Isocategorical Compact Groups

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Well known: $\exists$ finite groups $G_1 \not\cong G_2$ with isomorphic

(a) character tables

(b) representation rings

Example: $D_8 = Z_4 \rtimes Z_2$ and Q (quaternion group)
Remark:

- (a) and (b) contain the same information.
- This cannot happen for abelian groups since (a) and (b) encode $\widehat{G}$, thus G.

Around 2000: Much stronger result: $\exists$ finite groups $G_1 \not\cong G_2$ such that $\operatorname{Rep} G_1 \simeq \operatorname{Rep} G_2$ as tensor categories.

- Independently due to Etingof/Gelaki (IMRN 2001), Davydov (J.Alg. 2001), Izumi/Kosaki (Rev.Math.Phys. 2002).
- In this case, G_1, G_2 are called isocategorical.

Remarks

- Here: $\operatorname{Rep} G =$ fin. dim. representations of G over alg. cl. field of characteristic prime to $|G|$. (Izumi/Kosaki: fin. dim. unitary reps.)
- $\operatorname{Rep} G$ is considered as abstract k-linear tensor category (forgetting vector space structure of the objects, let alone the G-action).
- If $\operatorname{Rep} G_1 \simeq \operatorname{Rep} G_2$ as symmetric tensor categories then $G_1 \cong G_2$ (Doplicher/Roberts, Deligne).
- D_8 and Q are not isocategorical. Finding pairs of isocategorical groups is quite non-trivial. Lowest known order: 64. (Izumi/Kosaki)

Goal:

1. Generalize to compact (topol.) groups (whose repres. theory is just as nice as for finite groups).
2. (Somewhat) More categorical approach.
3. 2. is unavoidable since the arguments for finite groups do not generalize trivially.

- Etingof/Gelaki: G_1 isocateg. to G_2 $\Leftrightarrow$ $\mathrm{Rep}\,\mathbb{C}G_1 \simeq \mathrm{Rep}\,\mathbb{C}G_2$ (as $\otimes$-categories) $\Leftrightarrow$ $\exists$ Drinfeld twist J for $\mathbb{C}G_1$ s.th. $\mathbb{C}G_1^J \cong \mathbb{C}G_2$ (as Hopf algebras).
- Problem: The last implication $(\Rightarrow)$ does not hold in general for infinite dimensional Hopf algebras.
- We will see that $\mathbb{C}G_1^J \cong \mathbb{C}G_2$ does hold for G_1, G_2 isocategorical, if interpreted suitably (replace $\mathbb{C}G_i$ by Hopf von Neumann algebras $\mathcal{L}(G_i)$ and $J \in \mathcal{L}(G_1)\overline{\otimes}\mathcal{L}(G_1)$ (von Neumann tensor product)), but proof is quite indirect.
- More categorical approach: If G_1, G_2 are isocategorical but non-isomorphic, then $\mathrm{Rep}\,G_1 \simeq \mathrm{Rep}\,G_2$ as $\otimes$-categories, but not as symmetric tensor categories.
- Starting point: Classify symmetries on $\mathrm{Rep}\,G$ for G compact.
- NB: Must consider only even symmetries, i.e. those where $\theta(X) = \mathrm{id}_X \ \ \forall X$. (Symmetric categories where this fails correspond to “supergroups”.)
- From now on: $\mathrm{Rep}\,G =$ fin.dim. continuous unitary representations of G, all symmetries unitary.
- Def.: Symm(G) is the set of even symmetries on $\mathrm{Rep}\,G$.

