Coherent States And The Classical Limit In Quantum Mechanics

$\hbar \longrightarrow 0$

Bram Merten

Radboud University Nijmegen
Bachelor's Thesis Mathematics/Physics 2018
Department Of Mathematical Physics
Under Supervision of: Michael Mueger

Abstract

A thorough analysis is given of the academical paper titled "The Classical Limit for Quantum Mechanical Correlation Functions", written by the German physicist Klaus Hepp. This paper was published in 1974 in the journal of Communications in Mathematical Physics [1].

The part of the paper that is analyzed summarizes to the following: "Suppose expectation values of products of Weyl operators are translated in time by a quantum mechanical Hamiltonian and are in coherent states centered in phase space around the coordinates $(\hbar^{-1/2}\pi, \hbar^{-1/2}\xi)$, where (π, ξ) is an element of classical phase space, then, after one takes the classical limit $\hbar \longrightarrow 0$, the expectation values of products of Weyl operators become exponentials of coordinate functions of the classical orbit in phase space."

As will become clear in this thesis, authors tend to omit many non-trivial intermediate steps which I will precisely include. This could be of help to any undergraduate student who is willing to familiarize oneself with the reading of academical papers, but could also target any older student or professor who is doing research and interested in Klaus Hepp's work.

Preliminary chapters which will explain all the prerequisites to this paper are given as well.

Table of Contents

0 Preface

This thesis gives a full overview of "The Classical Limit for Quantum Mechanical Correlation Functions" by Klaus Hepp. Including all the prerequisites needed in order to understand the paper.

Chapter 1 'Introduction' quickly refreshes the most basic principles of quantum mechanics and from this it argues why the correspondence between classical and quantum mechanics is of interest to many (mathematical) physicists. It is not meant to give a rigorous description of the concepts used, but rather to sketch a background to the topic.

Chapter 2 'Operator theory' is heavily related to the field of functional analysis and is needed in order to understand the preliminaries to the paper. We start with the most basic definitions and properties of operators and quickly move on to quantum mechanical operators that are used in modern day quantum mechanics and in the paper. The last subsection of this chapter lists important and more advanced theorems from functional analysis that are constantly used later on.

Chapter 3 'Coherent states' are a vital concept in order to understand the classical limit. It is explained what coherent states are from the example of the quantum mechanical harmonic oscillator. Furthermore, various properties of coherent states are proven. Since it is assumed in the paper that operators of interest are taken in coherent states, understanding of this topic is crucial.

Chapter 4 'Weyl systems and the classical limit', starts off with a discussion of Weyl systems, the Weyl algebra and, from that, Weyl operators. First results of classical limits in Weyl systems are calculated in order to get familiarize with the idea of taking a classical limit. Subsequently, The correspondence of classical and quantum mechanics as discussed in chapter 1 is taken apart and understood in more mathematical detail.
In the last long subsection of this chapter, Hepp's theorem is introduced and thoroughly analyzed.

Chapter 5 'Conclusion' gives quick final thoughts about the thesis.

The reader who thinks he or she is well known with all of the mathematical preliminaries listed in the table of contents and is solely interested in the analysis of Hepp's paper may wish to immediately skip to Section (4.3).

Very basic knowledge of quantum mechanics and (functional) analysis is assumed.

1 Introduction

1.1 About Quantum Mechanics

Quantum mechanics is an extremely successful theory. It has in fact been such a successful theory that its predictions have never been shown to fail before, although some specific applications are beyond the reach of current calculational ability. Nevertheless, quantum mechanics has features that are strange compared to classical mechanics and uses concepts that some physicists have found difficult to accept.
Quantum mechanics is an *indeterministic* theory: even in principle, outcomes of experiments cannot possibly be predicted beforehand. It is postulated into the theory itself that this is the fundamental way in which nature works. Only *probabilities* for outcomes of those experiments can be predicted by calculation.
Another important postulate of quantum mechanics is the postulate of *wave-particle duality* which states that any object or any system consisting of particles exhibit both wave-like and particle-like behaviour.

1.2 About The Wave function

In quantum mechanics, (ensembles of) physical systems are characterized by a *wave function* $\psi(\mathbf{x}, t)$, which contains all information that can possibly be known about the system. We note that the wave function is a function that is solely dependent on the position variable $\mathbf{x} \in \mathbb{R}^n$ and time coordinate t.
Even though the wave function ψ encodes all possible information that can be known about the system, it is never directly observable. If ψ is properly normalized (as should be) then its interpretation is given by *Born's statistical interpretation*: $|\psi(\mathbf{x}, t)|^2$ is the probability distribution for the position of the particle.
From this interpretation it should be clear that it's nonsensical to talk about formulas determining direct variables such as position and momentum.
Rather, *expectation values* of dynamical variables are the only variables that can be calculated:

$$< A >= \frac{< \psi|\hat{A}|\psi >}{< \psi|\psi >}. \tag{1.2.1}$$

This should also further clarify the fact that all information of a system is encoded in ψ: if ψ is determined then so are the expectation values of dynamical variables.
As ψ encodes everything we would like to know, we would like to know how ψ evolves in time.

Time evolution of ψ is given by the well known *time-dependent Schrödinger equation (TDSE)*:

$$i\hbar\partial_t\psi = \hat{H}\psi, \tag{1.2.2}$$

and this time evolution of ψ determines how expectation values evolve in time according to (1.2.1).
The wave function ψ and its time-evolution give rise to the wave-like aspect of quantum theory, while the interpretation of the wave function gives rise to the particle-like aspect.

1.3 About The Correspondence of Classical and Quantum mechanics

The discussion between classical mechanics and quantum mechanics and its intersection is as old as quantum theory itself.
Even though nature seems to abide to the rules of quantum mechanics, classical mechanics has (almost) always given reasonable estimates. Therefore, if one gets rid of the assumption of quantization ($\hbar \longrightarrow 0$) introduced in quantum mechanics, one hopes to retrieve classical results or at least something close to it. Reality turns out to be a bit more difficult though.

Time evolution of classical orbits in phase space is governed by Hamilton's equations of motion and, given suitable initial conditions, this set of differential equations always has a locally unique solution (this will be treated more rigorously in chapter (4.2)).

Time evolution of quantum mechanical orbits in phase space is governed by the time evolution of the wave function ψ, which is given by the TDSE. The TDSE is a first order differential equation and, given suitable initial conditions, it can be shown that a global (but possibly non-unique) solution always exists. (This is also treated more thoroughly in chapter (4.2))

The correspondence between these two time evolutions is the one that is of interest. In general, a few assumptions are needed to obtain a rigorous transition from quantum mechanical orbits to classical orbits.
The correspondence is most closely linked by the *Ehrenfest relations* (discussed in (3.2)) and, indeed, when we assume $\hbar \longrightarrow 0$ in coherent states for q_h and p_h (see section (4.3)) and centered around the large mean of specific coordinates, the Ehrenfest relations give a rigorous transition back to the classical counterpart as described by Hamilton's equations (see for example [2]).
The paper written by Hepp exhibits a special case of this transition.

2 Operator Theory

2.1 Basic functional analysis

Physical quantities in quantum mechanics are represented by their respective operators acting on a Hilbert space $\mathscr{H}$. In this section we will build up knowledge on how operators act on Hilbert spaces using functional analysis and introduce quantum mechanical operators and their technicalities.

Definition 2.1. An **operator** A is a mapping $A : U \longrightarrow V$ where U and V are vector spaces over a field K.

Definition 2.2. An operator A is called a **linear operator** if $A(\alpha x_1 + \beta x_2) = \alpha A(x_1) + \beta A(x_2)$ for all $x_1, x_2 \in U$ and $\alpha, \beta \in K$.

For the remainder of this thesis, we will set $K = \mathbb{C}$, $U = V$ and equip this vector space with an inner product such that it is complete in the metric $d(x, y) = \|x - y\| = \langle x - y, x - y\rangle^{1/2}$, making it a complex Hilbert space $\mathscr{H}$.

Definition 2.3. A linear operator $A : \mathscr{H} \longrightarrow \mathscr{H}$ is called **bounded** if $\exists C \in \mathbb{R}$: $\|A\psi\| \le C\|\psi\|$ for all $\psi \in \mathscr{H}$.

Obviously, not every operator has to be bounded. Operators that are not bounded are called **unbounded operators**.
In fact, most quantum mechanical operators are unbounded and it is necessary to define tools to deal with those unbounded operators. A little more terminology is necessary before elaborating on this topic:

Definition 2.4. Let A be a bounded operator. The **adjoint operator** A^* of A is the unique operator satisfying $\langle\phi, A\psi\rangle = \langle A^*\phi, \psi\rangle$ for all $\psi, \phi \in \mathscr{H}$. A bounded operator A is called **self-adjoint** if $A^* = A$.

We demand our quantum mechanical operators to be self-adjoint. Reasons for this are, for example, that the spectral theorem is only defined for normal operators (and self-adjoint operators are normal) and Stone's theorem demands that the infinitesimal generator A of a one-parameter unitary group $U(t)$ is self-adjoint (See section (2.3) for more detail).
Quantum mechanical operators are most of the time unbounded and in order to talk about self-adjointness of unbounded operators we introduce more tools.

Theorem 2.1. *Let A be a linear, self-adjoint operator defined on all of $\mathcal{H}$, then A is bounded.*

Proof. The proof will be given shortly, after some more definitions. □

Remark. The negation of theorem (2.1)is important: If A is a linear, self-adjoint and unbounded operator, then it is impossible to define A on the entirety of the Hilbert space $\mathcal{H}$. Unbounded operators are subsequently only defined on a subspace of the relevant Hilbert space, which is called the **domain** of an operator. More formally:

Definition 2.5. An **unbounded operator** A on a Hilbert space $\mathcal{H}$ is a linear map $A : Dom(A) \longrightarrow \mathcal{H}$ with $Dom(A) \neq \mathcal{H}$ a dense subspace of $\mathcal{H}$ such that every element of $Dom(A)$ is properly mapped into $\mathcal{H}$ under A.

From this definition, we can define the adjoint operator of an unbounded operator, and subsequently self-adjointness of unbounded operators.

Definition 2.6. Let A be an unbounded operator on a Hilbert space $\mathcal{H}$.
We construct the **Adjoint Operator** A^* of A as follows: For any vector $\phi \in \mathcal{H}$, we say that $\phi \in Dom(A^*)$ if and only if the linear functional $\langle \phi, A\psi \rangle$, defined on $Dom(A)$, is bounded for all $\psi \in \mathcal{H}$.
Then, for all $\phi \in Dom(A^*)$, let $A^*\phi$ be the unique vector such that $\langle A^*\phi, \psi \rangle = \langle \phi, A\psi \rangle$ for all $\psi \in \mathcal{H}$. Note that Riesz theorem (if this theorem is unfamiliar, see for example [3, page 35]) guarantees the uniqueness and existence of the vector $A^*\phi$.
The operator A^* is the **Adjoint Operator** of A.

Definition 2.7. An unbounded operator A on a Hilbert space $\mathcal{H}$ is called **Symmetric** if $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ for all $\phi, \psi \in \mathcal{H}$.

Definition 2.8. An unbounded operator A is **self-adjoint** if $Dom(A^*) = Dom(A)$ and $A^*\psi = A\psi$ for all $\psi \in \mathcal{H}$ or, equivalently, $A^* = A$.
We note that by definition every self-adjoint operator is symmetric.

Definition 2.9. An unbounded operator A is called an **extension** of an unbounded operator B if $Dom(B) \subseteq Dom(A)$ and $A = B$ on $Dom(B)$.

Extensions of certain quantum mechanical operators will be handled in more depth in section (2.2). We now prove a small but important theorem about extensions of unbounded operators.

Theorem 2.2. *The operator A^* is an extension of the unbounded operator A if and only if A is symmetric.*

Proof. $\Longrightarrow$ If A^* is an extension of A then

$$\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle = \langle A\phi, \psi \rangle \tag{2.1.1}$$

for all $\phi, \psi \in Dom(A)$. Therefore A is symmetric.
$\Longleftarrow$ If A is symmetric then we use the Cauchy-Schwarz inequality:

$$|\langle \phi, A\psi \rangle| = |\langle A\phi, \psi \rangle| \leq \|A\phi\| \|\psi\| \tag{2.1.2}$$

for all $\psi, \phi \in Dom(A)$. Therefore $\phi \in Dom(A^*)$ and we see that the unique vector $A^*\phi$ for which $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$ is exactly $A\phi$. Hence they A^* coincides with A on $Dom(A)$. □

Definition 2.10. An unbounded operator A on a Hilbert space $\mathscr{H}$ is **closed** if the graph $\Gamma(A) = \{(\psi, A\psi), \psi \in \mathscr{H}\}$ of A is a closed subset of $\mathscr{H} \times \mathscr{H}$. The operator A is **closable** if the closure in $\mathscr{H} \times \mathscr{H}$ of the graph $\Gamma(A)$ is the graph of a function.

