Tensor Categories, Algebras and Quantum Fields

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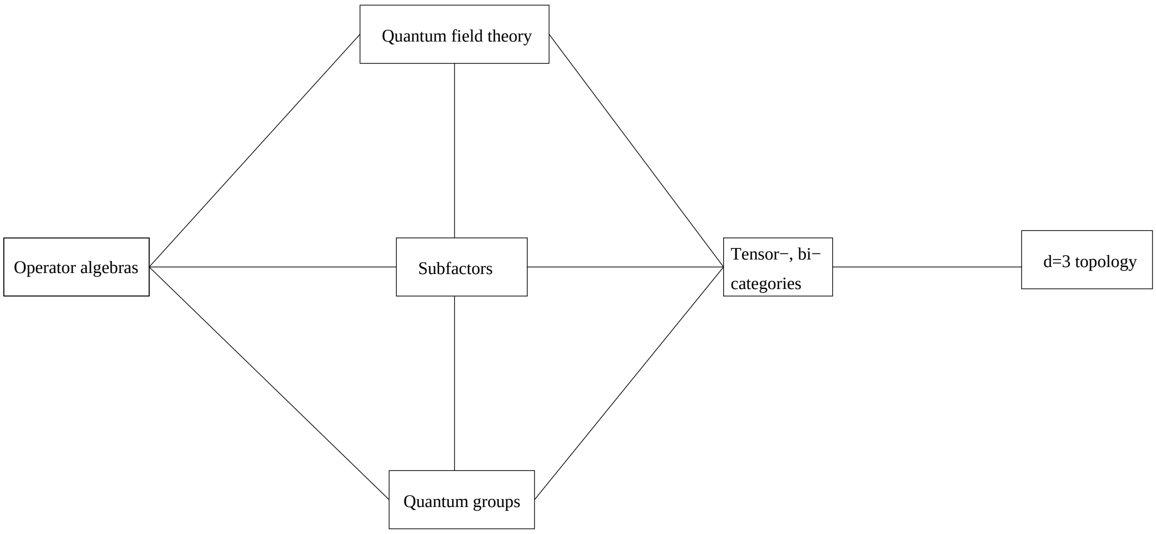
Contents

0.1 Introduction

The research I conducted since I obtained my PhD degree in 1997 involved three areas of interaction between the theory of operator algebras and category theory, as well as some applications of the latter to low dimensional topology. Pictorially the relations between the areas could be represented like this:



Since the theory of operator algebras, in the sense of C^*- and von Neumann algebras, is of a very analytical nature it may at first sight not appear to be amenable to the application of category theoretical methods beyond some very basic categorical notions, as e.g. in [58]. As it turns out, this is not quite the case. On the one hand, there are areas like quantum field theory and the theory of quantum groups, which (in some of their formulations, cf. e.g. [33, 44]) employ operator algebras. This is done in a very non-trivial way, insofar as deep results from the latter theory (modular theory, disintegration theory etc.) are used. In the mentioned areas very interesting monoidal categories of representations arise, often braided and rigid. While this may not be surprising in view of the relatively rich structures at hand, it is quite remarkable that subfactor theory, on the other hand, leads to non-trivial categorical considerations starting from very little.

My work, which is going to be described in more detail below, can be roughly organized as follows:

1. Identification and generalization of categorical structures implicit in subfactor theory and quantum field theory (QFT). Examples:
   - Subfactors: Frobenius algebras, bicategories and weak monoidal Morita equivalence {5}.
   - Subfactors: Longo/Rehren subfactor vs. categorical center $Z_1(\mathcal{C})$ {6}.
   - QFT: braided crossed G-categories arising from QFT with group action {10}.
2. Purely categorical considerations. Examples:
   - Galois theory of braided tensor categories, giving rise to braided crossed G-categories {2,9}. Modularization of braided categories {2}.
   - Structure of modular categories: double centralizer theorem, prime factorization {7}.
   - Modularity of $Z_1(\mathcal{C})$ and related results. {6}.
3. Applications of categorical notions/results to QFT, subfactors, quantum groups. Examples:
   - Modularity of representation category of completely rational chiral conformal QFTs {3}.
   - Conformal orbifold models: relations between $\mathrm{Rep}\, A$, $\mathrm{Rep}\, A^G$ and $G{-}\mathrm{Loc}\, A$. Holomorphic orbifold models and quasiabelian cohomology $H^3_{qa}(G,\mathbb{T})$ {10}.

- Local extensions of conformal models vs. Pareigis' category $\Gamma-\mathrm{Mod}_0$ of dyslexic modules. {10}
- Classification of modular invariants in categorical terms {11}.
- New proof of Tannaka theorem for discrete quantum groups {8}.
- Simplified and strengthened proof of abstract Tannaka theorem for symmetric categories {12}.

4. Applications to low dimensional topology. Examples:

- Morita invariance of state sum invariants of 3-manifolds {5}.
- Proof of a conjecture by Gelfand/Kazhdan on $Z_1(\mathcal{C})$ {6}.
- (Unfinished) Relation between state sum and surgery invariants.

Some of this work has been done in collaboration {3,8,12}. This survey is organized into two chapters. In the first I describe my work insofar as it does not involve quantum field theory. The aspects involving the latter are covered in a second chapter. For a more precise overview I refer to the table of contents.

Chapter 1

Categories

1.1 Tensor Categories

1.1.1 Preliminaries

The categories arising from operator algebraic structures are in general C^*- or even W^*-categories, cf. [23, 31, 48]. In particular, they are $\mathbb{C}$-linear with a positive involution on the morphisms, thus are unitary categories in the sense of [67]. Insofar as most of these categories actually have finite dimensional hom-sets we can usually ignore the topological aspects. (E.g., while this is not true for the category $\mathrm{Rep}\, A$ of all representations of a QFT, it does hold for the category $\mathrm{Rep}_f A$ of 'finite dimensional' representations.) The theory of algebraic quantum groups can be formulated over arbitrary ground fields, and the same holds for some of our purely categorical considerations. But with the exception of parts of {5} we will have no occasion to discuss non-k-linear categories.

We assume as known the standard notions of ((braided, symmetric) monoidal) categories as discussed in many references, e.g. [49, 41, 67], as well as rigid and spherical [5] categories. We also assume the notions of C^* and W^*-tensor categories and of conjugates in them. We only emphasize one important point: While in the definitions of rigid and spherical categories one assumes chosen duals $X^*, {}^*X$ or $\overline{X}$ and morphisms $e_X : X \otimes X^* \to \mathbf{1}$ etc. as part of the structure, there is no need to do so for $*$-categories. Basically this is due to the fact that the slightly different axiomatization of conjugates [48] eliminates all ambiguities which would arise in non-$*$ categories if one only assumed the existence of duals. For a detailed discussion we refer to {5, Section 2}.

By a semisimple category one usually means an abelian category in which all exact sequences split. If such a category is k-linear over an algebraically closed field and hom-sets are finite dimensional, every simple object X (this means every $s : Y \to X$ is either zero or an isomorphism) is absolutely simple, i.e. $\mathrm{End}\, X = k\mathrm{id}_X$. On some occasions we prefer to exclude zero objects. For our purposes it is thus more convenient to define a semisimple k-linear category as a category with direct sums, splitting idempotents, finite dimensional hom-sets and semisimple endomorphism algebras $\mathrm{End}\, X$. Thus every object is a finite direct sum of (absolutely) simple objects. Let us point out one observation in {5} to the effect that in a semisimple k-linear tensor category one can unambigously define the *square* of the dimension of any simple object even in the absence of a rigid/spherical or $*$-structure:

1.1.1 PROPOSITION *Let $\mathcal{C}$ be a k-linear tensor category with simple unit. If X is simple and has a two-sided dual then $d^2(X) = (\eta_X \circ e_X)(d_X \circ \varepsilon_X) \in k$ is a well defined quantity. If X, Y, XY are simple then $d^2(XY) = d^2(X)d^2(Y)$. Whenever $\mathcal{C}$ has a spherical or $*$- structure $d^2(X)$ coincides with $d(X)^2$ as defined using the latter.*

In order to avoid tedious repetition of the axioms we define:

1.1.2 DEFINITION *A fusion category is a semisimple k-linear tensor category over an algebraically* *closed field, satisfying* $\mathrm{End}\,\mathbf{1} = k$ *and being either spherical or a* $*$*-category with conjugates. A fusion category is finite if it has finitely many isomorphism classes of simple objects.*

1.1.3 Remark 1. Every $*$-category admits an essentially unique spherical structure (possibly passing to an equivalent category), cf. [78].

2. The definition of fusion categories in [25] includes finiteness, but assumes neither sphericity nor a $*$-structure. There are good reasons to do this, since one can prove remarkable results in the case $k = \mathbb{C}$. On the other hand, a large part of our considerations does not need finiteness, and the generality gained by assuming neither sphericity not a $*$-structure seems limited. □

1.1.4 Definition *Let* $\mathcal{C}$ *be a fusion category. If* $\mathcal{C}$ *has finitely many isomorphism classes of simple objects then we define*

$$\dim \mathcal{C} = \sum_X d^2(X) \ \in k,$$

where the summation is over the isomorphism classes of simple objects and $d^2(X)$ *is as in the proposition. If* $\mathcal{C}$ *has infinitely many simple objects then we formally posit* $\dim \mathcal{C} = \infty$.

The considerations in the remainder of this section are taken from {5}.

1.1.2 Frobenius Algebras in and Morita Equivalence of Tensor Categories

As is well known, if $G : \mathcal{D} \to \mathcal{C}$ is a right adjoint of a functor $F : \mathcal{C} \to \mathcal{D}$ then $T = GF$ is the part of a monad in $\mathcal{C}$, i.e. T is the object of a monoid in the (strict) tensor category $\mathrm{End}\,\mathcal{C}$. Similarly, $S = FG$ is part of a comonad in $\mathrm{End}\,\mathcal{D}$. Given any category $\mathcal{C}$ and a monad (T, m, η) in it, one can construct a category $\mathcal{D}$ and an adjoint pair $F \dashv G$ of functors inducing the given monad in the above fashion. The solutions to this problem, which are not unique, can be organized as a category with distinguished initial and final objects, the constructions due to Kleisli and Eilenberg/Moore, respectively.

The above considerations immediately generalize to any tensor category other than $\mathrm{End}\,\mathcal{C}$, where for notational simplicity we assume strictness. If $X \in \mathcal{C}$ has a right dual X^* (i.e. there exist $e : X \otimes X^* \to \mathbf{1}, d : \mathbf{1} \to X^* \otimes X$ satisfying the triangular equations) and $\Gamma = X^* \otimes X$ then $(\Gamma, \mathrm{id}_{X^*} \otimes e \otimes \mathrm{id}_X, d)$ is a monoid in $\mathcal{C}$. We are most interested in the case where X^* is also a left dual ${}^*X$ of X w.r.t. morphisms $e' : {}^*X \otimes X \to \mathbf{1}$, $d' : \mathbf{1} \to X \otimes {}^*X$, in which case we write $\overline{X}$. Then $(\Gamma, \mathrm{id}_{\overline{X}} \otimes e \otimes \mathrm{id}_X, d, \mathrm{id}_{\overline{X}} \otimes d' \otimes \mathrm{id}_X, e')$ is a Frobenius algebra in the sense of the following

1.1.5 Definition *Let* $\mathcal{C}$ *be a (strict) tensor category. A Frobenius algebra in* $\mathcal{C}$ *is a quintuple* $(\Gamma, m, \eta, \Delta, \varepsilon)$*, where* (Γ, m, η) *is a monoid in* $\mathcal{C}$*,* $(\Gamma, \Delta, \varepsilon)$ *is a comonoid in* $\mathcal{C}$ *and the following holds:*

$$m \otimes \mathrm{id}_\Gamma \,\circ\, \mathrm{id}_\Gamma \otimes \Delta = \Delta \,\circ\, m = \mathrm{id}_\Gamma \otimes m \,\circ\, \Delta \otimes \mathrm{id}_\Gamma.$$

If $\mathcal{C}$ *is* k*-linear then* $(\Gamma, m, \eta, \Delta, \varepsilon)$ *is strongly separable iff* $m \circ \Delta = \alpha\, \mathrm{id}_\Gamma$ *and* $\varepsilon \circ \eta = \beta\, \mathrm{id}_{\mathbf{1}}$ *with* $\alpha\beta \neq 0$.

1.1.6 Remark 1. Classically, a Frobenius algebra is a finite dimensional k-algebra A together with a linear functional $\varphi : A \to k$ such that the bilinear form $(a, b) \mapsto \varphi(ab)$ is non-degenerate. In [1] it is shown that for the category $\mathcal{C} = \mathrm{Vect}_k$ the two notions coincide. A classical Frobenius algebra is strongly separable iff $k\mathbf{1} \subset A$ is a Frobenius extension with index $[A : k\mathbf{1}]$ and $\varphi(1), [A : k\mathbf{1}] \in k^*$, cf. [41]. In {5} we show that the two notions of strong separability coincide in the case at hand.

2. In a semisimple category which is (left) rigid, all duals are two-sided. A spherical category [5] (not necessarily semisimple) has two-sided duals and the morphisms e, d, e', d' are part of the given structure, satisfying certain axioms. In such a category the Frobenius algebra $(\Gamma = \overline{X} \otimes X, \ldots)$ is always strongly separable with $\alpha\beta = d(\Gamma) = d(X)^2$.

3. If $\mathcal{C}$ is a $*$-category and $\overline{X}$ is a conjugate of X in the sense of [48] then the Frobenius algebra $\Gamma = \overline{X} \otimes X$ satisfies $\Delta = m^*, \varepsilon = \eta^*$. In this case we speak of a Frobenius $*$-algebra. Strongly separable Frobenius $*$-algebras were called ‘Q-systems’ in [46, 48]. □

Given a Frobenius algebra Γ in a tensor category $\mathcal{C}$ it is natural to ask whether it arises as above from an object X with two-sided dual $\overline{X}$. In general this is not true: In Vect_k, the Frobenius algebra $\overline{X} \otimes X$ is a matrix algebra $M_{\dim X}(k)$ whereas there are Frobenius algebras that are not of this form. There is, however, a positive answer if one generalizes the setting. If $\mathcal{E}$ is a bicategory, $\mathfrak{A}, \mathfrak{B} \in \mathrm{Obj}\,\mathcal{E}$ and $X : \mathfrak{A} \to \mathfrak{B}$ has a two-sided dual $\overline{X} : \mathfrak{B} \to \mathfrak{A}$ then $\Gamma = \overline{X}X \in \mathrm{End}\,\mathfrak{A}$ is a Frobenius algebra. Now we have the following converse {5}:

1.1.7 THEOREM *Let $\mathcal{A}$ be a strict tensor category and $(\Gamma, m, \eta, \Delta, \varepsilon)$ a Frobenius algebra in $\mathcal{A}$. Assume that one of the following conditions is satisfied:*

(a) $m \circ \Delta = \mathrm{id}_\Gamma$.

(b) $\mathcal{A}$ *is* $\mathrm{End}(\mathbf{1})$*-linear and*

$$m \circ \Delta = \alpha\, \mathrm{id}_\Gamma,$$

where α is an invertible element of the commutative monoid $\mathrm{End}(\mathbf{1})$.

(c) $\mathcal{A}$ has coequalizers.

Then there exists a bicategory $\mathcal{E}$ such that

1. $\mathrm{Obj}\,\mathcal{E} = \{\mathfrak{A}, \mathfrak{B}\}$.
2. *There is a fully faithful tensor functor $I : \mathcal{A} \to \mathrm{Hom}_\mathcal{E}(\mathfrak{A}, \mathfrak{A})$.*
3. *There are 1-morphisms $J : \mathfrak{A} \to \mathfrak{B}$ and $\overline{J} : \mathfrak{B} \to \mathfrak{A}$ which are mutual two-sided duals.*
4. *The Frobenius algebra in the tensor category $\mathrm{End}_\mathcal{E}(\mathfrak{A})$ arising from $J, \overline{J}$ is isomorphic to the image of $(\Gamma, \ldots)$ under I.*
5. *In case (c), I is an equivalence. In cases (a),(b), for every $Y \in \mathrm{End}_\mathcal{E}(\mathfrak{A})$ there is $X \in \mathcal{A}$ such that Y is a retract of $I(X)$. (Thus I is an equivalence if $\mathcal{A}$ has subobjects (=splitting idempotents).)*
6. *If $\mathcal{A}$ is a preadditive (k-linear) category then $\mathcal{E}$ is a preadditive (k-linear) 2-category.*
7. *If $\mathcal{A}$ has direct sums then $\mathcal{E}$ has direct sums of 1-morphisms.*
8. *If $\mathcal{A}$ is spherical, k-linear with $\mathrm{End}_\mathcal{A}(\mathbf{1}) = k$, and $\dim \mathrm{Hom}(\mathbf{1}, \Gamma) = 1$ then $\mathcal{A}$ is spherical and $\mathrm{End}_\mathcal{E}(\mathrm{id}_\mathfrak{B}) = k$.*
9. *If $\mathcal{A}$ is abelian (semisimple) then $\mathcal{E}$ is locally abelian (semisimple).*

Isomorphic Frobenius algebras $\Gamma, \tilde{\Gamma}$ give rise to isomorphic bicategories $\mathcal{E}, \tilde{\mathcal{E}}$.

1.1.8 REMARK In the case (c), which is only sketched in {5}, one defines $\mathcal{E}$ as the collection of the four categories $\mathcal{A}$, Γ–Mod, Mod–Γ and Γ–Mod–Γ. Here Γ–Mod has as objects pairs (X, μ), where $X \in \mathcal{A}$ and $\mu : \Gamma \otimes X \to X$ satisfies $\mu \circ \mathrm{id}_\Gamma \otimes \mu = \mu \circ m \otimes \mathrm{id}_X$ and $\mu \circ \eta \otimes \mathrm{id}_X = \mathrm{id}_X$, and the other categories are defined similarly. The composition of 1-morphisms being defined by a quotient procedure as in ring theory, this construction is analogous to the one of Eilenberg/Moore. In the cases (a),(b) one defines a bicategory $\mathcal{E}_0$ where $\mathrm{End}_{\mathcal{E}_0}(\mathfrak{A}) = \mathrm{Obj}\mathcal{A}$, $\mathrm{Hom}_{\mathcal{E}_0}(\mathfrak{A}, \mathfrak{B}) = \{\text{“}JX\text{”},\ X \in \mathcal{A}\}$, $\mathrm{Hom}_{\mathcal{E}_0}(\mathfrak{B}, \mathfrak{A}) = \{\text{“}X\overline{J}\text{”},\ X \in \mathcal{A}\}$ and $\mathrm{End}_{\mathcal{E}_0}(\mathfrak{B}) = \{\text{“}JX\overline{J}\text{”},\ X \in \mathcal{A}\}$. Composition of 1-morphisms is defined in the obvious way with the rule $\overline{J}J = \Gamma$. The 2-morphisms are defined by $\mathrm{Hom}_{\mathcal{E}_0}(\text{“}JX\text{”}, \text{“}JY\text{”}) = \mathrm{Hom}_\mathcal{A}(X, \Gamma Y)$ etc. as in Kleisli's constructions. One then defines $\mathcal{E}$ as the usual completion of $\mathcal{E}_0$ with splitting idempotents. The conditions (a),(b) then guarantee the existence of an identity morphism of $\mathfrak{B}$ as a retract of $J\overline{J} \in \mathrm{End}(\mathfrak{B})$. In the case where $\mathcal{A}$ is abelian and Γ is strongly separable, both constructions give equivalent bicategories $\mathcal{E}$. For further details we refer to {5}. □

There is a symmetry in the situation at hand: If $J, \overline{J}$ are two-sided duals in a bicategory $\mathcal{E}$ then there exists a Frobenius algebra $\tilde{\Gamma} = J\overline{J}$ in $\mathcal{B} = \mathrm{End}_{\mathcal{E}}(\mathfrak{B})$. Since this also holds when $\mathcal{E}$ is obtained from a Frobenius algebra Γ in $\mathcal{A}$, we can apply the same construction to $(\tilde{\Gamma}, \ldots)$ in $\mathcal{B}$. It turns out that we reobtain the initial data $\mathcal{A}$ and $(\Gamma, \ldots)$. This motivates the following

1.1.9 Definition *Two tensor categories $\mathcal{A}, \mathcal{B}$ are called weakly monoidally Morita equivalent $\mathcal{A} \approx \mathcal{B}$ iff there exists a bicategory $\mathcal{E}$ such that*

1. $\mathrm{Obj}\,\mathcal{E} = \{\mathfrak{A}, \mathfrak{B}\}$.
2. $\mathrm{End}_{\mathcal{E}}(\mathfrak{A})$ $(\mathrm{End}_{\mathcal{E}}(\mathfrak{B}))$ *is monoidally equivalent to the completion of $\mathcal{A}$ ($\mathcal{B}$) w.r.t. subobjects.*
3. *There are mutually two-sided dual 1-morphisms $J : \mathfrak{B} \to \mathfrak{A}$, $\overline{J} : \mathfrak{A} \to \mathfrak{B}$ such that the compositions $\eta_J \circ e_J \in \mathrm{id}_{\mathbf{1}_{\mathfrak{A}}}$ and $d_J \circ \varepsilon_J \in \mathrm{id}_{\mathbf{1}_{\mathfrak{B}}}$ are invertible.*

$\mathcal{E}$ is called a Morita context for $\mathcal{A}, \mathcal{B}$.

