta(X) = \omega(X)\mathrm{id}_X$ is the twist of the simple object X which is defined by the spherical structure [56].

We can now state our Main Theorem:

Theorem 1.2. *Let $\mathbb{F}$ be an algebraically closed field and $\mathscr{C}$ a spherical $\mathbb{F}$-linear tensor category with* $\mathrm{End}(\mathbf{1}) \cong \mathbb{F}$. *We assume that $\mathscr{C}$ is semisimple with finitely many simple* *objects and* $\dim\mathscr{C}\neq 0$. *Then also the center* $\mathscr{Z}_1(\mathscr{C})$ *has all these properties and is a modular category* [53]. *Furthermore, the dimension and the Gauss sums are given by*

$$\dim\mathscr{Z}_1(\mathscr{C})=(\dim\mathscr{C})^2,$$

$$\Delta_+(\mathscr{Z}_1(\mathscr{C}))=\Delta_-(\mathscr{Z}_1(\mathscr{C}))=\dim\mathscr{C}.$$

Defining the center $\mathscr{Z}_2(\mathscr{C})$ of a *braided* tensor category $\mathscr{C}$ to be the full subcategory whose objects are those X which satisfy

$$c(X,Y)=c(Y,X)^{-1}\quad\forall Y\in\mathrm{Obj}\,\mathscr{C},$$

one easily sees that $\mathscr{Z}_2(\mathscr{C})$ is stable w.r.t. isomorphisms (thus replete), direct sums, retractions, tensor products and duals, the inherited braiding obviously being symmetric. One can show that a braided category satisfying the properties in the theorem (i.e. a premodular category [8]) is modular iff the center $\mathscr{Z}_2(\mathscr{C})$ is trivial, in the sense that all objects of $\mathscr{Z}_2(\mathscr{C})$ are multiples of the tensor unit. (This was done in [47] for $*$-categories and in [7] for spherical categories with $\dim\mathscr{C}\neq 0$, see also Corollary 7.11 below.) Thus $\mathscr{Z}_2(\mathscr{Z}_1(\mathscr{C}))$ is trivial for all $\mathscr{C}$ as in the Main Theorem, which is the promised analogue of the 0-categorical observation $Z_1(Z_0(X))=\{\mathbf{1}\}$.

The Main Theorem can be generalized slightly: If $\mathscr{C}$ is as before except for $\mathbb{F}$ not being algebraically closed then there is a finite extension $\mathbb{F}'\supset\mathbb{F}$ such that $\mathscr{Z}_1(\mathscr{C}\otimes_{\mathbb{F}}\mathbb{F}')$ is modular. Concerning the prospects of further generalizations the author is not optimistic. There is little hope of proving semisimplicity of $\mathscr{Z}_1(\mathscr{C})$ without assuming $\dim\mathscr{C}\neq 0$. (Furthermore, it is known [53] that the dimension of a modular category must be non-zero.) In the non-semisimple case one might hope to prove that the center of a spherical noetherian category satisfies the non-degeneracy condition on the braiding introduced in [32]. But the methods of this paper will most likely not apply.

The results of the present work can be considered as generalizations of known results concerning Hopf algebras and we briefly comment on this in order to put our results into their context. We recall that the quantum double of a Hopf algebra was introduced, among many other things, in Drinfel'd's seminal work [10]. In the following discussion all Hopf algebras are finite dimensional over some field $\mathbb{F}$. The quantum double $D(H)$ of a Hopf algebra H is a certain Hopf algebra which contains H and the dual $\hat{H}$ as Hopf subalgebras and it is generated as an algebra by these. We refrain from repeating the well-known definition and refer to [21] for a nice treatment. We only remark that $D(H)\cong H\otimes_{\mathbb{F}}\hat{H}$ as a vector space, thus

$$\dim_{\mathbb{F}}D(H)=(\dim_{\mathbb{F}}H)^2.$$

Furthermore, $D(H)$ is quasitriangular, i.e. there is an invertible $R\in D(H)\otimes D(H)$ such that $\sigma\circ\Delta=R\Delta(\cdot)R^{-1}$ where σ is the flip automorphism of the tensor product. The constructions of the quantum double of a Hopf algebra and of the center of a monoidal category are linked by the equivalence

$$D(H)\text{-mod}\overset{\otimes}{\simeq}_{\mathrm{br}}\mathscr{Z}_1(H\text{-mod})$$

of braided monoidal categories, where H-mod and $D(H)$-mod are the categories of finite dimensional left H- and $D(H)$-modules, respectively, the braiding of $D(H)$-mod being provided by the R-matrix. Again, see [21, Chapter XIII.4] for a detailed account.

Now, the R-matrix of a quantum double $D(H)$ is non-degenerate in a certain sense, $D(H)$ being 'factorizable' [48]. If H is semisimple and cosemisimple then $D(H)$ is semisimple [46]. It then turns out to be also modular and the category $D(H)$-mod of finite dimensional left $D(H)$-modules is modular in the sense of Turaev [53]. (This was proved in [12] for algebras over algebraically closed fields of characteristic zero, but the latter condition can be dropped. In the appendix we give a general proof.) Furthermore, one clearly has

$$\dim \mathscr{Z}_1(H\text{-mod}) = \dim D(H)\text{-mod} = \dim D(H) \\ = (\dim H)^2 = (\dim H\text{-mod})^2, \tag{1.1}$$

where $\dim \mathscr{C}$ is the dimension of the monoidal category $\mathscr{C}$ as defined above.

It is now clear that our Main Theorem can be considered as an extension of the above results to tensor categories which are not necessarily representation categories of Hopf algebras. Here one remark on the notation is in order. In [22] Kassel and Turaev introduced a modified version of the construction of the center $\mathscr{Z}_1(\mathscr{C})$ and called it the quantum double $\mathscr{D}(\mathscr{C})$, see also [52]. Their category is the categorical version of a construction of Reshetikhin (which adjoins a certain square root θ to a quasitriangular Hopf algebra H in order to turn it into a ribbon algebra $H(\theta)$) applied to a quantum double, cf. [22, Theorem 5.4.1]. In the context of [22] the starting point was that even if $\mathscr{C}$ is rigid this need not be true for $\mathscr{Z}_1(\mathscr{C})$, whereas the category $\mathscr{D}(\mathscr{C})$ is rigid. As we will see, spherical categories (tensor categories with nice two-sided duals) are better behaved in the sense that their centers $\mathscr{Z}_1$ are again spherical. In addition, whereas $\mathscr{Z}_1(\mathscr{C})$ is modular for the categories satisfying the conditions of our Main Theorem, this is never true for $\mathscr{D}(\mathscr{C})$! This is why we stick to the original definition $\mathscr{Z}_1(\mathscr{C})$. Apart from writing $\mathscr{Z}_1(\mathscr{C})$ instead of $\mathscr{Z}(\mathscr{C})$, we do not attempt to change the established symbols, but we use the expression 'quantum double' as a synonym for $\mathscr{Z}_1(\mathscr{C})$ rather than $\mathscr{D}(\mathscr{C})$.

Unfortunately, the work on Hopf algebras mentioned above provides no clues on how to prove Theorem 1.2. This is where subfactor theory enters the present story. Starting from an inclusion $N \subset M$ of hyperfinite type II_1 factors of finite index and depth, Ocneanu [42] defined an 'asymptotic subfactor' $B \subset A$:

$$B = M \vee (M_\infty \cap M') \subset M_\infty = A.$$

(Here $N \subset M \subset M_1 \subset M_2 \subset \cdots$ is the Jones tower associated with $N \subset M$ and $M_\infty = \vee_i M_i$.) In [44] he argued that a certain monoidal category associated with $B \subset A$ is braided, concluding that the asymptotic subfactor is an 'analogue' of Drinfel'd's quantum double of a Hopf algebra. In fact, Ocneanu does not use category language and does not refer to the quantum double (center) of monoidal categories. In [14] Evans and Kawahigashi published proofs for most of Ocneanu's announcements. In the paper [29], which otherwise has little to do with the asymptotic subfactor, Longo and Rehren then constructed a subfactor $B \subset A$ from an infinite factor M and a—in our language—finite dimensional full monoidal subcategory $\mathscr{C}$ of $\mathrm{End}(M)$ and conjectured that it is related to Ocneanu's construction. This conjecture was made precise and proven in [36]. The author's involvement in the present story began when in 1998 he received a copy of a short preprint [17] by Izumi. In the meantime a full account of Izumi's results has appeared in [18]. In [17,18] Izumi gives an in-depth analysis of the LR-subfactor, in particular its $B-B$ sectors. Seeing [17] the present author was struck by the fact that its main theorem implicitly contained the definition of the center of a monoidal category. In fact properly formulated, Izumi's results provide a precise and completely general form of Ocneanu's 'analogy' between the asymptotic subfactor (or the LR-subfactor) and the quantum double, albeit the categorical one instead of the one for Hopf algebras. In Section 8.3 we will rephrase Izumi's results in categorical language to make this evident. Yet, this is not the main purpose of the present work.

In [44] it has been argued that the braided monoidal category associated with $B \subset A$ is modular and a complete proof has been provided in [18], where it was also shown that the dimension of the category in question is given by $(\dim \mathscr{C})^2$. As in our discussion of the Hopf algebra quantum double, it is again natural to ask whether a purely categorical version of these results can be proven. Here we have to face the problem that finite-index subfactors have a lot of 'in-built' categorical structure which is not a priori available in a purely categorical setting. (In particular, most of [18] strongly relies on this structure.) Yet this problem can be overcome once one realizes that the more algebraic part of subfactor theory can be cast into the language of 2-categories. This is the content of [37], which in a sense can be considered a continuation of [30], though in a somewhat more general setting.

The paper is organized as follows. In Section 2 we first recall some of the less standard definitions from [37]. We then summarize the main results of [37] on Frobenius algebras in tensor categories, related 2-categories and the notion of weak monoidal Morita equivalence of tensor categories. This section can by no means replace [37]. Our study of the quantum double $\mathscr{Z}_1(\mathscr{C})$ begins in Section 3, where we show that it preserves the closedness w.r.t. direct sums and subobjects and sphericity. Most importantly and least trivially, we prove the semisimplicity of $\mathscr{Z}_1(\mathscr{C})$. These results do not yet rely on the machinery of [37]. In Section 4 we prove the weak monoidal Morita equivalence $\mathscr{Z}_1(\mathscr{C}) \approx \mathscr{C} \otimes_{\mathbb{F}} \mathscr{C}^{\mathrm{op}}$, which in particular implies that the double construction squares the dimension of the category. Section 5 is devoted to the proof of modularity of $\mathscr{Z}_1(\mathscr{C})$, equivalent to triviality of the category $\mathscr{Z}_2(\mathscr{Z}_1(\mathscr{C}))$. As an important first step we analyze the structure of the simple objects of the double, providing an explanation for Ocneanu's 'tube algebra'. The next two section consider the case of categories with a positive $*$-structure (C^*-categories or unitary categories) and the special case where $\mathscr{C}$ is already braided. In Section 8 we consider applications to the invariants of 3-manifolds, proving a conjecture of Gelfand and Kazhdan and speculating about a far stronger result. Finally, we establish the link with subfactor theory, relying heavily on Izumi's work, improving on it only slightly.

2. Preliminaries

2.1. Some definitions and notations

We refer to [37, Section 2] for our general conventions and recall only a few less standard notations. A retract $Y \prec X$ is also called a subobject. A category has subobjects if all idempotents split, and every category $\mathscr{A}$ has a canonical completion $\bar{\mathscr{A}}^p$ for which this is the case. A $\mathbb{F}$-linear category is semisimple if it has direct sums and subobjects and every object is a finite direct sum of simple objects, X being simple iff $\mathrm{End}(X) \cong \mathbb{F}$. For monoidal categories we require in addition that $\mathbf{1}$ is simple. A subcategory of a semisimple category is called semisimple if it is closed w.r.t. direct sums and retractions, thus in particular replete (stable under isomorphisms).

Since all categories in question are $\mathbb{F}$-linear we understand the product $\mathscr{K} \otimes_{\mathbb{F}} \mathscr{L}$ of (tensor) categories in the sense of enriched category theory. Thus

$$\mathrm{Obj}\, \mathscr{K} \otimes_{\mathbb{F}} \mathscr{L} = \{K \boxtimes L,\ K \in \mathrm{Obj}\, \mathscr{K},\ L \in \mathrm{Obj}\, \mathscr{L}\},$$

where $X \boxtimes Y$ stands for the pair (X, Y), and

$$\mathrm{Hom}_{\mathscr{K} \otimes_{\mathbb{F}} \mathscr{L}}(K_1 \boxtimes L_1, K_2 \boxtimes L_2)) = \mathrm{Hom}_{\mathscr{K}}(K_1, K_2) \otimes_{\mathbb{F}} \mathrm{Hom}_{\mathscr{L}}(L_1, L_2)$$

with the obvious composition laws. We denote by $\mathscr{K} \boxtimes \mathscr{L} = \overline{\mathscr{K} \otimes_{\mathbb{F}} \mathscr{L}}^{\oplus}$ the completion w.r.t. finite direct sums. If $\mathscr{X}, \mathscr{Y}$ are monoidal categories the same holds for $\mathscr{X} \boxtimes \mathscr{Y}$. In order to save brackets we declare $\boxtimes$ to bind stronger than $\otimes$ but weaker than juxtaposition XY of objects (which abbreviates $X \otimes Y$). Note that $\otimes$ and $\boxtimes$ commute

$$X_1 \boxtimes Y_1 \otimes X_2 \boxtimes Y_2 = (X_1 \otimes X_2) \boxtimes (Y_1 \otimes Y_2) = X_1X_2 \boxtimes Y_1Y_2.$$

2.2. *Frobenius algebras and 2-categories*

Definition 2.1. Let $\mathscr{A}$ be a (strict) monoidal category. A Frobenius algebra in $\mathscr{A}$ is a quintuple $\mathsf{Q} = (Q, v, v', w, w')$, where Q is an object in $\mathscr{A}$ and $v : \mathbf{1} \to Q$, $v' : Q \to \mathbf{1}$, $w : Q \to Q^2$, $w' : Q^2 \to Q$ are morphisms satisfying the following conditions:

$$w \otimes \mathrm{id}_Q \circ w = \mathrm{id}_Q \otimes w \circ w, \tag{2.1}$$

$$w' \circ w' \otimes \mathrm{id}_Q = w' \circ \mathrm{id}_Q \otimes w', \tag{2.2}$$

$$v' \otimes \mathrm{id}_Q \circ w = \mathrm{id}_Q = \mathrm{id}_Q \otimes v' \circ w, \tag{2.3}$$

$$w' \circ v \otimes \mathrm{id}_Q = \mathrm{id}_Q = w' \circ \mathrm{id}_Q \otimes v, \tag{2.4}$$

$$w' \otimes \mathrm{id}_Q \circ \mathrm{id}_Q \otimes w = w \circ w' = \mathrm{id}_Q \otimes w' \circ w \otimes \mathrm{id}_Q. \tag{2.5}$$

A Frobenius algebra Q in a $\mathbb{F}$-linear category is strongly separable if

$$w' \circ w = \lambda_1\, \mathrm{id}_Q, \tag{2.6}$$

$$v' \circ v = \lambda_2\, \mathrm{id}_{\mathbf{1}} \tag{2.7}$$

with $\lambda_1, \lambda_2 \in \mathbb{F}^*$. Q is normalized if $\lambda_1 = \lambda_2$.

Let X be an object in a spherical category $\mathscr{A}$. Then the quintuple

$$(X\bar{X},\ \varepsilon(X),\ \bar{\varepsilon}(X), \mathrm{id}_X \otimes \varepsilon(\bar{X}) \otimes \mathrm{id}_{\bar{X}},\ \mathrm{id}_X \otimes \bar{\varepsilon}(\bar{X}) \otimes \mathrm{id}_{\bar{X}})$$

is easily seen to be a normalized strongly separable Frobenius algebra in $\mathscr{A}$. The following theorem, which combines the Theorems 3.12 and 5.13 from [37], shows that in fact every strongly separable Frobenius algebra in a tensor category arises in this way, provided one is ready to embed the category as a corner into a bicategory.

Theorem 2.2. *Let $\mathscr{A}$ be a strict $\mathbb{F}$-linear tensor category and $\mathsf{Q} = (Q, v, v', w, w')$ a strongly separable Frobenius algebra in $\mathscr{A}$. Then*:

(i) *There is a bicategory $\mathscr{E}$ such that*
   1. *The sets of 2-morphisms in $\mathscr{E}$ are finite dimensional $\mathbb{F}$-vector spaces and the horizontal and vertical compositions are bilinear.*
   2. *Idempotent 2-morphisms in $\mathscr{E}$ split.*
   3. $\mathrm{Obj}\,\mathscr{E} = \{\mathfrak{A}, \mathfrak{B}\}$.
   4. *There is an equivalence $\mathscr{HOM}_{\mathscr{E}}(\mathfrak{A},\mathfrak{A}) \overset{\otimes}{\simeq} \bar{\mathscr{A}}^p$ of tensor categories and therefore an equivalence $\mathscr{HOM}_{\mathscr{E}}(\mathfrak{A},\mathfrak{A}) \overset{\otimes}{\simeq} \mathscr{A}$ if $\mathscr{A}$ has subobjects.*
   5. *There are 1-morphisms $J : \mathfrak{B} \to \mathfrak{A}$ and $\bar{J} : \mathfrak{A} \to \mathfrak{B}$ such that $Q = J\bar{J}$.*
   6. *J and $\bar{J}$ are mutual two-sided duals, i.e. there are 2-morphisms*

$$e_J : \mathbf{1}_{\mathfrak{A}} \to J\bar{J}, \quad \varepsilon_J : \mathbf{1}_{\mathfrak{B}} \to \bar{J}J, \quad d_J : \bar{J}J \to \mathbf{1}_{\mathfrak{B}}, \quad \eta_J : J\bar{J} \to \mathbf{1}_{\mathfrak{A}}$$

   satisfying the usual equations.
   7. *We have*

$$v = e_J\colon\ \mathbf{1}_{\mathfrak{A}} \to Q = J\bar{J},$$

$$v' = \eta_J\colon\ Q = J\bar{J} \to \mathbf{1}_{\mathfrak{A}},$$

$$w = \mathrm{id}_J \otimes \varepsilon_J \otimes \mathrm{id}_{\bar{J}}\colon\ Q = J\bar{J} \to J\bar{J}J\bar{J} = Q^2,$$

$$w' = \mathrm{id}_J \otimes d_J \otimes \mathrm{id}_{\bar{J}}\colon\ Q^2 = J\bar{J}J\bar{J} \to J\bar{J} = Q$$

   and therefore $d_J \circ \varepsilon_J = \lambda_1\, \mathrm{id}_{\mathbf{1}_{\mathfrak{B}}}$, $\eta_J \circ e_J = \lambda_2 \mathrm{id}_{\mathbf{1}_{\mathfrak{A}}}$.
   8. *$\mathscr{E}$ is uniquely determined up to equivalence by the above properties. Isomorphic Frobenius algebras $\mathsf{Q}, \tilde{\mathsf{Q}}$ give rise to isomorphic bicategories $\mathscr{E}, \tilde{\mathscr{E}}$.*

(ii) *If $\mathscr{A}$ has direct sums then $\mathscr{E}$ has direct sums of 1-morphisms.*

(iii) *If the multiplicity of $\mathbf{1}$ in Q is exactly one* (*it is at least one due to the existence of v, v'*) *then $J, \bar{J}, \mathbf{1}_{\mathfrak{B}}$ are simple.* (*There is a weaker condition implying only simplicity of $\mathbf{1}_{\mathfrak{B}}$.*)

(iv) *If $\mathbb{F} = \mathbb{C}$ and $\mathscr{A}$ has a positive $*$-operation then $\mathscr{E}$ has a positive $*$-operation and is semisimple.*

(v) *If $\mathscr{A}$ is strict spherical and Q satisfies* (iii) *and is normalized then $\mathscr{E}$ is spherical. If, furthermore, $\mathscr{A}$ is semisimple and $\mathbb{F}$ is algebraically closed then $\mathscr{E}$ is semisimple.*

(vi) *If* (iv) *or* (v) *apply then the tensor category $\mathscr{B} = \mathscr{HOM}_{\mathscr{E}}(\mathfrak{B},\mathfrak{B})$ satisfies* $\dim \mathscr{B} = \dim \mathscr{A}$.

Remark 2.3. (1) If two tensor categories $\mathscr{A}, \mathscr{B}$ are ‘corners’ of a 2-category as above we call them weakly monoidally Morita equivalent. This is an equivalence relation which is considerably weaker than the usual equivalence, yet it implies that $\mathscr{A}$ and

$\mathscr{B}$ have the same dimension and define the same triangulation invariant [5,15] for 3-manifolds. See [37] for the details.

(2) Unfortunately, the above statement of the theorem will not be sufficient for our purposes since beginning in Section 4.2 we will make use of the concrete structure of the bicategory $\mathscr{E}$, which is explicitly constructed in the proof of Theorem 2.2. This is not the place to explain the latter which occupies the larger part of [37]. We can only hope that the above statement of the theorem and its role in this paper are sufficient to motivate the reader to acquire some familiarity with [37].

3. The quantum double of a tensor category

3.1. On half-braidings

We begin with the definition of the quantum double $\mathscr{Z}_1(\mathscr{C})$ of a (strict) monoidal category $\mathscr{C}$.

Definition 3.1. Let $\mathscr{C}$ be a strict monoidal category and let $X \in \mathscr{C}$. A half-braiding e_X for X is a family $\{e_X(Y) \in \mathrm{Hom}_{\mathscr{C}}(XY, YX),\ Y \in \mathscr{C}\}$ of morphisms satisfying

(i) Naturality w.r.t. the argument in brackets, i.e.

$$t \otimes \mathrm{id}_X \circ e_X(Y) = e_X(Z) \circ \mathrm{id}_X \otimes t \quad \forall t\colon Y \to Z. \tag{3.1}$$

(ii) The braid relation

$$e_X(Y \otimes Z) = \mathrm{id}_Y \otimes e_X(Z) \circ e_X(Y) \otimes \mathrm{id}_Z \quad \forall Y, Z \in \mathscr{C}. \tag{3.2}$$

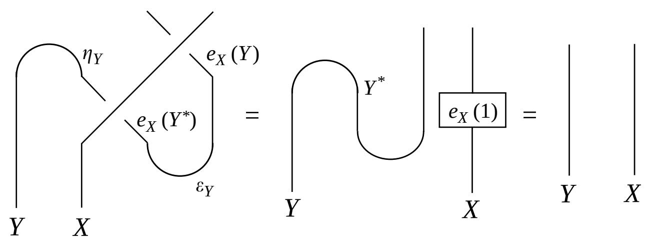
(iii) All $e_X(Z)$ are isomorphisms.

(iv) Unit property:

$$e_X(\mathbf{1}) = \mathrm{id}_X. \tag{3.3}$$

Lemma 3.2. *Let* $\{e_X(Y),\ Y \in \mathscr{C}\}$ *satisfy* (i) *and* (ii). *Then* (iii) $\Rightarrow$ (iv). *If* (iv) *holds and* Y *has a right dual* Y^* *then* $e_X(Y)$ *is invertible.*

Proof. Considering (3.2) with $Y = Z = \mathbf{1}$ gives $e_X(\mathbf{1}) = e_X(\mathbf{1})^2$. Thus (iii) implies $e_X(\mathbf{1}) = \mathrm{id}_X$. Let Y^* a right dual of Y with $\varepsilon_Y : \mathbf{1} \to Y^* \otimes Y$, $\eta_Y : Y \otimes Y^* \to \mathbf{1}$. Then using (i) and (iv) we find



Thus $e_X(Y)$ has a right inverse, which by a similar computation is seen to be also a left inverse. □

For later use we record the following alternative characterization of half-braidings.

Lemma 3.3. *Let $\mathscr{C}$ be semisimple and $\{X_i,\ i\in\Gamma\}$ a basis of simple objects. Let $Z\in\mathscr{C}$. Then there is a one-to-one correspondence between* (i) *families of morphisms $\{e_Z(X_i)\in\mathrm{Hom}_{\mathscr{C}}(ZX_i,X_iZ),\ i\in\Gamma\}$ such that*

$$t\otimes\mathrm{id}_Z\circ e_Z(X_k)=\mathrm{id}_{X_i}\otimes e_Z(X_j)\circ e_Z(X_i)\otimes\mathrm{id}_{X_j}\circ\mathrm{id}_Z\otimes t\quad\forall i,j,k\in\Gamma,\ t\in\mathrm{Hom}_{\mathscr{C}}(X_k,X_iX_j)\tag{3.4}$$

and (ii) *families of morphisms $\{e_Z(X)\in\mathrm{Hom}_{\mathscr{C}}(ZX,XZ),\ X\in\mathscr{C}\}$ satisfying* (i) *and* (ii) *from Definition* 3.1. *All $e_Z(X)$, $X\in\mathscr{C}$ are isomorphisms iff all $e_Z(X_i)$, $i\in\Gamma$ are isomorphisms.*

Proof. (ii) ⇒ (i). Obvious: just restrict $e_Z(\cdot)$ to $X\in\{X_i,\ i\in\Gamma\}$. Then (3.1), (3.2) imply (3.4).