- Clear: Some correspondence (even symmetries on $\mathrm{Rep}\, G$) $\leftrightarrow$ (Groups G' isocategorical to G).
- ($\rightarrow$): Let $\mathcal{C} = \mathrm{Rep}\, G$ and $c \in \mathrm{Symm}(G)$. Then there is a fiber functor $E : \mathcal{C} \rightarrow \mathrm{Hilb}$ (faithful linear $*$-preserving symmetric tensor functor), unique up to ntl. monoidal isomorphism. (Uniqueness fails w/o the the symmetry requirement!) Now $G' = \mathrm{Aut}_{\otimes} E$ is compact and $\mathrm{Rep}\, G' \simeq (\mathcal{C}, c)$ as symm. $\otimes$-categ.
- ($\leftarrow$): Let G' isocateg. to G. Let $(\mathcal{C}, c) = \mathrm{Rep}\, G,\ (\mathcal{C}', c') = \mathrm{Rep}\, G'$. Choose $\otimes$-equivalence $E : \mathcal{C}' \rightarrow \mathcal{C}$. Now $E(c')$ is even symmetry for $\mathcal{C} = \mathrm{Rep}\, G$.
- However: $\leftrightarrow$ is not bijection: The equivalence E is unique only up to composition with a monoidal self-equivalence $F \in \mathrm{Aut}_{\otimes} \mathrm{Rep}\, G$. On the other hand, if $F \in \mathrm{Aut}_{\otimes} \mathrm{Rep}\, G$ and $c' = F(c)$, then $F : (\mathcal{C}, c) \rightarrow (\mathcal{C}, c')$ is a symmetric monoidal equivalence, thus $G \cong G'$ for the corresponding groups.
- Theorem: For G compact, there is a bijection between $\{$isomorphism classes $[G']$ of compact groups isocateg. to $G\}$ and the orbit space $\mathrm{Symm}(G)/\,\mathrm{Aut}_{\otimes} \mathrm{Rep}\, G$.

- We omit the precise definition of $F(c)$, which is slightly tedious.
- Etingof/Gelaki: G is categorically rigid if every G' isocategorical to G is isomorphic to G.
- Thus: G is categorically rigid iff $\mathrm{Aut}_{\otimes} \mathrm{Rep}\, G$ acts transitively on Symm(G).
- Abelian compact groups are categ. rigid. (Pontryagin duality)
- Connected compact groups are categ. rigid. (McMullen (1984), Handelman (1993): Connected compact groups with isomorphic representation rings are isomorphic!)
- Thus: finite non-abelian groups provide simplest examples, and not much more is to be expected for infinite groups.
- Nevertheless: present approach provides new perspective, and the methods are pertinent to important open problems in operator algebra theory. (Ergodic compact group actions on C^*/von Neumann algebras)

- NB: $\mathrm{Aut}_\otimes \mathrm{Rep}\, G$ is a monoid, but not a group. Rather, it is a categorical group, i.e. tensor category where all objects are invertible (up to iso) and a groupoid, the morphisms being the natural monoidal isomorphisms of tensor functors.
- If $F, F' \in \mathrm{Aut}_\otimes \mathrm{Rep}\, G$ are natural monoidally isomorphic and $c \in \mathrm{Symm}(c)$ then $F(c) = F'(c)$.
- If $\mathcal{G}$ is a categorical group, we define

$$\pi_0(\mathcal{G}) = \mathrm{Obj}\, \mathcal{G}/\cong,$$

$$\pi_1(\mathcal{G}) = \mathrm{Aut}(\mathbf{1}_{\mathcal{G}}).$$

- Abbreviating $\widehat{\mathrm{Aut}}(G) := \pi_0(\mathrm{Aut}_\otimes \mathrm{Rep}\, G)$, we have an action of $\widehat{\mathrm{Aut}}(G)$ on Symm(G), and Symm$(G)/\widehat{\mathrm{Aut}}(G) = \{[G'] \mid G'$ isocateg. to $G\}$.
- Easy (not needed here): $\pi_1(\mathrm{Aut}_\otimes \mathrm{Rep}\, G) \cong Z(G)$ (as topol. groups).
- Essentially only one reference on $\widehat{\mathrm{Aut}}(G)$: Davydov (2001); still many open questions. Cf. also Davydov, arXiv:0708.2758.

TODO:

1. Determine Symm(G).
2. Determine $\widehat{\mathrm{Aut}}(G)$, its action on Symm$(G)$, and the orbit space. In particular: When is it transitive, i.e. G categ. rigid?
3. Given $c \in \mathrm{Symm}(G)$, compute the corresponding group G' (isocateg. to G) as explicitly as possible.

Rem: 1. is solved completely, 3. essentially satisfactorily, open questions w.r.t. 2.

1. Symmetries on $\mathrm{Rep}\, G$.

- $\mathcal{C} = \mathrm{Rep}\, G$ has a canonical symmetry c. Idea: rather than all symmetries c', classify the 'quotients' c'/c. ⇝ classification of 'twines' on $\mathrm{Rep}\, G$ (Bruguières).
- However, we will follow more conventional Hopf-algebraic approach, augmented by analytical considerations (Hopf-von Neumann algebras).

