Radboud University

BACHELOR'S THESIS

Fractal dimension of self-similar sets

Jesper van den Eijnden

supervised by
Dr. Michael Müger

July 18, 2018

Contents

1 Introduction

In the early 20$^{\text{th}}$ century, the Swedish mathematician Helge von Koch constructed a curve with some very unusual properties: it is bounded but of infinite length, and even though the curve is continuous, it has no tangents. This shape, called the von Koch curve, is one of the earliest examples of a *fractal* and it is constructed by a recursive procedure. The curve fills a significantly larger space than traditional curves, so calling it a one-dimensional shape seems to do it short. However, stating the curve is two-dimensional does also not accurately reflect its size.

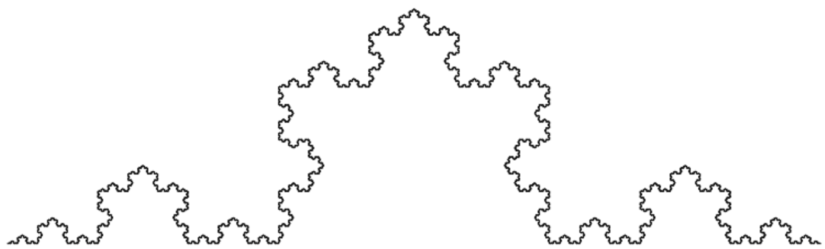

Figure 1: The von Koch curve.

This example nicely illustrates some features of fractals and the relevance of the concept *fractal dimension*. The von Koch curve has details at any scale and cannot be described by traditional methods, such as polynomial expressions or geometrical conditions. These qualities are generally attributed to fractals, and throughout this piece we keep them in mind when referring to a fractal $F \subset \mathbb{R}^d$. Roughly speaking, fractals are sets that have some of the following three typical characteristics.

First, self-similarity is a distinctive trait of many fractals. This means that portions of the shape resemble the bigger whole, somehow. Next, sets are often considered fractals if they have a very fine structure. This more or less indicates that zooming in on the shapes will keep on revealing new details, as with the von Koch curve. Last, fractals are generally not defined by classical mathematics, so they are not smooth shapes or familiar geometric objects.

Clearly a formal definition of the concept fractal is missing. This is no coincidence: explicit definitions of fractals have turned out to be too restrictive, as they exclude interesting shapes that ought to be considered a fractal. The result is that most theory on fractals is applicable to any (bounded) subset of $\mathbb{R}^d$, although the the study is only truly interesting for sets as described above.

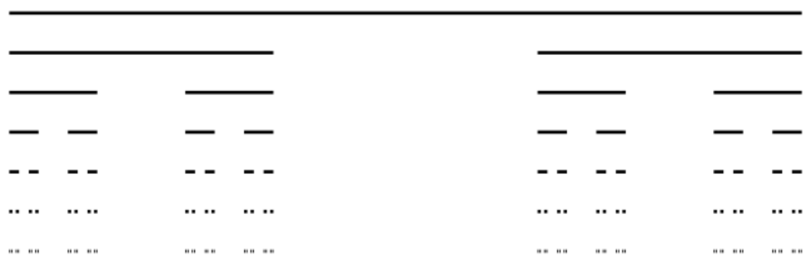

Figure 2: The first 7 stages in the construction of the Cantor set.

The archetypal example of a fractal is the Cantor set, which we will denote with C. It is defined as the intersection of a decreasing sequence of closed sets $C_0, C_1, C_2, ...$ which are constructed recursively. At stage 0, C_0 is defined as the unit interval $[0, 1]$. For $k > 0$, C_k is obtained by removing the open middle thirds of the intervals remaining in C_{k-1}, so C_k consists of 2^k intervals with length 3^{-k}. The resulting set $C = \bigcap_k C_k$ is an uncountable compact set with Lebesgue measure (length) zero. The set has the three mentioned properties, and one more that many fractals satisfy: it is defined recursively.

Fractals are no traditional shapes, so they do not necessarily fit traditional definitions. The two fractals we have seen so far display some shortcomings of usual concepts of dimension. Calling infinitely long but bounded curves one dimensional gives hardly any information about the curve itself. However, calling it two-dimensional is no solution either: in that case its measure with respect to the ambient space $\mathbb{R}^2$ is zero. Such sets are relatively negligible - which a fractal is usually not. Luckily, it is possible to generalize concepts in order to make them applicable to fractals, too. This thesis focuses on two types of fractal dimension, that is, notions of dimension that do retain information of fractals.

First, we explore *box-counting dimension*, which is rather easy to work with but has limited functionality. Next, Section 3 concerns *Hausdorff dimension*, a type of dimension that behaves the way one would hope, but that also takes more effort to use. After that, a specific class of fractals, the *self-similar sets*, is defined and the last section proves the main theorem of this thesis, which gives an elegant and simple expression for the box-counting and Hausdorff dimension of such fractals.

The main resource in writing this thesis has been the book *Fractal Geometry* by Kenneth Falconer. Thereby, the general approach and much content, like Example 2.3 and the proof of Theorem 5.8, is based on Falconer's text. However, this thesis covers full details concerning questions about measure theory that Falconer glosses over. A list of all sources can be found on page 34.

2 Box-counting dimension

In this section we consider box-counting dimension. Roughly speaking, a fractal's box-counting dimension reflects the behavior of the fractal's size under rescaling. This value is often easily determined or estimated heuristically. However, the concept fails to satisfy some desired elementary properties, which limits its mathematical usefulness. Those shortcomings are addressed at the end of this section. First, we define box-counting dimension and we explore several examples and box-counting dimensions basic properties.

2.1 Introducing box-counting dimension

To motivate the definition of box-counting dimension, we consider a line segment I of unit length. For $n \in \mathbb{N}$ we can cover I with line segments of length $1/n$. Any such covering consists of at least n segments. Similarly, we can cover a unit square I^2 with smaller squares of side length $1/n$. This will take at least n^2 little squares. Last, we may cover a unit cube I^3 with n^3 cubes of side length $1/n$. Using less cubes to cover I^3 would require using bigger cubes.

Clearly, the exponent in the required number of covering sets matches the more traditional notions of dimension of the covered shape. Box-counting dimension generalizes this idea. We cover a fractal F with arbitrarily small sets and determine the exponent s that links the amount of required covering sets to their diameter. The value s is the box-counting dimension of F.

More precisely, a δ-cover is a cover consisting of sets with diameter at most $\delta > 0$. With $N_\delta(F)$ we denote the smallest number of sets in a δ-cover of F. As δ decreases, $N_\delta(F)$ increases. If $N_\delta(F) \simeq c\delta^{-s}$ holds for some positive constants c and s, we call s the box-counting dimension of F. Solving for s gives the formal definition.

Definition 2.1. Box-counting dimension
Let F be a non-empty bounded subset of $\mathbb{R}^n$. For $\delta > 0$, we define $N_\delta(F)$ as the smallest number of sets in a δ-cover of F.
The lower box-counting dimension of F is defined as

$$\underline{\dim}_B F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

The upper box-counting dimension of F is defined as

$$\overline{\dim}_B F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

If these two limits coincide, their common value is called the ***box-counting dimension*** *of F:*

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

Not all fractals have a box-counting dimension, because the required limit may not exist. The next section gives an example of such a fractal. The term *box-counting* is justified by exploring an equivalent definition that uses a specific type of covering. Instead of considering arbitrary δ-covers, one might count the number of cubes that intersect F in a mesh of cubes with sides of length δ. This approach gives the same limits and provides box-counting dimension with its name. There are numerous such alternative, equivalent definitions of box-counting dimension. For example, one might also only consider covers with closed balls, which we will use in the proof of Lemma 5.5.

2.2 Examples of box-counting dimension

Example 2.2. The Sierpínski triangle
A fractal that is constructed in a way similar to the Cantor set, is the Sierpínski triangle. The starting point, stage 0, is a triangle S_0 with unit sides. In later stages, S_k is obtained by removing equilateral triangles from the remaining triangles in S_{k-1}, as depicted below. The shape S_k consists of 3^k triangles with side length 2^{-k}. The Sierpínski triangle is defined as $S = \bigcap_{i=0}^{\infty} S_i$.

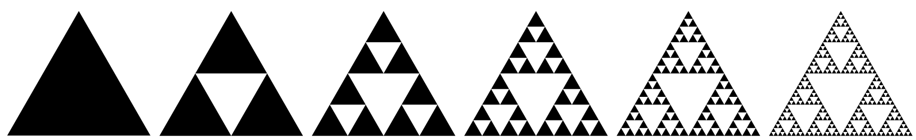

Figure 3: The first six stages in the construction of the Sierpínski triangle.

We determine $\dim_B(S)$ by estimating $\overline{\dim}_B(S)$ and $\underline{\dim}_B(S)$. Let $\delta > 0$ and choose $k \in \mathbb{N}$ such that $2^{-k} < \delta \leq 2^{-k+1}$. Then the triangles of S_k are a δ-cover of S, so $N_\delta(S) \leq 3^k$. This implies

$$\overline{\dim}(S) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(S)}{-\log \delta} \leq \overline{\lim_{k\to\infty}} \frac{\log 3^k}{-\log 2^{-k+1}} = \frac{\log 3}{\log 2}.$$

Alternatively, we can choose $l \in \mathbb{N}$ such that $2^{-l-1} \leq \delta < 2^{-l}$. Then a set of diameter δ can only intersect triangles in S_l that are less then 2^{-l} apart. Therefore, any such set intersects at most three triangles in S_l. This means that a δ-cover of S contains at least $3^l/3$ sets, which implies

$$\underline{\dim}_B F = \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta} \geq \underline{\lim}_{l\to\infty} \frac{\log 3^{l-1}}{-\log 2^{-l-1}} = \frac{\log 3}{\log 2}.$$

Hence, $\log 3/\log 2 \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F) \leq \log 3/\log 2$, which shows that the box-counting dimension of the Sierpínski triangle is $\log 3/\log 2$.

