On the structure of braided crossed G-categories (Appendix to book by Turaev)

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The main aim of this appendix is to discuss, for any finite group G, a close connection between braided crossed G-categories and ribbon categories containing the representation category of G as a full braided subcategory. In fact, in the context of finite semisimple categories, this will turn out to be a bijection (modulo suitable equivalences). As an application we prove that every braided G-crossed fusion category is equivalent to a strict monoidal category with strict G-action, justifying the strictness assumption made in Chapters VI and VII of this book. In the last section we touch upon the open problem of obtaining braided crossed G-categories as 'crossed products' of braided categories by finite group actions. The existence of such crossed products is shown to be equivalent to the conjectured existence of embeddings of braided fusion categories into modular categories of minimal size.

The original references on which this appendix is based are [4] by Bruguières, [20, 21] by Kirillov Jr. and [24, 27, 28] by the author. For a more extensive treatment of some of the matters discussed in this appendix, cf. [12], in particular Section 4, "Equivariantization and de-equivariantization".

1 Braided crossed G-categories

1.1 Definition *Let $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ be a monoidal category (with associativity constraint a and left and right unit constraints l, r). An automorphism of $\mathcal{C}$ is a monoidal functor (β, γ, σ) from $\mathcal{C}$ to itself, where β is a self-equivalence of $\mathcal{C}$, γ is a natural family $\{\gamma_{X,Y} : \beta(X \otimes Y) \to \beta(X) \otimes \beta(Y)\}$ of isomorphisms and $\sigma : \beta(\mathbf{1}) \to \mathbf{1}$ an isomorphism. For the exact conditions on β, γ, σ, cf. [23]. The composition $(\beta, \gamma, \sigma) \circ (\beta', \gamma', \sigma')$ is defined as $(\beta \circ \beta', \gamma'', \sigma'')$ with $\sigma'' = \sigma \circ \beta(\sigma')$ and γ'' defined by the composition*

$$\gamma''_{X,Y} : \beta(\beta'(X \otimes Y)) \xrightarrow{\beta(\gamma'_{X,Y})} \beta(\beta'(X) \otimes \beta'(Y)) \xrightarrow{\gamma_{\beta'(X),\beta'(Y)}} \beta(\beta'(X)) \otimes \beta(\beta'(Y)).$$

When $\mathcal{C}$ is braided, automorphisms of $\mathcal{C}$ are also required to respect the braiding, i.e. satisfy $\beta(c_{X,Y}) = c_{\beta(X),\beta(Y)}$. The set of (braided) automorphisms of $\mathcal{C}$ is denoted by Aut $\mathcal{C}$.

Now Aut $\mathcal{C}$ is the categorical group (i.e. monoidal category where every morphism is invertible and for every object X there is an object Y such that $X \otimes Y \cong \mathbf{1}$) having automorphisms of $\mathcal{C}$ as objects and natural monoidal isomorphisms (i.e. monoidal natural transformations all components of which are isomorphisms) of monoidal functors as morphisms.

The most concise way of defining an action of a (discrete) group on a monoidal category is as a monoidal functor:

1.2 Definition *If G is a group, let $\mathcal{G}$ be the discrete category (the only morphisms are the identity morphisms) with* $\mathrm{Obj}\ \mathcal{G} = G$ *and the obvious strict tensor product. An action β of G on a (braided) tensor category $\mathcal{C}$ is a monoidal functor $\beta : \mathcal{G} \to \mathrm{Aut}\ \mathcal{C}$, $\ g \mapsto \beta_g$. We usually abbreviate by writing ${}^gX = \beta_g(X)$, etc.*

Since each β_g is a monoidal functor, it comes with natural isomorphisms

$$\gamma_{g,X,Y} : \beta_g(X \otimes Y) \to \beta_g(X) \otimes \beta_g(Y) \quad \text{and} \quad \sigma_g : \beta_g(\mathbf{1}) \to \mathbf{1}.$$

On the other hand, the monoidality of the functor $\beta : \mathcal{G} \to \mathrm{Aut}\,\mathcal{C}$ provides natural monoidal isomorphisms $\delta_{g,h} : \beta_{gh} \to \beta_g \circ \beta_h$ and $\varepsilon : \beta_e \to \mathrm{id}_\mathcal{C}$, or in terms of components, $\delta_{g,h,X} : {}^{(gh)}X \to {}^g({}^hX)$ and $\varepsilon_X : {}^eX \to X$. One can easily unpack the definition to obtain the identities satisfied by these isomorphisms.

1.3 Definition *A G-action β on a strict monoidal category is called strict if all isomorphisms $\gamma_{g,X,Y}, \delta_{g,h,X}, \sigma_g, \varepsilon_X$ are identities.*

1.4 Definition *If $\mathcal{C}$ is a monoidal category and G a group, a G-grading on $\mathcal{C}$ is a map $\partial : \mathrm{Obj}\ \mathcal{C} \to G$ such that $\partial(X \otimes Y) = \partial X \cdot \partial Y$ and $\partial X = \partial Y$ whenever $X \cong Y$. The image of ∂ is called the G-spectrum of the G-graded monoidal category $\mathcal{C}$. The grading is called trivial or full if the G-spectrum equals $\{e\}$ or G, respectively.*

1.5 Remark 1. In the categorical literature, also G-gradings on the morphisms are considered. We, however, consider only gradings on the objects.

2. One could try to define a G grading to be a monoidal functor $\partial : \mathcal{C} \to \mathcal{G}$, but in the k-linear case, where $\mathrm{Hom}(X, Y)$ is never empty, this does not work since it would imply that all objects have degree e.

3. The above definition rules out direct sums of objects of different degrees. Often, however, it is desirable to work with semisimple categories, which includes having all direct sums. This can be accommodated by only requiring the existence of a full monoidal subcategory $\mathcal{C}_{\mathrm{hom}} \subset \mathcal{C}$ of *homogeneous* objects such that that (a) $\mathcal{C}_{\mathrm{hom}}$ satisfies Definition 1.4, and (b) every object in $\mathcal{C}$ is a finite direct sum of objects in $\mathcal{C}_{\mathrm{hom}}$. □

The following two definitions first appeared in [34], underlying Chapter VI of this book. Cf. also [5].