(i) ⇒ (ii). Let $X\cong\bigoplus_i n_iX_i$ and let $\{x_i^\alpha,\ \alpha=1,\dots,n_i\}$, $\{x_{i'}^\alpha,\ \alpha=1,\dots,n_i\}$ be dual bases in $\mathrm{Hom}_{\mathscr{C}}(X_i,X)$ and $\mathrm{Hom}_{\mathscr{C}}(X,X_i)$, respectively. Then define

$$e_Z(X)=\sum_{i\in\Gamma}\sum_{\alpha=1}^{n_i}x_i^\alpha\otimes\mathrm{id}_Z\circ e_Z(X_i)\circ\mathrm{id}_Z\otimes x_{i'}^\alpha.$$

Independence of $e_Z(X)\in\mathrm{Hom}_{\mathscr{C}}(ZX,XZ)$ of the choice of the x_i^α follows from duality of the bases $\{x_i^\alpha\},\{x_{i'}^\alpha\}$. In order to prove naturality (3.2) consider $Y\cong\bigoplus_i m_iX_i$ and corresponding intertwiners $y_i^\alpha,y_{i'}^\alpha$ and let $t\in\mathrm{Hom}_{\mathscr{C}}(X,Y)$. Then $y_{j'}^\beta t x_i^\alpha\in\mathrm{Hom}(X_i,X_j)$, which vanishes if $i\neq j$. Thus

$$t=\sum_{i\in\Gamma}\sum_{\alpha=1}^{n_i}\sum_{\beta=1}^{m_i}c(i,\alpha,\beta)y_i^\beta x_{i'}^\alpha,$$

where $c(i,\alpha,\beta)\in\mathbb{F}$. Therefore,

$$t\otimes\mathrm{id}_Z\circ e_Z(X)=\sum_{i\in\Gamma}\sum_{\alpha=1}^{n_i}\sum_{\beta=1}^{m_i}c(i,\alpha,\beta)y_i^\beta\otimes\mathrm{id}_Z\circ e_Z(X_i)\circ\mathrm{id}_Z\otimes x_{i'}^\alpha,$$

which coincides with $e_Z(Y)\circ\mathrm{id}_Z\otimes t$. If now $t\in\mathrm{Hom}_{\mathscr{C}}(X_k,X_iX_j)$ then naturality implies $t\otimes\mathrm{id}_Z\circ e_Z(X_k)=e_Z(X_iX_j)\circ\mathrm{id}_Z\otimes t$. Together with (3.4) this implies

$$e_Z(X_iX_j)\circ\mathrm{id}_Z\otimes t=\mathrm{id}_{X_i}\otimes e_Z(X_j)\circ e_Z(X_i)\otimes\mathrm{id}_{X_j}\circ\mathrm{id}_Z\otimes t$$

and since this holds for all $t\in\mathrm{Hom}_{\mathscr{C}}(X_k,X_iX_j)$ (3.2) follows. (This is a consequence of

$$\sum_{k\in\Gamma}\sum_{\alpha=1}^{N_{ij}^k}t_k^\alpha\circ t_{k'}^\alpha=\mathrm{id}_{X_iX_j},$$

where the $\{t_k^\alpha,\ \alpha=1,\dots,N_{ij}^k\}$ are bases in $\mathrm{Hom}_{\mathscr{C}}(X_k,X_iX_j)$.) □

3.2. Elementary properties of the quantum double

Definition 3.4. The center $\mathscr{Z}_1(\mathscr{C})$ of a strict monoidal category $\mathscr{C}$ has as objects pairs (X, e_X), where $X \in \mathscr{C}$ and e_X is a half-braiding. The morphisms are given by

$$\mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X), (Y, e_Y)) = \{t \in \mathrm{Hom}_{\mathscr{C}}(X, Y) \mid \mathrm{id}_X \otimes t \circ e_X(Z) = e_Y(Z) \circ t \otimes \mathrm{id}_X \;\; \forall Z \in \mathscr{C}\}. \tag{3.5}$$

The tensor product of objects is given by $(X, e_X) \otimes (Y, e_Y) = (XY, e_{XY})$, where

$$e_{XY}(Z) = e_X(Z) \otimes \mathrm{id}_Y \circ \mathrm{id}_X \otimes e_Y(Z). \tag{3.6}$$

The tensor unit is $(\mathbf{1}, e_{\mathbf{1}})$ where $e_{\mathbf{1}}(X) = \mathrm{id}_X$. The composition and tensor product of morphisms are inherited from $\mathscr{C}$. The braiding is given by

$$c((X, e_X), (Y, e_Y)) = e_X(Y).$$

For the proof that $\mathscr{Z}_1(\mathscr{C})$ is a strict braided tensor category we refer to [21]. The following is immediate from the definition of the center $\mathscr{Z}_1(\mathscr{C})$:

Lemma 3.5. *If $\mathscr{C}$ is $\mathbb{F}$-linear then so is $\mathscr{Z}_1(\mathscr{C})$. If the unit $\mathbf{1}$ of $\mathscr{C}$ is simple, then $\mathbf{1}_{\mathscr{Z}_1(\mathscr{C})}$ is simple.*

In [52, Proposition 1] it is proven that the center of an abelian monoidal category is abelian. In this paper we do not use the language of abelian categories since the notions of (co)kernels are not really needed. (Yet semisimple categories are abelian if we assume existence of a zero object.) Therefore, we prove two lemmas which show that the center construction behaves nicely w.r.t. direct sums and subobjects. The first result is contained in [52], but we repeat it for the sake of completeness.

Lemma 3.6. *If $\mathscr{C}$ has direct sums then also $\mathscr{Z}_1(\mathscr{C})$ has direct sums.*

Proof. Let $(Y, e_Y), (U, e_U)$ be objects in $\mathscr{Z}_1(\mathscr{C})$. Let $\mathscr{C} \ni Z \cong Y \oplus U$ with morphisms $v \in \mathrm{Hom}_{\mathscr{C}}(Y, Z), w \in \mathrm{Hom}_{\mathscr{C}}(U, Z)$, $v' \in \mathrm{Hom}_{\mathscr{C}}(Z, Y), w' \in \mathrm{Hom}_{\mathscr{C}}(Z, U)$ satisfying $v' \circ v = \mathrm{id}_Y, w' \circ w = \mathrm{id}_U, v \circ v' + w \circ w' = \mathrm{id}_Z$. Defining $e_Z(X) \in \mathrm{Hom}_{\mathscr{C}}(ZX, XZ)$ for all $X \in \mathscr{C}$ by

$$e_Z(X) = \mathrm{id}_X \otimes v \circ e_Y(X) \circ v' \otimes \mathrm{id}_X + \mathrm{id}_X \otimes w \circ e_U(X) \circ w' \otimes \mathrm{id}_X$$

we claim that (Z, e_Z) is an object of $\mathscr{Z}_1(\mathscr{C})$ and

$$(Z, e_Z) \cong (Y, e_Y) \oplus (U, e_U). \tag{3.7}$$

Naturality of $e_Z(X)$ w.r.t. X is obvious, and (3.2) is very easily verified using $v' \circ w = 0$. Finally, we have

$$e_Z(X) \circ v \otimes \mathrm{id}_X = \mathrm{id}_X \otimes v \circ e_Y(X),$$

which is just the statement that $v \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((Y, e_Y), (Z, e_Z))$. The analogous statement holding for v', w, w', (3.7) follows. □

Lemma 3.7. *If $\mathscr{C}$ has subobjects then also $\mathscr{Z}_1(\mathscr{C})$ has subobjects.*

Proof. Let $(Y,e_Y)\in\mathscr{Z}_1(\mathscr{C})$ and let e be an idempotent in $\mathrm{End}_{\mathscr{Z}_1(\mathscr{C})}((Y,e_Y))$. By definition of $\mathscr{Z}_1(\mathscr{C})$ this means that e is an idempotent in $\mathrm{End}_{\mathscr{C}}(Y)$ such that

$$\mathrm{id}_X\otimes e\circ e_Y(X)=e_Y(X)\circ e\otimes \mathrm{id}_X \quad \forall X\in\mathscr{C}. \tag{3.8}$$

Since $\mathscr{C}$ has subobjects there are $U\in\mathscr{C}$ and $v\in\mathrm{Hom}_{\mathscr{C}}(U,Y)$, $v'\in\mathrm{Hom}_{\mathscr{C}}(Y,U)$ such that $v\circ v'=e$ and $v'\circ v=\mathrm{id}_U$. Defining

$$e_U(X)=\mathrm{id}_X\otimes v'\circ e_Y(X)\circ v\otimes \mathrm{id}_X\in\mathrm{Hom}_{\mathscr{C}}(UX,XU),\quad X\in\mathscr{C},$$

naturality w.r.t. X is again obvious. Now,

$$\begin{aligned}
e_U(X_1X_2)&=\mathrm{id}_{X_1X_2}\otimes v'\circ e_Y(X_1X_2)\circ v\otimes \mathrm{id}_{X_1X_2}\\
&=\mathrm{id}_{X_1X_2}\otimes v'\circ \mathrm{id}_{X_1}\otimes e_Y(X_2)\circ e_Y(X_1)\otimes \mathrm{id}_{X_2}\circ v\otimes \mathrm{id}_{X_1X_2}\\
&=\mathrm{id}_{X_1X_2}\otimes v'\circ \mathrm{id}_{X_1}\otimes e_Y(X_2)\circ \mathrm{id}_{X_1}\otimes v\otimes \mathrm{id}_{X_2}\\
&\quad\circ \mathrm{id}_{X_1}\otimes v'\otimes \mathrm{id}_{X_2}\circ e_Y(X_1)\otimes \mathrm{id}_{X_2}\circ v\otimes \mathrm{id}_{X_1X_2}\\
&=\mathrm{id}_{X_1}\otimes e_U(X_2)\circ e_U(X_1)\otimes \mathrm{id}_{X_2},
\end{aligned}$$

whereby e_U is a half-braiding and (U,e_U) an object in $\mathscr{Z}_1(\mathscr{C})$. We used $v\circ v'=e$, (3.8) and $e\circ v=v\circ v'\circ v=v$. Using the same facts we finally compute

$$\mathrm{id}_X\otimes v\circ e_U(X)=\mathrm{id}_X\otimes v\circ \mathrm{id}_X\otimes v'\circ e_Y(X)\circ v\otimes \mathrm{id}_X=e_Y(X)\circ v\otimes \mathrm{id}_X.$$

Thus $v\in\mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((U,e_U),(Y,e_Y))$ and we have $(U,e_U)\prec(Y,e_Y)$. □

Lemma 3.8. *Let $\mathscr{C}$ be pivotal and e_Y a half-braiding satisfying* (i)–(iv). *Then*

$$e_Y(\bar X)=\mathrm{id}_{\bar XY}\otimes\bar\varepsilon(X)\circ \mathrm{id}_{\bar X}\otimes e_Y(X)^{-1}\otimes \mathrm{id}_{\bar X}\circ\varepsilon(\bar X)\otimes \mathrm{id}_{Y\bar X}. \tag{3.9}$$

Proof. By naturality and the braid relation we have

$$\varepsilon(\bar X)\otimes \mathrm{id}_Y=e_Y(\bar XX)\circ \mathrm{id}_Y\otimes\varepsilon(\bar X)=\mathrm{id}_{\bar X}\otimes e_Y(X)\circ e_Y(\bar X)\otimes \mathrm{id}_X\circ \mathrm{id}_Y\otimes\varepsilon(\bar X)$$

and using the invertibility of $e_Y(X)$ we get

$$\mathrm{id}_{\bar X}\otimes e_Y(X)^{-1}\circ\varepsilon(\bar X)\otimes \mathrm{id}_Y=e_Y(\bar X)\otimes \mathrm{id}_X\circ \mathrm{id}_Y\otimes\varepsilon(\bar X).$$

Now (3.9) follows by a use of the duality equations, see, e.g., [37, Section 2.3]. □

Proposition 3.9. *Let $\mathscr{C}$ be* (*strict*) *pivotal. Then also $\mathscr{Z}_1(\mathscr{C})$ is* (*strict*) *pivotal, the dual $\overline{(Y,e_Y)}$ being given by $(\bar Y,e_{\bar Y})$, where $e_{\bar Y}(X)$ is defined by*

$$\bar Y\otimes X\xrightarrow{\mathrm{id}_{\bar YX}\otimes\varepsilon(Y)}\bar Y\otimes X\otimes Y\otimes\bar Y\xrightarrow{\mathrm{id}_{\bar Y}\otimes e_Y(X)^{-1}\otimes \mathrm{id}_{\bar Y}}\bar Y\otimes Y\otimes X\otimes\bar Y\to$$
$$\xrightarrow{\bar\varepsilon(\bar Y)\otimes \mathrm{id}_{X\bar Y}}X\otimes\bar Y. \tag{3.10}$$

The evaluation and coevaluation maps are inherited from $\mathscr{C}$:

$$\varepsilon((Y,e_Y))=\varepsilon(Y),\quad \bar\varepsilon((Y,e_Y))=\bar\varepsilon(Y).$$

If $\mathscr{C}$ is spherical then also $\mathscr{Z}_1(\mathscr{C})$ is spherical.

Proof. We begin by showing that $e_{\bar{Y}}(\cdot)$ is a half-braiding for $\bar{Y}$. By construction we have $e_{\bar{Y}}(X) \in \mathrm{Hom}_{\mathscr{C}}(\bar{Y}X, X\bar{Y})$, and naturality w.r.t. X follows easily from the corresponding property for e_Y. Now

$$\begin{aligned}
e_{\bar{Y}}(X_1X_2) &= \bar{\varepsilon}(\bar{Y}) \otimes \mathrm{id}_{X_1X_2\bar{Y}} \circ \mathrm{id}_{\bar{Y}} \otimes e_Y(X_1X_2)^{-1} \otimes \mathrm{id}_{\bar{Y}} \circ \mathrm{id}_{\bar{Y}X_1X_2} \otimes \varepsilon(Y) \\
&= \bar{\varepsilon}(\bar{Y}) \otimes \mathrm{id}_{X_1X_2\bar{Y}} \circ \mathrm{id}_{\bar{Y}} \otimes e_Y(X_1)^{-1} \otimes \mathrm{id}_{X_2\bar{Y}} \\
&\quad \circ \mathrm{id}_{\bar{Y}X_1} \otimes e_Y(X_2)^{-1} \otimes \mathrm{id}_{\bar{Y}} \circ \mathrm{id}_{\bar{Y}X_1X_2} \otimes \varepsilon(Y) \\
&= \mathrm{id}_{X_1} \otimes \bar{\varepsilon}(\bar{Y}) \otimes \mathrm{id}_{X_2\bar{Y}} \circ \mathrm{id}_{X_1\bar{Y}} \otimes e_Y(X_2)^{-1} \otimes \mathrm{id}_{\bar{Y}} \circ \mathrm{id}_{X_1\bar{Y}X_2} \otimes \varepsilon(Y) \\
&\quad \circ \bar{\varepsilon}(\bar{Y}) \otimes \mathrm{id}_{X_1\bar{Y}X_2} \circ \mathrm{id}_{\bar{Y}} \otimes e_Y(X_1)^{-1} \otimes \mathrm{id}_{\bar{Y}X_2} \circ \mathrm{id}_{\bar{Y}X_1} \otimes \varepsilon(Y) \otimes \mathrm{id}_{X_2} \\
&= \mathrm{id}_{X_1} \otimes e_{\bar{Y}}(X_2) \circ e_{\bar{Y}}(X_1) \otimes \mathrm{id}_{X_2}.
\end{aligned}$$

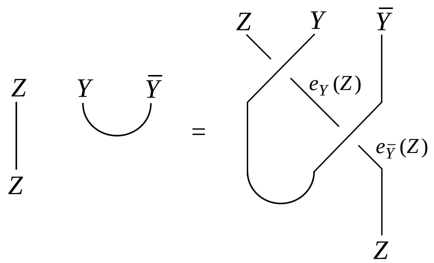
In the third equality we have used the duality equation $\mathrm{id}_Y \otimes \bar{\varepsilon}(\bar{Y}) \circ \varepsilon(Y) \otimes \mathrm{id}_Y = \mathrm{id}_Y$ and the interchange law.

In view of definition (3.10) of $e_{\bar{Y}}(X)$ together with $e_Y(\mathbf{1}) = \mathrm{id}_Y$ and the duality equation we have $e_{\bar{Y}}(\mathbf{1}) = \mathrm{id}_{\bar{Y}}$. Now Lemma 3.2 implies invertibility of $e_{\bar{Y}}(X)$ for all X.

It remains to show that $\varepsilon(Y) : \mathbf{1}_{\mathscr{C}} \to Y \otimes \bar{Y}$ is actually in

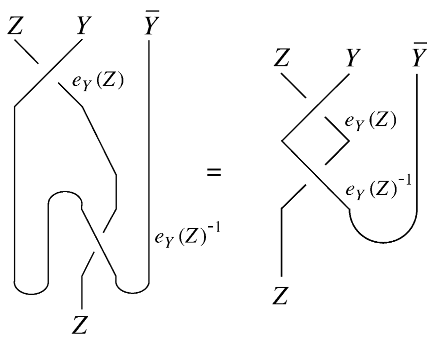
$$\mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}(\mathbf{1}_{\mathscr{Z}_1(\mathscr{C})}, (Y, e_Y) \otimes (\bar{Y}, e_{\bar{Y}})) = \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((\mathbf{1}, \mathrm{id}), (Y\bar{Y}, e_{Y\bar{Y}})),$$

which in view of (3.5) amounts to



(3.11)

With definition (3.10) of $e_{\bar{Y}}$ the right-hand side equals

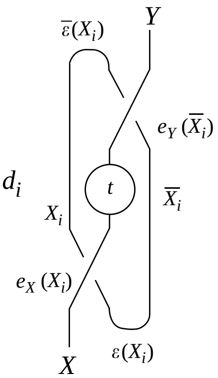


which coincides with the left-hand side of (3.11) as desired. That $\bar{\varepsilon}(Y)$ is a morphism in $\mathscr{Z}_1(\mathscr{C})$ is shown analogously. The composition of morphisms being the same in $\mathscr{Z}_1(\mathscr{C})$ as in $\mathscr{C}$, $\varepsilon(X)$, $\bar{\varepsilon}(X)$ inherit from $\mathscr{C}$ all equations needed to make $\mathscr{Z}_1(\mathscr{C})$ pivotal (spherical). If the pivotal structure of $\mathscr{C}$ is strict then the same clearly holds for $\mathscr{Z}_1(\mathscr{C})$. $\square$

3.3. Semisimplicity of $\mathscr{Z}_1(\mathscr{C})$

Lemma 3.10. *Let $\mathscr{C}$ be semisimple spherical with simple unit. We assume that there are only finitely many simple objects and that $\dim \mathscr{C} \neq 0$. Let $(X, e_X), (Y, e_Y) \in \mathscr{Z}_1(\mathscr{C})$. Then the map $E_{X,Y} : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$ defined by*

$$E_{X,Y}(t) = (\dim \mathscr{C})^{-1} \sum_{i \in \Gamma} d_i$$



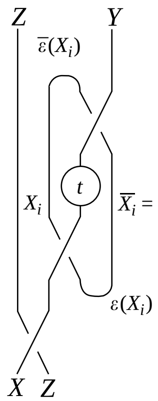
is a projection onto $\mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X), (Y, e_Y)) \subset \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Here $\{X_i,\ i \in \Gamma\}$ is a basis of simple objects and we abbreviate $d_i = d(X_i)$. The family of maps $E_{X,Y}$ is a conditional expectation in the sense that

$$E_{X,T}(c \circ b \circ a) = c \circ E_{Y,Z}(b) \circ a \tag{3.12}$$

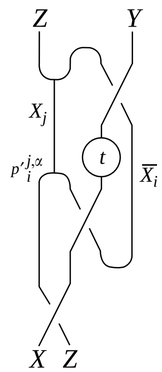
if $a \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X), (Y, e_Y))$, $b \in \mathrm{Hom}_{\mathscr{C}}(Y, Z)$, $c \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((Z, e_Z), (T, e_T))$.

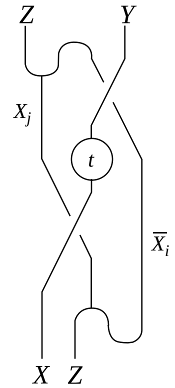
Proof. We compute

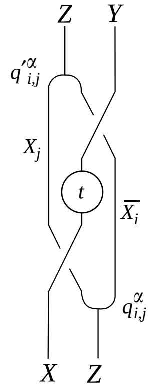
$$\dim \mathscr{C} \,.\, \mathrm{id}_Z \otimes E_{X,Y}(t) \circ e_X(Z) = \sum_{i \in \Gamma} d_i$$

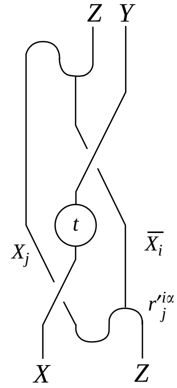


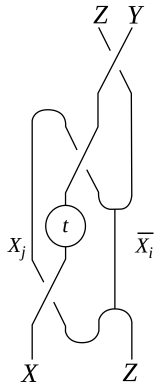
$$\sum_i \sum_{j,\alpha} d_i$$

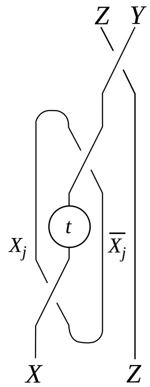


$= \sum_i \sum_{j,\alpha} d_i$ 

$= \sum_{i,j,\alpha} \frac{d_i d_j}{d(Z)}$ 

$= \sum_i \sum_{j,\alpha} d_j$ 

$= \sum_j \sum_{i,\alpha} d_j$ 

$= \sum_i d_j$ 

$= \dim\mathscr{C}\cdot e_Y(Z)\circ E_{X,Y}(t)\otimes \mathrm{id}_Z$

Here $\{p_i^{j,\alpha},\ \alpha=1,\dots,N_{Z,X_i}^{X_j}\}$ is, for every $j\in\Gamma$, a basis in $\mathrm{Hom}_{\mathscr{C}}(X_j, ZX_i)$ with dual basis $\{p_{i'}^{j,\alpha}\}$ such that $p_{i'}^{j,\alpha}\circ p_i^{k,\beta}=\delta_{j,k}\delta_{\alpha\beta}\,\mathrm{id}_{X_j}$ and $\mathrm{id}_{ZX_i}=\sum_{j,\alpha} p_i^{j,\alpha}\circ p_{i'}^{j,\alpha}$. We used the fact that $e_X(\cdot), e_Y(\cdot)$ are half-braidings, i.e. natural w.r.t. the second argument. Furthermore, the basis $\{q_{i,j}^{\alpha}\}$ in $\mathrm{Hom}_{\mathscr{C}}(Z, X_jX_{\bar{\imath}})$ and its dual basis are normalized such that $\mathrm{tr}_Z(q_{i,j'}^{\beta}\circ q_{i,j}^{\alpha})=d(Z)\delta_{\alpha\beta}$. We used that a basis together with its dual can be replaced by another one provided the normalizations are the same.

Since the above computation holds for all $Z\in\mathscr{C}$ we conclude that $E_{X,Y}(t)$ is in $\mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X,e_X),(Y,e_Y))$. Property (3.12) for morphisms a,c in $\mathscr{Z}_1(\mathscr{C})$ is obvious since by (3.5) a,c can be pulled through the half-braidings, changing the subscript of the conditional expectation E appropriately. In order to show that $E_{X,Y}$ is idempotent it thus suffices to show $E_{X,X}(\mathrm{id}_X)=\mathrm{id}_X$, which follows from the definition of $\dim\mathscr{C}$. □

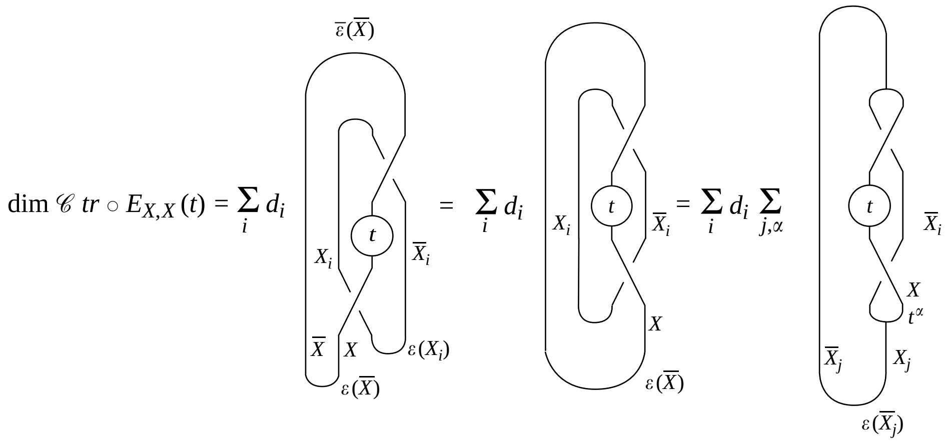
Remark 3.11. (1) Since the conditional expectations depend also on the half-braidings we should in principle denote them $E_{(X,e_X),(Y,e_Y)}$. We stick to $E_{X,Y}$ in order to keep the formulae simple.

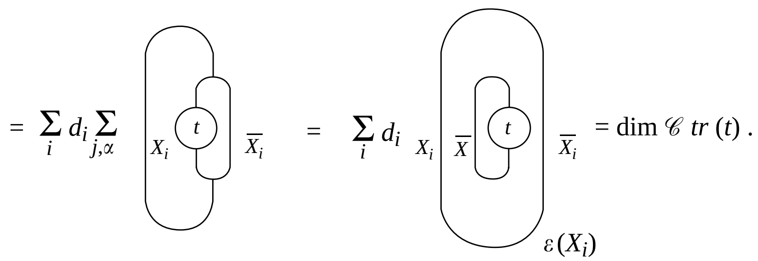
(2) The rôle of the assumption on the dimension is obvious: If $\dim\mathscr{C}=0$ then the map $E_{X,X}$ with the factor $(\dim\mathscr{C})^{-1}$ removed is identically zero on $\mathrm{End}_{\mathscr{Z}_1(\mathscr{C})}(X)$, thus we cannot use it to obtain a conditional expectation.

(3) The proof uses a special instance of the 'handle sliding' which has been formalized in [1], yet the present instance was discovered independently.

Lemma 3.12. *For every* $X \in \mathscr{C}$ *we have* $\mathrm{tr}_X \circ E_{X,X} = \mathrm{tr}_X$, *where* tr_X *is the trace on* $\mathrm{End}_{\mathscr{C}}(X)$ *provided by the spherical structure.*

Proof. Let $t \in \mathrm{Hom}_{\mathscr{C}}(X,X)$. Using the fact that the spherical structure of $\mathscr{Z}_1(\mathscr{C})$ is induced from $\mathscr{C}$ we compute





In the first step we have used Proposition 3.9, the second is based on standard properties of categories with duals. In the next step we use that, given a basis $\{t^\alpha\}$ in $\mathrm{Hom}_{\mathscr{C}}(X_j, \overline{X_i}X)$ with dual basis $\{\hat{t}^\alpha\}$, $\{e_X(\overline{X_i}) \circ t^\alpha\}$ is a basis in $\mathrm{Hom}_{\mathscr{C}}(X_j, X\overline{X_i})$ with dual basis $\{\hat{t}^\alpha \circ e_X(\overline{X_i})^{-1}\}$. Replacing one basis by the other leaves the expression invariant. □

A trace on a finite dimensional $\mathbb{F}$-algebra A is a $\mathbb{F}$-linear map $A \to \mathbb{F}$ such that $\mathrm{tr}(ab) = \mathrm{tr}(ba)$. It is non-degenerate if for every $a \neq 0$ there is b such that $\mathrm{tr}(ab) \neq 0$.