- G compact group $\rightsquigarrow$ Haar measure μ, normalized: $\mu(G) = 1$.
- Left regular representation on $H = L^2(G, \mu)$ (unitary):

$$(\pi_L(g)(f))(h) = f(g^{-1}h).$$

- $\mathcal{L}(G) = \pi_L(G)''$. (weak closure).
- Coproduct $\Delta : \pi_L(g) \mapsto \pi_L(g) \otimes \pi_L(g)\ \forall g$. (Existence and continuity must be proven; done in 60's.)
- $(\mathcal{L}(G), \Delta)$ is Hopf-von Neumann algebra (and Kac algebra). Has antipode, counit (only for G compact!), etc.
- Braidings c on $\mathrm{Rep}\, G \ \leftrightarrow$ R-matrices for $(\mathcal{L}(G), \Delta)$. (Well known for finite-dim Hopf algs; proof in the setting of discrete algebraic quantum groups: M/Roberts/Tuset, 2004. NB: G compact $\rightarrow (\mathcal{L}(G), \Delta)$ is discrete.)
- $\mathrm{Rep}\, G \simeq \mathrm{Rep}(\mathcal{L}(G), \Delta, R = \mathbf{1} \otimes \mathbf{1})$ as symmetric $\otimes$-category, $\mathrm{Rep}\, G' \simeq \mathrm{Rep}(\mathcal{L}(G), \Delta, R')$ for G' isocateg. to G.

- Let G be compact. A bicharacter of $(\mathcal{L}(G), \Delta)$ (or simply of $\widehat{G}$) is a unitary $\Psi \in \mathcal{L}(G)\overline{\otimes}\mathcal{L}(G)$ s.th.
  1. $(\Delta \otimes \mathrm{id})(\Psi) = \Psi_{13}\Psi_{23}$,
  2. $(\mathrm{id} \otimes \Delta)(\Psi) = \Psi_{13}\Psi_{12}$,
  3. $(\varepsilon \otimes \mathrm{id})(\Psi) = \mathbf{1} = (\mathrm{id} \otimes \varepsilon)(\Psi)$,
  4. $(S \otimes \mathrm{id})(\Psi) = (\mathrm{id} \otimes S)(\Psi) = \Psi^{-1}$.
- A bicharacter Ψ is called invariant if it commutes with $\Delta(G)$.
- A bicharacter Ψ is called antisymmetric if $\sigma(R) \equiv R_{21} = R^{-1}$.
- A bicharacter Ψ is called symplectic (⇒antisymm) if $m(R) = 1$.

Since Δ is cocommutative, an invariant bicharacter is an R-matrix. If it is also antisymmetric, it is a triangular R-matrix. (But: also non-invariant bicharacters are important.)

Prop.: For G compact, there are bijections

- braidings on $\mathrm{Rep}\, G \leftrightarrow$ invariant bicharacters (R-matrices),
- symmetries on $\mathrm{Rep}\, G \leftrightarrow$ invariant antisymmetric bicharacters (triangular R-matrices),
- even symmetries $\leftrightarrow$ invariant symplectic bicharacters.

Classification of symplectic invariant bicharacters

- Let $A \subset G$ closed abelian subgroup. (Thus A compact, $\widehat{A}$ discrete.) Let $\psi : \widehat{A} \times \widehat{A} \to \mathbb{T}$ be bicharacter of $\widehat{A}$ in usual sense (i.e. $\psi \in \mathrm{Hom}(\widehat{A} \wedge \widehat{A}, \mathbb{T})$).
- Spectral decomposition: $L^2(G) \cong \oplus_{\chi \in \widehat{A}} H_\chi$, where

$$H_\chi = \{x \in L^2(G) \mid \pi_L(a)x = \chi(a)x \ \forall a \in A\}.$$

  Let P_χ be orthogonal projection onto H_χ.
- $\Psi(A, \psi) := \sum_{\chi, \chi' \in \widehat{A}} \psi(\chi, \chi') \, P_\chi \otimes P_{\chi'}$ is bicharacter on $\widehat{G}$.
- $\Psi(A, \psi)$ is antisymmetric iff ψ is antisymmetric: $\psi(\chi, \chi')\psi(\chi', \chi) = 1$.
- $\Psi(A, \psi)$ is symplectic iff ψ is symplectic: $\psi(\chi, \chi) = 1$.
- $\Psi(A, \psi)$ is invariant iff $A \subset G$ is normal and ψ is G-invariant: $\psi({}^g\chi, {}^g\chi') = \psi(\chi, \chi')$.

