— Bachelor Thesis —

The Suslin Property

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Contents

1 Introduction

Research on the Suslin property started in 1920. In that year, one of Russian mathematician Mikhail Suslin's questions got published:

> "*Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel que tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?*" — [11], *Problème 3*

Is a (linearly) ordered set, without jumps or gaps and such that any set of its (non-empty and non-singleton) intervals that do not overlap each other is at most countable, necessarily an (ordinary) linear continuum? Unfortunately, Suslin died one year prior to this publication at the age of 24 during the Russian Civil War, so he had no chance to do more research on the answer of this question.

Over the years, the question was redefined to be described in terms and concepts from topology, and eventually the statement for topological spaces 'every family of mutually disjoint non-empty open sets is countable' became known as the Suslin property.

For this thesis, I have researched one of the aspects of the Suslin property, namely the question whether the product of two Suslin spaces is again a Suslin space. Chapters 2 and 5 are mainly about this.

It turns out that this question is undecidable in Zermelo-Fraenkel set theory. It needs an additional axiom, Martin's Axiom, before the question can be answered affirmatively. This axiom is introduced in Chapter 3. There are some interesting statements that are equivalent to Martin's Axiom, at the end of Chapter 3 and in Chapter 4 we will state three other notions and prove their equivalence. Most of chapter 3 and 4 are rewritten and more detailed versions of theorems and proofs in Sections II.2 and II.3 of [6], one of the primary resources for this thesis.

Finally, in Chapter 6, we give a set-up for a construction of two Suslin spaces with a non-Suslin product. The full proof of this is very elaborate and exceeds the level of a bachelor thesis, but the details can be found in [5].

Suslin's original question also turned out to be undecidable within ZFC. While this is an interesting subject on its own, I won't elaborate on that particular question any more in this thesis.

Preliminaries on cardinal numbers

This section is partly meant to give some preliminaries on cardinal numbers, and partly to explain certain notions centered around cardinal numbers that appear in this thesis and might be confusing. It is not very formal or mathematically complete, as there are enough resources for this, for example [1], Chapter 5 or [8], Section 1.1. Some knowledge of ordinal numbers is assumed, see for example Section 4.2 of [1].

- We say that sets A and B have the same cardinality if there exists a bijection $f : A \longrightarrow B$. 'Having the same cardinality' is an equivalence relation so we can we can speak of the smallest ordinal number κ such that A and κ have the same cardinality. We define this κ as *the* cardinality of A and denote this by $|A|$. An ordinal number κ is called a cardinal (number) if there is a set A with $|A| = \kappa$.

- We define $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}_{>0} := \{1, 2, 3, \dots\}$. These sets have the same cardinality (by the bijection $f(n) = n + 1$), which we denote by the cardinal ω. The set of real numbers has cardinality 2^ω and is strictly larger than ω by Cantor's diagonal argument. In this text, we mostly use ω and 2^ω to denote these two cardinalities. We may occasionally use $|\mathbb{N}|$ and $|\mathbb{R}|$ so that we can use $\mathbb{N}$ and $\mathbb{R}$ if we need concrete sets.

- We define ω_1 as the set of all countable ordinal numbers. This set itself is an ordinal number. If it were countable, then $\omega_1 + 1$ would also be countable and thus be an element of ω_1, which cannot be true since $\omega_1 < \omega_1 + 1$. Hence ω_1 must be uncountable. It is in fact the first uncountable ordinal number, and its cardinality is the first uncountable cardinal number. We use ω_1 to denote its cardinality, although some authors use $\aleph_1$.

- The continuum hypothesis (CH) is the statement 'there is no set whose cardinality is strictly between that of the natural numbers and that of the real numbers' and is independent of the ZFC axioms. It is equivalent to the statement that $\omega_1 = 2^\omega$, so if we assume the converse, then ω_1 would be one such cardinality that is strictly between that of the natural numbers and that of the real numbers.

- Often, we need to index a set that has some fixed cardinality, say κ. Instead of writing $\{x_\alpha\}_{\alpha \in I}$ for some index set I with $|I| = \kappa$, we simply write $\{x_\alpha\}_{\alpha < \kappa}$. This also is common practice in other texts. It makes sense since for every cardinal number κ there are exactly κ many ordinal numbers α with $\alpha < \kappa$. However, we do never use the fact that they are ordinal numbers so they can just be seen as index symbols. The notation $A \subseteq \kappa$ is used to denote a subset of all ordinal numbers less than κ. Sometimes a concrete index set is still used, if we need some structure on it or just for clarity.

2 The Suslin property

2.1 Definition and motivation

We begin with defining what the Suslin propery exactly is, and where it originated.

Definition 2.1. A topological space X has the Suslin property if every family of pairwise disjoint non-empty open subsets of X is countable.

Topological spaces with the Suslin property are also called Suslin spaces and were introduced in order to characterize the order type of the real numbers. It was well-known that any total order $(P, <)$ that satisfies

(1) P has no first or last element,

(2) P is connected in the order topology, and

(3) P is separable in the order topology

is isomorphic to $(\mathbb{R}, <)$.

Suslin[1] asked, as noted in the Introduction but now more modernly stated, whether it is possible to replace property (3) by the property

(3′) P has the Suslin property in the order topology.

In Proposition 2.3, we see that separability implies the Suslin property, but not vice versa, so it is not immediately clear whether the property could be exchanged. This leads to the following definition.

Definition 2.2. A Suslin line is a total order $(P, <)$ such that P, equipped with the order topology, is Suslin but not separable. The Suslin Hypothesis (SH) is the statement: 'There are no Suslin lines.'

It is clear that property (3) and (3′) are equivalent under assumption of SH. If P is a Suslin line, then there exists a Suslin line P' that satisfies (1) and (2). Hence SH is equivalent to the statement that properties (1), (2) and (3′) characterize the order $(\mathbb{R}, <)$.

Separability and the Suslin property must be somehow related. And they are, as we see in the following proposition.

Proposition 2.3. *Every separable space has the Suslin property, but there are Suslin spaces that are not separable.*

Proof. Assume X is a separable space. Then there is a countable dense $Y \subseteq X$. Since Y is dense, any non-empty open $U \subseteq X$ satisfies $Y \cap U \neq \emptyset$. Let $\mathcal{F}$ be a family of pairwise disjoint non-empty open subsets of X. For every $y \in Y$, there is at most one $U \in \mathcal{F}$ with $y \in U$ by pairwise disjointness. Since Y is countable, so is $\mathcal{F}$ and hence X is Suslin.

[1] Михайл Я́ковлевич Су́слин (1894–1919), transliterated in English to Mikhail Yakovlevich Suslin, was a Russian descriptive set theorist and topologist. His publications were in French, so the transliteration 'Souslin' can also be found in texts.

For the converse, we show that the cocountable topology on an uncountable set X is Suslin but not separable. Recall that the cocountable topology $\tau_{\text{cocnt}} := \{\emptyset\} \cup \{X \setminus Y \mid Y \subseteq X \text{ countable}\}$. Since countable subsets of X are closed and not equal to X, they can't be dense. So (X, τ_{cocnt}) is not separable.

Let $U, V \in \tau_{\text{cocnt}}$ be non-empty. Then there are countable $Y, Z \subseteq X$ such that $U = X \setminus Y$ and $V = X \setminus Z$. We have $U \cap V = (X \setminus Y) \cap (X \setminus Z) = X \setminus (Y \cup Z) \neq \emptyset$, since $Y \cup Z$ is countable and X is not. Thus every pair of non-empty open subsets of X is not disjoint, so every family of pairwise disjoint non-empty open subsets of X consists of at most one element. These are certainly countable families, so (X, τ_{cocnt}) is Suslin. □

One of the goals of research in pure mathematics is to generalize known concepts in order to view them from a higher level and make high-level connections. Suslin's question is not much different, since it attempts to describe the real number line in terms of the Suslin property, which is a weaker property than separability as we have now seen.

2.2 Products of separable spaces

As is usual with new properties on topological spaces, we want to know how derived spaces behave with respect to that property. Before we dive into products of Suslin spaces, we look at what we know about separability and products of separable spaces.

Theorem 2.4. *If $|I| \leq 2^\omega$ and X_i is separable for every $i \in I$, then $\prod_{i \in I} X_i$ is separable.*

Proof. Let τ_i be the topology on X_i for all $i \in I$ and define $X_\Pi := \prod_{i \in I} X_i$ with the product topology τ_Π. There is a simple proof for finite I and a more complicated one for infinite I, so we treat both cases separately.