1.6 Definition *A crossed G-category is a monoidal category together with a G-action β and a G-grading ∂ such that $\partial({}^gX) = g\,\partial X g^{-1}$. We define full subcategories by $\mathcal{C}_g = \partial^{-1}(g)$ and notice that the G-action leaves $\mathcal{C}_e$ stable.*

1.7 Remark The spectrum of a rigid G-graded monoidal category is a subgroup of G. In the case of a crossed G category it is a normal subgroup. □

1.8 Definition *A braiding on a crossed G-category $(\mathcal{C}, \beta, \partial)$ is a family of natural isomorphisms $\{c_{X,Y} : X \otimes Y \to {}^{\partial X}Y \otimes X\}_{X,Y \in \mathcal{C}_{\mathrm{hom}}}$ satisfying naturality in the sense that*

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{c_{X,Y}} & {}^{\partial X}Y \otimes X \\ \Big\downarrow{\scriptstyle s \otimes t} & & \Big\downarrow{\scriptstyle {}^{\partial X}t \otimes s} \\ X' \otimes Y' & \xrightarrow{c_{X',Y'}} & {}^{\partial X'}Y' \otimes X' \end{array}$$

commutes for all $s \in \mathrm{Hom}(X, X')$, $t \in \mathrm{Hom}(Y, Y')$, *as well as commutativity of*

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] \arrow[d, "a"'] & {}^{gh}Z \otimes (X \otimes Y) \arrow[r, "a^{-1}"] & ({}^{gh}Z \otimes X) \otimes Y \arrow[drr, "(\delta_{g,h,Z} \otimes \mathrm{id}) \otimes \mathrm{id}"] & & \\
X \otimes (Y \otimes Z) \arrow[r, "\mathrm{id} \otimes c_{Y,Z}"'] & X \otimes ({}^hZ \otimes Y) \arrow[r, "a^{-1}"] & (X \otimes {}^hZ) \otimes Y \arrow[rr, "c_{X,{}^hZ} \otimes \mathrm{id}"'] & & ({}^g({}^hZ) \otimes X) \otimes Y
\end{tikzcd}$$

for all $X \in \mathcal{C}_g, Y \in \mathcal{C}_h, Z \in \mathcal{C}_k$ *and of a similar diagram involving* $c_{X,Y\otimes Z}$.

1.9 Remark Imposing naturality, one can uniquely extend $c_{X,Y}$ to the situation where $X \in \mathcal{C}_{\mathrm{hom}}, Y \in \mathcal{C}$, but the requirement $X \in \mathcal{C}_{\mathrm{hom}}$ cannot be relaxed. □

2 The G-fixed category of a braided crossed G-category

The following construction is well known, but it is hard to find the first reference.

2.1 Definition/Proposition *Let* $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)$ *be a monoidal category. Let* β *be an action of the group* G *on* $\mathcal{C}$. *Then* $\mathcal{C}^G$ *is the monoidal category* $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r)^G = (\mathcal{C}^G, \otimes^G, \mathbf{1}^G, a^G, l^G, r^G)$ *defined as follows: Its objects are pairs* $(X, \{u_g\}_{g\in G})$, *where* $X \in \mathcal{C}_{\mathrm{hom}}$ *and, for each* $g \in G$, $u_g : {}^gX \to X$ *is an isomorphism such that the diagram*

$$\begin{tikzcd}
{}^{gh}X \arrow[r, "\delta_{g,h,X}"] \arrow[d, "u_{gh}"'] & {}^g({}^hX) \arrow[d, "{}^g(u_h)"] \\
X & {}^gX \arrow[l, "u_g"]
\end{tikzcd} \tag{2.1}$$

commutes for all $g, h \in G$. *The Hom-sets are defined by*

$$\mathrm{Hom}_{\mathcal{C}^G}((X,u),(Y,v)) = \{ s \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \mid \begin{tikzcd} {}^gX \arrow[r, "{}^gs"] \arrow[d, "u_g"'] & {}^gY \arrow[d, "v_g"] \\ X \arrow[r, "s"'] & Y \end{tikzcd} \text{ commutes } \forall g \in G. \}$$

The tensor product of objects is defined by $(X, u) \otimes (Y, v) = (X \otimes Y, w)$, *where* $w_\bullet$ *is given by*

$$w_g : {}^g(X \otimes Y) \xrightarrow{\gamma_{g,X,Y}} {}^gX \otimes {}^gY \xrightarrow{u_g \otimes v_g} X \otimes Y.$$

The tensor product of morphisms is inherited from $\mathcal{C}$. *The monoidal unit* $\mathbf{1}^G$ *is given by* $(\mathbf{1}, \{\sigma_g\})$, *where* $\sigma_g : {}^g\mathbf{1} \to \mathbf{1}$ *is the isomorphism* ${}^g\mathbf{1} \to \mathbf{1}$ *coming with the monoidal functor* $\beta : \mathcal{G} \to \mathrm{Aut}\,\mathcal{C}$. *The associativity constraint* a^G *is given by*

$$a^G((X, w^X), (Y, w^Y), (Z, w^Z)) = a(X, Y, Z),$$

and similarly for the unit constraints l^G, r^G.

2.1 Remark The correct name for $\mathcal{C}^G$ would be 'category of G-modules in $\mathcal{C}$', but it seems to be more customary to speak of the 'G-fixed' category. In [12], the passage $\mathcal{C} \rightsquigarrow \mathcal{C}^G$ is called 'equivariantization'. □

2.2 Proposition *Let $(\mathcal{C}, \beta, \partial, c)$ be a braided crossed G-category. Then $\mathcal{C}^G$ is braided with braiding c^G given by*

$$c^G_{(X,u),(Y,v)} : X \otimes Y \xrightarrow{c_{X,Y}} {}^{\partial X}Y \otimes X \xrightarrow{v_{\partial X} \otimes \mathrm{id}_X} Y \otimes X.$$

Proof. One must show that c_G is natural w.r.t. both variables and satisfies both braid (or hexagon) equations. This amounts to straightforward combinations of the properties of c and the definition of $\mathcal{C}^G$ and of c^G. We omit the details. ■

The notation ${}^gX$ for $\beta(g)(X)$ used above is convenient, but it hides the dependence on the choice of a functor $\beta : \mathcal{G} \to \mathrm{Aut}\ \mathcal{C}$. In principle, we should write $\mathcal{C}^{(G,\beta)}$ instead of $\mathcal{C}^G$. We will do so only in the formulation of the following result.

2.3 Lemma *Let $\beta_1, \beta_2 : \mathcal{G} \to \mathrm{Aut}\ \mathcal{C}$ be actions of a group G on a monoidal category $\mathcal{C}$. A natural monoidal isomorphism $\beta_1 \cong \beta_2$ of monoidal functors induces a monoidal equivalence $\mathcal{C}^{(G,\beta_1)} \simeq \mathcal{C}^{(G,\beta_2)}$ between the respective fixpoint categories.*

Defining functors of categories carrying a G-action is not entirely trivial, but for our purposes it is sufficient to have a notion of equivalence of monoidal G-categories. To this purpose, we observe that given a monoidal equivalence $F : \mathcal{C} \to \mathcal{D}$, there is an adjoint equivalence $G : \mathcal{D} \to \mathcal{C}$, unique up to natural monoidal isomorphism. Therefore, if $E \in \mathrm{Aut}\ \mathcal{C}$ then $F \circ E \circ G \in \mathrm{Aut}\,\mathcal{D}$, and this gives rise to a monoidal equivalence $\widetilde{F} : \mathrm{Aut}\ \mathcal{C} \to \mathrm{Aut}\ \mathcal{D}$.

2.4 Definition *Let $\mathcal{C}, \mathcal{D}$ be monoidal categories carrying G-actions β, β'. Then an equivalence of $\mathcal{C}, \mathcal{D}$ (as monoidal G-categories) is a monoidal equivalence $F : \mathcal{C} \to \mathcal{D}$ such that the monoidal functors $\widetilde{F} \circ \beta$ and β' (both from $\mathcal{G}$ to $\mathrm{Aut}\ \mathcal{D}$) are monoidally equivalent.*

A functor of G-graded monoidal categories is a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(\mathcal{C}_{\mathrm{hom}}) \subset \mathcal{D}_{\mathrm{hom}}$ and $\partial_{\mathcal{D}} F(X) = \partial_{\mathcal{C}} X\ \forall X \in \mathcal{C}_h$.