Example 2.3. Modified Cantor set with no box-counting dimension We construct a fractal of which the lower box-counting dimension is not equal to its upper box-counting dimension. For this purpose, we construct a modified Cantor set C' by taking the intersection of sets E'_k with $k \in \mathbb{N}$, where E'_k is defined as follows. Let $a_n = 10^n$ for $n \in \mathbb{N}$, then:

- $E'_0 = [0, 1]$;
- obtain E'_k by removing the middle 1/3 of the intervals in E_{k-1}, whenever $a_{2n} < k \leq a_{2n+1}$ for some $n \in \mathbb{N}$;
- obtain E'_k by removing the middle 3/5 of the intervals in E_{k-1}, whenever $a_{2n-1} < k \leq a_{2n}$ for some $n \in \mathbb{N}$.

Effectively this means that we construct E'_k by removing thirds if $1 < k \leq 10$, $100 < k \leq 1000$, $10.000 < k \leq 100.000$, and so on. For other values of k, E'_k is obtained by removing three fifths. C' is then defined as $\bigcap_{k=0}^{\infty} E'_k$.

Now we find an upper estimate and a lower estimate for respectively the lower and upper box-counting dimension of C'. Consider the length of the intervals that make up E'_{a_n} for some even $n \in \mathbb{N}$. There are 2^{a_n} such intervals. Fleshing out the construction of E_{a_n}, we find that the length of every remaining interval is given by

$$\delta_n = \left(\frac{1}{5}\right)^{a_0} \left(\frac{1}{3}\right)^{a_1 - a_0} \left(\frac{1}{5}\right)^{a_2 - a_1} \cdots \left(\frac{1}{3}\right)^{a_{n-1} - a_{n-2}} \left(\frac{1}{5}\right)^{a_n - a_{n-1}}.$$

Hence, the length of every interval at this stage is smaller than $\left(\frac{1}{5}\right)^{a_n - a_{n-1}}$. Using the corresponding intervals of E'_{a_n} as a cover of C', we find the following estimation.

$$\begin{aligned}
\underline{\dim}_B C' &\leq \varliminf_{n \to \infty} \frac{\log N_{\delta_n}(C')}{-\log \delta_n} \\
&\leq \varliminf_{n \to \infty} \frac{\log 2^{a_n}}{\log 5^{a_n - a_{n-1}}} \\
&= \varliminf_{n \to \infty} \frac{a_n \log 2}{(a_n - a_{n-1}) \log 5} \\
&= \varliminf_{n \to \infty} \frac{10 a_{n-1} \log 2}{9 a_{n-1} \log 5} \\
&= \frac{10 \log 2}{9 \log 5}
\end{aligned}$$

With a similar approach we can estimate the upper box-counting dimension. For this purpose, we consider the intervals of E'_{a_n} with n odd. Each of these intervals has length

$$\delta_n = \left(\frac{1}{5}\right)^{a_0} \left(\frac{1}{3}\right)^{a_1 - a_0} \left(\frac{1}{5}\right)^{a_2 - a_1} \cdots \left(\frac{1}{5}\right)^{a_{n-1} - a_{n-2}} \left(\frac{1}{3}\right)^{a_n - a_{n-1}}.$$

Therefore, the intervals at this stage have length $\left(\frac{1}{5}\right)^{a_{n-1}}\left(\frac{1}{3}\right)^{a_n - a_{n-1}}$ at least. Covering C' with intervals of length δ_n takes at least $2^{a_n}/2$ intervals, because every such interval can intersect at most two intervals of E'_{a_n}. Taking such a cover of C' and applying the estimate we just found, we find

$$\begin{aligned}
\overline{\dim}_B C' &\geq \overline{\lim}_{n\to\infty} \frac{\log N_{\delta_n}(C')}{-\log \delta_n} \\
&\geq \overline{\lim}_{n\to\infty} \frac{\log(2^{a_n}/2)}{\log(5^{a_{n-1}} 3^{a_n - a_{n-1}})} \\
&= \overline{\lim}_{n\to\infty} \frac{a_n \log 2 - \log 2}{a_{n-1}\log 5 + (a_n - a_{n-1})\log 3} \\
&= \overline{\lim}_{n\to\infty} \frac{10 a_{n-1}\log 2 - \log 2}{a_{n-1}\log 5 + 9a_{n-1}\log 3} \\
&= \frac{10\log 2}{\log 5 + 9\log 3}.
\end{aligned}$$

Comparing the estimates of the lower and upper box-counting dimension yields the desired result:

$$\underline{\dim}_B C' \leq \frac{10\log 2}{9\log 5} = 0,4785... < 0,6028... = \frac{10\log 2}{\log 5 + 9\log 3} \leq \overline{\dim}_B C'.$$

2.3 Properties of box-counting dimension

The following proposition covers some of box-counting dimensions basic properties, which will be useful in later sections.

Proposition 2.4. ***Properties of box-counting dimension***

(i) Box-counting dimension is monotonic. In other words, if $E \subset F \subset \mathbb{R}^d$, then $\dim_B(E) \leq \dim_B(F)$.

(ii) If F is a bounded, non-empty subset of $\mathbb{R}^d$, then $0 \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F) \leq d$.

(iii) If F is an open subset of $\mathbb{R}^d$, then $\dim_B(F) = d$.

Proof of Proposition 2.4
(i) Any δ-cover of E is also a δ-cover of F, so $N_\delta(E) \leq N_\delta(F)$ for all $\delta > 0$. Monotonicity of $\overline{\dim}_B$ and $\underline{\dim}_B$ follows, which implies the claim.

(ii) Only the last inequality requires a proof, for which we use a mentioned equivalent method of covering. We take the minimal amount of cubes with side δ in a mesh intersecting F as $N_\delta(F)$. F is contained in some large enough cube C, so applying (i) we find $N_\delta(F) \leq N_\delta(C) \leq c\delta^{-n}$ for some constant c. By construction, the exponent n gives $\dim_B(F)$.

(iii) F contains some cube C, so again considering a δ-mesh leads to $N_\delta(F) \geq N_\delta(C) \geq c\delta^{-n}$ for some constant c. Then $\dim_B(F) \geq n$ which, combined with (ii), proves the claim. $\square$

Unfortunately, box-counting dimension also has some undesired properties. For example, there are sets such as C', encountered in Example 2.3, that have no well-defined box-counting dimension. Yet also with sets for which the dimension is well-defined, problems arise. Countable sets may significantly alter a fractal's box-counting dimension, while such sets are usually negligible. This implies that a fractal's box-counting dimension does not necessarily reflect its size as one would hope. Furthermore, box-counting dimension does not always cope well with infinite unions. The following proposition may be short, but it illustrates these issues.

Proposition 2.5. ***Box-counting dimension and closures***
Let F be a bounded subset of $\mathbb{R}^d$, then $\dim_B(F) = \dim_B(\overline{F})$.

Proof of Proposition 2.5
Once again we use an equivalent covering method to determine $\dim_B$. Now we consider covers of F with closed δ-balls. A union $\bigcup_{i=1}^n B_i$ of such balls is closed and hence it contains F if and only if it contains $\overline{F}$. Therefore, $N_\delta(F) = N_\delta(\overline{F})$. The claim follows for $\overline{\dim}_B$ and $\underline{\dim}_B$, thus also for $\dim_B$. $\square$

Now we start running into problems as described. For example, $\dim_B(\mathbb{Q} \cap [0,1]) = \dim_B([0,1]) = 1$, while the first set is countable and has measure zero and the second set is uncountable and has a non-zero measure. Intuitively, one would surely not assign the same dimension to these sets. This example shows additionally that box-counting dimension does not behave well under infinite unions. Considering $\mathbb{Q} \cap [0,1]$ as a countable union of isolated rational points, each of which has box-counting dimension zero, we observe that $\dim_B(\bigcup_{i=1}^\infty F_i) = \sup_i\{\dim_B F_i\}$ does *not* generally hold. However, this property is desirable for a worthwhile concept of dimension.

3 Hausdorff dimension

In this section we explore another widely used notion of fractal dimension: Hausdorff dimension. Its construction involves some measure theory, in the shape of the so-called outer Hausdorff measure. Inspecting the behavior of a fractal's outer measure at various scales leads to its Hausdorff dimension. This approach results in a well-defined, well-behaving concept of fractal dimension. However, determining a fractal's Hausdorff dimension can be significantly more complicated than finding, for example, its box-counting dimension. After introducing the measure and the main definition, this section addresses an example and some properties of Hausdorff dimension.

3.1 Introducing outer Hausdorff measure

First, we need to define the general concept of outer measures and the outer measure that we use to define Hausdorff dimension. In this text, the diameter $|A|$ of any $A \subset \mathbb{R}^d$ is defined as $|A| = \sup\{|x-y| : x, y \in A\}$.

Definition 3.1. Outer measure
Let X be a set and let $\mathcal{P}(X)$ denote its power set. An ***outer measure*** *on X is a function $\mu^* : \mathcal{P}(X) \to [0,\infty]$ such that*

(i) $\mu^*(\emptyset) = 0$

(ii) If $A \subset B \subset X$, then $\mu^*(A) \le \mu^*(B)$.

(iii) If $\{A_n\}_{n\in\mathbb{N}}$ is an infinite sequence of subsets of X, then $\mu^*(\bigcup_{i=1}^{\infty} A_n) \le \sum_{i=1}^{\infty} \mu^*(A_n)$

Definition 3.2. Outer Hausdorff measure
Take $\delta > 0$ and $\alpha \ge 0$. For any $F \subset \mathbb{R}^d$ let

$$\mathcal{H}_\delta^\alpha(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^\alpha : F \subset \bigcup_{i=1}^{\infty} U_i \ , \ \forall i \ |U_i| < \delta\right\}.$$

Then the limit

$$m_\alpha^*(F) = \lim_{\delta\to 0} \mathcal{H}_\delta^\alpha(F)$$

exists and is called the α-dimensional ***outer Hausdorff measure****.*

Let us unravel this definition, starting with $\mathcal{H}_\delta^\alpha$. Here we cover the set F with sets U_i of diameter smaller than δ. Next, we take the infimum over the sums of powers of the covering sets diameters. Decreasing δ means permitting less covers, which results in an increase of $\mathcal{H}_\delta^\alpha(F)$. Therefore the limit $\lim_{\delta\to 0} \mathcal{H}_\delta^\alpha(F)$ always exists, although it may be infinite.

