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ON THE STRUCTURE OF MODULAR CATEGORIES

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1. *Introduction*

Braided tensor categories [**18**] play a central rôle in the representation theory of quantum groups [**9**, **19**], of Kac–Moody algebras [**21**], and of quantum field theories [**14**, **24**]. They also serve as input data for the construction of invariants of knots, links and 3-manifolds [**38**, **19**]. In both areas – representation theory and low-dimensional topology – a particular subclass of braided tensor categories is distinguished, that of modular categories. First formalized in [**37**], they are semisimple rigid ribbon categories that have finitely many isomorphism classes of simple objects and satisfy a non-degeneracy condition. Modular categories derive their name from the fact that they define [**38**] a (projective) finite-dimensional representation of $\mathrm{SL}(2, \mathbb{Z})$. At first sight mysterious, this modular representation is best understood in topological terms. Viz., a modular category gives rise, for every closed oriented surface M, to a finite-dimensional projective representation of the mapping class group of M. The modular representation associated with a modular category is then just the representation of the mapping class group of the torus. In the important special case of $*$-categories (or unitary categories) arising in quantum field theory this modular representation had been known earlier and rigorously studied in [**35**].

The non-degeneracy condition mentioned above amounts to non-degeneracy of a certain matrix, which is just the collection of invariants of the Hopf link for the possible labelings of the two components. Also the meaning of the non-degeneracy condition in the construction of the 3-manifold invariant is quite transparent. Yet, it is clearly desirable to have a more intrinsic understanding in purely algebraic terms. In the special case of unitary categories, it has long been known [**35**] that the modularity condition is equivalent to the absence of 'degenerate' objects. The latter property has a very satisfactory interpretation in terms of triviality of the center $\mathcal{Z}_2(\mathcal{C})$ of the braided category. This center is a canonical full symmetric subcategory of $\mathcal{C}$ and must not be confused with another notion of 'center' which is defined for any – not necessarily braided – tensor category. We denote the latter by $\mathcal{Z}_1(\mathcal{C})$. In the more general situation of a pre-modular category [**7**], the equivalence between modularity and triviality of the center has been proven only recently [**6**]. (In [**28**, Corollary 7.11] this proof is obtained as a byproduct.)

Since symmetric tensor categories are precisely those braided tensor categories which coincide with their center, modular categories may be seen as braided tensor categories diametrically opposed to the symmetric ones: modular categories are related to symmetric tensor categories in the same ways as groups with trivial

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center to abelian groups, or factors to commutative von Neumann algebras. Now, under some additional conditions, symmetric tensor categories are just representation categories of groups [**10**, **12**]; thus they should be considered as very basic algebraic objects. Our point of view is that modular categories merit being perceived similarly and being subjected to detailed scrutiny and, as far as feasible, classification. So far, very little has been known in the way of a general theory, our Theorem 4.5, according to which every modular category is (equivalent to) a finite direct product of prime modular categories, apparently being one of the first structural results.

On the other hand, many different constructions of modular categories are known; we briefly review these. There is a large class of constructions which go under the heading of quantum doubles, all of which yield modular categories under suitable assumptions. For the definition of the quantum double of finite-dimensional Hopf algebras (in particular, group algebras) and of tensor categories (the 'center' $\mathscr{Z}_1$ mentioned above) we refer to [**19**]. Proofs of modularity were given in [**2**] for quantum doubles of finite groups and in [**3**] for the twisted versions of [**11**], and in [**13**] and [**28**, Appendix] for semisimple cosemisimple Hopf algebras. These cases are subsumed by quantum groupoids and by semisimple spherical categories, for which modularity of the quantum double was proven in [**30**] and [**28**], respectively. (Actually, by results of Hayashi [**16**] and Ostrik [**34**], the categories considered in [**28**] are always representation categories of quantum groupoids. Thus, with some additional effort, the proof of the main theorem of [**28**] could also be deduced from the results of [**30**]. Conceptually, however, the direct proof seems more satisfactory.) Quantum groups at roots of unity give rise to modular categories if one considers appropriate quotients of their representation categories [**39**]. These categories are in fact $*$-categories [**40**] (or unitary categories [**38**]). (The categories obtained in this way and from quantum doubles, respectively, are not completely unrelated since the universal R-matrix of a deformed enveloping algebra A is computed by expressing A as a quotient of a quantum double. On the categorical side, every modular category $\mathscr{C}$ is a full tensor subcategory of its own quantum double $\mathscr{Z}_1(\mathscr{C})$; cf. [**28**]. This is remarkable insofar as the definition of the latter does not refer to the braiding of $\mathscr{C}$.) For some of the above modular categories a beautiful purely combinatorial construction is known [**6**]. Furthermore, there is an operation [**7**, **25**] which, heuristically, amounts to dividing a pre-modular category $\mathscr{C}$ by its center and which can be interpreted as a Galois completion [**25**, **8**]. This procedure is applicable whenever the objects of the center $\mathscr{Z}_2(\mathscr{C})$ have positive integer dimensions (automatic for $*$-categories) and trivial twists, and it yields a modular category that is non-trivial whenever $\mathscr{C}$ is not symmetric. Finally, for a suitable class of rational chiral conformal field theories, axiomatized using operator algebras, one can prove [**20**] that the category of representations is a modular $*$-category. (In WZW- and orbifold models the representation categories are those of [**40**] and of [**11**], respectively.) For a review of these results we refer to [**26**], where also some of the results of the present paper were announced.

In the next section we briefly review some of the formalism of modular categories, restricting ourselves to the facts that are needed in this paper. With the exception of Lemma 2.13 the results are well known, but we emphasize the rôle of centralizers in braided categories. In § 3 we prove our main technical result, a double centralizer theorem in modular categories. Section 4 gives several applications, the most important of which is that every modular subcategory of a modular category is a direct factor. This implies that every modular category factorizes as a finite direct product of prime modular categories. Here, a modular category $\mathscr{C}$ is prime if every full modular subcategory is either trivial or equivalent to $\mathscr{C}$. In § 5 we give some preliminary results about full embeddings of pre-modular categories into modular categories and conclude with a remark about the Galois theory for braided tensor categories developed in [**25**, **8**].

2. *Preliminaries on modular categories*

2.1. *Notation*

We assume known the standard definitions of (braided, symmetric) tensor categories; cf. [**23**, **18**] or [**4**, **19**, **38**]. All categories in this paper are supposed to be small and all tensor categories strict. (A tensor category is strict if the tensor product satisfies associativity $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ 'on the nose' and the unit object $\mathbf{1}$ satisfies $X \otimes \mathbf{1} = \mathbf{1} \otimes X = X$ for all X. By the coherence theorems, every tensor category is equivalent to a strict one.) Our notation is fairly standard, except that we omit the $\otimes$-symbol for the product of objects: $XY \equiv X \otimes Y$. Unit objects and unit morphisms in tensor categories are denoted by $\mathbf{1}$ and $\mathrm{id}_X \in \mathrm{End}(X)$, respectively. Our categories will be $\mathbb{F}$-linear over a field $\mathbb{F}$. We use capital letters $X, Y, \dots$ to denote both objects and isomorphism classes of simple objects. What is meant should be obvious from the context. As usual, we denote $N_{XY}^{Z} = \dim \mathrm{Hom}(Z, XY)$. A $*$-category [**15**, **22**] or unitary category [**38**] is a $\mathbb{C}$-linear category equipped with an antilinear involutive and contravariant endofunctor $*$ that leaves the objects fixed and such that $s^* \circ s = 0$ implies $s = 0$.

For the definitions of (left) rigid and ribbon categories see [**18**, **19**]. Every left rigid ribbon category is spherical [**5**]; that is, every object X has a two-sided dual $\overline{X}$ and left and right traces coincide. (Conversely, every spherical category with braiding automatically has a compatible ribbon structure [**41**].) We denote the twist by $\{\theta_X \mid X \in \mathscr{C}\}$. To every simple object X in a ribbon category we assign $\omega_X \in \mathbb{F}$ by $\omega_X \mathrm{id}_X = \theta_X$. One has $\omega_X = \omega_{\overline{X}}$ for all simple X, and in a $*$-category, $|\omega_X| = 1$ for all X. A pre-modular category [**7**] is a semisimple $\mathbb{F}$-linear rigid ribbon category with finitely many isomorphism classes of simple objects and tensor unit satisfying $\mathrm{End}\,\mathbf{1} \cong \mathbb{F}$.