We are now ready to state a lemma which we will use to prove theorem (2.1).

Lemma 2.3. *If A is an unbounded operator on a Hilbert space $\mathscr{H}$ then the graph $\Gamma(A^*)$ of the operator A^* is closed in $\mathscr{H} \times \mathscr{H}$. Moreover, a symmetric operator is always closable.*

Proof. Consider a sequence $\psi_n \in Dom(A^*)$ that converges to some $\psi \in \mathscr{H}$. Suppose also that $A^*\psi_n$ converges to some $\phi \in \mathscr{H}$ then $\langle \psi_n, A\Phi \rangle = \langle A^*\psi_n, \Phi \rangle$ for any $\Phi \in \mathscr{H}$ and, for any vector $\xi \in Dom(A)$, we have that:

$$\langle \psi, A\xi \rangle = \lim_{n \longrightarrow \infty} \langle \psi_n, A\xi \rangle = \lim_{n \longrightarrow \infty} \langle A^*\psi_n, \xi \rangle = \langle \phi, \xi \rangle. \tag{2.1.3}$$

Hence $\psi \in Dom(A^*)$ and $A^*\psi = \xi$ so that $\Gamma(A^*)$ is closed.
If the operator A is symmetric then by theorem (2.2) A^* is an extension of A. A^* is closed so A has a closed extension and is therefore also closable. □

We are finally able to give the proof of Theorem (2.1):

Proof. A is self-adjoint and therefore symmetric. By lemma (2.3), A is closable. Since A is bounded, $Dom(A) = \mathscr{H}$ so that the closure of A is A itself. Therefore, A is a closed operator on all of $\mathscr{H}$ and by the closed-graph theorem A is bounded. □

2.2 Operators in Quantum Mechanics

In quantum mechanics we demand probabilities calculated from the wave function to be finite and normalized to sum up to 1. We therefore demand that:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty \tag{2.2.1}$$

for all $\psi \in \mathcal{H}$. The space of measurable functions that satisfies this condition is Hilbert space, called the Hilbert space of **square integrable functions** and is denoted by $L^2(\mathbb{R})$. This space is therefore also called the **quantum Hilbert space** or "the arena" of quantum mechanics. Its inner product is simply $\langle\psi|\phi\rangle = \int_{-\infty}^{\infty} \overline{\psi(x)}\phi(x)dx$.

Definition 2.11. Let the quantum mechanical Hilbert space be $\mathcal{H} = L^2(\mathbb{R})$.
The **position operator X** and **momentum operator P** respectively are defined as:

$$(X\psi)(x) = x\psi(x), \qquad (P\psi)(x) = -i\hbar\frac{d\psi}{dx}.$$

We now immediately touch the relevancy of the previous subsection. If $\psi \in L^2(\mathbb{R})$ then the function $X\psi(x) = x\psi(x)$ can easily fail to be in $L^2(\mathbb{R})$. Similarly, a function $\psi \in L^2(\mathbb{R})$ doesn't have to be differentiable and even if it is differentiable then $\frac{d\psi}{dx}$ doesn't have to be in $L^2(\mathbb{R})$.
The position and momentum operators are therefore *unbounded operators* and they are only defined on their respective, suitable dense subspaces $Dom(X)$ and $Dom(P)$ of $L^2(\mathbb{R})$.
We immediately note that the operators X and P do not commute trivially. The **canonical commutation relation** of X and P is given by:

$$[X,P] \equiv XP - PX = i\hbar I. \tag{2.2.2}$$

Theorem 2.4. *The unbounded operators X and P are symmetric operators.*

Proof. Suppose $\phi, \psi, x\phi(x), x\psi(x) \in L^2(\mathbb{R})$. Then:

$$\langle\phi, X\psi\rangle = \int_{-\infty}^{\infty} \overline{\phi(x)}x\psi(x)dx = \int_{-\infty}^{\infty} \overline{x\phi(x)}\psi(x)dx = \langle X\phi, \psi\rangle.$$

Since $x \in \mathbb{R}$ and multiplications of elements in $L^2(\mathbb{R})$ are in $L^2(\mathbb{R})$ too.
Suppose $\phi, \psi, \frac{d\phi}{dx}, \frac{d\psi}{dx} \in L^2(\mathbb{R})$. Then:

$$\begin{aligned}\langle\phi, P\psi\rangle &= \int_{-\infty}^{\infty} \overline{\phi(x)}\Big(-i\hbar\frac{d\psi}{dx}dx\Big) = -i\hbar\overline{\phi(x)}\psi(x)|_{-\infty}^{\infty} + i\hbar\int_{-\infty}^{\infty}\overline{\frac{d\phi}{dx}}\psi(x)dx \\ &= \int_{-\infty}^{\infty}\Big(-i\hbar\frac{d\phi}{dx}\Big)\psi(x)dx = \langle P\phi, \psi\rangle.\end{aligned}$$

The second last step is true as $\psi(x)$ and $\phi(x)$ tend to 0 as $x \longrightarrow \infty$, therefore the first part of the partial integral vanishes. $\square$

We now turn our attention to several unbounded self-adjoint quantum mechanical operators and describe the dense domains on which they are properly defined.
We analyze each operator and combine them to form the dense domain of the Hamiltonian operator corresponding to the Schrödinger equation.
This will naturally lead us to define domains of sums of operators too.

Theorem 2.5. *Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function (such as the potential function). Let $V(X)$ be the unbounded operator given by:*

$$[V(X)\psi](\vec{x}) = V(\vec{x})\psi(\vec{x}), \tag{2.2.3}$$

where $\vec{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, with its respective domain:

$$Dom(V(X)) = \{\psi \in L^2(\mathbb{R}^n) \mid V(\vec{x})\psi(\vec{x}) \in L^2(\mathbb{R}^n)\} \subseteq L^2(\mathbb{R}^n). \tag{2.2.4}$$

Then $Dom(V(X))$ is dense in $L^2(\mathbb{R}^n)$ and $V(X)$ is self-adjoint on $Dom(V(X))$

Proof. We define subsets $A_m \subseteq \mathbb{R}^n$ by:

$$A_m = \{\vec{x} \in \mathbb{R}^n || V(\vec{x})| < m\} \tag{2.2.5}$$

Then $\cup_m A_m = \mathbb{R}^n$ and for any $\psi \in L^2(\mathbb{R}^n)$ we have $1_{A_m}\psi \in Dom(V(X))$ (Where 1_{A_m} is the indicator function). We use the dominated convergence theorem to conclude $1_{A_m}\psi \longrightarrow \psi$ as $m \longrightarrow \infty$. Therefore $Dom(V(X))$ is dense in $L^2(\mathbb{R}^n)$.
As $V(X)$ is a real-valued measurable function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ we trivially have:

$$\langle V(X)\psi, \phi\rangle = \langle \psi, V(X)\phi\rangle \tag{2.2.6}$$

for all $\psi, \phi \in Dom(V(X))$. Therefore $V(X)$ is symmetric on $Dom(V(X))$ such that $V(X)^*$ is an extension of $V(X)$.
Now let $\phi \in Dom(V(X)^*)$, recall that this means that the mapping:

$$\psi \mapsto \int_X \overline{\phi(x)} V(x)\psi(x)dx \tag{2.2.7}$$

is a bounded linear functional for all $\psi \in Dom(V(X))$. There exists a unique bounded extension for this linear functional onto $L^2(\mathbb{R}^n)$ such that there is a unique $\xi \in L^2(\mathbb{R}^n)$ that satisfies:

$$\int_X \overline{\psi(x)} V(x)\phi(x)dx = \int_X \overline{\xi(x)}\phi(x)dx. \tag{2.2.8}$$

Therefore, for all $\xi \in Dom(V(X))$:

$$\int_X [\overline{\psi(x)} V(x) - \overline{\xi(x)}]\phi(x)dx = 0. \tag{2.2.9}$$

If we let $\phi = (\psi V - \xi)1_{A_m}$ then $\psi V - \xi = 0$ almost everywhere on A_m and therefore almost everywhere on $\mathbb{R}^n$. This implies $\psi V = \xi \in Dom(V(X))$ so that $Dom(V(X)^*) = Dom(V(X))$.
We conclude that $V(X)$ is self-adjoint on $Dom(V(X))$. □

Remark. If we let $V(X) = x_j$ for some $j \in \{1, ..., n\}$ then we obtain the result for the position operator:

Corollary 2.5.1. *The position operator X_j is self-adjoint on:*

$$Dom(X_j) = \{\psi \in L^2(\mathbb{R}^n) \mid x_j\psi(\vec{x}) \in L^2(\mathbb{R}^n)\} \subseteq L^2(\mathbb{R}^n). \tag{2.2.10}$$

Theorem 2.6. *Let P_j be the momentum operator for some $j \in \{1, ..., n\}$. The domain of the momentum operator is then as follows:*

$$Dom(P_j) = \{\psi \in L^2(\mathbb{R}^n) \mid k_j\hat{\psi}(\vec{k}) \in L^2(\mathbb{R}^n)\} \subseteq L^2(\mathbb{R}^n). \tag{2.2.11}$$

Here, $\hat{\psi}$ is the Fourier transform of ψ. We define the momentum operator P_j on this domain as follows:

$$P_j\psi = \mathscr{F}^{-1}(\hbar k_j\hat{\psi}(\vec{k})). \tag{2.2.12}$$

Here $\mathscr{F}^{-1}$ is the inverse Fourier transform operator. Then P_j is self-adjoint on $Dom(P_j)$.

Proof. See for example [4, Section 9.8], note that the unitary of the Fourier transform is crucial here. □

Theorem 2.7. *Let $\Delta = \nabla^2$ be the Laplace operator. Then:*

$$Dom(\Delta) = \{\psi \in L^2(\mathbb{R}^n) \mid |\vec{k}|^2\hat{\psi}(\vec{k}) \in L^2(\mathbb{R}^n)\} \tag{2.2.13}$$

and define Delta on this domain as follows:

$$\Delta\psi = -\mathscr{F}^{-1}(|\vec{k}|^2\hat{\psi}(\vec{k})).$$

Here $\mathscr{F}^{-1}$ is the inverse Fourier transform operator and $\hat{\psi}$ is the Fourier transform of ψ. Then Δ is self-adjoint on $Dom(\Delta)$

Proof. Note that this proof has to be very similar to that of Theorem (2.6). See [4, Section 9.8] for more detail. □

Remark. The kinetic energy operator

$$\hat{K} = \frac{-\hbar^2}{2m}\Delta$$

is self-adjoint on the same domain as Δ, as can easily be seen by the previous theorem.

We have now defined the proper dense domains in $L^2(\mathbb{R}^n)$ for the relevant self-adjoint quantum mechanical operators which make up the Hamiltonian operator.
The only task remaining is to sum these operators to obtain the Hamiltonian operator itself.

Definition 2.12. Let A and B be unbounded operators on a Hilbert space $\mathscr{H}$, then $A+B$ is the operator given by $(A+B)\psi = A\psi + B\psi$ with the following domain:

$$Dom(A+B) = Dom(A) \cap Dom(B). \tag{2.2.14}$$

Theorem 2.8. *(Kato-Rellich Theorem):*
Suppose A, B to be unbounded self-adjoint operators on a Hilbert space $\mathscr{H}$ and $Dom(A) \subseteq Dom(B)$ and $\exists a, b \in \mathbb{R}_{>0}, a < 1$ such that the following condition holds for all $\psi \in Dom(A)$:

$$\|B\psi\| \leq a\|A\psi\| + b\|\psi\| \tag{2.2.15}$$

Then the operator $A+B$ is self-adjoint on $Dom(A)$.

Proof. See [4, Section 9.8] again for the advanced proof of this theorem. Some of these proofs are unfortunately a bit too technical for the relevancy to this thesis. □

However, this theorem will greatly help us to prove our 'final result' of this chapter, which we will see being used in Hepp's paper too.