- Prop.: Let Ψ be antisymmetric (symplectic) bicharacter on $\widehat{G}$. Then there are $A \subset G$ closed abelian and $\psi : \widehat{A} \times \widehat{A} \to \mathbb{T}$ antisymmetric (symplectic) bicharacter such that $\Psi = \Psi(A, \psi)$. (Ψ is invariant iff A is normal and ψ invariant.)
- Proof sketch: Define

$$H_R(\Psi) = \{(\phi \otimes \mathrm{id})(\Psi) \mid \phi \in \mathcal{L}(G)_*\},$$

$$H_L(\Psi) = \{(\mathrm{id} \otimes \phi)(\Psi) \mid \phi \in \mathcal{L}(G)_*\},$$

$M_{L/R} = H''_{L/R}$. Then $M_{L/R} \subset \mathcal{L}(G)$ are cocommutative Hopf von Neumann algebras. (D. Radford + arguments showing that $M_{L/R}$ are $*$-algebras.) Cocommutativity of $\Delta \;\Rightarrow\; M_{L/R}$ commutative. Ψ antisymmetric $\Rightarrow\; M_L = M_R =: M$. Thus M is commutative cocommutative Kac algebra.
Tatsuuma/Takesaki: $M_L = M_R \cong \mathcal{L}(A)$ where A is the compact abelian group of group-like unitaries of M. A is closed subgroup of G. Let H_χ be as earlier.
Now, $\Psi \in M \otimes M = \pi_L(A)'' \otimes \pi_L(A)''$ implies that Ψ acts by multiplication with a scalar on $H_\chi \otimes H_{\chi'}$. This defines a map $\psi : \widehat{A} \times \widehat{A} \to \mathbb{T}$ such that $\Psi = \Psi(A, \psi)$.

- Thus $(A,\psi) \mapsto \Psi(A,\psi)$ is surjective (to antisymm. bicharacters). It becomes injective if restricted to non-degenerate bicharacters. ($\forall \chi \neq e \ \exists \chi' : \psi(\chi,\chi') \neq 1$.)
- Thus: Even symmetries on $\mathrm{Rep}\, G$ bijectively correspond to pairs (A,ψ) where $A \subset G$ is closed normal abelian subgroup and ψ is non-deg. symplectic invariant bicharacter on $\widehat{A}$.
- Remark: Similar results exist for all braidings.

2. $\widehat{\mathrm{Aut}}(G) := \pi_0(\mathrm{Aut}_\otimes \mathrm{Rep}\, G) = ?$

Easy: If G is compact abelian then

$$\widehat{\mathrm{Aut}}(G) \cong H^2(\widehat{G},\mathbb{T}) \rtimes \mathrm{Aut}\ \widehat{G},$$

where $\alpha \in \widehat{G}$ acts on $H^2(\widehat{G},\mathbb{T})$ by $[\omega] \mapsto [\omega \circ \alpha \times \alpha]$.
Action on Symm(G): Let $c \in \mathrm{Symm}(G)$ correspond to (A,ψ) as above, and let $(\omega,\alpha) \in Z^2(\widehat{G},\mathbb{T}) \rtimes \mathrm{Aut}\ \widehat{G}$. Then $F_{(\omega,\alpha)}(c)$ corresponds to (A,ψ') with

$$(\chi,\chi') \mapsto \psi'(\chi,\chi') = \psi(\alpha^{-1}(\chi),\alpha^{-1}(\chi'))\frac{\omega(\alpha^{-1}(\chi),\alpha^{-1}(\chi'))}{\omega(\alpha^{-1}(\chi'),\alpha^{-1}(\chi))}.$$

In particular, when $\alpha = \mathrm{id}_{\widehat{G}}$ we have

$$(\chi, \chi') \mapsto \psi'(\chi, \chi') = \psi(\chi, \chi') \frac{\omega(\chi, \chi')}{\omega(\chi', \chi)}. \tag{1}$$

Well known facts:

- If K is abelian group and $\omega \in Z^2(K, \mathbb{T})$ (normalized) then

  $$\psi : (\chi, \chi') \mapsto \frac{\omega(\chi, \chi')}{\omega(\chi', \chi)}$$

  is symplectic bicharacter.
- This descends to a homomorphism
  $r : H^2(K, \mathbb{T}) \to \mathrm{Hom}(K \wedge K, \mathbb{T})$.
- r is an isomorphism.