A subcategory $\mathscr{S} \subset \mathscr{C}$ is full if and only if $\mathrm{Hom}_{\mathscr{S}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$ for all $X, Y \in \mathscr{S}$; thus it is determined by $\mathrm{Obj}\,\mathscr{S}$. A subcategory $\mathscr{S}$ is replete if and only if $X \in \mathscr{S}$ implies $Y \in \mathscr{S}$ for all $Y \in \mathscr{C}$ isomorphic to X. By a semisimple tensor subcategory of a semisimple spherical category $\mathscr{C}$ we mean a full subcategory which is stable with respect to direct sums, subobjects (thus in particular replete) and duals. If Γ denotes the set of isomorphism classes of simple objects of $\mathscr{C}$, the semisimple subcategories are in one-to-one correspondence with the subsets $\Gamma' \subset \Gamma$ which are closed under duals and satisfy

$$X, Y \in \Gamma', \ N_{XY}^{Z} \neq 0 \quad \Longrightarrow \quad Z \in \Gamma'.$$

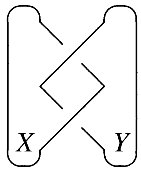
2.2. *Monodromies in braided tensor categories*

Definition 2.1. The *monodromy* of two objects X and Y in a tensor category with braiding c is defined by

$$c_M(X, Y) = c(Y, X) \circ c(X, Y) \in \mathrm{End}(XY).$$

Definition 2.2. For a braided spherical category $\mathscr{C}$ over $\mathbb{F}$ and $X, Y \in \mathscr{C}$ define $S(X, Y) \in \mathbb{F}$ by

$$S(X, Y)\mathrm{id}_{\mathbf{1}} = \mathrm{Tr}_{XY}(c_M(X, Y)) =$$



Remark 2.3. (1) The function $S(X, Y)$ depends only on the isomorphism classes $[X]$ and $[Y]$.

2. Note that we did not assume X and Y to be simple.

Lemma 2.4. *Let $\mathscr{C}$ be an $\mathbb{F}$-linear semisimple rigid ribbon category with* $\mathrm{End}\,\mathbf{1} \cong \mathbb{F}$. *The following identities hold:*

(i) $S(UX, Y) = (1/d(Y))S(U, Y)S(X, Y)$ *for all Y simple, and all U, X;*

(ii) $X \cong \bigoplus_i X_i, \;\; Y \cong \bigoplus_j Y_j \implies S(X, Y) = \sum_{i,j} S(X_i, Y_j)$;

(iii) $(1/d(X))S(X, Y)S(X, Z) = \sum_W N_{YZ}^W S(X, W)$ *for all X, Y, Z simple.*

Proof. The first claim follows from $\mathrm{Tr}_{UX} = \mathrm{Tr}_X \circ (\mathrm{Tr}_U \otimes \mathrm{id})$ together with the consequence $(\mathrm{Tr}_U \otimes \mathrm{id})(c_M(U, Y)) = d(Y)^{-1} S(U, Y)\mathrm{id}_Y$ of simplicity of Y. The second claim is immediate by cyclic invariance of the trace. Part (iii) follows by applying (i) and (ii) to $S(UV, Y)$. □

The next result is valid without the restriction to unitary (that is, $*$-) categories; cf. [**6**]. The proof uses a general result on handle slides [**1**], which we do not wish to enlarge upon. Rather than referring the result away completely, we give a simple argument for unitary categories which does not require $\mathscr{C}$ to be finite.

Proposition 2.5. *Let $\mathscr{C}$ be a unitary semisimple rigid ribbon category with* $\mathrm{End}\,\mathbf{1} \cong \mathbb{F}$ (*equivalently, a tensor $*$-category with direct sums, subobjects and conjugates*). *Let X and Y be simple. Then $S(X, Y) = d(X)d(Y)$ if and only if* $c_M(X, Y) = \mathrm{id}_{XY}$.

Proof. The 'if' statement is obvious from the definition of S. Thus assume $S(X, Y) = d(X)d(Y)$. Using the well-known equations [**38**, **35**]

$$d(X)d(Y) = \sum_Z N_{XY}^Z d(Z), \qquad S(X, Y) = \sum_Z N_{XY}^Z \frac{\omega_Z}{\omega_X \omega_Y} d(Z),$$

we see that this implies

$$\sum_Z N_{XY}^Z d(Z) \frac{\omega_Z}{\omega_X \omega_Y} = \sum_Z N_{XY}^Z d(Z).$$

Restricting the summation to those Z for which $N_{XY}^Z \neq 0$ we have

$$\sum_Z{}' N_{XY}^Z d(Z)\left(1 - \frac{\omega_Z}{\omega_X \omega_Y}\right) = 0. \tag{2.1}$$

Since ω_X, ω_Y and ω_Z all have absolute value 1, we have $\mathrm{Re}(1 - \omega_Z/\omega_X\omega_Y) \geqslant 0$, with equality if and only if $\omega_Z/\omega_X\omega_Y = 1$. Since $N_{XY}^Z d(Z) > 0$ in (2.1), we conclude that $\omega_Z = \omega_X \omega_Y$ whenever $N_{XY}^Z > 0$. Let Z be simple and consider $s\colon XY \to Z$ and $t\colon Z \to XY$. Then with the ribbon condition

$$\theta_{XY} = \theta_X \otimes \theta_Y \circ c_M(X, Y) = c_M(X, Y) \circ \theta_X \otimes \theta_Y$$

we have

$$\begin{aligned} c_M(X, Y) \circ t \circ s &= \theta_{XY} \circ (\theta_X \otimes \theta_Y)^{-1} \circ t \circ s \\ &= (\omega_X \omega_Y)^{-1} \theta_{XY} \circ t \circ s \\ &= (\omega_X \omega_Y)^{-1} t \circ \theta_Z \circ s = \frac{\omega_Z}{\omega_X \omega_Y} t \circ s. \end{aligned}$$

Since $\mathrm{End}(XY)$ is unital and spanned by morphisms $t \circ s$ as above, the fact that $\omega_Z = \omega_X \omega_Y$ for all Z contained in XY implies $c_M(X, Y) = \mathrm{id}_{XY}$. □

DEFINITION 2.6. Let $\mathscr{C}$ be a braided tensor category and $\mathscr{K}$ be a set of objects in $\mathscr{C}$, equivalently, a full subcategory of $\mathscr{C}$. Then we define the *centralizer* $C_{\mathscr{C}}(\mathscr{K})$ of $\mathscr{K}$ in $\mathscr{C}$ (or relative commutant $\mathscr{C} \cap \mathscr{K}'$) as the full subcategory defined by

$$\mathrm{Obj}\, C_{\mathscr{C}}(\mathscr{K}) = \{X \in \mathscr{C} \mid c_M(X, Y) = \mathrm{id}_{XY} \text{ for all } Y \in \mathscr{K}\}.$$

REMARK 2.7. (1) In [**25**, Subsection 5.2] the subcategory $C_{\mathscr{C}}(\mathscr{K})$ was called $\mathscr{C}_{\mathscr{K}}$. In subfactor theory, a related notion appears under the name 'permutant' in [**33**].

(2) If there is no danger of confusion concerning the ambient category $\mathscr{C}$, we will occasionally write $\mathscr{K}'$ instead of $C_{\mathscr{C}}(\mathscr{K})$.

If $\mathscr{K}_1$ and $\mathscr{K}_2$ are full subcategories of $\mathscr{C}$, by $\mathscr{K}_1 \vee \mathscr{K}_2$ we denote the smallest replete full subcategory of $\mathscr{C}$ containing $\mathscr{K}_1$ and $\mathscr{K}_2$ and stable under tensor products, direct sums and retractions.