Combining Lemma 2.3 with Definition 2.4, one finds:

2.5 Proposition *An equivalence $E : \mathcal{C} \to \mathcal{D}$ of braided crossed G-categories gives rise to an equivalence $E^G : \mathcal{C}^G \to \mathcal{D}^G$ of braided categories.*

Up to this point, our considerations were completely general in that we made no further assumptions on the categories or the groups. From now on we will restrict ourselves to finite groups and semisimple k-linear categories over an algebraically closed field k of characteristic zero.

2.6 Proposition *Let G be a finite group, k an algebraically closed field of characteristic zero and $\mathcal{C}$ a semisimple k-linear monoidal category carrying a G-action. Then $\mathcal{C}^G$ is a semisimple monoidal category having a full monoidal subcategory $\mathcal{S} \simeq \mathrm{Rep}_k G$. If $\mathcal{C}$ is braided or, more generally braided G-crossed, then $\mathcal{S}$ is a braided subcategory of $\mathcal{C}^G$. Cf. [20, 12].*

We see that a braided G-crossed category gives rise to a braided category $\mathcal{C}^G$ containing $\mathrm{Rep}_k G$ as full subcategory. In the next section, we will consider a construction that goes the opposite way. We will limit ourselves to the setting of the following definition:

2.7 Definition *Let k be an algebraically closed field of characteristic zero. A fusion category over k is a k-linear semisimple ribbon braided tensor category with simple unit, i.e.* $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k\mathrm{id}_{\mathbf{1}}$*, and finitely many isomorphism classes of simple objects.*

2.8 Remark 1. One may also consider non-braided fusion categories, in which case the definition of rigidity requires attention, one approach being the spherical categories of [1]. Cf. Appendix

2. In fact, braided spherical categories are the same as ribbon categories. □

3 From braided categories containing $\mathrm{Rep}\, G$ to braided G-crossed categories

The following definition is a straightforward generalization of notions from ordinary algebra:

3.1 Definition *Let $\mathcal{C}$ be a strict monoidal category. An algebra (=monoid) in $\mathcal{C}$ is a triple (A, m, η), where A is an object and $m : A \otimes A \to A$, $\eta : \mathbf{1} \to A$ are morphisms satisfying $m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m)$ and $m \circ (\eta \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A$. (In the non-strict case, one has to insert the associativity constraint at the obvious place.) If $\mathcal{C}$ has a braiding c, then an algebra in $\mathcal{C}$ is called commutative if $m \circ c_{A,A} = m$. A commutative algebra is called étale if there is a morphism $\Delta : A \to A \otimes A$ that satisfies $m \circ \Delta = \mathrm{id}_A$ and is a morphism of A-A bimodules, i.e.*

$$(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = m \circ \Delta = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta).$$

The study of (commutative) algebras in monoidal categories, e.g. those associated to quantum groups, is a very interesting subject and was used to great effect in [7]. However, we will only need the following example arising from representation categories of finite groups:

3.2 Proposition *Let G be a finite group and k an algebraically closed field of characteristic zero. The symmetric monoidal category $\mathcal{S} = \mathrm{Rep}_k G$ of finite dimensional representations of G on k-vector spaces contains a commutative algebra (A, m, η) with the following properties:*

(i) $\dim_k A = |G|$.

(ii) $\dim \mathrm{Hom}_{\mathcal{S}}(\mathbf{1}, A) = 1$.

(iii) The object A is 'absorbing': $A \otimes X \cong A^{\oplus \dim X} \quad \forall X \in \mathcal{S}$.

(iv) There is an isomorphism $G \xrightarrow{\cong} \mathrm{Aut}\,(A, m, \eta) \equiv \{s \in \mathrm{Aut}\; A \,|\, m \circ (s \otimes s) = s \circ m,\; s \circ \eta = \eta\}$.

(v) The algebra (A, m, η) is étale.

Proof. Let $A = \mathrm{Fun}(G, k)$ with algebra structure given by pointwise multiplication with the constant function 1 as unit. With the G-action $(\pi_l(g)f)(h) = f(g^{-1}h)$, this is the left regular representation π_l of G, which is well known to have properties (i) through (iii). ((i) is obvious, (ii) holds since the subspace of G-stable elements of A is one-dimensional (the constant functions), and (iii) follows from the fact that $A \cong \oplus_i \dim(X_i) \cdot X_i$, where X_i runs through the irreducible representations X_i.) For claim (iv), cf. e.g. [27, Remark 2.9]. With $\Delta(f)(g, h) = \delta_{g,h} f(g)$ for $f \in A = \mathrm{Fun}(G, k)$, the last statement holds by easy computations. ■

3.3 Remark For most applications in this appendix, Proposition 3.2 will be sufficient. In several other, but closely related, applications we are confronted by symmetric monoidal categories that are not a priori known to be of the form $\mathrm{Rep}\, G$. It is therefore important that every k-linear rigid symmetric monoidal category with simple unit, finitely many simple objects and trivial twists is equivalent to $\mathrm{Rep}\, G$ for a finite group G that is unique up to isomorphisms. (For stronger results without finiteness assumption cf. [7] and, in the case of $*$-categories, [10]. An exposition of the result for $*$-categories can be found in [29].) As to the last requirement, recall that every object in a symmetric ribbon category comes with a twist automorphism $\theta(X)$ of order two. In particular, for a simple object X we have $\theta(X) = \pm\mathrm{id}_X$, and the category is called even if all twists are identities. All these results have suitable generalizations to the non-even case. □

Again, as in commutative algebra, one defines

3.4 Definition *A (left) module over an algebra* (A, m, η) *is a pair* (X, μ) *where* $X \in \mathcal{C}$ *and* $\mu : A \otimes X \to X$ *satisfies* $\mu \circ (\mathrm{id}_A \otimes \mu) = \mu \circ (m \otimes \mathrm{id}_X)$. *The left modules form a category* ${}_A\mathcal{C} = A - \mathrm{Mod}_{\mathcal{C}}$ *with hom-sets defined by*

$$\mathrm{Hom}_{A-\mathrm{Mod}_{\mathcal{C}}}((X, \mu), (X', \mu')) = \{s \in \mathrm{Hom}(X, X') \mid s \circ \mu = \mu' \circ (\mathrm{id}_A \otimes s)\}.$$

Under a very weak condition on a braided category $\mathcal{C}$ (existence of coequalizers), which is satisfied in any abelian category, one finds that the category of modules over a commutative algebra in $\mathcal{C}$ is monoidal, the definition of the tensor product being a natural generalization of the usual one. The monoidal unit of ${}_A\mathcal{C}$ is (A, m), and there is a canonical monoidal functor $F : \mathcal{C} \to {}_A\mathcal{C}$ such that $X \mapsto (A \otimes X, m \otimes \mathrm{id}_X)$. Assuming $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$, one finds that $\mathrm{End}_{{}_A\mathcal{C}}(A) = k$ holds if and only if $\dim \mathrm{Hom}(\mathbf{1}, A) = 1$.