1.1.10 Proposition *$\approx$ is an equivalence relation. Given a tensor category $\mathcal{A}$ one has a bijection between tensor categories $\mathcal{B} \approx \mathcal{A}$ (together with a Morita context $\mathcal{E}$) and strongly separable Frobenius algebras in $\mathcal{A}$.*

The equivalence relation $\approx$ is much weaker than ordinary monoidal equivalence $\simeq$. Nevertheless it has remarkable consequences.

1.1.11 Proposition *If $\mathcal{A} \approx \mathcal{B}$ are finite fusion categories then* $\dim \mathcal{A} = \dim \mathcal{B}$.

For the center $Z_1(\mathcal{C})$ of tensor categories, cf. Section 1.5, the formalism of {5} together with [65] immediately implies the following.

1.1.12 Proposition *Let $\mathcal{C}_1, \mathcal{C}_2$ be weakly monoidally Morita equivalent tensor categories. Then there is an equivalence $Z_1(\mathcal{C}_1) \simeq Z_1(\mathcal{C}_2)$ of braided tensor categories.*

In [67] one finds a definition of invariants $TV(M, \mathcal{C})$ for oriented 3-manifolds based on triangulations for every unimodular category $\mathcal{C}$. (A unimodular category is a modular category, defined below, which is unimodal, i.e. every selfdual simple object is orthogonal, not symplectic.) In [4], a generalization $BW(M, \mathcal{C})$ to finite semisimple spherical categories [5], not necessarily braided, was given. Inspired by unpublished (and unfinished) work of Ocneanu [55], in {5} we outline a proof of the following result.

1.1.13 Theorem *Let $\mathcal{A}, \mathcal{B}$ be finite spherical fusion categories. If $\dim \mathcal{A} \neq 0$ and $\mathcal{A} \approx \mathcal{B}$ then we have $BW(M, \mathcal{A}) = BW(M, \mathcal{B})$ for all closed orientable 3-manifolds M.*

The proof is based on a generalization of the Barrett-Westbury invariant to k-linear bicategories. In the special case where the bicategory arises from a subfactor $A \subset B$ as in Subsection 1.1.4 this reduces to Ocneanu's invariant $Oc(M, A \subset B)$.

The research sketched above was motivated by its usefulness in the analysis {6} of the center $Z_1(\mathcal{C})$ of a finite fusion category $\mathcal{C}$, cf. Section 1.5. The following examples are somewhat simpler, but equally important.

1.1.3 Example: Finite Dimensional Hopf Algebras

Every finite dimensional Hopf algebra H is a (classical) Frobenius algebra, thus it gives rise to a Frobenius algebra (in the sense of Definition 1.1.5) in Vect_k. In {5} we show that there exists a less obvious Frobenius algebra Γ in the tensor category $\mathcal{C} = H{-}\mathrm{Mod}$. Contrary to H itself, this Frobenius algebra always satisfies $\dim \mathrm{Hom}(\mathbf{1}, \Gamma) = 1$. An important ingredient is the following

1.1.14 Lemma *Let (H,Δ) be a finite dimensional Hopf algebra. Then there exists a linear map $\tilde{m}: H\otimes H\to H$ which satisfies $\tilde{m}(\Delta(a)x)=a\tilde{m}(x)$ for all $a\in H,\ x\in H\otimes H$ and $\tilde{m}(\tilde{m}\otimes\mathrm{id})=\tilde{m}(\mathrm{id}\otimes\tilde{m})$.*

1.1.15 Theorem *Let H be a finite dimensional Hopf algebra over k. Let Λ,φ be left integrals in H and $\hat{H}$, respectively, normalized such that $\langle\varphi,\Lambda\rangle=1$. Let $\Gamma\in H-\mathrm{Mod}$ be the left regular representation, viz. H acting on itself by $\pi_\Gamma(a)b=ab$. The linear maps*

$$\begin{array}{rll}\tilde{\eta}: & k\to\Gamma, & c\mapsto c\Lambda,\\ \tilde{\varepsilon}: & \Gamma\to k, & x\mapsto\varepsilon(x),\\ \tilde{\Delta}: & \Gamma\to\Gamma\otimes\Gamma, & x\mapsto\Delta(x),\\ \tilde{m}: & \Gamma\otimes\Gamma\to\Gamma, & x\otimes y\mapsto\tilde{m}(x\otimes y)\end{array}$$

are morphisms in $H-\mathrm{Mod}$ and $(\Gamma,\tilde{m},\tilde{\eta},\tilde{\Delta},\tilde{\varepsilon})$ is a Frobenius algebra in $H-\mathrm{Mod}$ which satisfies $\dim\mathrm{Hom}(\mathbf{1},\Gamma)=1$. It is strongly separable iff H is semisimple and cosemisimple.

1.1.16 Theorem *Let H be a finite dimensional semisimple and cosemisimple Hopf algebra over an algebraically closed field k and let Γ be the associated strongly separable Frobenius algebra in $H-\mathrm{mod}$. If $\mathcal{E}$ is as in Theorem 1.1.7 and $\mathcal{B}=\mathrm{Hom}_{\mathcal{E}}(\mathfrak{B},\mathfrak{B})$ then we have the equivalence $\mathcal{B}\simeq\hat{H}-\mathrm{mod}$ of spherical tensor categories.*

Restating this theorem we have a weak monoidal Morita equivalence $H-\mathrm{Mod}\approx\hat{H}-\mathrm{Mod}$ whenever H is semisimple and cosemisimple. Together with Theorem 1.1.13 this implies $BW(M,H-\mathrm{Mod})=BW(M,\hat{H}-\mathrm{Mod})$. This was a known consequence of the fact that $BW(M,H-\mathrm{Mod})$ coincides with Kuperberg's invariant, but is quite non-obvious given only the definition of $BW(M,\mathcal{C})$. Of course, Theorem 1.1.13 is much more general, and another application will be given in Section 1.5.

1.1.4 Connections with Subfactor Theory

On the one hand, subfactor theory as initiated in [35] is a rather specialized field of functional analysis. On the other, however, it has applications in (the operator algebraic approach to) quantum field theory, cf. e.g. [45, 47, 10], and connections to various fields of pure mathematics, as is witnessed by Jones's work on knot theory [36, 37]. The latter connections are due to the categorical structure which is 'in-built' in subfactor theory. One aim of {5,6} was to elucidate these structures and to improve the communication between both fields. In doing so it becomes clear that several results first proved in subfactor theory have categorical versions which hold in considerably larger generality (ground fields other than $\mathbb{C}$, non-semisimple or infinite categories, etc.) By way of application, exhibiting clearly the connection between the center Z_1 of a tensor category and Ocneanu's asymptotic subfactor contributes to clarifying the significance of the latter. Ultimately, the hope is to make techniques from other areas of mathematics applicable to the classification programme of subfactors.

1.1.17 For any Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$, and for $M\subset\mathcal{B}(\mathcal{H})$ we write $M^*=\{x^*\mid x\in M\}$ and $M'=\{x\in\mathcal{B}(\mathcal{H})\mid xy=yx\ \forall y\in M\}$. A von Neumann algebra M on $\mathcal{H}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ satisfying $M=M^*=M''$. A factor is a von Neumann algebra with trivial center: $Z(M)\equiv M\cap M'=\mathbb{C}\mathbf{1}$. A factor M has type II_1 iff there exists a weakly continuous functional $tr: M\to\mathbb{C}$ such that $tr(ab)=tr(ba)$, $tr(\mathbf{1})=1$ and $tr(a^*a)\geq 0$ with equality iff $a=0$. A factor M (on a separable Hilbert space) is of type III iff for every orthogonal projection $p=p^2=p^*\in M$ there exists $v\in M$ such that $v^*v=\mathbf{1}$ and $vv^*=p$.

1.1.18 There exists a 2-category $\mathcal{E}$ whose objects are factors, whose 1-morphisms $\mathrm{Hom}(M,N)$ are unital weakly continuous $*$-algebra homomorphisms $\alpha: M\to N$ and where the 2-morphisms are given by

$$\mathrm{Hom}(\alpha,\beta)=\{s\in N\mid s\alpha(x)=\beta(x)s\ \ \forall x\in M\},\quad \alpha,\beta\in\mathrm{Hom}(M,N).$$

(By factoriality, all these 1-morphisms are injective maps.) Composition of 1-morphisms is just concatenation of maps and composition of 2-morphisms is multiplication in the image algebra. The $*$-operations of the factors give rise to a positive $*$-operation on $\mathcal{E}$. The full sub 2-category $\mathcal{E}_{III}$ whose objects are type III factors on separable Hilbert spaces has direct sums and subobjects.

Given an inclusion $A \subset B$ of factors ("A is a subfactor of B"), a conditional expectation is a map $E : B \to A$ that is $A - A$ bilinear ($E(abc) = aE(b)c \ \forall a, c \in A, b \in B$) and completely positive. For a conditional expectation E we define

$$[B : A]_E = (\sup\{\lambda \geq 0 \mid E(b^*b) - \lambda b^*b \geq 0 \ \forall b \in B\})^{-1}$$

and then

$$[B : A] = \inf_{E:B\to A} [B : A]_E.$$

Clearly, $[B : A] \in [1, \infty]$. The connection between subfactor theory and category theory is due to the following crucial result which (up to trivial reformulation) is due to Longo [45]:

1.1.19 Theorem *Let $A \subset B$ be an inclusion of type III factors (on separable Hilbert spaces). Then $[B : A] < \infty$ iff the inclusion $\iota : A \hookrightarrow B$ has a two-sided dual $\bar{\iota}$ in the bicategory $\mathcal{E}_{III}$.*

The subcategory $\mathcal{E}^f_{III} \subset \mathcal{E}_{III}$ retaining only the 1-morphisms $\alpha : A \to B$ for which $[B : \alpha(A)] < \infty$ has conjugates in the sense of [48] and is spherical [5]. For $\alpha : A \to B$ in $\mathcal{E}^f_{III}$ we have $[B : \alpha(A)] = d(\alpha)^2$, the right hand side being the categorical dimension in the sense of [48] or [5].

We briefly explain the results on subfactors which motivated the considerations in {5}. Let $A \subset B$ be an inclusion of type III factors with $[B : A] < \infty$. Then besides the embedding $\iota : A \to B$ there exists a dual morphism $\bar{\iota} : B \to A$ with $[A : \bar{\iota}(B)] = [B : A]$, thus there actually is a perfect symmetry between A and B. As in the general categorical discussion above, $\Gamma = \bar{\iota}\iota$ is the object of a Frobenius $*$-algebra in the tensor category $\mathrm{End}\, A$. Conversely, Longo has shown [46] that every strongly separable Frobenius $*$-algebra (or 'Q-system') in $\mathrm{End}\, A$ arises from a finite index inclusion $A \subset B$. On the other hand, by the work of Ocneanu [53, 54] (in the slightly different framework of type II_1 factors) every finite index subfactor gives rise to a $\mathbb{C}$-linear bicategory $\mathcal{E}_{A\subset B}$ with two objects $\mathfrak{A}, \mathfrak{B}$ in which every 1-morphism has a dual. In the type III context the 1-morphisms are just the retracts and direct sums of 1-morphisms obtained by considering all possible compositions of ι and $\bar{\iota}$. Thus there are two ways to obtain such a bicategory (a Morita context) from a Frobenius $*$-algebra in $\mathrm{End}\, A$: (a) construct the corresponding inclusion $A \subset B$ and consider Ocneanu's bicategory, (b) apply the categorical construction of Theorem 1.1.7 to $\mathrm{End}\, A$ and Γ and consider the full subcategory generated by J and $\overline{J}$. The following result shows that one obtains equivalent bicategories $\mathcal{E}$.

1.1.20 Theorem *Let A be a type III factor, let $(\Gamma, \ldots)$ be a strongly separable Frobenius $*$-algebra in $\mathrm{End}\, A$ and let $\mathcal{A}$ be the replete full subcategory with subobjects of $\mathrm{End}\, A$ generated by Γ. Let $\mathcal{E}_{A\subset B}$ be the bicategory associated with the subfactor $A \subset B$, where B is as constructed in [46]. If $\mathcal{E}$ is obtained from $\mathcal{A}$ and $(\Gamma, \ldots)$ by Theorem 1.1.7 then have an equivalence of bicategories $\mathcal{E} \simeq \mathcal{E}_{A\subset B}$.*

From this result we may conclude that Theorem 1.1.7 can act as a perfect replacement for the analytical constructions of subfactor theory insofar as only the bicategory $\mathcal{E}_{A\subset B}$ is of interest. This will allow us to apply ideas from subfactor theory [54, 26, 34] to the study of the center $Z_1(\mathcal{C})$ of a finite fusion category.

For more details on the previous matters see Sections 1.3 and 6.4 of {5}.

1.2 Tannaka Theory for Discrete Algebraic Quantum Groups

1.2.1 Discrete Algebraic Quantum Groups and Their Representations

As is well known, the dual vector space $\hat{H}$ of a finite dimensional Hopf algebra H can be equipped with a Hopf algebra structure, and there is a form of Pontryagin duality $\hat{\hat{H}} \cong H$. (If G is finite abelian and $H = kG$ then this is just Pontryagin's duality of locally compact groups.) This self duality of the category of finite dimensional Hopf algebras does not generalize to infinite dimensions, one reason being that it is not always possible to equip the algebraic dual $\hat{H}$ with a decent Hopf algebra structure. For this reason various other categories of Hopf algebra-like structures have been defined, e.g. Kac algebras [24] and locally compact quantum groups [44]. Both categories exhibit the desired selfduality and contain the category of locally compact abelian groups, thereby providing a true generalization of Pontryagin's theory. Kac algebras have turned out to be too restrictive, but the framework of [44] contains in particular the compact and discrete quantum groups defined by Woronowicz and others. While the existence of a Haar measure can be derived for locally compact groups, as well as for compact [75] and discrete [71] quantum groups, this does not seem to be the case for more general quantum groups. For this reason, the existence of a Haar functional is assumed in [44] as in most of the literature on the subject. Since the theory of locally compact quantum groups [24, 44] is rather analytical and technical, an alternative, purely algebraic formalism has been developed by Van Daele [72]. While less general, it still contains all compact and discrete quantum groups and is closed under duals and quantum doubles. In {8}, this framework is advocated as the most convenient environment for the proof of Tannaka-type reconstruction theorems. We begin with a few definitions, where for reasons of brevity we use somewhat non-standard terminology. All tensor products are algebraic.

If k is a field, a k-algebra A is called non-degenerate if $ab = 0\ \forall a \in A \Rightarrow b = 0$ and $ab = 0\ \forall b \in A \Rightarrow a = 0$. For every non-degenerate algebra A there exists a unital algebra $M(A)$ of two-sided multipliers, into which A embeds as an essential ideal, cf. [72] and references given there. A homomorphism $\alpha : A \to B$ of non-degenerate algebras is non-degenerate if $B = \alpha(A)B := \mathrm{span}_k\{\alpha(a)b,\ a \in A, b \in B\}$ and $B = B\alpha(A)$. A non-degenerate homomorphism α extends uniquely to a unital homomorphism $\overline{\alpha} : M(A) \to M(B)$.

A multiplier bialgebra is a non-degenerate algebra A together with a coproduct, i.e. a non-degenerate homomorphism $\Delta : A \to M(A \otimes A)$ that is coassociative (in a suitable sense). A multiplier bialgebra (A, Δ) is an algebraic quantum semigroup if it comes with a linear functional $\varphi : A \to k$ that is left invariant, i.e. $(\mathrm{id} \otimes \varphi)\Delta(a) = \varphi(a)1\ \forall a \in A$, and faithful, i.e. $\varphi(ab) = 0\ \forall a \in A \Rightarrow a = 0$ and $a \leftrightarrow b$. A multiplier bialgebra is a multiplier Hopf algebra iff the cancellation properties

$$\Delta(A)(A \otimes 1) = \Delta(A)(1 \otimes A) = (A \otimes 1)\Delta(A) = (1 \otimes A)\Delta(A) = A \otimes A$$

hold in $M(A \otimes A)$. Finally, a multiplier Hopf algebra with faithful left invariant functional is an algebraic quantum group (aqg). If $k = \mathbb{C}$ and A has a positive involution $*$, i.e. $a^*a = 0 \Rightarrow a = 0$, then we speak of multiplier $*$-algebras, $*$-algebraic quantum groups etc. Then φ is required to be positive.

A multiplier bialgebra gives rise to linear maps $\varepsilon : A \to k$ and $S : A \to M(A)$ having most of the expected properties. If A is a multiplier Hopf algebra then $S(A) = A$. A multiplier Hopf algebra with left invariant functional (A, Δ) is discrete if there exists $\Lambda \in A$ such that $a\Lambda = \Lambda a = \varepsilon(a)\Lambda$ for all $a \in A$ and compact if A has a unit (in which case A is a Hopf algebra). There exists a Pontryagin duality theory for algebraic quantum groups such that $\widehat{\widehat{(A, \Delta)}} \cong (A, \Delta)$. (We emphasize the difference from the duality of finite dimensional Hopf algebras: Instead of units for the algebras one assumes the existence of faithful left invariant functionals.) This duality establishes a bijection between agqs of compact and discrete type, respectively. In {8, Section 5.4} we prove

1.2.1 Proposition *Let (A, Δ) be compact. Then $\varphi(1) \neq 0$ iff $\widehat{(A, \Delta)}$ is semisimple, i.e. A is a direct sum of finite matrix algebras.*

1.2.2 Remark 1. Semisimple discrete quantum groups were first studied in [71] where they were called 'discrete quantum groups'. For semisimple multiplier Hopf algebras one can prove the existence of a faithful left invariant functional.