LEMMA 2.8. *For $\mathscr{C}$ and $\mathscr{K}$ as above, $C_{\mathscr{C}}(\mathscr{K})$ is replete and monoidal. If $\mathscr{C}$ is a semisimple category with direct sums and subobjects, then the same holds for $C_{\mathscr{C}}(\mathscr{K})$. If $\mathscr{C}$ has duals for all objects then also $C_{\mathscr{C}}(\mathscr{K})$ has duals. If $\mathscr{K}_1, \mathscr{K}_2 \subset \mathscr{C}$ then $C_{\mathscr{C}}(\mathscr{K}_1 \vee \mathscr{K}_2) = C_{\mathscr{C}}(\mathscr{K}_1) \cap C_{\mathscr{C}}(\mathscr{K}_2)$.*

Proof. Repleteness of $C_{\mathscr{C}}(\mathscr{K})$ follows from naturality of the braiding c. It is easy to see that $c_M(X_i, Y) = \mathrm{id}_{X_i Y}$ for $i = 1, 2$ implies $c_M(X_1 X_2, Y) = \mathrm{id}_{X_1 X_2 Y}$. Thus $C_{\mathscr{C}}(\mathscr{K})$ is closed under tensor products. In a semisimple category with $X = \bigoplus_{i \in I} X_i$ one has $c_M(X, Y) = \mathrm{id}_{XY}$ if and only if $c_M(X_i, Y) = \mathrm{id}_{X_i Y}$ for all $i \in I$. This implies that $C_{\mathscr{C}}(\mathscr{K})$ is closed under direct sums and subobjects. The statement concerning duals follows by the same argument as in the proof of [**25**, Proposition 2.7]. As to the last claim, the inclusion $C_{\mathscr{C}}(\mathscr{K}_1 \vee \mathscr{K}_2) \subset C_{\mathscr{C}}(\mathscr{K}_1) \cap C_{\mathscr{C}}(\mathscr{K}_2)$ is obvious. If X is in $C_{\mathscr{C}}(\mathscr{K}_1)$ and $C_{\mathscr{C}}(\mathscr{K}_2)$ then it also has trivial monodromy c_M with all tensor products of objects in $\mathscr{K}_1$ and $\mathscr{K}_2$, as well as direct sums and retracts of such. □

DEFINITION 2.9. The *center* of a braided tensor category $\mathscr{C}$ is

$$\mathscr{Z}_2(\mathscr{C}) = C_{\mathscr{C}}(\mathscr{C}).$$

We say that a semisimple braided tensor category has *trivial center* if every object of $\mathscr{Z}_2(\mathscr{C})$ is a direct sum of copies of the tensor unit $\mathbf{1}$ or if, equivalently, every simple object in $\mathscr{Z}_2(\mathscr{C})$ is isomorphic to $\mathbf{1}$.

REMARK 2.10. (1) Clearly, a braided tensor category is symmetric if and only if it coincides with its center.

(2) The objects of the center have previously been called *degenerate* [**35**, **25**], *transparent* [**7**] and *pseudotrivial* [**36**]. Yet, calling them *central* seems the most natural terminology, since the above definition is the correct analogue for braided tensor categories of the center of a monoid, as can be seen by appealing to the theory of n-categories.

(3) Proposition 2.5 now has the interpretation that the simple object X is central if and only if $S(X, Y) = d(X)d(Y)$ for all simple $Y \in \mathscr{C}$.

2.3. *Finite-dimensional categories*

In our considerations so far we have not made finiteness assumptions on $\mathscr{C}$. From now on we will work with categories which have finite dimension in the following sense.

DEFINITION 2.11. Let $\mathscr{C}$ be a semisimple $\mathbb{F}$-linear spherical category with $\mathrm{End}(\mathbf{1}) \cong \mathbb{F}$. If the set Γ of isomorphism classes of simple objects is finite, the dimension of $\mathscr{C}$ is defined by

$$\dim \mathscr{C} = \sum_{X \in \Gamma} d(X)^2;$$

otherwise it is ∞.

REMARK 2.12. (1) The sum over the squared dimensions appeared in [**38**], where its square roots are called ranks of the category. In subfactor theory [**31**, **32**] this number is called the 'global index'. The designation 'dimension' for this number has also been used in [**5**]. It is vindicated by the fact that $\dim \mathrm{Rep}(H) = \dim H$ for a finite-dimensional semisimple Hopf algebra, and in particular for group algebras. That $\dim \mathscr{C}$ is the correct generalization of $\dim H$ is corroborated by its behavior under various constructions like the quantum double, where $\dim \mathscr{Z}_1(\mathscr{C}) = (\dim \mathscr{C})^2$ [**28**].

(2) The dimension of a semisimple $\mathbb{F}$-linear category can be defined unambiguously whenever the category has two-sided duals; cf. [**27**, Subsection 2.2]. A sovereign/spherical or $*$-structure is not needed.

If $\mathscr{K}$ is a subcategory of $\mathscr{C}$, let $\chi_{\mathscr{K}}$ be the characteristic function of $\mathrm{Obj}\,\mathscr{K}$. Viz., $\chi_{\mathscr{K}}(X) = 1$ if $X \in \mathscr{K}$ and $\chi_{\mathscr{K}}(X) = 0$ otherwise.

LEMMA 2.13. *Let $\mathscr{C}$ be a pre-modular category and $\mathscr{K}$ a semisimple tensor subcategory. Then for all $X \in \mathscr{C}$ we have*

$$\sum_{Y \in \mathscr{K}} d(Y) S(X, Y) = d(X) \dim \mathscr{K}\, \chi_{C_{\mathscr{C}}(\mathscr{K})}(X). \tag{2.2}$$

Proof. If $X \in C_{\mathscr{C}}(\mathscr{K})$ then $S(X, Y) = d(X)d(Y)$ for all $Y \in \mathscr{K}$, and (2.2) follows immediately. Thus it remains to show that the left-hand side of (2.2) vanishes if $c_M(X, Y) \neq \mathrm{id}_{XY}$ for some $Y \in \mathscr{K}$. To this purpose, consider Lemma 2.4(c) with $Y, Z \in \mathscr{K}$. Since $\mathscr{K}$ is a sub-tensor category, the summation runs only over isomorphism classes of simple $W \in \mathscr{K}$. Multiplication with $d(Y)$

and summation over $Y \in \mathscr{K}$ yields

$$\frac{S(X,Z)}{d(X)} \sum_{Y \in \mathscr{K}} d(Y)S(X,Y) = \sum_{Y,W \in \mathscr{K}} d(Y)N_{YZ}^{W} S(X,W).$$

Now, $\sum_Y d(Y)N_{YZ}^W = \sum_Y d(Y)N_{WZ}^{\overline{Y}} = d(W)d(Z)$ and we obtain

$$\left(\sum_{Y \in \mathscr{K}} d(Y)S(X,Y)\right)[S(X,Z) - d(X)d(Z)] = 0 \quad \text{for all } X \in \mathscr{C}, Z \in \mathscr{K}.$$

If now $X \notin C_{\mathscr{C}}(\mathscr{K})$ then there exists a $Z \in \mathscr{K}$ which has non-trivial monodromy with X, and by Proposition 2.5 we have $S(X,Z) \neq d(X)d(Z)$. Thus the expression in the big brackets vanishes and we have finished. □

For $\mathscr{K} \in \mathscr{C}$, the lemma reduces to a known result, and together with Remark 2.10 it implies the following corollary; cf. for example, [**6**]. But for our purposes the generalization to arbitrary tensor subcategories $\mathscr{K}$ will be essential.

COROLLARY 2.14. *Let $\mathscr{C}$ be a pre-modular category with* $\dim \mathscr{C} \neq 0$ *and let X be a simple object. Then the following are equivalent:*

(i) *X is central;*

(ii) *$S(X,Y) = d(X)d(Y)$ for all simple $Y \in \mathscr{C}$;*

(iii) $\sum_Y S(X,Y)d(Y) \neq 0$.