The definition of m_α^* features two parameters: δ, which tends to zero, and a fixed value α. The arbitrarily small set diameter δ accounts for the roughness of fractals. The more detailed a shape is, the more impact decreasing δ will have. The value α incorporates the behavior of shapes under rescaling in α-dimensional space. Scaling a d-dimensional cube in $\mathbb{R}^d$ with factor r scales its volume by factor r^d. Scaling a set U with factor r will scale $|U|^\alpha$ with factor r^α. In this sense, α signifies from what perspective, or dimension, we consider our shape. This explains the term *α-dimensional*.

Proposition 3.3. ***Outer measure*** m_α^*
Outer Hausdorff measure m_α^* is an outer measure on $\mathbb{R}^d$.

Proof of Proposition 3.3
Clearly, the image under m_α^* of a set in $\mathbb{R}^d$ is non-negative, as required. We verify the three mentioned properties of outer measures.
(i) This follows immediately from the definition by covering the empty set with the empty set.
(ii) This holds because any cover of A is also a cover of B.
(iii) After fixing δ and taking $\epsilon > 0$, we may cover every A_n with sets $\{B_{n,k}\}_{k=1}^\infty$ such that $|B_{n,k}| < \delta$ for all k and $\sum_{k=1}^\infty |B_{n,k}|^\alpha \leq \mathcal{H}_\delta^\alpha(A_n) + \epsilon/2^n$. The latter is possible because $\mathcal{H}_\delta^\alpha$ is the infimum over such covers. We use this inequality to estimate $\mathcal{H}_\delta^\alpha(\bigcup_n A_n)$. Note that $\{F_{n,k}\}_{n,k\in\mathbb{N}}$ is a cover of $\bigcup_i A_n$.

$$\begin{aligned}
\mathcal{H}_\delta^\alpha\left(\bigcup_{n=1}^\infty A_n\right) &\leq \sum_{n,k} |B_{n,k}|^\alpha \\
&\leq \sum_n \mathcal{H}_\delta^\alpha(A_n) + \epsilon \\
&\leq \sum_n m_\alpha^*(A_n) + \epsilon
\end{aligned}$$

Letting ϵ and δ tend to zero leads to sub-additivity. $\square$

In fact, m_α^* is a measure when it is restricted to the Borel sets, which is proven as follows. Given any outer measure μ^* on $\mathbb{R}^d$, a set $B \subset \mathbb{R}^d$ is called μ^*-measurable if $\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \backslash B)$ for all $A \subset \mathbb{R}^d$. The collection of all measurable sets is a σ-algebra and the restriction of μ^* to this collection is a measure. Both statements are classic measure theory results, for example proven in Theorem 1.3.6 in [2]. Assuming these claims, we only need to proof that the Borel sets are m_α^*-measurable. For this purpose, we use another classic result, for example covered in Theorem 6.1.2 of [6]. If μ^* is an outer measure on a metric space X such that $m_\alpha^*(A \cup B) = m_\alpha^*(A) + m_\alpha^*(B)$ for all subsets A and B of X with $d(A, B) = \inf\{|a - b| : a \in A,\ b \in B\} > 0$, then the Borel sets in X are μ^*-measurable. Hence, we only need to verify that m_α^* is such an outer measure, called a *metric* outer measure.

Note that $m_\alpha^*(A \cup B) \leq m_\alpha^*(A) + m_\alpha^*(B)$ is implied by Property (iii). To prove the reverse inequality, we take $\delta > 0$ with $\delta < d(A, B)$. For any δ-cover $\{F_i\}$ of $A \cup B$, we define $F_i' = A \cap F_i$ and $F_i'' = B \cap F_i$. Then $\{F_i'\}_i$ and $\{F_i''\}_i$ are disjoint covers of A and B, respectively. We come to the following inequality:

$$\sum_{i=1}^{\infty} |F_i'|^\alpha + \sum_{j=1}^{\infty} |F_j''|^\alpha \leq \sum_{k=1}^{\infty} |F_k|^\alpha.$$

Taking infima over the covers in the equation above gives

$$\mathcal{H}_\delta^\alpha(A) + \mathcal{H}_\delta^\alpha(B) \leq \mathcal{H}_\delta^\alpha(A \cup B).$$

Letting δ tend to zero now yields the required inequality, with m_α^* instead of $\mathcal{H}_\delta^\alpha$.

This means that m_α^* restricted to the Borel sets defines a measure on $\mathbb{R}^d$, called *Hausdorff measure*. In this text, we do not actually need this measure, because the outer Hausdorff measure m_α^* will have all the necessary properties and is not restricted to the Borel sets.

3.2 Introducing Hausdorff dimension

In this section, we discover that a fractal's outer Hausdorff measure is either zero or infinity for almost all α. In fact, at one point the outer measure jumps from from one to the other. This unique value of α is the Hausdorff dimension of the fractal. The key idea is that α sets the surrounding space in which we determine the measure of the fractal F. Taking α too large means F will be of negligible size, in other words of measure zero. Taking α too small leads to a relatively over-sized fractal, in other words of infinite measure.

Proposition 3.4. ***Behavior of $\mathbf{m}_\alpha^*$ for small and large α***
Let F be a subset of $\mathbb{R}^d$, then

(i) m_0^* is the counting measure on $\mathbb{R}^d$, that is, $m_0^*(F)$ gives the number of points in F if F is finite, and ∞ otherwise.

(ii) If $s > d$, then $m_s^*(F) = 0$.

Proof of Proposition 3.4
(i) For any point $x \in \mathbb{R}^d$ and $\delta > 0$, it is quickly verified that $\mathcal{H}_\delta^0(\{x\}) = 1$. Hence, $m_0^*(\{x\}) = \lim_{\delta \to 0} \mathcal{H}_\delta^0(\{x\}) = 1$, which implies the claim.
(ii) Take $k \geq 1$. We can cover the unit cube Q in $\mathbb{R}^d$ with k^d cubes of side $\frac{1}{k}$ and thus of diameter $\epsilon_k = \frac{\sqrt{n}}{k}$. Then

$$\begin{aligned}
\mathcal{H}_{\epsilon_k}^s(Q) &= \inf \left\{ \sum_{i=1}^{\infty} |U_i|^\alpha : Q \subset \bigcup_{i=1}^{\infty} U_i \ , \ \forall i \ |U_i| < \epsilon_k \right\} \\
&\leq \sum_{i=1}^{k^d} \epsilon_k^s \\
&= k^{d-s} n^{\frac{s}{2}}
\end{aligned}$$

As k tends to infinity, the right hand side above will tend to 0, since $s > d$. This implies $m_s^*(Q) = 0$. By covering $\mathbb{R}^d$ with unit cubes like Q and using countable sub-additivity, we find $m_s^*(\mathbb{R}^d) = 0$, so especially $m_s^*(F) = 0$. $\square$

The general behavior of m_α^* is not all that different. Assume $m_\alpha^*(F) < +\infty$ for some α and take $\beta > \alpha$. Then for any δ-cover $\{U_i\}$ of F we know

$$\sum_{i=1}^{\infty} |U_i|^\beta = \sum_{i=1}^{\infty} |U_i|^{\beta-\alpha}|U_i|^\alpha \leq \delta^{\beta-\alpha} \sum_{i=1}^{\infty} |U_i|^\alpha. \tag{1}$$

Taking infima and letting δ tend to zero results in $m_\beta^*(F)$ on the left-hand side and 0 on the right-hand side. In other words, if $m_\alpha^*(F)$ is ever finite, the β-dimensional measure is zero for all $\beta > \alpha$.

Alternatively, if $m_\alpha^*(F) > 0$ and $\beta < \alpha$, we know that inequality (1) above holds in opposite direction, because $\beta - \alpha < 0$. Then a similar line of reasoning shows that whenever $m_\alpha^*(F)$ is non-zero, the β-dimensional measure is $+\infty$ for every $\beta < \alpha$.

We conclude that the α-dimensional outer Hausdorff measure of a fractal is almost always 0 or ∞. The outer measure can and will jump from 0 to ∞ at only one point, which leads us to the following definition.

Definition 3.5. Hausdorff dimension
For any set $F \subset \mathbb{R}$ its ***Hausdorff dimension*** $\dim_H(F)$ *is given by*

$$\begin{aligned} \dim_H(F) &= \inf\{\beta > 0 : m_\beta^*(F) = 0\} \\ &= \sup\{\beta > 0 : m_\beta^*(F) = \infty\} \end{aligned}$$

Note that the definition does not involve the actual value of $m_\alpha^*(F)$ at $\alpha = \dim_H(F)$. In fact, there are fractal examples of all cases, so when $m_\alpha^*(F)$ is zero, infinite, or non-zero and finite.

3.3 Example of Hausdorff dimension

If one can find a finite upper bound and a non-zero lower bound for the s-dimensional outer Hausdorff measure of a fractal F, it follows that s equals $\dim_H(F)$. Usually, finding a lower bound is more challenging, as is illustrated below.

Example: Cantor set
Here we prove that $s = \log 2/\log 3$ is the Hausdorff dimension of the Cantor set C. Remember that $C = \bigcap_{k=0}^{\infty} C_k$, where C_k is made of 2^k intervals of length 3^{-k}. For the upper bound, let $\delta > 0$ and choose $k \in \mathbb{N}$ such that $3^{-k} \leq \delta$. Then C_k is a δ-cover of C, so $\mathcal{H}_\delta^s(C) \leq 2^k 3^{-ks} = 1$, using the definition of s. Letting k tend to infinity gives $m_s^*(C) \leq 1$.

