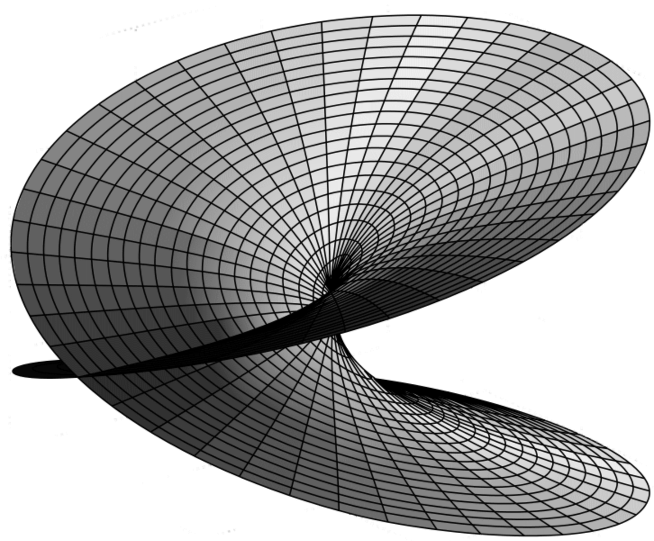
A topological proof of the Abel-Ruffini Theorem

Based on the method of V.I. Arnold



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Bachelorthesis

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January 2022

Abstract

This bachelor thesis gives a topological proof of the Abel-Ruffini theorem which states that there does not exist a general algebraic solution to polynomials of fifth order. The proof is loosely based on an essay by L. Goldmakher [9] who in turn based it on the proof of V.I. Arnold. [2]

Contents

1 A historical introduction

1.1 Old knowledge

This historical section of this document takes a lot of information from Burton's *"The history of mathematics: An introduction"*. [3].
Although it may pale in comparison to the age of counting, the notion of polynomials are still one of the oldest concepts in mathematics. A humble 18 millennia younger than the famous Ishango bone, the first archaeological evidence of polynomials was carved into Babylonian clay tablets dated at around 2000 BCE [4].

Yet, despite its age, it took most of the concept's life to find out which polynomials are, and aren't solve-able. The previously mentioned Babylonian tablets already described a method to obtain a solution to the variant of the problem of order two. The said method was not fully general, however, and restricted to finding the largest real root of the polynomial.

After a few millennia of refinement, the equation in the form we use it today was first recorded in 1637 in Descartes' La Géométrie. For a polynomial of the form $ax^2 + bx + c$, it gives the two roots as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The solution to the cubic equations followed a similar timeline. Although there is no evidence of the Babylonians knowing a method to tackle the problem, we know they had some knowledge on cube roots, as tablets have been uncovered which contained cubic root tables.

Initially, only a method was known for the specific class of cases where the quadratic term was zero. This later turned out to be enough though, since all third-order polynomials can be reduced to that form.

The solution to these "depressed" polynomials is that if we have a polynomial of the form $x^3 + ax + b$, it can be shown that

$$\sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

is one of the roots. Nowadays, the method is generalised to all polynomials of third degree. If one has a polynomial of the form $x^3 + ax^2 + bx + c$, the solutions are of the form [12]:

$$\sqrt[3]{\frac{S_1 \pm \sqrt{S_1^2 - \frac{4}{27} S_2^3}}{2}}.$$

Here we used

$$S_1 = \frac{3b - a^2}{3},$$
$$S_2 = \frac{2a^3 - 9ab + 27c}{27}.$$

1.2 Cardano, del Ferro, and Tartaglia

Oddly enough, the publication that gave the world the solution to the cubic polynomials goes an order further as well. Published in 1545, Cardano's Ars Magna contains an algebraic solution to both the Cubic and the Quartic polynomials.

Despite the value of the book, the ethics of its publication were debatable at best. The book is not solely the work of Cardano himself. Rather, it compiles the works of a handful of his competitors at the time. Three names are of note here. The first is Scipione del Ferro; an Italian mathematician who solved the previously mentioned class of depressed polynomials but chose to take his solutions to the grave. Despite Ferro's wishes for his work to remain private, Cardano went through the notes Ferro left his son-in-law to dig up the unreleased work.

This was not a one time incident either, as Cardano had actually already obtained the method to solve the depressed cubic polynomials from another Italian mathematician named Niccolò Fontana Tartaglia. Tartaglia had previously won a contest in which he was challenged to solve a few cubic polynomials. Cardano, curious how he did it, requested Tartanglia to explain his method to him. Tartanglia agreed, but only on the explicit condition that Cardano would not release the results but keep them to himself. Once Cardano got hold of Ferro's works however, he used that as an excuse to publish anyway.