In order that ${}_A\mathcal{C}$ be semisimple, some technical condition on the algebra A is needed. One suitable notion is the étaleness defined above, the terminology being motivated by the corresponding notion in commutative algebra, cf. [6]. Similar, but slightly differently formulated conditions were considered in [4, 22, 25]. In the context of fusion categories, these conditions are equivalent, and they imply in particular that the functor F is *dominant*, i.e. every simple object of ${}_A\mathcal{C}$ is a direct summand of $F(X)$ for some $X \in \mathcal{C}$.

However, the braiding of $\mathcal{C}$ does not unconditionally descend to a braiding on ${}_A\mathcal{C}$.

3.5 Definition *[24] The symmetric center* $\mathcal{Z}_2(\mathcal{C})$ *of a braided monoidal category* $\mathcal{C}$ *is the full subcategory consisting of those objects* X *that satisfy*

$$c(X, Y) \circ c(Y, X) = \mathrm{id}_{Y \otimes X} \quad \forall Y \in \mathcal{C}.$$

3.6 Remark 1. The symmetric center is a symmetric monoidal category. It coincides with $\mathcal{C}$ if and only if $\mathcal{C}$ is symmetric.

2. One can show that a braided fusion category is modular if and only if its symmetric center is trivial, in that it contains only direct multiples of the unit object. (The 'only if' direction is immediate; for the 'if', see [3]. Cf. also [32] for a similar result in the context of endomorphisms of a C^*-algebra.) □

As shown in [24], the obvious candidate for a braiding ${}_A\mathcal{C}$ actually is a braiding if and only if $A \in \mathcal{Z}_2(\mathcal{C})$. If this is not the case, naturality of the putative braiding w.r.t. one of the arguments fails.

Let now $\mathcal{C}$ be a braided monoidal category with a full braided monoidal subcategory $\mathcal{S} \simeq \mathrm{Rep}_k G$. Now Proposition 3.2 gives rise to a commutative étale algebra (A, m, η) in $\mathcal{S}$ and thus in $\mathcal{C}$. Since A has every simple object of $\mathcal{S}$ as a direct summand, we have $A \in \mathcal{Z}_2(\mathcal{C})$ if and only if $\mathcal{S} \subset \mathcal{Z}_2(\mathcal{C})$. Under this assumption, ${}_A\mathcal{C}$ is braided. The functor $F : \mathcal{C} \to {}_A\mathcal{C}$ has the property that it trivializes $\mathcal{S}$, in the sense that $F(X)$ is a direct multiple of $\mathbf{1}$ for every $X \in \mathcal{S}$. What is more, when $\mathcal{S} = \mathcal{Z}_2(\mathcal{C})$ then ${}_A\mathcal{C}$ has trivial center $\mathcal{Z}_2({}_A\mathcal{C})$ and thus is modular. For this reason, this ${}_A\mathcal{C}$ (which is non-trivial if and only if $\mathcal{Z}_2(\mathcal{C}) \neq \mathcal{C}$, i.e. $\mathcal{C}$ is not symmetric) is called the modularization of $\mathcal{C}$, cf. [4, 24].

For the purposes of this appendix, the case $\mathcal{S} \not\subset \mathcal{Z}_2(\mathcal{C})$ is more interesting:

3.7 Theorem *If $\mathcal{C}$ is a braided fusion category, $\mathcal{S} \simeq \mathrm{Rep}_k G$ a full monoidal subcategory and (A, m, η) the corresponding commutative étale algebra, then*

(i) ${}_A\mathcal{C}$ is a braided G-crossed fusion category, which we denote as $\mathcal{C} \rtimes \mathcal{S}$.

(ii) $(\mathcal{C} \rtimes \mathcal{S})^G \simeq \mathcal{C}$ as braided fusion category.

(iii) If $\mathcal{D}$ is a braided G-crossed fusion category and $\mathrm{Rep}\, G \simeq \mathcal{S} \subset \mathcal{D}^G$ as in Section 1 then $\mathcal{D}^G \rtimes \mathcal{S} \simeq \mathcal{D}$ as braided G-crossed fusion category.

The G-spectrum and the degree-zero subcategory of $\mathcal{C} \rtimes \mathcal{S}$ can be described quite explicitly:

3.8 Proposition *Under the assumptions of Theorem 3.7, we have:*

(i) The degree zero part of $\mathcal{C} \rtimes \mathcal{S}$ is given by $(\mathcal{C} \rtimes \mathcal{S})_e = \mathcal{S}' \rtimes \mathcal{S} = {}_A\mathcal{C}^{\mathrm{loc}}$. (Here $\mathcal{S}' \subset \mathcal{C}$ is the 'centralizer' of $\mathcal{S}$, i.e. the full subcategory of objects X such that $c(X,Y) \circ c(Y,X) = \mathrm{id}_{X \otimes Y}$ for all $Y \in \mathcal{S}$. In particular $\mathcal{C}' = \mathcal{Z}_2(\mathcal{C})$. Furthermore, ${}_A\mathcal{C}^{\mathrm{loc}} \subset {}_A\mathcal{C}$ is the full subcategory consisting of A-modules (X, μ) that are dyslexic or local, i.e. satisfy $\mu \circ c(X, A) \circ c(A, X) = \mu$. It is known [31] that ${}_A\mathcal{C}^{\mathrm{loc}}$ is always braided.) The braided category $(\mathcal{C} \rtimes \mathcal{S})_e$ is modular if and only if $\mathcal{Z}_2(\mathcal{S}') = \mathcal{S}$.

(ii) The G-spectrum of $\mathcal{C} \rtimes \mathcal{S}$ is given by

$$\mathrm{Spec}(\mathcal{C} \rtimes \mathcal{S}) = \{g \in G \mid \pi(g) = \mathrm{id}_V \ \forall (V, \pi) \in \mathcal{S} \cap \mathcal{Z}_2(\mathcal{C})\},$$

where we use $\mathcal{S} \simeq \mathrm{Rep}_k G$. In particular, the grading is trivial if and only if $\mathcal{S} \subset \mathcal{Z}_2(\mathcal{C})$ and full if and only if $\mathcal{S} \cap \mathcal{Z}_2(\mathcal{C})$ is trivial, i.e. contains only multiples of the identity.

3.9 Remark 1. In particular, if $\mathcal{C}$ is modular then $\mathcal{Z}_2(\mathcal{C})$ is trivial and thus $\mathcal{C} \rtimes \mathcal{S}$ has full G-spectrum for any $\mathcal{S}$. Furthermore, the double centralizer theorem [26] gives $\mathcal{S}'' = \mathcal{S}$, thus $\mathcal{Z}(\mathcal{S}') := \mathcal{S}' \cap \mathcal{S}'' = \mathcal{S}' \cap \mathcal{S} = \mathcal{S}$ (since $\mathcal{S} \subset \mathcal{S}'$), and therefore $\mathcal{S}' \rtimes \mathcal{S}$ is modular.

2. The proofs are too long to be given here. Cf. [20, 21, 27]. We only remark that statement 4 of Proposition 3.2 is crucial for obtaining both the G-action on and the G-grading of $\mathcal{C} \rtimes \mathcal{S}$ and for showing that the natural candidate for a braiding (which really is a braiding when $\mathcal{S} \subset \mathcal{Z}_2(\mathcal{C})$) actually is a braiding in the G-crossed sense.