Lemma 3.13. *Let* A *be a finite dimensional* $\mathbb{F}$*-algebra and* $\mathrm{tr} : A \to \mathbb{F}$ *a non-degenerate trace. If* tr *is vanishes on nilpotent elements then* A *is semisimple. Conversely, every trace (not necessarily non-degenerate) on a semisimple algebra vanishes on nilpotent elements.*

Proof. Well known, but see, e.g., [37]. □

Lemma 3.14. *Let A be a finite dimensional semisimple algebra over $\mathbb{F}$ with a non-degenerate trace* $\mathrm{tr} : A \to \mathbb{F}$. *Let B be a subalgebra containing the unit of A and assume there is a conditional expectation $E : A \to B$ (i.e. a linear map such that $E(bab') = bE(a)b'$ for $a \in A$, $b, b' \in B$) such that* $\mathrm{tr} \circ E = \mathrm{tr}$. *Then B is semisimple.*

Proof. Let $0 \neq x \in B$. By non-degeneracy of tr there is $y \in A$ such that $\mathrm{tr}(xy) \neq 0$. Now, using the properties of E we compute $0 \neq \mathrm{tr}(xy) = \mathrm{tr} \circ E(xy) = \mathrm{tr}(xE(y))$. Since $E(y) \in B$ we conclude that the restriction $\mathrm{tr}_B \equiv \mathrm{tr} \restriction B$ is non-degenerate, too. By Lemma 3.13 tr vanishes on nilpotent elements, thus the same trivially holds for tr_B. Now the other half of Lemma 3.13 applies and B is semisimple. □

Remark 3.15. Algebra extensions $A \supset B$ admitting a conditional expectation $E : A \to B$ (satisfying certain conditions) are well known as Frobenius extensions, cf., e.g., [20], and are called Markov extensions if there is an E-invariant trace on A.

Now we can put everything together:

Theorem 3.16. *Let $\mathbb{F}$ be algebraically closed and $\mathscr{C}$ a $\mathbb{F}$-linear, spherical and semisimple tensor category. We assume that there are only finitely many simple objects and that* $\dim \mathscr{C} \neq 0$. *Then the quantum double $\mathscr{Z}_1(\mathscr{C})$ is spherical and semisimple.*

Proof. Recall that by our definition of semisimplicity, $\mathscr{C}$ has direct sums, subobjects and a simple unit. By our earlier results also $\mathscr{Z}_1(\mathscr{C})$ has these properties and is spherical. It therefore only remains to show that the endomorphism algebra of every object of $\mathscr{Z}_1(\mathscr{C})$ is a multi matrix algebra.

Let $(X, e_X) \in \mathscr{Z}_1(\mathscr{C})$. Then $\mathrm{End}_{\mathscr{C}}(X)$ is a finite dimensional multi matrix algebra by semisimplicity of $\mathscr{C}$. The trace on $\mathrm{End}_{\mathscr{C}}(X)$ provided by the duality structure is non-degenerate, cf. e.g. [15, Lemma 3.1], and Lemmas 3.10, 3.12 provide us with a trace preserving conditional expectation $E_X : \mathrm{End}_{\mathscr{C}}(X) \to \mathrm{End}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X))$. Thus $\mathrm{End}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X))$ is semisimple by Lemma 3.14 and therefore a multi matrix algebra since $\mathbb{F}$ is assumed algebraically closed. □

Remark 3.17. (1) Even if (X, e_X) is simple as an object of $\mathscr{Z}_1(\mathscr{C})$ there is no reason why X should be simple in $\mathscr{C}$. Usually it is not. Since we do not know a priori which non-simple objects of $\mathscr{C}$ appear in the simple objects of $\mathscr{Z}_1(\mathscr{C})$ we cannot dispense with the assumption that $\mathscr{C}$ has all finite direct sums as done, e.g., in [53].

(2) We briefly remark on possibilities of generalization of the results of this section suggested by Etingof. Whenever a tensor category has left and right duals the considerations of Section 3.2 imply that also $\mathscr{Z}_1(\mathscr{C})$ has left and right duals. Viz., replacing $\bar{Y}$ in Proposition 3.9 by the left dual ${}^*Y$, one obtains a left dual $({}^*Y, e_{{}^*Y})$ of (Y, e_Y). Here $e_{{}^*Y}(\cdot)$ is invertible by virtue of the existence of the right dual Y^* in $\mathscr{C}$. Also the proof of semisimplicity generalizes, provided one makes suitable changes. E.g., in the

definition of $E_{X,Y}$ in Lemma 3.10 one replaces

$$\varepsilon(X_i) \to e_{X_i}, \quad \bar{\varepsilon}(X_i) \to \eta_{X_i}, \quad d_i \to d_{X_i} \circ \varepsilon_{X_i},$$

where the morphisms e_{X_i}, η_{X_i}, d_{X_i}, ε_{X_i} are as in [37, Proposition 2.4]. For the dimension $\dim \mathscr{C}$ one uses [37, Definition 2.5], which does not assume the existence of a pivotal/spherical structure, but only that $\mathscr{C}$ has two sided duals. (In a semisimple category a left dual ${}^*Y$ is automatically two sided.) Now the proof essentially goes through as before. Possibly also the results of the remainder of the paper hold in larger generality, but we do not pursue this.

(3) Note that we do net yet know that $\mathscr{Z}_1(\mathscr{C})$ has finitely many isomorphism classes of simple objects. To show this will be our next aim.

4. Weak Morita equivalence of $\mathscr{Z}_1(\mathscr{C})$ and $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$

4.1. A Frobenius Algebra in $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$

Throughout this section $\mathscr{C}$ will be a strict spherical tensor category with simple unit over an algebraically closed field $\mathbb{F}$. We require that $\mathscr{C}$ is semisimple with finite set Γ of isomorphism classes of simple objects and $\dim \mathscr{C} \neq 0$. The set Γ has a distinguished element 0 representing the tensor unit and an involution $i \mapsto \bar{\imath}$ which associates with every class the class of dual objects. We choose objects $\{X_i,\ i \in \Gamma\}$ in these classes, which are arbitrary except that we require $X_0 = \mathbf{1}$. We emphasize that we do not require $\overline{X_i} = X_{\bar{\imath}}$. This can be achieved by a suitable strictification of the category if and only if all self-dual objects are orthogonal [5]. (The terms real vs. pseudo-real do not seem appropriate if $\mathbb{F} \neq \mathbb{C}$.)) We choose once and for all square roots of the $d_i = d(X_i)$, as well as $\lambda = \sqrt{\dim \mathscr{C}}$ and $(\dim \mathscr{C})^{1/4} = \sqrt{\lambda}$. Let N_{ij}^k be the dimension of the space $\mathrm{Hom}(X_k, X_iX_j)$, let $\{t_{ij}^{k\alpha},\ \alpha = 1, \dots, N_{ij}^k\}$ be a basis in $\mathrm{Hom}(X_k, X_iX_j)$ and let $\{t_{ij'}^{k\alpha}\}$ be the basis in $\mathrm{Hom}(X_iX_j, X_k)$ which is dual in the sense of $t_{ij'}^{k\alpha} \circ t_{ij}^{k,\beta} = \delta_{\alpha\beta}$. Note that this normalization of the dual basis differs from the one provided by the trace by a factor of d_k. The present choice is more convenient since otherwise the dimensions would appear in the equation

$$\sum_{k,\alpha} t_{ij}^{k\alpha} \circ t_{ij'}^{k\alpha} = \mathrm{id}_{X_iX_j}.$$

The choice of the square roots, the X_i and of the bases $\{t_{ij}^{k\alpha}\}$ is immaterial but will be kept fixed throughout the rest of the paper, and the symbols Γ, X_i, N_{ij}^k, $t_{ij}^{k\alpha}$ will keep the above meanings.

With these preparations we can embark on the 2-categorical approach to the quantum double. We define

$$\mathscr{A} = \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}, \tag{4.1}$$

$$\hat{X}_i = X_i \boxtimes X_i^{\mathrm{op}} \in \mathrm{Obj}\, \mathscr{A}. \tag{4.2}$$

By [37, Lemma 2.9], $\mathscr{C}^{\rm op}, \mathscr{C} \otimes_{\mathbb{F}} \mathscr{C}^{\rm op}$ and $\mathscr{A}$ are strict spherical in a canonical way. Every $\hat{X}_i$, $i \in \Gamma$ is simple and if it is self-dual (i.e. if $i = \bar{\imath}$) then it is orthogonal irrespective of whether X_i is orthogonal or symplectic.

The following is a very slight generalization of [29, Proposition 4.10].

Proposition 4.1. *Let $\mathbb{F}$ be quadratically closed and let $\mathscr{C}$ be $\mathbb{F}$-linear semisimple spherical with* $\dim \mathscr{C} \neq 0$. *There is a normalized strongly separable Frobenius algebra* $\mathsf{Q} = (Q, v, v', w, w')$ *in* $\mathscr{A} = \mathscr{C} \boxtimes \mathscr{C}^{\rm op}$ *(with* $\lambda_1 = \lambda_2 = \lambda$*) such that*

$$Q \cong \bigoplus_{i \in \Gamma} \hat{X}_i. \tag{4.3}$$

Proof. Clearly, $d(Q) = \dim \mathscr{C}$. By definition of Q there are morphisms

$$v_i \in \mathrm{Hom}(\hat{X}_i, Q), \qquad v_i' \in \mathrm{Hom}(Q, \hat{X}_i), \quad i \in \Gamma,$$

such that

$$v_i' \circ v_j = \delta_{ij}\, \mathrm{id}_{\hat{X}_i}, \qquad \sum_i v_i \circ v_i' = \mathrm{id}_Q. \tag{4.4}$$

Defining $v = \lambda^{1/2} v_0$, $v' = \lambda^{1/2} v_0'$, (2.7) is trivial. With $t_{ij'}^{k\alpha} \in \mathrm{Hom}_{\mathscr{C}}(X_i X_j, X_k) \equiv \mathrm{Hom}_{\mathscr{C}^{\rm op}}(X_k^{\rm op}, X_i^{\rm op} X_j^{\rm op})$ the morphisms

$$\hat{t}_{ij}^k = \sum_{\alpha=1}^{N_{ij}^k} t_{ij}^{k\alpha} \boxtimes t_{ij'}^{k\alpha} \in \mathrm{Hom}_{\mathscr{A}}(\hat{X}_k, \hat{X}_i \hat{X}_j)$$

are independent of the choices of the bases $\{t_{ij}^{k\alpha}\}$. Then

$$w = \lambda^{-1/2} \sum_{i,j,k \in \Gamma} \sqrt{\frac{d_i d_j}{d_k}}\, v_i \otimes v_j \circ \hat{t}_{ij}^k \circ v_k' \tag{4.5}$$

is in $\mathrm{Hom}_{\mathscr{A}}(Q, Q^2)$, and $w' \in \mathrm{Hom}_{\mathscr{A}}(Q^2, Q)$ is defined dually. Eqs. (2.3) and (2.4) of Definition 2.1 are almost obvious. (Use $N_{0i}^j = \delta_{ij}$). The proof that w, w' satisfy (2.1), (2.2) and (2.5) is omitted since it is entirely analogous to the one in [29, p. 591]. Finally, $w' \circ w = \lambda \mathrm{id}_Q$ is proven by a simple computation observing $\hat{t}_{ij'}^k \circ \hat{t}_{ij}^k = N_{ij}^k\, \mathrm{id}_{\hat{X}_k}$ and using

$$\sum_{i,j} d_i d_j N_{ij}^k = \sum_{i,j} d_i d_j N_{j\bar{k}}^{\bar{\imath}} = \sum_j d_k d_j^2 = d_k \dim \mathscr{C} = d_k \lambda^2. \tag{4.6}$$

Thus (Q, v, v', w, w') is a strongly separable Frobenius algebra in $\mathscr{A}$. □

Thus Theorem 2.2 applies and yields a spherical bicategory $\mathscr{E}$. ($\mathscr{E}$ is strict as a bicategory except for the existence of non-trivial unit constraints for $\mathbf{1}_{\mathfrak{B}}$ and strict pivotal [6] except for isomorphisms $\gamma_{X,Y} : \bar{Y} \circ \bar{X} \to \overline{X \circ Y}$ which are non-trivial whenever $\mathrm{Ran}(Y) = \mathrm{Src}(X) = \mathfrak{B}$.) In particular, we have a spherical tensor category $\mathscr{B} = \mathscr{END}_{\mathscr{E}}(\mathfrak{B})$. In the rest of the paper $\mathscr{A}$, Q, $\mathscr{E}$ and $\mathscr{B}$ will have the above meanings. By construction Q contains the identity object of $\mathscr{A}$ with multiplicity 1, thus $J, \bar{J}$ and

$\mathbf{1}_B$ are simple by [37, Proposition 5.3] and $d(J) = d(\bar{J}) = \lambda$. (Condition (iii) of that proposition can also easily be verified directly.)

Lemma 4.2. $\dim \mathscr{B} = (\dim \mathscr{C})^2$.

Proof. Follows from $\dim \mathscr{B} = \dim \mathscr{A}$ and $\dim \mathscr{A} = (\dim \mathscr{C})^2$. The former is [37, Proposition 5.16] and the latter is obvious since the simple objects of $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ are those of the form $X \boxtimes Y^{\mathrm{op}}$ with X, Y simple. □

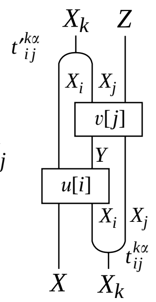
In the sequel we will write $\mathbf{1}$ instead of $\mathbf{1}^{\mathrm{op}}$ in order to alleviate the notation.

4.2. A fully faithful tensor functor $F : \mathscr{Z}_1(\mathscr{C}) \to \mathscr{B}$

In this subsection we will construct a functor $F : \mathscr{Z}_1(\mathscr{C}) \to \mathscr{B}$ and prove that it is fully faithful and monoidal. This already implies that $\mathscr{Z}_1(\mathscr{C})$ has finitely many isomorphism classes of simple objects, which is not at all obvious from Definition 3.4.

Lemma 4.3. *Let $X, Y \in \mathscr{C}$. There is a one-to-one correspondence between morphisms $u \in \mathrm{Hom}_{\mathscr{A}}((X \boxtimes \mathbf{1})Q, Q(Y \boxtimes \mathbf{1})) \equiv \mathrm{Hom}_{\mathscr{E}}(\bar{J}(X \boxtimes \mathbf{1})J, \bar{J}(Y \boxtimes \mathbf{1})J)$ and families $\{u[i] \in \mathrm{Hom}_{\mathscr{C}}(XX_i, X_iY),\ i \in \Gamma\}$. With $Z \in \mathscr{C}$ and $v \in \mathrm{Hom}_{\mathscr{A}}((Y \boxtimes \mathbf{1})Q, Q(Z \boxtimes \mathbf{1})) \equiv \mathrm{Hom}_{\mathscr{E}}(\bar{J}(Y \boxtimes \mathbf{1})J, \bar{J}(Z \boxtimes \mathbf{1})J)$ we have*

$$(v \bullet u)[k] = (d_k \lambda)^{-1} \sum_{i,j \in \Gamma} \sum_{\alpha=1}^{N_{ij}^k} d_i\, d_j \qquad (4.7)$$



Proof. Let $u \in \mathrm{Hom}_{\mathscr{A}}((X \boxtimes \mathbf{1})Q, Q(Y \boxtimes \mathbf{1}))$. Then

$$v'_j \otimes \mathrm{id}_{Y \boxtimes \mathbf{1}} \circ u \circ \mathrm{id}_{X \boxtimes \mathbf{1}} \otimes v_i$$

is in

$$\mathrm{Hom}_{\mathscr{A}}((X \boxtimes \mathbf{1})\hat{X}_i, \hat{X}_j(Y \boxtimes \mathbf{1})) = \mathrm{Hom}_{\mathscr{C}}(XX_i, X_jY) \otimes_{\mathbb{F}} \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X_i^{\mathrm{op}}, X_j^{\mathrm{op}}),$$

which vanishes if $i \neq j$. Thus

$$v'_i \otimes \mathrm{id}_{Y \boxtimes \mathbf{1}} \circ u \circ \mathrm{id}_{X \boxtimes \mathbf{1}} \otimes v_i = u[i] \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}}$$

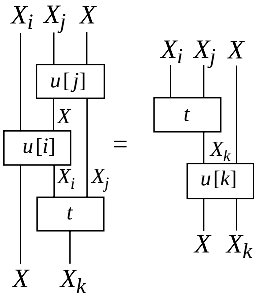
defines $u[i] \in \mathrm{Hom}_{\mathscr{C}}(XX_i, X_iY)$. Conversely, given $\{u[i] \in \mathrm{Hom}_{\mathscr{C}}(XX_i, X_iY),\ i \in \Gamma\}$,

$$u = \sum_i v_i \otimes \mathrm{id}_{Y \boxtimes \mathbf{1}} \circ u[i] \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}} \circ \mathrm{id}_{X \boxtimes \mathbf{1}} \otimes v'_i \qquad (4.8)$$

defines a morphism $u \in \mathrm{Hom}_{\mathscr{A}}((X \boxtimes \mathbf{1})Q, Q(Y \boxtimes \mathbf{1}))$.

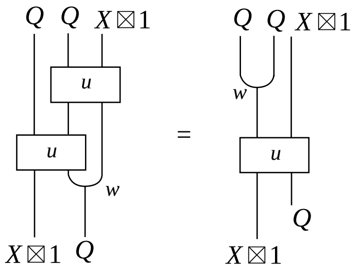
Eq. (4.7) follows easily from (4.8), the definition [37, Proposition 3.8] of the •-multiplication in $\mathscr{E}$ and the formula (4.5) for w, w'. □

Lemma 4.4. *Let* $u \in \mathrm{End}_{\mathscr{B}}(J(X \boxtimes \mathbf{1})\bar{J})$. *Then the associated family* $\{u[i]\}$ *satisfies the braiding fusion equation*



(4.9)

for all $i, j, k \in \Gamma$ *and all* $t \in \mathrm{Hom}_{\mathscr{C}}(X_k, X_iX_j)$ *iff* u *satisfies*



(4.10)

Proof. In view of definition (4.5) of $w \in \mathrm{Hom}_{\mathscr{A}}(Q, Q^2)$ and of (4.8), the left-hand side of (4.10) is seen to equal

$$\lambda^{-1/2}\sum_{i,j,k}\sqrt{\frac{d_id_j}{d_k}}\, v_i \otimes v_j \otimes \mathrm{id}_{X\boxtimes\mathbf{1}} \circ \mathrm{id}_{\hat{X}_i} \otimes u[j] \boxtimes \mathrm{id}_{X_j^{\mathrm{op}}} \circ u[i] \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}} \otimes \mathrm{id}_{\hat{X}_j}$$

$$\circ\, \mathrm{id}_{X\boxtimes\mathbf{1}} \otimes \hat{t}_{ij}^k \circ \mathrm{id}_{X\boxtimes\mathbf{1}} \otimes v_k',$$

whereas the right-hand side equals

$$\lambda^{-1/2}\sum_{i,j,k}\sqrt{\frac{d_id_j}{d_k}}\, v_i \otimes v_j \otimes \mathrm{id}_{X\boxtimes\mathbf{1}} \circ \hat{t}_{ij}^k \otimes \mathrm{id}_{X\boxtimes\mathbf{1}} \circ u[k] \boxtimes \mathrm{id}_{X_k^{\mathrm{op}}} \circ \mathrm{id}_{X\boxtimes\mathbf{1}} \otimes v_k'.$$

In view of the orthogonality relation satisfied by the $v's$, these two expressions are equal iff

$$\mathrm{id}_{\hat{X}_i} \otimes u[j] \boxtimes \mathrm{id}_{X_j^{\mathrm{op}}} \circ u[i] \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}} \otimes \mathrm{id}_{\hat{X}_i} \circ \mathrm{id}_{X\boxtimes\mathbf{1}} \otimes \hat{t}_{ij}^k$$

$$= \hat{t}_{ij}^k \otimes \mathrm{id}_{X\boxtimes\mathbf{1}} \circ u[k] \boxtimes \mathrm{id}_{X_k^{\mathrm{op}}} \quad \forall i, j, k \in \Gamma.$$

Inserting $\hat{t}^k_{ij} = \sum_\alpha t^{k\alpha}_{ij} \boxtimes t^{k\alpha}_{ij'}$, this becomes

$$\sum_{\alpha=1}^{N^k_{ij}} (\mathrm{id}_{X_i} \otimes u[j] \circ u[i] \otimes \mathrm{id}_{X_j} \circ \mathrm{id}_X \otimes t^{k\alpha}_{ij}) \boxtimes t^{k\alpha}_{ij'}$$

$$= \sum_{\alpha=1}^{N^k_{ij}} (t^{k\alpha}_{ij} \otimes \mathrm{id}_X \circ u[k]) \boxtimes t^{k\alpha}_{ij'}. \tag{4.11}$$

Multiplying from the right with $\mathrm{id}_{XX_k} \boxtimes t^{k\alpha}_{ij}$, we arrive at condition (4.9). Conversely, $\boxtimes$-tensoring (4.9) with $t^{k\alpha}_{ij'}$ and summing over α we obtain (4.11). □

Proposition 4.5. *There is a faithful functor* $F : \mathscr{Z}_1(\mathscr{C}) \to \mathscr{B}$.

Proof. Let $(X, e_X) \in \mathscr{Z}_1(\mathscr{C})$. By Lemma 4.3 the half braiding $\{e_X(Z), Z \in \mathscr{C}\}$ provides us with an element p^0_X in $\mathrm{End}_{\mathscr{B}}(\bar{J}(X \boxtimes \mathbf{1})J) \equiv \mathrm{Hom}_{\mathscr{A}}((X \boxtimes \mathbf{1})Q, Q(X \boxtimes \mathbf{1}))$. Since $e_X(\cdot)$ satisfies the braiding fusion relation (4.9), p^0_X satisfies (4.10). Now, multiplying (4.10) from the left with $w' \otimes \mathrm{id}_{X \boxtimes \mathbf{1}}$ and using (2.6) we obtain

$$w' \otimes \mathrm{id}_{X \boxtimes \mathbf{1}} \circ \mathrm{id}_Q \otimes p^0_X \circ p^0_X \otimes \mathrm{id}_Q \circ \mathrm{id}_{X \boxtimes \mathbf{1}} \otimes w = \lambda p^0_X, \tag{4.12}$$

which is just $p^0_X \bullet p^0_X = \lambda p^0_X$ in $\mathrm{End}_{\mathscr{E}}(\bar{J}(X \boxtimes \mathbf{1})J)$. Thus with $p_X = \lambda^{-1} p^0_X$,

$$F((X, e_X)) := (\bar{J}(X \boxtimes \mathbf{1})J, p_X) \tag{4.13}$$

is an object in $\mathscr{B}$, which defines the functor F on the objects. We will mostly write $F(X, e_X)$ instead of $F((X, e_X))$. Let $(X, e_X) \in \mathscr{Z}_1(\mathscr{C})$ with the above idempotent $p_X \in \mathrm{Hom}_{\mathscr{A}}((X \otimes \mathbf{1})Q, Q(X \otimes \mathbf{1})) \equiv \mathrm{End}_{\mathscr{E}}(\bar{J}(X \boxtimes \mathbf{1})J)$ and similarly (Y, e_Y), p_Y. Consider now $s \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X), (Y, e_Y)) \subset \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Then condition (3.5) implies

$$\mathrm{id}_Q \otimes (s \boxtimes \mathrm{id}_{\mathbf{1}}) \circ p_X = p_Y \circ (s \boxtimes \mathrm{id}_{\mathbf{1}}) \otimes \mathrm{id}_Q. \tag{4.14}$$

The element of $u \in \mathrm{Hom}_{\mathscr{A}}((X \boxtimes \mathbf{1})Q, Q(Y \boxtimes \mathbf{1})) \equiv \mathrm{Hom}_{\mathscr{E}}(\bar{J}(X \boxtimes \mathbf{1})J, \bar{J}(Y \boxtimes \mathbf{1})J)$ defined by (4.14) clearly satisfies $p_Y \bullet u \bullet p_X = u$ and is therefore a morphism in $\mathrm{Hom}_{\mathscr{E}}((\bar{J}(X \boxtimes \mathbf{1})J, p_X), (\bar{J}(Y \boxtimes \mathbf{1})J, p_Y))$. That the map $s \mapsto u$ is faithful follows from the first term in (4.14) and the fact that the $e_X(X_i)$, $i \in \Gamma$, and thus p_X (as a morphism in $\mathscr{A}$) are invertible. This defines F on the morphisms, and F is faithful. The simple argument proving that F respects the composition of morphisms is left to the reader. □

Proposition 4.6. *The functor* F *is full.*

Proof. We must show that every morphism in $\mathrm{Hom}_{\mathscr{B}}(F(X, e_X), F(Y, e_Y))$, where (X, e_X), $(Y, e_Y) \in \mathscr{Z}_1(\mathscr{C})$, is of the form $F(s)$ with $s \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X), (Y, e_Y))$. Now, the morphisms in $\mathrm{Hom}_{\mathscr{B}}((\bar{J}(X \boxtimes \mathbf{1})J, p_X), (\bar{J}(Y \boxtimes \mathbf{1})J, p_Y))$ are those elements s in $\mathrm{Hom}_{\mathscr{B}}(\bar{J}(X \boxtimes \mathbf{1})J, \bar{J}(Y \boxtimes \mathbf{1})J)$ which satisfy $s = p_Y \bullet s \bullet p_X$. p_X, p_Y being idempotents, every such s obviously is of the form $s = p_Y \bullet t \bullet p_X$ for some $t \in \mathrm{Hom}_{\mathscr{B}}(\bar{J}(X \boxtimes \mathbf{1})J, \bar{J}(Y \boxtimes \mathbf{1})J)$. By definition of $\mathscr{B}$ and by Lemma 4.3, s and t are represented by elements $\{s[i]\}, \{t[i]\}$ of $\bigoplus_{i \in \Gamma} \mathrm{Hom}_{\mathscr{C}}(XX_i, X_i Y)$. Given arbitrary t and setting $s = p_Y \bullet t \bullet p_X$ we will show that $s[0] \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is in fact in $\mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X), (Y, e_Y))$ and that