To find a lower bound, we will show that

$$\frac{1}{2} \leq \sum_{i=1}^{m} |V_i|^s \tag{2}$$

for any finite cover $\{V_i\}_{i\in\mathbb{N}}$ of C consisting of intervals in $[0,1]$. To see that this is sufficient, consider the following: certainly, covers worth considering consist of intervals in $[0,1]$. Given such a cover $\{U_i\}_{i\in\mathbb{N}}$, let $\epsilon > 0$ and define the sets V_i as follows.

- $U_i \subset V_i$ for all $i \in \mathbb{N}$.
- Every V_i is an open interval in $[0,1]$.
- $(\sum_{i=1}^{\infty} |U_i|^s) + \epsilon \geq \sum_{i=1}^{\infty} |V_i|^s$

The last property of V_i could be satisfied by setting $|V_i| = \sqrt[s]{|U_i|^s + \epsilon/(2^i)}$, for example. Then $\{V_i\}_{i\in\mathbb{N}}$ is an open cover of C, so by compactness of the Cantor set we can find a finite subcover $\bigcup_{j=1}^{m} V_j$. Assuming (2) we find

$$\left(\sum_{i=1}^{\infty} |U_i|^s\right) + \epsilon \geq \sum_{i=1}^{\infty} |V_i|^s \geq \sum_{j=1}^{m} |V_j|^s \geq \frac{1}{2}.$$

Letting ϵ tend to 0 gives a positive lower bound for $\sum_i |U_i|^s$, and thus also for $m_s^*(C)$. Hence, only proving (2) now remains.

We will prove the claim by counting intersections of the V_i with intervals in C_k. For every V_j choose k such that $3^{-(k+1)} \leq |V_j| < 3^{-k}$. Intervals in C_k are at least a distance 3^{-k} apart, so V_j intersects at most one interval of C_k. Thus, for $l \geq k$ there are at most $2^{l-k} = 2^l 3^{-sk}$ such intersections. Using $3^{-(k+1)} \leq |V_j|$, which implies $3^{-ks}3^{-s} \leq |V_j|^s$, we find $2^l 3^{-sk} \leq 2^l 3^s |V_j|^s$.

There are only finitely many V_i, so we can safely denote the largest k among all V_i with k_0. Every V_i intersects with at most $2^{k_0}3^s|V_j|^s$ intervals of C_{k_0}. In total, the V_i must intersect all 2^{k_0} intervals of C_k, as they form a cover of C. We use this to prove (2) as follows:

$$2^{k_0} \leq \sum_{j=1}^{m} 2^{k_0} 3^s |V_j|^s \implies 3^{-s} = \frac{1}{2} \leq \sum_{i=1}^{m} |V_i|^s.$$

3.4 Properties of Hausdorff dimension

In analogy of our treatment of box-counting dimension, we now consider some basic, useful properties of Hausdorff dimension.

Proposition 3.6. ***Properties of Hausdorff dimension***

(i) If $E \subset F \subset \mathbb{R}^d$, then $\dim_H(E) \leq \dim_H(F)$.

(ii) If $F \subset \mathbb{R}^d$, then $0 \leq \dim_H(F) \leq d$.

(iii) If $\{F_i\}_{i\in\mathbb{N}}$ is a countable family of sets in $\mathbb{R}^d$, then $\dim_H\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_{1\leq i\leq\infty}\{\dim_H(F_i)\}$.

Proof of Proposition 3.6
(i) This follows from the monotonicity of m_α^*.
(ii) By construction, Hausdorff dimension is non-negative. We established in Proposition 3.4 that $m_s^*(F) = 0$ for $s > d$, so $\dim_H(F) = \inf\{\beta > 0 : m_\beta^*(F) = 0\} \leq d$.
(iii) By monotonicity, $\dim_H\left(\bigcup_{i=1}^{\infty} F_i\right) \geq \dim_H(F_i)$ for all i. This implies that $\dim_H\left(\bigcup_{i=1}^{\infty} F_i\right) \geq \sup_i\{\dim_H(F_i)\}$. Alternatively, if $s > \sup_i \dim_H(F_i)$, then $m_s^*(F_i) = 0$ for all i. Hence, $m_s^*\left(\bigcup_i F_i\right) \leq \sum_i m_s^*(F_i) = 0$, so we also know that $\dim_H\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sup_i\{\dim_H(F_i)\}$. □

The next proposition covers a relation between a fractal's box-counting dimension and Hausdorff dimension. This will be useful in the proof of Theorem 5.9.

Proposition 3.7. ***Comparison of fractal dimensions***
If F is a non-empty, bounded subset of $\mathbb{R}^d$, then

$$\dim_H(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F).$$

Proof of Proposition 3.7
Let $s \geq 0$ such that $m_s^*(F) > 1$. Such an s always exists, because for small s we know that $m_s^*(F) = +\infty$. Using the definition of $\mathcal{H}_\delta^s(F)$, we find for δ sufficiently small that

$$1 < \mathcal{H}_\delta^s(F) \leq N_\delta(F)\delta^s.$$

Taking logarithms in the estimation above gives $0 < \log N_\delta(F) + s\log\delta$. Hence, $s \leq \underline{\lim}_{\delta\to 0} \log N_\delta(F)/-\log\delta = \underline{\dim}_B(F)$.
Note that $\dim_H(F) = \sup\{\beta : m_\beta^*(F) = +\infty\} = \sup\{\beta : m_\beta^*(F) > 1\}$. Thus, taking the supremum over admissible s gives $\dim_H(F)$ on the left hand side, as required. □

4 Iterated function systems

As mentioned before, many fractals resemble themselves on a smaller scale in some way. In this section, we discover a specific method of finding and describing fractals that strongly feature such resemblance. In short, any family of contracting maps gives rise to a unique fractal, which remains invariant under the contracting maps. This fractal can be obtained by an iterative procedure of repeatedly applying the maps to a large enough set. The collection of maps is called a *iterative function system* and this section addresses its direct link with fractals.

4.1 Introducing iterated function systems

The previous paragraph introduced a few new concepts which need to be defined explicitly.

Definition 4.1. Contraction
A ***contraction with ratio*** *r is a mapping $S : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$|S(x) - S(y)| \leq r|x - y| \ \ \forall x, y \in \mathbb{R}^d$$

with $0 < r < 1$ fixed.

Note that for a contraction S and set U we know that $|S(U)| \leq r|U|$. Also, the composition of contractions $S_1, ..., S_n$ with ratios $r_1, ..., r_n$ is again a contraction with ratio $r_1 \dots r_n$.

Definition 4.2. Iterated function system (IFS)
A finite collection $\{S_1, ..., S_n\}$ of contractions, with $n > 1$, is called an ***iterated function system (IFS)****.*

Definition 4.3. Attractor
Let $\{S_1, ..., S_n\}$ be an iterated function system. Then a non-empty compact set F is called an ***attractor*** *of the IFS if*

$$F = \tilde{S}(F) = \bigcup_{i=1}^{n} S_i(F).$$

A typical example of such an attractor is the von Koch curve of Section 1 (see Figure 1), which is constructed by repeatedly adding spikes to remaining line segments. Given α, β and γ as in Figure 4 and the rotation ρ centered at the origin of angle $\pi/3$, the corresponding contractions are given by

$$S_1(x) = \frac{x}{3}, \ S_2(x) = \rho\frac{x}{3} + \alpha, \ S_3(x) = \rho^{-1}\frac{x}{3} + \beta, \ S_2(x) = \frac{x}{3} + \gamma.$$

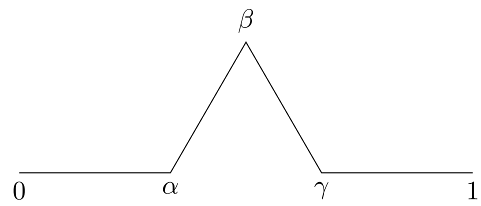

Figure 4: The points corresponding to the IFS of the von Koch curve.

However, this is a special case of an IFS because the involved contractions rescale by a constant factor of $\frac{1}{3}$, that is, the inequality in Definition 4.1 is in fact an equality. Such contractions are called similarities (see Definition 5.1) and are the main focus of the last section.

4.2 Hausdorff metric

Before we move on to this section's main result and proof, we need to do some groundwork. We define a distance function on the compact sets of $\mathbb{R}^d$, called the *Hausdorff metric*. For this purpose, we need the concept of a set's neighborhood. Given any set $A \subset \mathbb{R}^d$ and $\delta > 0$, let $A_\delta = \{x \in \mathbb{R}^d : |x-a| < \delta \text{ for some } a \in A\}$. The set A_δ is called the δ-neighborhood of A. It contains A and a little more, namely the points within a distance δ of A.

Definition 4.4. Hausdorff distance
*Let $A, B \subset \mathbb{R}^d$ be two non-empty compact sets. Then the **Hausdorff distance** between A and B is given by*

$$d_H(A, B) = \inf\{\delta : B \subset A_\delta \text{ and } A \subset B_\delta\}$$

In other words, the Hausdorff distance between two sets indicates how much both sets must be enlarged around their periphery in order to contain each other. This defines a metric, which is called the Hausdorff metric.

Proposition 4.5. *Properties of Hausdorff distance*
Let A, B, C be compact subsets of $\mathbb{R}^d$ and let $\{S_1, ..., S_n\}$ be an IFS. Then Hausdorff distance satisfies:

(i) $d_H(A, B) < \infty$;

(ii) $d_H(A, B) = 0 \iff A = B$;

(iii) $d_H(A, B) = d_H(B, A)$;

(iv) $d_H(A, C) \le d_H(A, B) + d_H(B, C)$;

(v) $d_H(\tilde{S}(A), \tilde{S}(B)) \le (\max_{1 \le i \le n} r_i) d_H(A, B)$.

The first four properties show that Hausdorff distance does indeed define a distance function, or metric, on the family of compact subsets of $\mathbb{R}^d$. Property (v) is important in the proof of uniqueness in Theorem 4.6.