3. In [12], the passage $\mathcal{C} \rightsquigarrow \mathcal{C} \rtimes \mathcal{S}$ is called 'de-equivariantization'. $\square$

The final unsurprising result shows that $\mathcal{C} \rtimes \mathcal{S}$ depends on $\mathcal{S} \subset \mathcal{C}$ only up to equivalence:

3.10 Proposition *Let $E : \mathcal{C} \to \mathcal{D}$ be an equivalence of braided fusion categories and $\mathrm{Rep}_k G \simeq \mathcal{S} \subset \mathcal{C}$ a full braided monoidal subcategory. Then there is an equivalence $\mathcal{C} \rtimes \mathcal{S} \simeq \mathcal{D} \rtimes E(\mathcal{S})$ of braided crossed G-categories.*

The proof relies on the fact that the commutative étale algebra in $\mathrm{Rep}\, G$ corresponding to the regular representation of G is unique up to isomorphism.

4 Classification and Coherence for Braided Crossed G-Categories

Combining the results of the two preceding sections we arrive at the following result:

4.1 Theorem *(i) The operations $\mathcal{D} \rightsquigarrow \mathcal{D}^G$ and $\mathcal{C} \rightsquigarrow \mathcal{C} \rtimes \mathcal{S}$ give rise to a bijection between {braided G-crossed fusion categories $\mathcal{D}$, modulo equivalence of braided G-crossed categories} and {braided fusion categories $\mathcal{C}$ containing a full symmetric subcategory $\mathcal{S} \simeq \mathrm{Rep}\, G$, modulo braided equivalence}.*

(ii) Under this correspondence, $\mathcal{C} \simeq \mathcal{D}^G$ is modular if and only if $\mathcal{D}_e$ is modular and $\mathcal{D}$ has full G-spectrum.

Proof. (i) is contained in the results of the preceding sections.

(ii): That modularity of $\mathcal{C}$ implies modularity of $(\mathcal{C} \rtimes \mathcal{S})_e$ and full G-spectrum of $\mathcal{C} \rtimes \mathcal{S}$ is contained in Theorem 3.7. As to the converse, let $\mathcal{D}$ have full G-spectrum and $\mathcal{D}_e$ be modular. Defining $\mathcal{C} = \mathcal{C}^G$ and $\mathcal{C}_0 = (\mathcal{D}_e)^G$ we have $\mathcal{C} \supset \mathcal{C}_0 \supset \mathcal{S} \simeq \mathrm{Rep}\, G$. Modularity of $\mathcal{D}_e \simeq \mathcal{C}_0 \rtimes \mathcal{S}$ implies

$$\mathcal{S} = \mathcal{Z}_2(\mathcal{C}_0) = \mathcal{C}_0 \cap \mathcal{C}_0'. \tag{4.1}$$

Since $\mathcal{C}_0 \subset \mathcal{C}$ is the maximal subcategory for which $\mathcal{C}_0 \rtimes \mathcal{S}$ has trivial grading, we have

$$\mathcal{C} \cap \mathcal{S}' = \mathcal{C}_0. \tag{4.2}$$

The fullness of the G spectrum of $\mathcal{D} \simeq \mathcal{C} \rtimes \mathcal{S}$ implies that

$$\mathcal{S} \cap \mathcal{Z}_2(\mathcal{C}) = \mathcal{S} \cap \mathcal{C}' \text{ is trivial.} \tag{4.3}$$

If now $X \in \mathcal{Z}_2(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'$ is simple, (4.2) implies $X \in \mathcal{C}_0$, upon which (4.1) implies $X \in \mathcal{S}$. But now (4.3) entails that X is trivial. Thus $\mathcal{Z}_2(\mathcal{C})$ is trivial, to wit $\mathcal{C}$ is modular. ■

4.2 Remark 1. A more satisfactory statement of the above correspondence would be in terms of a 2-equivalence of certain bicategories, cf. [12].

2. An interesting alternative characterization of braided G-crossed fusion categories $\mathcal{D}$ satisfying the two conditions of (ii) is given in [21]. □

We are now in a position to obtain a straightforward but useful application, which shows that no restriction of generality is entailed by the limitation to G-categories with strict G-action:

4.3 Theorem *Let G be a finite group and $((\mathcal{D}, \ldots), \beta, \partial, c)$ a braided G-crossed fusion category. Then there is a strict braided fusion category $(\mathcal{D}', \beta', \partial', c')$ with a strict G-action and an equivalence $F : \mathcal{D} \to \mathcal{D}'$ of braided crossed G-categories. (Thus in particular, F is G-equivariant.)*

Proof. Given a braided G-crossed fusion category $\mathcal{D}$, we have an equivalence $\mathcal{D} \simeq \mathcal{D}^G \rtimes \mathcal{S}$ of braided crossed G-categories, where $\mathrm{Rep}_k G \simeq \mathcal{S} \subset \mathcal{D}^G$. By the coherence theorem for braided monoidal categories, there is a strict braided monoidal category $\widetilde{\mathcal{C}} \simeq \mathcal{D}^G$ with a distinguished strict symmetric subcategory $\widetilde{\mathcal{S}}$. By Proposition 3.10, we have $\mathcal{D} \simeq \widetilde{\mathcal{C}} \rtimes \widetilde{\mathcal{S}}$ as braided crossed G-categories. The claim now follows from the fact that there is a model for $\widetilde{\mathcal{C}} \rtimes \widetilde{\mathcal{S}}$ that is strict as a monoidal category and has a strict G-action. This is the category $\widetilde{\mathcal{C}}_A$ ($A \in \widetilde{\mathcal{S}}$ again being the left regular representation) discussed in [24, 27], where also the equivalence with ${}_A\mathcal{C}$ was proven. ■

It would be quite interesting to prove the theorem in a more direct way, hopefully extending its domain of validity.

5 Braided crossed G-categories as crossed products

The axioms of a crossed G-category $\mathcal{D}$ imply that the part $\mathcal{D}_e$ in trivial degree is a monoidal category with G-action β. In the case where $\mathcal{D}$ is a fusion category one can prove that $\dim \mathcal{D}_g \equiv \sum_{X \in \mathcal{D}_g} d(X)^2 = \dim \mathcal{D}_e$ (the sum being over the classes of simple objects in $\mathcal{D}_g$), whenever $\mathcal{D}_g$ is non-trivial, see Chapter VII, Section 1.7. This makes it reasonable to consider $\mathcal{D}$ as a crossed product of $\mathcal{D}_e$ with G: "$\mathcal{D} \simeq \mathcal{D}_e \rtimes_\beta G$". The question therefore arises whether, given a monoidal category with G-action there exists a crossed G-category $\mathcal{D}$ with full G-spectrum (i.e. $\mathcal{D}_g \neq \emptyset \ \forall g$) and a G-equivariant equivalence $\mathcal{C} \to \mathcal{D}_e$. Similarly, if $\mathcal{C}$ is braided and β an action $G \curvearrowright \mathcal{C}$ by braided automorphisms, one can ask for $\mathcal{D}$ to be a braided crossed G-category.