Theorem 2.9. *Suppose $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a measurable function where $n \in \{1,2,3\}$ that can be decomposed as $V = V_1 \cup V_2$. Here, $V_1 \in L^2(\mathbb{R}^n)$ is a real-valued measurable function and V_2 a bounded, real valued measurable function. Then the Hamiltonian operator $\hat{H} = \frac{\hbar^2}{2m}\Delta + V(X)$ is self-adjoint on $Dom(\Delta)$*

Proof. Let $A = \frac{-\hbar^2}{2m}\Delta$ and $B = V(X)$. Let $\epsilon > 0$ and $\psi \in Dom(\Delta)$, according to [5, Vol.2 Theorem X.20-X.29], there exists a constant $c_\epsilon > 0$ such that for all $x \in \mathbb{R}^n$:

$$|\psi(\vec{x})| \leq \epsilon\|\Delta\psi\| + c_\epsilon\|\psi\| \tag{2.2.16}$$

Therefore we have:

$$\|V\psi\| \leq \sup|\psi(\vec{x})|\|V_1\| + \sup|V_2(\vec{x})|\|\psi\| \leq \epsilon\|V_1\|\|\Delta\psi\| + (c_\epsilon\|V_1\| + \sup|V_2(\vec{x})|)\|\psi\| \tag{2.2.17}$$

Such that $Dom(\Delta) \subseteq Dom(V(X))$. It is implicit in the statement of the theorem and definition of $V(X)$ that $Dom(V(X)) \subseteq Dom(\Delta)$. If we choose $\epsilon < 1$ then the Kato-Rellich theorem concludes the proof. □

2.3 Important Results from Functional Analysis

We conclude this section with some essential results from functional analysis whose proofs are too technical and long to give, but are very important and constantly used in quantum mechanics and in this paper.

Definition 2.13. A **one-parameter group** on $\mathscr{H}$ is a collection of unitary operators $U(t)$ with $t \in \mathbb{R}$ such that $U(0) = I$ and $U(x+y) = U(x)U(y)$ for all $x, y \in \mathbb{R}$.
A one-parameter group is called **strongly continuous** if $\lim_{y\to x} \|U(x)\psi - U(y)\psi\| = 0$ for every $\psi \in \mathscr{H}$ and $x, y \in \mathbb{R}$.

Definition 2.14. If $U(t)$ is a strongly continuous one-parameter group of unitary operators on a Hilbert space $\mathscr{H}$, then the **infinitesimal generator** A of $U(t)$ is defined as the operator:

$$A\psi = \lim_{t \longrightarrow 0} \frac{1}{i} \frac{U(t)\psi - \psi}{t}. \tag{2.3.1}$$

The dense domain $Dom(A)$ of the infinitesimal operator is:

$$Dom(A) = \{\psi \in \mathscr{H} \mid \lim_{t \longrightarrow 0} \frac{1}{i} \frac{U(t)\psi - \psi}{t} \text{ exists in the norm topology on } \mathscr{H}\}. \tag{2.3.2}$$

Theorem 2.10. *(Stone's theorem):*
Let $U(t)$ be a strongly continuous one-parameter group of unitary operators on a Hilbert space $\mathscr{H}$, then for the infinitesimal operator A of $U(t)$: $U(t) = \exp(itA)$ for all $t \in \mathbb{R}$ and A is densely defined and self-adjoint.

Theorem 2.11. *(The spectral theorem for unbounded self-adjoint operators):*
Let A be a self-adjoint operator on $\mathscr{H}$. Then there exists a σ-finite measure space $(X, \mathscr{M}, \mu)$, a unitary map $U : \mathscr{H} \longrightarrow L(X, \mathscr{M}, \mu)$ and a measurable function f on X such that:

$$U(Dom(A)) = \{\psi \in L^2(X, \mathscr{M}, \mu) \mid f\psi \in L^2(X, \mathscr{M}, \mu)\} \tag{2.3.3}$$

and

$$(UAU^{-1}(\psi)) = f(x)\psi(x) \tag{2.3.4}$$

for all $\psi \in U(Dom(A))$.

Theorem 2.12. *(Baker-Campbell-Hausdorff's theorem): If operators X and Y are in some lie algebra $\mathfrak{g}$ over a field of characteristic 0, then the expression:*

$$Z = log(e^X e^Y) \tag{2.3.5}$$

can be written as an infinite sum of elements of $\mathfrak{g}$.

We consider a special case of the Baker-Campbell-Hausdorff theorem, which will be an important tool to work with exponents of operators.

Corollary 2.12.1. *Let $ad_X Y = [X, Y]$ be a linear operator on a lie algebra $\mathfrak{g}$ for some fixed $X \in \mathfrak{g}$. Then let Ad_A be the linear transformation of $\mathfrak{g}$ given by $Ad_A Y = AYA^{-1}$ for some matrix Lie group G and $A \in G$. Then:*

$$Ad_{e^X} Y = e^X Y e^{-X} = e^{ad_X} Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + ... \tag{2.3.6}$$

Specially, if $[X, Y]$ is central (that is, commuting with both X and Y), then:

$$e^{sX} Y e^{-sX} = Y + s[X, Y]. \tag{2.3.7}$$

Moreover:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{2.3.8}$$

or, equivalently:

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B. \tag{2.3.9}$$

In the special case that A, B are unbounded self-adjoint operators such that $[A, B] = i\hbar I$ then:

$$e^{i(sA+tB)} = e^{ist\hbar/2} e^{isA} e^{itB} = e^{-ist\hbar/2} e^{itB} e^{isA}, \tag{2.3.10}$$

such that:

$$e^{isA} e^{itB} = e^{-ist\hbar} e^{itB} e^{isA}. \tag{2.3.11}$$

Which is a commutation relation that is constantly used in quantum mechanics.
See for example [6, Prop. 2.20] for more reading.

Theorem 2.13. *(Duhamel's Formula):*
If $[A_{i,j}(t)]_{1\leq i,j\leq n}$ is a matrix-valued function of $t \in \mathbb{R}$ that is C^∞ for every matrix element of $A_{i,j}(t)$ then:

$$\frac{d}{dt} e^{A(t)} = \int_0^1 e^{sA} A'(t) e^{(1-s)A(t)} ds. \tag{2.3.12}$$

See for example [7] for more reading.

3 Coherent States

Coherent states are a vital part of understanding classical limits in quantum mechanics. Loosely stated coherent states are the states of the quantum mechanical harmonic oscillator that most closely resemble the oscillatory behaviour of the classical harmonic oscillator.
Intuitively it should then be clear that these are the states in which quantum mechanical operators are in states of minimal uncertainty.
We shall see that expectation values of quantum mechanical operators taken in coherent states obey classical equations of motion, which could undoubtedly be helpful for examining classical limits.

3.1 The Quantum Mechanical Harmonic Oscillator

We motivate the idea of coherent states from the quantum mechanical linear harmonic oscillator.
Let m be the mass of a 1-dimensional particle attracted to the origin by a force proportional to the displacement from the centre. This force is given by Hooke's law: $F = -kx$ with k the force constant, therefore:

$$V(x) = -\int F dx = \frac{1}{2}kx^2. \tag{3.1.1}$$

It should be noted that this potential is of great importance in both classical and quantum physics, as it can be used as an approximation of an arbitrary continuous potential $W(x)$ which is nearby a stable equilibrium position of a point $x = a$ (this can informally be visualized as a well in a potential).
To elaborate on this small intermezzo quickly we note that if we expand a sufficiently regular potential $W(x)$ around a then:

$$W(x) = W(a) + W'(a)(x-a) + \frac{1}{2}W''(a)(x-a)^2 + ... \tag{3.1.2}$$

Since $W(x)$ is at stable equilibrium at $x = a$ we have $W'(a) = 0$ and $W''(a) > 0$.
If we let a be the origin of coordinates and $W(a)$ the origin of the energy scale then the potential of the harmonic oscillator with $k = W''(a) =$ constant is the first order approximation to $W(x)$.
The quantum mechanical linear harmonic oscillator is therefore of great importance to quantum systems for which there exist small vibrations about a point of stable equilibrium such as vibrational motion in molecules.

It is convenient to write use $\omega = \sqrt{\frac{k}{m}}$ and $V(x) = \frac{1}{2m}(m\omega x)^2$ so that the Hamiltonian operator can be written as:

$$\hat{H} = \frac{1}{2m}[P^2 + (m\omega X)^2]. \tag{3.1.3}$$

The basic idea in order to evaluate the Hamiltonian and thereafter the Schrödinger equation is to factor the Hamiltonian operator, this will be done using so-called ladder operators.

Definition 3.1. The **ladder operators** $a_\pm$ are defined as:

$$a_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp iP + m\omega X). \tag{3.1.4}$$

a_+ is called the **raising operator** and a_- is called the **lowering operator**.

We immediately note that we can write the X and P operator in terms of ladder operators:

$$X = \sqrt{\frac{\hbar}{2m\omega}}(a_- + a_+), \qquad P = \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}(a_- - a_+). \tag{3.1.5}$$

And similarly:

$$X^2 = \frac{\hbar}{2m\omega}(a_- + a_+)^2, \qquad P^2 = -\frac{\hbar m\omega}{2}(a_- - a_+)^2. \tag{3.1.6}$$

Which will be relevant for determining uncertainties of operators as defined in the next subsection.
The commutation relations between ladder operators can easily be calculated from their definitions:

$$a_- a_+ = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}I, \qquad a_+ a_- = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}I. \tag{3.1.7}$$

From that we easily note the relations:

$$[a_-, a_+] = I, \qquad \hat{H} = \hbar\omega\Big(a_- a_+ - \frac{1}{2}I\Big) = \hbar\omega\Big(a_+ a_- + \frac{1}{2}I\Big). \tag{3.1.8}$$

And also the following commutation relations:

$$[\hat{H}, a_-] = \hbar\omega a_-[a_+, a_-] = -\hbar\omega a_-, \qquad [\hat{H}, a_+] = \hbar\omega a_+[a_-, a_+] = \hbar\omega a_+. \tag{3.1.9}$$

The time-independent Schrödinger equation (TISE) for the harmonic oscillator then takes the following form:

$$\hbar\omega\Big(a_\pm a_\mp \pm \frac{1}{2}I\Big)\psi = E\psi. \tag{3.1.10}$$

Now follows the first important result, we also switch to the usual Dirac bra-ket notation to stress the effect of the operators.

Theorem 3.1. *If $\psi \in L^2(\mathbb{R})$ satisfies the TISE with energy E, that is: $\hat{H}|\psi\rangle = E|\psi\rangle$ then $a_+|\psi\rangle$ and $a_-|\psi\rangle$ satisfy the TISE with respective energy levels $(E+\hbar\omega)$ and $(E-\hbar\omega)$.*

Proof.

$$\hat{H}a_+|\psi\rangle = (a_+\hat{H}+\hbar\omega a_+)|\psi\rangle = a_+(E+\hbar\omega)|\psi\rangle = (E+\hbar\omega)a_+|\psi\rangle,$$
$$\hat{H}a_-|\psi\rangle = (a_-\hat{H}-\hbar\omega a_-)|\psi\rangle = a_-(E-\hbar\omega)|\psi\rangle = (E+\hbar\omega)a_-|\psi\rangle.$$

□

After applying the lowering operator repeatedly, there must be some state after which the application of the lowering operator has no more effect, as states with negative energies don't exist. This is then the state of lowest energy and is called the ground state.

Definition 3.2. The **ground state** of a quantum mechanical system is the state $|0\rangle$ such that if the lowering operator is applied on $|0\rangle$, then it gives a 0 solution to the TISE, that is: $a_-|0\rangle = 0$.

It is possible to determine the explicit form of this ground state, and from the ground state construct all possible stationary states of the harmonic oscillator by repeatedly applying the raising operator.

Theorem 3.2. *The ground state $|0\rangle$ is given by:*

$$|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \in L^2(\mathbb{R}). \tag{3.1.11}$$

Proof. Since

$$a_-|0\rangle = \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx}+m\omega X\right)\Big|0\Big\rangle = |0\rangle, \tag{3.1.12}$$

we have that:

$$\frac{d|0\rangle}{dx} = \frac{-m\omega}{\hbar}x|0\rangle. \tag{3.1.13}$$

Which is a differential equation that is easily solved by integrating both sides and noting that the solution must be an exponential function with integration constant A:

$$|0\rangle = Ae^{-\frac{m\omega}{2\hbar}x^2}.$$

We normalize this state to unity:

$$1 = |A|^2\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} = |A|^2\sqrt{\frac{\pi\hbar}{m\omega}} \tag{3.1.14}$$

such that $A^2 = \sqrt{\frac{m\omega}{\pi\hbar}}$ and therefore:

$$|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}. \tag{3.1.15}$$

□

We can now find any stationary excited state $|n\rangle$ of the harmonic oscillator by repeatedly applying the raising operator:

$$|n\rangle = A_n(a_+)^n|0\rangle \in L^2(\mathbb{R}).$$

With $A_n = \frac{1}{\sqrt{n}}$ the normalization constant as can easily be checked from the well known relations $a_+|n\rangle = \sqrt{n+1}|n+1\rangle$ and $a_-|n\rangle = \sqrt{n}|n-1\rangle$.