(Very well known for K finite abelian, but true for all (discrete) abelian K, e.g. Olesen/Pedersen/Takesaki 1980. Davydov's proof also works without change.)

In view of this and (1): $\widehat{\mathrm{Aut}}(G)$ acts transitively on Symm(G), thus G is categorically rigid! (Highbrow proof, but illuminating.)

2-cocycles on $\widehat{G}$

- For G compact, a 2-cocycle on $\widehat{G}$ is a unitary $\Omega \in \mathcal{L}(G) \otimes \mathcal{L}(G)$ satisfying

$$(\Delta \otimes \mathrm{id})(\Omega)\Omega_{12} = (\mathrm{id} \otimes \Delta)(\Omega)\Omega_{23}.$$

  The set of 2-cocycles is denoted $Z^2(\widehat{G})$.
- G abelian $\Rightarrow$ $Z^2(\widehat{G})$ (as above) $\cong$ ordinary $Z^2(\widehat{G}, \mathbb{T})$ (where $\widehat{G}$ is discrete abelian group).
- $\Omega \in Z^2(\widehat{G})$ is normalized if $\Omega(e_0 \otimes \mathbf{1}) = (e_0 \otimes \mathbf{1})$ and $\leftrightarrow$, where $e_0 = \int \pi_L(g) d\mu(g)$ is the two-sided integral in $\mathcal{L}(G)$.
- $\Omega \in Z^2(\widehat{G})$ is invariant if it commutes with $\Delta(\mathcal{L}(G))$.
- If $A \subset G$ is closed abelian subgroup and $\omega \in Z^2(\widehat{A}, \mathbb{T})$ then

$$\Omega(A, \omega) := \sum_{\chi, \chi' \in \widehat{A}} \omega(\chi, \chi')\, P_\chi \otimes P_{\chi'} \ \in Z^2(\widehat{G}).$$

  Ω is normalized iff ω is, and invariant iff $A \subset G$ is normal and ω is invariant.

$\widehat{\mathrm{Aut}}(G)$ for general compact G

- In general: Every $F \in \mathrm{Aut}_{\otimes} \operatorname{Rep} G$ is naturally isomorphic to $F_{(U,V)}$, where $V \in Z^2(\widehat{G})^G$ and

$$U \in \mathcal{U}(L^2(G)) \text{ s.th. } U\mathcal{L}(G)U^* = \mathcal{L}(G),\ Ue_0U^* = e_0.$$

- The set of such pairs (U, V) carries an associative composition, an inverse operation and an equivalence relation $\sim$ s.th.

$$\widehat{\mathrm{Aut}}(G) \cong \{(U, V)\}/\sim.$$

- No time for details.
- $F_{(U,V)}(c_\Psi) = c_{\Psi'}$, where $\Psi' = \sigma(V)^*\Psi V$.
- Thus c_Ψ is in the $\mathrm{Aut}_{\otimes} \operatorname{Rep} G$-orbit of $c_G = c_{\mathbf{1}}$ iff there is $V \in Z^2(\widehat{G})^G$ such that $\Psi = \sigma(V)V^*$.

- Recall:
  1. $\Psi = \Psi(A, \psi)$ with $\psi \in \mathrm{Hom}(\widehat{A} \wedge \widehat{A}, \mathbb{T})^G$,
  2. There is $\omega \in Z^2(\widehat{A}, \mathbb{T})$ s.th. $\psi(\chi, \chi') = \omega(\chi', \chi)/\omega(\chi, \chi')$.
- Now $\Omega(A, \omega) \in Z^2(\widehat{G})$ and $\Psi = \sigma(\Omega(A, \omega))\Omega(A, \omega)^*$.
- $\Omega(A, \omega) \in Z^2(\widehat{G})^G$ iff ω is G-invariant.
- Thus: If $\omega \in Z^2(\widehat{A}, \mathbb{T})$ with $r([\omega]) = \psi$ can be chosen G-invariant then c_Ψ is in the $\mathrm{Aut}_\otimes \mathrm{Rep}\, G$ orbit of $c_{\mathbf{1}}$.
- Thm: G admits $G' \ncong G$ isocategorical to G iff $\exists (A, \psi)$ with $A \subset G$ closed normal abelian and ψ symplectic invariant bicharacter not admitting $\omega \in Z^2(\widehat{A}, \mathbb{T})^G$ with $r([\omega]) = \psi$.
- Such groups G can be constructed using $Sp(V)$ where V is fin.dim. vector space over $\mathbb{Z}_2$. (Etingof/Gelaki, Davydov, Izumi/Kosaki). In fact: If G is finite and every closed normal abelian subgroup has trivial 2-torsion then G is categ. rigid (Davydov).