LEMMA 2.15. *Let $\mathscr{C}$ be pre-modular and Y and Z simple. Then*

$$\sum_X S(X,Y)S(X,Z) = \dim \mathscr{C} \sum_{W \in \mathscr{Z}_2(\mathscr{C})} N_{YZ}^W d(W).$$

If $\mathscr{Z}_2(\mathscr{C})$ is trivial then $S^2 = \dim \mathscr{C}\, C$, where $C = (C_{XY})$ and $C_{XY} = \delta_{X,\overline{Y}}$.

Proof. Multiplying (iii) of Lemma 2.4 with $d(X)$ and summing over X we obtain

$$\begin{aligned}\sum_X S(X,Y)S(X,Z) &= \sum_{X,W} N_{YZ}^W S(X,W)d(X)\\ &= \dim \mathscr{C} \sum_{W \in \mathscr{Z}_2(\mathscr{C})} N_{YZ}^W d(W),\end{aligned}$$

where we have used Lemma 2.13 with $\mathscr{K} = \mathscr{C}$. The last claim follows from $N_{XY}^0 = \delta_{X,\overline{Y}}$. □

COROLLARY 2.16. *Let $\mathscr{C}$ be pre-modular with* $\dim \mathscr{C} \neq 0$. *Then the following are equivalent:*

(i) *the center $\mathscr{Z}_2(\mathscr{C})$ is trivial;*

(ii) *the matrix $(S(X,Y))$, indeed by isomorphism classes of simple objects, is invertible.*

Proof. The implication (ii) ⇒ (i) is obvious. Conversely, if $\mathscr{Z}_2(\mathscr{C})$ is trivial, the lemma gives $S^2 = \dim \mathscr{C}\, C$. Part (ii) then follows from the invertibility of C and because $\dim \mathscr{C} \neq 0$. □

REMARK 2.17. (1) Alternatively, the statement of the corollary can be obtained directly from a handle slide argument; cf. [**6**, **28**].

(2) If $\mathscr{C}$ is a $*$-category, one easily shows that $\overline{S(X,Y)} = S(X,\overline{Y}) = S(\overline{X},Y)$. In the case where $\mathscr{Z}_2(\mathscr{C})$ is trivial, Lemma 2.15 implies that the columns (or rows) of S are mutually orthogonal. Thus $\widetilde{S} = (\dim\mathscr{C})^{-1/2} S$ is unitary. Without modularity one can still show that for simple X and Y the columns S_X and S_Y (or rows) of the S-matrix are either orthogonal or parallel; cf. [**35**]. If $\overline{\overline{\mathscr{C}}}$ is the modularization and $F(X)$ the image of X in $\overline{\overline{\mathscr{C}}}$, the following are equivalent:

(i) $S_X \parallel S_Y$,

(ii) there exists $Z \in \mathscr{Z}_2(\mathscr{C})$ such that $\mathrm{Hom}(X, ZY) \neq \{0\}$,

(iii) $\mathrm{Hom}_{\overline{\overline{\mathscr{C}}}}(F(X), F(Y)) \neq \{0\}$,

(iv) $F(X)$ and $F(Y)$ have the same simple summands; cf. [**25**].

3. *Centralizers in modular categories*

DEFINITION 3.1 [**37**]. A modular category is a pre-modular category satisfying the (equivalent) conditions of Corollary 2.16.

We are now in a position to prove our first main result.

THEOREM 3.2. *Let $\mathscr{C}$ be a modular category and let $\mathscr{K} \subset \mathscr{C}$ be a semisimple tensor subcategory. Then we have*

(i) $C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K})) = \mathscr{K}$,

(ii) $\dim\mathscr{K} \cdot \dim C_{\mathscr{C}}(\mathscr{K}) = \dim\mathscr{C}$.

(*We also simply write* $\mathscr{K}'' = \mathscr{K}$ *and* $\dim\mathscr{K} \cdot \dim\mathscr{K}' = \dim\mathscr{C}$.)

Proof. We apply Lemma 2.13 to compute the characteristic function of $C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K}))$:

$$\chi_{C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K}))}(X) = \frac{1}{d(X)\dim C_{\mathscr{C}}(\mathscr{K})} \sum_{Z \in C_{\mathscr{C}}(\mathscr{K})} S(X,Z) d(Z)$$
$$= \frac{1}{d(X)\dim C_{\mathscr{C}}(\mathscr{K})} \sum_{Z \in \mathscr{C}} \chi_{C_{\mathscr{C}}(\mathscr{K})}(Z) S(X,Z) d(Z).$$

We use the lemma once again to compute $\chi_{C_{\mathscr{C}}(\mathscr{K})}(Z)$, obtaining

$$\chi_{C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K}))}(X)$$
$$= \frac{1}{d(X)\dim C_{\mathscr{C}}(\mathscr{K})} \sum_{Z \in \mathscr{C}} S(X,Z) d(Z) \frac{1}{d(Z)\dim\mathscr{K}} \sum_{U \in \mathscr{K}} S(Z,U) d(U)$$
$$= \frac{1}{d(X)\dim C_{\mathscr{C}}(\mathscr{K})\dim\mathscr{K}} \sum_{U \in \mathscr{K}} d(U) \sum_{Z \in \mathscr{C}} S(X,Z) S(Z,U).$$

The summation over $Z \in \mathscr{C}$ can be performed using Lemma 2.15, and since the center of $\mathscr{C}$ is trivial, by Corollary 2.16 we have

$$\sum_{Z \in \mathscr{C}} S(X,Z) S(Z,U) = \dim\mathscr{C}\, \delta_{[X],\overline{[U]}}.$$

Using $d(U) = d(\overline{U})$ and the fact that $\mathcal{K}$ is closed with respect to duals we obtain

$$\chi_{C_{\mathcal{C}}(C_{\mathcal{C}}(\mathcal{K}))}(X) = \frac{\dim \mathcal{C}}{d(X) \dim \mathcal{K} \dim C_{\mathcal{C}}(\mathcal{K})} \sum_{U \in \mathcal{K}} d(U) \delta_{[X], \overline{[U]}}$$

$$= \frac{\dim \mathcal{C}}{\dim \mathcal{K} \dim C_{\mathcal{C}}(\mathcal{K})} \chi_{\mathcal{K}}(X). \tag{3.1}$$

Since the tensor unit $\mathbf{1}$ is contained in any tensor subcategory, we have $\chi_{\mathcal{K}}(\mathbf{1}) = \chi_{\mathcal{K}''}(\mathbf{1}) = 1$. Thus for $X = \mathbf{1}$, (3.1) proves claim (ii), and plugging this back into (3.1), we see that claim (i) ensues. □

REMARK 3.3. (1) Using (i), one easily verifies the following. If $\mathcal{K}$ is any subcategory closed with respect to duals then $C_{\mathcal{C}}(C_{\mathcal{C}}(\mathcal{K}))$ is the semisimple tensor subcategory generated by $\mathcal{K}$, that is, the completion with respect to direct sums and subobjects.

(2) In a subfactor context, the double centralizer property $\mathcal{K}'' = \mathcal{K}$ was stated by A. Ocneanu [**33**] without published proof. Subfactor analogues of both (i) and (ii) were proved by Izumi [**17**] using considerable machinery. By contrast, the above proof uses only well-known properties of modular categories.

COROLLARY 3.4. *Let $\mathcal{C}$ be a modular category and let $\mathcal{K} \subset \mathcal{C}$ be a semisimple tensor subcategory. Then*

$$\mathcal{Z}_2(C_{\mathcal{C}}(\mathcal{K})) = \mathcal{Z}_2(\mathcal{K}).$$

Proof. We compute

$$\mathcal{Z}_2(C_{\mathcal{C}}(\mathcal{K})) = C_{C_{\mathcal{C}}(\mathcal{K})}(C_{\mathcal{C}}(\mathcal{K})) = (\mathcal{C} \cap \mathcal{K}') \cap (\mathcal{C} \cap \mathcal{K}')'$$

$$= (\mathcal{C} \cap (\mathcal{C} \cap \mathcal{K}')') \cap \mathcal{K}' = C_{C_{\mathcal{C}}(C_{\mathcal{C}}(\mathcal{K}))}(\mathcal{K}) = C_{\mathcal{K}}(\mathcal{K}) = \mathcal{Z}_2(\mathcal{K}),$$

where we have used $C_{\mathcal{C}}(C_{\mathcal{C}}(\mathcal{K})) = \mathcal{K}$. □

The two most interesting cases are where $\mathcal{K}$ is modular or symmetric.