2. In the $*$-case, compact quantum groups always satisfy $\varphi(1) \neq 0$ and therefore discrete $*$-aqgs are always semisimple. □

A non-degenerate representation of an aqg is a ($*$-)homomorphism $\pi : A \to \mathrm{End}(V)$ such that $\pi(A)V = V$. The category of these representations and their intertwiners is denoted by $\mathrm{Rep}(A, \Delta)$. By $\mathrm{Rep}_f(A, \Delta)$ we denote the full subcategory of finite dimensional representations. The representation theory of the above structures has been studied by Van Daele and coauthors and in {8}. In particular,

1.2.3 Proposition {8} *Let (A, Δ) be a discrete $*$-aqg. Then the category $\mathrm{Rep}_f(A, \Delta)$ is a semisimple tensor $*$-category with conjugates (in the sense of [48]).*

More generally, if (A, Δ) is discrete and semisimple then $\mathrm{Rep}_f(A, \Delta)$ is a semisimple spherical tensor category in the sense of [5].

1.2.2 Tannaka Theorems for Discrete Quantum Groups

$\mathrm{Rep}_f(A, \Delta)$ is a concrete tensor category, in that its objects are vector spaces with additional structure. It is often profitable to consider $\mathrm{Rep}_f(A, \Delta)$ as an abstract tensor category $\mathcal{C}$ together with a faithful tensor functor $E : \mathcal{C} \to \mathrm{Vect}_k$ and representations π_X of A on $E(X)$. A faithful tensor functor E is called a fiber (or embedding) functor. If $\mathcal{C}$ is a $*$-category then E is required to be $*$-preserving. In [74], Woronowicz characterized those C^*-tensor categories that are categories of corepresentations of 'compact matrix pseudogroups'. (The latter are related to compact quantum groups as compact Lie groups to compact groups.)

We denote by $\mathcal{H}$ the category of finite dimensional Hilbert spaces and by Σ its obvious symmetry. In the joint paper {8}, which is partly of a review character, we argued that it is more natural to prove Tannaka-type theorems for discrete quantum groups and we gave simple proofs of the following results:

1.2.4 Theorem *Let $\mathcal{C}$ be a semisimple tensor $*$-category and $E : \mathcal{C} \to \mathcal{H}$ an embedding functor. Then there exists a discrete $*$-algebraic quantum semigroup (A, Δ) and an equivalence $F : \mathcal{C} \to \mathrm{Rep}_f(A, \Delta)$ of tensor $*$-categories such that $K \circ F = E$, where $K : \mathrm{Rep}_f(A, \Delta) \to \mathrm{Vect}_{\mathbb{C}}$ is the forgetful functor. If $\mathcal{C}$ has conjugates then (A, Δ) is a discrete $*$-aqg.*

An R-matrix for an aqg (A, Δ) is an invertible element, satisfying the usual relations, of $M(A \otimes A)$ such that $R\Delta(a)R^{-1} = \sigma\Delta(a)$ for all $a \in A$, where σ is the flip automorphism. An R-matrix gives rise to a braiding on $\mathrm{Rep}_{(f)}(A, \Delta)$. The converse also holds:

1.2.5 Theorem *Let $\mathcal{C}, E$ and the derived data $(A, \Delta), F$ be as above. Then there is a bijection between braidings on $\mathcal{C}$ and a R-matrices for (A, Δ) such that $\mathcal{C} \simeq \mathrm{Rep}_f(A, \Delta)$ as braided tensor categories. Unitary braidings correspond to unitary R-matrices.*

All these results generalize to the case of semisimple categories without $*$-operation, cf. {8, Section 5.4}. If $\mathcal{C}$ is braided and E respects the braiding, i.e. $c(X, Y) = \Sigma_{E(X),E(Y)}$, then $\mathcal{C}$ must be symmetric and (A, Δ) is cocommutative. In this case we recover the classical Tannaka theorem:

1.2.6 Theorem *Let $\mathcal{C}$ be a semisimple symmetric tensor $*$-category with conjugates and $E : \mathcal{C} \to \mathcal{H}$ an embedding functor such that $c(X, Y) = \Sigma_{E(X),E(Y)}$. Then there exist a compact group G and an equivalence $F : \mathcal{C} \to \mathrm{Rep}_f G$ of tensor $*$-categories such that $K \circ F = E$, where $K : \mathrm{Rep}_f G \to \mathrm{Vect}_{\mathbb{C}}$ is the forgetful functor.*

1.2.3 The Regular Monoid

In Subsection 1.1.3 we have seen that the left regular represenation of a finite dimensional Hopf algebra H gives rise to a Frobenius algebra $(\Gamma, m, \eta, \Delta, \varepsilon)$ in $\mathrm{Rep}_f H$. Obviously, in the infinite dimensional case, the regular representation does not live in $\mathrm{Rep}_f H$ anymore, but the above results also fails for other reasons. Yet the following results from {12, Section 3} show that some of the structure survives.

1.2.7 Lemma *Let Γ be the left regular representation of the $*$-aqg (A,Δ). Then there are unitary isomorphisms $\Gamma \otimes \theta \cong \Gamma \otimes I_\theta$, natural in θ for all $\theta \in \mathrm{Rep}(A,\Delta)$, and I_θ is the representation $a \mapsto \varepsilon(a)\mathrm{id}_\theta$. Γ contains ε as a subrepresentation iff (A,Δ) is discrete, in which case the multiplicity is one.*

1.2.8 Proposition *Let (A,Δ) be a $*$-aqg. Then there exists a linear map $\tilde{m} : A \otimes A \to A$ which satisfies $\tilde{m}(\Delta(a)x) = a\tilde{m}(x)$ for all $a \in A,\ x \in A \otimes A$ and $\tilde{m}(\tilde{m} \otimes \mathrm{id}) = \tilde{m}(\mathrm{id} \otimes \tilde{m})$.*

The former condition means that $\tilde{m}$ is a morphism $\Gamma \otimes \Gamma \to \Gamma$ in $\mathrm{Rep}(A,\Delta)$ and the latter that $(\Gamma, \tilde{m})$ is a semigroup in $\mathrm{Rep}(A,\Delta)$. If (A,Δ) is discrete then $c \mapsto c\Lambda$ defines a morphism $\eta \in \mathrm{Hom}(\varepsilon, \Gamma)$ and one verifies that $(\Gamma, \tilde{m}, \eta)$ is a monoid in the tensor category $\mathrm{Rep}(A,\Delta)$. Furthermore, we have $\Delta(A) \subset A \otimes A$ iff (A,Δ) is compact, equivalently A is unital. In that case we find $\Delta \in \mathrm{Hom}(\Gamma, \Gamma \otimes \Gamma)$ and $(\Gamma, \Delta, \varepsilon)$ is a comonoid in $\mathrm{Rep}(A,\Delta)$. The monoid and comonoid structures exist both iff (A,Δ) is finite dimensional, in which case they coincide with those in Subsection 1.1.3. We summarize the part that will be relevant in the next subsection:

1.2.9 Theorem *Let (A,Δ) be a discrete $*$-aqg with left regular representation Γ. Then there exists a monoid $(\Gamma, \tilde{m}, \eta)$ in $\mathrm{Rep}(A,\Delta)$ where Γ has the absorbing property $\Gamma \otimes \theta \cong \Gamma \otimes I_\theta \cong \dim V_\theta \Gamma$ and satisfies $\dim \mathrm{Hom}(\varepsilon, \Gamma) = 1$. If (A,Δ) is cocommutative (thus $\mathrm{Rep}_f(A,\Delta) \simeq \mathrm{Rep}_f G$ for a compact group G) then $\mathrm{Rep}(A,\Delta)$ is symmetric and the monoid is commutative: $\tilde{m} \circ c_{\Gamma,\Gamma} = \tilde{m}$.*

Again the results generalize to (semisimple discrete) aqg without $*$-operation.

1.3 Symmetric Tensor Categories: Abstract Reconstruction

The Tannaka theory of Woronowicz [74] as well as the one of {8} described above rely on the existence of a fiber functor, i.e. a faithful tensor functor $E : \mathcal{C} \to \mathrm{Vect}_k$. A tensor category, usually but not always assumed symmetric, together with a fiber functor is called tannakian (or concrete or embedded). In the general formalism, the braiding of $\mathcal{C}$, if it exists, plays a minor role. It just corresponds to an R-matrix for (A,Δ). If both $\mathcal{C}$ and E are symmetric then cocommutativity of (A,Δ) and thus $\mathrm{Rep}_f(A,\Delta) \simeq \mathrm{Rep}_f G$ follow.

The works [23, 15] go beyond Tannakian reconstruction in that they characterize classes of symmetric tensor categories that admit a fiber functor and actually construct the latter. Here the symmetry plays a very crucial rôle. The frameworks in which the cited works are located are somewhat different. Working with C^*-tensor categories with unitary symmetry and assuming that all objects have conjugates and twist $+1$, Doplicher and Roberts [23] first prove that the dimension of every object is in $\mathbb{N}$. They then construct a compact group G and an equivalence $F : \mathcal{C} \to \mathrm{Rep}_f G$. (Finally they generalize to the case including twists -1, giving rise to a supergroup.) Deligne [15] considers k-linear symmetric abelian tensor categories where k is an algebraically closed field of characteristic zero. *Assuming* that the dimensions of all objects are in $\mathbb{N}$ he constructs a fiber functor E and then appeals to the Tannaka-type theorem of Saavedra Rivano, as completed by Deligne and Milne, to conclude that $\mathcal{C}$ is equivalent to the category of representations of an algebraic group scheme. We begin with a brief description of the idea of his proof, which is closer in spirit to the present chapter.

We assume throughout that k is algebraically closed of characteristic zero, and that $\mathcal{C}$ is k-linear symmetric. We also require $\mathcal{C}$ to have a generator, i.e. an object Y such that every $X \in \mathcal{C}$ is a direct summand of Y^n for some $n \in \mathbb{N}$, and that $\mathcal{C}$ be even, i.e. all objects have twist $+1$. (Concerning the generalizations to categories without generators and/or allowing objects with twist -1 we have nothing new to say, cf. [23, 15, 16].) The basic idea in [15] is to embed $\mathcal{C}$ into the associated category $\hat{\mathcal{C}} = \mathrm{Ind}\,\mathcal{C}$ and to construct a commutative monoid (Γ, m, η) in $\hat{\mathcal{C}}$ such that $\Gamma \otimes X \cong d(X)\Gamma$ for all $X \in \mathcal{C}$. The k-vector space $K = \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathbf{1}, \Gamma)$ is a commutative algebra w.r.t. the multiplication $(a, b) \mapsto m \circ a \otimes b$ and the unit $1_K = \eta$. Defining a functor $E_K : \mathcal{C} \to \mathrm{Vect}_k$ by

$$\begin{aligned} E_K(X) &= \mathrm{Hom}_{\hat{\mathcal{C}}}(\mathbf{1}, \Gamma \otimes X), \\ E_K(s)\phi &= \mathrm{id} \otimes s \circ \phi, \quad s : X \to Y,\ \phi \in E_K(X), \end{aligned}$$

one finds that $E_K(X)$ is a K-module for every X, the K-action being given by

$$K \times E_K(X) \to E_K(X), \quad (a, \phi) \mapsto m \otimes \mathrm{id}_X \circ a \otimes \phi.$$

In fact, E_K is a tensor functor $\mathcal{C} \to K-\mathrm{Mod}$. Picking a maximal ideal M in K, we can construct a tensor functor $E_{\hat{k}} : \mathcal{C} \to \hat{k}-\mathrm{Mod}$, where $\hat{k} = K/M$ is a field extension of k. The latter can be shown to be algebraic. Thus if k is algebraically closed, $E_{\hat{k}}$ is a fiber functor into Vect_k. Now the classic result of Saavedra Rivano implies that there exists an algebraic group G and an equivalence $F : \mathcal{C} \to \mathrm{Rep}_f G$ such that $E = K \circ F$, where $K : \mathrm{Rep}_f G \to \mathrm{Vect}_k$ is the forgetful functor. On the other hand, if $\mathcal{C}$ is a $*$-category one may apply an argument in [8] to obtain a fiber functor that is $*$-preserving, and then Theorem 1.2.6 implies an analogous result for a compact Lie group G.

The construction as described above is not entirely satisfactory. Once the category $\mathcal{C}$ is shown to be equivalent to $\mathrm{Rep}_f G$, the discussion in Subsection 1.2.3 implies that $\mathrm{Ind}\,\mathcal{C} \simeq \mathrm{Rep}(A, \Delta)$ contains a commutative monoid (Γ, m, η) which has the absorbing property and contains the tensor unit as direct summand with multiplicity one. If one applies the above prescription to *this* monoid to define a fiber functor E, the latter property implies $\dim_k K = 1$, thus E automatically is a fiber functor into Vect_k. Defining $G = \mathrm{Aut}^{\otimes} E$, following Saavedra Rivano, one finds easily

$$G \cong \mathrm{Aut}(\Gamma, m, \eta) := \{g \in \mathrm{Aut}\Gamma \mid g \circ m = m \circ g \otimes g,\ g \circ \eta = \eta\}.$$

Thus the group G for which $\mathcal{C} \simeq \mathrm{Rep}_f G$ holds can be read off the regular monoid (Γ, m, η) without considering the fiber functor that the latter gives rise to. Therefore it would be highly desirable to directly construct the regular monoid in $\hat{\mathcal{C}}$. (The monoid constructed in [15] fails to satisfy $\dim \mathrm{Hom}(\mathbf{1}, \Gamma) = 1$, thus it is at best an uncontrolled multiple of the regular monoid.) This is done in {12, Section 3}, where we limit ourselves to the semisimple case and use simplifications of Deligne's work in [7, 64].

1.3.1 THEOREM *Let k be algebraically closed of characteristic zero and let $\mathcal{C}$ be a k-linear semisimple rigid symmetric tensor category with* $\mathrm{End}\,\mathbf{1} = k\mathrm{id}_{\mathbf{1}}$ *and such that $d(X) \in \mathbb{N}$ for all $X \in \mathcal{C}$. Then*

1. *There exists a commutative monoid (Γ, m, η) in $\mathrm{Ind}\,\mathcal{C}$ such that $\Gamma \otimes X \cong d(X)\Gamma$ for all $X \in \mathcal{C}$ and* $\dim \mathrm{Hom}(\mathbf{1}, \Gamma) = 1$.

2. *The group $G = \mathrm{Aut}(\Gamma, m, \eta)$ is proalgebraic and $\mathcal{C} \simeq \mathrm{Rep}_f G$.*

3. *If $\mathcal{C}$ is a $*$-category then G is a compact topological group.*

The new result here is 1, from which the statements 2 and 3 follow as explained above. (Of course, 1 also follows from 2 or 3 via the results from Subsection 1.2.3, but our proof is much more direct and purely categorical.) We briefly sketch the construction. For every object $X \in \mathcal{C}$ and every $n \in \mathbb{N}$, the symmetry gives rise to a homomorphism $\pi_n^X : \Sigma_n \to \mathrm{End}X^n$, and the idempotents

$$P_n^{\pm}(X) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \left\{ \begin{array}{c} 1 \\ \mathrm{sgn}(\sigma) \end{array} \right\} \pi_X^n(\sigma)$$

project onto subobjects $S_n(X)$ and $A_n(X)$, respectively, of X^n. In particular, $d(A_{d(X)}(X)) = 1$ and $A_{d(X)} \cong \mathbf{1}$ if $X \cong \overline{X}$. The object $S(X) = \oplus_{n\in\mathbb{Z}_+} S_n(X)$ is easily seen to give rise to a commutative monoid $(S(X), m, \eta)$ in $\hat{\mathcal{C}}$, the symmetric algebra over X.

For any commutative monoid (Γ, m, η) in a k-linear tensor category $\mathcal{D}$, (Γ, m) is a (left) Γ-module. An ideal in (Γ, m, η) consists of $(X, \mu) \in \Gamma-\mathrm{Mod}$ and a monomorphism $\iota \in \mathrm{Hom}_{\Gamma-\mathrm{Mod}}((X, \mu), (\Gamma, m))$. Considering the cokernel $p : (\Gamma, m) \to (\Gamma', m_1)$ of ι, there exists a commutative monoid (Γ', m', η') in $\mathcal{D}$ such that $p \circ m = m \circ p \otimes p, p \circ \eta = \eta'$. This monoid is considered the quotient of (Γ, m, η) by the ideal $((X, \mu), \iota)$. One has a notion of maximal ideals, and as in commutative algebra one shows that every ideal is contained in a maximal ideal and that the commutative quotient monoid (Γ', m', η') is a 'field'. The latter now means that the commutative k-algebra $\mathrm{Hom}(\mathbf{1}, \Gamma')$ is a field, thus an extension of k.

Let Z be a generator for $\mathcal{C}$ satisfying $\overline{Z} \cong Z$, and consider the commutative monoid $(S(Z), m, \eta)^{\times d}$, where $d = d(Z)$. Using ideas from [7], see also [64], one finds an ideal $((X, \mu), \iota)$ in $(S(Z), m, \eta)^{\times d}$ such that the quotient of $(S(Z), m, \eta)^{\times d}$ by that ideal (or any ideal containing the latter) has the absorbing property for all $X \in \mathcal{C}$. Quotienting by a maximal ideal containing $((X, \mu), \iota)$ we thus obtain a commutative absorbing monoid in $\hat{\mathcal{C}}$ where $K = \mathrm{Hom}(\mathbf{1}, \Gamma)$ is a field extension of k. Showing that the latter is algebraic, thus trivial, completes the construction of a monoid with all desired properties. Now the previous considerations show that $\mathcal{C} \simeq \mathrm{Rep}_f G$ where G is the automorphism group of this monoid, and the uniqueness of fiber functors proves that this monoid is in fact the one arising from the regular representation. In particular, one concludes $\Gamma \cong \oplus_i d(X_i) X_i$. (Note that it seems hopeless to directly define a commutative monoid structure on this object Γ.)