In the non-braided case it is easy to give an affirmative answer, discovered independently in the preprint [34], cf. Section 2.1 of Chapter VIII of this book, and in [33]. For simplicity of exposition, we assume $\mathcal{C}$ and the G-action to be strict. (As we know by Theorem 4.3, this is justified in the case of fusion categories, but everything can also be done without strictness assumptions. Cf. in particular [15].) We define a monoidal category $\mathcal{D}$ by $\mathrm{Obj}\,\mathcal{D} = \mathrm{Obj}\,\mathcal{C} \times G$ with tensor product $(X,g) \otimes (Y,h) = (X \otimes {}^gY, gh)$. The Hom-set $\mathrm{Hom}_\mathcal{D}((X,g),(Y,h))$ is defined as $\mathrm{Hom}_\mathcal{C}(X,Y)$ when $g = h$ and by $\emptyset$ or $\{0\}$ (in the k-linear case) for $g \neq h$, composition being inherited from $\mathcal{C}$. Finally, if $s \in \mathrm{Hom}((X,g),(X',g))$ and $t \in \mathrm{Hom}((Y,h),(Y',h))$ then $s \otimes t := s \otimes {}^gt \in \mathrm{Hom}_\mathcal{C}(X \otimes {}^gY, X' \otimes {}^gY') = \mathrm{Hom}_\mathcal{D}((X,g) \otimes (Y,h), (X',g) \otimes (Y',h))$. $\mathcal{D}$ has an obvious G-grading $\partial : (X,g) \mapsto g$ and G-action ${}^g(X,h) = ({}^gX, ghg^{-1})$ w.r.t. which it is a crossed G-category. (Notice that $\mathcal{C} \rtimes G$ is not closed under direct sums of objects of different degrees. While this can easily be remedied, this is not needed for the discussion that follows.) The question arises whether a braiding for $\mathcal{C}$ can be lifted to a braiding for $\mathcal{D} = \mathcal{C} \rtimes G$. The following simple observation in [28] provides an obstruction:

5.1 Lemma *Let $\mathcal{D}$ be a braided crossed G-category. If there exists an invertible object in $\mathcal{D}_g$ then ${}^gX \cong X$ for every $X \in \mathcal{D}_e$.*

Proof. Let $Y \in \mathcal{D}_g$ be invertible and $X \in \mathcal{D}_e$. The braiding of $\mathcal{D}$ provides isomorphisms $Y \otimes X \cong {}^gX \otimes Y \cong Y \otimes {}^gX$. Invertibility of Y implies $X \cong {}^gX$. ■

Since $\mathcal{C} \rtimes G$ has invertible objects $(\mathbf{1}, g)$ for every $g \in G$, the lemma implies that the crossed G-category $\mathcal{C} \rtimes G$ can have a braiding only if $X \cong {}^gX$ for all $g \in G$, $X \in \mathcal{C}$. In many situations, this is an unacceptable restriction. Nevertheless, it is interesting to say a bit more about braidings on $\mathcal{C} \rtimes G$. The following result is essentially a converse of the construction of a braiding on $\mathcal{C} \rtimes G$ given in Chapter VIII, Theorem 2.3.1.

5.2 Proposition *A braiding on $\mathcal{C} \rtimes G$ gives rise to a full and faithful monoidal functor $F : \mathcal{C} \to \mathcal{C}^G$ such that $K \circ F = \mathrm{id}_{\mathcal{C}}$, where $K : \mathcal{C}^G \to \mathcal{C}$ is the forgetful functor $(X, \{u_g\}) \mapsto X$, and therefore to an identification of $\mathcal{C}$ with a full monoidal subcategory of $\mathcal{C}^G$.*

Proof. In $\mathcal{C} \rtimes G$, we have $(\mathbf{1}, g) \otimes (X, e) = ({}^gX, g)$ and $(X, e) \otimes (\mathbf{1}, e) = (X, g)$. Thus the braiding $c_{(\mathbf{1},g),(X,e)} : ({}^gX, g) \xrightarrow{\cong} (X, g)$ provides an isomorphism $u_{X,g} : {}^gX \to X$. The braid identity for $c_{(\mathbf{1},g)\otimes(\mathbf{1},h),(X,e)}$ implies that $\{u_{X,g}\}_{g\in G}$ satisfies (2.1), thus $(X, \{u_{X,g}\})$ is an object in $\mathcal{C}^G$. The naturality of the braiding c implies that $F : X \mapsto (X, \{u_{X,g}\})$ is a functor $\mathcal{C} \to \mathcal{C}^G$. This functor is faithful by construction, and it is full since by definition $\mathrm{Hom}_{\mathcal{C}^G}((X, \{u_g\}), (Y, \{v_g\})) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$. By construction, it is clear that $K \circ F = \mathrm{id}_{\mathcal{C}}$. Finally, the braid identity for $c_{(\mathbf{1},g),(X,e)\otimes(Y,e)}$ is equivalent to $u_{X\otimes Y,g} = u_{X,g} \otimes u_{Y,g}$, which implies that F is strict monoidal. ■

Lemma 5.1 shows that the straightforward crossed product $\mathcal{C} \rtimes G$ in general cannot be equipped with a braiding. (For a much more extensive discussion of $\mathcal{C} \rtimes G$, including the non-strict case, and braidings on it, cf. [15].) In order to construct braided crossed G-categories, one needs to adopt a more sophisticated approach, starting from the observation that each category $\mathcal{E}_g$ is a bimodule category over $\mathcal{E}_e$. However, this is not the place to do so. Instead, we point out that the problem of defining *braided* crossed products $\mathcal{C} \rtimes G$ is closely related to one raised in an earlier conjecture of the author. In the remainder of this section, we assume the ground field to be $\mathbb{C}$, which implies $d(X)^2 \geq 0$ for every object X, cf. [13]. Thus, if $\mathcal{C} \subset \mathcal{D}$ is a full subcategory, we have $\dim \mathcal{C} \leq \dim \mathcal{D}$, and equality holds if and only if the categories are equivalent.

In [26], it was proven that if $\mathcal{D}$ is a modular category and $\mathcal{C} \subset \mathcal{D}$ a full monoidal subcategory, then $\dim \mathcal{D} \geq \dim \mathcal{C} \cdot \dim \mathcal{Z}_2(\mathcal{C})$ holds. Thus, there is a lower bound on the size (as measured by the dimension) of a modular category containing a given braided fusion category as a full subcategory. Furthermore, the following general conjecture was formulated, motivated by situations where it is true:

5.3 Conjecture *[27] Every braided fusion category $\mathcal{C}$ embeds fully into a modular category $\mathcal{D}$ with $\dim \mathcal{D} = \dim \mathcal{C} \cdot \dim \mathcal{Z}_2(\mathcal{C})$.*

Now one has:

5.4 Theorem *The following are equivalent:*

(i) Conjecture 5.3 holds for every braided fusion category $\mathcal{C}$ whose symmetric center $\mathcal{Z}_2(\mathcal{C})$ is even (and therefore equivalent to $\mathrm{Rep}\, G$ for a finite group G).

(ii) For every modular category $\mathcal{M}$ acted upon by a finite group G there is a braided crossed G-category $\mathcal{E}$ with full G-spectrum and a G-equivariant equivalence $\mathcal{E}_e \simeq \mathcal{M}$.