COROLLARY 3.5. *Let $\mathcal{C}$ be a modular category and let $\mathcal{K} \subset \mathcal{C}$ be a semisimple tensor subcategory that is modular. Then $\mathcal{L} \equiv C_{\mathcal{C}}(\mathcal{K})$ is modular too.*

Proof. By Corollary 2.16, modularity of $\mathcal{K}$ is equivalent to triviality of $\mathcal{Z}_2(\mathcal{K}) = C_{\mathcal{K}}(\mathcal{K})$. By Corollary 3.4, also $C_{\mathcal{L}}(\mathcal{L}) = C_{\mathcal{K}}(\mathcal{K})$ is trivial, and thus $\mathcal{L}$ is modular. □

COROLLARY 3.6. *Let $\mathcal{C}$ be a modular category and let $\mathcal{K} \subset \mathcal{C}$ be a semisimple tensor subcategory that is symmetric. Let $\mathcal{L} \equiv C_{\mathcal{C}}(\mathcal{K})$. Then $\mathcal{Z}_2(\mathcal{L}) = \mathcal{K}$.*

Proof. This is obvious because $\mathcal{Z}_2(\mathcal{K}) = \mathcal{K}$. □

4. *On the structure of modular categories*

4.1. *Prime factorization of modular categories*

Our first application of the double centralizer theorem in modular categories is also the most striking one. It illustrates how different modular categories are from their opposite extreme case, viz. the symmetric categories and the group duals (at least in characteristic zero [**12**, **10**]).

The following was proved in [**28**] as Corollary 7.8. (The proof uses only two preceding results and is independent of the rest of [**28**].) Here $\mathscr{A} \boxtimes \mathscr{B}$ is the completion with respect to direct sums of the product of $\mathscr{A}$ and $\mathscr{B}$ as $\mathbb{F}$-linear categories. Note that $\mathscr{A} \boxtimes \mathscr{B}$ has an obvious tensor structure if $\mathscr{A}$ and $\mathscr{B}$ do.

PROPOSITION 4.1. *Let $\mathscr{C}$ be a braided tensor category that is $\mathbb{F}$-linear semisimple with* $\mathrm{End}\,\mathbf{1} \cong \mathbb{F}$ *and two-sided duals. Let $\mathscr{K} \subset \mathscr{C}$ be a semisimple tensor subcategory which has trivial center $\mathscr{Z}_2(\mathscr{K})$. Then we have the equivalence*

$$\mathscr{K} \boxtimes C_{\mathscr{C}}(\mathscr{K}) \simeq \mathscr{K} \vee C_{\mathscr{C}}(\mathscr{K})$$

of braided tensor categories. If $\mathscr{C}$ is spherical then by restriction $\mathscr{K}$ and $C_{\mathscr{C}}(\mathscr{K})$ are also spherical, and the above equivalence is one of spherical categories. A similar result holds if $\mathscr{C}$ is a $*$-category.

Writing $\mathscr{L} = C_{\mathscr{C}}(\mathscr{K})$, we naturally ask whether $\mathscr{K} \vee \mathscr{L} = \mathscr{C}$, since this would imply $\mathscr{C} \simeq \mathscr{K} \boxtimes \mathscr{L}$. If $\mathscr{C}$ is modular, the double centralizer theorem provides the missing step.

THEOREM 4.2. *Let $\mathscr{C}$ and $\mathscr{K}$ be modular categories where $\mathscr{K}$ is identified with a full* (*tensor*) *subcategory of $\mathscr{C}$. Let $\mathscr{L} = C_{\mathscr{C}}(\mathscr{K})$. Then there is an equivalence of ribbon categories:*

$$\mathscr{C} \simeq \mathscr{K} \boxtimes \mathscr{L}.$$

Proof. Modularity of $\mathscr{L}$ has been proved in Lemma 3.5. If we can show that the full subcategories $\mathscr{K}$ and $\mathscr{L}$ of $\mathscr{C}$ generate $\mathscr{C}$, then Proposition 4.1 provides an equivalence $\mathscr{C} \simeq \mathscr{K} \boxtimes \mathscr{L}$ of braided tensor categories. The equivalence is automatically an equivalence of ribbon categories since $\mathscr{K}$ and $\mathscr{L}$ commute. (Alternatively, one appeals to the compatibility of the spherical structures.) For the remaining fact, $\mathscr{K} \vee \mathscr{L} = \mathscr{C}$, we give two proofs, the first of which works only for unitary modular categories.

Unitary categories. By Proposition 4.1, we have $\dim \mathscr{K} \vee \mathscr{L} = \dim \mathscr{K} \cdot \dim \mathscr{L}$, which, by Theorem 3.2, coincides with $\dim \mathscr{C}$. Since $\mathscr{K} \vee \mathscr{L}$ is a full subcategory of $\mathscr{C}$ and the numbers $d(X)^2 \in \mathbb{R}$ are non-negative, the equality

$$\dim \mathscr{C} = \sum_{X \in \mathscr{C}} d(X)^2 = \sum_{X \in \mathscr{K} \vee \mathscr{L}} d(X)^2 = \dim \mathscr{K} \vee \mathscr{L}$$

implies that all simple objects of $\mathscr{C}$ are contained in $\mathscr{K} \vee \mathscr{L}$ and therefore $\mathscr{C} = \mathscr{K} \vee \mathscr{L} \simeq \mathscr{K} \boxtimes \mathscr{L}$, as desired.

General case. The preceding argument to the effect that $\mathscr{K} \vee \mathscr{L}$ exhausts $\mathscr{C}$, does not work if we are not dealing with $*$-categories, since $\dim \mathscr{A} = \dim \mathscr{B}$ does not imply that a replete full inclusion $\mathscr{A} \subset \mathscr{B}$ is an identity. Yet, we can argue as follows:

$$\begin{aligned}
\mathscr{K} \vee \mathscr{L} &= C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K} \vee \mathscr{L})) = C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K}) \cap C_{\mathscr{C}}(\mathscr{L})) \\
&= C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K}) \cap C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K}))) = C_{\mathscr{C}}(C_{\mathscr{C}}(\mathscr{K}) \cap \mathscr{K}) \\
&= C_{\mathscr{C}}(\mathscr{C} \cap \mathscr{K}' \cap \mathscr{K}) = C_{\mathscr{C}}(C_{\mathscr{K}}(\mathscr{K})) = C_{\mathscr{C}}(\mathscr{Z}_2(\mathscr{K})) = \mathscr{C}.
\end{aligned}$$

Hence we have used Lemma 2.8 and the fact that $\mathscr{Z}_2(\mathscr{K})$ is trivial. □

REMARK 4.3. (1) That $\mathscr{L}$ is modular can be derived alternatively from the easy fact that a direct product $\mathscr{C} \simeq \mathscr{K} \boxtimes \mathscr{L}$ is modular if and only if both $\mathscr{K}$ and $\mathscr{L}$ are modular.

(2) An interesting special case is provided by the quantum double of a modular category $\mathscr{K}$. If we put $\mathscr{C} = \mathscr{Z}_1(\mathscr{K})$, there exists a braided monoidal embedding functor I: $\mathscr{K} \hookrightarrow \mathscr{C}$. The above theorem implies that $\mathscr{C} \simeq I(\mathscr{K}) \boxtimes C_{\mathscr{C}}(I(\mathscr{K}))$. In this situation, $C_{\mathscr{C}}(I(\mathscr{K}))$ can be computed explicitly and is given by $\widetilde{I}(\widetilde{\mathscr{K}})$, where $\widetilde{\mathscr{K}}$ coincides with $\mathscr{K}$ as a tensor category, but has the opposite braiding, and $\widetilde{I}$ is a braided monoidal embedding. Thus $\mathscr{C} \simeq \mathscr{K} \boxtimes \widetilde{\mathscr{K}}$, which is the statement of [**28**, Theorem 7.9]. Even if $\mathscr{K}$ is not modular, $I(\mathscr{K})$ and $\widetilde{I}(\widetilde{\mathscr{K}})$ are each other's centralizers in $\mathscr{C}$, and we have $I(\mathscr{K}) \cap \widetilde{I}(\widetilde{\mathscr{K}}) = I(\mathscr{Z}_2(\mathscr{K}))$. See [**28**, § 7] for the details.