1.4 Structure of Modular Categories

1.4.1 Preliminaries

Adopting the perspective of 'categorification' [3] it is not surprising that on the level of (1-)categories we find analogues of notions that are familiar from 0-categories, i.e. sets, monoids etc. In particular, we have centralizers and centers:

1.4.1 Definition *Let $\mathcal{C}$ be a braided tensor category and $\mathcal{K}$ a set of objects in $\mathcal{C}$, equivalently, a full subcategory of $\mathcal{C}$. The centralizer $C_{\mathcal{C}}(\mathcal{K})$ of $\mathcal{K}$ in $\mathcal{C}$ (or relative commutant $\mathcal{C} \cap \mathcal{K}'$) is the full subcategory defined by*

$$\mathrm{Obj}\, C_{\mathcal{C}}(\mathcal{K}) = \{X \in \mathcal{C} \mid c(Y, X) \circ c(X, Y) = \mathrm{id}_{XY} \quad \forall Y \in \mathcal{K}\}.$$

1.4.2 Definition *The center of a braided category $\mathcal{C}$ is defined as $Z_2(\mathcal{C}) = C_{\mathcal{C}}(\mathcal{C})$. It clearly is a symmetric category. We say a semisimple braided tensor category has trivial center, if every object of $Z_2(\mathcal{C})$ is a direct sum of copies of the tensor unit $\mathbf{1}$ or if, equivalently, every simple object in $Z_2(\mathcal{C})$ is isomorphic to $\mathbf{1}$.*

1.4.3 Lemma *Let $\mathcal{C}$ be braided monoidal and $\mathcal{K}$ a subcategory. Then $C_{\mathcal{C}}(\mathcal{K})$ is replete, monoidal and closed w.r.t. direct sums and retractions in $\mathcal{C}$. If $\mathcal{C}$ is k-linear semisimple then the same holds for $C_{\mathcal{C}}(\mathcal{K})$. If $\mathcal{C}$ has duals for all objects then also $C_{\mathcal{C}}(\mathcal{K})$ has duals. If $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{C}$ then $C_{\mathcal{C}}(\mathcal{K}_1 \vee \mathcal{K}_2) = C_{\mathcal{C}}(\mathcal{K}_1) \cap C_{\mathcal{C}}(\mathcal{K}_2)$, where $\mathcal{K}_1 \vee \mathcal{K}_2 \subset \mathcal{C}$ is the smallest full monoidal subcategory containing $\mathcal{K}_1, \mathcal{K}_2$.*

Now we restrict our attention to semisimple categories with finitely many classes of simple objects.

1.4.4 Definition *A premodular category [11] is a finite braided fusion category. A modular category [67] is a premodular category where the matrix $(S(X, Y)) = Tr_{XY}(c(Y, X) \circ c(X, Y))$, indexed by isomorphism classes of simple objects, is invertible.*

The following was proven by Rehren [59] for $*$-categories and in full generality by Bruguières and others, cf. [6].

1.4.5 Theorem *Let $\mathcal{C}$ be pre-modular with $\dim \mathcal{C} \neq 0$. Then the following are equivalent:*

(i) The center $Z_2(\mathcal{C})$ is trivial.

(ii) $\mathcal{C}$ is modular.

The importance of modular categories is due to the fact that they serve as input in the construction of certain topological quantum field theories in 2+1 dimensions, cf. [67]. On the other hand, they arise from quantum groups at roots of unity [69] and in conformal field theory {3}. The latter situation will be discussed in Subsection 2.1.2.

1.4.2 Modular Categories: Double Centralizer Theorem and Prime Factorization

In {7} I argued that modular categories appear in sufficiently many different contexts to deserve some attention for their own sake, and I took some steps towards a structure theory of modular categories. All results in the remainder of this subsection are from {7}. All subcategories considered here are replete full tensor subcategories closed w.r.t. direct sums, retractions and duals. The crucial ingredient is the following 'double centralizer theorem' (DCT).

1.4.6 Theorem *Let $\mathcal{C}$ be a modular category and let $\mathcal{K} \subset \mathcal{C}$. Then we have*

(i) $C_\mathcal{C}(C_\mathcal{C}(\mathcal{K})) = \mathcal{K}$.

(ii) $\dim \mathcal{K} \cdot \dim C_\mathcal{C}(\mathcal{K}) = \dim \mathcal{C}$.

1.4.7 Corollary *Let $\mathcal{C}$ be a modular category and let $\mathcal{K} \subset \mathcal{C}$. Then*

$$Z_2(C_\mathcal{C}(\mathcal{K})) = Z_2(\mathcal{K}).$$

In particular, if $\mathcal{K}$ is modular then so is $C_\mathcal{C}(\mathcal{K})$. If $\mathcal{K}$ is symmetric then $Z_2(C_\mathcal{C}(\mathcal{K})) = \mathcal{K}$.

1.4.8 Proposition *Let $\mathcal{C}$ and $\mathcal{K}$ be modular categories where $\mathcal{K}$ is identified with a full (tensor) subcategory of $\mathcal{C}$. Let $\mathcal{L} = C_\mathcal{C}(\mathcal{K})$. Then there is an equivalence of modular categories:*

$$\mathcal{C} \simeq \mathcal{K} \boxtimes \mathcal{L}.$$

(Here $\mathcal{K} \boxtimes \mathcal{L}$ is the completion w.r.t. direct sums of the tensor product $\mathcal{K} \otimes_k \mathcal{L}$, whose objects are pairs of objects and whose hom-spaces are tensor products over k of the respective hom-spaces in $\mathcal{K}, \mathcal{L}$.)

1.4.9 Definition *A modular category $\mathcal{C}$ is prime if every modular subcategory is equivalent either to $\mathcal{C}$ or the trivial modular category $Vect_k$.*

1.4.10 Theorem *Every modular category is equivalent to a finite direct product of prime ones.*

The theorem implies that it suffices to classify prime modular categories. Note however that the factorization of a modular category into prime ones is not necessarily unique. Examples for this non-uniqueness will be presented below.

1.4.3 Examples and Applications

A natural class of modular categories to consider is provided by quantum doubles of finite groups. Whenever $G = G_1 \times G_2$ we have $D(G)-\mathrm{Mod} \simeq D(G_1)-\mathrm{Mod} \boxtimes D(G_1)-\mathrm{Mod}$. It is thus sufficient to consider prime groups. The abelian case, where G is prime iff it is cyclic of prime power order, has been worked out completely:

1.4.11 Theorem *Let p be prime, $G = \mathbb{Z}/p^n\mathbb{Z}$ and $\mathcal{C} = D(G)-\mathrm{Mod}$.*

(i) $\mathcal{C}$ is prime iff $p = 2$.

(ii) If p is odd, there is a one-to-one correspondence between isomorphisms $\alpha : G \to \hat{G}$ and modular subcategories $\mathcal{K}_\alpha \subset \mathcal{C}$ given by $K_\alpha = \{(g, \alpha(g)),\ g \in G\}$. (Recall that the iso-classes of simple objects in $D(G) - \mathrm{Mod}$ are labeled by elements of $G \times \hat{G}$.) The categories $\mathcal{K}_\alpha$ are prime, and $C_\mathcal{C}(\mathcal{C}_\alpha) = \mathcal{C}_{\overline{\alpha}}$, where $\overline{\alpha}(\cdot) = \alpha(\cdot)^{-1}$. The prime factorizations of $\mathcal{C}$ are thus given by $\mathcal{C} \simeq \mathcal{C}_\alpha \boxtimes \mathcal{C}_{\overline{\alpha}},\ \alpha \in Isom(G, \hat{G})$.

This result implies, already for finite abelian groups, that there is no correlation between simplicity of G and primality of $D(G)-\mathrm{Mod} \simeq Z_2(\mathrm{Rep}\, G)$!

For unitary modular categories (i.e. $*$-categories) the following is an easy consequence of the DCT:

1.4.12 Proposition *Let $\mathcal{C}$ be a unitary modular category and $\mathcal{K} \subset \mathcal{C}$. Then*

$$\dim \mathcal{C} \geq \dim \mathcal{K} \cdot \dim Z_2(\mathcal{K}).$$

Equality holds iff $C_\mathcal{C}(\mathcal{K}) = C_\mathcal{K}(\mathcal{K})(= Z_2(\mathcal{K}))$, in which case we say that $\mathcal{C}$ is a minimal modular full extension of $\mathcal{K}$.

1.4.13 Example Consider $\mathcal{S} \subset \mathcal{C}$ where $\mathcal{C}$ is modular and $\mathcal{S}$ is symmetric. With $\mathcal{K} = C_\mathcal{C}(\mathcal{S})$ we have $Z_2(\mathcal{K}) = Z_2(\mathcal{S}) = \mathcal{S}$. Thus $\mathcal{C}$ is a minimal modular full extension of $\mathcal{K}$. By the DCT this situation is generic: If $\mathcal{C} \supset \mathcal{K}$ is a minimal modular extension and we set $\mathcal{S} = Z_2(\mathcal{K})$ then $\mathcal{K} = C_\mathcal{C}(\mathcal{S})$. □

It is natural to expect that this bound is optimal:

1.4.14 Conjecture *Let $\mathcal{K}$ be a unitary pre-modular category. Then there exists a unitary modular category $\mathcal{C}$ and a full and faithful tensor functor $I : \mathcal{K} \to \mathcal{C}$ such that*

$$\dim \mathcal{C} = \dim \mathcal{K} \cdot\ Z_2(\mathcal{K}).$$

A direct approach to the proof does not seem promising. See Remark 1.6.20 for a possible approach.

1.5 The Center Z_1 of a Fusion Category

1.5.1 Definition and Basic Properties

It seems desirable to have means to produce modular categories that are simpler than those via quantum groups at roots of unity and rational conformal field theories. (We note in passing that these two subjects are closely related.) The approach of modularization {2},[11] will be mentioned in Subsection 1.6.4.

It has long been known that the quantum doubles $D(G)$ of finite groups have modular representation categories, cf. e.g. [2]. The latter construction has a generalization to arbitrary tensor categories, independently discovered by Drinfel'd (unpublished), Majid [51] and Joyal and Street [38]. See also [41].

1.5.1 DEFINITION/PROPOSITION *The center $Z_1(\mathcal{C})$ of a strict monoidal category $\mathcal{C}$ has as objects pairs (X, e_X), where $X \in \mathcal{C}$ and e_X is a half braiding, i.e. a family $\{e_X(Y) \in \mathrm{Hom}_{\mathcal{C}}(XY, YX),\ Y \in \mathcal{C}\}$ of isomorphisms, natural in Y and satisfying the braid relation*

$$e_X(Y \otimes Z) = \mathrm{id}_Y \otimes e_X(Z) \circ e_X(Y) \otimes \mathrm{id}_Z \quad \forall Y, Z \in \mathcal{C}.$$

The morphisms are given by

$$\mathrm{Hom}_{Z_1(\mathcal{C})}((X, e_X), (Y, e_Y) = \{t \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid \quad \mathrm{id}_X \otimes t \circ e_X(Z) = e_Y(Z) \circ t \otimes \mathrm{id}_X \quad \forall Z \in \mathcal{C}\}.$$

The tensor product of objects is given by $(X, e_X) \otimes (Y, e_Y) = (XY, e_{XY})$, where

$$e_{XY}(Z) = e_X(Z) \otimes \mathrm{id}_Y \circ \mathrm{id}_X \otimes e_Y(Z).$$

The tensor unit is $(\mathbf{1}, e_{\mathbf{1}})$ where $e_{\mathbf{1}}(X) = \mathrm{id}_X$. The composition and tensor product of morphisms are inherited from $\mathcal{C}$. With

$$c_{(X,e_X),(Y,e_Y)} = e_X(Y)$$

$Z_1(\mathcal{C})$ is a braided tensor category.

The following are some quite trivial observations.

1.5.2 LEMMA *If $\mathcal{C}$ is k-linear then so is $Z_1(\mathcal{C})$. If the unit $\mathbf{1}$ of $\mathcal{C}$ is absolutely simple, then $\mathbf{1}_{Z_1(\mathcal{C})}$ is absolutely simple. If $\mathcal{C}$ has direct sums then also $Z_1(\mathcal{C})$ has direct sums. If $\mathcal{C}$ has subobjects then also $Z_1(\mathcal{C})$ has subobjects.*

1.5.3 PROPOSITION *Let $\mathcal{C}$ be (strict) pivotal. Then also $Z_1(\mathcal{C})$ is (strict) pivotal, the dual $\overline{(Y, e_Y)}$ being given by $(\overline{Y}, e_{\overline{Y}})$, where $e_{\overline{Y}}(X)$ is defined by*

$$\overline{Y} \otimes X \xrightarrow{\mathrm{id}_{\overline{Y}X} \otimes \varepsilon(Y)} \overline{Y} \otimes X \otimes Y \otimes \overline{Y} \xrightarrow{\mathrm{id}_{\overline{Y}} \otimes e_Y(X)^{-1} \otimes \mathrm{id}_{\overline{Y}}} \overline{Y} \otimes Y \otimes X \otimes \overline{Y} \longrightarrow$$
$$\xrightarrow{\overline{\varepsilon}(\overline{Y}) \otimes \mathrm{id}_{X\overline{Y}}} X \otimes \overline{Y}$$

The evaluation and coevaluation maps are inherited from $\mathcal{C}$:

$$\varepsilon((Y, e_Y)) = \varepsilon(Y), \quad \overline{\varepsilon}((Y, e_Y)) = \overline{\varepsilon}(Y).$$

If $\mathcal{C}$ is spherical then also $Z_1(\mathcal{C})$ is spherical.

The question of semisimplicity of $Z_1(\mathcal{C})$ is less obvious. We can answer it only assuming that $\mathcal{C}$ is a finite fusion category with non-zero dimension. The following and the results in the next subsection are taken from {6}.

1.5.4 THEOREM *Let k be algebraically closed and $\mathcal{C}$ a finite fusion category over k with $\dim \mathcal{C} \neq 0$. Then the quantum double $Z_1(\mathcal{C})$ is spherical and semisimple.*

1.5.2 Morita Equivalence $Z_1(\mathcal{C}) \approx \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$ and Modularity

So far we know that $Z_1(\mathcal{C})$ is semisimple if $\mathcal{C}$ is a finite fusion category. In order to obtain results about the 'size' of $Z_1(\mathcal{C})$ we need the more sophisticated machinery described in Subsection 1.1.2. We define

$$\mathcal{A} = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}},$$
$$\hat{X}_i = X_i \boxtimes X_i^{\mathrm{op}} \in \mathrm{Obj}\,\mathcal{A}.$$

By {5, Lemma 2.9}, $\mathcal{C}^{\mathrm{op}}, \mathcal{C} \otimes_k \mathcal{C}^{\mathrm{op}}$ and $\mathcal{A}$ are strict spherical in a canonical way. Every $\hat{X}_i,\ i \in \Gamma$ is simple.

The following is a very slight generalization of [47, Proposition 4.10].

1.5.5 LEMMA *Let k be quadratically closed and let $\mathcal{C}$ be a finite fusion category over k with* $\dim \mathcal{C} \neq 0$. *There is a normalized strongly separable Frobenius algebra* $(\Gamma, m, \eta, \Delta, \varepsilon))$ *in* $\mathcal{A} = \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{op}}$ *(with* $\alpha = \beta$*) such that*

$$\Gamma \cong \bigoplus_{i \in \Gamma} \hat{X}_i.$$

1.5.6 PROPOSITION *Let $\mathcal{E}$ be the Morita context arising from the Frobenius algebra* $(\Gamma, \ldots)$ *in the tensor category $\mathcal{A}$, and let* $\mathcal{B} = \mathrm{End}_{\mathcal{E}}(\mathfrak{B})$. *There exists an equivalence* $\mathcal{B} \simeq Z_1(\mathcal{C})$. *Thus* $\dim Z_1(\mathcal{C}) = (\dim \mathcal{C})^2$.

Since we have fairly good control over the bicategory $\mathcal{E}$, the same follows for $Z_1(\mathcal{C})$. For example we can prove that the number of isomorphism classes of simple objects in $Z_1(\mathcal{C})$ is bounded by $\sum_{i,j} \dim \mathrm{Hom}_{\mathcal{C}}(X_i X_j, X_j X_i)$. In the case where $\mathcal{C} = \mathrm{Rep}\, G$ for G finite abelian this bound is attained. Furthermore,

1.5.7 THEOREM *Let k be an algebraically closed field and $\mathcal{C}$ a finite fusion category over k with* $\dim \mathcal{C} \neq 0$. *Then also the center $Z_1(\mathcal{C})$ has all these properties and is a modular category [67]. Furthermore, the dimension and the Gauss sums* $\Delta_{\pm}(\mathcal{D}) = \sum_i \theta(X_i)^{\pm 1} d(X_i)^2$ *are given by*

$$\begin{aligned} \dim Z_1(\mathcal{C}) &= (\dim \mathcal{C})^2, \\ \Delta_{\pm}(Z_1(\mathcal{C})) &= \dim \mathcal{C}. \end{aligned}$$

If $\mathcal{C}$ has a braiding c then $\mathcal{C}$ can be considered as a full subcategory of $Z_1(\mathcal{C})$ via the fully faithful braided tensor functor $F_1 : X \mapsto (X, e_X)$, where $e_X(Y) = c_{X,Y}$. If $\tilde{\mathcal{C}}$ denotes the tensor category $\mathcal{C}$ with the braiding $\tilde{c}_{X,Y} = c(Y, X)^{-1}$ then there exists a similar braided tensor functor $F_2 : \tilde{\mathcal{C}} \to Z_1(\mathcal{C})$ such that $C_{Z_1(\mathcal{C})}(F_1(\mathcal{C})) = F_2(\tilde{\mathcal{C}})$ and $C_{Z_1(\mathcal{C})}(F_2(\tilde{\mathcal{C}})) = F_1(\mathcal{C})$. (This holds in complete generality, independently of the DCT.) If now $\mathcal{C}$ satisfies the assumptions of Theorem 1.5.7 then modularity of $Z_1(\mathcal{C})$ and Proposition 1.4.8 imply

1.5.8 THEOREM *If $\mathcal{C}$ is modular then* $Z_1(\mathcal{C}) \simeq \mathcal{C} \boxtimes \tilde{\mathcal{C}}$ *as modular categories.*

In particular, every modular category appears as direct factor of $Z_1(\mathcal{C})$ for some fusion category $\mathcal{C}$.

In view of Theorem 1.1.13, the weak monoidal Morita equivalence $Z_1(\mathcal{C}) \approx \mathcal{C} \boxtimes \tilde{\mathcal{C}}$ implies

$$TV(M, Z_1(\mathcal{C})) = BW(M, Z_1(\mathcal{C})) = BW(M, \mathcal{C} \boxtimes \tilde{\mathcal{C}}) = BW(M, \mathcal{C}) \cdot BW(-M, \mathcal{C})$$

for every fusion category.

One can show {6} that the number of simple objects in $Z_1(\mathcal{C})$ coincides with the dimension of the state space $H_{S^1 \times S^1}$ of the torus in the $d = 2 + 1$ TQFT defined by triangulation, as conjectured in [30]. On the other hand, in the surgery TQFT RT [67] based on a modular category $\mathcal{D}$, $\dim H_{S^1 \times S^1}$ coincides with the number of simple objects. This provides support to the *conjecture* that $BW(M, \mathcal{C}) = RT(M, Z_1(\mathcal{C}))$ for every oriented closed 3-manifold M and every unimodal finite fusion category $\mathcal{C}$. In a formalism involving subfactos it has recently been shown [42] that the surgery TQFT associated with Ocneanu's asymptotic subfactor coincides with the triangulation TQFT [55] of the original (finite index and depth) subfactor. In view of the relation [34],{6} between the asymptotic subfactor and $Z_1(\mathcal{C})$ it seems to be a simple exercise to give a proof of the conjecture involving only topological and categorical arguments.