Assume I is finite, say $|I| = n$. For all $i \in I$, X_i is separable, so there is a countable dense subset $A_i \subseteq X_i$. Then $\prod_{i=1}^n A_i$ is dense in X_Π and $|\prod_{i=1}^n A_i| \leq |\mathbb{N}|^n = |\mathbb{N}|$. So X is separable.

Now assume I is not finite. Since $|I| \leq 2^\omega = |\mathbb{R}|$, we can say without loss of generality that we have $\mathbb{Q} \subseteq I \subseteq \mathbb{R}$, so I is dense in $\mathbb{R}$. Define the set of tuples T by

$$T := \{(r_1, r_2, \dots, r_{n-1}, k_1, k_2, \dots, k_n) \mid n \geq 2, r_m \in \mathbb{Q} \text{ with } r_m < r_{m+1} \text{ and } k_m \in \mathbb{N}_{>0}\}.$$

Now $|T| \leq \sum_{n=2}^\infty |\mathbb{Q}^{n-1} \times \mathbb{N}^n| \leq \sum_{n=2}^\infty |\mathbb{N}| \leq |\mathbb{N}|$, so T is countable. For each $i \in I$ let $A_i = \{x_k^i\}_{k=1}^\infty$ be a countable dense subset of X_i and for each $t \in T$ define $x^t \in X_\Pi$ by

$$x_i^t := \begin{cases} x_{k_1}^i & \text{if } i \leq r_1 \\ x_{k_m}^i & \text{if } r_{m-1} < i \leq r_m \\ x_{k_n}^i & \text{if } r_{n-1} < i \end{cases}$$

for each $i \in I$.

Now $A := \{x^t \mid t \in T\}$ is a countable subset of X_Π. We show that it is dense. Let $\mathcal{B}_\Pi$ be the product topology base for X_Π and take $B \in \mathcal{B}_\Pi$. Then there is a fixed finite $F \subseteq I$ and $U_i \in \tau_i$ such that $B = \prod_{i \in I} U_i$ with $U_i \neq X_i \iff i \in F$.

Take $\{i_1, i_2, \ldots, i_n\} = F$ such that $i_1 < i_2 < \ldots < i_n$. Then we can take $r_j \in \mathbb{Q}$ for $1 \leq j \leq n-1$ such that $i_1 < r_1 < i_2 < r_2 < \ldots < i_{n-1} < r_{n-1} < i_n$. For all $1 \leq m \leq n$, the set $A_{i_m} \subseteq X_{i_m}$ is dense, so there is a positive integer k_m such that $x_{k_m}^{i_m} \in U_{i_m}$. For $t = (r_1, r_2, \ldots, r_{n-1}, k_1, k_2, \ldots, k_n)$, we have $x^t \in B$. Every $U \in \tau_\Pi$ contains a base element B, so $U \cap A \supseteq B \cap A \neq \emptyset$. This makes A a countable dense subset of X_Π, so X_Π is separable. □

Theorem 2.4 is a special case of the Hewitt-Marczewski-Pondiczery Theorem (Theorem 2.3.15 in [4]). The given proof is from [10], Theorem 1.

2.3 Products of Suslin spaces

Since separability and the Suslin property are related, Theorem 2.4 makes us believe that something similar can be said about how products of Suslin spaces behave. To state and prove this clearly, we need to introduce a lemma.

Lemma 2.5. *Let A be an uncountable index set and F_α be finite for every $\alpha \in A$. Then there is an uncountable $B \subseteq A$ and a finite set F such that $F_\alpha \cap F_\beta = F$ for every pair of distinct $\alpha, \beta \in B$.*

Lemma 2.5 is often called the Δ-system lemma and is in fact useful for creating spaces that are *not* Suslin, as we will see when we use it for creating one in the proof of the next theorem, and again in Chapter 6. Its proof can be found in [9], Lemma A.3.29.

Theorem 2.6. *Assume $\{X_i\}_{i \in I}$ are spaces such that $\prod_{i \in J} X_i$ is Suslin for every finite $J \subseteq I$. Then $\prod_{i \in I} X_i$ is Suslin.*

Proof. Suppose $\{U_\alpha\}_{\alpha < \omega_1}$ is a family of pairwise disjoint non-empty open subsets of $\prod_{i \in I} X_i$. Without loss of generality, all U_α are elements of the product basis of $\prod_{i \in I} X_i$.

For each $\alpha < \omega_1$, there exists a finite $J_\alpha \subseteq I$ such that $\pi_i(U_\alpha) \neq X_i \Leftrightarrow i \in J_\alpha$. By Lemma 2.5, there is an uncountable $B \subseteq \omega_1$ and a finite $J \subseteq I$ such that $J_\alpha \cap J_\beta = J$ for every pair of distinct $\alpha, \beta \in B$.

Suppose J is empty. Take $\alpha \neq \beta$ arbitrarily from B. Then $J_\alpha \cap J_\beta = J = \emptyset$, so we can say that there exists an $x \in \prod_{i \in I} X_i$ with $x_i \in \pi_i(U_\alpha)$ if $i \in J_\alpha$ and $x_i \in \pi_i(U_\beta)$ if $i \in J_\beta$. Then $x \in U_\alpha \cap U_\beta$, but U_α and U_β are disjoint ↯.

J is non-empty. Let $\pi : \prod_{i \in I} X_i \longrightarrow \prod_{i \in J} X_i$ be the projection map. Then $\{U_\alpha\}_{\alpha \in B}$ is an uncountable family of pairwise disjoint non-empty open subsets of $\prod_{i \in J} X_i$, which is a Suslin space ↯.

So $\prod_{i \in I} X_i$ has the Suslin property. □

But what about these finite products themselves? Or products of just two Suslin spaces, do they always have the Suslin property? It turns out that this question cannot be answered using only the Zermelo-Fraenkel axioms! Even the addition of the Axiom of Choice does not help. We have to use additional axioms to determine the answers to these questions.

If we assume the Continuum Hypothesis (CH), then we can construct two Suslin spaces with a non-Suslin product. This construction is postponed to Chapter 6. Also, if P is a Suslin line (Definition 2.2), then P is a Suslin space but P^2 is not. This depends on the assumption of $\neg\mathsf{SH}$ and we will not prove it, a proof can be found in [6], Theorem II.4.3.

Unfortunately, assuming $\neg\mathsf{CH}$ does not give us anything useful. We have to add another axiom before we can say that every product of Suslin spaces is Suslin. We could introduce the statement 'the product of any two Suslin spaces is Suslin' as an axiom and finish this text, but it turns out that there is a more usual axiom in set theory that suffices. We will introduce it in the next Chapter.

3 Martin's Axiom

The goal of this chapter is to introduce Martin's Axiom. One of the intermediate steps towards the definition of Martin's Axiom is also an interesting statement, at the end of this chapter and in the next chapter we will give some equivalent statements of this.

3.1 Partial orders

Martin's Axiom is defined in terms of partial orders. A partial order is a non-empty set P with an order relation $\leq$ which is reflexive, transitive and antisymmetric.

Example 3.1. Let (X, τ) be a non-empty topological space and define $P := \tau \setminus \{\emptyset\}$ with $U \leq V$ iff $U \subseteq V$. Then P is non-empty and the relation $\leq$ is reflexive, transitive and antisymmetric, thus $(\tau \setminus \{\emptyset\}, \subseteq)$ is a partial order.

We define a list of properties on partial orders.

Definition 3.2. Here, $(P, \leq)$ is a partial order.

- A subset $D \subseteq P$ is called dense if for every $p \in P$ there is a $q \in D$ with $q \leq p$. This q is called an extension for p in D.
- $p, q \in P$ are called compatible if there is an $r \in P$ with $r \leq p$ and $r \leq q$.
- A subset $A \subseteq P$ is called an antichain if for every $p, q \in A$, p and q are not compatible.
- P has the countable chain condition or is c.c.c. if every antichain $A \subseteq P$ is countable.

Since there are multiple properties of partial orders that are called *antichains*, the one in Definition 3.2 is also called strong antichain. We, however, do not use any of these other properties and call the defined one just an antichain.