The solution to the quartic equation contained in the book is not Cardano's own work either. It was his student Lodovico Ferrari who managed to reduce solving fourth-order problems to solving third-order problems. Since Cardano had already previously obtained the solution to those, the Ars Magna taught its readers how to solve either power.

In his later years, Cardano published a work explaining how to optimally cheat at chance games [7].

The solution to the quartics is quite a monster that takes up multiple sheets of paper to note down. It is, therefore, neither worth nor practical to write it out here.

Sadly, right after the fourth-order, equations that solve polynomials cease to exist. For three centuries after the publication of the Ars Magna, no progress on studying higher-order problems was made. This is where the topic of this document will be coming into view.

1.3 Abel and Ruffini

Joseph-Louis Lagrange is a name history has deemed too valuable to be granted the right to be forgotten. Ironically most famous for another problem that turned out to be impossible to solve, he had laid the foundation for the works of Paolo Ruffini and Niels Henrik Abel which proved that no algebraic equation solving the quintic polynomials can be constructed.

Lagrange was Euler's successor in a handful of ways. He took his seat as the director of mathematics in the Prussian Academy of Sciences for example. Similarly, Euler gave an alternative method to reduce the Quartics to the Cubics in his Elements of Algebra which Lagrange was inspired by to try and reduce the quintics to the quartics.

His first published work on the subject was printed by the Berlin royal academy of sciences in 1770. In it, he spends over two hundred pages covering known methods to study the previous two orders. Time and time again, he had to conclude that they seemingly couldn't be extended to the higher case. Defeated, he spends the final few pages expressing how unlikely he deemed it for a solution to the quintics to exist.

Although seeming mostly fruitless, the work he published in this document laid the groundwork for the entire field of Galois theory. By covering a multitude of established solutions, Lagrange found a common thread between them in the form of permutations. Though, he did not refer to them explicitly. These ideas ended up as the basis for the works of Ruffini and Abel.
Information given about the lives of both individuals is based mainly on [10, 5, 11].

There is a cruel irony to the tale of the unsolvability of quintics. The founders of the cubic equation wished for their work to remain hidden. It took a third party to show it to the world. In contrast, both Abel and Ruffini tried to make their theorem know but were mostly ignored by the mathematical community.

Despite Lagrange's disappointed conclusion in his previously mentioned publication, he still promised to return to the problem, seemingly holding up hope that a solution exists. This seemed to have been the general opinion at the time. When Ruffini solved the centuries old problem by showing that no solution exists, he did not receive the praise one would expect. Instead, he had sent a copy of his proof to Lagrange on three different occasions and never received a reply. He wasn't taken much more seriously by other colleagues either.

One of the few mathematicians to recognise the proof was Cauchy, who not only praised him for it but went on to generalise some of the material Ruffini had established. This praise was just in time as well, as Ruffini died less than a year later.

Cauchy, in turn, leads us to Abel. At first, a young Abel actually thought to have found a solution to the quintics but found an error in his work when trying to work out an example. This effort of his to still find a solution shows how little light Ruffini's work had gotten. A few years earlier when studying Cauchy's works, he came across a reference to that of Ruffini. However, he seemingly didn't know that the problem was solved by the Italian, only that he worked on it.

Skip a few years, and Abel also found a proof of the result. His version also patched a minor gap that was present in that of his predecessor. Paid out of his own pocket, he got it in print on a short six-page pamphlet. He had sent this pamphlet to Gauss, but got an even worse reaction than Ruffini received from Lagrange. Instead of simply being ignored, Abel was informed that Gauss was highly displeased with receiving his work. Abel's pamflet was later found unopened after Gauss's death.

In the end, the work reached the hands of mathematicians that were willing to accept the result. Abel's friend Crelle founded a journal devoted entirely to mathematics in 1826. Several articles in the first volume were entirely dedicated to Abel's work.

Nowadays, multiple people have presented alternative proofs for the same theorem. The field of group theory that Ruffini helped invent for it is now one of the fundamental branches of mathematics. Although their own time could not reward them with it, their works deserve the wide recognition they today have.

2 The idea of the proof

The Abel–Ruffini theorem states that there is no algebraic solution to the general polynomials of degree five. More rigorously, we aim to prove the following claim:

Theorem 2.1. *There is no closed-form expression F(p) containing only addition, subtraction, division, multiplication, integer powers, and finite integer radicals, such that F(p) provides the roots of any algebraic equation p of the form*

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0.$$

The proof presented in this document is somewhat lengthy, but allows itself to easily be cut into sections. For the sake of clarity, we will first provide a very rough and informal sketch of the proof before we will work it out rigorously in later sections. This means that any theorem and terminology mentioned in this section will be proven and properly formulated in the relevant later sections.