Proof of Proposition 4.5
(i) The definition of Hausdorff metric implies the following equation:

$$d_H(A, B) = \sup_{a \in A} d_H(\{a\}, B) = \sup_{a \in A}(\inf_{b \in B} |a - b|). \tag{3}$$

Both sets are bounded because they are compact, so the expression on the right hand side above is finite, as required
(ii) Assume that $\mathrm{d}_H(A, B) = 0$, so $A \subset B_\delta$ for every $\delta > 0$. B is compact so it is closed, which shows

$$A \subset \bigcap_{\delta > 0} B_\delta = \overline{B} = B$$

The reverse inclusion follows similarly and thus $A = B$ whenever $\mathrm{d}_H(A, B) = 0$. The other implication is evident.

(iii) Symmetry follows instantly from the definition.

(iv) We use Equation (3). For any $a \in A$

$$\begin{aligned} d_H(\{a\}, C) &= \inf_{c \in C} |a - c| \\ &\leq \inf_{c \in C}(|a - b| + |b - c|) && \forall b \in B \\ &= |a - b| + d_H(\{b\}, C) && \forall b \in B \\ &\leq |a - b| + d_H(B, C) && \forall b \in B. \end{aligned}$$

Taking the infimum over all $b \in B$, we obtain

$$\begin{aligned} d_H(\{a\}, C) &\leq \inf_{b \in B} |a - b| + d_H(B, C) \\ &= d_H(a, B) + d_H(B, C). \end{aligned}$$

Taking the supremum over $a \in A$ then shows $\mathrm{d}_H(A, C) \leq d_H(A, B) + d_H(B, C)$.

(v) First, we let $\delta_0 = \max_{1 \leq i \leq n} \mathrm{d}_H(S_i(A), S_i(B))$. Then we know that $S_i(A) \subset S_i(B)_{\delta_0}$ and $S_i(B) \subset S_i(A)_{\delta_0}$ for all $i \in \{1, \ldots, n\}$. This implies $\tilde{S}(A) \subset \tilde{S}(B)_{\delta_0}$ and $\tilde{S}(B) \subset \tilde{S}(A)_{\delta_0}$, so

$$\mathrm{d}_H(\tilde{S}(A), \tilde{S}(B)) \leq \max_{1 \leq i \leq n} \mathrm{d}_H(S_i(A), S_i(B)). \tag{4}$$

Hausdorff metric is defined as an infimum over distances and a contraction S_i reduce distances, at least with factor r_i. This means we can state

$$\max_{1 \leq i \leq n} \mathrm{d}_H(S_i(A), S_i(B)) \leq (\max_{1 \leq i \leq n} r_i)\mathrm{d}_H(A, B).$$

In combination with (4) this proves the claim. $\square$

4.3 Unique attractors

Theorem 4.6. ***Unique attractors of iterated function systems***
Let $\{S_1, ..., S_n\}$ be an IFS. Then this system has a unique attractor. In other words, there is a unique non-empty compact set F such that

$$F = \bigcup_{i=1}^{n} S_i(F).$$

If B is any non-empty compact set such that $S_i(B) \subset B$ for all i with $1 \leq i \leq n$, then the attractor F can be expressed as

$$F = \bigcap_{k=0}^{\infty} \tilde{S}^k(B).$$

Proof of Theorem 4.6
The following proof consists of three steps: first, we establish the existence of a set B that contains its image under any S_i. Next, we prove that iteratively applying $\tilde{S}$ to such a set results in an attractor F. Last, we use Hausdorff metric to prove uniqueness of F.

Take a contraction S_i with corresponding ratio r_i from the IFS. We will determine a radius R_i such that the ball B_i with radius R_i centered around the origin satisfies $S_i(B_i) \subset B_i$. By the triangle inequality, we know that $|S_i(x)| \leq |S_i(x) - S_i(0)| + |S_i(0)|$ for any $x \in \mathbb{R}^d$. The definition of contractions then leads to

$$|S_i(x)| \leq r_i|x| + |S_i(0)|.$$

We want that $|x| \leq R_i$ implies $|S_i(x)| \leq R_i$. The inequality above shows it suffices to have $r_i R_i + |S(0)| \leq R_i$. In other words, a ball with a radius R_i larger than $|S(0)|/(1-r)$ will do. If we let B be the ball among all B_i with the largest radius, we have found a set that satisfies $S_i(B) \subset B$ for all i.

We now know that $\tilde{S}(B) \subset B$ and consequently that $\tilde{S}^{k+1}(B) \subset \tilde{S}^k(B)$ for all $k \in \mathbb{N}$. Hence, $\{\tilde{S}^k(B)\}_k$ forms a decreasing sequence of non-empty compact sets. The non-empty compact intersection $F = \bigcap_{k=0}^{\infty} \tilde{S}^k(B)$ of these sets is an attractor as desired, which is clarified below.

$$\begin{aligned}
\tilde{S}(F) &= \tilde{S}\left(\bigcap_{k=0}^{\infty} \tilde{S}^k(B)\right) \\
&= \bigcap_{k=0}^{\infty} \tilde{S}\left(\tilde{S}^k(B)\right) \\
&= \bigcap_{k=1}^{\infty} \tilde{S}^k(B) \\
&= F
\end{aligned}$$

Only checking the uniqueness of F remains. Suppose there is another attractor G, then we have both $\tilde{S}(F) = F$ and $\tilde{S}(G) = G$. Using property (v) of Hausdorff distance, we find

$$\mathrm{d}_H(F, G) = \mathrm{d}_H(\tilde{S}(F), \tilde{S}(G)) \leq (\max_{1 \leq i \leq n} r_i)\mathrm{d}_H(F, G).$$

Because $0 < \max_{1 \leq i \leq n} r_i < 1$, we know that $\mathrm{d}_H(F, G) = 0$ and we conclude that $F = G$. $\square$

4.4 Encoding fractals

The previous theorem provides us with a new way to approach many fractals. We have established that when a compact set B is big enough, $\{\tilde{S}^k(B)\}_k$ is a decreasing sequence of sets that converges to the attractor $F = \bigcap_{k=0}^{\infty} S^k(B)$. Further inspection of this intersection leads to a method of describing the points in F.

Proposition 4.7. ***Encoding points of attractors***
Let $\{S_1, ..., S_n\}$ be an IFS and let F be its unique attractor, so $F = \bigcap_{k=0}^{\infty} \tilde{S}^k(B)$ for any large enough compact set B. We define the function

$$\phi : \{(i_1, i_2, ...) : 1 \leq i_j \leq n\} \to F$$

that sends $(i_1, i_2, ...)$ to

$$\{x_{i_1, i_2, ...}\} = \bigcap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(B).$$

Then ϕ is surjective.

Proof of Proposition 4.7
First, we verify that ϕ is well-defined. For a given sequence $(i_1, i_2, ...)$, we know that $\{S_{i_1} \circ \cdots \circ S_{i_k}(B)\}_{k \in \mathbb{N}}$ is a decreasing sequence of compact sets of which the diameters tend to zero, in a non-empty complete metric space. Hence, the intersection contains precisely one point, which we define as $x_{i_1, i_2, ...}$. Furthermore, this intersection is contained in F because

$$\bigcap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(B) \subset \bigcap_{k=0}^{\infty} \tilde{S}^k(B) = F.$$

Next, we prove surjectivity. For $k \in \mathbb{N}$ we use $\mathcal{I}_k$ to denote the set of all sequences $(i_1, ..., i_k)$ with $1 \leq i_k \leq n$. Then we can rewrite $\tilde{S}^k(B)$ as follows:

$$\begin{aligned}
\tilde{S}^k(B) &= \tilde{S} \circ \cdots \circ \tilde{S}\,(B) \\
&= \bigcup_{i=1}^{m} S_i \left(\bigcup_{i=1}^{m} S_i \left(\ldots \left(\bigcup_{i=1}^{m} S_i(B) \right) \ldots \right) \right) \\
&= \bigcup_{I_k} S_{i_1} \circ \cdots \circ S_{i_k}(B).
\end{aligned}$$

Any point $x \in F$ is contained in $\tilde{S}^k(B)$ for every $k \in \mathbb{N}$. Hence, for every $k \in \mathbb{N}$ there is a sequence $(i_1, ..., i_k) \in \mathcal{I}_k$ such that $x \in S_{i_1} \circ \cdots \circ S_{i_k}(B)$. This leads to a sequence $(i_1, i_2, ...)$ with the desired property. $\square$

The function ϕ allows us to notate points x as $x_{i_1,i_2,...}$, where $(i_1, i_2, ...)$ is a corresponding sequence. Note that this sequence is not necessarily unique, but this is enough to be useful in the next section.

5 Fractal dimension of self-similar sets

Now we focus on certain special iterated function systems, namely those of which the contractions scale with a constant factor. The attractors of such systems are known as self-similar sets, because of their high degree of self-similarity. The box-counting and Hausdorff dimension of these attractors can be expressed elegantly in terms of the systems' contraction ratios. This section first introduces some necessary background and then we prove the main theorem. Last, we see some examples of using this theorem to determine fractal dimensions.

Definition 5.1. Similarity
*A contracting **similarity with ratio** r is a mapping $S : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$|S(x) - S(y)| = r|x - y| \ \ \forall x, y \in \mathbb{R}^d$$

with $0 < r < 1$ fixed.

It is clear that similarities are just contractions with the added condition that distance decreases consistently. After all, the requirement for ratios is only that $|S(x) - S(y)| \leq r|x - y|$. Consequently, Theorem 4.6 holds for iterated function systems consisting of similarities, which justifies the following definition.