3.2 Uncertainty Principles and Coherent States

We now introduce the generalized uncertainty principle, from which coherent states readily follow.

Definition 3.3. Let A be a symmetric operator on $\mathscr{H}$, then the uncertainty $\Delta_\psi A$ of A in a state ψ with $\psi \in Dom(A)$ is given by:

$$(\Delta_\psi A)^2 = \left\langle (A - \langle A\rangle_\psi I)^2 \right\rangle = \left\langle (A - \langle A\rangle_\psi I)\psi, (A - \langle A\rangle_\psi I)\psi \right\rangle = \langle A\psi, A\psi\rangle - (\langle\psi, A\psi\rangle)^2. \tag{3.2.1}$$

Theorem 3.3. *Suppose A and B are symmetric operators in a state ψ such that $\psi \in Dom(AB) \cap Dom(BA)$ where:*

$$Dom(AB) = \{\psi \in Dom(B) | B\psi \in Dom(A)\} \tag{3.2.2}$$

then

$$(\Delta_\psi A)^2 (\Delta_\psi B)^2 \geq \frac{1}{4} |\langle [A,B]\rangle_\psi|^2. \tag{3.2.3}$$

Proof. Define the operators $A' = A - \langle\psi, A\psi\rangle I$ and $B' = B - \langle\psi, B\psi\rangle I$. Clearly A' and B' are symmetric. We use the Cauchy-Schwartz inequality to note:

$$\begin{aligned}
\langle A'\psi, A'\psi\rangle\langle B'\psi, B'\psi\rangle &\geq |\langle A'\psi, B'\psi\rangle|^2 \\
&\geq |\Im\langle A'\psi, B'\psi\rangle|^2 \\
&= \frac{1}{4}|\langle A'\psi, B'\psi\rangle - \langle B'\psi, A'\psi\rangle|^2 \\
&\geq \frac{1}{4}|\langle\psi, A'B'\psi\rangle - \langle\psi, B'A'\psi\rangle|^2 \\
&= \frac{1}{4}|\langle\psi, [A', B'], \psi\rangle|^2 = \frac{1}{4}|\langle\psi, [A,B], \psi\rangle|^2.
\end{aligned}$$

□

Corollary 3.3.1. *Since P and X are symmetric operators satisfying :*

$$[X,P] = i\hbar I, \tag{3.2.4}$$

we have ***Heisenberg's uncertainty principle****:*

$$(\Delta_\psi X)(\Delta_\psi P) \geq \frac{\hbar}{2}, \tag{3.2.5}$$

for states $\psi \in Dom(XP) \cap Dom(PX) \subseteq L^2(\mathbb{R})$.

Definition 3.4. States in which the uncertainty of operators satisfy Heisenberg's uncertainty principle with equality are called **states of minimal uncertainty**.

Theorem 3.4. *The ground-state $|0\rangle$ is a state of minimal uncertainty.*

Proof. Since:

$$\begin{aligned}\langle 0|(a_- + a_+)^2|0\rangle &= \langle 0|a_-a_- + a_-a_+ + a_-a_+ + a_+a_+|0\rangle = \langle 0|a_-a_+|0\rangle = 1,\\ \langle 0|(a_- - a_+)^2|0\rangle &= \langle 0|a_-a_- - a_-a_+ - a_-a_+ + a_+a_+|0\rangle = \langle 0| - a_-a_+|0\rangle = -1,\end{aligned} \tag{3.2.6}$$

we have:

$$\begin{aligned}\langle X\rangle_0 &= \sqrt{\frac{\hbar}{2m\omega}}\langle 0|(a_- + a_+)|0\rangle = 0,\\ \langle P\rangle_0 &= \frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\langle 0|(a_- + a_+)|0\rangle = 0.\end{aligned} \tag{3.2.7}$$

And also:

$$\begin{aligned}\langle X^2\rangle_0 &= \frac{\hbar}{2m\omega}\langle 0|(a_- + a_+)^2|0\rangle = \frac{\hbar}{2m\omega},\\ \langle P^2\rangle_0 &= -\frac{\hbar m\omega}{2}\langle 0|(a_- + a_+)^2|0\rangle = -\frac{\hbar m\omega}{2}.\end{aligned} \tag{3.2.8}$$

Therefore:

$$\langle(\Delta_0 X)^2\rangle\langle(\Delta_0 P)^2\rangle = \frac{\hbar^2}{4}. \tag{3.2.9}$$

□

Definition 3.5. Coherent states are states $|\alpha\rangle$ such that $a_-|\alpha\rangle = \alpha|\alpha\rangle$. These coherent states have an explicit form in terms of wavefunctions of the harmonic oscillator, as these form a basis for $L^2(\mathbb{R})$. This is called the n-representation for coherent states:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}.\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{3.2.10}$$

Corollary 3.4.1. *The ground state* $|0\rangle$ *is a coherent state.*

Theorem 3.5. *Coherent states are states of minimal uncertainty.*

Proof. $a_-|\alpha\rangle = \alpha|\alpha\rangle$ implies that $\langle\alpha|a_+ = \langle\alpha|\overline{\alpha}$ therefore:

$$\begin{aligned}
&\langle\alpha|a_-a_+|\alpha\rangle = |\alpha|^2,\\
&\langle\alpha|a_- + a_+|\alpha\rangle = \alpha+\overline{\alpha},\\
&\langle\alpha|a_- - a_+|\alpha\rangle = \alpha-\overline{\alpha},\\
&\langle\alpha|(a_- + a_+)^2|\alpha\rangle = \alpha^2+\overline{\alpha}^2+1+2\alpha\overline{\alpha} = (\alpha+\overline{\alpha})^2+1,\\
&\langle\alpha|(a_- - a_+)^2|\alpha\rangle = \alpha^2+\overline{\alpha}^2-1-2\alpha\overline{\alpha} = (\alpha-\overline{\alpha})^2-1,
\end{aligned} \tag{3.2.11}$$

so that:

$$\begin{aligned}
(\Delta_\alpha X)^2 = \langle X^2\rangle_\alpha - \langle X\rangle_\alpha^2 &= \frac{\hbar}{2m\omega}(\langle\alpha|(a_-+a_+)^2|\alpha\rangle - \langle\alpha|(a_-+a_+)|\alpha\rangle\\
&= \frac{\hbar}{2m\omega}[(\alpha+\overline{\alpha})^2+1-(\alpha+\overline{\alpha})^2] = \frac{\hbar}{2m\omega},\\
(\Delta_\alpha P)^2 = \langle P^2\rangle_\alpha - \langle P\rangle_\alpha^2 &= -\frac{\hbar m\omega}{2}(\langle\alpha|(a_--a_+)^2|\alpha\rangle - \langle\alpha|(a_--a_+)|\alpha\rangle\\
&= -\frac{\hbar m\omega}{2}[(\alpha-\overline{\alpha})^2-1-(\alpha-\overline{\alpha})^2] = \frac{\hbar m\omega}{2}.
\end{aligned} \tag{3.2.12}$$

And therefore:

$$(\Delta_\alpha X)^2(\Delta_\alpha P)^2 = \frac{\hbar^2}{4}. \tag{3.2.13}$$

□

Theorem 3.6. ***Ehrenfest's theorem****: Let A be a symmetric operator in state* $\psi \in L^2(\mathbb{R})$*. Then the time evolution of the expectation value of A is given by:*

$$\frac{d}{dt}\langle\hat{A}\rangle_\psi = \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle_\psi + \langle\frac{\partial\hat{A}}{\partial t}\rangle_\psi. \tag{3.2.14}$$

Proof. We have:

$$\frac{d}{dt}\langle\hat{A}\rangle_\psi = \frac{d}{dt}\langle\psi|\hat{A}|\psi\rangle = \langle\frac{d\psi}{dt}|\hat{A}|\psi\rangle + \langle\psi|\frac{\partial\hat{A}}{\partial t}|\psi\rangle + \langle\psi|\hat{A}|\frac{d\psi}{dt}\rangle. \tag{3.2.15}$$

substituting the TDSE $\partial_t\psi = -\frac{i}{\hbar}\hat{H}\psi$ gives:

$$
\begin{aligned}
&= \langle -\frac{i}{\hbar}\hat{H}\psi|\hat{A}|\psi\rangle + \langle\psi|\hat{A}| - \frac{i}{\hbar}\hat{H}\psi\rangle + \langle\psi|\frac{\partial\hat{A}}{\partial t}|\psi\rangle \\
&= \frac{i}{\hbar}\langle\psi|\hat{H}\hat{A} - \hat{A}\hat{H}|\psi\rangle + \langle\frac{\partial\hat{A}}{\partial t}\rangle_\psi \\
&= \frac{1}{i\hbar}\langle\psi[\hat{A},\hat{H}]|\psi\rangle + \langle\frac{\partial\hat{A}}{\partial t}\rangle_\psi \\
&= \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle_\psi + \langle\frac{\partial\hat{A}}{\partial t}\rangle_\psi.
\end{aligned}
\tag{3.2.16}
$$

□

We now conclude with the main result of this chapter which rambles all previous knowledge together:

Theorem 3.7. *The expectation values of the symmetric operators P and X taken in coherent states of the harmonic oscillator satisfy the classical equations of motion.*

Proof. We use Ehrenfest's theorem and properties of coherent states:

$$
\begin{aligned}
&\frac{d}{dt}\langle X\rangle_0 = \frac{1}{i\hbar}\langle[X, \frac{1}{2m}(P^2 + (m\omega X)^2)]\rangle_0 + \langle\frac{\partial X}{\partial t}\rangle_0 = \frac{1}{i\hbar}\langle[X, \frac{P^2}{2m}]\rangle_0 = \frac{1}{m}\langle P\rangle_0 \\
&\frac{d}{dt}\langle P\rangle_0 = \frac{1}{i\hbar}\langle[P, \frac{1}{2m}(P^2 + (m\omega X)^2)]\rangle_0 + \langle\frac{\partial P}{\partial t}\rangle_0 = \frac{1}{i\hbar}\langle[P, \frac{m\omega^2 X^2}{2}]\rangle_0 = m\omega\langle X\rangle_0 = -k\langle X\rangle_0.
\end{aligned}
\tag{3.2.17}
$$

Which can be compared to their classical counterparts $F = \frac{dp}{dt} = -kx$ and $p = mv = m\frac{dx}{dt}$. □

We have shown that expectation values of P and X in coherent states of the harmonic oscillator satisfy classical equations of motion. As the harmonic oscillator can be used to approximate arbitrary continuous potentials, this can be of great use to study the classical limit in many cases.

Time-evolution of coherent states are given by a simple rotation in α space:

$$
|\alpha(t)\rangle \equiv \exp\frac{-i\hat{H}t}{\hbar}|\alpha\rangle = e^{\frac{-i\omega t}{2}}|\alpha e^{-i\omega t}\rangle. \tag{3.2.18}
$$

Therefore coherent states remain coherent under time evolution, and thus coherent states remain to obey classical equations of motion throughout time.

4 Weyl Systems And The Classical Limit

Consider a system of n particles ($j = 1, ..., n$) and let $\hbar = 1$. A postulate in quantum mechanics is that classical displacements in position and momenta correspond to the one-parameter groups associated with the position and momentum coordinates $\mathbf{x}_j$ and $\mathbf{p}_j$.

4.1 Weyl systems

We have seen in chapter 2 that in quantum mechanics, using Stone's theorem, there is a one-to-one correspondence between self-adjoint operators and strongly continuous one-parameter unitary groups.
The position and momentum operators are unbounded so that $\mathbf{x}_j$ and $\mathbf{p}_j$ do not have bounded spectra. In order not to worry about the domain we work in, we introduce their corresponding unitary groups defined by the means of the spectral theorem:

$$e^{i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j}} \quad \text{and} \quad e^{i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j}} \quad \text{with} \quad \boldsymbol{s_j}, \boldsymbol{r_j} \in \mathbb{R}^3. \tag{4.1.1}$$

For the operators x_i and p_i, these correspond to multiplication by $\exp(itx_i)$ and pull-back by translation $x \longrightarrow x + r$.
The one-parameter group of automorphisms corresponding to these bounded operators under a certain multiplication law generate an algebra.