Proof. (ii)⇒(i): Let $\mathcal{C}$ be a braided fusion category with even center. By the reconstruction theorem, there is a finite group G such that $\mathcal{S} = \mathcal{Z}_2(\mathcal{C}) \simeq \mathrm{Rep}_k G$. Being the modularization [4, 24] of $\mathcal{C}$, $\mathcal{M} = \mathcal{C} \rtimes \mathcal{S}$ is modular, and it carries a G-action such that $\mathcal{M}^G \simeq \mathcal{C}$. By assumption (i), there is a braided crossed G-category $\mathcal{E}$ with full G-spectrum and G-equivariant equivalence $\mathcal{E}_e \simeq \mathcal{M}$. This implies $\dim \mathcal{E} = |G| \dim \mathcal{M} = \dim \mathcal{C}$. Now $\mathcal{D} = \mathcal{E}^G$ is a braided fusion category containing $\mathcal{M}^G \simeq \mathcal{C}$ as a full braided subcategory. We have $\dim \mathcal{D} = |G| \dim \mathcal{E} = |G| \dim \mathcal{C} = \dim \mathcal{C} \cdot \dim \mathcal{Z}_2(G)$. Finally, $\mathcal{D}$ is modular by Theorem 4.1.ii.

(i)⇒(ii): Let $\mathcal{M}$ be a modular category with G-action. Then $\mathcal{C} = \mathcal{M}^G$ is a braided fusion category with $\mathcal{S} \simeq \mathrm{Rep}_k G$ as full braided subcategory. Since $\mathcal{M} \simeq \mathcal{C} \rtimes \mathcal{S}$ has trivial G-grading and is modular, we have $\mathcal{Z}_2(\mathcal{C}) = \mathcal{S}$. By assumption (ii), there is a full braided embedding $\mathcal{C} \hookrightarrow \mathcal{D}$ with $\mathcal{D}$ modular of dimension $\dim \mathcal{C} \cdot \dim \mathcal{Z}_2(\mathcal{C}) = |G| \dim \mathcal{C} = |G|^2 \dim \mathcal{M}$. In view of $\mathcal{S} \subset \mathcal{C} \subset \mathcal{D}$, we can consider the braided crossed G-category $\mathcal{E} = \mathcal{D} \rtimes \mathcal{S}$. Since $\mathcal{D}$ is modular, $\mathcal{E}$ has full G-spectrum, and we have the G-equivariant equivalence $\mathcal{E}_e = (\mathcal{D} \cap \mathcal{S}') \rtimes \mathcal{S} = \mathcal{C} \rtimes \mathcal{S} \simeq \mathcal{M}$. ■

The significance of this result is that the problem of minimal embeddings into modular categories, for which no direct approach is in sight, can be reduced to the crossed product problem which appears more amenable, if by no means easy. However it seems that this problem does not always have a solution: According to V. Ostrik and collaborators (private communication concerning as-yet-unpublished work), there exists a cohomological obstruction. However this may turn out, there is a special situation where there is reason to believe that the mentioned obstruction vanishes:

5.5 Conjecture *Let $\mathcal{C}$ be a modular category and N a positive integer. Then there exists a distinguished braided crossed G-category $\mathcal{C} \wr S_N$ with $G = S_N$, full spectrum and an equivalence $(\mathcal{C} \wr S_N)_e \simeq \mathcal{C}^{\boxtimes N}$ that is equivariant w.r.t. the obvious S_N-action on $\mathcal{C}^{\boxtimes N}$.*

Unfortunately, no good formulation in terms of a universal property is known. However, there is a hypothetical application to quantum field theory, which we will state at the end of the next section.

6 Remarks on applications in conformal field theory

In the concluding section of this appendix, we briefly outline the connection of the results described in this appendix to conformal field theory. In fact, much of the author's work was motivated by such applications, and the same is probably true of [20, 21]. Since it is not possible to go into technical details, we limit ourselves to indicating the broad line of ideas and giving some pertinent references. A complete account can be found in [28].

In 1971, Doplicher, Haag and Roberts (DHR) studied [9] a class of representations of a quantum field theory defined on 3+1 dimensional Minkowski space in the context of the operator algebraic approach to axiomatic quantum field theory. (The latter had been founded by Haag and others around 1960. For a recent review of some aspects, cf. [16].) DHR showed that the category of the representations under consideration is a rigid semisimple unitary symmetric monoidal category, and they conjectured that such a category always is equivalent to the representation category of a compact (super)group. In the late 1980s, this was proven by Doplicher and Roberts [10], and Deligne independently arrived at an analogous result [7] for pro-algebraic groups. (He did not require the categories to be unitary.) Doplicher and Roberts also proved [11] that, given a quantum field theory $\mathcal{A}$ and the compact group G such that $\mathrm{Rep}\,\mathcal{A} \simeq \mathrm{Rep}\,G$, there exists an extended quantum field theory $\mathcal{F}$ acted upon by G such that $\mathcal{F}^G \cong \mathcal{A}$ and such that $\mathrm{Rep}\,\mathcal{F}$ is trivial (at least when G is second countable and $\mathrm{Rep}\,\mathcal{A}$ is even, i.e. the supergroup is a group). Thus, the existence of non-trivial representations of a quantum field theory $\mathcal{A}$ can be understood as a consequence of $\mathcal{A}$ being the G-fixed subtheory $\mathcal{F}^G$ of some 'bigger' quantum field theory $\mathcal{F}$. Furthermore, there is a one-to-one correspondence between quantum field theories $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}$ and closed subgroups $H \subset G$, given by $H \mapsto \mathcal{F}^H$ and

$$\mathcal{B} \mapsto H^{\mathcal{B}} = \{g \in G \mid g \upharpoonright \mathcal{B} = \mathrm{id}\}.$$

These results amount to a beautiful Galois theory for local fields, where the Doplicher-Roberts extension $\mathcal{F}$ corresponds to the algebraic closure. As a consequence of this theory, the representation categories of the extensions $\mathcal{B} \supset \mathcal{A}$ can be understood in purely group theoretic terms, without any rôle for the dynamics of the quantum fields.

All the results described above remain valid in $2+1$ spacetime dimensions, but in $1+1$ dimensions (or on 'the lightray' $\mathbb{R}$) the situation changes considerably. As shown in [14], a quantum field theory still gives rise to a rigid semisimple unitary braided monoidal category of representations, but one can no more prove that the braiding is a symmetry, quite in line with the theoretical physics literature. A host of 'rational' models studied in conformal field theory suggested that the representation category $\mathrm{Rep}\,\mathcal{A}$ should be a modular category under suitable assumptions. In [19], a very simple and natural set of axioms for a chiral conformal field theory, known to be satisfied by several infinite families of interesting examples associated with loop groups, was shown to imply modularity of the representation category. (A similar result was also proven in the context of vertex operator algebras, cf. [17].) Since a non-trivial modular category cannot be the representation category of a group, it is clear that analogues of the results of [10, 11] cannot be expected. Not even replacing the group by a more general algebraic structure, e.g. Hopf algebra, is very promising. Cf. [30] for a discussion of this issue.