Definition 5.2. Self-similar set
Let F be the attractor of an IFS $\{S_1, ..., S_n\}$ consisting of similarities. In other words, F is non-emtpy, compact, and

$$F = \bigcup_{i=1}^{n} S_i(F).$$

*Then F is called a **self-similar set**.*

The von Koch curve is a self-similar set, corresponding to four similarities with contraction ratios $\frac{1}{3}$ (see Section 4.1). Also, the Sierpínski triangle and the Cantor set are self-similar sets. For example, the Cantor set is the attractor of the similarities $S_1, S_2 : [0, 1] \to [0, 1]$ with $S_1 : x \mapsto \frac{1}{3}x$ and $S_2 : x \mapsto \frac{1}{3}x + \frac{2}{3}$.

5.1 Groundwork

This subsection focuses on the technical lemmas and definitions that are necessary to understand Theorem 5.9. The first result, Lemma 5.3, will be crucial in determining a lower bound for self-similar sets' Hausdorff dimension.

Lemma 5.3. *Outer measure estimate*
Let μ be an outer measure on a bounded set $F \subset \mathbb{R}^d$. Let $s > 0$ and suppose that there are numbers $q > 0$ and $r > 0$ such that

$$\mu(U) \leq q|U|^s$$

for all sets $U \subset \mathbb{R}^d$ with $|U| \leq r$. Then the following inequality holds:

$$\frac{\mu(F)}{q} \leq m_s(F).$$

Proof of lemma 5.3
As seen before, first considering any cover of F and then taking infima leads to the required estimate of the Hausdorff measure. Take a positive $\delta < r$ and let U_i be any δ-cover of F, then

$$\mu(F) \leq \mu\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq q \sum_{i=1}^{\infty} |U_i|^s.$$

The last inequality follows from the lemma's assumptions. Taking infima and letting δ tend to zero results in $\mu(F) \leq q m_s(F)$, hence the claim. $\square$

To use the lemma above, we need an appropriate outer measure on the given fractal. The following theorem constructs an outer measure on the infinite product space $X^{\mathbb{N}}$ step by step, which later leads to the required outer measure.

Proposition 5.4. *Outer measure on an infinite product space*
Let X be a finite set and let $p : X \to [0,1], x \mapsto p_x$ be a function such that $\sum_{x\in X} p_x = 1$.

(i) For $k \in \mathbb{N}$ and $C \subset X^k$ define

$$I_C = \{(x_1, x_2, \ldots) \in X^{\mathbb{N}} \mid (x_1, \ldots, x_k) \in C\}.$$

For $k = 0$, define $I_\emptyset = X^{\mathbb{N}}$.
Then $\mathcal{A} = \{I_C : k \in \mathbb{N}_0, C \subset X^k\}$ is an algebra.

(ii) For $k \in \mathbb{N}_0,\ C \subseteq X^k$ define

$$\mu(I_C) = \sum_{(i_1,\ldots,i_k)\in C} p_{i_1} \cdots p_{i_k}.$$

Then μ is a finitely additive measure on $\mathcal{A}$ with $\mu(X^{\mathbb{N}}) = 1$.

(iii) The measure μ has the following properties.

(α) μ is continuous at $\emptyset$: if $\{A_k \in \mathcal{A}\}_{n\in\mathbb{N}}$ satisfies $A_{k+1} \subseteq A_k\ \forall k$ and $\bigcap_k A_k = \emptyset$, then $\lim_{k\to\infty} \mu(A_k) = 0$.

(β) μ is continuous from below: if $B \in \mathcal{A}$ and $\{B_n \in \mathcal{A}\}_{n\in\mathcal{A}}$ satisfies $B_{n+1} \supseteq B_n\ \forall n$ with $\bigcup_n B_n = B$, then $\lim_{n\to\infty} \mu(B_n) = \mu(B)$.

(γ) μ is countably sub-additive, as in, if $A \in \mathcal{A}$ and $\{A_n \in \mathcal{A}\}_{n\in\mathbb{N}}$ with $A \subseteq \bigcup_n A_n$, then $\mu(A) \leq \sum_n \mu(A_n)$.

(iv) For a any set $A \subseteq X^{\mathbb{N}}$ define

$$\tilde{\mu}(A) = \inf\{\sum_{i=1}^{\infty} \mu(B_i) \mid B_i \in \mathcal{A}\ \forall i, \quad A \subseteq \bigcup_{i=1}^{\infty} B_i\}.$$

Then $\tilde{\mu}$ is an outer measure op $X^{\mathbb{N}}$.

(v) The restriction of $\tilde{\mu}$ to $\mathcal{A}$ equals μ, in other words $\tilde{\mu}|_{\mathcal{A}} = \mu$

Proof of Proposition 5.4
(i) Let $C \subset X^k$ and $D \subset X^l$ with $l > k$. We write $C' = C \times X^{l-k}$. Then $I_C \cup I_D = I_{C' \cup D}$, where $C' \cup D \subset X^l$. The case where $k = l$ speaks for itself. Hence, $\mathcal{A}$ is closed under finite unions. By construction it contains $\emptyset$ and $X^{\mathbb{N}}$, so only checking that $\mathcal{A}$ is closed under complements remains. Take subsets $C_1, ..., C_m$ in X^k, then $(\bigcup_{i=1}^m I_{C_i})^c = \bigcap_{i=1}^m (I_{C_i})^c = \bigcap_{i=1}^m I_{C_i^c}$. This intersection can be expressed as a set of sequences with specified starting terms, namely the points in $\bigcap_{i=1}^m C_i^c$. If the sets are subsets of different X^k, X^l, and so on, the previous procedure with C' shows our reasoning still holds.

(ii) Let C and D be disjoint subsets of X^k, then

$$\mu(I_C \cup I_D) = \sum_{C \cup D} p_{i_1} \cdots p_{i_k} = \sum_{C} p_{i_1} \cdots p_{i_k} + \sum_{D} p_{j_1} \cdots p_{j_k} = \mu(I_C) + \mu(I_D).$$

If $C \subset X^k$ and $D \subset X^l$ with $l > k$, consider $C \times X^{l-k} \subset X^l$. The equation above then still holds, since $\sum_{x \in X} p_x = 1$. Finite additivity follows inductively.

(iii) (α) We prove the claim by by contradiction, so we assume there is an $\epsilon > 0$ such that $\mu(A_k) > \epsilon$ for all k and prove that this implies $\bigcap_k A_k \neq \emptyset$. For $A \in \mathcal{A}$ and $(x_1, ..., x_n) \in X^n$, we define the section

$$A^{x_1,...,x_n} = \left\{ (z_{n+1}, z_{n+2}, ...) \in X^{\mathbb{N}} : (x_1, ..., x_n, z_{n+1}, ...) \in A \right\}.$$

Also, we define $B_k^1 = \{x \in X : \mu(A_k^x) > \epsilon/2\}$. Using that $\mu(X^{\mathbb{N}}) = 1$ and Fubini's theorem, we find the following estimation:

$$\begin{aligned} \epsilon < \mu(A_k) &= \sum_{x \in X} \mu(A_k^x) p_x \\ &= \sum_{x \in B_k^1} \mu(A_k^x) p_x + \sum_{x \in X \setminus B_k^1} \mu(A_k^x) p_x \\ &\leq \mu(B_k^1) + \frac{\epsilon}{2}. \end{aligned}$$

This means we have a decreasing sequence $\{B_k^1\}_{k \in \mathbb{N}}$ in the finite set X, with $\mu(B_k^1) > \epsilon/2$ for all k. Hence, we can fix y_1 in the non-empty intersection of all B_k^1. So far, we have found points $x \in B_k^1$ such that $\mu(A_k^x) > \epsilon/2$, where the A_k^x are sections of sets in the decreasing sequence $\{A_k\}$. Similarly, we can now define B_k^2 as the set of points x such that $\mu(A_k^{y_1,x}) > \epsilon/4$, where the $A_k^{y_1,x}$ are sections of sets in the decreasing sequence $\{A_k^{y_1}\}$. Here we have $\mu(A_k^{y_1}) > \epsilon/2$ for all k, instead of $\mu(A_k) > \epsilon$. In analogy with y_1, we can then pick a $y_2 \in \bigcap_k B_k^2$. Inductively repeating this process, we obtain a sequence $\{y_1, y_2, y_3, ...\}$.
We verify that $\{y_1, y_2, ...\} \in \bigcap_k A_k$. Consider A_j for some $j \in \mathbb{N}$, and write $A_j = C \times X^{\mathbb{N}}$ for some $C \subset X^l$. By construction, $\mu(A_j^{y_1,...,y_l}) > \epsilon 2^{-l}$, so $A_j^{y_1,...,y_l} \neq \emptyset$. This means that for some $\{z_{l+1}, z_{l+2}, ...\}$ we have $\{y_1, ..., y_l, z_{l+1}, ...\} \in A_j$. Note

that the elements of A_j have no restrictions from term $l+1$ and on. Hence, $\{y_1, y_2, y_3, ...\} \in A_j$ for this (arbitrary) j, which proves that $\bigcap_k A_k$ is not empty.

(β) This follows from (α) by defining $A_n = B \backslash B_n$, so $\{A_n\}$ is a decreasing sequence of sets with empty intersection. Hence, $\mu(B) - \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty}(\mu(B) - \mu(B_n)) = \lim_{n\to\infty} \mu(B\backslash B_n) = \lim_{n\to\infty} A_n = 0$.

(γ) A finitely additive measure is automatically finitely sub-additive. Combining this with (β) leads to

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} A_i) \leq \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

(iv) Clearly $\tilde{\mu}(\emptyset) = 0$. For $A_1, A_2 \in \mathcal{A}$ with $A_1 \subset A_2$, the inequality $\tilde{\mu}(A_1) \leq \tilde{\mu}(A_2)$ holds because any cover of A_1 is also a cover of A_2. Only verifying countable sub-additivity remains. Let $\{A_n \subset X^{\mathbb{N}}\}_n \in \mathbb{N}$ be any sequence of sets in $X^{\mathbb{N}}$ and take $\epsilon > 0$. By definition of $\tilde{\mu}$, there are sequences $\{B_{n,i}\}_{i\in\mathbb{N}}$ in $\mathcal{A}$ such that

$$\forall n \in \mathbb{N} : A_n \subset \bigcup_{i=1}^{\infty} B_{n,i} \text{ and } \sum_{i=1}^{\infty} \mu(B_{n,i}) \leq \tilde{\mu}(A_n) + \epsilon 2^{-n}.$$

Observe that $\{B_{n,i}\}_{n,i\in\mathbb{N}}$ is a countable cover of $\bigcup_{n=1}^{\infty} A_n$, so

$$\tilde{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n,i} \mu(B_{n,i}) \leq \sum_{n} (\tilde{\mu}(A_n) + \epsilon 2^{-n}) = \epsilon + \sum_{n=1}^{\infty} \tilde{\mu}(A_n).$$

Letting $\epsilon \to 0$ gives the required estimation.