The central idea of the argument is that we can show it to be impossible to construct an algebraic expression that behaves ugly enough for said expression to give the requested roots.

Now, by the fundamental theorem of algebra, an algebraic solution to the quintics should return 5 different values for any p that has no repeated roots. Therefore, $F(p)$ has to be a set of 5 distinct elements. It is possible to pick one element from this set and track how it changes as we move p around in polynomial space. *Section 3* and the start of *section 4* will show that there are certain loops p can traverse whose image under $F(p)$ is no longer a loop. More precisely it will prove the following:

Corollary 2.1.1. *Let p be a polynomial with 5 distinct roots $F(p) = (r_1, \ldots, r_5)$. For any permutation σ of this tuple there is a corresponding loop in polynomial space γ centred at p such that σ is induced by γ.*

The next step of the proof is showing that no closed-form algebraic expression $\hat{f}$ behaves like this. For summation, subtraction, multiplication, division, and integer powers, this is clear. They are well defined operations and can never return multiple values for a single input.

The last allowed building block in algebraic expressions, however, forms the central obstacle in our proof. Radicals, as previously mentioned, are also not functions and can "break" loops. An easy example is the loop $\gamma(t) = e^{2\pi it}$ which runs around the unit circle in $\mathbb{C}$ once. If we take the square root of this loop, we get $\sqrt{\gamma(t)} = e^{\pi it}$, which only traverses half the same path and therefore is no longer a loop.

This means that any possible algebraic solution to the general polynomials of degree five would be forced to contain radicals. Ofcourse, considering the equation $x^n + c = 0$ leads to the same conclusion.

To seemingly take a small detour, consider how we would actually define "loop breaking". We seek to describe the phenomena where a loop in one space is associated with a path that is explicitly not a loop in another space. This turns out to be challenging to do rigorously however. For example, let's say we naively define it as follows:

> "A function between two topological spaces $F : X \to Y$ is said to **break** a loop γ in X if $F \circ \gamma$ is not a loop."

With this definition, we immediately run into a problem. A function can never do this. Functions are well-defined after all. One could try to apply multi-valued functions to tackle this further, but we will instead establish some tools from topology so that we can use covering spaces instead.
Just as how the solutions to the cubics and quartics contain nested radicals, it turns out that for the quintics a single radical does not suffice. We will give an argument that shows that a single radical is not strong enough to break all loops that the needed expression would have to break. The example that will be used in the rest of the proof for this is commutator loops.

A commutator loop $[\gamma_1, \gamma_2]$ is of the form $\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ where γ_1 and γ_2 are loops, the multiplication is loop composition and the inverse loops are defined as walking the loop's path in reverse. This commutator loop is still a loop as it is a path that starts and ends at the same point.

This brings us to two important claims.

1. Under our hypothetical solution to the quintics $F(p)$, we can find some commutator loops that would change which root the expression gives, just as we could with general loops.

2. The difference here is that the radicals of commutator loops are still loops, which proves that an equation that only contains a single radical is not strong enough for our cause.

Two sections are spent examining these claims.

Section 4 will properly introduce the required terminology and then end by proving the following lemma:

Lemma 2.2. *There are loops of the form $[\gamma_1, \gamma_2]$ that induce non-trivial permutations on the set $F(p)$.*

Section 5 will prove that algebraic operations aside from radicals are unable to break loops and finishes by proving the following corollary:

Corollary 2.2.1. *An algebraic equation containing only a single radical can't break all loops.*

In the final step of our proof, we repeat the previous argument inductively. If we have an expression with a nested radical in it, we can construct a commutator loop of commutator loops that should still permute which root the expression gives. It has to do this despite the nested radical not being strong enough to destroy the loop. In the case of the quintics, we can keep doing this indefinitely, and can thus not find a general solution with finitely many radicals.

To rephrase this; We found a concrete difference between how a solution to the quintics *would have to* behave, and how algebraic solutions *can* behave.

Section 7 will show that the set of commutator permutations of $\mathcal{A}$ is also its own set of commutator permutations. This means that for any N there are permutations that take the form of N levels of nested permutations.

Section 6 will then show that for any such N, an algebraic equation would require N+1 levels of nested radicals. Combined, this concludes the proof.

3 Loop permutations

In this section, we seek to study how loops in polynomial space correspond to permuting the roots of a polynomial. Let us introduce the following terminology:

3.1 Towards loop breaking

Definition 3.1 (paths and loops)**.** *A **path** γ in a topological space X is a continous function from the closed unit interval into said space: $\gamma : [0,1] \to X$. A **Loop** is a path γ with the propery that $\gamma(0) = \gamma(1)$. The loop γ is then said to be based at $\gamma(0)$.*

Definition 3.2 (Permutation)**.** *A **Permutation** is a bijective function from a set to itself.*
*A **Transposition** is a Permutation that acts as the identity on all but two elements.*
S_n is the set of all permutations of n elements.