Example 3.3. Again assume (X, τ) is a topological space and take $(\tau \setminus \{\emptyset\}, \subseteq)$ as partial order. U and V are compatible if and only if there is a non-empty $W \in \tau$ with $W \subseteq U \cap V$. So U and V are not compatible if and only if their intersection is empty. This order thus has the countable chain condition if and only if every family of pairwise disjoint open subsets of X is countable. This is exactly the Suslin property! Because of this, some authors do not differentiate between the two names and call Suslin spaces c.c.c. spaces. We will keep using the terms c.c.c. for partial orders and Suslin for topological spaces in order to minimize confusion

Definition 3.4. Let $(P, \leq)$ be a partial order. A subset $F \subseteq P$ is called a filter if

1. For all $p, q \in F$ there is an $r \in F$ with $r \leq p$ and $r \leq q$, and
2. If $p \in P, q \in F$ and $p \geq q$, then $p \in F$.

The first property makes sure that every pair of elements of a filter are compatible. The second property could make one intuitively think that filters should be 'large' sets in some sense and that the entire partial order itself is a filter, but that is misleading. Partial orders in general are not such that every pair of elements is compatible, so the enire partial order itself is in general not a filter.

Example 3.5. In topology, there are objects that are called filters too. If X is a topological space, a topological filter is defined as a non-empty family $\mathcal{F}$ of non-empty subsets of X such that the following holds:

1. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and
2. If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.

If we take the ordering $\subseteq$ again, this time not on $\tau \setminus \{\emptyset\}$ but on $\mathcal{P}(X) \setminus \{\emptyset\}$, then a topological filter coincides with the definition of a filter on a partial order.

A concrete example of a topological filter is the neighbourhood set $\mathcal{N}_x$ of a point $x \in X$.

3.2 The axiom

Now we almost know enough about partial orders to give the definition of Martin's Axiom. For this, we first need the following axiom scheme. Let κ be a cardinal number.

Definition 3.6. Martin's Axiom for κ (MA_κ) is the statement: 'Whenever $(P, \leq)$ is a non-empty c.c.c. partial order and $\mathcal{D}$ a family of dense subsets of P with $|\mathcal{D}| \leq \kappa$, then there is a filter $F \subseteq P$ such that $F \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.'

One should pay close attention to what this statement is actually saying, since we will use it a lot in proving other statements or as a conclusion from other statements. Just like one could say that the Axiom of Choice *gives* (the existence of) a choice function for every set of sets or that Zorn's Lemma *gives* a maximal element for some partial orders, Martin's Axiom for κ *gives* a filter with certain properties for some partial orders and families of dense subsets.

It is clear that if $\kappa < \lambda$, then MA_λ implies MA_κ. To understand the axiom even better, we want to delimit the cardinalities κ for which MA_κ is useful as an addition to ZFC and for which it is not.

Proposition 3.7. *MA_ω is true.*

Proof. Let $(P, \leq)$ be a non-empty partial order and $\{D_n\}_{n \in \mathbb{N}}$ a family of dense subsets. Choose an arbitrary $p_0 \in P$ and choose $p_{n+1} \in D_n$ such that $p_{n+1} \leq p_n$ for each $n \in \mathbb{N}$. This is possible, since D_n is dense, and it gives $p_0 \geq p_1 \geq p_2 \geq \cdots$.

Define the following set:

$$G := \{q \in P \mid \exists n \text{ such that } q \geq p_n\}.$$

Then G is a filter (the filter generated by $\{p_n\}_{n \in \mathbb{N}}$) and $G \cap D_n \neq \emptyset$ for all $n \in \mathbb{N}$. Since $\mathbb{N}$ has cardinality ω, MA_ω is true. □

The above proposition is known as the Rasiowa-Sikorski lemma. Of course, MA_n is also true for finite n. Just end the p_i sequence at the n-th entry or conclude it from the remark above Proposition 3.7.

Proposition 3.8. *CH implies MA_κ for $\kappa < 2^\omega$.*

Proof. If CH holds, then $2^\omega = \omega_1$, so for every $\kappa < 2^\omega$ we have $\kappa \leq \omega$. From Proposition 3.7 we know that MA_κ is true for $\kappa \leq \omega$, so MA_κ is true for all $\kappa < 2^\omega$ $\square$

Proposition 3.9. *MA_{2^ω} is false.*

The actual proof of Proposition 3.9 is not important for this text, it can be found in [6], Section II.2, Example 5. The important bit is that MA_κ is not interesting for $\kappa = 2^\omega$ or larger cardinalities, since these are false statements within ZFC.

If we assume CH, we saw that MA_κ is either provably true or provably false, for each κ. In this case, it is not correct to refer to MA_κ as an axiom. MA_κ becomes interesting when we reject CH. In ZFC+$\neg$CH, it is not provable whether MA_κ for $\omega < \kappa < 2^\omega$ is true or not. The assumption of MA_κ for these values of κ is so useful that it deserves its own definition.

Definition 3.10. Martin's Axiom (MA) is the statement: 'For all $\kappa < 2^\omega$, MA_κ holds.'

Martin's Axiom was introduced by Martin[2] and Solovay[3] in [7] in order to generalize the Suslin Hypothesis. It leads to the Rasiowa-Sikorski lemma (Lemma 3.7), which is used for *forcing*, a technique to prove consistency of axioms.

MA and MA_κ are the main topics of this text. In this chapter and the next we will define three more statements that are equivalent to MA_κ and in Chapter 5 we will state the main result about the product of Suslin spaces, using MA.

We have seen that CH implies MA and it is known that CH is consistent with ZFC. This means that MA also is consistent with ZFC. It turns out that MA+$\neg$CH too is consistent with ZFC. A proof of this is very advanced and uses techniques like the mentioned forcing, far beyond the scope of this text. It can be found in [6], Section VIII.6. We assume that MA+$\neg$CH is consistent with ZFC in order to use it in the remainder of this text.

3.3 Smaller partial orders

In this section, we introduce a statement that seems weaker than Martin's Axiom for κ, namely MA_κ restricted to partial orders of cardinality less or equal to κ. It turns out though, that these two statements are equivalent. In the next chapter, we use this to introduce two more statements that are equivalent to Martin's Axiom for κ. But first, we have to give a few new definitions.

Definition 3.11. Let A be a set.

- A function f is n-ary if it is of the form $f : A^n \longrightarrow A$ for $n \in \mathbb{N}_{>0}$.
- A function is finitary if it is n-ary for some $n \in \mathbb{N}_{>0}$.
- $B \subseteq A$ is closed under f if f is an n-ary function and $f(B^n) \subseteq B$.

[2]Donald A. Martin (1940–), American set theorist and philosopher of mathematics.
[3]Robert M. Solovay (1938–), American set theorist.

- If $\mathcal{S}$ is a family of finitary functions and B a subset of A, then the closure of B under $\mathcal{S}$ is the smallest set C such that $B \subseteq C$ and C is closed under all functions in $\mathcal{S}$.

Note that the closure in de last definition always exists, since A is closed under $\mathcal{S}$ and since we remain closed under $\mathcal{S}$ if we take the intersection of all subsets of A that are closed under $\mathcal{S}$ and contain B.

Some texts define 0-ary functions (which are just constants) separately and include them in the set of finitary functions. We however do not need 0-ary functions and to make the definition a bit simpler, they are not included in our definition.

Lemma 3.12. *Let $\kappa \geq \omega$ be a cardinality, $\{X_\alpha\}_{\alpha<\kappa}$ be a family of sets with $|X_\alpha| \leq \kappa$ for each $\alpha < \kappa$. Then $|\bigcup_{\alpha<\kappa} X_\alpha| \leq \kappa$.*

Proof. For each $\alpha < \kappa$, choose an injection $f_\alpha : X_\alpha \longrightarrow \kappa$. This can be done using the Well-Ordering Theorem, which is equivalent to AC. Now define $f : \bigcup_{\alpha<\kappa} X_\alpha \longrightarrow \kappa \times \kappa$ by choosing $\alpha < \kappa$ such that $x \in X_\alpha$ and setting $f(x) := (\alpha, f_\alpha(x))$.