1.6 Galois Theory of Braided Tensor Categories

1.6.1 Tensor Categories of Modules

On several occasions we have considered (bi)module categories of a monoid in a tensor category $\mathcal{C}$. In the absence of a braiding on $\mathcal{C}$ only $\Gamma-\mathrm{Mod}-\Gamma$ is monoidal. For symmetric $\mathcal{C}$ it has long been known that $\Gamma-\mathrm{Mod}$ is again symmetric monoidal, cf. e.g. [15]. The braided case was considered in [57]:

1.6.1 Definition/Proposition *If $\mathcal{C}$ is braided and has coequalizers then Γ–Mod is a tensor category with $(X,\mu)\otimes(Y,\eta)=\mathrm{coeq}(\alpha,\beta)$ where $\alpha,\beta:\Gamma\otimes X\otimes Y\to X\otimes Y$ are given by*

$$\alpha=\mu\otimes\mathrm{id}_Y,\qquad \beta=\mathrm{id}_X\otimes\eta\,\circ\, c_{\Gamma,X}\otimes\mathrm{id}_Y.$$

There is a tensor functor $F:\mathcal{C}\to\Gamma-\mathrm{Mod}_\mathcal{C}$ such that $F(X)=(\Gamma\otimes X, m\otimes\mathrm{id}_X)$. The full subcategory $\Gamma-\mathrm{Mod}^0_\mathcal{C}\subset\Gamma-\mathrm{Mod}_\mathcal{C}$ consisting of the objects (X,μ) satisfying $\mu\circ c_{X,\Gamma}\circ c_{\Gamma,X}=\mu$ is monoidal and braided.

These results were rediscovered in [11] for the special case where $\Gamma\in Z_2(\mathcal{C})$, implying $\Gamma-\mathrm{Mod}^0_\mathcal{C}=\Gamma-\mathrm{Mod}_\mathcal{C}$, and in generality in [43]. The material in the remainder of this subsection is from {9}.

1.6.2 Proposition *Let $\mathcal{C}$ be a finite braided fusion category and let $(\Gamma,m,\eta,\Delta,\varepsilon)$ be a strongly separable Frobenius algebra in $\mathcal{C}$ satisfying $\dim\mathrm{Hom}(\mathbf{1},\Gamma)=1$. Then $\Gamma-\mathrm{Mod}_\mathcal{C}$ is a semisimple k-linear spherical tensor category with $\mathrm{End}_\Gamma\mathbf{1}=k\mathrm{id}_\mathbf{1}$, and*

$$\dim\Gamma-\mathrm{Mod}_\mathcal{C}=(\dim\Gamma)^{-1}\dim\mathcal{C}.$$

A certain nuisance of the tensor category $\Gamma-\mathrm{Mod}$ is the somewhat implicit definition of the tensor product. Therefore the following alternative description is useful:

1.6.3 Definition/Proposition *Let $\mathcal{C}$ be a strict braided fusion category and $(\Gamma,m,\eta,\Delta,\varepsilon)$ a strongly separable Frobenius algebra in $\mathcal{C}$. Then the following defines a tensor category $\tilde{\mathcal{C}}_\Gamma$.*

- $\mathrm{Obj}\tilde{\mathcal{C}}_\Gamma=\mathrm{Obj}\,\mathcal{C}$.
- $X\tilde{\otimes}Y=X\otimes Y$.
- $\mathrm{Hom}_{\tilde{\mathcal{C}}_\Gamma}(X,Y)=\mathrm{Hom}_\mathcal{C}(\Gamma X,Y)$.
- *Let $s\in\mathrm{Hom}_{\tilde{\mathcal{C}}_\Gamma}(X,Y)=\mathrm{Hom}_\mathcal{C}(\Gamma X,Y)$ and $t\in\mathrm{Hom}_{\tilde{\mathcal{C}}_\Gamma}(Y,Z)=\mathrm{Hom}_\mathcal{C}(\Gamma Y,Z)$. Then $t\tilde{\circ}s=t\circ\mathrm{id}_\Gamma\otimes s\circ\Delta\otimes X$ in $\mathrm{Hom}_{\tilde{\mathcal{C}}_\Gamma}(X,Z)=\mathrm{Hom}_\mathcal{C}(\Gamma X,Z)$.*
- *Let $s\in\mathrm{Hom}_{\tilde{\mathcal{C}}_\Gamma}(X,Y)=\mathrm{Hom}_\mathcal{C}(\Gamma X,Y)$ and $t\in\mathrm{Hom}_{\tilde{\mathcal{C}}_\Gamma}(Z,T)=\mathrm{Hom}_\mathcal{C}(\Gamma Z,T)$. Then $s\tilde{\otimes}t=s\otimes t\,\circ\,\mathrm{id}_\Gamma\otimes c_{\Gamma,X}\otimes\mathrm{id}_Z\,\circ\,\Delta\otimes\mathrm{id}_X\otimes\mathrm{id}_Z$ in $\mathrm{Hom}_{\tilde{\mathcal{C}}_\Gamma}(XZ,YT)=\mathrm{Hom}_\mathcal{C}(\Gamma XZ,YT)$.*

The canonical completion $\hat{\mathcal{C}}_\Gamma=\tilde{\mathcal{C}}_\Gamma^p$ of $\tilde{\mathcal{C}}_\Gamma$ to a category with splitting idempotents is semisimple. (Recall that $\mathrm{Obj}\,\hat{\mathcal{C}}_\Gamma=\{(X,p),\ X\in\mathrm{Obj}\tilde{\mathcal{C}}_\Gamma,\ p=p^2\in\mathrm{End}_{\tilde{\mathcal{C}}_\Gamma}X\}$ etc. Instead of $(X,\mathrm{id}_X)\in\hat{\mathcal{C}}_\Gamma$ we simply write X.) If $\mathcal{C}$ is a $*$-category and $\Delta=m^*,\varepsilon=\eta^*$ then $\tilde{\mathcal{C}}_\Gamma,\hat{\mathcal{C}}_\Gamma$ are $*$-categories. The functor $\iota:\mathcal{C}\to\tilde{\mathcal{C}}_\Gamma$ given by $X\mapsto X,\ s\mapsto\varepsilon\otimes s$ is monoidal and faithful. The composite of ι with the full embedding $\tilde{\mathcal{C}}_\Gamma\to\hat{\mathcal{C}}_\Gamma$ is also denoted by ι.

1.6.4 Proposition *Let $\mathcal{C}$ and $(\Gamma,m,\eta,\Delta,\varepsilon)$ be as before. Then there exists a monoidal equivalence $K:\hat{\mathcal{C}}_\Gamma\to\Gamma-\mathrm{Mod}_\mathcal{C}$ such that $K\circ\iota\cong F$ as tensor functors.*

1.6.2 Galois Extensions of Braided Tensor Categories

1.6.5 Definition *Let $\mathcal{C}$ be a strict braided fusion category and $\mathcal{S} \subset \mathcal{C}$ a finite full even symmetric fusion subcategory. Let $(\Gamma, \ldots)$ be the Frobenius algebra in $\mathcal{C}$ arising from Theorem 1.3.1 and Theorem 1.1.15. Then we write $\mathcal{C} \rtimes_0 \mathcal{S} := \tilde{\mathcal{C}}_\Gamma$ and $\mathcal{C} \rtimes \mathcal{S} := \hat{\mathcal{C}}_\Gamma$.*

1.6.6 Proposition *$\mathcal{C} \rtimes_0 \mathcal{S}$ and $\mathcal{C} \rtimes \mathcal{S}$ are strict spherical tensor categories and $\mathcal{C} \rtimes \mathcal{S}$ is semisimple. If $\mathcal{C}$ is a $*$-category then $\mathcal{C} \rtimes_0 \mathcal{S}$ and $\mathcal{C} \rtimes \mathcal{S}$ have $*$-structures extending that of $\mathcal{C}$. There exists a tensor functor $\iota : \mathcal{C} \to \mathcal{C} \rtimes \mathcal{S}$ which is faithful and injective on the objects. The group $G = \mathrm{Aut}(\Gamma, m, \eta)$ acts on $\mathcal{C} \rtimes \mathcal{S}$ via $\gamma_g(s) = s \circ g^{-1} \otimes \mathrm{id}_X$ for $s \in \mathrm{Hom}_{\mathcal{C}\rtimes\mathcal{S}}(X, Y) = \mathrm{Hom}(\Gamma X, Y)$ and $\gamma_g((X, p)) = (X, \gamma_g(p))$. We have $(\mathcal{C} \rtimes_0 \mathcal{S})^G \cong \mathcal{C}$ and $(\mathcal{C} \rtimes \mathcal{S})^G \simeq \mathcal{C}$. If $\mathcal{C}$ is finite then $\dim \mathcal{C} \rtimes \mathcal{S} = \dim \mathcal{C}/|G| = \dim \mathcal{C}/ \dim \mathcal{S}$.*

1.6.7 Theorem *Let $\mathcal{S} \subset \mathcal{C}$ be as before. The tensor functor $\iota : \mathcal{C} \to \mathcal{C} \rtimes \mathcal{S}$ has the following universal property.*

1. *For every simple object $Y \in \mathcal{C} \rtimes \mathcal{S}$ there exists $X \in \mathcal{C}$ such that Y is a direct summand of $Y \prec \iota(X)$.*
2. *For every $X \in \mathcal{S}$ we have $\iota(X) \cong d(X)\mathbf{1}$ in $\mathcal{C} \rtimes \mathcal{S}$.*
3. *If $\mathcal{D}$ is semisimple and $\iota' : \mathcal{C} \to \mathcal{D}$ satisfies 1-2 then there exists a tensor functor $\iota'' : \mathcal{C} \rtimes \mathcal{S} \to \mathcal{D}$, unique up to monoidal natural isomorphism, such that $\iota' = \iota'' \circ \iota$.*

We call $\mathcal{C} \rtimes \mathcal{S}$ the Galois extension of $\mathcal{C}$ by the full symmetric subcategory $\mathcal{S}$.
We now identify $\mathcal{C}$ with the subcategory $\iota(\mathcal{C})$ of $\mathcal{C} \rtimes \mathcal{S}$.

1.6.8 Theorem *Let $\mathcal{C}$ be a fusion category and $\mathcal{S} \subset \mathcal{C}$ a finite full even symmetric subcategory such that $\mathcal{S} \simeq \mathrm{Rep}\, G$. Then there is a one-to-one correspondence between subgroups $H \subset G$ and categories $\mathcal{E}$ such that $\mathcal{C} \subset \mathcal{E} \subset \mathcal{C} \rtimes \mathcal{S}$ and $\mathcal{E}$ is the completion w.r.t. subobjects of the category $(\mathcal{C} \rtimes_0 \mathcal{S}) \cap \mathcal{E}$. The correspondence is given by $\mathcal{E} = (\mathcal{C} \rtimes \mathcal{S})^H$ and $H = \mathrm{Aut}_{\mathcal{E}}(\mathcal{C} \rtimes \mathcal{S})$.*

1.6.9 Proposition *In the above correspondence the subgroup $H \subset G$ is normal iff there is a even symmetric subcategory $\mathcal{T} \subset \mathcal{S}$ such that $\mathcal{E} \cong \mathcal{C} \rtimes \mathcal{T}$. In this case $\mathrm{Aut}_{\mathcal{E}}(\mathcal{C} \rtimes \mathcal{S}) = H$ and $\mathrm{Aut}_{\mathcal{C}}(\mathcal{E}) \cong G/H$.*

1.6.3 Ramification Theory

We briefly discuss the known facts concerning the decomposition of $\iota(X) \in \mathcal{C} \rtimes \mathcal{S}$ for simple $X \in \mathcal{C}$.

1.6.10 Definition *For $X, Y \in \mathcal{C}$ we write $X \sim Y$ iff $\mathrm{Hom}_{\mathcal{C}}(\Gamma X, Y) \neq \{0\}$.*

1.6.11 Theorem {9} *Restricted to simple objects, the relation $\sim$ is an equivalence relation. Let $X, Y \in \mathcal{C}$ be simple. If $X \not\sim Y$ then $\iota(X), \iota(Y)$ are disjoint, to wit $\iota(X), \iota(Y)$ have no isomorphic subobjects. For every equivalence class σ there exist a finite set I_σ, mutually non-isomorphic simple objects $Z_i \in \mathcal{C} \rtimes \mathcal{S}$, $i \in I_\sigma$, and natural numbers $N_X, X \in \sigma$, such that*

$$\iota(X) \cong N_X \bigoplus_{i \in I_\sigma} Z_i \quad \forall X \in \sigma$$

In the case where G is abelian (i.e. every simple object of $\mathcal{S}$ is invertible) a more precise description of the simple objects of $\mathcal{C} \rtimes \mathcal{S}$ can be given, cf. {2}. Let Δ and K be the sets of isomorphism classes of simple objects in $\mathcal{C}$ and $\mathcal{S}$, respectively. Then K is an abelian group which acts on Δ in the obvious way (if X is simple and Y is invertible then $X \otimes Y$ is simple), and the equivalence classes considered above are just the orbits of this action. Defining, for every $[X] \in \Delta$,

$$K_X = \{[Y] \in K \mid Y \otimes X \cong X\},$$

K_X is a subgroup of K and one proves that there exists $\alpha \in Z^2(K_X, k^*)$ such that $\mathrm{End}_{\mathcal{C}\rtimes\mathcal{S}}(\iota(X))$ is isomorphic to the twisted group algebra $k^\alpha K_X$. Now $L_X = \{k \in K_X \mid \alpha(k,l) = \alpha(l,k)\ \forall l \in K_X\}$ is a subgroup of K_X spanning the center of $k^\alpha K_X$. Thus there is a one-to-one correspondence between the characters of L_X and the non-isomorphic simple subobjects of $\iota(X)$.

1.6.4 $\mathcal{C} \rtimes \mathcal{S}$ as braided crossed G-category

Now we turn to the question whether (or in which generalized sense) $\mathcal{C} \rtimes \mathcal{S}$ is braided. Since the category $\mathcal{C} \rtimes_0 \mathcal{S}$ has the same objects as $\mathcal{C}$, a natural candidate for a braiding is just the braiding c of $\mathcal{C}$. However, $\mathcal{C} \rtimes_0 \mathcal{S}$ has more morphisms, thus naturality of this putative braiding must be verified. In fact, the following was proven in {2}.

1.6.12 Lemma *The braiding c of $\mathcal{C}$ lifts to a braiding for $\mathcal{C} \rtimes_0 \mathcal{S}$ iff $\mathcal{S} \subset Z_1(\mathcal{C})$. In that case also $\mathcal{C} \rtimes \mathcal{S}$ is braided.*

1.6.13 Proposition *If $\mathcal{S} \subset Z_1(\mathcal{C})$ then $Z_1(\mathcal{C} \rtimes \mathcal{S}) \cong Z_1(\mathcal{C}) \rtimes \mathcal{S}$. In particular, $\mathcal{C} \rtimes \mathcal{S}$ has trivial center iff $\mathcal{S} = Z_1(\mathcal{C})$.*

Thus $\hat{\mathcal{C}} = \mathcal{C} \rtimes Z_1(\mathcal{C})$ has trivial center. If $\mathcal{C}$ is finite then also $\hat{\mathcal{C}}$ is finite and thus modular by Theorem 1.4.5. Therefore $\hat{\mathcal{C}}$ was called the modular closure in {2}. (In [11], the equivalent category $\Gamma - \mathrm{Mod}$, where (Γ, m, η) is the regular monoid in the symmetric category $Z_1(\mathcal{C})$, was called the modularization of $\mathcal{C}$.)

When $\mathcal{S}$ is not contained in $Z_1(\mathcal{C})$ it turns out that $\mathcal{C} \rtimes \mathcal{S}$ is still braided, but one needs to generalize the notion. Such a generalization was introduced in [68], and below we give a variant of that definition.

1.6.14 Definition *Let G be a (discrete) group. A strict crossed G-category is a strict tensor category $\mathcal{D}$ together with*

- *a full tensor subcategory $\mathcal{D}_G \subset \mathcal{D}$ of homogeneous objects,*
- *a map $\partial : \mathrm{Obj}\,\mathcal{D}_G \to G$ constant on isomorphism classes,*
- *a homomorphism $\gamma : G \to \mathrm{Aut}\mathcal{D}$ (monoidal self-isomorphisms of $\mathcal{D}$)*

such that

1. *$\partial(X \otimes Y) = \partial X\, \partial Y$ for all $X, Y \in \mathcal{D}_G$.*
2. *$\gamma_g(\mathcal{D}_h) \subset \mathcal{D}_{ghg^{-1}}$, where $\mathcal{D}_g \subset \mathcal{D}_G$ is the full subcategory $\partial^{-1}(g)$.*

If $\mathcal{D}$ is additive we require that every object of $\mathcal{D}$ be a direct sum of objects in $\mathcal{D}_G$. A braiding for a crossed G-category $\mathcal{D}$ is a family of isomorphisms $c_{X,Y} : X \otimes Y \to {}^XY \otimes X$, defined for all $X \in \mathcal{D}_G,\ Y \in \mathcal{D}$, such that

$$\begin{array}{ccc} X \otimes Y & \xrightarrow{s \otimes t} & X' \otimes Y' \\ \downarrow c_{X,Y} & & \downarrow c_{X',Y'} \\ {}^XY \otimes X & \xrightarrow[{}^Xt \otimes s]{} & {}^{X'}Y' \otimes X' \end{array}$$

commutes for all $s : X \to X', t : Y \to Y'$, and

$$\begin{aligned} c_{X,Z\otimes T} &= \mathrm{id}_{{}^XZ} \otimes c_{X,T} \circ c_{X,Z} \otimes \mathrm{id}_T, \\ c_{X\otimes Y,Z} &= c_{X,{}^YZ} \otimes \mathrm{id}_Y \circ \mathrm{id}_X \otimes c_{Y,Z}, \end{aligned}$$

for all $X, Y \in \mathcal{D}_G,\ Z, T \in \mathcal{D}$

1.6.15 THEOREM *$\mathcal{C} \rtimes \mathcal{S}$ is a rigid braided crossed G-category, where $\mathcal{S} \simeq \mathrm{Rep}\, G$.*

1.6.16 DEFINITION/PROPOSITION *The spectrum of a crossed G-category $\mathcal{D}$ is set $\mathrm{Spec}\,\mathcal{D} = \{g \in G \mid \mathcal{D}_g \neq \emptyset\}$. $\mathrm{Spec}\,\mathcal{D}$ is closed under multiplication and under conjugation with elements of G. It is closed under inverses, thus a normal subgroup, if $\mathcal{D}$ has duals.*

The spectrum is called full if it coincides with G and trivial if it is $\{e\}$.

1.6.17 PROPOSITION *The embedding $(\mathcal{C} \cap \mathcal{S}') \rtimes \mathcal{S} \hookrightarrow \mathcal{C} \rtimes \mathcal{S}$ gives rise to an isomorphism $(\mathcal{C} \rtimes \mathcal{S})_e \cong (\mathcal{C} \cap \mathcal{S}') \rtimes \mathcal{S}$. $\mathcal{C} \rtimes \mathcal{S}$ has trivial spectrum iff $\mathcal{S} \subset Z_2(\mathcal{C})$.*

Let $\mathcal{S}_0 \subset \mathcal{S}$ be a full inclusion of finite even symmetric fusion categories. Let $(\Gamma, \ldots)$, $(\Gamma_0, \ldots)$ be the corresponding Frobenius algebras in $\mathcal{S}_0, \mathcal{S}$, respectively, with automorphism groups G_0, G. Then $\Gamma \cong \Gamma_0 \oplus Z$ and $\mathrm{Hom}(\Gamma_0, Z) = \{0\}$, thus the projector $q \in \mathrm{End}\,\Gamma$ onto Γ_0 is central. The group

$$N = \{g \in G \mid g \circ q = q\}$$

is a normal subgroup of $G = \mathrm{Aut}(\Gamma, m, \eta)$. It coincides with

$$N = \{g \in G \mid \pi_X(g) = \mathrm{id}_{E(X)}\ \forall X \in \mathcal{S}_0\},$$

where $E : \mathcal{S} \to \mathrm{Vect}_{\mathbb{C}}$ is the fiber functor and π_X is the representation of G on $E(X)$. (This is easily deduced from $E(X) = \mathrm{Hom}(\mathbf{1}, \Gamma \otimes X)$ and the fact that $g \in G$ acts on $E(X)$ by $\pi_X(g) : \phi \mapsto g \otimes \mathrm{id}_X \circ \phi$.) This implies $G_0 \cong G/N$.