More explicit construction of G'

- So far: $(A, \psi) \rightsquigarrow$ triang. R-matrix $\rightsquigarrow$ symmetry c' on $\mathrm{Rep}\, G \rightsquigarrow G'$ with $\mathrm{Rep}\, G_1 \simeq (\mathrm{Rep}\, G, c')$.
- Last step: quite obscure, since it depends on the reconstruction theorem (DR/Deligne).
- Recall: (H, R) quasi-triang. Hopf algebra, $\Omega \in Z^2(H)$ (as earlier). With $\Delta' = \Omega\Delta(\cdot)\Omega^{-1},\ R' = \sigma(\Omega)R\Omega^{-1}$, one finds that
  1. (H, Δ', R') is quasi-triang. Hopf algebra,
  2. $\mathrm{Rep}(H, \Delta', R') \simeq \mathrm{Rep}(H, \Delta, R)$ as braided $\otimes$-categories.
- Same holds for discrete quantum groups, thus for $(\mathcal{L}(G), \Delta)$ with G compact.

- Let Ψ be sympl. bichar. on $\widehat{G}$.
- Then $\exists(A,\psi)$ (ψ sympl. inv. bichar. on $\widehat{A}$) s.th. $\Psi = \Psi(A,\psi)$
- Can choose $\omega \in Z^2(\widehat{A},\mathbb{T})$ s.th. $\psi(\chi,\chi') = \omega(\chi',\chi)/\omega(\chi,\chi')$.
- Then $\Omega(A,\omega) \in Z^2(\widehat{G})$ and $\sigma(\Omega(A,\omega))\Omega(A,\psi)^* = \Psi(A,\psi)$.
- Thus: For every Ψ (sympl. inv. bichar.) we can find $\Omega \in Z^2(\widehat{G})$ s.th. $\sigma(\Omega)\Omega^* = \Psi$.
- Twisting $(\mathcal{L}(G),\Delta,\Psi)$ by this Ω^*, we get $(\mathcal{L}(G),\Delta',\mathbf{1})$. Now, $\mathrm{Rep}\, G_1 \simeq \mathrm{Rep}(\mathcal{L}(G),\Delta',\mathbf{1})$, thus:

  $$G_1 \cong \{x \in \mathcal{L}(G) \mid xx^* = x^*x = 1,\ \Omega^*\Delta(x)\Omega = x \otimes x\}.$$

  NB: If ω is G-invariant then $\Omega(A,\omega)$ is G-invariant, thus $\Delta' = \Delta$. We see again that the non-existence of G-invariant ω is crucial.
- Thus $(\mathcal{L}(G'),\Delta_{G'}) = (\mathcal{L}(G),\Delta_G)^\Omega$. Thus group von Neumann algebras of isocategorical groups are related by twist!

More group theoretical construction of G'

- The above is (somewhat) more explicit description of G_1, but still not purely group theoretical.
- If G (thus G') is second countable (=metrizable) one can do even better:
- Etingof/Gelaki: homomorphism $\gamma : H^2(\widehat{A}, \mathbb{T})^G \to H^2(K, A)^K$, where $K = G/A$. Now the group G_1 corresponding to (A, ψ) is isomorphic to the extension of K by A belonging to $\gamma(r^{-1}(\psi)) \in H^2(K, A)^K$.
- For infinite, but second countable, compact groups similar approach works:
  1. One can find Borel measurable element of $Z^2(K, A)^K$.
  2. Use results by Mackey on correspondence of extensions of K by A with Borel measurable $\omega \in Z^2(K, A)^K$ to obtain G'.