Take $x, y \in \bigcup_{\alpha<\kappa} X_\alpha$ such that $f(x) = f(y)$. Then there is an $\alpha < \kappa$ such that $x, y \in X_\alpha$ and then $f_\alpha(x) = f_\alpha(y)$. We have chosen f_α to be injective, so $x = y$ and f is injective. By this injectivity, we have $|\bigcup_{\alpha<\kappa} X_\alpha| \leq |\kappa \times \kappa| = |\kappa| = \kappa$. □

Lemma 3.13. *Let $\kappa \geq \omega$ be a cardinality, A a set, $B \subseteq A$ with $|B| \leq \kappa$, $\mathcal{S}$ a family of at most κ finitary functions on A. Then the closure of B under $\mathcal{S}$ has cardinality at most κ.*

Proof. Take $D \subseteq A$ and $f \in \mathcal{S}$, then f is n-ary for some $n \in \mathbb{N}_{>0}$. Note that $|D| \leq \kappa$ implies $|D^n| \leq \kappa$, which implies $|f(D^n)| \leq \kappa$.

Define $C_0 := B$ and

$$C_{m+1} := C_m \cup \bigcup_{f \in S} f(C_m)$$

for $m \in \mathbb{N}$. Since $|C_0| = |B| \leq \kappa$ and by applying induction to m and Lemma 3.12, we have $|C_m| \leq \kappa$ for all $m \in \mathbb{N}$. Now define $C := \bigcup_{m\in\mathbb{N}} C_m$. Then we have

$$C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C.$$

Take $f \in \mathcal{S}$. Then f is n-ary for some $n \in \mathbb{N}$. Take $(x_1, \ldots, x_n) \in C^n$. Then there is an $m \in \mathbb{N}$ such that $x_i \in C_m$ for $1 \leq i \leq n$. Now $f(x_1, \ldots x_n) \in C_{m+1} \subseteq C$, so C is closed under f and since f was arbitrary, C is closed under all functions in $\mathcal{S}$.

Since no element can be removed from C, it is the closure of B under $\mathcal{S}$ and by Lemma 3.12, we have $|C| \leq \kappa$. □

Theorem 3.14. *MA_κ is equivalent to MA_κ restricted to partial orders of cardinality at most κ.*

Proof. If we assume MA_κ for all partial orders, then it is also true for the subclass of partial orders with cardinality at most κ.

Assume MA_κ restricted to partial orders of cardinality at most κ and let $(Q, \leq)$ be a c.c.c. partial order of arbitrary cardinality and $\mathcal{D}$ a family of at most κ dense subsets of Q. We want to find a filter that has non-empty intersection with all sets in $\mathcal{D}$, by applying the restricted form of MA_κ to a suitable subset of Q.

For each $D \in \mathcal{D}$, let $f_D : Q \longrightarrow Q$ be a function such that $f_D(p) \in D$ and $f_D(p) \leq p$ for each $p \in Q$. This function exists because D is dense. Define $g : Q \times Q \longrightarrow Q$ such that for every compatible $p, q \in Q$ we have $g(p,q) \leq p$ and $g(p,q) \leq q$. By Lemma 3.13 there is a $P \subseteq Q$ with $|P| \leq \kappa$ which is closed under g and every f_D.

Now we have a partial order P with cardinality at most κ. Since P is closed under every f_D, every $p \in P$ has an extension $f_D(p)$ in $D \cap P$, so $D \cap P$ is dense for every $D \in \mathcal{D}$. Since P is closed under g, any two elements are compatible in P if and only if they are compatible in Q. This last result implies that P is c.c.c., so we can apply the restricted form of MA_κ, giving us a filter $G \subseteq P$ with $G \cap D \cap P \neq \emptyset$ for each $D \in \mathcal{D}$. Define $H := \{q \in Q \mid \exists p \in G \text{ with } p \leq q\}$, the filter in Q generated by G. Then $\emptyset \neq G \cap D \cap P \subseteq G \cap D \subseteq H \cap D$ for each $D \in \mathcal{D}$. $\square$

We could have stopped immediately after the second paragraph, since the downwards Löwenheim-Skolem theorem ([8], Theorem 2.8.3) tells us that such a subset $P \subseteq Q$ must exist. However, the given proof of Theorem 3.14 is a more direct construction of P.

4 More equivalents of MA_κ

We have seen that MA_κ is equivalent to MA_κ restricted to partial orders with cardinality at most κ. The goal of this chapter is to give two more statements that are equivalent to MA_κ. One of the statements has to do with Boolean algebras, which we define in the first section, and the other is a statement about topological spaces. The statements are combined into one theorem at the end of this chapter, Theorem 4.12. This is also Theorem II.3.4 from [6].

4.1 Boolean algebras

Definition 4.1. A Boolean algebra is a structure $(\mathcal{B}, \vee, \wedge, \overline{\cdot}, 0, 1)$ such that $\mathcal{B}$ is a set, $\vee$ and $\wedge$ are binary operations on $\mathcal{B}$, $b \mapsto \overline{b}$ is a unary operation on $\mathcal{B}$ and $0 \neq 1 \in \mathcal{B}$ are constants. The structure must also satisfy the following properties for all $b, c, d \in \mathcal{B}$:

1. $b \vee (c \vee d) = (b \vee c) \vee d$ and $b \wedge (c \wedge d) = (b \wedge c) \wedge d$ (Associativity).
2. $b \vee c = c \vee b$ and $b \wedge c = c \wedge b$ (Commutativity).
3. $b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$ and $b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d)$ (Distributivity).
4. $b \vee b = b = b \wedge b$ (Idempotency).
5. $b \vee (b \wedge c) = b = b \wedge (b \vee c)$ (Absorption).
6. $\overline{\overline{b}} = b$ (Double negation).
7. $\overline{b \vee c} = \overline{b} \wedge \overline{c}$ and $\overline{b \wedge c} = \overline{b} \vee \overline{c}$ (De Morgan's laws).
8. $b \vee \overline{b} = 1$ and $b \wedge \overline{b} = 0$.
9. $b \vee 0 = b = b \wedge 1$, $b \vee 1 = 1$ and $b \wedge 0 = 0$.

Note that some of these properties are redundant. Since we will work with only one Boolean algebra at a time, $(\mathcal{B}, \vee, \wedge, \overline{\cdot}, 0, 1)$ is often shortened to just $\mathcal{B}$.

Example 4.2.

1. Fix $n \in N$ positive and define $\mathcal{B}_n := \{\mathsf{True}, \mathsf{False}\}^n$, the set of all n-tuples of truth values (bits). If we define $\vee$ as bitwise *or*, $\wedge$ as *and*, $\overline{\cdot}$ as negation, $0 := \{\mathsf{False}, \mathsf{False}, \dots, \mathsf{False}\}$ and $1 := \{\mathsf{True}, \mathsf{True}, \dots, \mathsf{True}\}$, then we get the Boolean algebra of truth values. It is heavily used in computer science, since it can be represented by the presence of absence of a voltage in a memory cell in order to store data and to perform arithmetic.
2. Fix a set X, then $(\mathcal{P}(X), \cup, \cap, X \setminus \cdot, \emptyset, X)$ forms a Boolean algebra. If X is finite, then the Boolean algebra is isomorphic to $\mathcal{B}_{|X|}$ of the previous example.
3. Recall that a subset U of a topological space X is <u>regular open</u> if and only if $U = \mathrm{cl}(U)^\circ$. Also recall that intersections of regular open sets are regular open and that $\mathrm{cl}(Y)^\circ$ is regular open for each $Y \subseteq X$. So for every pair of regular open sets $U, V \subseteq X$, we have that $U \cap V$, $\mathrm{cl}(U \cup V)^\circ$ and $\mathrm{cl}(X \setminus U)^\circ$ are regular open. Now define $U \wedge V := U \cap V$, $U \vee V := \mathrm{cl}(U \cup V)^\circ$ and $\overline{U} := \mathrm{cl}(X \setminus U)^\circ$. Then $(\mathcal{B}, \vee, \wedge, \overline{\cdot}, \emptyset, X)$ is a Boolean algebra.

For $b, c \in \mathcal{B}$, we can define $b \leq c$ if and only if $b \wedge c = b$. Then $(\mathcal{B}, \leq)$ is a partial order with $b, c \leq b \vee c$ and $b, c \geq b \wedge c$. We implicitly include this partial order relation in every Boolean algebra. This also means that all definitions in Definition 3.2, as well as the definition of a filter and of Martin's Axiom (for κ), can be applied to Boolean algebras using this implicit partial order.

There are multiple ways to define Boolean algebras. Our definition of a Boolean algebra is based on Definition 11.1.66 in [9]. Instead of starting with a structure and then defining an implicit partial order on it like we do, one could start with a partial order and then add operators like $\wedge$ and $\vee$, defined in terms of $\leq$, and narrowing the class of partial orders down to partial orders where certain elements exist. This approach is used in for example [3].