Theorem 4.1. *Under the following multiplication law:*

$$e^{i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j}}e^{i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j}}e^{-i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j}} = e^{i\sum_{j=1}^{n}(\boldsymbol{x_j}+\boldsymbol{r_j})\cdot\boldsymbol{s_j}} \quad \forall \boldsymbol{s_j}, \boldsymbol{r_j} \in \mathbb{R}^3. \tag{4.1.2}$$

The operators form an algebra $\mathscr{W}$ called the ***Weyl algebra****.*

Proof. We check the that the given multiplication law satisfies the three properties of an algebra.
Let $\alpha \in \mathbb{C}$, $a \equiv \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})$, $b \equiv \exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})$, $c \equiv \exp(i\sum_{j=1}^{n}\boldsymbol{x_j'}\cdot\boldsymbol{s_j'})$ then:

$$\begin{aligned}
(A1)\, a*(b+c) &= \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})\Big(\exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j}) + \exp(i\sum_{j=1}^{n}\boldsymbol{x_j'}\cdot\boldsymbol{s_j'})\Big)\exp(-i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j}) \\
&= \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})(\exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\exp(-i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})+ \\
&\quad \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})(\exp(i\sum_{j=1}^{n}\boldsymbol{x_j'}\cdot\boldsymbol{s_j'})\exp(-i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j}) \\
&= exp(i\sum_{j=1}^{n}(\boldsymbol{x_j}+\boldsymbol{r_j})\cdot\boldsymbol{s_j}) + exp(i\sum_{j=1}^{n}(\boldsymbol{x_j'}+\boldsymbol{r_j})\cdot\boldsymbol{s_j'}) \\
&= a*b + a*c.
\end{aligned}$$

$$
\begin{aligned}
(A2)\, a*(b*c) =& \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})*(\exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})(\exp(i\sum_{j=1}^{n}\boldsymbol{x_j^{'}}\cdot\boldsymbol{s_j^{'}})\exp(-i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\\
=& \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})*\exp(i\sum_{j=1}^{n}(\boldsymbol{x_j^{'}}+\boldsymbol{s_j})\cdot\boldsymbol{s_j^{'}})\\
=& \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})\exp\left(i\sum_{j=1}^{n}(\boldsymbol{x_j^{'}}+\boldsymbol{s_j})\cdot\boldsymbol{s_j^{'}}\right)\exp(-i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})\\
=& \exp\left(i\sum_{j=1}^{n}((\boldsymbol{x_j^{'}}+\boldsymbol{s_j})+\boldsymbol{r_j})\cdot\boldsymbol{s_j^{'}}\right)\\
=& \exp\left(i\sum_{j=1}^{n}((\boldsymbol{x_j^{'}}+\boldsymbol{r_j})+\boldsymbol{s_j})\cdot\boldsymbol{s_j^{'}}\right)\\
=& \exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\exp\left(i\sum_{j=1}^{n}(\boldsymbol{x_j^{'}}+\boldsymbol{r_j})\cdot\boldsymbol{s_j^{'}}\right)\exp(-i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\\
=& \exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\exp(i\sum_{j=1}^{n}\boldsymbol{s_j}\cdot\boldsymbol{r_j})\exp(i\sum_{j=1}^{n}\boldsymbol{x_j^{'}}\cdot\boldsymbol{s_j^{'}})\exp(-i\sum_{j=1}^{n}\boldsymbol{s_j}\cdot\boldsymbol{r_j})\exp(-i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\\
=& \exp\left(i\sum_{j=1}^{n}(\boldsymbol{x_j}+\boldsymbol{r_j})\cdot\boldsymbol{s_j}\right)\exp(i\sum_{j=1}^{n}\boldsymbol{x_j^{'}}\cdot\boldsymbol{s_j^{'}})\exp\left(-i\sum_{j=1}^{n}(\boldsymbol{x_j}+\boldsymbol{r_j})\cdot\boldsymbol{s_j}\right)\\
=& \exp\left(i\sum_{j=1}^{n}(\boldsymbol{x_j}+\boldsymbol{r_j})\cdot\boldsymbol{s_j}\right)*\exp(i\sum_{j=1}^{n}\boldsymbol{x_j^{'}}\cdot\boldsymbol{s_j^{'}})\\
=& \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})\exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\exp(-i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})*\exp(i\sum_{j=1}^{n}\boldsymbol{x_j^{'}}\cdot\boldsymbol{s_j^{'}})\\
=& (a*b)*c.\\
(A3)\, a*(\alpha b) =& \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})*\alpha\exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\\
=& \exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})\alpha\exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\exp(-i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})\\
=& \alpha\exp(i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})\exp(i\sum_{j=1}^{n}\boldsymbol{x_j}\cdot\boldsymbol{s_j})\exp(-i\sum_{j=1}^{n}\boldsymbol{p_j}\cdot\boldsymbol{r_j})\\
=& \alpha(a*b).
\end{aligned}
$$

The unit element obviously is just 1. Hence $\mathscr{W}$ is an algebra. □

Definition 4.1. For $\boldsymbol{z_j} \equiv \boldsymbol{r_j} + i\boldsymbol{s_j} \in \mathbb{C}^{3n}$ over a Hilbert space $\mathscr{H}$, we define the **Weyl operator** by:

$$
W(z) \equiv e^{-\frac{i}{2}\sum_{j=1}^{n}\boldsymbol{r_j}\cdot\boldsymbol{s_j}}e^{i\sum_{j=1}^{n}\boldsymbol{r_j}\cdot\boldsymbol{p_j}}e^{i\sum_{j=1}^{n}\boldsymbol{s_j}\cdot\boldsymbol{x_j}}. \tag{4.1.3}
$$

Where the inner product is given $\langle z, z'\rangle = \sum_{j=1}^{n} \boldsymbol{z_j^*} \cdot \boldsymbol{z_j'}$.

We quickly check the following properties:

Corollary 4.1.1. *The multiplication law of the Weyl algebra* $\mathcal{W}$ *can be written more compactly as* $W(z)W(z') = e^{(\frac{i}{2}\Im\langle z,z'\rangle)}W(z+z')$ *and* $W(z) = W^*(-z)$.

Proof. To avoid confusion, the first part is evaluated piece-wise:

$$\begin{aligned}
\exp(\frac{i}{2}\Im\langle z, z'\rangle) = & \exp\Big(\frac{i}{2}\Im(\sum_{j=1}^{n}\boldsymbol{z_j^*}\cdot\boldsymbol{z_j'})\Big) \\
&= \exp\Big[\frac{i}{2}\Im\Big(\sum_{j=1}^{n}(\boldsymbol{r_j} - i\boldsymbol{s_j})\cdot(\boldsymbol{r_j'} + \boldsymbol{i}\boldsymbol{s_j'})\Big)\Big] \\
&= \exp\Big[\frac{i}{2}\Im\Big(\sum_{j=1}^{n}(\boldsymbol{r_j}\boldsymbol{r_j'} + \boldsymbol{s_j}\boldsymbol{s_j'}) + i(\boldsymbol{r_j}\boldsymbol{s_j'} - \boldsymbol{r_j'}\boldsymbol{s_j})\Big)\Big] \\
&= \exp\Big(\frac{i}{2}\sum_{j=1}^{n}(\boldsymbol{r_j}\boldsymbol{s_j'} - \boldsymbol{r_j'}\boldsymbol{s_j})\Big).
\end{aligned}$$

$$\begin{aligned}
W(z+z') &= \exp\Big(-\frac{i}{2}\sum_{j=1}^{n}(\boldsymbol{r_j}+\boldsymbol{r_j'})\cdot(\boldsymbol{s_j}+\boldsymbol{s_j'})\Big)\exp\Big(i\sum_{j=1}^{n}(\boldsymbol{r_j}+\boldsymbol{r_j'})\cdot\boldsymbol{p_j}\Big)\exp\Big(i\sum_{j=1}^{n}(\boldsymbol{s_j}+\boldsymbol{s_j'})\cdot\boldsymbol{x_j}\Big) \\
&= \exp\Big(-\frac{i}{2}\sum_{j=1}^{n}(\boldsymbol{r_j}\cdot\boldsymbol{s_j} + \boldsymbol{r_j}\cdot\boldsymbol{s_j'} + \boldsymbol{r_j'}\cdot\boldsymbol{s_j} + \boldsymbol{r_j'}\cdot\boldsymbol{s_j'})\Big)\exp\Big(i\sum_{j=1}^{n}(\boldsymbol{r_j}+\boldsymbol{r_j'})\cdot\boldsymbol{p_j}\Big)\exp\Big(i\sum_{j=1}^{n}(\boldsymbol{s_j}+\boldsymbol{s_j'})\cdot\boldsymbol{x_j}\Big).
\end{aligned}$$

Such that:

$$\begin{aligned}
&\exp\Big(\frac{i}{2}\Im\langle z, z'\rangle\Big)W(z+z') = \\
&\quad \exp\Big(\frac{i}{2}\sum_{j=1}^{n}(\boldsymbol{r_j}\boldsymbol{s_j'} - \boldsymbol{r_j'}\boldsymbol{s_j})\Big)\exp(-\frac{i}{2}\sum_{j=1}^{n}(\boldsymbol{r_j}\cdot\boldsymbol{s_j} + \boldsymbol{r_j}\cdot\boldsymbol{s_j'} + \boldsymbol{r_j'}\cdot\boldsymbol{s_j} + \boldsymbol{r_j'}\cdot\boldsymbol{s_j'})\Big) \\
&\quad \exp\Big(i\sum_{j=1}^{n}(\boldsymbol{r_j}+\boldsymbol{r_j'})\cdot\boldsymbol{p_j}\Big)\exp\Big(i\sum_{j=1}^{n}(\boldsymbol{s_j}+\boldsymbol{s_j'})\cdot\boldsymbol{x_j}\Big) \\
&\quad = \exp\Big(-\frac{i}{2}\sum_{j=1}^{n}(\boldsymbol{r_j}\cdot\boldsymbol{s_j} + \boldsymbol{r_j'}\cdot\boldsymbol{s_j'})\Big)\exp\Big(i\sum_{j=1}^{n}(\boldsymbol{r_j}+\boldsymbol{r_j'})\cdot\boldsymbol{p_j}\Big)\exp\Big(i\sum_{j=1}^{n}(\boldsymbol{s_j}+\boldsymbol{s_j'})\cdot\boldsymbol{x_j}\Big) \\
&\quad = \exp(-\frac{i}{2}\sum_{j=1}^{n}\boldsymbol{r_j}\cdot\boldsymbol{s_j})\exp(i\sum_{j=1}^{n}\boldsymbol{r_j}\cdot\boldsymbol{p_j})\exp(i\sum_{j=1}^{n}\boldsymbol{s_j}\cdot\boldsymbol{x_j})\exp(-\frac{i}{2}\sum_{j=1}^{n}\boldsymbol{r_j'}\cdot\boldsymbol{s_j'}) \\
&\quad \exp(i\sum_{j=1}^{n}\boldsymbol{r_j'}\cdot\boldsymbol{p_j})\exp(i\sum_{j=1}^{n}\boldsymbol{s_j'}\cdot\boldsymbol{x_j}) \\
&\quad = W(z)W(z').
\end{aligned}$$

Moreover, by formula (2.3.11):

$$
\begin{aligned}
W^*(-z) &= \exp\Big(-i\sum_{j=1}^{n}(-\boldsymbol{s_j})\cdot\boldsymbol{x_j}\Big)\exp\Big(-i\sum_{j=1}^{n}(-\boldsymbol{r_j})\cdot\boldsymbol{p_j}\Big)\Big(\frac{i}{2}\sum_{j=1}^{n}(-\boldsymbol{r_j})\cdot(-\boldsymbol{s_j})\Big)\\
&= \exp(\frac{i}{2}\sum_{j=1}^{n}\boldsymbol{r_j}\cdot\boldsymbol{s_j})\exp(i\sum_{j=1}^{n}\boldsymbol{r_j}\cdot\boldsymbol{p_j})\exp(i\sum_{j=1}^{n}\boldsymbol{s_j}\cdot\boldsymbol{x_j})\\
&= \exp(\frac{i}{2}\sum_{j=1}^{n}\boldsymbol{r_j}\cdot\boldsymbol{s_j})\exp(i\sum_{j=1}^{n}\boldsymbol{s_j}\cdot\boldsymbol{x_j})\exp(-i\sum_{j=1}^{n}\boldsymbol{r_j}\cdot\boldsymbol{p_j})\\
&= W(z).
\end{aligned}
$$

□

To see what happens when we take the classical limit $\hbar \longrightarrow 0$, we define $q_\hbar \equiv q\sqrt{\hbar}$ and $p_\hbar \equiv p\sqrt{\hbar} = i\sqrt{\hbar}\frac{d}{dq}$.