1.6.18 THEOREM *Let $\mathcal{S} \subset \mathcal{C}$ with $\mathcal{S} \simeq \mathrm{Rep}\, G$. Let N be the normal subgroup of G corresponding to the full inclusion $\mathcal{S} \cap Z_2(\mathcal{C}) \subset \mathcal{S}$ as above. Then $\mathrm{Spec}\,\mathcal{C} \rtimes \mathcal{S} = N$. In particular, $\mathcal{C} \rtimes \mathcal{S}$ has full spectrum iff $\mathcal{S} \cap Z_2(\mathcal{C})$ is trivial, i.e. consists only of multiples of $\mathbf{1}$.*

1.6.19 COROLLARY *If $\mathcal{C}$ is modular then $\mathcal{C} \rtimes \mathcal{S}$ has full spectrum and $(\mathcal{C} \rtimes \mathcal{S})_e$ is modular.*

1.6.20 REMARK We briefly comment on an approach to the construction of minimal modular extensions as defined in Subsection 1.4.3. Let $\mathcal{K}$ be a premodular category and $\mathcal{C} \supset \mathcal{K}$ a minimal modular extension. Let $\mathcal{S} = Z_2(\mathcal{K})$ and consider the full inclusion $\mathcal{K} \rtimes \mathcal{S} \subset \mathcal{C} \rtimes \mathcal{S}$. The left hand side is a braided category carrying a G-action (where $\mathcal{S} \simeq \mathrm{Rep}\, G$) – in fact it is just the modularization of $\mathcal{K}$ as in {2},[11] – whereas the right hand side is a braided crossed G-category with full spectrum. In a sense, $\mathcal{C} \rtimes \mathcal{S}$ is a crossed product of its grade-0 subcategory $\mathcal{K} \rtimes \mathcal{S}$ by the action of G. In not-too-different contexts such crossed products have been considered before, cf. e.g. [70]. The idea for the *construction* of $\mathcal{C}$ from $\mathcal{K}$ thus is: (i) consider the G-category $\mathcal{D} = \mathcal{K} \rtimes Z_2(\mathcal{K})$, define a braided crossed G-category $\mathcal{E} =$"$\mathcal{D} \rtimes G$" and define $\mathcal{C} = \mathcal{E}^G$. □

Chapter 2

Quantum Fields

2.1 QFT on $\mathbb{R}$ and S^1

2.1.1 Braided Crossed G-Categories from QFT on $\mathbb{R}$ with G-Symmetry

In this subsection we consider quantum field theories (QFTs) living on the line $\mathbb{R}$ and acted upon by a group G of internal symmetries. In {10} we show that this setting gives rise to a braided crossed G-category $G\text{–Loc}\, A$ and a full subcategory $G\text{–Loc}_f A$ that in addition is rigid and semisimple. These considerations take place in the rigorous setting of (operator) algebraic QFT, cf. [33]. In the case where $G = \{e\}$ our analysis reduces to well known results, cf. [22, 27, 32]. Here we limit ourselves to stating the most important definitions and results.

Let $\mathcal{K}$ be the set of intervals in $\mathbb{R}$, i.e. the bounded connected open subsets of $\mathbb{R}$. We write $I^\perp = \mathbb{R} - \overline{I}$, and for $I, J \in \mathcal{K}$ we write $I < J$ and $I > J$ if $I \subset (-\infty, \inf J)$ or $I \subset (\sup J, +\infty)$, respectively. The following is the one-dimensional version of a very classical definition, cf. [33].

2.1.1 Definition *A net of algebras on $\mathbb{R}$ is a triple $(\mathcal{H}_0, A, \Omega)$, usually simply denoted by A, where $\mathcal{H}_0$ is a separable Hilbert space with a distinguished non-zero vector Ω, and A is an assignment $\mathcal{K} \ni I \mapsto A(I) \subset \mathcal{B}(\mathcal{H}_0)$, where $A(I)$ is a type III factor, cf. 1.1.17. These data must satisfy*

- *Isotony:* $I \subset J \;\Rightarrow\; A(I) \subset A(J)$,
- *Locality:* $I \subset J^\perp \;\Rightarrow\; A(I) \subset A(J)'$,
- *Irreducibility:* $\vee_{I\in\mathcal{K}} A(I) = \mathcal{B}(\mathcal{H}_0)$ *(equivalently,* $\cap_{I\in\mathcal{K}} A(I)' = \mathbb{C}\mathbf{1}$*),*
- *Strong additivity:* $A(I) \vee A(J) = A(\overline{I \cup J}^0)$ *whenever $I, J \in \mathcal{K}$ are adjacent, i.e. their closures intersect in exactly one point.*
- *Haag duality: For every $I \in \mathcal{K}$ we have $A(I) = A(I^\perp)'$, where*

$$A(I^\perp) = \mathrm{Alg}\,\{A(J),\ J \in \mathcal{K},\ J \subset I'\}.$$

Given these data we also define

$$A_\infty = \bigcup_{I\in\mathcal{K}} A(I) \ \subset \mathcal{B}(\mathcal{H}_0).$$

As a consequence of the fact that the $A(I)$ are factors we have $Z(A_\infty) = \mathbb{C}\mathbf{1} = Z(A(I^\perp))\ \forall I \in \mathcal{K}$.

2.1.2 Remark The axioms of quantum field theory also require covariance w.r.t. a representation of the Poincaré group with positivity of the generator of time translation. These axioms are not needed in our considerations, which is why we prefer to speak of nets of algebras. □

2.1.3 DEFINITION *Let $(\mathcal{H}_0, \Omega, A)$ be a net of algebras on $\mathbb{R}$. A topological group G acts on A (by inner symmetries) if there is a strongly continuous unitary representation $V : G \to \mathcal{U}(\mathcal{H}_0)$ such that*

1. *$\beta_g(A(I)) = A(I)\ \forall g \in G,\ I \in \mathcal{K}$, where $\beta_g(x) = V(g)xV(g)^*$.*
2. *$V(g)\Omega = \Omega$.*
3. *If $\beta_g \restriction A(I) = \mathrm{id}$ for some $I \in \mathcal{K}$ then $g = e$.*

2.1.4 REMARK 1. In most of what follows the topology of G is not taken into account. In fact, we will mostly be interested in finite groups, but many results generalize to compact groups without difficulty, cf. {10}.

2. Condition 3 is crucial for the definition of the G-grading on $G-\mathrm{Loc}\,A$. It entails no serious loss of generality. □

The following is well known.

2.1.5 DEFINITION/PROPOSITION *Let B be a unital $*$-algebra. Let $\mathrm{End}\,B$ be the category whose objects $\rho, \sigma, \ldots$ are unital $*$-algebra homomorphisms from B into itself. With*

$$\begin{aligned} \mathrm{Hom}(\rho,\sigma) &= \{s \in B \mid s\rho(x) = \sigma(x)s \quad \forall x \in B\}, \\ t \circ s &= ts, \quad s \in \mathrm{Hom}(\rho,\sigma), t \in \mathrm{Hom}(\sigma,\eta), \\ \rho \otimes \sigma &= \rho(\sigma(\cdot)), \\ s \otimes t &= s\rho(t) = \rho'(t)s, \quad s \in \mathrm{Hom}(\rho,\rho'),\ t \in \mathrm{Hom}(\sigma,\sigma'), \end{aligned}$$

$\mathrm{End}\,B$ is a $\mathbb{C}$-linear strict tensor category with unit $\mathbf{1} = \mathrm{id}_B$ and positive $*$-operation. We have $\mathrm{End}\mathbf{1} = Z(B)$.

We now turn to the definition of $G-\mathrm{Loc}\,A$ as a full subcategory of $\mathrm{End}\,A_\infty$.

2.1.6 DEFINITION *Let $I \in \mathcal{K},\ g \in G$. An object $\rho \in \mathrm{End}\,A_\infty$ is called g-localized in I if*

$$\begin{aligned} \rho(x) &= x & \forall J < I,\ x \in A(J), \\ \rho(x) &= \beta_g(x) & \forall J > I,\ x \in A(J). \end{aligned}$$

ρ is g-localized if it is g-localized in some $I \in \mathcal{K}$. ρ is G-localized (in I) if it is g-localized (in I) for some $g \in G$. A g-localized $\rho \in \mathrm{End}\,A_\infty$ is transportable if for every $J \in \mathcal{K}$ there exists $\rho' \in \mathrm{End}\,A_\infty$, g-localized in J, such that $\rho \cong \rho'$ (in the sense of unitary equivalence).

2.1.7 REMARK 1. If ρ is g-localized in I and $J \supset I$ then ρ is g-localized in J.

2. Direct sums of transportable morphisms are transportable.

3. Requirement 3 in Definition 2.1.3 implies that if ρ is g-localized and h-localized then $g = h$. □

2.1.8 DEFINITION *$G-\mathrm{Loc}\,A$ is the full subcategory of $\mathrm{End}\,A_\infty$ whose objects are finite direct sums of G-localized transportable objects of $\mathrm{End}\,A_\infty$. Thus $\rho \in \mathrm{End}\,A_\infty$ is in $G-\mathrm{Loc}\,A$ iff there exists a finite set Δ and, for all $i \in \Delta$, there exist $g_i \in G$, $\rho_i \in \mathrm{End}\,A_\infty$ g_i-localized transportable, and $v_i \in A_\infty$ such that $v_i^* \circ v_j = \delta_{ij}$ and*

$$\rho = \sum_i v_i\, \rho_i(\cdot)\, v_i^*.$$

We say $\rho \in G-\mathrm{Loc}\,A$ is G-localized in $I \in \mathcal{K}$ if there exists a decomposition as above where all ρ_i are g_i-localized in I and transportable and $v_i \in A(I)\ \forall i$.

For $g \in G$, let $(G-\mathrm{Loc}\,A)_g$ be the full subcategory of $G-\mathrm{Loc}\,A$ consisting of those ρ that are g-localized, and let $(G-\mathrm{Loc}\,A)_G$ be the union of the $(G-\mathrm{Loc}\,A)_g, g \in G$.

For $g \in G$ define $\gamma_g \in \mathrm{Aut}(G-\mathrm{Loc}\, A)$ by

$$\begin{aligned} \gamma_g(\rho) &= \beta_g \rho \beta_g^{-1}, \\ \gamma_g(s) &= \beta_g(s), \quad s \in \mathrm{Hom}(\rho, \sigma) \subset A_\infty. \end{aligned}$$

2.1.9 LEMMA *The map $\partial : \mathrm{Obj}\,(G\text{–}\mathrm{Loc}\, A)_G \to G$ defined by $\partial\rho = g$ if $\rho \in (G\text{–}\mathrm{Loc}\, A)_g$ is a G-grading, and $G-\mathrm{Loc}\, A$ is a $\mathbb{C}$-linear crossed G-category with $\mathrm{End}\mathbf{1} = \mathbb{C}\mathrm{id}_{\mathbf{1}}$, positive $*$-operation, direct sums and subobjects (i.e. orthogonal projections split).*

2.1.10 REMARK The categories $\mathrm{Rep}\, A$ and G–$\mathrm{Loc}\, A$ do not have zero objects, thus cannot be additive or abelian. This is due to the fact that we consider only $\pi = (\pi_I)$ where all π_I are unital and unital $\rho \in \mathrm{End} A_\infty$, respectively, and could be remedied by dropping these conditions. We refrain from doing so since it would unnecessarily complicate the analysis without any real gains. □

Before we can construct a braiding for $G-\mathrm{Loc}\, A$ some preparations are needed.

2.1.11 LEMMA *If ρ is g-localized in I then $\rho(A(I)) \subset A(I)$ and $\rho \restriction A(I)$ is ultraweakly continuous.*

2.1.12 LEMMA *Let ρ, σ be g-localized in I. Then $\mathrm{Hom}(\rho, \sigma) \subset A(I)$, implying that $G-\mathrm{Loc}\, A$ is a W^*-category in the sense of [31].*

2.1.13 LEMMA *Let $\rho_i \in G-\mathrm{Loc}\, A,\ i = 1, 2$ be g_i-localized in I_i, where $I_1 < I_2$. Then*

$$\rho_1 \otimes \rho_2 = \gamma_{g_1}(\rho_2) \otimes \rho_1.$$

Recall that for homogeneous σ we write ${}^{\sigma}\!\rho = \gamma_{\partial\sigma}(\rho)$ as in [68].

2.1.14 PROPOSITION *G–$\mathrm{Loc}\, A$ admits a braiding, i.e. a family of isomorphisms $c_{\rho,\sigma} : \rho \otimes \sigma \to {}^{\rho}\sigma \otimes \rho$, for all $\rho \in (G-\mathrm{Loc}\, A)_G$, $\sigma \in G-\mathrm{Loc}\, A$, satisfying the conditions in Definition 1.6.14. If ρ_1, ρ_2 are as in Lemma 2.1.13 then $c_{\rho_1,\rho_2} = \mathrm{id}_{\rho_1 \otimes \rho_2} = \mathrm{id}_{{}^{\rho_1}\rho_2 \otimes \rho_1}$.*

We only indicate how c is defined. Let $\rho \in (G\text{–}\mathrm{Loc}\, A)_g$, $\sigma \in G\text{–}\mathrm{Loc}\, A$ be G-localized in $I, J \in \mathcal{K}$, respectively. Let $\tilde{I} < J$. By transportability we can find $\tilde{\rho} \in (G\text{–}\mathrm{Loc}\, A)_g$ localized in $\tilde{I}$ and a unitary $u \in \mathrm{Hom}(\rho, \tilde{\rho})$. By Lemma 2.1.13 we have $\tilde{\rho} \otimes \sigma = \gamma_g(\sigma) \otimes \tilde{\rho}$, thus the composite

$$c_{\rho,\sigma} : \ \rho \otimes \sigma \xrightarrow{u \otimes \mathrm{id}_\sigma} \tilde{\rho} \otimes \sigma = \ \gamma_g(\sigma) \otimes \tilde{\rho} \xrightarrow{\mathrm{id}_{\gamma_g(\sigma)} \otimes u^*} \gamma_g(\sigma) \otimes \rho$$

is unitary and a candidate for the braiding. As an element of A_∞, $c_{\rho,\sigma} = \gamma_g(\sigma)(u^*)u = \beta_g \sigma \beta_g^{-1}(u^*)u$. Next one shows that $c_{\rho,\sigma}$ is independent of the choices involved and that it satisfies the conditions in Definition 1.6.14. For the details see {10}.

In view of Lemma 2.1.11 we can define

2.1.15 DEFINITION *G–$\mathrm{Loc}_f A$ is the full tensor subcategory of G–$\mathrm{Loc}\, A$ of those objects ρ satisfying $[A(I) : \rho(A(I))] < \infty$ whenever ρ is G-localized in I.*

By an adaptation of the approach of [32] one proves the following:

2.1.16 PROPOSITION *$G-\mathrm{Loc}_f A$ is semisimple (in the sense that every object is a finite direct sum of (absolutely) simple objects). Every object of $G-\mathrm{Loc}_f A$ has a conjugate in the sense of [48] and $G-\mathrm{Loc}_f A$ is rigid.*

Summarizing the preceding discussion we have

2.1.17 THEOREM *Let A be a net of algebras on $\mathbb{R}$, acted upon by the group G of internal symmetries. Then $G-\mathrm{Loc}\, A$ is a braided crossed G-category and $G-\mathrm{Loc}_f A$ is a semisimple rigid braided crossed G-category.*

It is obvious that for any braided crossed G-category $\mathcal{D}$, the grade zero subcategory $\mathcal{D}_e$ is a braided tensor category. In the case of $G-\mathrm{Loc}\, A$ these subcategories have been studied for a long time, and we write

2.1.18 DEFINITION $\mathrm{Loc}\, A = (G-\mathrm{Loc}\, A)_e$ *and* $\mathrm{Loc}_f A = (G-\mathrm{Loc}_f A)_e$.

2.1.19 REMARK 1. $\mathrm{Loc}\, A$ is just the familiar category of transportable localized morphisms defined in [27]. (If there is no group acting, put $G = \{e\}$.) It is equivalent to the full subcategory $\mathrm{DHR}(A)$ of $\mathrm{Rep}\, A_\infty$ whose objects are those representations π satisfying the DHR criterion: For every $I \in \mathcal{K}$ we have $\pi \upharpoonright A(I^\perp) \cong \pi_0 \upharpoonright A(I^\perp)$, i.e. there exists a unitary $u_I : \mathcal{H}_0 \to \mathcal{H}_\pi$ such that $ux = \pi(x)u$ for all $x \in A(I^\perp)$. All this is very classical, cf. e.g. [22, 27].

2. For non-trivial symmetries G, the category $G-\mathrm{Loc}\, A$ contains information that cannot be obtained from $\mathrm{Loc}\, A$. The closest precedent to our above considerations can be found in [60]. There, however, several restrictive assumptions were made, under which the G-crossed structure essentially trivializes. In particular only abelian groups G were considered. □

2.1.2 Completely Rational CQFT on S^1: Modularity and Full Spectrum

While the constructions in the preceding section are quite general, we are typically interested in a more specific situation, viz. chiral conformal field theories on S^1. We will give the definitions and show how they give rise to a net of algebras on $\mathbb{R}$. We then outline the results of {3} where it is proved, under very natural assumptions, that $\mathrm{Loc}\, A$ is a unitary modular category. A more complete and fairly self-contained review is in preparation {13}.

Let $\mathcal{I}$ be the set of intervals in S^1, i.e. connected open non-empty and non-dense subsets of S^1. (Equivalently, the set $\{(x,y) \in S^1 \times S^1 \mid x \neq y\}$.) For every $J \subset S^1$, J' is the interior of the complement of J.

The following definition is classical, see e.g. [13, 28].