In order to work with Boolean algebras, there are some more definitions that are useful.

Definition 4.3. Let $\mathcal{B}$ be a Boolean algebra.

- If $C \subseteq \mathcal{B}$, then $b \in \mathcal{B}$ is a supremum of C if it is the smallest element such that $b \geq c$ for each $c \in C$. A supremum might not exist, but if it exists, it is unique. Dually, $b' \in \mathcal{B}$ is the infimum of C if it is the largest element such that $b' \leq c$ for each $c \in C$.
- $\mathcal{B}$ is called complete if the supremum and infimum both exist for every $C \subseteq \mathcal{B}$.
- $G \subsetneq \mathcal{B}$ is called an ultrafilter if it is a filter and if the only filters that contain G are $\mathcal{B}$ and G itself.

For filters F on Boolean algebras, the first property of Definition 3.4 could be interchanged for the statement: 'For all $b, c \in F$, we have $b \wedge c \in F$'. One could check that these definitions of filters are equivalent, but sometimes this new definition is easier to use, like in the following proposition.

Proposition 4.4. *Let $\mathcal{B}$ be a Boolean algebra, $G \subseteq \mathcal{B}$ an ultrafilter and $b \in \mathcal{B}$ an element. Then either $b \in G$ or $\overline{b} \in G$.*

Proof. Suppose both $b \in G$ and $\overline{b} \in G$. Then we must have $b \wedge \overline{b} = 0 \in G$, and since $c \geq 0$ for all $c \in \mathcal{B}$, we have $G = \mathcal{B}$ ↯.

If there exists a $c \in G$ such that $b \wedge c = 0$, then we have $\overline{c} = \overline{c} \vee 0 = \overline{c} \vee (b \wedge c) = (\overline{c} \vee b) \wedge (\overline{c} \vee c) = (\overline{c} \vee b) \wedge 1 = \overline{c} \vee b$, which implies $c = \overline{\overline{c}} = \overline{\overline{c} \vee b} = c \wedge \overline{b}$. So $c \leq \overline{b}$, and since G is a filter, $\overline{b} \in G$.

If no such c exists, so $b \wedge c \neq 0$ for each $c \in G$, then there exists a filter $F \subsetneq \mathcal{B}$ with $b \in F$ and $G \subseteq F$. Since G is an ultrafilter, we must have $F = G$, so we have $b \in G$. □

4.2 MA_κ and Boolean algebras

Lemma 4.5. *Let $(P, \leq)$ be a partial order. Then there exists a complete Boolean algebra $\mathcal{B}$ and a map $f : P \longrightarrow \mathcal{B} \setminus \{0\}$ such that*

(1) $f(P)$ is dense in $\mathcal{B} \setminus \{0\}$,

(2) If $p, q \in P$ and $p \leq q$, then $f(p) \leq f(q)$, and

(3) $p, q \in P$ are not compatible if and only if $f(p) \wedge f(q) = 0$.

Proof. For each $p \in P$, consider $N_p := \{q \in P \mid q \leq p\}$ and then define $\mathcal{N} := \{N_p\}_{p \in P}$. For each $p \in P$ we have $p \in N_p$, so $\mathcal{N}$ covers P. Take $p, q \in P$ such that $N_p \cap N_q \neq \emptyset$, and take $s \in N_p \cap N_q$. Then $s \leq p$ and $s \leq q$. We have

$$s \in N_s = \{r \in P \mid r \leq s\} \subseteq \{r \in P \mid r \leq p\} \cap \{r \in P \mid r \leq q\} = N_p \cap N_q.$$

This implies that $\mathcal{N}$ forms a base for a topology, which we equip P with. In this topology, N_p is the smallest open set containing p.

Let $\mathcal{B}$ be the Boolean algebra of regular open sets of P, as defined in Example 4.2.3. The induced partial order is equal to the subset order. Define f by $f(p) = \text{cl}(N_p)^\circ$. We show that f satisfies (1), (2) and (3).

(1) Take $U \in \mathcal{B} \setminus \{0\}$. Then U is a non-empty regular open subset of P. Take $p \in U$, then $N_p \subseteq U$, so $f(p) = \text{cl}(N_p)^\circ \subseteq \text{cl}(U)^\circ = U$.

(2) If $p \leq q$, then $N_p \subseteq N_q$, so $\text{cl}(N_p)^\circ \subseteq \text{cl}(N_q)^\circ$, giving $f(p) \subseteq f(q)$.

(3) Suppose $p, q \in P$ are compatible. Take r such that $r \leq p$ and $r \leq q$. We know $f(r) \leq f(p)$ and $f(r) \leq f(q)$ by (2), so $f(p) \cap f(q) \supseteq f(r) \neq \emptyset$, giving $f(p) \wedge f(q) \neq 0$. For the converse, suppose that $p, q \in P$ are not compatible. Then there is no $r \in P$ with $r \leq p$ and $r \leq q$, so N_p and N_q are disjoint. N_q is open, so $\text{cl}(N_p) \cap N_q = \emptyset$, so $f(p) \cap N_q = \text{cl}(N_p)^\circ \cap N_q = \emptyset$. The same holds for N_q, as $f(p)$ is open: $f(p) \cap \text{cl}(N_q) = \emptyset \Rightarrow f(p) \cap f(q) = f(p) \cap \text{cl}(N_q)^\circ = \emptyset$. So $f(p) \wedge f(q) = 0$. □

Since each Boolean algebra is a partial order, and since we now have a way to map partial orders back to Boolean algebras, we could look at MA_κ where we restrict the partial orders we look at to Boolean algebras. As it turns out, this is also equivalent to the unrestricted form of MA_κ.

Theorem 4.6. *For any $\kappa \geq \omega$, MA_κ is equivalent to MA_κ restricted to complete Boolean algebras.*

Proof. If we assume MA_κ for all partial orders, then it is also true for the subclass of partial orders that are Boolean algebras.

Assume MA_κ restricted to complete Boolean algebras. Let $(P, \leq)$ be a c.c.c. partial order and $\mathcal{D}$ a family of at most κ dense subsets of P. Let $\mathcal{B}$ and f be as in Lemma 4.5.

<u>Suppose</u> that $\{U_\alpha\}_{\alpha<\omega_1}$ is an antichain in $\mathcal{B}$. By property (1) of Lemma 4.5, there is a $p_\alpha \in P$ for each $\alpha < \omega_1$ such that $f(p_\alpha) \subseteq U_\alpha$. By property (3), $\{p_\alpha\}_{\alpha<\omega_1}$ is an antichain in P. But P is c.c.c. ↯. So $\mathcal{B}$ is c.c.c..

Take $D \in \mathcal{D}$. For every non-empty $U \in \mathcal{B}$, there is a $p \in P$ with $f(p) \subseteq U$. We know D is dense, so there is a $q \in D$ with $q \leq p$, so $f(q) \subseteq U$ and $f(q) \in f(D)$. Thus $f(D)$ is dense in $\mathcal{B} \setminus \{\emptyset\}$ for every $D \in \mathcal{D}$.

Since we assume MA_κ restricted to Boolean algebras, there is a filter $G \subseteq \mathcal{B}$ such that $G \cap f(D) \neq \emptyset$ for every $D \in \mathcal{D}$. Define $H := f^{-1}(G)$, then $H \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

If H is a filter, we are done. H meets the second requirement in Definition 3.4: Take $p \in H, q \in P$ such that $q \geq p$, then $p \in f^{-1}(G)$, so $f(p) \in G$. We have $f(q) \supseteq f(p) \in G$ and since G is a filter in $\mathcal{B}$, $f(q) \in G$ so $q \in f^{-1}(G) = H$.

For the first requirement: Take $p, q \in H$. Then $f(p), f(q) \in G$. G is a filter, so there is a non-empty $U \in \mathcal{B}$ with $U \subseteq f(p)$ and $U \subseteq f(q)$, so $f(p) \cap f(q) \neq \emptyset$. By property (3) of Lemma 4.5, p and q are compatible in P, so there is an $r \in P$ with $r \leq p$ and $r \leq q$. Unfortunately, $f(r)$ does not have to be a member of G, so $r \in H$ does not have to hold and H does not need to be a filter.

To fix this, we will add dense sets to $\mathcal{D}$ in such a way that its cardinality does not change, but it does force G to be such that $f^{-1}(G)$ is a filter.