Theorem 4.2. *If we use $W(z\hbar^{-1/2})$ to cause a respective displacement $r\hbar^{-1/2}$ and $s\hbar^{-1/2}$ on the Hilbert spaces $\mathscr{H} = L^2(\mathbb{R}, dq)$ and $\mathscr{H} = L^2(\mathbb{R}, dp)$ respectively, then, after taking the classical limit $\hbar \longrightarrow 0$, $q_\hbar$ and $p_\hbar$ converge respectively to $\boldsymbol{r}\cdot\boldsymbol{1}$ and $\boldsymbol{s}\cdot\boldsymbol{1}$.*

Proof.

$$
\begin{aligned}
&W(z\hbar^{-1/2})\exp(is(q_\hbar - r))W(-z\hbar^{-1/2}) = W(z\hbar^{-1/2})\exp(is(q_\hbar - r))W^*(z\hbar^{-1/2})\\
&= \exp\Big(\frac{-i}{2\sqrt{\hbar}}r\cdot s\Big)\exp\Big(\frac{i}{\sqrt{\hbar}}r\cdot p\Big)\exp\Big(\frac{i}{\sqrt{\hbar}}s\cdot q\Big)\exp(is(q_\hbar - r))\\
&\quad \exp\Big(\frac{-i}{\sqrt{\hbar}}s\cdot q\Big)\exp\Big(\frac{-i}{\sqrt{\hbar}}r\cdot p\Big)\exp\Big(\frac{i}{2\sqrt{\hbar}}r\cdot s\Big)\\
&= \exp(-isr)\exp\Big(\frac{i}{\sqrt{\hbar}}r\cdot p\Big)\exp\Big(\frac{i}{\sqrt{\hbar}}s\cdot q\Big)\exp(isq_\hbar)\exp\Big(\frac{-i}{\sqrt{\hbar}}s\cdot q\Big)\exp\Big(\frac{-i}{\sqrt{\hbar}}r\cdot p\Big)\\
&= \exp(-isr)\exp\Big(i\frac{r}{\sqrt{\hbar}}\cdot p\Big)\exp(isq_\hbar)\exp\Big(-i\frac{r}{\sqrt{\hbar}}\cdot p\Big)\\
&= \exp(-isr)\exp(is(q_\hbar + r))\\
&= \exp(isq_\hbar).
\end{aligned}
$$

As $\hbar \longrightarrow 0$, $\exp(isq_\hbar) = (\exp(isq))^{\sqrt{\hbar}} \longrightarrow 1$, therefore:

$$
W(z\hbar^{-1/2})\exp(is(q_\hbar))W(-z\hbar^{-1/2}) \longrightarrow \exp(isr). \tag{4.1.4}
$$

Very similarly:

$$
\begin{aligned}
&W(z\hbar^{-1/2})\exp(is(p_\hbar - r))W(-z\hbar^{-1/2}) = W(z\hbar^{-1/2})\exp(is(p_\hbar - r))W^*(z\hbar^{-1/2})\\
&= \exp\Big(\frac{-i}{2\sqrt{\hbar}} r\cdot s\Big)\exp\Big(\frac{i}{\sqrt{\hbar}} r\cdot p\Big)\exp\Big(\frac{i}{\sqrt{\hbar}} s\cdot q\Big)\exp(is(p_\hbar - r))\\
&\quad \exp\Big(\frac{-i}{\sqrt{\hbar}} s\cdot q\Big)\exp\Big(\frac{-i}{\sqrt{\hbar}} r\cdot p\Big)\exp\Big(\frac{i}{2\sqrt{\hbar}} r\cdot s\Big)\\
&= \exp(-isr)\exp\Big(\frac{i}{\sqrt{\hbar}} r\cdot p\Big)\exp\Big(\frac{i}{\sqrt{\hbar}} s\cdot q\Big)\exp(isp_\hbar)\exp\Big(\frac{-i}{\sqrt{\hbar}} s\cdot q\Big)\exp\Big(\frac{-i}{\sqrt{\hbar}} r\cdot p\Big)\\
&= \exp(-isr)\exp\Big(i\frac{r}{\sqrt{\hbar}}\cdot p\Big)\exp(isp_\hbar)\exp\Big(-i\frac{r}{\sqrt{\hbar}}\cdot p\Big)\\
&= \exp(-isr)\exp(is(p_\hbar + r))\\
&= \exp(isp_\hbar).
\end{aligned}
$$

As $\hbar \longrightarrow 0$, $exp(itp_\hbar) = (exp(itp))^{\sqrt{\hbar}} \longrightarrow 1$, therefore:

$$
W(z\hbar^{-1/2})\exp(it(p_\hbar))W(-z\hbar^{-1/2}) \longrightarrow \exp(its). \tag{4.1.5}
$$

□

Convergence of a sequence of self-adjoint operators $a_n \longrightarrow a$ means that all sequences of bounded functions f of a_n converge: $f(a_n) \longrightarrow f(a)$. It suffices to have convergence for the class of functions $f(a) = \exp(iat)$ for all $t \in \mathbb{R}$. This shows that $Wx_\hbar W^{-1}$ and $Wp_\hbar W^{-1}$ converge strongly to respectively r and s.
We have therefore shown that quantum mechanical time-automorphisms for linear equations of motion are equivalent to their classical linear equations of motion.

4.2 Correspondence Of Classical and Quantum Mechanical Time Evolution

Because the Schrödinger equation is a first order differential equation in the time variable t, once $\psi(t_0)$ is known at a fixed time t_0 (Initial condition of the DE) then $\psi(t)$ is determined at all times. $\psi(t)$ and $\psi(t_0)$ are related by the unitary time-evolution operator $U(t, t_0)$:

$$
\psi(t) = U(t,t_0)\psi(t_0), \qquad U(t_0,t_0) = I. \tag{4.2.1}
$$

Substituting $\psi(t) = U(t,t_0)\psi(t_0)$ into the time-dependent Schrödinger equation yields:

$$\begin{aligned}\hat{H}\psi(x,y,z,t) =& \Big(\frac{-\hbar^2}{2m}\nabla^2 + V(t)\Big)\psi(x,y,z,t) = i\hbar\frac{\partial\psi(x,y,z,t)}{\partial t},\\ \longrightarrow & i\hbar\frac{\partial}{\partial t}U(t,t_0) = \hat{H}U(t,t_0).\end{aligned} \tag{4.2.2}$$

Using the condition that $U(t_0,t_0) = I$, and assuming the Hamiltonian H is time-independent, the unique solution of this differential equation is:

$$U(t,t_0) = \exp\Big(-\frac{i}{\hbar}\hat{H}(t-t_0)\Big), \tag{4.2.3}$$

as is very easily seen by substitution into (4.2.2).
When the Hamiltonian is time-dependent on the other hand, the solution becomes slightly more difficult. Suppose H is time-dependent, then the unique solution of (4.2.2) is:

$$\begin{aligned}U(t,t_0) =& 1 + \sum_{n=1}^{\infty}(-i)^n\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2...\int_{t_0}^{t_{n-1}}dt_nH(t_1)H(t_2)...H(t_n)\\ =& \hat{T}\Big[\exp\Big(-\frac{i}{\hbar}\int_{t_0}^{t}dt^{'}H(t')\Big)\Big].\end{aligned} \tag{4.2.4}$$

Where $\hat{T}$ is the time ordering operator defined by:

$$\hat{T}(a(t_1)a(t_2)...a(t_n)) \equiv a(t_{i1})a(t_{i2})...a(t_{in})$$

with $t_{i1} \geq t_{i2} \geq ... \geq t_{in}$, as is also easily seen by substitution into (4.2.2).
This solution will be relevant for further investigation into the classical limit.

Let

$$H(\pi(t),\xi(t)) = \frac{\pi^2(t)}{2m} + V(\xi(t)) \tag{4.2.5}$$

be the Hamiltonian of a classical system where $(\pi,\xi) \in \mathbb{R}^{2f}$, the usual 2f-dimensional phase space in which all possible states of the system are represented.
The classical time evolution of this system is uniquely defined by Hamilton's classical equations of motion:

$$\dot{\pi}(t) = \frac{\delta\pi(t)}{\delta t} = -\frac{\delta H(\pi,\xi)}{\delta\xi(t)} = -\nabla V(\xi(t)), \qquad m\dot{\xi}(t) = m\frac{\delta\xi(t)}{\delta t} = m\frac{\delta H(\pi,\xi)}{\delta\pi(t)} = m\frac{2\pi(t)}{2m} = \pi(t).$$

(4.2.6)

If we consider initial data

$$\xi(\alpha,0) \equiv \xi, \qquad \pi(\alpha,0) \equiv \pi, \qquad \alpha \equiv \frac{\xi + i\pi}{\sqrt{2}}, \tag{4.2.7}$$

and if we consider ∇V to be Lipschitz around ξ, then it follows from the local version of the Picard-Lindelöf theorem (taught in any undergraduate ODE course) that there is a unique solution to (4.2.6) namely $(\xi(\alpha,t), \pi(\alpha,t))$ for $t < T(\alpha) \leq \infty$ where we assume H to be continuous on the interval $[t, T(\alpha)]$. This unique solution then describes the classical time evolution of the system completely.
The quantum mechanical equivalent of this problem is given by promoting the classical variables to their corresponding quantum mechanical operators, giving the famous TDSE:

$$i\hbar \frac{\delta \psi(x,t)}{\delta t} = -\frac{\hbar}{2m}\nabla^2 \psi(x,t) + V(x)\psi(x,t). \tag{4.2.8}$$

In $\mathscr{H} = L^2(\mathbb{R}^f)$ under certain assumptions on the domains of the operators, this ODE always has (non-unique) global solutions for ψ_t, which describes the complete quantum mechanical time evolution of the state of the system it is in.

The relationship between the quantum mechanical solutions of (4.2.8) and the classical solutions of (4.2.5) is the one that is of main interest. It has also been a vivid topic of discussion between quantum physicists ever since the birth of quantum mechanics itself.

We have worked with Ehrenfest's theorem in chapter 3 in order to derive the **Ehrenfest relations**, which gives the simplest connection between classical and quantum mechanics:

$$\frac{d}{dt}\langle q \rangle = \frac{\langle p \rangle}{m}, \qquad \frac{d}{dt}\langle p \rangle = -\langle V'(q) \rangle, \tag{4.2.9}$$

however, the Ehrenfest relations do, in general, not offer a solution to (4.2.6) since $-\langle V'(x)\rangle \neq V'(\langle q \rangle)$ unless ∇V is linear as is the case of the harmonic oscillator.

4.3 Classical Limit and Hepp's theorem

There is hope that classical time evolution can be re-obtained as we let $\hbar \longrightarrow 0$ because then fluctuations in the potential can be neglected.
Indeed, if we recall the operators:

$$p_\hbar \equiv \sqrt{\hbar} p = -\sqrt{\hbar} i \frac{d}{dx}, \qquad q_\hbar \equiv \sqrt{\hbar} q = \sqrt{\hbar} x, \qquad a \equiv \frac{q + ip}{\sqrt{2}}, \tag{4.3.1}$$

then, if we let $p_\hbar$ and $q_\hbar$ respectively be in coherent states, centered around the large mean values of $\frac{\pi}{\sqrt{\hbar}}$ and $\frac{\xi}{\sqrt{\hbar}}$ respectively, we re-obtain the time evolution of classical systems in classical orbits, as will be proven shortly.