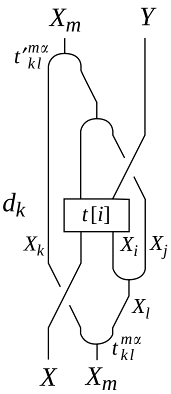
$$s[m] = \mathrm{id}_{X_m} \otimes s[0] \circ e_X(X_m) \quad \forall m \in \Gamma,$$

which is equivalent to

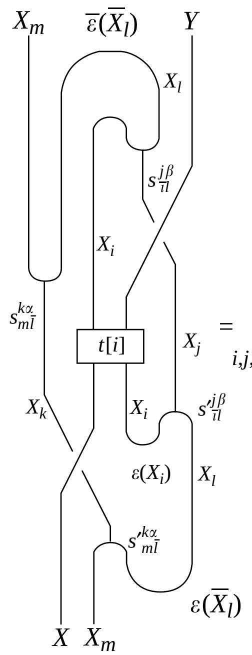
$$s = \mathrm{id}_Q \otimes (s[0] \boxtimes \mathrm{id}_{\mathbf{1}}) \circ p_X = F(s[0]).$$

Starting from the explicit statement of $s = p_Y \bullet t \bullet p_X$ we compute:

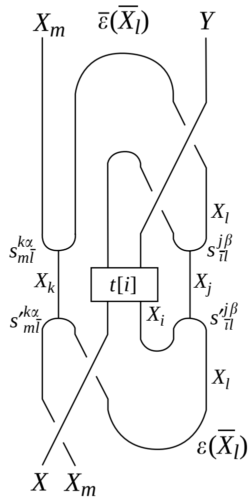
$$d_m \lambda^4 \cdot s[m] = \sum_{i,j,k,l,\in\Gamma} \sum_{\alpha,\beta} d_i\, d_j\, d_k$$

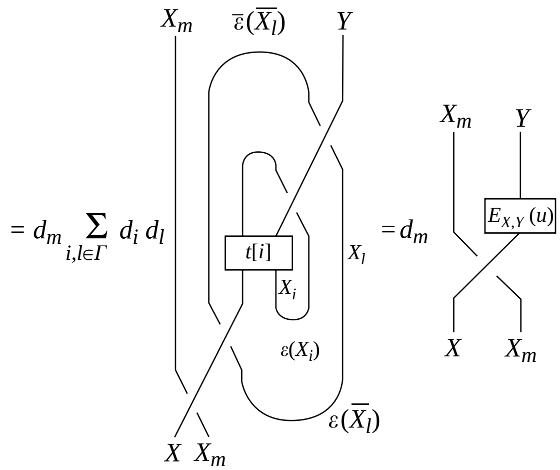


$$= \sum_{\substack{i,j,k,l\in\Gamma\\ \alpha,\beta}} d_i\, d_j\, d_k\, \frac{d_m}{d_k}\frac{d_l}{d_j}$$



$$= \sum_{\substack{i,j,k,l\in\Gamma\\ \alpha,\beta}} d_i\, d_m\, d_l$$





where $u\in\mathrm{Hom}_{\mathscr{C}}(X,Y)$ does not depend on m. (On the very left, p_X, p_Y and each of the $\bullet$-operations contribute one factor λ. Furthermore, $s^{k\alpha}_{m\bar{l}}$, $\alpha=1,\ldots,N^k_{m\bar{l}}$ is a basis in $\mathrm{Hom}(X_k, X_m\overline{X_l})$ with dual basis s'. We do not use $t^{k\alpha}_{m\bar{l}}$ since we cannot assume $\overline{X_l}=X_{\bar{l}}$ without losing generality.) For $m=0$ we have $X_m=\mathbf{1}$ and thus $s[0]=\lambda^{-4}E_{X,Y}(u)$, thus $s[0]\in\mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X,e_X),(Y,e_Y))$. Plugging this into the above equation for $m\neq 0$ we obtain $s[m]=\mathrm{id}_{X_m}\otimes s[0]\circ e_X(X_m)$ and therefore $s=F(\lambda s[0])$. We conclude that the functor F is full. □

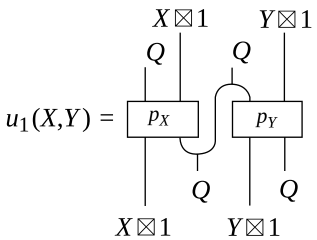
Proposition 4.7. *The functor F is strong monoidal.*

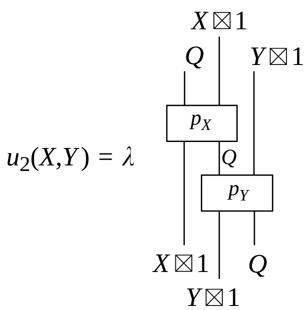
Proof. First we observe that $F(\mathbf{1}_{\mathscr{Z}_1(\mathscr{C})})=F(\mathbf{1}_{\mathscr{C}},\mathrm{id}_X)=(\bar{J}J,\lambda^{-1}\mathrm{id}_Q)$, which follows from (4.13) by putting $X=\mathbf{1}$ and $e_X(X_i)=\mathrm{id}_{X_i}$ $\forall i$. Comparing with [37, Theorem 3.12] we see $F(\mathbf{1}_{\mathscr{Z}_1(\mathscr{C})})=\mathbf{1}_{\mathfrak{B}}$.

Now we have to show that $F(X,e_X)F(Y,e_Y)$ and $F((X,e_X)(Y,e_Y))$ are naturally isomorphic. We compute

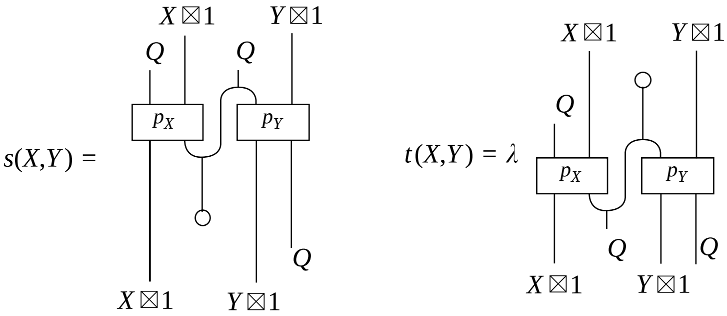
$$F(X,e_X)F(Y,e_Y)=(\bar{J}(X\boxtimes\mathbf{1})J,\,p_X)(\bar{J}(Y\boxtimes\mathbf{1})J,\,p_Y)$$
$$=(\bar{J}(X\boxtimes\mathbf{1})Q(Y\boxtimes\mathbf{1})J,u_1(X,Y)),$$
$$F((X,e_X)(Y,e_Y))=F(XY,e_{XY})=(\bar{J}(XY\boxtimes\mathbf{1})J,u_2(X,Y)),$$

where

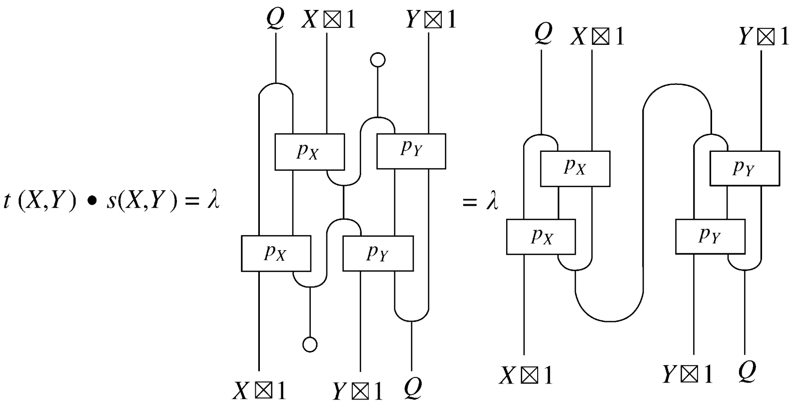




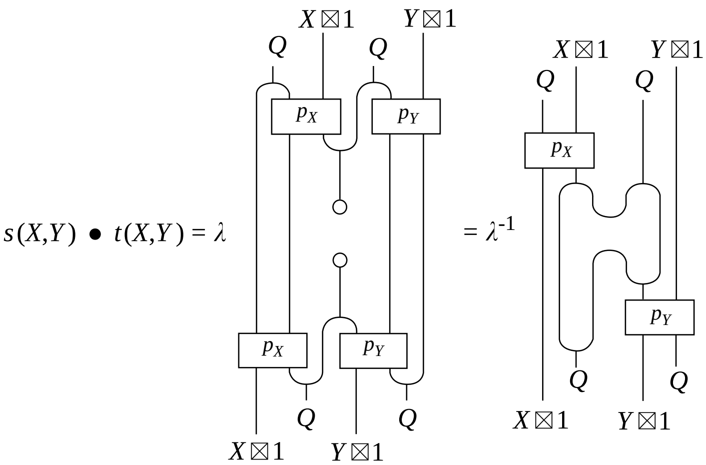
Obviously, F is not strict. But with $s(X,Y) \in \mathrm{Hom}_{\mathscr{E}}((X \boxtimes \mathbf{1})(Y \boxtimes \mathbf{1}), (X \boxtimes \mathbf{1})Q(Y \boxtimes \mathbf{1}))$ and $t(X,Y) \in \mathrm{Hom}_{\mathscr{E}}((X \boxtimes \mathbf{1})Q(Y \boxtimes \mathbf{1}), (X \boxtimes \mathbf{1})(Y \boxtimes \mathbf{1}))$ defined by



we compute



Here the second equality follows (verify!) by repeated use of Eqs. (2.1)–(2.5). Using $p_X \bullet p_X = p_X$, $p_Y \bullet p_Y = p_Y$ and the duality equation for Q we obtain $t(X,Y) \bullet s(X,Y) = u_2(X,Y)$. Now,



Here we used that p_X, p_Y come from half-braidings, implying that we have (4.10) and its dual version by Lemma 4.4. (Take into account two factors of λ which come from the normalization of $p_{X/Y}$.) It is easy to see that the last expression equals $u_1(X,Y)$.

It remains to verify that the functor F is coherent in the sense of [34, XI.2]. The computations present no difficulties and are simplified by the fact that the categories $\mathscr{Z}_1(\mathscr{C})$ and $\mathscr{B}$ are strict except for the unit in $\mathscr{B}=\mathrm{End}_{\mathscr{E}}(\mathfrak{B})$. We refrain from spelling out the details. □

4.3. *F is essentially surjective*

In order to conclude that F establishes an equivalence $\mathscr{B}\overset{\otimes}{\cong}\mathscr{Z}_1(\mathscr{C})$ of tensor categories it remains to prove that F is essentially surjective, viz. that for every object Y of $\mathscr{B}$ there is $(X,e_X)\in\mathscr{Z}_1(\mathscr{C})$ such that $F(X,e_X)\cong Y$. We begin with a result due to Izumi [18].

Lemma 4.8. *Let $Y\in\mathscr{C}$ be simple. Then the 1-morphisms $(Y\boxtimes\mathbf{1})J$, $(\mathbf{1}\boxtimes Y^{\rm op})J:\mathfrak{B}\to\mathfrak{A}$ and $\bar{J}(Y\boxtimes\mathbf{1})$, $\bar{J}(\mathbf{1}\boxtimes Y^{\rm op}):\mathfrak{A}\to\mathfrak{B}$ are simple. Furthermore,*

$$(Y\boxtimes\mathbf{1})J\cong(\mathbf{1}\boxtimes\bar{Y}^{\rm op})J,$$

$$\bar{J}(Y\boxtimes\mathbf{1})\cong\bar{J}(\mathbf{1}\boxtimes\bar{Y}^{\rm op}).$$

Proof. Let $Y,Z\in\mathscr{C}$. By duality be have the isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathscr{E}}((Y\boxtimes\mathbf{1})J,(Z\boxtimes\mathbf{1})J)&\cong\mathrm{Hom}_{\mathscr{E}}((Y\boxtimes\mathbf{1})J\bar{J},Z\boxtimes\mathbf{1})\\&=\mathrm{Hom}_{\mathscr{A}}((Y\boxtimes\mathbf{1})Q,Z\boxtimes\mathbf{1})\end{aligned}$$

of vector spaces. In view of $Q\cong\bigoplus_i X_i\boxtimes X_i^{\rm op}$ this implies

$$\mathrm{Hom}_{\mathscr{E}}((Y\boxtimes\mathbf{1})J,(Z\boxtimes\mathbf{1})J)\cong\mathrm{Hom}_{\mathscr{C}}(Y,Z).$$

In particular, if $Y\in\mathscr{C}$ is simple then $(Y\boxtimes\mathbf{1})J\in\mathrm{Hom}_{\mathscr{E}}(\mathfrak{B},\mathfrak{A})$ is simple, and so is $(\mathbf{1}\boxtimes Y^{\rm op})J$ by a similar argument. Furthermore,

$$\mathrm{Hom}_{\mathscr{E}}((Y\boxtimes\mathbf{1})J,(\mathbf{1}\boxtimes\bar{Y}^{\rm op})J)\cong\mathrm{Hom}_{\mathscr{E}}(Y\boxtimes Y^{\rm op},J\bar{J})=\mathrm{Hom}_{\mathscr{A}}(Y\boxtimes Y^{\rm op},Q).$$

Now, $Y\boxtimes Y^{\rm op}$ is simple and contained in Q with multiplicity one, thus these spaces are one dimensional and

$$(Y\boxtimes\mathbf{1})J\cong(\mathbf{1}\boxtimes\bar{Y}^{\rm op})J.$$

Similar arguments apply to the $\mathfrak{A}-\mathfrak{B}$-morphisms. □

Corollary 4.9. *Let $X,Y\in\mathscr{C}$. Then there is $Z\in\mathscr{C}$ such that*

$$\bar{J}(X\boxtimes Y^{\rm op})J\cong\bar{J}(Z\boxtimes\mathbf{1})J$$

and such that the isomorphisms

$$e\in\mathrm{Hom}_{\mathscr{E}}(\bar{J}(X\boxtimes Y^{\rm op})J,\bar{J}(Z\boxtimes\mathbf{1})J)\equiv\mathrm{Hom}_{\mathscr{A}}((X\boxtimes Y^{\rm op})Q,Q(Z\boxtimes\mathbf{1})),$$

$$f\in\mathrm{Hom}_{\mathscr{E}}(\bar{J}(Z\boxtimes\mathbf{1})J,\bar{J}(X\boxtimes Y^{\rm op})J)\equiv\mathrm{Hom}_{\mathscr{A}}((Z\boxtimes\mathbf{1})Q,Q(X\boxtimes Y^{\rm op}))$$

can be chosen such that

$$e = v \otimes \tilde{e}, \quad f = v \otimes \tilde{f}$$

with $\tilde{e} \in \mathrm{Hom}_{\mathscr{A}}((X \boxtimes Y^{\mathrm{op}})Q, Z \boxtimes \mathbf{1})$ *and* $\tilde{f} \in \mathrm{Hom}_{\mathscr{A}}((Z \boxtimes \mathbf{1})Q, X \boxtimes Y^{\mathrm{op}})$. (*Alternatively, one can find morphisms of the form* $e = \tilde{e} \otimes v'$, $f = \tilde{f} \otimes v'$.)

Proof. Using the lemma we compute

$$\bar{J}(X \boxtimes Y^{\mathrm{op}})J = \bar{J}(X \boxtimes \mathbf{1})(\mathbf{1} \boxtimes Y^{\mathrm{op}})J \cong \bar{J}(X \boxtimes \mathbf{1})(\bar{Y} \boxtimes \mathbf{1})J = \bar{J}(X\bar{Y} \boxtimes \mathbf{1})J.$$

We put $Z = X\bar{Y}$ and denote by $\hat{e}$ the isomorphism $(\mathbf{1} \boxtimes Y^{\mathrm{op}})J \to (\bar{Y} \boxtimes \mathbf{1})J$ provided by the preceding lemma. Now the claim follows with $\tilde{e} = \mathrm{id}_{X \boxtimes \mathbf{1}} \times \hat{e}$ if we keep in mind that tensoring $\tilde{e}$ with $\mathrm{id}_{\bar{J}}$ (in $\mathscr{E}$) amounts to tensoring with v in $\mathscr{A}$, as follows from the definition of $\mathscr{E}$. $\tilde{f}$ is defined similarly. Alternatively, using the isomorphism $\bar{J}(X \boxtimes \mathbf{1}) \cong \bar{J}(\mathbf{1} \boxtimes \bar{X}^{\mathrm{op}})$ one obtains a solution with $e = \tilde{e} \otimes v'$, etc. □

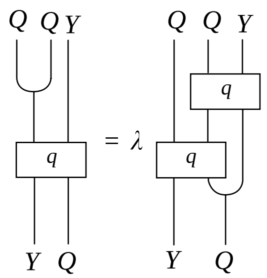
The lemma implies that every object of $\mathscr{B}$ is isomorphic to one of the form $(\bar{J}(X \otimes \mathbf{1})J, p_X)$. This looks quite promising since also $F(X, e_X)$ has this form. In fact, by Lemma 4.3 and Lemma 3.3 we obtain a family of morphisms $\{e_X(Y): XY \to YX, Y \in \mathscr{C}\}$ natural w.r.t. Y. Yet, in order to conclude that this is a half-braiding (and therefore $(\bar{J}(X \otimes \mathbf{1})J, p) = F(X, e_X)$) we need that p satisfies (4.10) and $p[0] = \mathrm{id}_X$. Not every object of $\mathscr{B}$ satisfies these conditions as is exemplified, e.g., the object $\bar{J}(X \boxtimes \mathbf{1})J = (\bar{J}(X \boxtimes \mathbf{1})J, p)$ where

$$p = \mathrm{id}_{\bar{J}(X \boxtimes \mathbf{1})J} = v \otimes \mathrm{id}_{X \boxtimes \mathbf{1}} \otimes v' \in \mathrm{Hom}_{\mathscr{A}}((X \boxtimes \mathbf{1})Q, Q(X \boxtimes \mathbf{1})).$$

One easily verifies that p does not satisfy (4.10). In view of $p[i] = \delta_{i0}\mathrm{id}_X$ it is also clear that the corresponding $e_X(Y)$ fails to be invertible for all Y.

The following result on the 2-category $\mathscr{E}$ is quite general in that it does not rely on $\mathscr{A} = \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$.

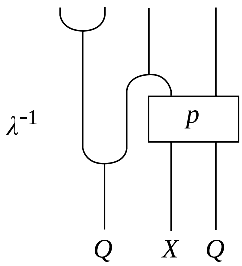
Lemma 4.10. *Let* $X \in \mathscr{A}$ *and* $p = p \bullet p \in \mathrm{End}_{\mathscr{E}}(\bar{J}XJ)$. *Then there is* $Y \in \mathscr{A}$, $q = q \bullet q \in \mathrm{End}_{\mathscr{E}}(\bar{J}YJ)$ *such that* $(\bar{J}YJ, q) \cong (\bar{J}XJ, p)$ *and in addition*



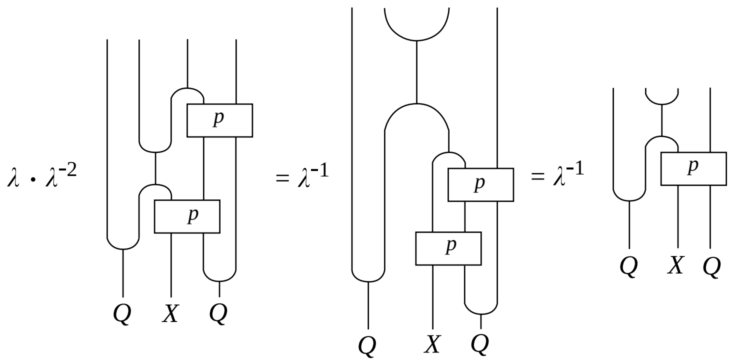
(4.15)

Remark 4.11. Condition (4.15) implies $q \bullet q = q$ as is seen by multiplication with $w' \otimes \mathrm{id}_Y$ from the left.

Proof. Using the (non-strict) unit $\mathfrak{B}-\mathfrak{B}$ morphism $\mathbf{1}_{\mathfrak{B}}=(\bar{J}J,\lambda^{-1}\,\mathrm{id}_Q)$ we put $(Y,q)=\mathbf{1}_{\mathfrak{B}}(X,p)=(QX,\lambda^{-1}\,\mathrm{id}_Q\times p)$. The isomorphism $(\bar{J}YJ,q)\cong(\bar{J}XJ,p)$ was proven in [37, Theorem 3.12]. We claim that q satisfies (4.15). In terms of (Y,q) (and keeping in mind that $Y=QX$!) the left-hand side of (4.15) is given by



For the right-hand side we compute

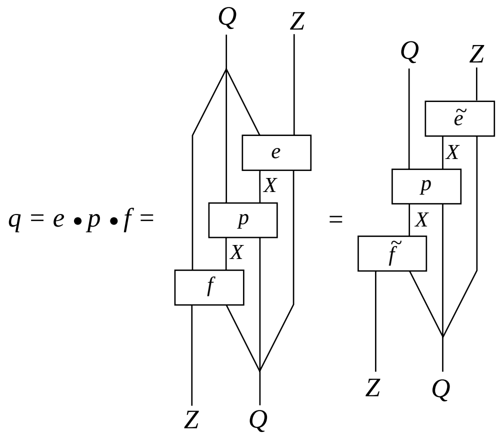


In the last step we have used $p\bullet p=p$. That the result coincides with the left-hand side follows now from a standard computation using the properties of a Frobenius algebra. □

Proposition 4.12. *Every object of* $\mathscr{B}$ *is isomorphic to one of the form* $(\bar{J}(Z\boxtimes\mathbf{1})J,q)$ *where* $q\in\mathrm{End}_{\mathscr{B}}(\bar{J}(Z\boxtimes\mathbf{1})J)$ *satisfies* (4.15) (*with* $Y=Z\boxtimes\mathbf{1}$).

Proof. By the preceding lemma every object $(\bar{J}(X\boxtimes Y^{\mathrm{op}})J,p)$ of $\mathscr{B}$ is isomorphic to one which satisfies (4.15), which allows us to assume this property in the rest of the proof. By Corollary 4.9 there is $Z\in\mathscr{C}$ such that $\bar{J}(X\boxtimes Y^{\mathrm{op}})J\cong\bar{J}(Z\boxtimes\mathbf{1})J$. Let $e:\bar{J}(X\boxtimes Y^{\mathrm{op}})J\to\bar{J}(Z\boxtimes\mathbf{1})J$, $f:\bar{J}(Z\boxtimes\mathbf{1})J\to\bar{J}(X\boxtimes Y^{\mathrm{op}})J$ be a pair of mutually inverse isomorphisms. Then with $q=e\bullet p\bullet f$ we have $(\bar{J}(Z\boxtimes\mathbf{1})J,q)\cong(\bar{J}(X\boxtimes Y^{\mathrm{op}})J,p)$. If we can show that also q satisfies (4.15) Lemma 4.4 applies and the claim follows. Now by Corollary 4.9 Z,e,f can be chosen such that $e=v\otimes\tilde{e}$, $f=v\otimes\tilde{f}$, where

$\tilde{e}, \tilde{f}$ are mutually inverse 2-morphisms between $(X \otimes Y)J$ and $(Z \otimes \mathbf{1})J$. Therefore,



where the four-fold vertices denote triple (co)products. That q satisfies (4.15) is now obvious from the respective property of p and $\tilde{f} \bullet \tilde{e} = \mathrm{id}$. □

Proposition 4.13. *The preceding proposition remains true if one adds the requirement that* $q[0] = \lambda^{-1}\mathrm{id}_Z$ (*notation of Lemma* 4.3).

Proof. Let Z, q be as in the preceding proposition. Multiplying (4.15) with $v' \otimes v' \otimes \mathrm{id}_{Z\boxtimes\mathbf{1}}$ and using $v' \otimes \mathrm{id}_{Z\boxtimes\mathbf{1}} \circ q = q[0] \otimes v$ we obtain $\lambda q[0]^2 = q[0] \in \mathrm{End}_{\mathscr{C}}(Z)$. Let $f : \tilde{Z} \to Z,\ g : Z \to \tilde{Z}$ be a splitting of the idempotent $\lambda q[0]$. Then it is easy to verify that with

$$\tilde{q} = \mathrm{id}_Q \otimes (g \boxtimes \mathrm{id}_{\mathbf{1}}) \circ q \circ (f \boxtimes \mathrm{id}_{\mathbf{1}}) \otimes \mathrm{id}_Q$$

we have $(\bar{J}(\tilde{Z} \boxtimes \mathbf{1})J, \tilde{q}) \cong (\bar{J}(Z \boxtimes \mathbf{1})J, q)$. This $\tilde{q}$ still verifies (4.15) and, in addition, $\tilde{q}[0] = \lambda^{-1}\mathrm{id}_{\tilde{Z}}$. □

Now we are ready to state our first main result.

Theorem 4.14. *The tensor categories* $\mathscr{B}$ *and* $\mathscr{Z}_1(\mathscr{C})$ *are equivalent as spherical categories, thus we have the weak monoidal Morita equivalence* (*in the sense of* [37]) $\mathscr{Z}_1(\mathscr{C}) \approx \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$. *In particular*,

$$\dim \mathscr{Z}_1(\mathscr{C}) = (\dim \mathscr{C})^2.$$

Proof. We have shown that every simple object in $\mathscr{B}$ isomorphic to the image under F of a simple object in $\mathscr{Z}_1(\mathscr{C})$. Since $\mathscr{Z}_1(\mathscr{C})$ and $\mathscr{B}$ are both semisimple (in particular closed under direct sums and subobjects) we conclude that F is essentially surjective. Since F is also fully faithful we have an equivalence of categories by [34, Theorem IV.4.1]. F being monoidal we have an equivalence of monoidal categories by [50, I.4.4]. (This already implies that $\mathscr{B}$ and $\mathscr{Z}_1(\mathscr{C})$ have the same dimension, since by

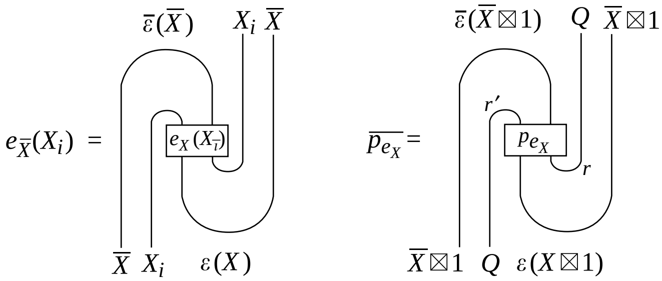


Fig. 1. $e_{\bar{X}}(X_i)$ and $\overline{p_{e_X}}$.

[37, Proposition 2.4] the latter are well-defined independently of the chosen spherical or $*$-structure and, of course, invariant under monoidal equivalence.)