We use the shorthand notation $\perp$ to indicate incompatibility. Define

$$D_{pq} := \{r \in P \mid (r \leq p \text{ and } r \leq q) \text{ or } r \perp p \text{ or } r \perp q\}.$$

Take $r_0 \in P$. If r_0 has an extension that is not compatible with p or q, then that extension is an element of D_{pq}. If this is not the case, i.e. if all extensions of r_0 are compatible with p and with q, then r_0 itself is compatible with p, so there is an $r_1 \in P$ with $r_1 \leq r_0$ and $r_1 \leq p$. This r_1 is compatible with q, so there is an $r_2 \in P$ with $r_2 \leq r_1$ and $r_2 \leq q$. Now $r_2 \leq p$, so $r_2 \in D_{pq}$, which proves that D_{pq} is dense in P.

If we assume $|P| \leq \kappa$, then $|\{D_{pq} \mid p, q \in P\}| \leq |\kappa \times \kappa| = |\kappa| = \kappa$. Adding D_{pq} to $\mathcal{D}$ for all p and q does not change the cardinality of $\mathcal{D}$, so without loss of generality we may assume that they are elements of $\mathcal{D}$.

For $p, q \in H$, fix $r \in H \cap D_{pq}$. Since all elements of a filter are pairwise compatible, $r \perp p$ and $r \perp q$ do *not* hold, so $r \leq p$ and $r \leq q$ must hold, so the first requirement of definition 3.4 is met and H is a filter.

We can conclude that MA_κ restricted to Boolean algebras implies MA_κ restricted to partial orders with cardinality of at most κ, and by Theorem 3.14, this is equivalent to MA_κ. $\square$

4.3 MA_κ and compact Hausdorff spaces

The last equivalent statement is of a more topological nature. We need to use the notions of Hausdorffness and compactness from our topological toolbox.

Definition 4.7. Let X be a set, κ a cardinality and $\mathcal{A} := \{A_\alpha\}_{\alpha<\kappa}$ a family of subsets of X. Then $\mathcal{A}$ has the finite intersection property if every subfamily $\{A_\alpha\}_{\alpha\in J}$ with $J \subseteq \kappa$ finite has a non-empty intersection.

Example 4.8. Consider a set X and the partial order on the powerset of X, with $\subseteq$ as relation and a filter $\mathcal{F}$ on X. If $A, B \in \mathcal{F}$, then there is a $C \in \mathcal{F}$ with $C \subseteq A$ and $C \subseteq B$ and every superset of C is also an element of $\mathcal{F}$, all by definition of a filter. This means that $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$. By induction, the intersection of any finite subfamily of $\mathcal{F}$ has an intersection in $\mathcal{F}$. Since $\emptyset \notin \mathcal{F}$, this intersection is non-empty, so powerset filters have the finite intersection property.

Theorem 4.9. *Assume MA_κ. Let X be a compact Suslin Hausdorff space and let U_α be dense open subsets of X for each $\alpha < \kappa$. Then $\bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$.*

Proof. Take $P := \{U \subseteq X \mid U \text{ open and non-empty}\}$. Together with the relation $\subseteq$, this is the same partial order as in Example 3.1. Since X is a Suslin space, P is c.c.c.. Define a family $\mathcal{D} := \{D_\alpha\}_{\alpha<\kappa}$ of subsets of P by $D_\alpha := \{U \in P \mid \text{cl}(U) \subseteq U_\alpha\}$.

Take $\alpha < \kappa$, $U \in P$ and consider $U \cap U_\alpha$. Since $U_\alpha \subseteq X$ is dense, $U \cap U_\alpha$ is non-empty. X is compact Hausdorff, thus normal, so there is a non-empty open $V \subseteq X$ with $V \subseteq \mathrm{cl}(V) \subseteq U \cap U_\alpha$. This means $\mathrm{cl}(V) \subseteq U_\alpha$, so V is an extension of U in D_α, making D_α dense in P for every $\alpha < \kappa$.

We now have a non-empty c.c.c. partial order P and a family $\mathcal{D}$ of dense subsets of P with $|\mathcal{D}| < \kappa$. The assumption MA_κ gives us a filter $F \subseteq P$ such that $F \cap D_\alpha \neq \emptyset$ for all $\alpha < \kappa$.

Suppose $\bigcap_{U \in F} \mathrm{cl}(U)$ is empty. Then

$$X = X \setminus \emptyset = X \setminus \left(\bigcap_{U \in F} \mathrm{cl}(U) \right) = \bigcup_{U \in F} (X \setminus \mathrm{cl}(U)).$$

We see that $\{X \setminus \mathrm{cl}(U)\}_{U \in F}$ is an open cover of X and since X is compact, there exists a finite subcover that we can index by $G \subseteq F$, so

$$X = \bigcup_{U \in G} (X \setminus \mathrm{cl}(U)) = X \setminus \left(\bigcap_{U \in G} \mathrm{cl}(U) \right)$$

and we see $\bigcap_{U \in G} \mathrm{cl}(U) = \emptyset$. Since F is a filter, it has the finite intersection property, so

$$\emptyset \neq \bigcap_{U \in G} U \subseteq \bigcap_{U \in G} \mathrm{cl}(U) = \emptyset \quad \lightning.$$

So $\bigcap_{U \in F} \mathrm{cl}(U)$ is non-empty.

For every $\alpha < \kappa$, we know $F \cap D_\alpha \neq \emptyset$, so there is an open V_α with $V_\alpha \in F$ and with $\mathrm{cl}(V_\alpha) \subseteq U_\alpha$. We know $\bigcap_{U \in F} \mathrm{cl}(U) \subseteq \mathrm{cl}(V_\alpha) \subseteq U_\alpha$ for every $\alpha < \kappa$, so $\bigcap_{U \in F} \mathrm{cl}(U) \subseteq \bigcap_{\alpha < \kappa} U_\alpha$ and $\bigcap_{\alpha < \kappa} U_\alpha$ is non-empty, which completes the proof. □

We are not quite done with Boolean algebras yet. We need one more theorem involving them to be able to prove the final all-encompassing theorem of this chapter. We say that a filter of a Boolean algebra is proper if it is not equal to the entire Boolean algebra itself. Ultrafilters are proper by definition.

Lemma 4.10. *Let $\mathcal{B}$ be a Boolean algebra. Then every proper filter is contained in an ultrafilter.*

Proof. Let $F \subsetneq \mathcal{B}$ be a filter and consider the partial order $(P, \subseteq)$ where P is the set of proper filters in $\mathcal{B}$ that contain F. In particular, $F \in P$. Take a non-empty subset $\mathcal{C}$ of P such that $G \subseteq G'$ or $G' \subseteq G$ for every $G, G' \in \mathcal{C}$. Then $\bigcup_{C \in \mathcal{C}} C$ is a proper filter that contains F, so it is an element of P. It also contains every element of $\mathcal{C}$, so it is an upper bound for $\mathcal{C}$. Using Zorn's Lemma, there is a maximal element G in P. A maximal element in P is a proper filter that contains F and cannot be expanded to a bigger proper filter. So G is an ultrafilter that contains F. □

Theorem 4.11. *If for every compact Suslin Hausdorff space X and every family of dense open subsets $\{U_\alpha\}_{\alpha < \kappa}$ of X we have $\bigcap_\alpha U_\alpha \neq \emptyset$, then MA_κ restricted to complete Boolean algebras holds.*

Proof. Let $\mathcal{B}$ be a c.c.c. Boolean algebra and $\mathcal{D}$ a family of dense subsets of $\mathcal{B} \setminus \{0\}$ with $|\mathcal{D}| \leq \kappa$.

Define X as the set of ultrafilters of $\mathcal{B}$, $N_b := \{G \in X \mid b \in G\}$ for some $b \in \mathcal{B}$ and $\mathcal{N} := \{N_b\}_{b \in \mathcal{B}}$. Note that $b \leq c$ implies $N_b \subseteq N_c$, since all filters that contain b also contain c. Also note that since 1 is contained in every ultrafilter and 0 in none of them, we have $N_0 = \{G \in X \mid 0 \in G\} = \emptyset$ and $N_1 = \{G \in X \mid 1 \in G\} = X$. Thirdly, note that $N_c \cup N_d = N_{c \vee d}$ for $b, c \in \mathcal{B}$, since ultrafilters contain $c \vee d$ if and only if they contain c or d.