(3) Finally, it is interesting to note the analogy with the following result from classical non-commutative algebra. Let $A \subset B$ be an inclusion of matrix algebras. Then

(i) $C_B(A)$ is a matrix algebra,

(ii) $C_B(C_B(A)) = A$, and

(iii) $B \cong A \otimes C_B(A)$.

These results also hold in the infinite-dimensional case if A and B are Type I factors (that is, isomorphic to the algebras of bounded linear operators on some Hilbert spaces).

DEFINITION 4.4. A modular category $\mathscr{C}$ is *prime* if every semisimple modular subcategory is equivalent to either $\mathscr{C}$ or the trivial modular category $\mathrm{Vect}_{\mathbb{F}}$.

Now we can state our second main result.

THEOREM 4.5. *Every modular category is equivalent to a finite direct product of prime modular categories.*

Proof. This is an obvious consequence of Theorem 4.2 since modular categories have finitely many (equivalence classes of) simple objects, and proper replete full subcategories have strictly fewer simple objects. □

REMARK 4.6. (1) A. Bruguières (private communication, September 2000) first observed that the above factorization is in general non-unique. Yet, uniqueness does hold if every simple object except the unit has dimension not equal to 1. In the next subsection we will find many examples for the non-uniqueness.

(2) For the classification of modular categories, Theorem 4.5 has the obvious consequence that it is sufficient to consider prime modular categories. As in other respects, modular categories are better behaved than finite groups since there are no non-trivial exact sequences to be considered.

4.2. *Quantum doubles of finite abelian groups*

Quantum doubles of finite groups provide a large and relatively easy-to-analyze class of modular categories; cf. for example, [**2**]. As an exploratory step towards a classification of modular ($*$-)categories it is natural to find the prime factorizations of the categories $D(G)$-mod ($\equiv D(\mathbb{C}G)$-mod). If G is a direct product of subgroups K and L then it is easy to see that $D(G) \cong D(K) \otimes D(L)$ as a quasitriangular Hopf algebra. Thus $D(G)\text{-mod} \simeq D(K)\text{-mod} \boxtimes D(L)\text{-mod}$. Therefore, in order for $D(G)$-mod to be prime, G must be prime (not a direct product of non-trivial subgroups). This condition is, however, not sufficient, as is shown by the complete analysis of the abelian case given below.

If G is abelian, the simple objects of $\mathscr{C} = D(G)$-mod have dimension 1 and are invertible, the isomorphism classes of simple objects and their tensor product being given by the abelian group $\Gamma(\mathscr{C}) = G \times \widehat{G}$. There is an obvious one-to-one correspondence between replete full tensor subcategories $\mathscr{K} \subset \mathscr{C}$ and subgroups $K \subset \Gamma(\mathscr{C})$. Apart from this we only need the fact that

$$S((g, \chi), (h, \sigma)) = \langle \sigma, g\rangle\langle \chi, h\rangle.$$

It is well known that every finite abelian group is isomorphic to a direct product of cyclic groups of prime power order,

$$G \cong \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z},$$

where the pairs (p_i, n_i) are unique up to permutation. By the above remark we can therefore restrict ourselves to a consideration of $G = \mathbb{Z}/p^n\mathbb{Z}$.

THEOREM 4.7. *Let p be prime, $G = \mathbb{Z}/p^n\mathbb{Z}$ and $\mathscr{C} = D(G)$-mod.*

(i) *If $p = 2$ then $\mathscr{C}$ is prime.*

(ii) *If p is odd, there is a one-to-one correspondence between isomorphisms $\alpha: G \to \widehat{G}$ and replete full modular subcategories $\mathscr{K}_\alpha \subset \mathscr{C}$ given by*

$$K_\alpha = \{(g, \alpha(g)) \mid g \in G\}.$$

The categories $\mathscr{K}_\alpha$ are prime, and $C_{\mathscr{C}}(\mathscr{K}_\alpha) = \mathscr{K}_{\overline{\alpha}}$, where $\overline{\alpha}(\cdot) = \alpha(\cdot)^{-1}$. The prime factorizations of $\mathscr{C}$ are thus given by $\mathscr{C} \simeq \mathscr{K}_\alpha \boxtimes \mathscr{K}_{\overline{\alpha}}$, where $\alpha \in \mathrm{Isom}(G, \widehat{G})$.

Proof. Let $\mathscr{K} \subset \mathscr{C}$ be a replete full tensor subcategory corresponding to the subgroup $K \subset \Gamma$. If $\mathscr{K}$ is modular then, by Theorem 4.2, $\Gamma \cong K \times L$, where L corresponds to the centralizer $\mathscr{L} = C_{\mathscr{C}}(\mathscr{K})$. By the uniqueness result for the factorization of finite abelian groups, a (non-trivial) factorization $\Gamma \cong K \times L$ of $\Gamma \cong (\mathbb{Z}/p^n\mathbb{Z})^2$ is possible only if $K \cong L \cong \mathbb{Z}/p^n\mathbb{Z}$. Identifying G and $\widehat{G}$ with $\mathbb{Z}/p^n\mathbb{Z}$, we see that the S-matrix is given by

$$S((a, b), (c, d)) = \exp\left\{\frac{2\pi i}{p^n}(ad + bc)\right\}.$$

Furthermore, every cyclic subgroup $K \subset \Gamma$ of order p^n is of the form $\{(ja, jb) \mid j \in \mathbb{Z}/p^n\mathbb{Z}\}$, where a and b are not both multiples of p. Now $\mathscr{K}$ is modular if and only if $j \neq 0$ implies $(ja, jb) \notin \mathscr{Z}_2(\mathscr{K})$, which by Proposition 2.14 is equivalent to

$$\sum_{k=0}^{p^n - 1} \exp\left\{\frac{2\pi i}{p^n} 2jkab\right\} = 0.$$

Since

$$\sum_{k=0}^{N-1} a^k = \begin{cases} (a^N - 1)/(a - 1) & \text{for } a \neq 1, \\ N & \text{otherwise,} \end{cases}$$

this is the case if and only if

$$\exp\left\{\frac{2\pi i}{p^n}2jab\right\}\neq 1 \quad\text{and}\quad \exp\left\{\frac{2\pi i}{p^n}2jabp^n\right\}=1.$$

The latter condition is always satisfied, and the former leads to

$$(ja, jb)\in\mathscr{Z}_2(\mathscr{K}) \iff 2jab\in p^n\mathbb{Z}. \tag{4.1}$$

If $p=2$ then (4.1) is satisfied by $j=p^{n-1}$ irrespective of a and b. Thus $(p^{n-1}a, p^{n-1}b)\in\mathscr{Z}_2(\mathscr{K})$ and $\mathscr{K}$ is not modular, proving (i). From now on let p be odd. If $p\mid a$ or $p\mid b$ then again $j=p^{n-1}$ solves (4.1) and $\mathscr{K}$ is not modular. If $p\nmid a$ and $p\nmid b$ then (4.1) is satisfied if and only if $j\equiv 0 \pmod{p^n}$, which implies modularity of $\mathscr{K}$. Rephrasing this in an invariant fashion leads to the first part of statement (ii). Proper subgroups of K_α have order smaller than p^n and therefore cannot give rise to modular subcategories by the argument at the beginning of the proof. Thus the $\mathscr{K}_\alpha$, for $\alpha\in\mathrm{Isom}(G,\widehat{G})$, are prime, and they exhaust the prime factors of $\mathscr{C}$. With $\alpha,\alpha'\in\mathrm{Isom}(G,\widehat{G})$, Proposition 2.5 finally implies that $\mathscr{K}_\alpha'=\mathscr{K}_{\alpha'}$ if and only if

$$S((g,\alpha(g)),(h,\alpha'(h)))=\langle\alpha(g),h\rangle\langle\alpha'(h),g\rangle=1 \quad\text{for all } g,h\in G.$$

For $G=\mathbb{Z}/p^n\mathbb{Z}$ and $\alpha\in\mathrm{Isom}(G,\widehat{G})$ one easily shows $\langle\alpha(g),h\rangle=\langle\alpha(h),g\rangle$ for all g and h, which evidently entails $\mathscr{K}_\alpha'=\mathscr{K}_{\overline{\alpha}}$. □

REMARK 4.8. (1) Since $\#\mathrm{Isom}(G,\widehat{G})=p^n-p^{n-1}$, the theorem nicely illustrates to what extent the factorization of modular categories into primes can be non-unique.