2.1.20 DEFINITION *A chiral conformal field theory is a quadruple $(\mathcal{H}_0, A, U, \Omega)$, usually simply denoted by A, where*

1. *$\mathcal{H}_0$ is a separable Hilbert space with a distinguished non-zero vector Ω,*
2. *A is an assignment $\mathcal{I} \ni I \mapsto A(I)$, where $A(I)$ is a von Neumann algebra on $\mathcal{H}_0$.*
3. *U is a strongly continuous unitary representation on $\mathcal{H}_0$ of the Möbius group $PSU(1,1) = SU(1,1)/\{\mathbf{1}, -\mathbf{1}\}$, i.e. the group of those fractional linear maps $\mathbb{C} \to \mathbb{C}$ which map the circle into itself.*

These data must satisfy

- *Isotony:* $I \subset J \;\Rightarrow\; A(I) \subset A(J)$,
- *Locality:* $I \subset J' \;\Rightarrow\; A(I) \subset A(J)'$,
- *Irreducibility:* $\vee_{I \subset S^1} A(I) = \mathcal{B}(\mathcal{H}_0)$ *(equivalently,* $\cap_{I \in \mathcal{I}} A(I)' = \mathbb{C}\mathbf{1}$*)*,
- *Covariance:* $U(a)A(I)U(a)^* = A(aI) \quad \forall a \in PSU(1,1),\ I \in \mathcal{I}$,
- *Positive energy:* $L_0 \geq 0$*, where* L_0 *is the generator of the rotation subgroup of* $PSU(1,1)$,

- *Vacuum: every vector in $\mathcal{H}_0$ which is invariant under the action of $PSU(1,1)$ is a multiple of Ω.*

2.1.21 For consequences of these axioms see, e.g., [29]. We limit ourselves to listing some facts:

1. Type: The von Neumann algebra $A(I)$ is a factor of type III_1 for every $I \in \mathcal{I}$.
2. Haag duality: $A(I)' = A(I') \quad \forall I \in \mathcal{I}$.
3. Reeh-Schlieder property: $\overline{A(I)\Omega} = \overline{A(I)'\Omega} = \mathcal{H}_0 \quad \forall I \in \mathcal{I}$.
4. The modular groups and conjugations associated with $(A(I), \Omega)$ have a geometric meaning, cf. [12, 29] for the precise statements.
5. Additivity: If $I, J \in \mathcal{I}$ are such that $I \cap J, I \cup J \in \mathcal{I}$ then $A(I) \vee A(J) = A(I \cup J)$.

2.1.22 Remark Note that strong additivity, defined in the same way as for nets of algebras on $\mathbb{R}$, is not implied by the other axioms. By Möbius covariance strong additivity holds in general if it holds for one pair I, J of adjacent intervals. Furthermore, every CQFT can be extended canonically to one satisfying strong additivity. □

2.1.23 Definition *A representation π of A consists of a Hilbert space $\mathcal{H}$ and a family $\{\pi_I, I \in \mathcal{I}\}$, where π_I is a unital $*$-representation of $A(I)$ on $\mathcal{H}$ such that*

$$I \subset J \quad \Rightarrow \quad \pi_J \restriction A(I) = \pi_I.$$

π is called covariant if there is a positive energy representation U_π of the universal covering group $\widehat{PSU(1,1)}$ of the Möbius group on $\mathcal{H}$ such that

$$U_\pi(a)\pi_I(x)U_\pi(a)^* = \pi_{aI}(U(a)xU(a)^*) \quad \forall a \in \widehat{PSU(1,1)},\ I \in \mathcal{I}.$$

We denote by $\mathrm{Rep}\, A$ the W^*-category of all representations on separable Hilbert spaces, with bounded intertwiners as morphisms.

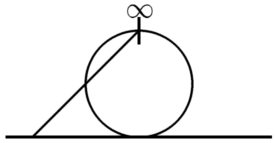
2.1.24 Definition/Proposition *If A satisfies strong additivity and π is a representation then the Jones index of the inclusion $\pi_I(A(I)) \subset \pi_{I'}(A(I'))$ does not depend on $I \in \mathcal{I}$ and we define the dimension*

$$d(\pi) = [\pi_{I'}(A(I')) : \pi_I(A(I))]^{1/2} \in [1, \infty].$$

We define $\mathrm{Rep}_f A$ to be the the full subcategory of $\mathrm{Rep}\, A$ of those representations satisfying $d(\pi) < \infty$.

2.1.25 Proposition *Every chiral CQFT $(\mathcal{H}_0, A, U, \Omega)$ satisfying strong additivity gives rise to a net of algebras on $\mathbb{R}$.*

The proof proceeds by arbitrarily picking a point $\infty \in S^1$ and identifying $S^1 - \{\infty\}$ with $\mathbb{R}$ by stereographic projection



With $\mathcal{I}_\infty = \{I \in \mathcal{I} \mid \infty \notin \overline{I}\}$ there is a bijection between $\mathcal{I}_\infty$ and $\mathcal{K}$. The family $A(I), I \in \mathcal{K}$ is just the restriction of $I \mapsto A(I), I \in \mathcal{I}$ to $I \in \mathcal{I}_\infty \equiv \mathcal{K}$. Strong additivity of the theory on S^1 together with Haag duality on S^1 implies Haag duality in the sense of Definition 2.1.1.

The following result was proven in {3, Appendix B}. On the one hand, it provides a very satisfactory interpretation of the category $\mathrm{Loc}\, A$ and on the other, it equips the category $\mathrm{Rep}\, A$ with a monoidal structure, which is quite non-obvious a priori.

2.1.26 PROPOSITION *Let $(\mathcal{H}_0, A, U, \Omega)$ be a chiral CQFT satisfying strong additivity. (A possibly existing symmetry G is ignored.) Then there are equivalences of $*$-categories*

$$\begin{aligned} \mathrm{Loc}\, A &\simeq \mathrm{Rep}\, A, \\ \mathrm{Loc}_f A &\simeq \mathrm{Rep}_f A, \end{aligned}$$

where $\mathrm{Rep}_{(f)}A$ refers to the chiral CQFT and Definition 2.1.23, whereas $\mathrm{Loc}_{(f)}A$ refers to the net of algebras on $\mathbb{R}$ obtained by restriction and Definition 2.1.8.

Now we are interested in rational models, i.e. chiral CQFTs which admit only finitely many equivalence classes of irreducible representations. Our aim is to identify additional axioms that single out such models without eliminating the known rational models.

2.1.27 DEFINITION *A chiral CQFT satisfies the split property if the map*

$$m: \ A(I) \otimes_{alg} A(J) \to A(I) \vee A(J), \quad x \otimes y \mapsto xy$$

extends to an isomorphism of von Neumann algebras whenever $I, J \in \mathcal{I}$ satisfy $\overline{I} \cap \overline{J} = \emptyset$.

2.1.28 REMARK 1. The split property is implied by the property $Tr e^{-\tau L_0} < \infty \ \forall \tau > 0$. The latter property and strong additivity have been verified in all known rational models.

2. There are models, like the $U(1)$ current algebra [13], that satisfy the standard axioms, strong additivity and the split property and that have infinitely many inequivalent irreducible representations. This means that we need another axiom to single out the rational theories. □

2.1.29 DEFINITION/PROPOSITION *Let A satisfy strong additivity and the split property. For $E = I \cup J$, where $I, J \in \mathcal{I}$ satisfy $\overline{I} \cap \overline{J} = \emptyset$, we define $A(E) = A(I) \vee A(J)$. Then the index of the inclusion $A(E) \subset A(E')'$ does not depend on I, J and we define*

$$\mu(A) = [A(E')' : A(E)] \ \in [1, \infty].$$

A chiral CQFT is completely rational if it satisfies (a) strong additivity, (b) the split property and (c) $\mu(A) < \infty$.

2.1.30 REMARK 1. By the proposition, every chiral CQFT satisfying strong additivity and the split property defines a numerical invariant $\mu(A) \in [1, \infty]$. The models where the latter is finite – the completely rational ones – are arguably the best behaved non-trivial quantum field theories, in that very strong results on both their structure and representation theory have been proven {3}, cf. also {13}. In particular the invariant $\mu(A)$ has a nice interpretation, see below.

2. All known classes of rational chiral CQFTs are completely rational in the above sense. For the WZW models connected to loop groups this is proven in [73, 76]. Even more importantly, the class of completely rational models is stable under tensor products and finite extensions and subtheories. This has applications to orbifold (and coset) models, cf. the next section. □

2.1.31 THEOREM *{3} Let A be a completely rational CQFT. Then*

- *Every representation of A on a separable Hilbert space is completely reducible, i.e. a direct sum of irreducible representations. (For non-separable representations this is also true if one assumes local normality, which is automatic in the separable case, or equivalently covariance.)*
- *Every representation in $\mathrm{Rep}_f A$ is covariant.*
- *Every irreducible separable representation π has finite dimension $d(\pi)$, thus $\mathrm{Rep}_f A$ is just the category of finite direct sums of irreducible representations.*

- *The number of unitary equivalence classes of separable irreducible representations is finite and*

$$\mu(A) = \dim \mathrm{Rep}_f A \equiv \sum_i d(\pi_i)^2.$$

- *The center $Z_2(\mathrm{Rep}_f A)$ is trivial, thus $\mathrm{Rep}_f A$ is a unitary modular category.*

If A admits a symmetry group G, defined as in Definition 2.1.3, the above of course applies to $(G-\mathrm{Loc}_f A)_e = \mathrm{Loc}_f A \simeq \mathrm{Rep}_f A$. Concerning the categories $(G-\mathrm{Loc}\, A)_g$, $g \neq e$ we prove in {10}:

2.1.32 THEOREM *Let A be a completely rational chiral CQFT with finite symmetry group G of inner symmetries. Then*

1. *The braided crossed G-category $G-\mathrm{Loc}_f A$ has full spectrum, i.e. $(G-\mathrm{Loc}_f A)_g \neq \emptyset$ for every $g \in G$.*
2. *Every object $\rho \in G-\mathrm{Loc}\, A$ is a direct sum (possibly infinite) of objects in $G-\mathrm{Loc}_f A$. Thus every simple object in $G-\mathrm{Loc}\, A$ has finite dimension.*
3. $\dim G-\mathrm{Loc}_f A = |G| \dim \mathrm{Rep}_f A$.

2.1.33 REMARK 1. The proof relies on the relationship between $G-\mathrm{Loc}\, A$ and $\mathrm{Loc}\, A^G$, where A^G is the G-invariant subtheory of A, the 'orbifold theory', cf. Section 2.2.2. It would be desirable to find a more direct argument, but this seems very difficult.

2. In the physics literature and the mathematical literature on vertex operator algebras, the objects of $G-\mathrm{Loc}\, A$ with grade $g \neq e$ appear as 'twisted representations of A'. There are results on the existence of such representations for nice VOAs, cf. e.g. [20], but in the latter context there seems to be no proof that the twisted representations form a braided crossed G-category. □

2.1.34 Combining Theorem 2.1.31 (from {3}) with [67] we have a chain of constructions

$$\text{Completely rational CQFT} \rightsquigarrow \text{modular category} \rightsquigarrow 3-\text{manifold invariant}.$$

Now Theorem 2.1.32 (from {10}) together with [68], where modular crossed G-categories are used to define invariants of G-manifolds, i.e. manifolds together with a principal G-bundle on them, amounts to an equivariant version of this chain of constructions.

2.2 Orbifold Theories

2.2.1 General Results on Local Extensions

In this subsection we consider finite local extensions of nets of algebras on $\mathbb{R}$. The primary references are [47, 9].

2.2.1 DEFINITION *Let $I \mapsto A(I) \subset \mathcal{B}(\mathcal{H}_0^A)$ be a net of algebras on $\mathbb{R}$. A subtheory $B \subset A$ is a family of sub-von Neumann algebras $B(I) \subset A(I)$ such that, defining*

$$\mathcal{H}_0^B = \overline{\bigcup_{I \in \mathcal{K}} B(I)\Omega},$$

the triple $(\mathcal{H}_0^B, (B(I) \upharpoonright \mathcal{H}_0^B), \Omega)$ is a net of algebras on $\mathbb{R}$. If a group $\mathcal{P}$ of spacetime symmetries acts on A via $\mathrm{Ad}\, U(a)$ then we require $U(a)B(I)U(a)^* = B(aI)$ for all $a \in \mathcal{P}, I \in \mathcal{K}$. We also say that A is a local extension of B.

One can show [47] that the Jones index $[A(I) : B(I)]$ does not depend on $I \in \mathcal{K}$. This value is taken as the definition of the index $[A : B]$ of the local extension. If $[A : B] < \infty$ then A is a finite local extension of B. In this case one can prove $\mu(B) = [A : B]^2 \mu(A)$ and thus $\dim \mathrm{Rep}_f B = [A : B]^2 \dim \mathrm{Rep}_f A$, cf. {3}. The following is essentially contained in [47].

2.2.2 THEOREM *Let B be a net of algebras on $\mathbb{R}$. Then there is a one-to-one correspondence between finite local extensions $A \supset B$ (modulo unitary equivalence) and strongly separable commutative Frobenius algebras $(\Gamma, m, \eta, \Delta, \varepsilon)$ in $\mathrm{Loc}_f B$ (modulo isomorphism). Under this correspondence, one has $[A : B] = d(\Gamma)$.*

In [47, 9] a tensor functor $\alpha : \mathrm{Loc}\, B \to \mathrm{End}\, A_\infty$ is defined. Building on that one can prove {10}:

2.2.3 PROPOSITION *Let B be a net of algebras on $\mathbb{R}$ and $A \supset B$ a finite local extension corresponding to the Frobenius algebra $(\Gamma, m, \eta, \Delta, \varepsilon)$ in $\mathcal{C} = \mathrm{Loc}_f B$. Let $F : \mathcal{C} \to \Gamma - \mathrm{Mod}_\mathcal{C}$ be the canonical tensor functor from Definition/Proposition 1.6.1. Then there exists a full and faithful tensor functor $K : \Gamma - \mathrm{Mod}_\mathcal{C} \to \mathrm{End}\, A_\infty$ such that*

$$\begin{array}{ccc} \mathrm{Loc}_f B & \xrightarrow{F} & \Gamma - \mathrm{Mod}_\mathcal{C} \\ & \searrow^{\alpha} & \downarrow K \\ & & \mathrm{End}\, A_\infty \end{array}$$

commutes. The restriction of K to the full subcategory $\Gamma - \mathrm{Mod}^0_\mathcal{C}$ maps into $\mathrm{Loc}_f A$.

For finite index extensions of completely rational models one has $\dim \mathrm{Rep}_f A = \dim \mathrm{Rep}_f B / [A : B]^2 = \dim \mathrm{Rep}_f A / d(\Gamma)^2$. Comparing with $\dim \Gamma - \mathrm{Mod}^0_\mathcal{C} = \dim \mathcal{C} / d(\Gamma)^2$ [43] one deduces that K is essentially surjective and thus obtains {10}:

2.2.4 THEOREM *Let B be the restriction to $\mathbb{R}$ of a completely rational CQFT on S^1, and let A be a finite local extension corresponding to the Frobenius algebra $(\Gamma, \ldots)$ in $\mathcal{C} = \mathrm{Loc}_f B$. Then $K : \Gamma - \mathrm{Mod}^0_\mathcal{C} \to \mathrm{Loc}_f A$ is an equivalence of braided tensor categories.*

Results similar to Theorems 2.2.2 and 2.2.4 have been formulated in [43] in the context of vertex operator algebras. There, however, they are much more difficult to prove, and no complete proof has appeared yet.

2.2.2 Orbifold Theories

If A is a QFT with symmetry group G then the 'orbifold theory' is given by $I \mapsto A(I)^G \upharpoonright \mathcal{H}_0^G$. Now A is a local extension of A^G with index $[A : A^G] = |G|$, and the considerations of the preceding subsection apply. If A lives on S^1 and is completely rational then A^G is completely rational iff G is finite, cf. [77].

By a classical result [21], $\mathrm{Loc}\, A^G$ has a full symmetric subcategory $\mathcal{S} \simeq \mathrm{Rep}\, G$, consisting of direct sums of the irreducible subrepresentations of $\pi_0^A \upharpoonright A^G$. Furthermore, the Frobenius algebra in $\mathcal{C} = \mathrm{Loc}_f A^G$ corresponding (by Theorem 2.2.2) to the extension $A \supset A^G$ is precisely the one of the regular representation of G, cf. Theorem 1.1.15, thus $\Gamma - \mathrm{mod}_\mathcal{C} \simeq \mathcal{C} \rtimes \mathcal{S}$. It is therefore natural to compare the braided crossed G-categories $\mathcal{C} \rtimes \mathcal{S}$ and $G - \mathrm{Loc}\, A$.

2.2.5 LEMMA *Let A be a net of algebras on $\mathbb{R}$ and G a finite group of symmetries. Write $\mathcal{C} = \mathrm{Loc}_f A^G$ and let $(\rho, p) \in \mathcal{C} \rtimes \mathcal{S}$ be simple (thus $\rho \in \mathcal{C}$ and p is a minimal idempotent in $\mathrm{End}_{\mathcal{C} \rtimes_0 \mathcal{S}}(\rho)$). With $g = \partial(\rho, p) \in G$ we have $K((\rho, p)) \in (G - \mathrm{Loc}_f A)_g$. Thus the functor $K : \mathcal{C} \rtimes \mathcal{S} \to \mathrm{End}\, A_\infty$ (and a fortiori $\alpha : \mathrm{Loc}_f A^G \to \mathrm{End}\, A_\infty$) has its image in $G - \mathrm{Loc}_f A$.*

2.2.6 PROPOSITION *The functor* $K : \mathrm{Loc}_f A^G \rtimes \mathcal{S} \to G\!-\!\mathrm{Loc}_f A$ *is a functor of braided crossed G-categories.*

When G is finite we have

2.2.7 THEOREM *Let* A *be a net of algebras on* $\mathbb{R}$ *and* G *a finite group of symmetries. Then* $K : \mathrm{Loc}_f A^G \rtimes \mathcal{S} \to G\!-\!\mathrm{Loc}_f A$ *is essentially surjective, thus gives rise to the equivalence*

$$G\!-\!\mathrm{Loc}_f A \;\simeq\; \mathrm{Loc}_f A^G \rtimes \mathcal{S},$$

of braided crossed G-categories.

In view of Propositions 1.6.17 and 1.6.6 this implies the braided equivalences

$$\begin{aligned} \mathrm{Loc}_f A &\simeq (\mathrm{Loc}_f A^G \cap \mathcal{S}') \rtimes \mathcal{S}, \\ G\!-\!\mathrm{Loc}_f A &\simeq \mathrm{Loc}_f A^G \rtimes \mathcal{S} \end{aligned}$$

and therefore the relation

$$|G| = \frac{\dim \mathrm{Loc}_f A^G}{\dim G\!-\!\mathrm{Loc}_f A} = \frac{\dim(\mathrm{Loc}_f A^G \cap \mathcal{S}')}{\dim \mathrm{Loc}_f A} \tag{2.1}$$

between the dimensions. Diagrammatically, the categories in question are related as follows:

$$\begin{array}{ccccccc} & & \xleftarrow{\quad\text{grade } 0\quad} & & & & \\ & & \mathrm{Loc}_f A & \subset & G\!-\!\mathrm{Loc}_f A & & \\ G - \text{fixpoints} & \Big\downarrow & \cup & & \cup & \Big\uparrow & \cdot \rtimes \mathcal{S} \\ & & \mathrm{Loc}_f A^G \cap \mathcal{S}' & \subset & \mathrm{Loc}_f A^G & & \end{array}$$

Here $G - \mathrm{Loc}\, A$ is a braided crossed G-category whereas the other categories are braided in the usual sense. The vertical inclusions are the G-fixed subcategories, the horizontal inclusions are full. The G-grading on $G - \mathrm{Loc}\, A$ descends to $(G - \mathrm{Loc}\, A)^G \simeq \mathrm{Loc}\, A^G$ iff G is abelian, in which case $(\mathrm{Loc}\, A^G)_e = \mathrm{Loc}\, A^G \cap \mathcal{S}'$.

The strongest results are achieved when A arises from a completely rational theory on S^1:

2.2.8 THEOREM *If* A *is completely rational and* G *is a finite group of symmetries then the orbifold theory* A^G *is completely rational with* $\mu(A^G) = |G|^2 \mu(A)$. *The categories* $\mathrm{Loc}_f A^G$ *and* $\mathrm{Loc}_f A \simeq (G\text{–}\mathrm{Loc}\, A)_e$ *are modular.* $G\text{–}\mathrm{Loc}_f A \simeq \mathrm{Loc}_f A \rtimes \mathcal{S}$ *has full spectrum and* $\dim G\text{–}\mathrm{Loc}_f A = |G| \dim \mathrm{Loc}_f A$.