It remains to show that the spherical structures are compatible. As to the conjugation maps, we have

$$\overline{F(X,e_X)} = \overline{(\bar{J}(X \boxtimes \mathbf{1})J,\, p_{e_X})} = (\bar{J}(\bar{X} \boxtimes \mathbf{1})J,\, \overline{p_{e_X}}),$$

$$F(\overline{(X,e_X)}) = F((\bar{X}, e_{\bar{X}})) = (\bar{J}(\bar{X} \boxtimes \mathbf{1})J,\, p_{e_{\bar{X}}}).$$

Putting together Proposition 3.9 and Lemma 3.2, $e_{\bar{X}}(X_i)$ is as in Fig. 1, where the pair of unlabeled morphisms is any solution of the duality equations. In view of the definition of p_X in Proposition 4.5 and of $\overline{p_{e_X}}$ in [37, Theorem 5.14], cf. Fig. 1, it is clear that $\overline{p_{e_X}} = p_{e_{\bar{X}}}$ and therefore

$$\overline{F(X,e_X)} = F(\overline{(X,e_X)}).$$

Now by Proposition 3.9, the spherical structure of $\mathscr{Z}_1(\mathscr{C})$ is inherited from $\mathscr{C}$, concretely $\varepsilon_{\mathscr{Z}_1(\mathscr{C})}(X,e_X) = \varepsilon_{\mathscr{C}}(X)$. Considering how the spherical structures of $\mathscr{E}_0$ and $\mathscr{E}$ arise from that of $\mathscr{A}$ in [37, Theorem 5.13] it is essentially obvious that $F : \mathscr{Z}_1(\mathscr{C}) \to \mathscr{B}$ is an equivalence of spherical categories irrespective of the fact that the latter is neither strict monoidal nor strict spherical. We omit the easy details. □

Remark 4.15. (1) In the case where $\mathscr{C}$ is the representation category of a finite dimensional involutive semisimple and cosemisimple Hopf algebra H, $\mathscr{Z}_1(\mathscr{C})$ is equivalent [21, Theorem XIII.5.1] to the representation category of the quantum double $D(H)$ and our result is just the fact $\dim D(H) = (\dim H)^2$.

(2) It seems likely that a simpler proof of the theorem can be given using the interpretation of the tensor category $\mathscr{B} = \mathscr{E}\mathscr{N}\mathscr{D}_{\mathscr{E}}(\mathfrak{B})$ as bimodule category $Q-\mathrm{Mod}-Q$, together with the recent work [45].

5. Modularity of the quantum double

5.1. The ‘Tube algebra’

By definition [37] of $\mathscr{E}$, every simple $\mathfrak{B}-\mathfrak{B}$ morphism is contained in $\bar{J}(X \boxtimes Y^{\mathrm{op}})J$ for some simple X, Y. In view of Lemma 4.8 every simple $\mathfrak{B}-\mathfrak{B}$-morphism is in fact contained in $\bar{J}(X \boxtimes \mathbf{1})J$ for some simple $X \in \mathscr{C}$ (as well as in $\bar{J}(\mathbf{1} \boxtimes \bar{X}^{\mathrm{op}})J$). Defining

$$\hat{Y}_L = \bigoplus_{i \in \Gamma} X_i \boxtimes \mathbf{1}, \qquad \hat{Y}_R = \bigoplus_{i \in \Gamma} \mathbf{1} \boxtimes X_i^{\mathrm{op}},$$

we conclude that either of $\bar{J}\hat{Y}_L J$, $\bar{J}\hat{Y}_R J$ contains all simple $\mathfrak{B}-\mathfrak{B}$-morphisms. With

$$\Xi_L = \mathrm{End}_{\mathscr{E}}(\bar{J}\hat{Y}_L J), \qquad \Xi_R = \mathrm{End}_{\mathscr{E}}(\bar{J}\hat{Y}_R J)$$

we thus have a one-to-one correspondence between isomorphism classes of simple $\mathfrak{B}-\mathfrak{B}$-morphisms and minimal central idempotents in Ξ_L or, equivalently, in Ξ_R. From now on we will stick to Ξ_L. By construction of $\mathscr{E}$ we have $\Xi_L = \mathrm{Hom}_{\mathscr{A}}(\hat{Y}_L Q, Q\hat{Y}_L)$ as a vector space. Thus

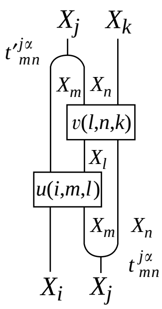
$$\begin{aligned}\Xi_L &\cong \bigoplus_{i,j,k,l} \mathrm{Hom}_{\mathscr{A}}(X_iX_j \boxtimes X_j^{\mathrm{op}}, X_kX_l \boxtimes X_k^{\mathrm{op}}) \\ &\cong \bigoplus_{i,j,k,l} \mathrm{Hom}_{\mathscr{C}}(X_iX_j, X_kX_l) \otimes_{\mathbb{F}} \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X_j^{\mathrm{op}}, X_k^{\mathrm{op}}) \\ &\cong \bigoplus_{i,j,l} \mathrm{Hom}_{\mathscr{C}}(X_iX_j, X_jX_l).\end{aligned}$$

We therefore have

$$\mathrm{End}_{\mathscr{E}}(\bar{J}\hat{Y}_L J) \equiv \mathrm{Hom}_{\mathscr{A}}(\hat{Y}_L Q, Q\hat{Y}_L) \cong \bigoplus_{i,j,k} \mathrm{Hom}_{\mathscr{C}}(X_iX_j, X_jX_k) \tag{5.1}$$

and in complete analogy to the proof of (4.7) one shows that the multiplication in Ξ_L is given by

$$(v \bullet u)\,(i,j,k) = (d_j\,\lambda)^{-1} \sum_{l,m,n \in \Gamma} \sum_{\alpha=1}^{N_{mn}^j} d_m\, d_n \tag{5.2}$$



Remark 5.1. (1) We observe that up to a different normalization (5.1) and (5.2) coincide with Ocneanu's definition of the 'tube algebra', cf. [44,14,18]. (The $(ij|X|jk)$ of Izumi corresponds to our $u[i,j,k]\cdot d_j/\lambda$.) Note, however, that we *derive* (5.1), (5.2) from an intrinsic definition of the algebra $\Xi_L = \mathrm{End}_{\mathscr{E}}(\bar{J}\hat{Y}_L J)$, which makes the correspondence between the minimal central idempotents of Ξ_L and the isomorphism classes of simple objects in $\mathscr{B}$ completely obvious. (Compare this to the laborious proof in [18].) The above considerations therefore completely clarify the role of the tube algebra. We suspect that Ocneanu arrived at his definition of the tube algebra by similar considerations.

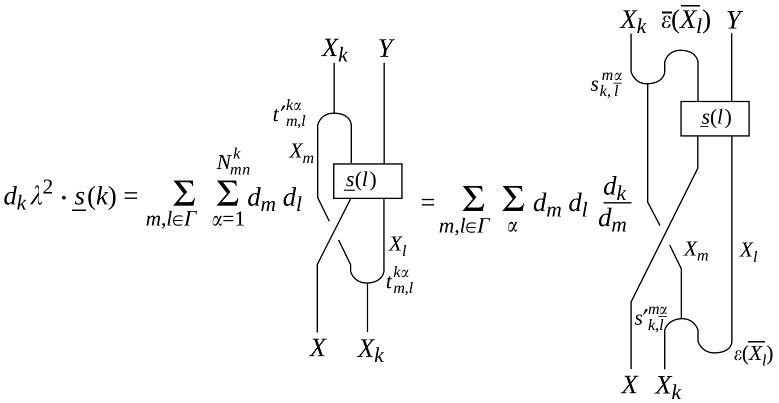
(2) Note that in the definition of Ξ_L we could replace $\hat{Y}_L$ by

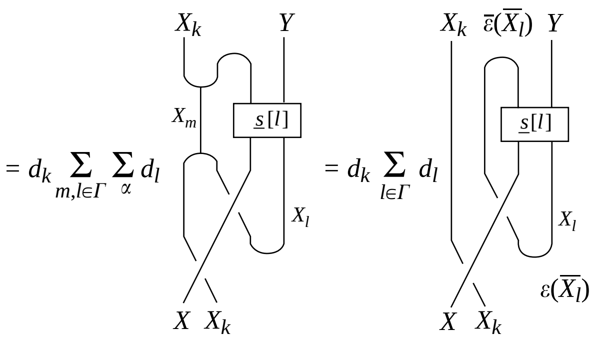
$$\hat{Y}_L^N = \bigoplus_{i \in \Gamma} N_i\,(X_i \boxtimes \mathbf{1})$$

with arbitrary $\{N_i\} \in \mathbb{N}^\Gamma$. The algebras $\mathrm{End}_{\mathscr{E}}(\bar{J}\hat{Y}_L^N J)$, of which the tube algebra happens to be the smallest, all have the same center, thus are Morita equivalent. We emphasize that only this common center has an invariant meaning, and in fact it has a well-known interpretation in terms of TQFTs, see Subsection 8.2.

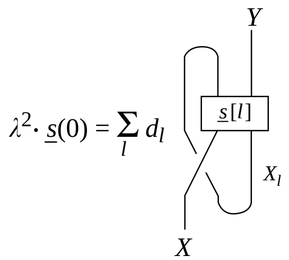
Lemma 5.2. *Let $(X, e_X) \in \mathscr{Z}_1(\mathscr{C})$ and $Y \in \mathscr{C}$. Then there is a isomorphism between the vector spaces* $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ *and* $\mathrm{Hom}_{\mathscr{B}}(F(X, e_X), \bar{J}(Y \boxtimes \mathbf{1})J)$.

Proof. The proof is similar to the one of Proposition 4.6, but simpler. Let $s \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and let $\underline{t} \in \mathrm{Hom}_{\mathscr{B}}(\bar{J}(X \boxtimes \mathbf{1})J, \bar{J}(Y \boxtimes \mathbf{1})J)$ be defined (using Lemma 4.3) by $\underline{t}[i] = \delta_{i0}s$. Then the map $\pi : s \mapsto \underline{s} = \underline{t} \bullet p_{(X,e_X)} \in \mathrm{Hom}_{\mathscr{B}}(F(X, e_X), \bar{J}(Y \boxtimes \mathbf{1})J)$ is injective since $(\underline{t} \bullet p_{(X,e_X)})[i] = \lambda^{-1}\mathrm{id}_{X_i} \otimes s \circ e_X(X_i)$, in particular $(\underline{t} \bullet p_{(X,e_X)})[0] = \lambda^{-1}s$. Let, conversely, $\underline{s} \in \mathrm{Hom}_{\mathscr{B}}(F(X, e_X), \bar{J}(Y \boxtimes \mathbf{1})J)$, i.e. $\underline{s} = \underline{s} \bullet p_{(X,e_X)} \in \mathrm{Hom}_{\mathscr{B}}(\bar{J}(X \boxtimes \mathbf{1})J, \bar{J}(Y \boxtimes \mathbf{1})J)$. Then





Putting $k = 0$ we obtain



and plugging this back into the preceding equation we obtain

$$\underline{s}[k] = \mathrm{id}_{X_i} \otimes \underline{s}[0] \circ e_X(X_i).$$

Thus $\underline{s}$ is in the image of π, which proves that π is an isomorphism. □

Proposition 5.3. *Let* $(X, e_X) \in \mathscr{Z}_1(\mathscr{C})$ *be simple and let* $N_i^X = \dim \mathrm{Hom}_{\mathscr{C}}(X_i, X)$. *Then the (simple) object* $F(X, e_X) \in \mathscr{B}$ *is contained in* $\bar{J}(X_i \boxtimes \mathbf{1})J$ *with multiplicity* N_i^X. *Let* $\{p_i^\alpha\}$, $\{p_{i'}^\alpha\}$ *be bases in* $\mathrm{Hom}_{\mathscr{C}}(X_i, X)$, $\mathrm{Hom}_{\mathscr{C}}(X, X_i)$, *respectively, normalized by* $p_{i'}^\alpha \circ p_i^\beta = \delta_{\alpha\beta}\, \mathrm{id}_{X_i}$. *Then* $q_i^\alpha \in \mathrm{Hom}_{\mathscr{E}}(F(X, e_X), \bar{J}(X_i \boxtimes \mathbf{1})J)$, $q_{i'}^\alpha \in \mathrm{Hom}_{\mathscr{E}}(\bar{J}(X_i \boxtimes \mathbf{1})J, F(X, e_X))$ *defined by*

$$q_i^\alpha[k] = \left(\frac{d(X)}{\lambda^2\, d_i}\right)^{1/2} \mathrm{id}_{X_k} \otimes p_{i'}^\alpha \circ e_X(X_k),$$

$$q_{i'}^\alpha[k] = \left(\frac{d(X)}{\lambda^2\, d_i}\right)^{1/2} e_X(X_k) \circ p_i^\alpha \otimes \mathrm{id}_{X_k}$$

satisfy $q_{i'}^\alpha \bullet q_i^\beta = \delta_{\alpha\beta}\, \mathrm{id}_{F(X,e_X)}$. *The idempotent* $z^i_{(X,e_X)} = \sum_{\alpha=1}^{N_i^X} q_i^\alpha \bullet q_{i'}^\alpha$ *in* $\mathrm{End}_{\mathscr{E}}(\bar{J}(X_i \boxtimes \mathbf{1})J)$ *corresponding to the isotypic component of* (X, e_X) *is given by*

$$z^i_{(X,e_X)}[k] = \frac{d(X)}{\lambda d_i} \sum_{\alpha=1}^{N_i^X} \mathrm{id}_{X_k} \otimes p_{i'}^\alpha \circ e_X(X_k) \circ p_i^\alpha \otimes \mathrm{id}_{X_k} \tag{5.3}$$

Remark 5.4. The choice of square root of $d(X)$ is immaterial, but it must be the same in the equations defining q_i^α and $q_{i'}^\alpha$.

Proof. In view of the preceding lemma all that remains to be verified is the normalization. Since (X, e_X) is simple we have $q_{i'}^\alpha \bullet q_i^\beta = c_{\alpha\beta}\, \mathrm{id}_{F(X,e_X)}$. Plugging $q_i^\alpha[k], q_{i'}^\alpha[k]$ into (4.7) and comparing with the middle term of the computation in Lemma 3.10 (with $Z = X_k$) we see that

$$(q_{i'}^\alpha \bullet q_i^\beta)[k] = \frac{d(X)}{\lambda d_i}\, \mathrm{id}_{X_k} \otimes E_{X,X}(p_i^\alpha \circ p_{i'}^\beta) \circ e_X(X_k)$$

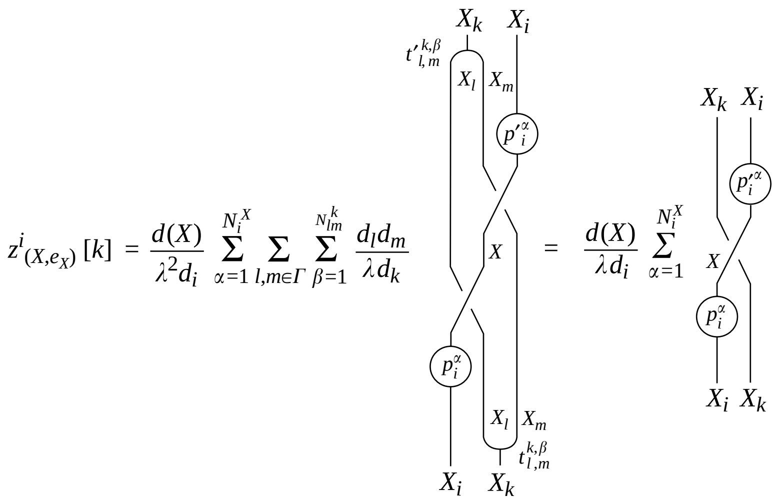
$$= \frac{\mathrm{tr}_X \circ E_{X,X}(p_i^\alpha \circ p_{i'}^\beta)}{\lambda d_i}\, e_X(X_k),$$

since $E_{X,X}(p_i^\alpha \circ p_{i'}^\beta)$ is a scalar multiple of id_X due to the simplicity of (X, e_X). Now, by definition of the functor F we have $\mathrm{id}_{F(X,e_X)}[k] = \lambda^{-1} e_X(X_k)$, thus by comparison we find $c_{\alpha\beta} = d_i^{-1}\,\mathrm{tr}_X \circ E_{X,X}(p_i^\alpha \circ p_{i'}^\beta)$. Computing

$$\mathrm{tr}_X \circ E_{X,X}(p_i^\alpha \circ p_{i'}^\beta) = \mathrm{tr}_X(p_i^\alpha \circ p_{i'}^\beta) = \mathrm{tr}_{X_i}(p_{i'}^\alpha \circ p_i^\beta) = d_i \delta_{\alpha\beta},$$

where we used the invariance of the trace under the conditional expectation and cyclic permutations, we obtain $c_{\alpha\beta} = \delta_{\alpha\beta}$ as claimed.

Now we can compute $z^i_{(X,e_X)} = \sum_\alpha q_i^\alpha \bullet q_{i'}^\alpha$ as follows:



We have pulled $t^{k\beta}_{lm'}$ through the braiding and used (4.6). □

Proposition 5.5. *Let $(X, e_X) \in \mathscr{Z}_1(\mathscr{C})$ be simple. The minimal central idempotent $z_{(X,e_X)}$ in Ξ_L corresponding to $F(X, e_X)$ is given by*

$$z_{(X,e_X)}[i,j,k] = \delta_{ik}\,\frac{d(X)}{\lambda d_i} \sum_{\alpha=1}^{N_i^X} \mathrm{id}_{X_j} \otimes p_{i'}^\alpha \circ e_X(X_j) \circ p_i^\alpha \otimes \mathrm{id}_{X_j}, \tag{5.4}$$

where the $\{p_i^\alpha\}$, $\{p_{i'}^\alpha\}$, $i \in \Gamma$, $\alpha = 1, \ldots, N_i^X$ are bases as in Proposition 5.3.

Proof. Since $\bar{J}\hat{Y}_L J$ is a direct sum $\bigoplus_i \bar{J}(X_i \boxtimes \mathbf{1})J$ we only need to add up the idempotents in $\mathrm{End}_{\mathscr{E}}(\bar{J}(X_i \boxtimes \mathbf{1})J)$, which we identified in Proposition 5.3, inside $\Xi_L = \mathrm{End}_{\mathscr{E}}(\bar{J}\hat{Y}_L J)$. With isomorphism (5.1) the claimed identity follows. □

As a first application of the tube algebra we can give an easy bound on the 'size' of the quantum double:

Corollary 5.6. *The number $\#\mathscr{Z}_1(\mathscr{C})$ of isomorphism classes of simple objects of $\mathscr{Z}_1(\mathscr{C})$ satisfies*

$$\#\mathscr{Z}_1(\mathscr{C}) \leqslant \sum_{i,j \in \Gamma} \dim \mathrm{Hom}_{\mathscr{C}}(X_i X_j, X_j X_i).$$

Proof. By the equivalence $\mathscr{Z}_1(\mathscr{C}) \overset{\otimes}{\simeq} \mathscr{B}$ and the above considerations we have $\#\mathscr{Z}_1(\mathscr{C}) = \dim Z(\Xi_L)$. Since the center of Ξ_L is spanned by the $z_{(X,e_X)}$ constructed above and since $z_{(X,e_X)}[i,j,k] = 0$ if $i \neq k$ we have

$$Z(\Xi_L) \subset \bigoplus_{i,j} \mathrm{Hom}_{\mathscr{C}}(X_iX_j, X_jX_i),$$

which implies the bound. □

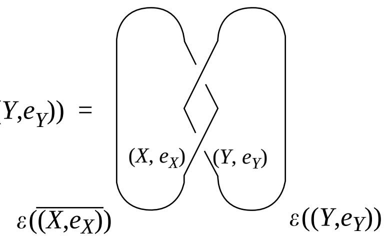
Remark 5.7. If G is a finite abelian group whose order is non-zero in $\mathbb{F}$ then $G-$mod is semisimple, symmetric and all simple objects have dimension one. Thus the right-hand side of the above inequality equals $|G|^2$. In view of $\mathscr{Z}_1(G-\mathrm{mod}) \simeq D(G)-\mathrm{mod}$ we have $\#\mathscr{Z}_1(G-\mathrm{mod}) = |G|^2$, which proves that the bound is optimal.

The next two subsections, which do not pretend much originality, will follow [18] quite closely except for shortcuts in the proofs.

5.2. Invertibility of the S-matrix

In this subsection we will prove that the S-matrix

$S((X,e_X),(Y,e_Y)) =$

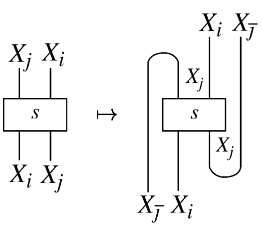


of $\mathscr{Z}_1(\mathscr{C})$ is invertible, thus $\mathscr{Z}_1(\mathscr{C})$ is modular in the sense of Turaev [53]. The strategy will be to define a vector space isomorphism $\mathfrak{S}$ of the subspace

$$\Xi_0 = \bigoplus_{i,j\in\Gamma} \mathrm{Hom}_{\mathscr{C}}(X_iX_j, X_jX_i)$$

of Ξ_L which we have seen to contain the center of Ξ_L. We will prove that $\mathfrak{S}$ leaves $Z(\Xi_L)$ stable and that the S-matrix of $\mathscr{Z}_1(\mathscr{C})$ is the matrix representation of $\mathfrak{S} \upharpoonright Z(\Xi_L)$ w.r.t. the basis $\{d(X)^{-1} z_{(X,e_X)},\ (X,e_X) \text{ simple}\}$.

Lemma 5.8. *The application* $\mathfrak{S}: \Xi_0 \to \Xi_0$ *defined by*



(5.5)

on the direct summands, where $\frown$, $\smile$ *are any solution of the duality equation for* X_j, $X_{\bar{\jmath}}$, *is a vector space isomorphism of order four.*

Proof. The above map $\mathrm{Hom}_{\mathscr{C}}(X_iX_j, X_jX_i) \to \mathrm{Hom}_{\mathscr{C}}(X_{\bar{\jmath}}X_i, X_iX_{\bar{\jmath}})$ is an isomorphism by duality. The same holds for $\mathfrak{S}$ which is just a direct sum of such isomorphisms, since the map $(i,j) \mapsto (\bar{\jmath}, i)$ is a permutation of $\Gamma \times \Gamma$. That $\mathfrak{S}$ has order four is an obvious consequence of sphericity of $\mathscr{C}$. □

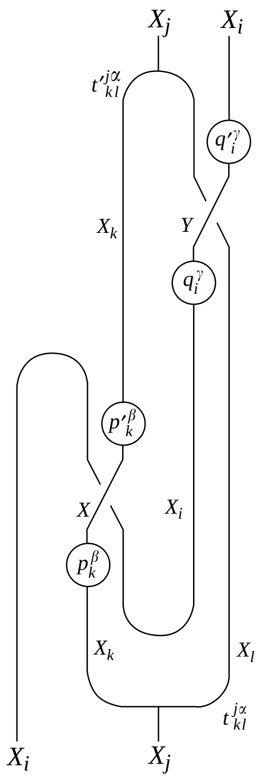
Lemma 5.9. *Let* $(X, e_X), (Y, e_Y)$ *be simple objects in* $\mathscr{Z}_1(\mathscr{C})$. *Then*

$$z_{(Y,e_Y)}\mathfrak{S}(z_{(X,e_X)}) = \frac{d(X)}{d(Y)\lambda^2} S(\overline{(X,e_X)}, (Y,e_Y)) \cdot z_{(Y,e_Y)}. \tag{5.6}$$

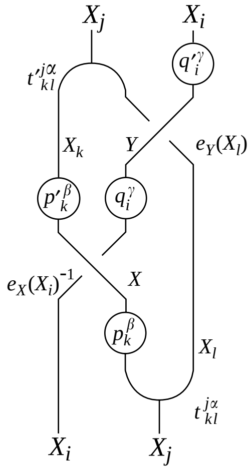
Proof. With (5.4), (5.5), and (5.2) we compute

$$(z_{(Y,e_Y)}\mathfrak{S}(z_{(X,e_X)}))[i,j,i]$$

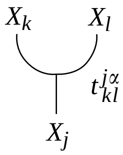
$$= (d_j\,\lambda)^{-1} \sum_{k,l} \sum_{\alpha=1}^{N_{kl}^j} \sum_{\beta=1}^{N_k^X} \sum_{\gamma=1}^{N_i^Y} d_k\, d_l\, \frac{d(X)}{\lambda d_k}\, \frac{d(Y)}{\lambda d_i}$$



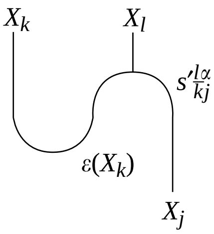
$$= (d_j\,\lambda)^{-1} \sum_{k,l} \sum_{\alpha=1}^{N_{kl}^j} \sum_{\beta=1}^{N_k^X} \sum_{\gamma=1}^{N_i^Y} d_k\, d_l\, \frac{d(X)}{\lambda d_k}\, \frac{d(Y)}{\lambda d_i}$$



where we have used Lemma 3.8. Replacing

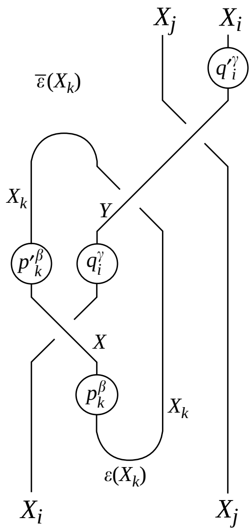


by $\left(\frac{d_j}{d_l}\right)^{1/2}$



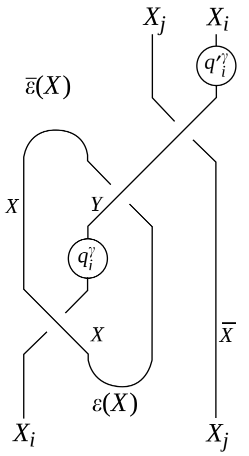
where $\{s^{l\alpha}_{\bar{k}j'}\}$ is a basis in $\mathrm{Hom}(\overline{X_k}X_j, X_l)$, and correspondingly for the dual basis, pulling $s^{l\alpha}_{\bar{k}j'}$ through the half-braiding $e_Y(\cdot)$ and summing over l, α we obtain

$$= \frac{d(X)\, d(Y)}{d_i\, \lambda^3} \sum_k \sum_{\beta=1}^{N_k^X} \sum_{\gamma=1}^{N_i^Y}$$

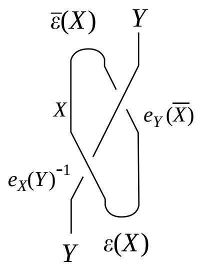


By sphericity of $\mathscr{C}$, naturality of the $e_Y(\cdot)$ and $\sum_{k,\beta} p_k^\beta \circ p_{k'}^\beta = \mathrm{id}_X$ this equals

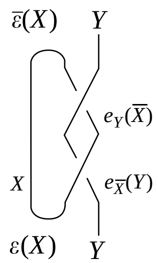
$$= \frac{d(X)\, d(Y)}{d_i\, \lambda^3} \sum_{\gamma=1}^{N_i^Y}$$



Using naturality of $e_X(\cdot)$ we can pull q_i^γ through $e_X(X_i)^{-1}$. Furthermore, since (Y, e_Y) is simple we have



$=$



$$= \frac{S(\overline{(X,e_X)},(Y,e_Y))}{d(Y)}\ \mathrm{id}_Y,$$

and (5.6) follows by comparison with (5.4). □

Proposition 5.10. $\mathfrak{S}$ *maps the center of* Ξ_L *into itself. The modular matrix* S *is invertible.*

Proof. Summing (5.6) over all classes of simple (Y, e_Y) and using $\sum_{(X,e_X)} z_{(X,e_X)} = 1_{\Xi_L}$ we obtain

$$\mathfrak{S}\left(\frac{z_{(X,e_X)}}{d(X)}\right) = \sum_{(Y,e_Y)} \lambda^{-2} S(\overline{(X,e_X)}, (Y,e_Y))\, \frac{z_{(Y,e_Y)}}{d(Y)}, \tag{5.7}$$

whence the first claim. Therefore the isomorphism $\mathfrak{S} : \Xi_0 \to \Xi_0$ restricts to $Z(\Xi_L)$ and the matrix $\lambda^{-1} S(\bar{\cdot}, \cdot)$ expresses the action of $\mathfrak{S} \upharpoonright Z(\Xi_L)$ in terms of the basis $\{d(X)^{-1} z_{(X,e_X)}\}$. Thus S is invertible. □

Remark 5.11. (1) Note that $\lambda^2 = \dim \mathscr{C} = \sqrt{\dim \mathscr{Z}_1(\mathscr{C})}$. This is the correct normalization since $(\dim \mathscr{M})^{-1/2} S$ is known to be of order four in every modular category $\mathscr{M}$ [53,47].