(2) Table 1 makes clear that for an abelian group G there is no relation between simplicity of G and primality of $D(G)$-mod.

TABLE 1.

G	G simple?	$D(G)$-mod prime?
$\mathbb{Z}/2\mathbb{Z}$	Yes	Yes
$\mathbb{Z}/p\mathbb{Z}$, $p\neq 2$	Yes	No
$\mathbb{Z}/2^n\mathbb{Z}$, $n\geqslant 2$	No	Yes
$\mathbb{Z}/p^n\mathbb{Z}$, $p\neq 2$, $n\geqslant 2$	No	No

5. *Further applications*

5.1. *Modular extensions of braided categories*

The theory of Galois extensions of braided tensor categories developed in [**7**, **25**] provides a means to construct a modular category $\overline{\overline{\mathscr{C}}}$ from a given pre-modular category $\mathscr{C}$. In the language of [**25**], this modular closure is given by $\overline{\overline{\mathscr{C}}}=\mathscr{C}\rtimes\mathscr{Z}_2(\mathscr{C})$, that is, by adding morphisms which turn the objects in $\mathscr{Z}_2(\mathscr{C})$ into multiples of the tensor unit $\mathbf{1}$. The dimension of the modular closure is given by

$$\dim\overline{\overline{\mathscr{C}}}=\frac{\dim\mathscr{C}}{\dim\mathscr{Z}_2(\mathscr{C})}, \tag{5.1}$$

cf. [**7**, **29**]. Thus $\overline{\overline{\mathscr{C}}}$ is, in a sense, a quotient of $\mathscr{C}$ by $\mathscr{Z}_2(\mathscr{C})$, in fact it is trivial if and only if $\mathscr{C}$ is symmetric.

On the other hand, one may wish to find a modular category related to $\mathscr{C}$ without modifying the latter. More precisely, the problem is to find a modular category $\widehat{\mathscr{C}}$, into which $\mathscr{C}$ embeds as a full subcategory. If we restrict ourselves to $*$-categories, for which the dimensions take values in $\mathbb{R}_+$, we obtain a lower bound on the dimension of $\widehat{\mathscr{C}}$ as an immediate corollary of Theorem 3.2.

PROPOSITION 5.1. *Let $\mathscr{C}$ be a unitary modular category and $\mathscr{K} \subset \mathscr{C}$ a semisimple tensor subcategory. Then*

$$\dim \mathscr{C} \geqslant \dim \mathscr{K} \cdot \dim \mathscr{Z}_2(\mathscr{K}). \tag{5.2}$$

Equality holds if and only if $C_{\mathscr{C}}(\mathscr{K}) = C_{\mathscr{K}}(\mathscr{K}) = \mathscr{Z}_2(\mathscr{K})$.

Proof. The obvious inclusion (of replete full tensor subcategories)

$$C_{\mathscr{C}}(\mathscr{K}) \supset C_{\mathscr{K}}(\mathscr{K}) = \mathscr{Z}_2(\mathscr{K})$$

implies that $\dim C_{\mathscr{C}}(\mathscr{K}) \geqslant \dim \mathscr{Z}_2(\mathscr{K})$ and therefore

$$\dim \mathscr{C} = \dim \mathscr{K} \cdot \dim C_{\mathscr{C}}(\mathscr{K}) \geqslant \dim \mathscr{K} \cdot \dim \mathscr{Z}_2(\mathscr{K}).$$

Equality in (5.2) is equivalent to $\dim C_{\mathscr{C}}(\mathscr{K}) = \dim \mathscr{Z}_2(\mathscr{K})$ and thus to

$$C_{\mathscr{C}}(\mathscr{K}) = \mathscr{Z}_2(\mathscr{K}). \qquad \square$$

It is natural to expect that this bound is optimal.

CONJECTURE 5.2. Let $\mathscr{C}$ be a unitary pre-modular category. Then there exist a unitary modular category $\widehat{\mathscr{C}}$ and a full and faithful tensor functor $I: \mathscr{C} \to \widehat{\mathscr{C}}$ such that

$$\dim \widehat{\mathscr{C}} = \dim \mathscr{C} \cdot \mathscr{Z}_2(\mathscr{C}).$$

Such a category $\widehat{\mathscr{C}}$ is called a *minimal modular extension* of $\mathscr{C}$.

REMARK 5.3. (1) It is instructive to compare this equation with (5.1). In fact, the appearance of the factor $\dim \mathscr{Z}_2(\mathscr{C})$ in the conjecture and in (5.1) is not accidental. This is evident from the orbifold construction in conformal field theory, where the Galois closure $\overline{\overline{\mathscr{C}}}$ and the minimal modular extension $\widehat{\mathscr{C}}$ of a pre-modular category $\mathscr{C}$ both appear naturally; cf. [**26**].

(2) The conjecture makes sense also without the unitarity assumption. But without the latter it is not clear in which sense $\widehat{\mathscr{C}}$ is minimal.

(3) If $\mathscr{C}$ is symmetric, that is, $\mathscr{C} = \mathscr{Z}_2(\mathscr{C})$, (5.2) reduces to $\dim \widehat{\mathscr{C}} \geqslant (\dim \mathscr{C})^2$. If $\omega_X = 1$ for all simple X, by [**12**] we have a finite group G such that $\mathscr{C} \simeq G$-mod and $\dim \mathscr{C} = |G|$. Since the modular category $D(G)$-mod satisfies $\dim D(G)$-mod $= |G|^2$ and contains G-mod as a full subcategory, we see that pre-modular $*$-categories that are symmetric with trivial twists in fact admit a minimal modular extension.

(4) The example $\mathscr{C} = G$-mod also shows that $\widehat{\mathscr{C}}$, if it exists, will in general not be unique. The modular categories $D^{\omega}(G)$-mod [**11**], where $\omega \in \mathbb{Z}^3(G, \mathbb{T})$, are minimal modular extensions of G-mod which are inequivalent for different $[\omega] \in H^3(G, \mathbb{T})$.

(5) In a subfactor context, the existence (and 'essential uniqueness'!) of such a $\mathcal{C}$ is claimed by Ocneanu [**33**]. To the best of our knowledge a proof has not appeared.

Despite the fact that we do not know how to construct a minimal modular extension for a given pre-modular category, using our results [**28**] on the categorical quantum double we can construct many examples of unitary pre-modular categories which do possess a minimal modular extension.

Let $\mathcal{C}$ be a pre-modular category with $\dim \mathcal{C} \neq 0$ and let $\mathcal{Z}_1(\mathcal{C})$ be its quantum double. It is known [**28**] that $\mathcal{C}$ is modular and satisfies $\dim \mathcal{Z}_1(\mathcal{C}) = (\dim \mathcal{C})^2$. There are fully faithful braided tensor functors

$$I\colon \mathcal{C} \to \mathcal{Z}_1(\mathcal{C}) \quad \text{and} \quad \widetilde{I}\colon \widetilde{\mathcal{C}} \to \mathcal{Z}_1(\mathcal{C}),$$

where $\widetilde{\mathcal{C}}$ equals $\mathcal{C}$ as a tensor category, the braiding given by

$$\widetilde{c}(X,Y) = c(Y,X)^{-1}.$$

Let

$$\mathcal{E} = I(\mathcal{C}) \vee \widetilde{I}(\widetilde{\mathcal{C}})$$

be the replete full subcategory of $\mathcal{Z}_1(\mathcal{C})$ generated by $I(\mathcal{C})$ and $\widetilde{I}(\widetilde{\mathcal{C}})$.