While groups thus lose the distinguished rôle they played in higher spacetime dimensions, it is perfectly natural to study group actions on conformal field theories and the corresponding fixpoint theories $\mathcal{F}^G$, called 'orbifold theories'. One classical result of DHR remains true, namely the representation category $\mathrm{Rep}\,\mathcal{F}^G$ still contains a full symmetric monoidal subcategory equivalent to $\mathrm{Rep}\,G$. On the other hand, given a full symmetric subcategory $\mathcal{S} \subset \mathrm{Rep}\,\mathcal{A}$, the construction in [11] applies and provides an extension $\mathcal{F} = \mathcal{A} \rtimes \mathcal{S} \supset \mathcal{A}$ acted upon by the compact group dual to the symmetric category $\mathcal{S}$. As in high dimensions, the passages $\mathcal{A} \rightsquigarrow \mathcal{A} \rtimes \mathcal{S}$ to the extended theory and the orbifolding $\mathcal{F} \rightsquigarrow \mathcal{F}^G$ are essentially inverses of each other, i.e. $(\mathcal{A} \rtimes \mathcal{S})^G \cong \mathcal{A}$ and $\mathcal{F}^G \rtimes \mathcal{S} \cong \mathcal{F}$.

However, the relationship between the categories $\mathrm{Rep}\,\mathcal{A}$ and $\mathrm{Rep}\,\mathcal{A}^G$ on the one hand and between $\mathrm{Rep}\,\mathcal{A}$ and $\mathrm{Rep}(\mathcal{A} \rtimes \mathcal{S})$ will be more complicated than in the high dimensional situation. In the context of completely rational conformal field theories [19], it was shown in [28] that the representation category of an extension $\mathcal{A} \rtimes \mathcal{S}$ of a quantum field theory $\mathcal{A}$ by a symmetric subcategory of $\mathrm{Rep}\,\mathcal{A}$ is given by

$$\mathrm{Rep}(\mathcal{A} \rtimes \mathcal{S}) \simeq (\mathrm{Rep}(\mathcal{A}) \cap \mathcal{S}') \rtimes \mathcal{S}. \tag{6.1}$$

(Here as in all that follows, we must assume that G is finite. Otherwise the theories $\mathcal{A}$ and $\mathcal{A} \rtimes \mathcal{S}$ cannot both be completely rational.) While we saw in Section 3 that that a braided fusion category $\mathcal{C}$ can be recovered from its extension $\mathcal{C} \rtimes \mathcal{S}$ by a full symmetric subcategory via $(\mathcal{C} \rtimes \mathcal{S})^G \simeq \mathcal{C}$, there is little reason to hope that $\mathrm{Rep}\,\mathcal{A}$ can be recovered from $\mathrm{Rep}(\mathcal{A} \rtimes \mathcal{S})$, since this is given by (6.1) and some information was lost in the passage from $\mathrm{Rep}\,\mathcal{A}$ to $\mathrm{Rep}(\mathcal{A}) \cap \mathcal{S}'$. Since the Doplicher-Roberts construction and orbifolding are inverse operations, it follows that also the representation category $\mathrm{Rep}\,\mathcal{F}^G$ of an orbifold quantum field theory is not determined by $\mathrm{Rep}\,\mathcal{F}$. This was already understood in the early works on orbifolds, e.g. [8]. (This phenomenon can be seen even in the simplest case, the one where $\mathrm{Rep}\,\mathcal{F}$ is trivial, i.e. equivalent to $\mathrm{Vect}_{\mathbb{C}}$. In this specific situation, it turns out that $\mathrm{Rep}\,\mathcal{F}^G \simeq D^\omega(G)$-Mod, where $D^\omega(G)$ is the twisted quantum double. The cohomology class $[\omega] \in H^3(G, U(1))$ is encoded in $\mathcal{F}$, but clearly not in the trivial category $\mathrm{Rep}\,\mathcal{F}$.)

The solution to the problem of computing $\mathrm{Rep}\,\mathcal{F}^G$ in terms of categorical information pertaining to $\mathcal{F}$ was found in [28]:

6.1 THEOREM *To a completely rational conformal field theory $\mathcal{F}$ acted upon freely by a finite group G, one can associate a braided crossed G-category $G-\mathrm{Rep}\,\mathcal{F}$ with full G-spectrum. The degree zero subcategory is the category of ordinary representations as considered by Doplicher, Haag and Roberts [9] (known to be modular by [19]).*

The non-trivially graded objects correspond to 'twisted representations' of $\mathcal{F}$. While they do not satisfy the DHR criterion, they do so upon restiction to the orbifold theory $\mathcal{F}^G$, which explains their relevance for the determination of $\mathrm{Rep}(\mathcal{F}^G)$. With these preparations, one can prove [28]:

6.2 THEOREM *If $\mathcal{F}$ is a completely rational CFT carrying a free action of a finite group G, then there is an equivalence*

$$\mathrm{Rep}(\mathcal{F}^G) \simeq (G-\mathrm{Rep}\,\mathcal{F})^G$$

of braided monoidal categories. Conversely, one has the equivalence

$$G-\mathrm{Rep}\,\mathcal{F} \simeq \mathrm{Rep}(\mathcal{F}^G) \rtimes \mathcal{S}$$

of braided crossed G-categories, where $\mathcal{S} \subset \mathrm{Rep}(\mathcal{F}^G)$ is the symmetric subcategory of representations of $\mathcal{F}^G$ arising from the vacuum representations of $\mathcal{F}$.

Thus, the braided G-crossed category of twisted representations of $\mathcal{F}$ and the representation category $\mathrm{Rep}\,\mathcal{F}^G$, together with its symmetric subcategory $\mathcal{S} \subset \mathrm{Rep}\,\mathcal{F}^G$, contain the same information. In particular, this clarifies the phenomenon [8] that $\mathrm{Rep}\,\mathcal{F}$ does not determine $\mathrm{Rep}\,\mathcal{F}^G$.

Now we are in a position to connect Conjecture 5.5 to conformal field theory. Let $\mathcal{A}$ be a completely rational conformal field theory and N a positive integer. Now the N-fold direct tensor power $\mathcal{B} = \mathcal{A}^{\boxtimes N}$ of $\mathcal{A}$ carries an obvious S_N-action and it is natural to conjecture the following:

6.3 CONJECTURE *If $\mathcal{A}$ is a completely rational CFT (thus $\mathrm{Rep}\,\mathcal{A}$ is modular) then the (braided S_N-crossed) category of S_N-twisted representations of $\mathcal{A}^{\boxtimes N}$ is equivalent to the category $(\mathrm{Rep}\,A)\wr S_N$ of Conjecture 5.5. (Thus S_N-$\mathrm{Rep}(\mathcal{A}^{\boxtimes N})$ depends on $\mathcal{A}$ only via $\mathrm{Rep}\,\mathcal{A}$.)*

This conjecture is compatible with the rigorous work that has been done on CFTs of the form $\mathcal{A}^{\boxtimes N}$ and the fixpoint theories ('orbifolds') $(\mathcal{A}^{\boxtimes N})^G$ for $G \subset S_N$, cf. e.g. [2, 18], but to our understanding the Conjectures 5.5 and 6.3 are still open.

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