Theorem 4.3. *Let $\alpha \in \mathbb{C}$ and $U(\alpha) \equiv \exp(\alpha a^* - \alpha^* a)$, then*

$$U(\alpha) = \exp i(\pi q - \xi p), \qquad U(\alpha) a U^*(\alpha) = a - \alpha \tag{4.3.2}$$

and, taken in the coherent state $|\alpha\rangle = U(\alpha)|0\rangle$, for an arbitrary monomial (polynomial consisting of only 1 term) in the p's and q's we have:

$$\Big\langle \frac{\alpha}{\sqrt{\hbar}} | \Big(q - \frac{\xi}{\sqrt{\hbar}}\Big) ... \Big(p - \frac{\pi}{\sqrt{\hbar}}\Big) | \frac{\alpha}{\sqrt{\hbar}} \Big\rangle = \Big\langle 0 \Big| q...p \Big| 0 \Big\rangle. \tag{4.3.3}$$

And therefore

$$\lim_{\hbar \longrightarrow 0} \Big\langle \frac{\alpha}{\sqrt{\hbar}} \Big| (q_\hbar)...(p_\hbar) \Big| \frac{\alpha}{\sqrt{\hbar}} \Big\rangle = \xi ... \pi. \tag{4.3.4}$$

Proof.

$$\begin{aligned}
U(\alpha) \equiv \exp(\alpha a^* - \alpha^* a) &= \exp\Big(\frac{\xi + i\pi}{\sqrt{2}} \frac{q - ip}{\sqrt{2}} - \frac{\xi - i\pi}{\sqrt{2}} \frac{q + ip}{\sqrt{2}}\Big) \\
&= \exp\Big(\frac{1}{2}(\xi q - \xi i p + i\pi q + \pi p - \xi q - \xi i p + i\pi q - \pi p)\Big) \\
&= \exp\Big(\frac{1}{2}(2i(\pi q - \xi p))\Big) = \exp i(\pi q - \xi p).
\end{aligned}$$

As $[a,[a,\alpha]] = [\alpha,[a,\alpha]] = 0$, we use Baker-Campbell-Hausdorff's theorem (2.3.7):

$$\begin{aligned}
U(\alpha) a U^*(\alpha) &= \exp i(\pi q - \xi p) \frac{q + ip}{\sqrt{2}} \exp -i(\pi q - \xi p) \\
&= \frac{\xi + i\pi}{\sqrt{2}} + i\Big[\pi q - \xi p, \frac{q + ip}{\sqrt{2}}\Big] \\
&= a + i\Big(\Big[\pi q, \frac{q}{\sqrt{2}}\Big] + \Big[\pi q, \frac{ip}{\sqrt{2}}\Big] + \Big[\frac{q}{\sqrt{2}}, \xi p\Big] + \Big[\frac{ip}{\sqrt{2}}, \xi p\Big]\Big) \\
&= a + i\Big(\frac{1}{\sqrt{2}}\Big(\pi \cancel{[q,q]} + \cancel{[\pi,q]} q\Big) + \frac{i}{\sqrt{2}}\Big(\pi [q,p] + \cancel{[\pi,p]} q\Big) \\
&+ \frac{1}{\sqrt{2}}\Big(\xi [q,p] + \cancel{[q,\xi]} p\Big) + \frac{i}{\sqrt{2}}\Big(\xi \cancel{[p,p]} + \cancel{[p,\xi]} p\Big) \\
&= a + i\Big(\frac{i}{\sqrt{2}}\pi i + \frac{1}{\sqrt{2}}\xi i\Big) = a - \Big(\frac{\xi - i\pi}{\sqrt{2}}\Big) = a - \alpha.
\end{aligned}$$

We quickly note that $|\alpha\rangle = U(\alpha)|0\rangle$ is indeed a coherent state. As $U(\alpha)aU^*(\alpha) = a - \alpha$ we have $U(\alpha)a = aU(\alpha) - \alpha U(\alpha)$ which implies $aU(\alpha) = U(\alpha)a + \alpha U(\alpha)$. Therefore:

$$a|\alpha\rangle = aU(\alpha)|0\rangle = \quad U(\alpha)a|0\rangle + \alpha U(\alpha)|0\rangle = \alpha|\alpha\rangle \tag{4.3.5}$$

as $a|0\rangle = 0$.
We also have that:

$$\begin{aligned}\Big\langle \frac{\alpha}{\sqrt{\hbar}}\Big|\Big(q - \frac{\xi}{\sqrt{\hbar}}\Big)...\Big(p - \frac{\pi}{\sqrt{\hbar}}\Big)\Big|\frac{\alpha}{\sqrt{\hbar}}\Big\rangle &= \Big\langle \frac{\alpha}{\sqrt{\hbar}}|U\Big(\frac{\xi}{\sqrt{\hbar}}\Big)qU^*\Big(\frac{\xi}{\sqrt{\hbar}}\Big)...U\Big(\frac{\pi}{\sqrt{\hbar}}\Big)pU^*\Big(\frac{\pi}{\sqrt{\hbar}}\Big)\Big|\frac{\alpha}{\sqrt{\hbar}}\Big\rangle \\ &= \Big\langle 0\Big|U^*\Big(\frac{\xi}{\sqrt{\hbar}}\Big)U\Big(\frac{\xi}{\sqrt{\hbar}}\Big)qU^*\Big(\frac{\xi}{\sqrt{\hbar}}\Big)U\Big(\frac{\xi}{\sqrt{\hbar}}\Big)...U^*\Big(\frac{\pi}{\sqrt{\hbar}}\Big)U\Big(\frac{\pi}{\sqrt{\hbar}}\Big)pU^*\Big(\frac{\pi}{\sqrt{\hbar}}\Big)U\Big(\frac{\pi}{\sqrt{\hbar}}\Big)\Big|0\Big\rangle \\ &= \langle 0|q...p|0\rangle.\end{aligned} \tag{4.3.6}$$

And hence

$$\lim_{\hbar \longrightarrow 0} \Big\langle \frac{\alpha}{\sqrt{\hbar}}\Big|(q_\hbar)...(p_\hbar)\Big|\frac{\alpha}{\sqrt{\hbar}}\Big\rangle = \xi...\pi. \tag{4.3.7}$$

As $p_\hbar$ and $q_\hbar$ are centered around large mean values of $\hbar^{-1/2}\pi$ and $\hbar^{-1/2}\xi$.

□

If we let $\xi(\alpha, s), \pi(\alpha, t)$ denote the solutions of the classical equations of motion with initial data α then we will show that (4.3.4) in Weyl form is preserved under time evolution of any self adjoint extension $H_\hbar$ of $\frac{p_\hbar^2}{2m} + V_\hbar$:

$$\lim_{\hbar \longrightarrow 0} \Big\langle \frac{\alpha}{\sqrt{\hbar}}\Big|(q_\hbar(s))...(p_\hbar(t))\Big|\frac{\alpha}{\sqrt{\hbar}}\Big\rangle = \xi(\alpha, s)...\pi(\alpha, t). \tag{4.3.8}$$

in which the unitary time evolution operator of the equations of motion is linearized around the classical path:

$$U_f = T\Big[\exp\Big(-i\int_0^t dt'\Big(\frac{p^2}{2m} + \frac{q^2}{2}V''(\xi(\alpha, t'))\Big)\Big)\Big]. \tag{4.3.9}$$

Indeed, this equation does hold under certain assumptions. Its mathematically precise statement, which I shall call Hepp's theorem, is as follows:

Theorem 4.4. *(Hepp's theorem) Let $V(\xi) \in C^3(\mathbb{R})$ and $\xi(\alpha,t)$ be a solution of the initial classical equations (4.2.6) with initial data α. Assume that $\int |V(x)|^2 exp(-\rho x^2) < \infty$ for some $\rho < \infty$. Let $H_\hbar = H(x_\hbar, p_\hbar)$ be any self-adjoint extension of the Hermitian operator $\frac{\hbar p^2}{2m} + V(x\sqrt{\hbar}) \in L^2(\mathbb{R}^1)$ and $U_\hbar(t) \equiv exp(-iH_\hbar t/\hbar)$. Then, for all t for which the classical trajectory $(\xi(t), \pi(t))$ continues to exist and for all $(r,s) \in \mathbb{R}^2$,*

$$\begin{aligned}\operatorname*{s-lim}_{\hbar\to 0} U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^*(t)\exp i\Big[r\Big(q-\frac{\xi(\alpha,t)}{\sqrt{\hbar}}\Big)+s\Big(p-\frac{\pi(\alpha,t)}{\sqrt{\hbar}}\Big)\Big]U_\hbar(t)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)\\ = \exp i[rq(\alpha,t)+sp(\alpha,t)]\end{aligned} \tag{4.3.10}$$

and

$$\begin{aligned}\operatorname*{s-lim}_{\hbar\to 0} U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^*(t)\exp i[rq_\hbar + sp_\hbar]U_\hbar(t)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)\\ = \exp i[r\xi(\alpha,t)+s\pi(\alpha,t)].\end{aligned} \tag{4.3.11}$$

Where the limits are taken in the strong order topology and $(p(\alpha,t), q(\alpha,t))$ are solutions of (4.2.6) linearized around $\xi(\alpha,t)$ with initial data (p,q) that arise from the self-adjoint classical hamiltonian:

$$H(t) = \frac{p^2}{2m} + \frac{q^2}{2}V''(\xi(\alpha,t)). \tag{4.3.12}$$

Before giving the proof, we shall introduce and prove a small lemma.

Lemma 4.5. *The set of normalized states $K = \{\psi_a(x) = \pi^{-1/4}e^{-\frac{(x-a)^2}{2}} | a \in \mathbb{R}\}$ spans $L^2(\mathbb{R})$. We claim $\forall k \in \mathbb{R} : 0 < k < T$, $\exists \hbar_k > 0 : \forall \hbar < \hbar_k$ and $\forall |s| \le k$, the total set of states satisfies:*

$$\{\psi_a^{\hbar s} \equiv U_\hbar^1(s)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)W(s,0)\psi_a\} \subseteq Dom(p^2)\cap Dom\Big(\frac{V(\sqrt{\hbar}q)}{\hbar}\Big) = Dom\Big(p^2 + \frac{V(\sqrt{\hbar}q)}{\hbar}\Big). \tag{4.3.13}$$

Proof. We recall that the selfadjoint Hamiltonian $H(t)$ is defined as:

$$H(r) = \frac{p^2}{2m} + \frac{q^2}{2}V''(\xi(\alpha,r)).$$

Since this hamiltonian is quadratic in $H(r)$ and $V''(\xi_r)$ is continuous in r, we have that the Dyson series $W(t,s)$ converges for small $|t-s|$ and

$$\begin{aligned}W(s,0)qW^*(s,0) &= \alpha q + \beta p,\\ W(s,0)pW^*(s,0) &= \gamma q + \delta p,\end{aligned} \tag{4.3.14}$$

With continuous dependence on s. This can be seen as follows:

$$
\begin{aligned}
&W(s,0)q = \hat{T}\exp\Big(-i\int_0^s dr\frac{p^2}{2m} + \frac{q^2}{2}V''(\xi(\alpha,r)\Big)q = \\
&\Big(1+\sum_{n=1}^{\infty}(-i)^n\int_0^s ds_1\int_0^{s_1}ds_2...\int_0^{s_{n-1}}ds_n H(r_1)H(r_2)...H(r_n)\Big)q \\
=&(\alpha q+\beta p)\Big(1+\sum_{n=1}^{\infty}(-i)^n\int_0^s ds_1\int_0^{s_1}ds_2...\int_0^{s_{n-1}}ds_n H(r_1)H(r_2)...H(r_n)\Big) \\
=&(\alpha q+\beta p)W(s,0).
\end{aligned}
$$

Here α and β are terms that arise from the the commutation relations with q on $H(r)$ by bringing it to the left.
The proof for $W(s,0)pW^*(s,0) = \gamma q + \delta p$ is exactly the same.

Next we note that ψ_a satisfies:

$$
[q-a+ip]\psi_a = [x-a+\frac{d}{dx}]\pi^{-1/4}e^{-\frac{-(x-a)^2}{2}} = [x-a+(a-x)]\psi_a = 0 \tag{4.3.15}
$$

and therefore also:

$$
0 = U_\hbar^1(s)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)W(s,0)[q-a+ip]\psi_a. \tag{4.3.16}
$$

We will now bring the term $[q-a+ip]$ to the left using the commutation relations of (4.3.14) and (4.3.2) and recall that $\psi_a^{\hbar s} = U_\hbar^1(s)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)W(s,0)\psi_a$:

$$
\begin{aligned}
0 =&U_\hbar^1(s)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)\Big[\alpha q+\beta p - a + i\gamma q + i\delta p\Big]W(s,0)\psi_a \\
=&\Big[(\alpha+i\gamma)(q-\frac{\xi_s}{\sqrt{\hbar}}) - a + i(\delta - i\beta)(p-\frac{\pi_s}{\sqrt{\hbar}})\Big]\psi_a^{\hbar s}.
\end{aligned}
$$

Therefore:

$$
\psi_a^{\hbar s} = C\exp\Big[-\frac{(\alpha+i\gamma)}{2(\delta-i\beta)}\Big(x-\frac{\xi_s}{\sqrt{\hbar}}-\frac{a}{(\alpha+i\gamma)}\Big)^2 + i\pi_s\frac{x}{\sqrt{\hbar}}\Big].
$$

We note that:

$$
\Re\Big(\frac{(\alpha+i\gamma)}{2(\delta-i\beta)}\Big) = \Re\Big(\frac{(\alpha+i\gamma)}{2(\delta-i\beta)}\frac{(\delta+i\beta)}{\delta+i\beta}\Big) = \frac{1}{2(\delta^2+\beta^2)} > \eta_k > 0
$$

for all $|s| \le k$ and since we assumed that $\int |V(x)|^2 exp(-\rho x^2) < \infty$ for some $\rho < \infty$.We obtain the result of formula (4.3.13) for $\hbar_k = \frac{2\eta_k}{\rho}$. $\square$

From the previous lemma we conclude that $W_\hbar(t,s)W(s,r)\psi_a$ is strongly differentiable with respect to s.
We now start the proof of the main theorem.