(v) Let $A \in \mathcal{A}$, then the definition of $\tilde{\mu}$ directly implies $\tilde{\mu}(A) \leq \mu(A)$. For the reverse inequality, take $\{A_n \in \mathcal{A}\}_{n\in\mathbb{N}}$ such that $A \subset \bigcup_n A_n$. Then $A = A \cap (\bigcup_n A_n) = \bigcup_n (A \cap A_n)$, with $A \cap A_n \in \mathcal{A}$. Monotonicity and the countable sub-additivity proven in (iii)(γ) lead to

$$\mu(A) = \mu(\bigcup_n (A \cap A_n)) \leq \sum_n \mu(A \cap A_n) \leq \sum_n \mu(A_n).$$

This holds for any countable cover of A, so $\mu(A) \leq \tilde{\mu}(A)$. □

Last, we will need the following technical lemma.

Lemma 5.5. ***Intersection of balls and closures***
Take $a_1, a_2, r > 0$ and let $\{V_i\}$ be a family of disjoint open sets in $\mathbb{R}^d$. Suppose that every V_i contains a ball $B_{a_1 r}$ and is contained in a ball $B_{a_2 r}$, of radius $a_1 r$ and $a_2 r$, respectively. Then any ball B_r of radius r intersects at most $(1+2a_2)^d a_1^{-d}$ of the closures $\overline{V}_i$.

Proof of lemma 5.5
Based on the assumptions, we know that $|V_i| \leq 2a_2 r$ for all V_i. Hence, if some $\overline{V}_i$ intersects B_r, then this $\overline{V}_i$ is contained in the ball of radius $(1+2a_2)r$ concentric with B_r. Also, we know that $(a_1 r)^n \leq \mathrm{vol}(V_i)$ for all V_i.

Let n be the number of closures $\overline{V}_i$ intersecting B_r. Then n disjoint balls of radius $a_1 r$ must be contained in the aforementioned ball of radius $(1+2a_2)r$. Translating to volumes, we obtain $n(a_1 r)^d \leq (1+2a_2)^d r^d$, which directly implies the claim. □

5.2 Fractal dimension of self-similar sets

This subsection proves an elegant expression for the box-counting and Hausdorff dimension of many self-similar fractals. In its proof, we find a lower and upper bound for the fractal's outer Hausdorff measure. The lower bound is found with the outer measure estimate of Lemma 5.3. The construction of this outer measure will be based on the outer measure $\tilde{\mu}$ defined in Proposition 5.4, and the technical Lemma 5.5 will be used to show this measure can indeed be estimated as required. However, we first need to specify one condition that the iterated function systems of our self-similar fractals must satisfy.

Definition 5.6. Open set condition
*Let $\{S_1, ..., S_n\}$ be an IFS. Then this system satisfies the **open set condition** if there is a non-empty, bounded and open set V such that*

$$\bigcup_{i=1}^{n} S_i(V) \subset V$$

with this union disjoint.

Definition 5.7. Outer measure on self-similar fractals
Let $\{S_1, ..., S_n\}$ be an IFS consisting of similarities with corresponding ratios $\{r_1, ..., r_n\}$ and attractor F. Suppose that this system satisfies the open set condition and let s be the value such that

$$\sum_{i=1}^{n} r_i^s = 1.$$

We define an outer measure ν on F with $\nu(F) = 1$ by

$$\nu(A) = \tilde{\mu}(\{(i_1, i_2, ...) : x_{i_1, i_2, ...} \in A\})$$

for $A \subset F$ and $\tilde{\mu}$ constructed as in Proposition 5.4, taking $X = \{1, ..., n\}$ and $p : i \mapsto r_i^s$. Here $x_{i_1, i_2, ...}$ denotes $\phi(i_1, i_2, ...)$ as defined in Proposition 4.7.

Proposition 5.8. ***Outer measure estimate of*** ν
The outer measure ν, as defined in Definition 5.7, satisfies the conditions of Lemma 5.3, the outer measure estimate. In other words, there are values $q > 0$, $s > 0$ and $r > 0$ such that $\nu(U) \leq q|U|^s$ for all sets U with $|U| \leq r$.

Proof of Proposition 5.8

We prove the claim by first focusing on one specific case: let $B \subset \mathbb{R}^d$ be a ball of radius $r < 1$. Since ν is an outer measure *on* F, it is zero elsewhere, so $\nu(B) = \nu(B \cap F)$. We will cover $B \cap F$ with sets $V_{i_1,\ldots,i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(V)$, where $1 \leq i_j \leq n$, and then estimate the outer measure of this cover. The challenge is selecting appropriate corresponding sequences $(i_1, ..., i_k)$. Mainly, we demand that the conditions of Lemma 5.5 are satisfied for every $V_{i_1,\ldots,i_k}$. In short, we want to find finite sequences $(i_1, ..., i_k)$ that satisfy the following properties.

1. Every $V_{i_1,\ldots,i_k}$ contains a ball of radius $a_1 r$ and is contained in a ball of radius $a_2 r$, for fixed values a_1 and a_2.

2. All $V_{i_1,\ldots,i_k}$ are pairwise disjoint.

3. $F \cap B \subset \bigcup V_{i,\ldots,i_k}$

When we find an admissible cover, the final estimation, step 4, follows quickly because the measure of every $V_{i_1,\ldots,i_k}$ is easily estimated, and their number will be given by Lemma 5.5.

Step 1

Let $\mathcal{I} = \{(i_1, i_2, ...) : 1 \leq i_j \leq n\}$. We cut off every $(i_1, i_2, ...) \in \mathcal{I}$ after the first term k for which

$$\left(\min_{1\leq i\leq n} r_i\right) r \leq r_{i_1} \ldots r_{i_k} \leq r. \tag{5}$$

This unique term always exists: the first term r_{1_1} can never be smaller than $(\min_i r_i)r$ and the product of enough terms will eventually get smaller than r. We only need to verify that the product is ever between the two bounds. Suppose that for some sequence $(i_1, i_2, ...)$ there is a l such that $r_{i_1} \ldots r_{i_l} > r$ and $r_{i_1} \ldots r_{i_{l+1}} < (\min_i r_i)r$. Since $r_{l+1} \geq \min_i r_i$, this implies $r_{i_1} \ldots r_{i_l} < r$, which contradicts the assumptions. Thus, for every sequence in $\mathcal{I}$ there is a unique k^{th} term that puts the product between the two bounds for the first time. We let Q denote the set of finite sequences obtained by cutting of every infinite sequence as described above.

Now we pick a_1 and a_2 such that the open set V contains a ball of radius a_1 and is contained in a ball of radius a_2. For $(i_1, ..., i_k) \in Q$, we know $S_{i_1} \circ ... \circ S_{i_k}$ is a similarity of ratio $r_{i_1} \ldots r_{i_k}$. Therefore, the corresponding set $V_{i_1,\ldots,i_k}$ contains a ball of radius $r_{i_1} \ldots r_{i_k} a_1$ and is contained in a ball of radius $r_{i_1} \ldots r_{i_k} a_2$. Hence, (5) tells us that any $V_{i_1,\ldots,i_k}$, with $(i_1, .., i_k) \in Q$, contains a ball of radius $(\min_i r_i)a_1 r$ and is contained in a ball of radius $a_2 r$.

Step 2

Here it is important to note that when $(i_1, ..., i_k) \in Q$, no other sequences in Q start with the exact terms $i_1, ..., i_k$ in that order. Hence, two sequences in Q differ always at one term at least. The open set condition holds for V, so the image of (subsets of) V under different S_i is disjoint. Two sets $V_{i_1,\ldots,i_k}$ and $V_{j_1,\ldots,j_l}$, corresponding to different sequences in Q, are images under different S_i, as we just established. Hence, any pair of such sets is disjoint.

Step 3
Now we verify that

$$F = \bigcup_Q F_{i_1,\ldots,i_k} \subset \bigcup_Q \overline{V}_{i_1,\ldots,i_k}. \tag{6}$$

By Theorem 4.6, we know that $F = \bigcap_{k=0}^{\infty} S^k(\overline{V})$. This implies that $F \subset \overline{V}$, so $F_{i_1,\ldots,i_k} \subset \overline{V}_{i_1,\ldots,i_k}$ for all $(i_1, ..., i_k) \in Q$. This shows that $\bigcup_Q F_{i_1,\ldots,i_k} \subset \bigcup_Q \overline{V}_{i_1,\ldots,i_k}$.

For the first equality in (6), fix $j_1 \in \{1, ..., n\}$. If $(j_1) \in Q$, no other sequences in Q start with j_1. If $(j_1) \notin Q$, then either $(j_1, j_2) \in Q$ for all j_2, or the admissible sequences will branch off further, that is, some of such two-term sequences may be contained in Q, and some need at least a third term. Eventually, this branching must stop, since all sequences in Q are finite. At this point, there are n types of the longest obtained sequence(s), as they may end with all $j_l \in \{1, ..., n\}$. Using $F = \bigcup_{i=1}^n F_i$ repeatedly, starting at the end of every branch, it follows that the union $\bigcup_Q F_{j_1,\ldots,j_k}$ over sequences starting with j_i equals F_{j_1}. Taking the union over all possible starting terms $j_1 = 1, 2, ..., n$, we find $F = \bigcup_Q F_{i_1,\ldots,i_k}$.