Proof. Modularity of $\mathrm{Loc}_f A$ and $\mathrm{Loc}_f A^G$ follows by complete rationality and {3}. Thus $G\!-\!\mathrm{Loc}_f A \simeq \mathrm{Loc}_f A^G \rtimes \mathcal{S}$ has full spectrum by Corollary 1.6.19. By Theorem 1.4.6 we have $\dim \mathrm{Loc}_f A^G \cap \mathcal{S}' = \dim \mathrm{Loc}_f A^G / |G|$, and $\dim G\!-\!\mathrm{Loc}_f A = |G| \dim \mathrm{Loc}_f A$ follows from (2.1). ■

2.2.3 The Holomorphic Case

An interesting special case of orbifold models arises when A has trivial representation category: $\mathrm{Rep}\, A \simeq \mathrm{Vect}_{\mathbb{C}}$. For historical reasons, one speaks of holomorphic orbifold models [18]. In our framework, this amounts to considering completely rational chiral CQFTs with $\mu(A) = 1$. (One can show that it suffices to assume that A satisfies (a) the split property and (b) Haag duality for 2-intervals, i.e. $A(E)' = A(E')$ whenever $E = I \cap J$ with $I, J \in \mathcal{I}, \overline{I} \cap \overline{J} = \emptyset$.) The preceding results {9,10} immediately imply:

2.2.9 THEOREM *Let A be a completely rational chiral CQFT whose representation category* $\mathrm{Rep}\,A$ *is trivial. Let G be a finite group acting on A. Then $G-\mathrm{Loc}\,A$ has, up to isomorphism, exactly one simple object X_g of grade g for every $g \in G$, and we have $d(X_g) = 1$.*

Let $\mathcal{D}$ be a semisimple braided crossed G-category having exactly one isoclass of simples object of every grade. If $\mathcal{D}_0$ is the full subcategory consisting only of the simple objects then every non-zero morphism in $\mathcal{D}_0$ is an isomorphism. Ignoring the zero-morphism this shows that $\mathcal{D}_0$ is a categorical group. As shown in [68], categorical groups that are also braided crossed G-categories are classified, up to equivalence, by $H^3_{qa}(G,k^*)$. The latter is a quasiabelian cohomology group as introduced by Ospel [56]. As usual, one has $H^3_{qa}(G,A) = Z^3_{qa}(G,A)/B^3_{qa}(G,A)$. The elements of $Z^3_{qa}(G,A)$ are pairs (ω,σ) with $\omega \in Z^3(G,A)$ and $\sigma : G\times G \to A$ satisfying

$$\begin{aligned} \omega(x,y,z) + \omega(xyx^{-1}, xzx^{-1}, x) + \sigma(x, y+z) &= \omega(xyx^{-1}, x, z) + \sigma(x,z) + \sigma(x,y), \\ \omega(x,y,z) + \omega(xyzy^{-1}x^{-1}, x, y) - \sigma(x+y, z) &= \omega(x, yzy^{-1}, y) - \sigma(x,z) - \sigma(y,z) \end{aligned}$$

and

$$\begin{aligned} \omega(uxu^{-1}, uyu^{-1}, uzu^{-1}) &= \omega(x,y,z), \\ \sigma(uxu^{-1}, uyu^{-1}) &= \sigma(x,y) \end{aligned}$$

for all $x,y,z,u \in G$. Now $(\omega,\sigma) \in B^3_{qa}(G,A)$ if there exists $\eta : G\times G \to A$ such that

$$\begin{aligned} \omega(x,y,z) &= \eta(y,z) - \eta(x+y,z) + \eta(x,y+z) - \eta(x,y), \\ \sigma(x,y) &= \eta(x,y) - \eta(y,x). \end{aligned}$$

For abelian G one obviously recovers the abelian cohomology from [51]: $H^3_{qa}(G,A) = H^3_{ab}(G,A)$.
From these considerations one concludes:

2.2.10 COROLLARY *A completely rational chiral CQFT A with trivial* $\mathrm{Rep}\,A$ *defines an element of $H^3_{qa}(G,\mathbb{C}^*)$, namely the invariant corresponding to the braided G-crossed categorical group* $(G-\mathrm{Loc}\,A)_{00}$.

A determination of those $(\omega,\sigma) \in H^3_{qa}(G,\mathbb{C}^*)$ which can arise from a CQFT with symmetry G seems out of reach. At least we know that not every $[\omega] \in H^3(G,\mathbb{C}^*)$ can appear, but only those for which there exists a compatible $\sigma : G\times G \to \mathbb{C}^*$.
The question arises, whether the passage from a semisimple braided crossed G-category $\mathcal{D}$ of the above type to the categorical group $\mathcal{D}_{00}$ entails a loss of information or whether $\mathcal{D}$ can be recovered (up to equivalence) from the element of $H^3_{qa}(G,k^*)$ associated with $\mathcal{D}_{00}$.

2.2.11 CONJECTURE *For every $[(\omega,\sigma)] \in H^3(G,k^*)$ there exists a semisimple k-linear braided crossed G-category $\mathcal{C}(\omega,\sigma)$ such that*

1. *Up to isomorphism, there is precisely one simple object X_g for every $g \in G$, and $d(X_g) = 1$.*
2. *$[(\omega,\sigma)]$ is the invariant associated with $\mathcal{C}(\omega,\sigma)_{00}$.*
3. *$\mathcal{C}(\omega,\sigma)^G$ has a full subcategory $\mathcal{S} \simeq \mathrm{Rep}\,G$ such that $\mathcal{C}(\omega,\sigma) \simeq \mathcal{C}(\omega,\sigma)^G \rtimes \mathcal{S}$.*
4. *As a tensor category $\mathcal{C}(\omega,\sigma)^G \simeq D^\omega(G)-\mathrm{Mod}$, where $D^\omega(G)$ is the twisted quantum double [17] of G. The braiding of $\mathcal{C}(\omega,\sigma)^G$ depends on σ and therefore may differ from that of $D^\omega(G)-\mathrm{Mod}$.*

2.2.12 REMARK This conjecture, work on which is in progress, would demonstrate the precise way in which $H^3_{qa}(G,k^*)$, rather than $H^3(G,k^*)$, is relevant in the context of holomorphic orbifold models.

The main problem in the proof is requirement 3. If it were not for the latter, it would suffice to k-linearize the braided G-crossed categorical group associated with (ω, σ) and to complete w.r.t. direct sums. But in order for 3 to hold, every $X \in \mathcal{C}(\omega, \sigma)$ must be a direct summand of a G-invariant object and for every irreducible $\pi \in \mathrm{Rep}\, G$ there must exist $X_\pi \in \mathcal{C}(\omega, \sigma)^G$ such that $\mathrm{Hom}(\mathbf{1}, X_\pi)$ together with the G-action is equivalent to π as a representation. □

2.3 Modular Invariants

In what follows, a two-dimensional CQFT is understood to be a functor from a category of Riemann surfaces to the category of (infinite dimensional!) Hilbert spaces, satisfying axioms similar to those of Segal [66]. A fundamental problem in conformal field theory is the construction of a two-dimensional theory starting from two (possibly identical) chiral conformal theories A^L, A^R. In general, there will not be a unique way to do this, and an additional piece of input data is needed, which we call the 'modular invariant'. In the physics literature it is widely believed that this datum consists of a matrix (Z_{ij}) where i, j run through the sets Δ^L, Δ^R of irreducible representations of A^L, A^R, respectively. Z is subject to the conditions $Z_{ij} \in \mathbb{Z}_+$, $Z_{00} = 1$, where $0 \in \Delta^{L/R}$ corresponds to the vacuum representation $\pi_0^{L/R}$, and $ZT^R = T^L Z$, $ZS^R = S^L Z$. (The latter conditions imply that Z intertwines the representations of $SL(2, \mathbb{Z})$ associated with the modular categories $\mathrm{Rep}\, A^L$ and $\mathrm{Rep}\, A^R$, respectively.) In order to avoid confusion, we will speak of modular invariant matrices.

While considerable work has been invested in the classification of modular invariant matrices Z for certain modular categories, in particular in the case $A^L = A^R$, it has become clear that a more sophisticated approach is called for. One such approach has been proposed by Rehren [61, 62]. We recall that $d = 1 + 1$ Minkowski space is the pseudo-Riemannian manifold $M = \mathbb{R}^2$ with constant metric $g = \mathrm{diag}(1, -1)$. Two points $(x_0, x_1), (y_0, y_1)$ are mutually space-like ($x \perp y$) iff $d^2(x, y) = (x_0 - y_0)^2 - (x_1^2 - y_1^2) < 0$. For $S \subset M$ we define the space-like complement $S^\perp = \{x \in M \mid x \perp y \ \forall y \in S\}$. A quantum field theory on M is an assignment $O \mapsto A(O) \subset \mathcal{B}(\mathcal{H}_0)$ defined for suitable $O \subset M$ and satisfying axioms similar to those in Definition 2.1.1. (Locality now means $O_1 \subset O_2^\perp \Rightarrow [A(O_1), A(O_2)] = \{0\}$.) We define a bijection $\phi : \mathbb{R} \times \mathbb{R} \to M$ by $(x_L, x_R) \mapsto (x_L + x_R, x_L - x_R)$ and write $\mathcal{O} = \{\phi(I_L \times I_R),\ I_L, I_R \in \mathcal{K}\}$. There is a theory of completely rational QFTs over M mirroring the one for theories over $\mathbb{R}$.

Given QFTs $(\mathcal{H}_0^L, A^L, \Omega^L), (\mathcal{H}_0^R, A^R, \Omega^R)$ on $\mathbb{R}$, we obtain a QFT A over M on the Hilbert space $\mathcal{H}_0^L \otimes \mathcal{H}_0^R$ with vacuum $\Omega^L \otimes \Omega^R$ as the assignment $\mathcal{O} \ni O \mapsto A^L(I_L) \otimes A^R(I_R) \subset \mathcal{B}(H_0^A \otimes \mathcal{H}_0^B)$ where $I_L \times I_R = \phi^{-1}(O)$. If A^L, A^R are completely rational, $B \supset A$ is a finite local extension and $\pi \in \mathrm{Rep}_f B$ then the restriction $\pi \restriction A$ decomposes as

$$\pi \restriction A \cong \bigoplus_{i \in \Delta^L, j \in \Delta^R} Z_{ij}\ \pi_i^L \otimes \pi_j^R,$$

where $Z_{ij} \in \mathbb{Z}_+$ and π_i^L (π_i^R) are the irreducible representations of A^L (A^R). In [62], Rehren proved that $Z_{00} = 1$ and $ZT^R = T^L Z$, but the question when $ZS^R = S^L Z$ holds remained open. The results in the remainder of this section are extracted from {11}.

2.3.1 THEOREM *Let A^L, A^R be completely rational and B a finite local extension of $A = A^L \otimes A^R$. Then B is completely rational, and the following are equivalent:*

(i) $ZS^R = S^L Z$, thus Z is a modular invariant for the pair A^L, A^R of chiral theories.

(ii) $\mu(B) = 1$.

(iii) B has trivial representation category.

2.3.2 REMARK We are convinced that the modular invariants identified in the above approach are precisely those which define a two dimensional CQFT in the sense of Segal. While this has not yet been shown for any rigorous definition of d=2 CQFT, we give a heuristic argument why this should be true. Namely, the conditions (ii), (iii) in the above theorem are equivalent to the vanishing of Roberts' local 1-cohomology of the Minkowski space QFT B. Considering the latter as a sort of obstruction, it does not seem unreasonable to expect that its vanishing is necessary and sufficient for the theory to make sense on any Riemann surface, as required by Segal's axioms. □

Theorem 2.3.1 raises the question of classifying the local extensions $B \supset A^L \otimes A^R$ with trivial $\mathrm{Rep}\, B$. This essentially reduces to the following purely categorical statement which is interesting in itself:

2.3.3 PROPOSITION *(i) Let k be an algebraically closed field and let $\mathcal{C}, \mathcal{D}$ be finite fusion categories over k. Let $\mathcal{E} = \mathcal{C} \boxtimes \mathcal{D}^{\mathrm{op}}$. Then there is a bijection between*

1. *Monoidal equivalences $F : \mathcal{C} \to \mathcal{D}$ modulo monoidal natural isomorphisms.*
2. *Isomorphism classes of Frobenius algebras $(\Gamma, m, \eta, \Delta, \varepsilon)$ in $\mathcal{E}$ such that*

$$\Gamma \cong \bigoplus_{i \in I} X_i \boxtimes Y_i^{\mathrm{op}},$$

where $\{X_i,\ i \in I\}$ and $\{Y_i,\ i \in I\}$ are complete sets of simple objects in $\mathcal{C}$ and $\mathcal{D}$, respectively.

(ii) If $\mathcal{C}$ and $\mathcal{D}$ are braided then braided equivalences $F : \mathcal{C} \to \mathcal{D}$ correspond to Frobenius algebras that are commutative w.r.t. the braiding on $\mathcal{E}$ given by

$$c_{\mathcal{E}}(U \boxtimes X, V \boxtimes Y) = c_{\mathcal{C}}(U, V) \boxtimes c_{\mathcal{D}^{\mathrm{op}}}(X, Y),$$

where $c_{\mathcal{D}^{\mathrm{op}}}(X, Y) = c_{\mathcal{D}}(Y, X)$.
(iii) If $\mathcal{C}$ and $\mathcal{D}$ are $*$-categories then monoidal equivalences F of $*$-categories (for which the isomorphisms $d^F_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ are unitaries) correspond to Frobenius algebras satisfying $\Delta = m^*, \varepsilon = \eta^*$. Furthermore, the braidings in (ii) are unitary.

Together with some observations in [61] this allows to prove the following.

2.3.4 THEOREM *Let A^L, A^R be completely rational chiral theories. There is a one-to-one correspondence between unitary equivalence classes of local extensions $B \supset A^L \otimes A^R$ such that $\mathrm{Rep}\, B$ is trivial — and therefore modular invariants of the type considered in [62] — and triples $([\hat{A}^L], [\hat{A}^R], [F])$, where $[\hat{A}^L], [\hat{A}^R]$ are equivalence classes of local extensions of A^L and A^R, respectively, and $[F]$ is the isomorphism class of a functor $F : \mathrm{Rep}(\hat{A}^L) \to \mathrm{Rep}(\hat{A}^R)$ establishing an equivalence of braided tensor $*$-categories.*

In view of Theorem 2.3.1, this result provides a classification in terms of triples $([\hat{A}^L], [\hat{A}^R], [F])$ of precisely those finite local extensions $B \supset A^L \otimes A^R$ for which Z is a modular invariant matrix. If we recall that the local extensions $\hat{A}^{L/R} \supset A^{L/R}$ are classified by the commutative Frobenius algebras in $\mathrm{Rep}\, A^{L/R}$, we see that the above classification can be formulated in purely categorical terms: consider (equivalence classes of) triples (Γ^L, Γ^R, F) where Γ^L (Γ^R) is a commutative Frobenius algebra in $\mathrm{Rep}\, A^L$ $(\mathrm{Rep}\, A^R)$ and $F : \Gamma^L - \mathrm{Mod}^0_{\mathrm{Rep}\, A^L} \to \Gamma^R - \mathrm{Mod}^0_{\mathrm{Rep}\, A^R}$ is a braided equivalence.

2.3.5 CONJECTURE *Let A^L, A^R be completely rational chiral CQFTs. The the $d = 2$ conformal field theories in the sense of Segal 'associated with (A^L, A^R)' (to be made precise!) are classified by the triples $([\Gamma^L], [\Gamma^R], [F])$ (or $([\hat{A}^L], [\hat{A}^R], [F])$) as above*

2.3.6 Remark We conclude by remarking that some support for the conjecture can be drawn from the heuristic considerations in [52], which we refrain from repeating. These authors concluded that the 2-dimensional conformal field theories associated with (A^L, A^R) are classified by triples $(\hat{A}^L, \hat{A}^R, \sigma)$, where $\hat{A}^{L/R} \supset A^{L/R}$ are finite local extensions and $\sigma : \Delta(\hat{A}^L) \mapsto \Delta(\hat{A}^R)$ is an isomorphism of the fusion rings of $\mathrm{Rep}\,\hat{A}^L$ and $\mathrm{Rep}\,\hat{A}^L$. Obviously, an equivalence of tensor categories gives rise to such an isomorphism, but the converse is in general not true. Given the categorical nature of the whole subject of conformal QFT, it seems mathematically much more natural to require an equivalence F of braided categories as done above rather than just an isomorphism of fusion rings in [52]. □

Author's Publications

PhD Thesis and papers derived from it:

{A} Superselection structure of quantum field theories in $1 + 1$ dimensions. 117pp. Doctoral dissertation. University of Hamburg, April 1997. DESY-preprint 97-073.

{A1} Disorder operators, quantum doubles, and Haag duality in $1 + 1$ dimensions. 8pp. *In:* Proceedings of the NATO Advanced Study Institute on *Quantum fields and quantum spacetime*, Cargèse 1996. Plenum Press, New York 1997.

{A2} Quantum double actions on operator algebras and orbifold quantum field theories. Commun. Math. Phys. **181**, 137–181 (1998).

{A3} The superselection structure of massive quantum field theories in $1 + 1$ dimensions. Rev. Math. Phys. **10**, 1147–1170 (1998).

{A4} On charged fields with group symmetry and degeneracies of Verlinde's matrix S. Ann. Inst. Henri Poincaré B (Phys. Théor.) **7**, 359–394 (1999).

Postdoctoral Work:

{1} On soliton automorphisms in massive and conformal theories. Rev. Math. Phys. **11**, 337–359 (1999).

{2} Galois theory for braided tensor categories and the modular closure. Adv. Math. **150**, 151–201 (2000).

{3} Multi-interval subfactors and modularity of representations in conformal field theory. With Y. Kawahigashi and R. Longo. Commun. Math. Phys. **219**, 631–669 (2001).

{4} Conformal field theory and Doplicher-Roberts reconstruction. *In:* R. Longo (ed.): *Mathematical physics in mathematics and physics. Quantum and operator algebraic aspects.* Fields Inst. Commun. **30**, 297–319 (2001).

{5} From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories. To appear in J. Pure Appl. Alg. (MSRI preprint 2002-003, `math.CT/0111204`).

{6} From subfactors to categories and topology II. The quantum double of tensor categories and subfactors. To appear in J. Pure Appl. Alg. (MSRI preprint 2002-004, `math.CT/0111205`).

{7} On the structure of modular categories. To appear in Proc. Lond. Math. Soc. (`math.CT/0201017`)

{8} Representations of algebraic quantum groups and reconstruction theorems for tensor categories. With L. Tuset and J. E. Roberts. `math.QA/0203206`. Submitted.

{9} Galois extensions of braided tensor categories and braided crossed G-categories. `math.CT/0209093`.

{10} Conformal orbifold theories, crossed G-categories and quasiabelian cohomology. In preparation.

{11} Extensions and modular invariants of rational conformal field theories. In preparation.

{12} Regular representations of algebraic quantum groups and embedding theorems. With L. Tuset. In preparation.

{13} On the structure and representation theory of rational chiral conformal theories. In preparation.

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