(2) An alternative proof of the modularity of $\mathscr{Z}_1(\mathscr{C})$ and of $\dim \mathscr{Z}_1(\mathscr{C}) = (\dim \mathscr{C})^2$ could be given as follows. If $\mathscr{C}$ satisfies the assumptions of our Theorem 1.2, there exists a finite dimensional quantum groupoid H such that $\mathscr{C}$ is monoidally equivalent to the category $H-\mathrm{Mod}$ of left modules over H. In [40], the quantum double of finite dimensional quantum groupoids was defined, and the category $D(H)-\mathrm{Mod}$ was shown to be modular. Modularity of $\mathscr{Z}_1(\mathscr{C})$ follows, provided one proves the equivalence $\mathscr{Z}_1(\mathscr{C}) \simeq D(H)-\mathrm{Mod}$ of braided tensor categories. Proceeding in analogy to the Hopf algebra case [21], this should not present any serious difficulty. Yet, we think that a direct categorical proof which avoids weak Hopf algebras is more satisfactory.

(3) The tensor category $\mathscr{B} = \mathscr{E}\mathscr{N}\mathscr{D}_{\mathscr{E}}(\mathfrak{B})$ defined in [37] is known to be equivalent to the category of $Q-Q$-bimodules, cf. [37, Remark 3.18]. Combining this with the ideas of [45], it should be possible to give a considerably simpler proof of the braided equivalence $\mathscr{Z}_1(\mathscr{C}) \simeq \mathscr{B}$.

In order to give the promised analogue of the (rather trivial) observation $Z_1(Z_0(S)) = \{\mathrm{id}_S\}$ from the Introduction we need the following

Definition 5.12. The center $\mathscr{Z}_2(\mathscr{C})$ of a braided monoidal category $\mathscr{C}$ is the full subcategory defined by

$$\mathrm{Obj}\, \mathscr{Z}_2(\mathscr{C}) = \{X \in \mathrm{Obj}\, \mathscr{C} \mid c(X,Y) = c(Y,X)^{-1} \ \ \forall Y \in \mathrm{Obj}\, \mathscr{C}\}.$$

Obviously the subcategory $\mathscr{Z}_2(\mathscr{C})$ is symmetric, contains the monoidal unit and is stable w.r.t. direct sums, retractions (in particular isomorphisms, thus replete) and duals.

Corollary 5.13. *The category $\mathscr{Z}_2(\mathscr{Z}_1(\mathscr{C}))$ is trivial, i.e. all objects are direct multiples of the monoidal unit.*

Proof. It is well known that a semisimple braided category $\mathscr{A}$ containing a simple object $X \ncong \mathbf{1}$ in $\mathscr{Z}_2(\mathscr{A})$ is not modular. ($X \in \mathscr{Z}_2(\mathscr{A})$ implies $S(X,Y) = d(X)d(Y)$ for all Y. This is colinear to $S(\mathbf{1},Y) = d(Y)$.) □

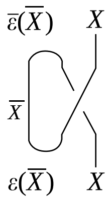
Remark 5.14. One can in fact prove [7] that $\mathscr{C}$ is modular iff $\dim \mathscr{C} \neq 0$ and the center $\mathscr{Z}_2(\mathscr{C})$ consists only of the direct multiples of the unit or, equivalently, iff all simple objects of $\mathscr{Z}_2(\mathscr{C})$ are isomorphic to the unit object. We will show this in Section 7 as a byproduct of a more general computation.

Remark 5.15. There is little doubt that a more conceptual understanding of the above proof (and of the subsequent subsection) can be gained by looking at them in the light of Lyubashenko's works [31,32]. The latter also raise the question whether there is a generalization to non-semisimple Noetherian categories. We hope to pursue this elsewhere.

Remark 5.16. It is natural to ask whether there are higher dimensional analogues to the above result in $d=1$ and the trivial case $d=0$ mentioned in the Introduction. (See [3] for a review of the theory of n-categories.) Thus, considering the center constructions in $d=2$ [4,9], can one show that $\mathscr{Z}_3^{(2)}(\mathscr{Z}_2^{(2)}(\mathscr{C}))$ is trivial? Here $\mathscr{C}, \mathscr{Z}_2^{(2)}(\mathscr{C}), \mathscr{Z}_3^{(2)}(\mathscr{Z}_2^{(2)}(\mathscr{C}))$ are (semisimple spherical) braided, sylleptic and symmetric 2-categories, respectively.

5.3. Computation of the Gauss sums

If $\mathscr{C}$ is a braided spherical tensor category a theorem of Deligne, cf. [56, Proposition 2.11], implies that $\mathscr{Z}_1(\mathscr{C})$ is a ribbon category (or balanced). Namely,

θ_X = 

for $X \in \mathscr{C}$

defines a natural automorphism $\{\theta_X, X \in \mathrm{Obj}\,\mathscr{C}\}$ of the identity functor which satisfies

$$\theta_{XY} = \theta_X \otimes \theta_Y \circ c(Y,X) \circ c(X,Y) \quad \forall X, Y,$$

$$\theta_{\bar{X}} = \overline{\theta_X}, \quad \forall X.$$

(A similar results hold for $*$-categories, cf. [30].) For the simple objects we have $\theta_X = \omega_X \,\mathrm{id}_X$ with $\omega_X \in \mathbb{F}^*$.

The quantum double $\mathscr{Z}_1(\mathscr{C})$ is braided and by the arguments in Section 3 we know that it has a spherical structure which is induced by the one on $\mathscr{C}$. We will show that the numbers $\omega_{(X,e_X)}$ can be computed in terms of the tube algebra and will compute the Gauss sum

$$\Delta_\pm(\mathscr{Z}_1(\mathscr{C})) = \sum_{(X,e_X)} \omega_{(X,e_X)}^{\pm 1} d(X,e_X)^2,$$

which plays an important role in the construction of topological invariants.

Following [18] we consider the element $t \in \Xi_L$ defined by

$$t[i,j,k] = \frac{\lambda}{d_i}\,\delta_{ik}\delta_{ij}\,\mathrm{id}_{X_i^2}. \tag{5.8}$$

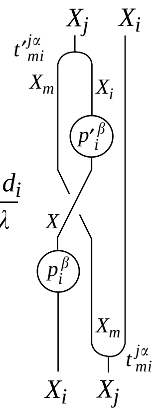
It will turn out that t is in the center of Ξ_L.

Lemma 5.17. *For simple* $(X,e_x) \in \mathscr{Z}_1(\mathscr{C})$ *we have*

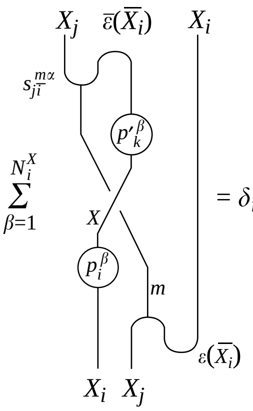
$$tz_{(X,e_X)} = \omega_{(X,e_X)}^{-1} z_{(X,e_X)}.$$

Proof. From (5.2), (5.4) and (5.8) we obtain

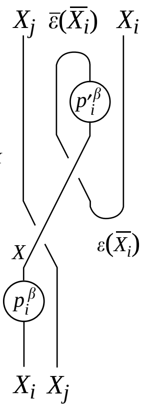
$$(tz_{(X,e_X)})\,[i,j,k] \;=\; \delta_{ik}\,\frac{\lambda}{d_i}\,\frac{d(X)}{\lambda d_i}\sum_{m\in\Gamma}\sum_{\alpha=1}^{N^j_{mi}}\sum_{\beta=1}^{N^X_i}\frac{d_m\,d_i}{d_j\lambda}$$



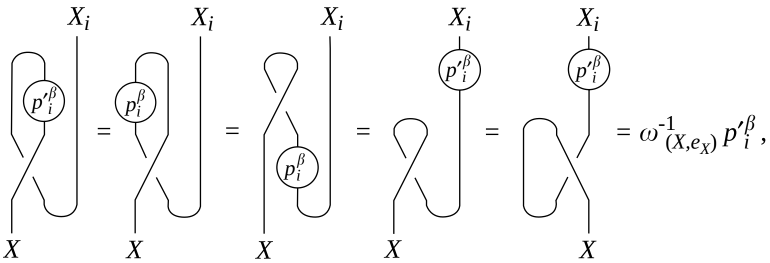
$$=\delta_{ik}\,\frac{d(X)}{\lambda d_i}\sum_{m\in\Gamma}\sum_{\alpha=1}^{N^j_{mi}}\sum_{\beta=1}^{N^X_i}$$



$$=\delta_{ik}\,\frac{d(X)}{\lambda d_i}\sum_{\beta=1}^{N^X_i}$$



Now the claim is a consequence of the following computation:



which is justified by the same arguments as in the proof of Lemma 5.9. Here we used standard properties of the spherical structure in the first and third equalities and naturality of the half-braiding $e_X(\cdot)$ w.r.t. the second argument in the second equality. The rest follows since (X,e_X) is simple. □

Proposition 5.18. *We have*

$$\Delta_\pm(\mathscr{Z}_1(\mathscr{C})) = \dim \mathscr{C}.$$

Proof. In view of $\sum z_{(X,e_X)} = \mathbf{1}_{\Xi_L}$ the lemma implies

$$t = \sum_{(X,e_X)} \omega^{-1}_{(X,e_X)} z_{(X,e_X)},$$

which proves that t is central in Ξ_L. To this equation we apply the linear form $\phi \in \Xi_L^*$

$$\phi(x) = \lambda \sum_{i\in\Gamma} d_i \,\mathrm{tr}_{X_i}(x[i,0,i]).$$

On one hand by (5.8) we clearly have $\phi(t) = \lambda^2$. On the other hand with (5.4) we compute

$$\phi(z_{(X,e_X)}) = d(X) \sum_{i\in\Gamma} \mathrm{tr}_{X_i}\left(\sum_{\alpha=1}^{N_i^X} p_{i'}^{\alpha} \circ p_i^{\alpha}\right) = d(X)\sum_{i\in\Gamma} d_i N_i^X = d(X)^2.$$

Putting everything together we obtain $\Delta_-(\mathscr{Z}_1(\mathscr{C})) = \lambda^2 = \dim \mathscr{C}$. The equality for Δ_+ follows from $\dim \mathscr{Z}_1(\mathscr{C}) = (\dim \mathscr{C})^2$ and the fact $\Delta_+(\mathscr{M})\Delta_-(\mathscr{M}) = \dim \mathscr{M}$, which holds for every modular category $\mathscr{M}$ [53,47]. □

This completes the proof of Theorem 1.2.

Remark 5.19. (1) A modular category satisfying $\Delta_+(\mathscr{C}) = \Delta_-(\mathscr{C})$ gives rise to an anomaly-free surgery TQFT, cf. [53]. Thus for quantum doubles the construction of the associated TQFTs simplifies considerably.

(2) The representation category of a rational conformal quantum field theory is a braided $*$-category and the central charge $c \in \mathbb{R}$ of the CQFT is related, cf. e.g. [47], to the Gauss sums $\Delta_-(\mathscr{C})$ by

$$\frac{\Delta_-(\mathscr{C})}{|\Delta_-(\mathscr{C})|} = \exp\left(\frac{2\pi i c_{\mathscr{C}}}{8}\right).$$

Since the Gauss sum of a quantum double is given by $\Delta_-(\mathscr{Z}_1(\mathscr{C})) = \dim \mathscr{C}$, thus positive, we conclude that the ‘central charge’ of a double satisfies

$$c_{\mathscr{Z}_1(\mathscr{C})} \equiv 0 \,(\mathrm{mod}\, 8).$$

6. The quantum double of a $*$-category

Consider the quantum double $\mathscr{Z}_1(\mathscr{C})$ of a $*$-category $\mathscr{C}$. If $s \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X,e_X),(Y,e_Y)) \subset \mathrm{Hom}_{\mathscr{C}}(X,Y)$ then clearly $s^* \in \mathrm{Hom}_{\mathscr{C}}(Y,X)$. It does, not, however, follow that $s^* \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((Y,e_Y),(X,e_X))$. But there is a suitable full subcategory of $\mathscr{Z}_1(\mathscr{C})$ which is a $*$-category.

Definition 6.1. Let $\mathscr{C}$ be a tensor $*$-category. Then the unitary quantum double $\mathscr{Z}_1^*(\mathscr{C})$ is defined as $\mathscr{Z}_1(\mathscr{C})$ except that the half-braidings $e_X(Y)$ are required to be unitary, not just invertible.

Lemma 6.2. *Let $\mathscr{C}$ be a tensor $*$-category. Then $\mathscr{Z}_1^*(\mathscr{C})$ is a $*$-category.*

Proof. For $s \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X,e_X),(Y,e_Y)) \subset \mathrm{Hom}_{\mathscr{C}}(X,Y)$ we have

$$\mathrm{id}_Z \otimes s \circ e_X(Z) = e_Y(Z) \circ s \otimes \mathrm{id}_Z \quad \forall Z.$$

Starring this equation and using $e_X(Z)^* = e_X(Z)^{-1}$ we obtain

$$\mathrm{id}_Z \otimes s^* \circ e_Y(Z) = e_X(Z) \circ s^* \otimes \mathrm{id}_Z \quad \forall Z,$$

thus $s^* \in \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((Y,e_Y),(X,e_X))$. $\square$

In the applications of the quantum double to operator algebras, like to the asymptotic subfactor [18] or quantum field theory [24], one is mainly interested in the unitary quantum double. In order for the results of Theorem 1.2 to remain valid for $\mathscr{Z}_1^*(\mathscr{C}) \subset \mathscr{Z}_1(\mathscr{C})$ one must show $\mathscr{Z}_1^*(\mathscr{C})$ that is equivalent to $\mathscr{Z}_1(\mathscr{C})$ as a tensor category. Given an isomorphism $s: X \to Y$ in a W^*-category we can use polar decomposition [16] to obtain a unitary morphism $\tilde{s}: X \to Y$. But we cannot construct a unitary half-braiding in this way since it is not clear that the unitaries $e_X(Z)$, $Z \in \mathscr{C}$ can be chosen such that naturality (3.1) and the braid relation (3.2) hold. Therefore a global approach is needed, which we develop using our machinery from Section 4.

Let $\mathscr{C}$ be a $*$-category with conjugates, simple unit and finitely many simple objects. All dimensions $d(X)$ are positive, and we choose the square roots of the latter and of $\dim \mathscr{C}$ to be positive. Reconsidering the constructions of Section 4 we now choose the bases $\{t_{ij}^{k\alpha}\}$ in $\mathrm{Hom}_{\mathscr{C}}(X_k, X_iX_j)$ to be orthonormal, i.e. $t_{ij'}^{k\alpha} = t_{ij}^{*k\alpha}$, and similarly $v_i' = v_i^*$. Then $v' = v^*$ and $w' = w^*$, such that the considerations of [37, Section 5.3] apply. We thus obtain a $*$-bicategory $\mathscr{E}_* \subset \mathscr{E}$ which is equivalent to $\mathscr{E}$. The considerations in Sections 4 and 5 of this paper remain essentially unchanged except for replacing $\varepsilon(X), \bar{\varepsilon}(X)$ by standard solutions $r_X, \bar{r}_X$ of the conjugate equations [30] everywhere.

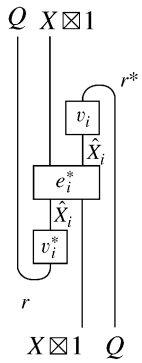
Lemma 6.3. *Let $\mathscr{C}$ be a $*$-category. Let $(X,e_X) \in \mathscr{Z}_1(\mathscr{C})$ and $F(X,e_X) = (\bar{J}(X \boxtimes \mathbf{1})J, p_X)$. Then the idempotent $p_X \in \mathrm{End}_{\mathscr{E}}(\bar{J}(X \boxtimes \mathbf{1})J)$ satisfies $p_X = p_X^\#$ iff $e_X(Z)$ is unitary for all Z.*

Proof. We recall from Proposition 4.5 that p_X is given by

$$p_X = \lambda^{-1} \sum_i v_i \otimes \mathrm{id}_{X \boxtimes \mathbf{1}} \circ e_X(X_i) \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}} \circ \mathrm{id}_{X \boxtimes \mathbf{1}} \otimes v_i'.$$

In view of the definition [37, Section 5.3] of the involution # on $\mathrm{End}_{\mathscr{E}}(\bar{J}(X \boxtimes \mathbf{1})J) \equiv \mathrm{Hom}_{\mathscr{A}}(\bar{J}(X \boxtimes \mathbf{1})JQ, Q\bar{J}(X \boxtimes \mathbf{1})J)$ we have

$p_X^{\#} = \lambda^{-1} \sum_i$



where $e_i^* \equiv e_X(X_i)^* \boxtimes \mathrm{id}_{X_i^{\mathrm{op}}}$ and $r = w \circ v : \mathbf{1} \to Q^2$. In view of (4.3) it is clear that there are uniquely determined $\hat{r}_i : \mathbf{1} \to \hat{X}_{\bar{\imath}} \otimes \hat{X}_i,\ i \in \Gamma$ such that

$$r = \sum_i v_{\bar{\imath}} \otimes v_i \circ \hat{r}_i.$$

Using $\mathrm{id}_Q \otimes r^* \circ r \otimes \mathrm{id}_Q = \mathrm{id}_Q$ one easily shows

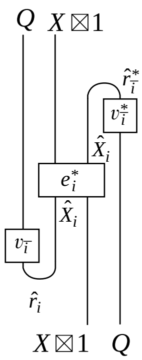
$$\mathrm{id}_{\hat{X}_i} \otimes \hat{r}_i^* \circ \hat{r}_{\bar{\imath}} \otimes \mathrm{id}_{\hat{X}_i} = \mathrm{id}_{\hat{X}_i}.$$

(This amounts to the identification $\overline{r_i} = \hat{r}_{\bar{\imath}}$ which is possible since all the self-conjugate $\hat{X}_i,\ i \in \Gamma$ are orthogonal.) Thus

$$v_i^* \otimes \mathrm{id}_Q \circ r = \mathrm{id}_{\hat{X}_i} \otimes v_{\bar{\imath}} \circ \hat{r}_{\bar{\imath}}, \qquad \mathrm{id}_Q \otimes v_i^* \circ r = v_{\bar{\imath}} \otimes \mathrm{id}_{\hat{X}_i} \circ \hat{r}_i$$

and we obtain

$p_X^{\#} = \lambda^{-1} \sum_i$



This equals p_X iff

$$\mathrm{id}_{X_{\bar{\imath}}X} \otimes \overline{r}_i^* \circ \mathrm{id}_{X_{\bar{\imath}}} \otimes e_X(X_i)^* \otimes \mathrm{id}_{X_{\bar{\imath}}} \circ r_i \otimes \mathrm{id}_{XX_{\bar{\imath}}} = e_X(X_{\bar{\imath}}) \quad \forall i \in \Gamma.$$

Considering Lemma 3.8, this is the case iff $e_X(X_i)^* = e_X(X_i)^{-1}$ for all $i \in \Gamma$. In view of Lemma 3.3 and the fact that the x_i^α occurring in its proof are automatically isometries, this is equivalent to unitarity of $e_X(Z)$ for all Z. □

Theorem 6.4. *Let $\mathscr{C}$ be a tensor $*$-category with simple unit, finitely many simple objects, conjugates, direct sums and subobjects. Then $\mathscr{Z}_1^*(\mathscr{C})$ is monoidally equivalent to $\mathscr{Z}_1(\mathscr{C})$, thus modular.*

Proof. Let $(X,e_X)\in\mathscr{Z}_1(\mathscr{C})$ and $(\bar{J}(X\otimes\mathbf{1})J,\, p_X)=F(X,e_X)$. Since $\mathrm{End}_{\mathscr{E}}(\bar{J}(X\otimes\mathbf{1})J)$ is a finite dimensional von Neumann algebra it contains an orthogonal projection $q_X=q_X^2=q_X^*$ and an invertible element s such that $s\,p_X s^{-1}=q_X$. It is clear that $(\bar{J}(X\boxtimes\mathbf{1})J,q_X)\cong(\bar{J}(X\boxtimes\mathbf{1})J,p_X)$, and by the lemma there is a unitary half-braiding $\tilde{e}_X(\cdot)$ such that $F(X,\tilde{e}_X)=(\bar{J}(X\boxtimes\mathbf{1})J,q_X)$. Thus

$$\mathscr{Z}_1^*(\mathscr{C})\overset{\otimes}{\simeq}\mathscr{B}_*\overset{\otimes}{\simeq}\mathscr{B}\overset{\otimes}{\simeq}\mathscr{Z}_1(\mathscr{C}),$$

where $\mathscr{B}_*\equiv\mathrm{End}_{\mathscr{E}_*}(\mathfrak{B})$. (The equivalence $\mathscr{B}_*\cong\mathscr{B}$ has already been demonstrated in [37, Proposition 5.6].) $\square$

7. The quantum double of a braided category

For the moment, let $\mathscr{C}$ be any (strict) braided monoidal category. Given such a category $\mathscr{C}$ we denote by $\tilde{\mathscr{C}}$ the braided monoidal category which coincides with $\mathscr{C}$ as a monoidal category, but has the braiding

$$\tilde{c}(X,Y)=c(Y,X)^{-1}.$$

It is well known (e.g., [21, Proposition XIII.4.3]) that for a braided monoidal category $\mathscr{C}$ there is a strict braided monoidal functor $I:\mathscr{C}\to\mathscr{Z}_1(\mathscr{C})$ given by

$$I(X)=(X,e_X)\quad\text{with } e_X(\cdot)=c(X,\cdot)$$

$$I(f)=f$$

on the objects and morphisms, respectively. I is full, faithful and injective on the objects, thus an embedding of $\mathscr{C}$ into $\mathscr{Z}_1(\mathscr{C})$.

Now, also $\tilde{\mathscr{C}}$ embeds into $\mathscr{Z}_1(\mathscr{C})$ via the functor $\tilde{I}$ defined by

$$\tilde{I}(X)=(X,\tilde{e}_X)\quad\text{with } \tilde{e}_X(\cdot)=\tilde{c}(X,\cdot),$$

$$\tilde{I}(f)=f.$$

Lemma 7.1. *$I(\mathscr{C})$ and $\tilde{I}(\tilde{\mathscr{C}})$ are replete full subcategories of $\mathscr{Z}_1(\mathscr{C})$.*

Proof. By definition, $(Y,e_Y)\in\mathscr{Z}_1(\mathscr{C})$ being isomorphic to $I(X)=(X,e_X)$ (where $e_X(Z)=c(X,Z)$) means that there is an isomorphism $u:X\to Y$ in $\mathscr{C}$ such that $e_Y(Z)=id_Z\otimes u\circ e_x(Z)\circ u^{-1}\otimes\mathrm{id}_Z$. With $e_X(Z)=c(X,Z)$ and naturality of the braiding c this implies $e_Y(Z)=c(Y,Z)$ and thus $(Y,e_Y)=I(Y)$. Thus $I(\mathscr{C})$ is replete. The proof for $\tilde{I}(\tilde{\mathscr{C}})$ clearly is the same. That $I(\mathscr{C})$ and $\tilde{I}(\tilde{\mathscr{C}})$ are full subcategories of $\mathscr{Z}_1(\mathscr{C})$ follows from naturality of the braiding in $\mathscr{C}$, which implies that every morphism $u:X\to Y$ in $\mathscr{C}$ automatically satisfies condition (3.1) in Definition 3.1 and thus is a morphism from $(X,c(X,\cdot))$ to $(Y,c(Y,\cdot))$. $\square$

Definition 7.2. Two subcategories $\mathscr{A},\mathscr{B}$ of a braided tensor category are said to commute iff $c(X,Y)\circ c(Y,X)=\mathrm{id}_{YX}$ for all $X\in\mathscr{A},\ Y\in\mathscr{B}$. For a braided monoidal category $\mathscr{C}$ and a subcategory $\mathscr{A}$ the relative commutant $\mathscr{C}\cap\mathscr{A}'$ is the full subcategory defined by

$$\mathrm{Obj}\,\mathscr{C}\cap\mathscr{A}'=\{X\in\mathrm{Obj}\,\mathscr{C}\mid c(X,Y)\circ c(Y,X)=\mathrm{id}_{YX}\ \ \forall Y\in\mathrm{Obj}\,\mathscr{A}\}.$$

The properties of the braiding imply that $\mathscr{C}\cap\mathscr{A}'$ is monoidal and stable under isomorphisms (thus replete), direct sums, retractions and two-sided duals. When there is no danger of confusion about the ambient category $\mathscr{C}$ we write also simply $\mathscr{A}'$. Note that $\mathscr{Z}_2(\mathscr{C})=\mathscr{C}\cap\mathscr{C}'$, which justifies the terminology center.