Proof. We begin by expanding $\frac{H_\hbar}{\hbar}$ around the classical trajectory $\xi(\alpha,t)\equiv\xi_t$ in powers of $q_\hbar-\xi(\alpha,t)$ and $p_\hbar-\pi(\alpha,t)$:

$$\begin{aligned}\frac{p^2}{2m}+\frac{1}{\hbar}V(x\sqrt{\hbar}) =& \frac{1}{\hbar}\Big(\frac{\pi(t)^2}{2}+V(\xi(t))\Big)+\frac{1}{\sqrt{\hbar}}[\pi_t(p-\frac{\pi_t}{\sqrt{\hbar}})+V^{'}(\xi_t)(q-\frac{\xi_t}{\sqrt{\hbar}})]\\ &+\Big(\frac{p-\frac{\pi_t}{\sqrt{\hbar}}}{2}\Big)+V^{''}(\xi_t)\Big(\frac{q-\frac{\xi_t}{\sqrt{\hbar}}}{2}\Big)+\mathcal{O}(\sqrt{\hbar})\\ &\equiv H^0_\hbar(t)+H^1_\hbar(t)+H^2_\hbar(t)+\mathcal{O}(\sqrt{\hbar})\end{aligned}$$

in which

$$\begin{aligned}H^0_\hbar(t) &\equiv \frac{1}{\hbar}H(\pi,\xi)\\ H^1_\hbar(t) &\equiv \frac{1}{\sqrt{\hbar}}\Big[\pi_t\Big(p-\frac{\pi_t}{\sqrt{\hbar}}\Big)+V^{'}(\xi_t)\Big(q-\frac{\xi_t}{\sqrt{\hbar}}\Big)\Big]\\ H^2_\hbar(t) &\equiv \Big(\frac{p-\frac{\pi_t}{\sqrt{\hbar}}}{2}\Big)+V^{''}(\xi_t)\Big(\frac{q-\frac{\xi_t}{\sqrt{\hbar}}}{2}\Big)\\ &\quad .....\end{aligned}$$

We note that there arises a linear term $H^{1'}_\hbar(t)\equiv\frac{1}{\sqrt{\hbar}}[\pi_t p+V^{'}(\xi_t)q)]$ which is the generator of displacement by $\frac{1}{\sqrt{\hbar}}(\xi(\alpha,t)-\xi(\alpha,0))$ in position and $\frac{1}{\sqrt{\hbar}}(\pi(\alpha,t)-\pi(\alpha,0))$ in momentum.
The left hand side of (4.3.10) can therefore be written in Weyl form:

$$W^*_\hbar(t,0)\exp i(rq+sp)W_\hbar(t,0) \tag{4.3.17}$$

where:

$$W_\hbar(t,s)\equiv U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U^{1*}_\hbar(t)U_\hbar(t-s)U^1_\hbar(s)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)\exp\Big(i\int_s^t dr\,H^0_\hbar(r)\Big) \tag{4.3.18}$$

with:

$$U^0_\hbar(t)=\hat{T}\exp\Big(-i\int_0^t dt^{'}H^0_\hbar(t')\Big),\quad U^1_\hbar(t)=\hat{T}\exp\Big(-i\int_0^t dt^{'}H^1_\hbar(t')\Big),\quad ...$$

such that:

$$\begin{aligned}W_\hbar(t,0) =& U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^{1*}(t)U_\hbar(t)U_\hbar^1(0)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)\exp\Big(i\int_0^t dr H_\hbar^0(r)\Big)\\ =& U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^{1*}(t)U_\hbar(t)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)\hat{T}\exp\Big(i\int_0^t dr H_\hbar^0(r)\Big)\\ =& U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^{1*}(t)U_\hbar(t)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^0(t).\end{aligned}$$

Indeed, because the propagator $U_\hbar^1(t)$ defines an automorphism of the Weyl algebra (see 4.3.2):

$$U_\hbar^{1*}(t)\Big(a^\# - \frac{\alpha_t^\#}{\sqrt{\hbar}}\Big)U_\hbar^1(t) = a^\# - \frac{\alpha^\#}{\sqrt{\hbar}}$$

we have that

$$\begin{aligned}&W_\hbar^*(t,0)\exp i(rq+sp)W_\hbar(t,0) =\\ &U_\hbar^{0*}(t)U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^*(t)U_\hbar^1(t)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)\exp i[rq+sp]U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^{1*}(t)U_\hbar(t)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^0(t)\\ =& U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^*(t)\exp i\Big[r\Big(q-\frac{\xi}{\sqrt{\hbar}}\Big)+s\Big(p-\frac{\pi}{\sqrt{\hbar}}\Big)\Big]U_\hbar(t)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big).\end{aligned}$$

Hepp's theorem can then be interpreted as follows: As s-lim$_{\hbar\longrightarrow 0}$, the time evolution of $\frac{H_\hbar}{\hbar}$ differs from $H_\hbar^1$ by a factor $U_\hbar^2$.
If on a dense subspace s-lim$_{\hbar\longrightarrow 0}\, W_\hbar(t,s) = W(t,s) \equiv \hat{T}\exp\Big(-i\int_s^t dr H(r)\Big)$ holds, then so do (4.3.17) and (4.3.10).

If $0<k<T, |s|,|t|\le k$ and if $\hbar<\hbar_k$ then for any self-adjoint extension $H_\hbar$ of $\frac{p_\hbar^2}{2m}+V_\hbar$ we obtain a Duhamel formula (see (2.3.12)):

$$W(t,0)\psi_a - W_\hbar(t,0)\psi_a = \int_0^t ds\frac{d}{ds}W_\hbar(t,s)W(s,0)\psi_a \tag{4.3.19}$$

and we note that:

$$\frac{d}{ds}W_\hbar(t,s)W(s,0)\psi_a = iW_\hbar(t,s)\Big[\frac{1}{\hbar}V(\xi_s+\sqrt{\hbar}q)-\frac{1}{\hbar}V(\xi_s)-\frac{1}{\sqrt{\hbar}}V'(\xi_s)q-V''(\xi_s)\frac{q^2}{2}\Big]\psi_a. \tag{4.3.20}$$

Our last step will be to estimate the norm of (4.3.19).
We will say that there exists some $\sigma>0$ such that $V(\xi_s+x)\in C^3$ for all $|s|\le k$ and $|x|\le\sigma$. We consider the above term again:

$$\int dx\Big|\frac{1}{\hbar}V(\xi_s+\sqrt{\hbar}q)-\frac{1}{\hbar}V(\xi_s)-\frac{1}{\sqrt{\hbar}}V'(\xi_s)q-V''(\xi_s)\frac{q^2}{2}\Big|^2|W(s,0)\psi_a)(x)|^2. \tag{4.3.21}$$

Clearly, if $|x| \geq \frac{\sigma}{\sqrt{\hbar}}$ then $|W(s,0)\psi_a)(x)|^2$ and $|V(\xi_s + \sqrt{\hbar}q)|^2$ increases at most like $\exp(\hbar\rho x^2)$, as follows from $\int |V(x)|^2 exp(-\rho x^2) < \infty$. Therefore each term in (4.3.21) is $\mathcal{O}(\hbar^N)$ for every choice of N.
If $|x| \geq \frac{\sigma}{\sqrt{\hbar}}$ then:

$$
\begin{aligned}
&\Big|\frac{1}{\hbar}V(\xi_s + \sqrt{\hbar}q) - \frac{1}{\hbar}V(\xi_s) - \frac{1}{\sqrt{\hbar}}V'(\xi_s)q - V''(\xi_s)\frac{q^2}{2}\Big| \\
\leq& x^2(\int_0^1 dy|V''(\xi_s + \sqrt{\hbar}xy) - V''(\xi_s)| - \int_0^1 y|V''(\xi_s + \sqrt{\hbar}xy) - V''(\xi_s)| \\
=& x^2 \int_0^1 dy(1-y)|V''(\xi_s + \sqrt{\hbar}xy) - V''(\xi_s)| \leq Cx^3\hbar^{3/2},
\end{aligned}
$$

where Hölder continuity of V'' is used.
Therefore:

$$
\|W(t,0)\psi_a - W_\hbar(t,0)\psi_a\| = \mathcal{O}(\hbar^{3/2}), \tag{4.3.22}
$$

such that s-lim $W_\hbar(t,s) = W(t,s)$ holds and this proves (4.3.10).
By the exact same argument as above we note that

$$
\begin{aligned}
&\|U^*\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)U_\hbar^*(t)\exp i[rq_\hbar + sp_\hbar]U_\hbar(t)U\Big(\frac{\alpha}{\sqrt{\hbar}}\Big)\psi - \exp i[r\xi_t + s\pi_t]\psi\| \\
&= \|W_\hbar^*(t,0)\exp i\sqrt{h}(rq + sp)W_\hbar(t,0)\psi - \psi\|
\end{aligned} \tag{4.3.23}
$$

and, since s-lim $W_\hbar(t,0) = W(t,0)$ and s-lim $\exp i\sqrt{h}(rq + sp) = I$, (4.3.11) follows.

□

5 Conclusion

We have shown that expectation values of products of Weyl operators that are translated in time by a quantum mechanical Hamiltonian and are in coherent states centered in phase space around the coordinates $(\hbar^{-1/2}q, \hbar^{-1/2}p)$, where (q,p) is an element of classical phase space, then, after we take the classical limit $\hbar \longrightarrow 0$, the expectation values of products of Weyl operators become exponentials of coordinate functions of the classical orbit in phase space.

in Chapter 1 we have started off by giving a background discussion as to why this topic is interesting, formulating some basic ideas and background thoughts.

In Chapter 2 we have developed some of the formalism of the mathematical description of quantum mechanics that mostly stems from functional analysis. Understanding this mathematical description allows for a much more complete and rigorous description of operators and their description as some parts of chapter 4 show.

In Chapter 3 we developed the idea of coherent states from the idea of the quantum mechanical harmonic oscillator. Coherent states of the harmonic oscillator obey classical equations of motion and maintain to obey these classical equations of motion under time evolution. This is a first example of an analogy between classical and quantum mechanics.

In Chapter 4 we discuss some ideas of the classical limit and give a complete analysis of Klaus Hepp's paper. We introduce the Weyl algebra and Weyl operator to investigate the general setting of the classical limit and show examples. We discuss correspondence of classical and quantum mechanical time-evolution and give a specific correspondence in the sense of Hepp's paper.

I hope that the reader has learnt something new from this thesis. Even if he or she hasn't, then I still hope that some new connections of relevance have formed between the various topics in quantum theory and functional analysis such as the broad idea of the classical limit, but also assumptions under which classical limits are taken and a discrete example of the classical limit from the paper of Hepp.

References

[1] Hepp, K.: Commun. Math. Phys. Vol.35 Nr.4 (1974) The classical limit for quantum mechanical correlation functions.

[2] Klauder,J.R.: Journal of Mathematical Physics 4, 1058 (1963); 5, 177 (1964); 8, 2392 (1967).

[3] Barbera MacCluer, Elementary Functional Analysis.

[4] Brian C. Hall, Quantum Theory for Mathematicians.

[5] Reed and Simon, Methods of Modern Mathematical Physics Vol 1,2,3,4.

[6] Brian C. Hall, Lie groups, Lie algebras and representations.

[7] UBC, Duhamel's formula www.math.ubc.ca/ feldman/m428/duhamel.pdf

Miscellaneous literature used for background reading and forming thoughts:

1. Bonneau and Faraut, Self-adjoint extensions of operators and the teaching of quantum mechanics.
2. Walter Thirring, Quantum Mathematical Physics.
3. Bransden and Joachain, Quantum Mechanics.
4. David J. Griffiths, Introduction to Quantum Mechanics.