PROPOSITION 5.4. *The subcategory $\mathcal{E} \subset \mathcal{Z}_1(\mathcal{C})$ satisfies*
(i) $\mathcal{Z}_2(\mathcal{E}) = I(\mathcal{Z}_2(\mathcal{C})) = \widetilde{I}(\mathcal{Z}_2(\widetilde{\mathcal{C}}))$,
(ii) $\dim \mathcal{E} = (\dim \mathcal{C})^2 / \dim \mathcal{Z}_2(\mathcal{C})$.

Proof. We compute the centralizer of $\mathcal{E}$ in $\mathcal{Z}_1(\mathcal{C})$:

$$C_{\mathcal{Z}_1(\mathcal{C})}(\mathcal{E}) = C_{\mathcal{Z}_1(\mathcal{C})}(I(\mathcal{C}) \vee \widetilde{I}(\widetilde{\mathcal{C}})) = C_{\mathcal{Z}_1(\mathcal{C})}(I(\mathcal{C})) \cap C_{\mathcal{Z}_1(\mathcal{C})}(\widetilde{I}(\widetilde{\mathcal{C}}))$$
$$= \widetilde{I}(\widetilde{\mathcal{C}}) \cap I(\mathcal{C}) = I(\mathcal{Z}_2(\mathcal{C})).$$

(We have used Lemma 2.8 and the observations from [**28**, § 4] that $I(\mathcal{C})$ and $\widetilde{I}(\widetilde{\mathcal{C}})$ are each other's centralizer in $\mathcal{Z}_1(\mathcal{C})$ and satisfy $I(\mathcal{C}) \cap \widetilde{I}(\widetilde{\mathcal{C}}) = I(\mathcal{Z}_2(\mathcal{C}))$.) This computation obviously implies that $\mathcal{Z}_2(\mathcal{E}) = \mathcal{E} \cap \mathcal{E}' = I(\mathcal{Z}_2(\mathcal{C}))$. Furthermore, we have $\dim C_{\mathcal{Z}_1(\mathcal{C})}(\mathcal{E}) = \dim \mathcal{Z}_2(\mathcal{C})$, and using part (b) of the double centralizer theorem we obtain

$$\dim \mathcal{E} = (\dim \mathcal{C})^2 / \dim \mathcal{Z}_2(\mathcal{C}). \qquad \square$$

REMARK 5.5. Fact (ii) is intuitively clear: if $\mathcal{Z}_2(\mathcal{C})$ is trivial then $\mathcal{E}$ is equivalent to the direct product of $I(\mathcal{C}) \cong \mathcal{C}$ and $\widetilde{I}(\widetilde{\mathcal{C}}) \cong \widetilde{\mathcal{C}}$; cf. [**28**, Theorem 7.10]. Thus for non-trivial $\mathcal{Z}_2(\mathcal{C})$, $\mathcal{E}$ should be a product of $I(\mathcal{C})$ and $\widetilde{I}(\widetilde{\mathcal{C}})$ amalgamated over the common subcategory $I(\mathcal{Z}_2(\mathcal{C}))$.

COROLLARY 5.6. *The category $\mathcal{Z}_1(\mathcal{C}) \supset \mathcal{E}$ is a minimal modular extension of $\mathcal{E}$.*

Proof. By construction, $\mathcal{Z}_1(\mathcal{C})$ is modular and contains $\mathcal{E}$ as a replete full braided subcategory. In view of the preceding results and the known facts on the double we have

$$\dim \mathcal{Z}_1(\mathcal{C}) = (\dim \mathcal{C})^2 = \frac{(\dim \mathcal{C})^2}{\dim \mathcal{Z}_2(\mathcal{C})} \dim \mathcal{Z}_2(\mathcal{C}) = \dim \mathcal{E} \cdot \dim \mathcal{Z}_2(\mathcal{E}).$$

Thus the bound in Proposition 5.1 is attained. $\square$

5.2. *The inverse problem for Galois extensions of premodular categories*

In this subsection we will solve the inverse problem of the Galois theory for braided tensor categories which was developed in [**25**]; see also [**8**]. To begin with, given an arbitrary compact group G, we have $\mathrm{Gal}(\mathrm{Rep}(G)) \cong G$. Thus every compact group is the Galois group of some braided tensor category. This is, however, not very interesting since symmetric tensor categories have trivial modular closure. We will therefore show that, for every finite group, there is a finite-dimensional unitary pre-modular category with $\mathrm{Gal}(\mathscr{C}) \cong G$ and which is not symmetric, and which thus has non-trivial modular closure. To this purpose we will construct modular categories $\mathscr{C}$ containing $\mathscr{S} = \mathrm{Rep}(G)$ as a full subcategory. Defining $\mathscr{D} = C_{\mathscr{C}}(\mathscr{S})$, we see that Corollary 3.6 implies $\mathscr{Z}_2(\mathscr{D}) = \mathscr{S}$. Thus we have $\mathrm{Gal}(\mathscr{D}) \cong G$ as desired. Non-triviality of $\mathscr{D} \rtimes \mathscr{Z}_2(\mathscr{D}) = \mathscr{D} \rtimes \mathscr{S}$ is equivalent to $\mathscr{D} \supsetneq \mathscr{S}$, and thus $\dim \mathscr{C} > (\dim \mathscr{S})^2$.

We exhibit two ways of constructing such a $\mathscr{C}$, both of which involve the center $\mathscr{Z}_1(\mathscr{C})$ or quantum double of a tensor category $\mathscr{C}$, which has already been referred to in § 5.1. See [**19**] for a nice discussion of the center construction and [**28**] for additional results which we will need.

The first construction is $\mathscr{C} = \mathscr{Z}_1(\mathscr{Z}_1(\mathscr{S}))$. Since $\mathscr{S}$ is braided, $\mathscr{Z}_1(\mathscr{S})$ and $\mathscr{Z}_1(\mathscr{Z}_1(\mathscr{S}))$ contain $\mathscr{S}$ as a full subcategory. By the results of [**28**], $\mathscr{C}$ is modular and $\dim \mathscr{C} = (\dim \mathscr{S})^4$. In particular, $\dim \mathscr{C} > (\dim \mathscr{S})^2$ as required. Since we obtain

$$\dim(C_{\mathscr{C}}(\mathscr{S}) \rtimes \mathscr{S}) = (\dim \mathscr{S})^2 = |G|^2,$$

it is natural to conjecture that the modular closure of $\mathscr{Z}_1 = C_{\mathscr{C}}(\mathscr{S})$ is equivalent to $\mathscr{Z}_1(\mathscr{S}) \cong D(G)\text{-mod}$.

The other procedure is as follows. Let

$$\mathbf{1} \longrightarrow H \longrightarrow K \longrightarrow G \longrightarrow \mathbf{1}$$

be an exact sequence of finite groups with $H \neq \{e\}$. With $\mathscr{C} = \mathscr{Z}_1(\mathrm{Rep}(K))$, $\mathscr{C}$ is modular and $\dim \mathscr{C} = |K|^2$. Furthermore, $\mathscr{S} = \mathrm{Rep}(G)$ is contained as a full subcategory in $\mathrm{Rep}(K)$ and thus in $\mathscr{Z}_1(\mathrm{Rep}(K))$. We obtain

$$\dim(C_{\mathscr{C}}(\mathscr{S}) \rtimes \mathscr{S}) = \frac{|K|^2}{|G|^2} = |H|^2.$$

We therefore expect that $\mathscr{D} \rtimes \mathscr{S} = C_{\mathscr{C}}(\mathscr{S}) \rtimes \mathscr{S}$ is equivalent to

$$\mathscr{Z}_1(\mathrm{Rep}(H)) \cong D(H)\text{-mod}.$$

Thus we have given two different proofs of the following result.

THEOREM 5.7. *Let G be a finite group. Then there is a unitary pre-modular category $\mathscr{C}$ such that $\mathrm{Gal}(\mathscr{C}) \cong G$ and such that $\mathscr{C}$ is not symmetric, thus having a non-trivial modular closure.*

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