Proposition 7.3. *Let $\mathscr{C}$ be braided monoidal. Then*

$$\mathscr{Z}_1(\mathscr{C})\cap I(\mathscr{C})'=\tilde{I}(\tilde{\mathscr{C}}),$$

$$\mathscr{Z}_1(\mathscr{C})\cap \tilde{I}(\tilde{\mathscr{C}})'=I(\mathscr{C}).$$

Proof. By definition, $\mathscr{Z}_1(\mathscr{C})\cap I(\mathscr{C})'$ is the full subcategory of $\mathscr{Z}_1(\mathscr{C})$ whose objects (X,e_X) satisfy

$$c((X,e_X),(Y,e_Y))\circ c((Y,e_Y),(X,e_X))=\mathrm{id}_{(Y,e_Y)(X,e_X)}\qquad \forall (Y,e_Y)\in\Gamma(\mathscr{C}).$$

Using the definition of I and the definition of the braiding in $\mathscr{Z}_1(\mathscr{C})$ by $c((X,e_X),(Y,e_Y))=e_X(Y)$ we obtain

$$\mathrm{Obj}\,\mathscr{Z}_1(\mathscr{C})\cap I(\mathscr{C})'=\{(X,e_X)\in\mathscr{Z}_1(\mathscr{C})\ \mid\ e_X(Y)\circ c(Y,X)=\mathrm{id}_{YX}\}.$$

But this amounts to

$$\mathrm{Obj}\,\mathscr{Z}_1(\mathscr{C})\cap I(\mathscr{C})'=\{(X,e_X)\mid X\in\mathscr{C},\ e_X(Y)=c(Y,X)^{-1}\},$$

which is nothing but $\mathrm{Obj}\,\tilde{I}(\tilde{\mathscr{C}})$. The second equality is proven in the same way. □

Remark 7.4. As an obvious consequence we see that the subcategories $I(\mathscr{C})$ and $\tilde{I}(\tilde{\mathscr{C}})$ of $\mathscr{Z}_1(\mathscr{C})$ are equal to their second commutants: $I(\mathscr{C})''=I(\mathscr{C})$. Note that this holds without any technical assumptions on $\mathscr{C}$. See Remark 7.9 below for remarks on a general double commutant theorem.

The next observation provides another link between the centers $\mathscr{Z}_1$ and $\mathscr{Z}_2$ (besides the triviality of $\mathscr{Z}_2(\mathscr{Z}_1(\mathscr{C}))$ stated by Theorem 1.2). It can be interpreted as saying that $I(\mathscr{C})\vee\tilde{I}(\tilde{\mathscr{C}})$, the monoidal subcategory of $\mathscr{Z}_1(\mathscr{C})$ generated by $I(\mathscr{C})\cong\mathscr{C}$ and $\tilde{I}(\tilde{\mathscr{C}})\cong\tilde{\mathscr{C}}$, is an amalgamated product over their intersection $\mathscr{Z}_2(\mathscr{C})$.

Lemma 7.5. *Let $\mathscr{C}$ be braided. Then in $\mathscr{Z}_1(\mathscr{C})=\mathscr{Z}_1(\mathscr{C})$ we have*

$$I(\mathscr{C})\cap\tilde{I}(\tilde{\mathscr{C}})=I(\mathscr{Z}_2(\mathscr{C})).$$

Proof. Obviously, $I(X)=\tilde{I}(Y)$ is equivalent to $X=Y$ and $c(X,\cdot)=\tilde{c}(X,\cdot)$ and thus to $X\in\mathscr{Z}_2(\mathscr{C})$. □

The following results are stated in somewhat greater generality than needed here since we have other applications in mind, cf. Remark 7.9.

Lemma 7.6. *Let $\mathscr{C}$ be monoidal and semisimple with two-sided duals. Let $\mathscr{K}$, $\mathscr{L}$ be full monoidal subcategories which are semisimple (i.e. closed under direct sums and retractions) and have trivial intersection $\mathscr{K}\cap\mathscr{L}$ in the sense that every object contained both in $\mathscr{K}$ and $\mathscr{L}$ is a multiple of the tensor unit. If $K_1,K_2\in\mathscr{K}$ and $L_1,L_2\in\mathscr{L}$ then*

$$\mathrm{Hom}_{\mathscr{C}}(K_1L_1,K_2L_2)\cong\mathrm{Hom}_{\mathscr{K}}(K_1,K_2)\otimes_{\mathbb{F}}\mathrm{Hom}_{\mathscr{L}}(L_1,L_2). \tag{7.1}$$

More precisely, the linear maps

$$\otimes:\ \mathrm{Hom}_{\mathscr{K}}(K_1,K_2)\otimes_{\mathbb{F}}\mathrm{Hom}_{\mathscr{L}}(L_1,L_2)\to\mathrm{Hom}_{\mathscr{C}}(K_1L_1,K_2L_2)$$

induced by $(k,l)\mapsto k\otimes l$ are isomorphisms for all $K_1,K_2,\ L_1,L_2$. If $K_1,K_2\in\mathscr{K}$, $L_1,L_2\in\mathscr{L}$ are simple then $K_1L_1,K_2L_2\in\mathscr{C}$ are simple. They are isomorphic iff $K_1\cong K_2$, $L_1\cong L_2$.

Proof. By duality we have

$$\mathrm{Hom}_{\mathscr{C}}(K_1L_1,K_2L_2)\cong\mathrm{Hom}_{\mathscr{C}}(\bar{K}_2K_1,L_2\bar{L}_1).$$

Now $\bar{K}_2K_1\in\mathscr{K}$ and $L_2\bar{L}_1\in\mathscr{L}$, and since $\mathscr{K},\mathscr{L}$ are *monoidal* subcategories and closed w.r.t. retractions, all subobjects of $\bar{K}_1K_1$, $L_2\bar{L}_1$ are in $\mathscr{K}$ and $\mathscr{L}$, respectively. Since the monoidal unit $\mathbf{1}$ is (up to isomorphism) the only simple object common to $\mathscr{K}$ and our categories are semisimple, all morphisms $f:\bar{K}_2K_1\to L_2\bar{L}_1$ thus factorize through the monoidal unit:

$$\mathrm{Hom}_{\mathscr{C}}(\bar{K}_2K_1,L_2\bar{L}_1)\cong\mathrm{Hom}_{\mathscr{C}}(\bar{K}_2K_1,\mathbf{1})\otimes_{\mathbb{F}}\mathrm{Hom}_{\mathscr{C}}(\mathbf{1},L_2\bar{L}_1).$$

Using duality again we obtain (7.1). Thus $\mathrm{Hom}_{\mathscr{C}}(K_1L_1,K_2L_2)$ and $\mathrm{Hom}_{\mathscr{K}}(K_1,K_2)\otimes_{\mathbb{F}}\mathrm{Hom}_{\mathscr{L}}(L_1,L_2)$ have the same dimension and the $\otimes$-product on $\mathrm{Hom}(K_1,K_2)\times\mathrm{Hom}(L_1,L_2)$ extends to an isomorphism. The remaining claims are obvious consequences. □

Proposition 7.7. *Let $\mathscr{C}$ be braided monoidal and semisimple with two-sided duals. Let $\mathscr{K}$, $\mathscr{L}$ be semisimple full monoidal subcategories which commute and have trivial intersection. Then the full monoidal subcategory $\mathscr{K}\vee\mathscr{L}$ of $\mathscr{C}$ generated by $\mathscr{K}$ and $\mathscr{L}$ (by tensor products and direct sums) is equivalent as a braided monoidal category to $\mathscr{K}\boxtimes\mathscr{L}$. If $\mathscr{C}$ is spherical then this is an equivalence of spherical categories.*

Proof. Consider the functor $T:\mathscr{K}\otimes_{\mathbb{F}}\mathscr{L}\to\mathscr{K}\vee\mathscr{L}$ defined by $X\boxtimes Y\mapsto X\otimes Y$. By the above it is full and faithful. In order to prove that T is strong monoidal we compute

$$T(X \boxtimes Y) \otimes T(Z \boxtimes W) = X \otimes Y \otimes Z \otimes W,$$

$$T((X \boxtimes Y) \otimes (Z \boxtimes W)) = T((X \otimes Z) \boxtimes (Y \otimes W)) = X \otimes Z \otimes Y \otimes W.$$

Now the family

$$F_2(X \boxtimes Y, Z \boxtimes W) = \mathrm{id}_X \otimes c(Y,Z) \otimes \mathrm{id}_W$$

of morphisms $T(X \boxtimes Y) \otimes T(Z \boxtimes W) \to T((X \boxtimes Y) \otimes (Z \boxtimes W))$ clearly is natural and makes T strong monoidal. The easy proof of the coherence condition is left to the reader. In order to show that T is a braided tensor functor we must prove that the diagram

$$\begin{array}{ccc} T(X \boxtimes Y) \otimes T(Z \boxtimes W) & \xrightarrow{c_{\mathscr{C}}} & T(Z \boxtimes W) \otimes T(X \boxtimes Y) \\ \downarrow F_2 & & \downarrow F_2 \\ T(X \boxtimes Y \otimes Z \boxtimes W) & \xrightarrow[T(c_{\mathscr{K} \boxtimes \mathscr{L}})]{} & T(Z \boxtimes W \otimes X \boxtimes Y) \end{array}$$

commutes, where $c_{\mathscr{K} \boxtimes_{\mathbb{F}} \mathscr{L}} = c_{\mathscr{K}} \boxtimes c_{\mathscr{L}}$ is the direct product braiding. Using the definition of T and F_2 and taking into account that $\mathscr{K}$ and $\mathscr{L}$ commute this is an easy exercise. Now the functor T extends uniquely (up to natural isomorphism) to $\mathscr{K} \boxtimes \mathscr{L} = \overline{\mathscr{K} \otimes_{\mathbb{F}} \mathscr{L}}^{\oplus}$, remaining braided monoidal by naturality of the braiding. This extension is essentially surjective, thus an equivalence of braided spherical categories. That the equivalence respects spherical structures (if present) is obvious. □

Corollary 7.8. *Let $\mathscr{C}$ be braided monoidal and semisimple with two-sided duals (and spherical structure). Let $\mathscr{K} \subset \mathscr{C}$ be a semisimple full monoidal subcategory which has trivial center $\mathscr{Z}_2(\mathscr{K}) = \mathscr{K} \cap \mathscr{K}'$. Then we have the equivalence*

$$\mathscr{K} \boxtimes (\mathscr{C} \cap \mathscr{K}') \overset{\otimes}{\simeq}_{\mathrm{br}} \mathscr{K} \vee (\mathscr{C} \cap \mathscr{K}') \subset \mathscr{C}$$

of braided (spherical) categories.

Proof. The subcategory $\mathscr{L} = \mathscr{C} \cap \mathscr{K}'$ commutes with $\mathscr{K}$. Furthermore, $\mathscr{K} \cap \mathscr{L} = \mathscr{K} \cap \mathscr{K}' = \mathscr{Z}_2(\mathscr{K})$ is trivial by assumption. Thus the proposition applies. □

Remark 7.9. If we knew that $\mathscr{K} \vee (\mathscr{C} \cap \mathscr{K}') = \mathscr{C}$, we could conclude that $\mathscr{C}$ is equivalent, as a braided tensor category, to the direct product $\mathscr{K} \boxtimes (\mathscr{C} \cap \mathscr{K}')$. In [39] we will prove that this is indeed the case if $\mathscr{C}$ is a modular category. Thus whenever a modular category $\mathscr{C}$ contains a modular category $\mathscr{K}$ as a full tensor subcategory then $\mathscr{C} \overset{\otimes}{\simeq}_{\mathrm{br}} \mathscr{K} \boxtimes \mathscr{L}$, where $\mathscr{L} = \mathscr{C} \cap \mathscr{K}'$ is also modular. As a consequence, every modular category is a (finite) direct product of prime (or simple) ones, usually in a non-unique way. The proof relies on the following double commutant theorem: if $\mathscr{C}$ is a modular category and $\mathscr{K}$ is a semisimple monoidal subcategory closed under duality then (i) $\mathscr{K}''=\mathscr{K}$ and (ii) $\dim\mathscr{K}\cdot\dim\mathscr{K}'=\dim\mathscr{C}$. Here we are interested only in the full inclusion $I(\mathscr{C})\subset\mathscr{Z}_1(\mathscr{C})$ where $\mathscr{C}$ is modular, which can be treated without the full strength of the double commutant theorem. (Recall that we have proven $\mathscr{Z}_1(\mathscr{C})\cap(\mathscr{Z}_1(\mathscr{C})\cap I(\mathscr{C})')'=I(\mathscr{C})$.)

Theorem 7.10. *Let $\mathscr{C}$ be a modular category. Then there is a canonical equivalence*

$$\mathscr{Z}_1(\mathscr{C}) \stackrel{\otimes}{\simeq}_{\rm br} \mathscr{C}\boxtimes\tilde{\mathscr{C}}$$

of braided tensor categories.

Proof. We apply the corollary to $\mathscr{Z}_1(\mathscr{C})$ and the subcategory $I(\mathscr{C})$. The latter is braided isomorphic to $\mathscr{C}$, thus has trivial center. Therefore, as braided spherical categories

$$\mathscr{C}\boxtimes\tilde{\mathscr{C}}\cong I(\mathscr{C})\boxtimes\tilde{I}(\tilde{\mathscr{C}})=I(\mathscr{C})\boxtimes\mathscr{Z}_1(\mathscr{C})\cap I(\mathscr{C})'\simeq I(\mathscr{C})\vee I(\mathscr{C})'.$$

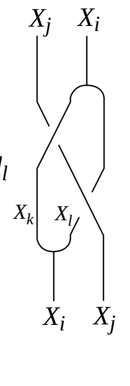
Thus, we are done if we can show that the full subcategory $I(\mathscr{C})\vee\tilde{I}(\tilde{\mathscr{C}})$ exhausts $\mathscr{Z}_1(\mathscr{C})$. If we assume $\mathscr{C}$ to be a $*$-category then also $\mathscr{Z}_1^*(\mathscr{C})\simeq\mathscr{Z}_1(\mathscr{C})$ is. By the above, we have $\dim(I(\mathscr{C})\vee\tilde{I}(\tilde{\mathscr{C}}))=\dim(\mathscr{C}\boxtimes\tilde{\mathscr{C}})=(\dim\mathscr{C})^2$, which coincides with the dimension of $\mathscr{Z}_1(\mathscr{C})$ by our main theorem. Since $\dim(I(\mathscr{C})\vee\tilde{I}(\tilde{\mathscr{C}}))$ is a full semisimple subcategory of $\mathscr{Z}_1(\mathscr{C})$ the categories must coincide. This argument does not work if $\mathscr{C}$ is not a $*$-category. We therefore give another proof which works in generality.

To this purpose we show that the minimal central idempotents of Ξ_L corresponding to the simple objects $I(X_k)\tilde{I}(X_l)\in\mathscr{Z}_1(\mathscr{C})$, $k,l\in\Gamma$ sum up to the unit of Ξ_L. By the definitions of $I,\tilde{I}$ we have $I(X_k)\tilde{I}(X_l)=(X_kX_l,e_{X_kX_l}(\cdot))$ with

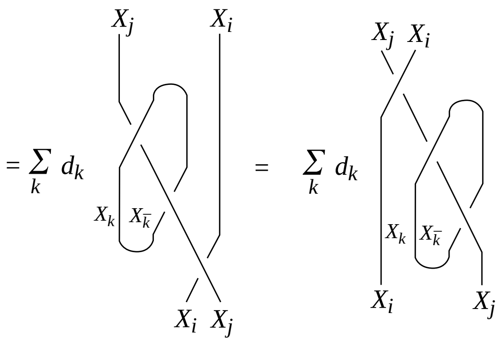
$$e_{X_kX_l}(Z)=c(X_k,Z)\otimes{\rm id}_{X_l}\circ{\rm id}_{X_k}\otimes c(Z,X_l)^{-1}.$$

Thus according to Proposition 5.4 the sum over the corresponding minimal central idempotents in Ξ_L is given by

$$\left(\sum_{k,l} z(I(X_k)\,\tilde{I}\,(X_l))\right)[i,j,n]=\frac{\delta_{in}}{d_i}\sum_{k,l}\sum_{\alpha=1}^{N_{kl}^i} d_k d_l$$



Computations which are identical to those in Lemma 3.10 (except for turning an over- into an under-crossing) show that the right-hand side equals



But this is nothing else than

$$= \left(\sum_k d_k S(X_k, X_j) \right) c(X_j, X_i)^{-1} = \left(\sum_k d_k S(X_k, X_j) \right) c(X_i, X_j). \tag{7.2}$$

Since $\mathscr{C}$ is assumed modular we have $\sum_k d_k S(X_k, X_j) = \delta_{j,0} \dim \mathscr{C}$ and thus

$$\left(\sum_{k,l} z(I(X_k)\tilde{I}(X_l)) \right) [i, j, n] = \delta_{in}\delta_{j,0} \dim \mathscr{C} \, \mathrm{id}_{X_i}.$$

The reader is invited to convince himself that this is the unit of the tube algebra Ξ_L by plugging it into (5.2). □

The method used in the proof allows to prove the following characterization of modular categories, which appeared in [7].

Corollary 7.11 (of proof). *Let $\mathscr{C}$ be a $\mathbb{F}$-linear semisimple spherical braided tensor category with finitely many simple objects. Then $\mathscr{C}$ is modular iff $\dim \mathscr{C} \neq 0$ and $\mathscr{Z}_2(\mathscr{C})$ is trivial.*

Proof. If $\mathscr{C}$ is modular then $\dim \mathscr{C} \neq 0$ [53] and $\mathscr{Z}_2(\mathscr{C})$ is trivial. If, conversely, $\mathscr{Z}_2(\mathscr{C})$ is trivial then for every $j \neq 0$ there exists i such that $c(X_i, X_j) \neq c(X_j, X_i)^{-1}$. But then (7.2) implies $\sum_k d_k S(X_k, X_j) = 0 \ \forall j \neq 0$, which is known to be equivalent to invertibility of S. □

Remark 7.12. It is well known that a braided tensor category $\mathscr{C}$ is monoidally isomorphic to its reverse $\mathscr{C}^{\mathrm{rev}}$ which coincides with $\mathscr{C}$ as a category but has the tensor product reversed: $X \otimes_{\mathrm{rev}} Y := Y \otimes X$. On the other hand the duality functor $X \mapsto \bar{X}$ provides a monoidal equivalence $\mathscr{C}^{\mathrm{op}} \overset{\otimes}{\simeq} \mathscr{C}^{rev}$. Putting this together we have $\tilde{\mathscr{C}} \overset{\otimes}{\simeq} \mathscr{C} \overset{\otimes}{\simeq} \mathscr{C}^{\mathrm{rev}} \overset{\otimes}{\simeq} \mathscr{C}^{\mathrm{op}}$. Thus for modular $\mathscr{C}$ we actually have an equivalence $\mathscr{Z}_1(\mathscr{C}) \overset{\otimes}{\simeq} \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ of tensor categories, not just weak monoidal Morita equivalence.

8. Applications

8.1. An adjoint for the forgetful functor $\mathscr{Z}_1(\mathscr{C}) \to \mathscr{C}$

In Section 4 we proved that the functor $F : \mathscr{Z}_1(\mathscr{C}) \to \mathscr{B}$ is fully faithful and essentially surjective. By [34, Theorem IV.4.1] this implies that F has a two-sided adjoint $G : \mathscr{B} \to \mathscr{Z}_1(\mathscr{C})$. Together with Lemma 5.2 this implies

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(X, Y) &\cong \mathrm{Hom}_{\mathscr{B}}(F(X, e_X), \bar{J}(Y \boxtimes \mathbf{1})J)\\ &\cong \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X), G(\bar{J}(Y \boxtimes \mathbf{1})J)),\end{aligned}$$

where $(X, e_X) \in \mathscr{Z}_1(\mathscr{C}), Y \in \mathscr{C}$. With the forgetful functor $H : \mathscr{Z}_1(\mathscr{C}) \to \mathscr{C}, (X, e_X) \mapsto X$ this becomes

$$\mathrm{Hom}_{\mathscr{C}}(H(X, e_X), Y) \cong \mathrm{Hom}_{\mathscr{Z}_1(\mathscr{C})}((X, e_X), G(\bar{J}(Y \boxtimes \mathbf{1})J)). \tag{8.1}$$

We thus have

Proposition 8.1. *The forgetful functor $H : \mathscr{Z}_1(\mathscr{C}) \to \mathscr{C}$, $(X, e_X) \mapsto X$ has a two-sided adjoint $K : \mathscr{C} \to \mathscr{Z}_1(\mathscr{C})$, $X \mapsto G(\bar{J}(X \boxtimes \mathbf{1})J)$. On the objects one has*

$$K(Y) \cong \bigoplus_{(X,e_X)} \dim \mathrm{Hom}_{\mathscr{C}}(X, Y)(X, e_X), \tag{8.2}$$

where the summation is over the isomorphism classes of simple objects in $\mathscr{Z}_1(\mathscr{C})$.

Proof. Eq. (8.1) just says that K is a right adjoint of H. That K is also a left adjoint of H is proven in the same way. One must also show that the isomorphisms in (8.1) are natural w.r.t. (X, e_X) and Y. We leave this to the reader. For $Y = X_i$ and (X, e_X) simple, (8.1) implies that $K(X_i)$ contains (X, e_X) with multiplicity $\dim \mathrm{Hom}_{\mathscr{C}}(X, X_i)$. For general Y we have

$$K(Y) \cong \bigoplus_{i \in \Gamma} \dim \mathrm{Hom}(X_i, Y) \bigoplus_{(X,e_X)} \dim \mathrm{Hom}_{\mathscr{C}}(X, X_i)(X, e_X),$$

and (8.2) follows by semisimplicity of $\mathscr{C}$. □

Remark 8.2. By the general theory [37] there is a dual Frobenius algebra $\hat{\mathsf{Q}} = (\hat{Q}, \ldots)$ in $\mathscr{B}$, where $\hat{Q} = \bar{J}J$. Under the equivalence $\mathscr{Z}_1(\mathscr{C}) \stackrel{\otimes}{\simeq} \mathscr{B}$, $\hat{Q}$ corresponds to

$$G(\bar{J}J) = K(\mathbf{1}) \cong \bigoplus_{(X,e_X)} 1 \dim \mathrm{Hom}_{\mathscr{C}}(X, \mathbf{1})\,(X, e_X). \tag{8.3}$$

Thus this object is part of the Frobenius algebra in $\mathscr{Z}_1(\mathscr{C})$ which establishes the weak monoidal Morita equivalence $\mathscr{Z}_1(\mathscr{C}) \approx \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$. $K(\mathbf{1})$ clearly contains the unit $(\mathbf{1}, \mathrm{id})$ of $\mathscr{Z}_1(\mathscr{C})$ with multiplicity one. The reader might find it amusing to identify explicitly the morphisms w, w' in $\mathscr{Z}_1(\mathscr{C})$ which come with the strongly separable Frobenius algebra.

8.2. Invariants of 3-manifolds

There are two classes of invariants of 3-manifolds associated with a modular tensor category $\mathscr{C}$, cf. [53]. On the one hand we have the surgery invariants $\mathrm{RT}(M,\mathscr{C})$ of Reshetikhin and Turaev [49] which are based on the fact that every connected oriented closed 3-manifold can be obtained from S^3 by surgery along a framed link. It turned out [8] that modularity of the category $\mathscr{C}$ is not really necessary, since it suffices that $\mathscr{C}$ be 'modularizable'. Yet, the invariant of the manifold being defined in terms of a link invariant, the existence of a (non-symmetric) braiding is essential. On the other hand, there are the state sum invariants based on a triangulation of the manifold. Generalizing on [54], an invariant $\mathrm{TV}(M,\mathscr{C})$ associated with any modular category $\mathscr{C}$ was defined in [53]. Later it was understood that in fact no braiding is necessary for the construction of a triangulation invariant, cf. [5,15], provided $\mathscr{C}$ has two-sided duals. (This had been anticipated in [43], which was never published.) We denote the corresponding invariant by $\mathrm{Tr}(M,\mathscr{C})$. Gelfand and Kazhdan formulated a conjecture [15, Conjecture 1] pointing towards a link between the two invariants being provided by the quantum double. Our results on the quantum double of semisimple tensor categories allow us to prove this conjecture.

Proposition 8.3. *Let $\mathscr{C}$ satisfy the assumptions of Theorem* 1.2 *and consider the state-sum TQFT associated with $\mathscr{C}$, as defined in* [15]. *Then the dimension of the vector space $\mathscr{H}_{S^1\times S^1}$ associated to the two-dimensional torus equals the number $\#\mathscr{Z}_1(\mathscr{C})$ of isomorphism classes of simple objects of $\mathscr{Z}_1(\mathscr{C})$.*

Proof (Sketch). By the considerations of Section 5.1, $\#\mathscr{Z}_1(\mathscr{C})$ coincides with the dimension of the center of the tube algebra Ξ_L. But this center is isomorphic to $\mathscr{H}_{S^1\times S^1}$, as discovered by Ocneanu [44] and explained in more detail in [14, Theorem 3.1]. □

Remark 8.4. The above argument is only a sketch because the triangulation TQFT in $2+1$ dimensions considered in [44,14] is derived from a subfactor, see [26] for a detailed exposition. Here as in [37, Section 7] we use the fact that the latter is equivalent to the invariant defined in [5,15]. This is more or less clear, but certainly deserves being made precise, as we plan to do in [38]. Note also that in order for a spherical category to give rise to a triangulation TQFT—as opposed to just the invariant—one must assume that it does not contain symplectic self-dual simple objects. This is done in [53] and [6, p. 4018], but unfortunately ignored in the bulk of the literature on the subject.