Let Q_1 denote the sequences $(i_1, ..., i_k) \in Q$ such that $B \cap \overline{V}_{i_1,\ldots,i_k} \neq \emptyset$. We verified in Step 1 and Step 2 that the $V_{i_1,\ldots,i_k}$ satisfy the conditions of Lemma 5.5, so the lemma shows that there are

$$q = (1 + 2a_2)^d a_1^{-d} (\min_i r_i)^{-d}$$

sequences in Q_1. Moreover, the sets $\overline{V}_{i_1,\ldots,i_k}$ corresponding to sequences in Q_1 cover $F \cap B$.

Step 4
So far, we have found a cover $\bigcup_{Q_1} \overline{V}_{i_1,\ldots,i_k}$ of $B \cap F$, consisting of at most q disjoint closed balls. This leads to the estimation of $\nu(B)$.

$$\begin{aligned}
\nu(B) &= \nu(F \cap B) \\
&\leq \nu\left(\bigcup_{Q_1} \overline{V}_{i_1,\ldots,i_k}\right) \\
&= \tilde{\mu}\left(\left\{(i_1, i_2, ...) : x_{i_1,i_2,\ldots} \in \bigcup_{Q_1} \overline{V}_{i_1,\ldots,i_k}\right\}\right) \\
&\leq \tilde{\mu}\left(\{(i_1, i_2, ...) : (i_1, i_2, ...) \in I_{Q_1}\}\right)
\end{aligned}$$

The last estimation, where I_{Q_1} is defined as in Proposition 5.4, holds because the sequence corresponding to $\{x_{i_1,i_2,...}\} = \bigcap_k S_{i_1} \circ \cdots \circ S_{i_k}(V)$ can be curtailed to be an element of Q_1. In other words, the full sequence starts with this finite sequence of Q_1. We continue the estimation as follows:

$$\nu(B) \leq \sum_{Q_1} \tilde{\mu}(I_{\{(i_1,...,i_k)\}}) = \sum_{Q_1} (r_{i_1} \ldots r_{i_k})^s \leq \sum_{Q_1} r^s \leq qr^s.$$

So, we have found positive values q and s such that $\nu(B) \leq qr^s$ for all balls with a radius of some $r < 1$. Now let U be any set of radius $r_0 < r$. Then U is contained in a ball B_U of radius r_0, and we conclude

$$\nu(U) \leq \nu(B_U) \leq \nu(B) \leq qr^s.$$

Hence, ν satisfies the conditions of the Lemma 5.3, as claimed. □

After this preliminary work, we are finally ready to state and proof the main result.

Theorem 5.9. ***Fractal dimension of self-similar sets***
Let $\{S_1, ..., S_n\}$ be an IFS consisting of similarities with corresponding ratios $\{r_1, ..., r_n\}$ and attractor F. Suppose that this system satisfies the open set condition and let s be the unique value such that

$$\sum_{i=1}^{n} r_i^s = 1.$$

Then $\dim_H(F) = \dim_B(F) = s$.
Especially, if $r_1 = ... = r_n = r$ for some r, then $\dim_H(F) = \dim_B(F) = \frac{\log n}{\log(1/r)}$.

Proof of Theorem 5.9
To prove this theorem, we determine a finite upper and a non-zero lower bound for $m_s(F)$, which then implies that s is $\dim_H(F)$. The box-counting dimension of F is determined afterwards. First, we verify the uniqueness of s.

The function $s \mapsto \sum_i r_i^s$ is monotonic because it is the sum of monotonic functions $s \mapsto r_i^s$. For large values of s, the sum will be smaller than 1 because $0 < r_i < 1$, and for small (negative) s the sum will be larger then 1. The intermediate value theorem shows existence of s, which is unique due to the mentioned monotonicity.

To find an upper bound for $m_s(F)$, we cover F with the images of itself under repeated application of various S_i. As in the proof of Proposition 4.7, we write $\mathcal{I}_k = \{(i_1, ..., i_k) : 1 \leq i_j \leq n\}$. Also, again we use the notation $F_{i_1,...,i_k} = S_{i_1} \circ ... \circ S_{i_k}(F)$ for $(i_1, ..., i_k) \in \mathcal{I}_k$. By induction we prove that

$$F = \bigcup_{\mathcal{I}_k} F_{i_1,...,i_k} \tag{7}$$

for any $k \in \mathbb{N}$. We know $F = \bigcup_{i=1}^n S_i(F) = \bigcup_{\mathcal{I}_1} S_{i_1}(F)$, which proves the case $k = 1$. Assuming (7) for some $k \in \mathbb{N}$, we find that equality also holds for $k + 1$ as follows.

$$\begin{aligned}
F &= \bigcup_{i=1}^{n} S_i(F) \\
&= \bigcup_{\mathcal{I}_1} S_{i_1}\left(\bigcup_{\mathcal{I}_k} F_{j_1,\dots,j_k}\right) \\
&= \bigcup_{\mathcal{I}_{k+1}} F_{i_1,\dots,i_{k+1}}
\end{aligned}$$

This proves (7). For $\delta > 0$, we can choose k large enough so that

$$|F_{i_1,\dots,i_k}| \leq (\max_{1\leq i\leq n} r_i)^k |F| \leq \delta.$$

Hence, for any $\delta > 0$ we have the δ-cover $\{F_{i_1,\dots,i_k}\}_{\mathcal{I}_k}$ of F. Using this cover and the main property of s, we find the following estimation.

$$\begin{aligned}
\mathcal{H}^s_\delta(F) &= \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : F \subset \bigcup_{i=1}^{\infty} U_i \ , \ \forall i \ |U_i| < \delta\right\} \\
&\leq \sum_{\mathcal{I}_k} |F_{i_1,\dots,i_k}|^s \\
&= \sum_{\mathcal{I}_k} (r_{i_1} \dots r_{i_k})^s |F|^s \\
&= \left(\sum_{i_1=1}^{m} r_{i_1}^s\right) \dots \left(\sum_{i_k=1}^{m} r_{i_k}^s\right) |F|^s \\
&= |F|^s.
\end{aligned}$$

Letting δ tend to zero gives $m_s(F) \leq |F|^s < \infty$.

To determine a lower bound for $\mathcal{H}^s_\delta(F)$, there is not much work left to do. We use Lemma 5.3 applied to the outer measure ν of Definition 5.7. The previous proposition verified that this is possible. This means that we know there is a $q > 0$ such that

$$0 < \frac{\nu(F)}{q} \leq m_s(F).$$

This gives the required positive lower bound. Hence, we have found that $0 < m_s(F) < \infty$, so $\dim_H(F) = s$.

For box-counting dimension, we will cover F with $\bigcup_Q \overline{V}_{i_1,\dots,i_k}$ as in the proof of Proposition 5.8. However, now we directly estimate the number and size of the covering sets. So, we have chosen a positive $r < 1$ and defined Q as in what followed (5). Again using the branching pattern of sequences in Q and $\sum_i r_i^s = 1$, one can verify that $\sum_Q (r_{i_1} \dots r_{i_k})^s = 1$. Combining this with $(\min_{1\le i\le n} r_i)\, r \le r_{i_1} \dots r_{i_k}$, we find that Q contains at most $(\min_i r_i)^{-s} r^{-s}$ sequences. Also, for every $\{i_1, ..., i_k\} \in Q$ we know $r_{i_1} \dots r_{i_k} \le r$, so $|\overline{V}_{i_1,\dots,i_k}| = r_{i_1} \dots r_{i_k} |V| \le r|\overline{V}|$. This allows us to estimate $\overline{\dim}_B(F)$.

$$\begin{aligned}
\overline{\dim}_B F &= \varlimsup_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta} \\
&\le \varlimsup_{r\to 0} \frac{\log\left((\min_i r_i)^{-s} r^{-s}\right)}{-\log r|V|} \\
&= \varlimsup_{r\to 0} s\, \frac{(\min_i r_i) + \log r}{\log |V| + \log r} \\
&= s
\end{aligned}$$

By Proposition 3.7, we conclude that $s = \dim_H(F) \le \underline{\dim}_B(F) \le \overline{\dim}_B(F) \le s$, which proves the claim. $\square$

5.3 Examples of determining fractal dimension

Theorem 5.9 radically simplifies determining the dimensions of fractals that we have seen earlier. Instead of struggling to find lower and upper bounds, we can now even apply a visual approach, which can be formalized easily. For example, consider once again the Cantor set C.

Figure 5: The first seven stages of the Cantor set construction.

As mentioned before, this fractal is the attractor of the similarities $S_1, S_2 : [0,1] \to [0,1]$ with $S_1 : x \mapsto \frac{1}{3}x$ and $S_2 : x \mapsto \frac{1}{3}x + \frac{2}{3}$. The open set condition holds, which follows by considering the open interval $(0,1)$. All contraction ratios are equal, so Theorem 5.9 directly gives $\dim_H(C) = \dim_B(C) = \log 2/\log 3$.

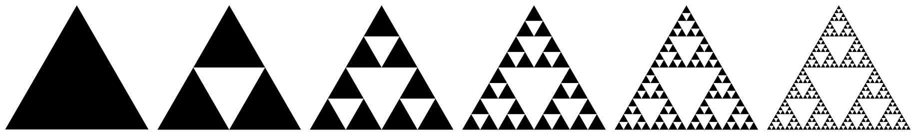

Figure 6: The first six stages in the construction of the Sierpínski triangle.

In case of the Sierpínski triangle S, we see the fractal is the attractor of three similarities with ratio $1/2$, even without knowing the explicit formulas of the IFS. By considering the interior of the first-stage triangle S_0, we verify that the open set condition is satisfied. Again, all ratios are equal, so we conclude that $\dim_H(S) = \dim_B(S) = \log 3/\log 2$.

Last, we take another look at the von Koch curve K. We saw in Section 4.1 that this is the attractor of four similarities with contraction ratio $\frac{1}{3}$. Therefore, $\dim_H(K) = \dim_B(K) = \log(4)/\log(3)$.

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