Take $G \in X$ and $b \in G$, which is possible because filters are non-empty. We then have $G \in N_b$, so $\mathcal{N}$ covers X. Now take $b, c \in \mathcal{B}$ such that $N_b \cap N_c \neq \emptyset$. Then there is a $G \in N_b \cap N_c$, so for this G we have $b, c \in G$. Since G is a filter, there is a $d \in G$ with $d \leq b$ and $d \leq c$, so $N_d \subseteq N_b \cap N_c$. We conclude that $\mathcal{N}$ forms a base for a topology on X. For the rest of the proof, we equip X with this topology.

For Hausdorffness, take two distinct ultrafilters $G, F \subseteq \mathcal{B}$. Without loss of generality, we can take $b \in G \setminus F$. Suppose b is compatible with c for each $c \in F$. Then there is a $b_c \in \mathcal{B} \setminus \{0\}$ with $b_c \leq b$ and $b_c \leq c$ for each $c \in F$. Consider the filter generated by $\{b_c\}_{c \in F}$, b and F. This is a filter that is strictly larger than F. But F is an ultrafilter $\lightning$. So there is a $c \in F$ that is not compatible with b. Since all elements of a filter are pairwise compatible, we have $c \notin G$. Now consider the open sets N_b and N_c. We have $b \in G$ so $G \in N_b$ and likewise $F \in N_c$. Suppose there is a filter $H \in N_b \cap N_c$. Then $b, c \in H$, so b and c are compatible $\lightning$. So N_b and N_c are two disjunct open sets containing G and F respectively and since we can do this for any two distinct ultrafilters of $\mathcal{B} \setminus \{0\}$, X is Hausdorff.

For compactness, let $\mathcal{C} \subseteq \mathcal{B}$ be such that $\bigcup_{c \in \mathcal{C}} N_c = X$, so $\{N_c\}_{c \in \mathcal{C}}$ is a cover of X. Define

$$F := \{b \in \mathcal{B} \mid \text{there is an } n \in \mathbb{N} \text{ and there are } c_1, \ldots, c_n \in \mathcal{C} \text{ such that } b \geq \bigwedge_{i=1}^{n} \overline{c_i}\}.$$

If $b, b' \in F$, then there are $n, n' \in N$ and $c_1, \ldots, c_n, c'_1, \ldots, c'_{n'} \in \mathcal{C}$ such that $b \geq \bigwedge_{i=1}^{n} \overline{c_i}$ and $b' \geq \bigwedge_{i=1}^{n'} \overline{c'_i}$. Take $c := \bigwedge_{i=1}^{n} \overline{c_i} \wedge \bigwedge_{i=1}^{n'} \overline{c'_i}$. Then $c \leq b$, $c \leq b'$ and $c \in F$. If $b'' \geq b$, then $b'' \geq \bigwedge_{i=1}^{n} \overline{c_i}$ and thus $b'' \in F$. This proves that F is a filter on $\mathcal{B}$.

Suppose that $F \neq \mathcal{B}$. By Lemma 4.10, there is an ultrafilter $G \in X$ that contains F. We have $\overline{c} \in F \subseteq G$ for every $c \in \mathcal{C}$, so by Lemma 4.4, $c \notin G$ for every $c \in \mathcal{C}$. This means that $G \notin \bigcup_{c \in \mathcal{C}} N_c = X$ $\lightning$, so F is equal to $\mathcal{B}$. In particular $0 \in F$, so there are $c_1, \ldots, c_n \in \mathcal{C}$ such that $0 = \bigwedge_{i=1}^{n} \overline{c_i}$. We have $1 = \overline{0} = \overline{\bigwedge_{i=1}^{n} \overline{c_i}} = \bigvee_{i=1}^{n} c_i$ and this implies $\bigcup_{i=1}^{n} N_c = N_{\bigvee_{i=1}^{n} c_i} = N_1 = X$, so $\{N_{c_i}\}_{i=1}^{n}$ is a finite subcover of X. This means that every cover of X by elements of the base has a finite subcover, so X is compact.

We have $N_b \cap N_c = \emptyset$ if and only if $b \wedge c = 0$, so X is Suslin. For each $D \in \mathcal{D}$, define $W_D := \bigcup_{b \in D} N_b$. Since D is dense in P, we have that for every $c \in \mathcal{B}$ there is a $b \in D$ with $b \leq c$, so $N_b \subseteq N_c$ and $N_c \cap W_D \supseteq N_c \cap N_b = N_b \neq \emptyset$. This means that $W_D \cap U \neq \emptyset$ for every non-empty open $U \subseteq X$, so W_D is open and dense in the topological sense for each $D \in \mathcal{D}$.

By assumption, $\bigcap_{D \in \mathcal{D}} W_D$ is non-empty, so fix a filter G from this set and take an arbitrary $D \in \mathcal{D}$. Then $G \in W_D$, so there is a $b \in D$ with $G \in N_b$ and thus $b \in G$, so $b \in G \cap D \neq \emptyset$. $\square$

4.4 The MA_κ equivalence theorem

We now have four equivalent statements.

Theorem 4.12. *For any $\kappa \geq \omega$, the following are equivalent:*

(1) MA_κ.

(2) MA_κ*, restricted to partial orders with cardinality at most κ.*

(3) MA_κ*, restricted to complete Boolean algebras.*

(4) If X is a compact Suslin Hausdorff space and U_α are dense open subsets of X for each $\alpha < \kappa$, then $\bigcap_\alpha U_\alpha \neq \emptyset$.

Proof. We have seen that (1) and (2) are equivalent by Theorem3.14 and that (1) and (3) are equivalent by Theorem 4.6. Furthermore, (1) implies (4) by Theorem 4.9 and (4) implies (3) by Theorem 4.11, giving us that all four statements are equivalent. □

5 The Suslin property again

We now return the question of whether a product of two Suslin spaces is necessarily Suslin. Recall that ω_1 is (the cardinality of) the first uncountable ordinal. If we assume $\neg\mathsf{CH}$, this is a set that is strictly between $\mathbb{N}$ and $\mathbb{R}$ with respect to cardinality.

Lemma 5.1. *Assume MA_{ω_1}. Let X be a space with the Suslin property and $\{U_\alpha\}_{\alpha<\omega_1}$ be a family of non-empty open subsets of X. Then there is an uncountable $A \subseteq \omega_1$ such that $\{U_\alpha\}_{\alpha\in A}$ has the finite intersection property.*

Proof. Define $V_\alpha := \bigcup_{\gamma>\alpha} U_\gamma$. Then $V_\alpha \subseteq V_\beta$ if $\beta < \alpha$. We want an α such that $\mathrm{cl}(V_\alpha) = \mathrm{cl}(V_\beta)$ whenever $\beta > \alpha$. Suppose there is no such α. Then we have an increasing sequence of ordinal numbers $\{\alpha_\xi\}_{\xi<\omega_1}$ with $\mathrm{cl}(V_{\alpha_{\xi+1}}) \neq \mathrm{cl}(V_{\alpha_\xi})$. Then the sequence $\{V_{\alpha_\xi} \setminus \mathrm{cl}(V_{\alpha_{\xi+1}})\}_{\xi<\omega_1}$ is an uncountable sequence of non-empty disjoint open subsets of the Suslin space X ↯.

Take an α with $\mathrm{cl}(V_\alpha) = \mathrm{cl}(V_\beta)$ whenever $\beta > \alpha$. Since $V_\alpha \subseteq V_\beta$ if $\beta < \alpha$ and $\mathrm{cl}(V_\alpha) = \mathrm{cl}(V_\beta)$ if $\beta \geq \alpha$, we have $\mathrm{cl}(V_\alpha) \subseteq \mathrm{cl}(V_\beta)$ for all $\beta < \omega_1$. Define the partial order $P := \{U \subseteq V_\alpha \mid U \text{ open and non-empty}\}$ with inclusion as relation. X is a Suslin space, thus P is a c.c.c. partial order. For each $\beta < \omega_1$, define $D_\beta := \{U \in P \mid \exists \gamma > \beta \text{ with } U \subseteq U_\gamma\}$.

Take $U \in P$. Then $\emptyset \neq U \subseteq V_\alpha \subseteq \mathrm{cl}(V_\alpha) \subseteq \mathrm{cl}(V_\beta)$. The set V_β is of course dense in its closure and U is open, so $\emptyset \neq U \cap V_\beta = U \cap \bigcup_{\gamma>\beta} U_\gamma$. This means that there is a $\gamma > \beta$ with $U \cap U_\gamma \neq \emptyset$, so $U \cap U_\gamma \in D_\beta$. Thus $U \cap U_\gamma$ is an extension of U in D_β, making D_β dense in P for every $\beta < \omega_1$.