By Theorem 1.2, $\mathscr{Z}_1(\mathscr{C})$ is modular, thus gives rise to a surgery TQFT in $2+1$ dimensions, cf. [53]. For these TQFTs it is known (by construction) that the dimension of the vector space $\mathscr{H}_{S^1\times S^1}$ associated with the torus equals the number of isomorphism classes of simple objects in the category. Thus the above result provides support for the conjecture that the triangulation and surgery TQFTs associated with $\mathscr{C}$ and $\mathscr{Z}_1(\mathscr{C})$, respectively, are isomorphic. (This conjecture, while very natural, seems to have appeared in print only in [25, Question 5].) In particular, the corresponding invariants of closed oriented 3-manifolds should coincide

$$\mathrm{RT}(M, \mathscr{Z}_1(\mathscr{C})) = \mathrm{Tr}(M, \mathscr{C}) \quad \forall M. \tag{8.4}$$

Presently, we have no proof for this, but we note that Kawahigashi Sato and Wakui recently provided a proof [51] in the setting of unitary categories arising from a subfactor. If $\mathscr{C}$ is modular, the braided equivalence $\mathscr{Z}_1(\mathscr{C}) \overset{\otimes}{\simeq}_{\mathrm{br}} \mathscr{C} \boxtimes \tilde{\mathscr{C}}$ proven in Section 7 implies

$$\begin{aligned}\mathrm{RT}(M, \mathscr{Z}_1(\mathscr{C})) &= \mathrm{RT}(M, \mathscr{C} \boxtimes \tilde{\mathscr{C}}) = \mathrm{RT}(M, \mathscr{C}) \cdot \mathrm{RT}(M, \tilde{\mathscr{C}}) \\ &= \mathrm{RT}(M, \mathscr{C}) \cdot \mathrm{RT}(-M, \mathscr{C}),\end{aligned}$$

and (8.4) follows from [53, Theorem VII.4.1.1], according to which $\mathrm{TV}(M, \mathscr{C}) = \mathrm{RT}(M, \mathscr{C}) \cdot \mathrm{RT}(-M, \mathscr{C})$. For non-modular $\mathscr{C}$ we only have the following weaker result:

Proposition 8.5. *Let $\mathscr{C}$ be as in Theorem* 1.2 *and M an oriented closed* 3-*manifold. Then*

$$\mathrm{RT}(M, \mathscr{Z}_1(\mathscr{C})) \cdot \mathrm{RT}(-M, \mathscr{Z}_1(\mathscr{C})) = \mathrm{Tr}(M, \mathscr{C}) \cdot \mathrm{Tr}(-M, \mathscr{C}).$$

If $\mathscr{C}$ is unitary then $|\mathrm{RT}(M, \mathscr{Z}_1(\mathscr{C}))| = |\mathrm{Tr}(M, \mathscr{C})|$.

Proof. We compute

$$\begin{aligned}\mathrm{RT}(M, \mathscr{Z}_1(\mathscr{C})) \cdot \mathrm{RT}(-M, \mathscr{Z}_1(\mathscr{C})) &= \mathrm{TV}(M, \mathscr{Z}_1(\mathscr{C})) \\ &= \mathrm{Tr}(M, \mathscr{Z}_1(\mathscr{C})) \\ &= \mathrm{Tr}(M, \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}) \\ &= \mathrm{Tr}(M, \mathscr{C}) \cdot \mathrm{Tr}(M, \mathscr{C}^{\mathrm{op}}) \\ &= \mathrm{Tr}(M, \mathscr{C}) \cdot \mathrm{Tr}(-M, \mathscr{C}).\end{aligned}$$

Here the first equality is due to Turaev's theorem, which applies since $\mathscr{Z}_1(\mathscr{C})$ is modular. The second is the equality [6] of TV and Tr for spherical $\mathscr{C}$. The third equality follows from the weak monoidal Morita equivalence $\mathscr{Z}_1(\mathscr{C}) \approx \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ together with the Morita invariance of the invariant Tr, cf. [37, Theorem 7.1]. The last two equalities follow from general properties of the invariant Tr [6].

If $\mathscr{C}$ is unitary we have $\mathrm{RT}(-M, \mathscr{Z}_1(\mathscr{C})) = \overline{\mathrm{RT}(M, \mathscr{Z}_1(\mathscr{C}))}$ and $\mathrm{Tr}(-M, \mathscr{C}) = \overline{\mathrm{Tr}(M, \mathscr{C})}$, and we are done. □

8.3. Subfactor theory: the Longo–Rehren subfactor

As stated in the introduction, the present project originated in the author's observation that the quantum double of monoidal categories appears implicitly in Izumi's preprint [17]. Therefore it seems reasonable to briefly comment on the subfactor setting.

Let M be a type III factor with separable predual. Then the tensor category $\mathrm{End}_f(M)$ of (normal unital $*$-) endomorphisms ρ of M such that $[M:\rho(M))] < \infty$ is a $*$-category with duals, direct sums and subobjects. (Here one uses that every orthogonal projection $p = p^2 = p^*$ in M is equivalent to $\mathbf{1}$, i.e. there is $V \in M$ such that $V^*V = \mathbf{1}$, $VV^* = p$.) Let $\mathscr{C} \subset \mathrm{End}_f(M)$ be a full monoidal subcategory with the same completeness properties and finite dimension. Choosing objects $\{\rho_i,\ i \in \Gamma\}$ in the classes of simple objects, defining

$$A = M \bar{\otimes} M^{\mathrm{op}}$$

and picking a direct sum

$$\gamma = \bigoplus_{i \in \Gamma} \rho_i \otimes \rho_i^{\mathrm{op}}$$

one shows [29] γ to be part of a Frobenius algebra (‘Q-system’) (Q, v, v^*, w, w^*) in $\mathrm{End}_f(A)$. At this point one applies a beautiful and fundamental result due to Longo [28], which implies that there is a subfactor $B \subset A$ such that γ is a canonical endomorphism for the inclusion $B \subset A$. This means that there is a normal morphism $\bar{\imath} : A \to B$ which is a dual (in the 2-category of factors, morphisms and intertwiners) of the embedding morphism $\imath = \mathrm{id} : B \to A$, such that $\gamma = \imath \circ \bar{\imath}$. The subfactor B is simply given by

$$B = w^* \gamma(A) w. \tag{8.5}$$

(The verification that this really gives a subalgebra is easy.) We call the subfactor thus obtained from M and $\mathscr{C}$ the Longo–Rehren subfactor.

Among the objects of interest in subfactor theory are the monoidal subcategories $\mathrm{Hom}_{B\subset A}(A,A) \subset \mathrm{End}_f(A)$ and $\mathrm{Hom}_{B\subset A}(B,B) \subset \mathrm{End}_f(B)$ generated by $\gamma = \imath \circ \bar{\imath}$ and $\hat{\gamma} = \bar{\imath} \circ \imath$, respectively, and the categories $\mathrm{Hom}_{B\subset A}(A,B)$, $\mathrm{Hom}_{B\subset A}(B,A)$ of morphisms which are contained in $\bar{\imath} \circ (\imath \circ \bar{\imath})^n$ and $\imath \circ (\bar{\imath} \circ \imath)^n$, respectively, for some $n \in \mathbb{Z}_{\geqslant}$. The reader should appreciate that in this way every subfactor with finite index provides us with a $\mathbb{C}$-linear $*$-2-category with two objects and with non-strict spherical structure, thus in particular with a Morita context for the tensor categories $\mathrm{Hom}_{B\subset A}(A,A)$ and $\mathrm{Hom}_{B\subset A}(B,B)$. (The dimension of the four categories of 1-morphisms is finite iff the subfactor has finite depth.) Our painful construction in [37] just models the categorical structure implicit in subfactor theory, where thanks to the inbuilt structure one just needs the simple formula (8.5)!

Alas, the above construction does not necessarily yield (tensor) categories which are equivalent to the $\mathrm{Hom}_{\mathscr{E}}(\mathfrak{A},\mathfrak{A})$, $\mathrm{Hom}_{\mathscr{E}}(\mathfrak{B},\mathfrak{B})$, $\mathrm{Hom}_{\mathscr{E}}(\mathfrak{A},\mathfrak{B})$, $\mathrm{Hom}_{\mathscr{E}}(\mathfrak{B},\mathfrak{A})$ of Section 4.1. This becomes clear already by comparing our $\mathscr{A} = \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ with

$$\mathrm{Hom}_{B\subset A}(A,A) = \{\rho \in \mathrm{End}_f(A) \mid \rho \prec \gamma^n, n \in \mathbb{Z}_{\geqslant}\}.$$

The latter obviously is (equivalent to) a full subcategory of $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$, but they coincide only if every $\rho_i \otimes \rho_j$, $i, j \in \Gamma$ is contained in γ^n for some n. This condition can be shown to be equivalent to connectedness of a certain graph, the fusion graph of $\mathscr{C}$.

With these preparations a short inspection of Izumi's work [18], where the categorical double does not appear explicitly, shows that essentially he has proven the following theorem:

Theorem 8.6. *Let M be a type III factor with separable predual and let $\mathscr{C}$ be a full monoidal subcategory of* $\mathrm{End}_f(M)$ *which is closed under duals, direct sums, subobjects and is finite dimensional, and let $B \subset A$ be the corresponding LR subfactor. If the fusion graph of $\mathscr{C}$ is connected then we have the following equivalences of tensor categories*:

$$\mathrm{Hom}_{B\subset A}(A,A) \simeq \mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}},$$

$$\mathrm{Hom}_{B\subset A}(B,B) \simeq \mathscr{Z}_1(\mathscr{C}).$$

Proof. The fusion graph is connected iff the objects $X \boxtimes X^{\mathrm{op}}$ generate all of $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$. Thus the statement on $\mathrm{Hom}_{B\subset A}(A,A)$ is contained in [18, Theorem 4.1]. Under the connectedness assumption Izumi's 'quantum double of Δ' coincides with the $B-B$-morphisms $\mathrm{Hom}_{B\subset A}(B,B)$. Then our second claim follows from [18, Theorem 4.6], where the quantum double appears in only slightly disguised form. Instead of half-braidings $Z \mapsto e_X(Z)$ satisfying the braid relation and naturality Izumi uses maps $I \ni i \mapsto e_X(X_i)$ satisfying the braiding fusion relation. These two pictures are equivalent by our Lemma 3.3. □

Remark 8.7. The above theorem is the precise formulation of Ocneanu's remarkable intuitive insight [44] that his asymptotic subfactor [42] (which is strongly related [36] to the Longo–Rehren subfactor $B \subset A$) is 'the subfactor analogue of Drinfel'd's quantum double'. In view of the fact that irreducible depth-two subfactors are precisely the subfactors arising from outer actions of a Hopf algebra, the most natural way to make Ocneanu's claim precise would be the following: The asymptotic subfactor of $M^H \subset M$ is isomorphic to $P^{D(H)} \subset P$ or its dual. Yet, this clearly cannot be the case since the index of the asymptotic inclusion coincides with the global index of the original subfactor, which for depth two coincides with the index. Thus for $N = M^H \subset M$, $[A:B]=[M:N]$ and $B \subset A$ cannot arise from a $D(H)$-action.

Using the results of Section 8.1 we can remove the connectedness condition:

Corollary 8.8. *Let M, $\mathscr{C}$ be as in the theorem, but with possibly disconnected fusion graph. Then the category* $\mathrm{Hom}_{B\subset A}(A,A)$ *is equivalent to the monoidal subcategory of $\mathscr{C} \boxtimes \mathscr{C}^{\mathrm{op}}$ generated by the $X \boxtimes X^{\mathrm{op}}$, where X runs through the simple objects of $\mathscr{C}$.* $\mathrm{Hom}_{B\subset A}(B,B)$ *is braided equivalent to the sub-tensor category of $\mathscr{Z}_1(\mathscr{C})$ generated by those simple objects $(X,e_X) \in \mathscr{Z}_1(\mathscr{C})$ for which X contains the tensor unit $\mathbf{1}$ of $\mathscr{C}$.*

Proof. The first statement is well known. By definition, $\mathrm{Hom}_{B\subset A}(B,B)$ is generated by the dual Frobenius object $\hat{Q}$. For the LR-subfactor this is the $K(\mathbf{1})$ given in (8.3), from which the second claim follows. □

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Appendix A. On quantum doubles of finite dimensional Hopf algebras

The core of this paper was the proof that the quantum doubles of certain tensor categories are modular. That $\mathrm{Rep}\, D(H)$ is modular has been proven for $H = \mathbb{C}G$ [2], where G is a finite group, and for semisimple H over an algebraically closed field k of characteristic zero [12]. A proof which covers also weak Hopf algebras (or finite quantum groupoids) is given in [40]. Our aim in this appendix is to give a proof which uses the ideas of Lyubashenko [31,32] and Majid [33] and therefore is more in the spirit of our proof in the categorical situation. In the sequel H will always be a finite dimensional Hopf algebra. Since the main application will be to quantum doubles the following will be useful.

Lemma A.1. *Let H be a finite dimensional Hopf algebra H over the field k and let $D(H)$ be the quantum double. The following are equivalent*:

(i) *H is semisimple and cosemisimple.*
(ii) *The antipode of H is involutive and char $k \nmid \dim H$.*
(iii) *$D(H)$ is semisimple.*
(iv) *$D(H)$ is cosemisimple.*

Remark A.2. If the characteristic of k is zero then H is semisimple iff it is cosemisimple, and the second condition in (ii) is vacuous.

Proof. For the equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv), see [46] and for (i) $\Leftrightarrow$ (ii) see [13]. □

In order for the category $\mathrm{Rep}\, D(H)$ to be modular it must be semisimple, which by the lemma reduces us to the case where H satisfies (i) and (ii).

Lemma A.3. *Let H satisfy the (equivalent) conditions of Lemma* A.1. *Then there are two-sided integrals $\Lambda \in H$, $\mu \in \hat{H}$ which are traces in the sense that*

$$\langle \mu, ab \rangle = \langle \mu, ba \rangle, \qquad \langle \alpha\beta, \Lambda \rangle = \langle \beta\alpha, \Lambda \rangle$$

for all $a, b \in H$, $\alpha\beta \in \hat{H}$. The category $\mathrm{Rep}\, D(H)$ *is a spherical category.*

Proof. Semisimple Hopf algebras are unimodular [27], which by definition means that there are two-sided integrals. By [27, Proposition 8], unimodular Hopf algebras satisfy

$$\langle \mu, ab \rangle = \langle \mu, bS^2(a) \rangle \quad \forall a, b \in H,$$

thus $\langle \mu, \cdot \rangle$ is tracial by involutivity of S. Sphericity of $\mathrm{Rep}\, D(H)$ is now an obvious consequence of [6] where it was shown under the weaker assumption that S^2 is inner. □

We briefly recall some results on quasitriangular Hopf algebras. As shown be Drinfel'd [11], the antipode of a finite dimensional quasitriangular Hopf algebra H is inner, i.e. there is an invertible $u \in H$ such that $S^2(A) = uAu^{-1}$. One has the explicit formulae

$$u = m \circ (S \otimes \mathrm{id})(R_{21}),$$

$$u^{-1} = m \circ (\mathrm{id} \otimes S^2)(R_{21}),$$

(i.e., $u = \sum_i S(f_i)e_i$ if $R = \sum_i e_i \otimes f_i$). Furthermore, Drinfel'd proved

$$uS(u) = S(u)u \in Z(H), \qquad \varepsilon(u) = 1, \qquad \Delta(u) = (R_{21}R)^{-1}(u \otimes u).$$

Recall [53] that a ribbon Hopf algebra is a quasitriangular Hopf algebra H together with $\theta \in Z(H)$ satisfying

$$\theta^2 = uS(u), \qquad S(\theta) = \theta, \qquad \varepsilon(\theta) = 1, \qquad \Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta). \tag{A.1}$$

Proposition A.4. *Let H be a quasitriangular semisimple and cosemisimple Hopf algebra. Then H is a ribbon Hopf algebra with $\theta = u$.*

Proof. Since $S^2 = \mathrm{id}$ it follows that Drinfel'd's u is central. Now [23, Proposition 4] implies that u is a ribbon element. □

Remark A.5. For a ribbon Hopf algebra H with $\theta = u$ the quantum trace, which for a representation π on a vector space V is defined by

$$\mathrm{Tr}^q_\pi(X) := \mathrm{Tr} \circ \pi(u\theta^{-1}X)$$

coincides with the usual trace Tr on End V. In particular, all quantum dimensions $d(\pi)$ coincide with the classical dimensions $\dim V_\pi$. Therefore,

$$\dim \mathrm{Rep}\, H = \sum_i d(\pi_i)^2 = \sum_i (\dim V_{\pi_i})^2 = \dim_k H.$$

In order to conclude that Rep H is modular it remains to prove that the S-matrix of the ribbon category Rep H is invertible. A related notion of non-degeneracy was introduced in [48], where a quasitriangular Hopf algebra was called *factorizable* if the map

$$\hat{H} \to H: z \mapsto \langle z \otimes \mathrm{id}, I \rangle, \quad I = R_{21}R$$

is injective, thus invertible. Furthermore, it was shown that every quantum double $D(H)$ is factorizable. The notion of factorizability plays an important role in the works [33,25] where an action of $SL(2,\mathbb{Z})$ on ribbon Hopf algebras is defined and studied.

Definition A.6 (Lyubashenko and Majid [33]). For a quasitriangular Hopf algebra the selfdual Fourier transforms $\mathscr{S}_+$, $\mathscr{S}_-$ are defined by the linear endomorphisms of H

$$\mathscr{S}_+(b) = (\mathrm{id} \otimes \mu)(R_{21}(\mathbf{1} \otimes b)R_{12}),$$

$$\mathscr{S}_-(b) = (\mathrm{id} \otimes \mu)(R_{12}^{-1}(\mathbf{1} \otimes b)R_{21}^{-1}),$$

where μ is a left integral in $\hat{H}$. If H is ribbon the map $\mathscr{T}: H \to H$ is defined by $\mathscr{T}(b) = \theta b$.

Theorem A.7 (Lyubashenko and Majid [33]). *For a factorizable ribbon Hopf algebra the following modular relations hold*:

$$\mathscr{S}_+ \circ \mathscr{S}_- = \mathrm{id} = \mathscr{S}_- \circ \mathscr{S}_+, \quad (\mathscr{S}_+ \circ \mathscr{T})^3 = \lambda \mathscr{S}_+^2, \quad \mathscr{S}_+^2 = \underline{S}, \tag{A.2}$$

where $\underline{S}(x) = R^{(2)}S(Ad\, R^{(1)}(x))$ *with* $Ad\, Y(x) = Y_{(1)}xS(Y_{(2)})$ *is the braided antipode, and* $\lambda \neq 0$ *is defined by* $\mathscr{S}_+(\theta) = \lambda\theta^{-1}$.

Lemma A.8. *The following decompositions hold in every finite dimensional Hopf algebra.*

$$a \otimes \mathbf{1} = \sum_i (\mathbf{1} \otimes x_i)\Delta(y_i) = \sum_i \Delta(y_i)(\mathbf{1} \otimes S^2(x_i)),$$

where

$$\sum_i y_i \otimes x_i = (\mathrm{id} \otimes S^{-1})\Delta(a).$$

Proof. Inserting $\sum_i y_i \otimes x_i = a_{(1)} \otimes S^{-1}(a_{(2)})$ into $\sum_i (\mathbf{1} \otimes x_i)\Delta(y_i)$ we obtain

$$\begin{aligned}(\mathbf{1} \otimes S^{-1}(a_{(2)}))\Delta(a_{(1)}) &= (\mathbf{1} \otimes S^{-1}(a_{(3)}))(a_{(1)} \otimes a_{(2)}) \\ &= a_{(1)} \otimes S^{-1}(a_{(3)})a_{(2)} = a \otimes \mathbf{1},\end{aligned} \tag{A.3}$$

and the other equality is verified similarly. □

Proposition A.9. *Let H be a quasitriangular semisimple, cosemisimple Hopf algebra and $\mu \in \hat{H}$ a left integral. Then the self-dual Fourier transforms $\mathscr{S}_\pm$ map the center of H into itself*:

$$\mathscr{S}_\pm(Z(H)) \subset Z(H). \tag{A.4}$$

Proof. By [27, Proposition 8] unimodularity is equivalent to the identity

$$\mu(ab) = \mu(bS^2(a)) \quad \forall a, b \in H, \tag{A.5}$$

which will be used in the sequel. Let $a \in H$, $b \in Z(H)$. Then

$$\begin{aligned} a\mathscr{S}_+(b) &= a(\mathrm{id} \otimes \mu)(R_{21}(\mathbf{1} \otimes b)R_{12}) \\ &= (\mathrm{id} \otimes \mu)((a \otimes \mathbf{1})R_{21}R_{12}(\mathbf{1} \otimes b)) \\ &= \sum_i (\mathrm{id} \otimes \mu)((\mathbf{1} \otimes x_i)\Delta(y_i)R_{21}R_{12}(\mathbf{1} \otimes b)) \\ &= \sum_i (\mathrm{id} \otimes \mu)((\mathbf{1} \otimes x_i)R_{21}R_{12}(\mathbf{1} \otimes b)\Delta(y_i)) \\ &= \sum_i (\mathrm{id} \otimes \mu)(R_{21}R_{12}(\mathbf{1} \otimes b)\Delta(y_i)(\mathbf{1} \otimes S^2(x_i))) \\ &= (\mathrm{id} \otimes \mu)(R_{21}R_{12}(\mathbf{1} \otimes b)(a \otimes \mathbf{1})) \\ &= (\mathrm{id} \otimes \mu)(R_{21}R_{12}(\mathbf{1} \otimes b))a = \mathscr{S}_+(b)a, \end{aligned} \tag{A.6}$$

thus $\mathscr{S}_+(b) \in Z(H)$. We have used $b \in Z(H)$, $[\Delta(\cdot), R_{21}R_{12}] = 0$, (A.5) and the lemma. □

Remark A.10. (1) In restriction to the center, the braided antipode $\underline{S}$ appearing in (A.2) equals the antipode S.

(2) For a ribbon algebra H it is trivial that $\mathscr{T}$ maps $Z(H)$ into itself, since $\theta \in Z(H)$.

The modularity condition requires invertibility of the matrix

$$S_{i,j} = (\mathrm{Tr}^q_{\pi_i} \otimes \mathrm{Tr}^q_{\pi_j})(R_{21}R_{12}), \tag{A.7}$$

where i, j range over the equivalence classes of irreducible representations of H. Now, in the semisimple case the representations are in one-to-one correspondence with the minimal projections in $Z(H)$, which leads to the following result.

Theorem A.11. *Let H be a factorizable quasitriangular semisimple and cosemisimple Hopf algebra. Then the category* Rep H *is modular.*

Proof. We have already proven that Rep H is a spherical ribbon category. Thus by Proposition A.9 the center of H is stable under the Fourier transform $\mathscr{S}_+$. By factorizability $\mathscr{S}_+$ is invertible, and the same holds for the restriction $\mathscr{S}_+ \restriction Z(H)$. By the remark after Lemma A.4 the quantum traces on H-modules V, in terms of which the S-matrix is defined, coincide with the usual traces on End V. In terms of the basis for $Z(H)$ given by the minimal idempotents P_i we obtain

$$\mathscr{S}_+(P_j) = \sum_i \mathscr{S}_{ij} P_i,$$

where the matrix $\mathscr{S} = (\mathscr{S}_{ij})$ is invertible. But $\mathscr{S}$ is nothing but the modular matrix (A.7) as we have

$$\mathscr{S}_{ij} = d_i\,\mu(P_i\,\mathscr{S}_+(P_j)) = d_i\,(\mu \otimes \mu)(R_{21}R_{12}\,(P_i \otimes P_j))$$
$$= \frac{1}{d_j}(\mathrm{Tr}_i^q \otimes \mathrm{Tr}_j^q)(R_{21}R_{12}).$$

We have used $\mu(P_i) = 1/d_i$ and $\mu(x\,P_i) = 1/d_i\,\mathrm{Tr}_i(x)$ [55]. □

Note that the proof is conceptually quite similar to our proof of modularity for general semisimple spherical categories. In view of Lemma A.1 the following is now immediate:

Corollary A.12. *Let H be semisimple and cosemisimple. Then* Rep $D(H)$ *is modular.*

We close the appendix with a remark which is meant to aid the reader in appreciating the ‘self-duality’ of a quantum double $D(H)$, in particular since in general it is not self-dual in the sense of Hopf algebras: $D(H)^* \not\simeq D(H)$. (For a finite abelian group G we in fact have $D(G) \simeq D(G)^* \simeq \mathbb{C}G \otimes \mathbb{C}(G)$.) For any finite dimensional Hopf algebra one can use the integrals to define ‘Fourier transforms’ $H \to \hat{H}$. In [41] Fourier transforms $\mathscr{F}_{\sigma,\sigma'}$, σ, $\sigma' = \pm$, defined as linear maps $H \to \hat{H}$ which intertwine certain actions of H on H and $\hat{H}$ by multiplication and translation, respectively, were studied systematically and used to give a new proof of the invertibility of the antipode. There is a beautiful relation between these more conventional Fourier transforms, relating H and $\hat{H}$, and the selfdual Fourier transforms [33], which map $D(H)$ onto itself. For simplicity we restrict ourselves to finite dimensional Kac algebras, where things are easier since the Haar measures are two-sided invariant traces and since there are unique Fourier transforms $\mathscr{F}: H \to \hat{H}$ and $\hat{\mathscr{F}}: \hat{H} \to H$:

$$\langle \mathscr{F}(x), y\rangle = \langle \mu, xS(y)\rangle, \quad \forall x, y \in H, \tag{A.8}$$

$$\langle \alpha, \hat{\mathscr{F}}(\beta)\rangle = \langle \hat{S}(\alpha)\beta, \Lambda\rangle, \quad \forall \alpha, \beta \in \hat{H}, \tag{A.9}$$

where Λ, μ are the integrals in $H, \hat{H}$, respectively. If $\imath: H \to D(H)$, $\hat{\imath}: \hat{H} \to D(H)$ are the canonical embedding maps then the following diagram commutes:

$$\begin{array}{ccccc}
\hat{H} \otimes H & \xrightarrow{\hat{\imath}\otimes\imath} & D(H) \otimes D(H) & \xrightarrow{m} & D(H) \\
\Big\downarrow{\scriptstyle \hat{\mathscr{F}}\otimes\mathscr{F}} & & & & \Big\downarrow{\scriptstyle \mathscr{S}_-} \\
H \otimes \hat{H} & \xrightarrow[\imath\otimes\hat{\imath}]{} & D(H) \otimes D(H) & \xrightarrow[m]{} & D(H)
\end{array}$$

This nice observation is due to Kerler [25, Proposition 9] (with different conventions), who, however, did not emphasize the interpretation of $\mathscr{F}$, $\hat{\mathscr{F}}$ as conventional Fourier transforms.

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