P is a non-empty c.c.c. partial order and $\{D_\beta\}_{\beta<\omega_1}$ is a family of at most ω_1 dense subsets of P, so MA_{ω_1} gives us a filter F in P such that $F \cap D_\beta \neq \emptyset$ for every $\beta < \omega_1$.

Define $A := \{\gamma < \omega_1 \mid \exists U \in F \text{ with } U \subseteq U_\gamma\} \subseteq \omega_1$. Take $\beta < \omega_1$, then there is a $U \in F$ with $U \in D_\beta$, so there is a $\gamma > \beta$ with $U \subseteq U_\gamma$. This γ then is an element of A. For every $\beta < \omega_1$, we can find a $\gamma > \beta$ with $\gamma \in A$ this way, so A is uncountable.

Take $B \subseteq A$ finite and consider $\{U_\gamma\}_{\gamma\in B}$. For every $\gamma \in B$ there exists a $U'_\gamma \in F$ with $U'_\gamma \subseteq U_\gamma$. Since F is a filter, it has the finite intersection property and since $\{U'_\gamma\}_{\gamma\in B}$ is finite, we have

$$\emptyset \neq \bigcap_{\gamma\in B} U'_\gamma \subseteq \bigcap_{\gamma\in B} U_\gamma,$$

so $\{U_\gamma\}_{\gamma\in A}$ has the finite intersection property. □

Using this, we can prove the following.

Theorem 5.2. *Assume MA_{ω_1}. Then a product of two Suslin spaces is Suslin.*

Proof. Take (X, τ_X) and (Y, τ_Y) two Suslin spaces and suppose $X \times Y$ is not a Suslin space. Then there is an (uncountable) family of non-empty open disjoint subsets $\{W_\alpha\}_{\alpha<\omega_1}$ of $X \times Y$. For every $\alpha < \omega_1$, there is an open rectangle $U_\alpha \times V_\alpha \subseteq W_\alpha$ with $U_\alpha \in \tau_X$ and $V_\alpha \in \tau_Y$ both non-empty. This gives us a family $\{U_\alpha\}_{\alpha<\omega_1}$ of non-empty open subsets of X. Since we assume MA_{ω_1}, we can use Lemma 5.1 and there is an uncountable $A \subseteq \omega_1$ such that $\{U_\alpha\}_{\alpha\in A}$ has the finite intersection property.

By this property, $U_\alpha \cap U_\beta \neq \emptyset$ when $\alpha \neq \beta \in A$. But $U_\alpha \times V_\alpha \cap U_\beta \times V_\beta \subseteq W_\alpha \cap W_\beta = \emptyset$, so $V_\alpha \cap V_\beta = \emptyset$ whenever $\alpha \neq \beta \in A$. This gives us an uncountable family $\{V_\alpha\}_{\alpha \in A}$ of pairwise disjoint non-empty open subsets of the Suslin space Y ↯. So $X \times Y$ is a Suslin space. □

Corollary 5.3. *Assume MA_{ω_1}. Then every product of Suslin spaces is Suslin.*

Proof. We have seen this for the product of two Suslin spaces in Theorem 5.2. By induction, we can extend this to finite products of Suslin spaces, and by Theorem 2.6 we can extend this to arbitrary products. □

6 The Galvin-Laver counterexample

The goal of this chapter is to find two Suslin spaces such that their product is not Suslin. We have already seen that this cannot be done with only the ZFC axioms, we need to include the Continuum Hypothesis as well to obtain the Galvin-Laver (counter)example. Laver[4] first introduced this result in unpublished work, Galvin[5] completed it into [5] in 1980. While the article refers to its content as 'a simple proof of Laver's result', giving the entire proof is beyond the scope of this text, and we will state some generalized results without proof to obtain the counterexample. The whole proof can be found in [5] if one is interested. Note that in the paper, the c.c.c. and the Suslin property are both called 'ω_1-c.c.' and that the definition of compatibility is slightly different.

The first useful result reduces our problem to partial orders instead of topological spaces. We need to define a partial order on a product of partial orders before this makes sense.

Definition 6.1. If $(P, \leq_P)$ and $(Q, \leq_Q)$ are partial orders, then $P \times Q$ is partially ordered by the rule $(p, q) \leq (p', q') \Leftrightarrow p \leq_P p'$ and $q \leq_Q q'$.

Using this definition, there exist two Suslin spaces whose product is not Suslin if and only if there are two c.c.c. partially ordered sets whose product does not satisfy the c.c.c. property. The proof of this uses Gleason spaces and forcing and is the proof of Theorem 2.8 in [5]. For the construction of these partial orders, we need some notational shorthands.

Definition 6.2. Let X, Y be sets and κ be a cardinal number.

- $[X]^\kappa := \{A \subseteq X \mid |A| = \kappa\}$.
- $[X]^{<\kappa} := \{A \subseteq X \mid |A| < \kappa\}$.
- If $X \subseteq [\kappa]^2$, then $P(\kappa, X) := \{A \in [\kappa]^{<\omega} \mid [A]^2 \subseteq X\}$. This is a partial order with *reverse* (!) inclusion as relation. So $A \leq B$ if and only if $A \supseteq B$.

Note that $[A]^2$ is just the set of all subsets of A with exactly 2 elements, so X in the last bullet is a set of sets of exactly two ordinal numbers less than κ. Also note that $[B]^\kappa = \emptyset$ whenever $|B| < \kappa$.

Lemma 6.3. *Let $X, Y \subseteq [\omega_1]^2$ be disjoint. Then $P(\omega_1, X) \times P(\omega_1, Y)$ is not c.c.c..*

Proof. Take $\alpha < \omega_1$. Since $|\{\alpha\}| = 1 < 2$, we have $[\{\alpha\}]^2 = \emptyset$ and thus $[\{\alpha\}]^2 \subseteq X$ holds, so $\{\alpha\} \in P(\omega_1, X)$ for each $\alpha < \omega_1$. Likewise we have $\{\alpha\} \in P(\omega_1, Y)$ for each $\alpha < \omega_1$.

Take $\alpha, \beta < \omega_1$ with $\alpha \neq \beta$ and <u>suppose</u> that $(\{\alpha\}, \{\alpha\})$ and $(\{\beta\}, \{\beta\})$ are compatible. Then there are $A \in P(\omega_1, X)$ and $B \in P(\omega_1, Y)$ such that $(A, B) \leq (\{\alpha\}, \{\alpha\})$ and $(A, B) \leq (\{\beta\}, \{\beta\})$. Then we have $A \leq \{\alpha\}$, $B \leq \{\alpha\}$, $A \leq \{\beta\}$ and $B \leq \{\beta\}$, so $A \supseteq \{\alpha, \beta\}$ and $B \supseteq \{\alpha, \beta\}$ hold. We have $A \in P(\omega_1, X)$, so $\{\alpha, \beta\} \in [A]^2 \subseteq X$ and $B \in P(\omega_1, Y)$, so $\{\alpha, \beta\} \in [B]^2 \subseteq Y$. Now $\{\alpha, \beta\} \in X \cap Y$, but X and Y were disjoint $\lightning$.

This means that $\{(\{\alpha\}, \{\alpha\})\}_{\alpha < \omega_1}$ is an uncountable antichain and $P(\omega_1, X) \times P(\omega_1, Y)$ is not c.c.c.. $\square$

[4] Richard Laver (1942–2012), American set theorist.
[5] Fred Galvin, (1936–), American set theorist and combinatorist.

Now the conclusion follows from the following lemma.

Lemma 6.4. *Assume* CH*. Then there are disjoint* $X, Y \subseteq [\omega_1]^2$ *such that* $P(\omega_1, X)$ *and* $P(\omega_1, Y)$ *have the c.c.c. property.*

The proof of this lemma uses a very elaborate way to define X and Y inductively. It is part of the proof of Theorem 3.3 in [5] which uses the Δ-system lemma (Lemma 2.5) again. Combining Lemma 6.3 with 6.4, we see that $P(\omega_1, X)$ and $P(\omega_1, Y)$ are c.c.c. partial orders with a non-c.c.c. product. This proves that there exist Suslin spaces with a non-Suslin product if we assume CH.

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