Definition 3.3 (Quotient map)**.** *A **quotient map** is a surjective map $f\colon (X, \tau_X) \to (Y, \tau_Y)$ with $U \in \tau_Y \iff f^{-1}(U) \in \tau_X$.*

Definition 3.4 (Quotient topology)**.** *Let X be a topological space, Y as set, and $f : X \to Y$ a surjective map. The **quotient topology** on Y with respect to f is the finest topology on Y which makes f continuous.*

Proposition 3.5. *Every quotient map $q : X \to Y$ induces an equivalence relation $\sim_q$ on X defined by $x \sim_q \tilde{x} \iff q(x) = q(\tilde{y})$. Conversely, every equivalence relation $\sim$ on X induces a quotient map*

$$p : X \to X/\!\sim \qquad x \mapsto [x].$$

Here $X/\!\sim$ is the set of all equivalence classes under $\sim$, equipped with the quotient topology with respect to p.

Proof. Since the second claim follows by definition, We only need to prove that $\sim_q$ is an equivalence relation. But since q is a function, $\sim_q$ has the reflexivity, symmetry, and transitivity properties immediately. □

If we have a topological space X and an equivalence relation $\sim$, we refer to $X/\!\sim$ equipped with the quotient topology as the quotient space of X under $\sim$.

There are multiple ways to specify polynomials. The two most natural ones are by giving their roots or their coefficients. If we're restricting ourselves to monic polynomials, a polynomial of degree n can be determined by its n roots in $\mathbb{C}$ or by n coefficients. Thus, there are at least two ways to describe order n polynomials as elements in $\mathbb{C}^n$. To avoid confusion we'll refer to root spaces and coefficient spaces as $\mathbb{C}^n_r$ and $\mathbb{C}^n_c$ respectively. There is a natural way of going from the former to the latter representation. If we are given the roots of a polynomial, we can find its coefficients by writing the polynomial in the form $(z - r_1)(z - r_2)\dots$ and then simply expanding the terms. Equivalently, one could directly use Vieta's formulas. This gives us a map $\alpha : \mathbb{C}_r^n \to \mathbb{C}_c^n$ from root space to coefficient space.

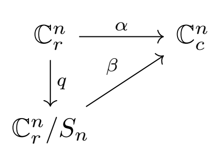
The map α is continuous since the coefficients are found by multiplying and adding roots. Both of which are continuous operations. Furthermore α also is surjective since for all $(z_1, \dots, z_n)$ there is a polynomial $x^n + z_1 x^{n-1} \dots + z_n$, and by the fundamental theorem of algebra each such polynomial has a root representation.

The map α is not injective as multiple permutations of roots correspond to the same polynomials. For example, roots (r_1, r_2) and (r_2, r_1) both map to $(-r_1 - r_2, r_1 \cdot r_2)$. We can construct a bijective version of α by dividing root space by all permutations of n elements.
To make this more concrete, take $p_1, p_2 \in \mathbb{C}_r^n$ such that $\alpha(p_1) = \alpha(p_2)$. Since the fundamental theorem of algebra states that a polynomial has a unique set of roots this implies that p_2 is just a permutation of p_1. Thus all sets of roots mapping to the same element are permutations of each other.
With this in mind we will define the relation $\sim_\alpha$ as the equivalence relation induced by the map α.

Proposition 3.6. *$x \sim_\alpha y$ implies that there is a $\sigma \in S_n$ such that $x = \sigma(y)$*

Proof. By the fundamental theorem of algebra, a polynomial has a unique set of roots. This implies that for a polynomial $p \in \mathbb{C}_c^n$, all elements in $\alpha^{-1}(p)$ are permutations of each other. Since $x \sim_\alpha y \iff \alpha(x) = \alpha(y)$, this means $x = \sigma(y)$ for some permutation σ □

For ease of use, we'll refer to the quotient space $\mathbb{C}_r^n / \sim_\alpha$ as $\mathbb{C}_r^n / S_n$. This quotient space comes with a quotient map $q : \mathbb{C}_r^n \to \mathbb{C}_r^n / S_n$ which is surjective and continuous. Since a (monic) polynomial is uniquely identified by either its set of roots or by its coefficients, we have a bijection $\beta : \mathbb{C}_r^n / S_n \to \mathbb{C}_c^n$ which is the composition of the bijection between root space and the set of polynomial; and the bijection between the set of polynomials and coefficient space. We furthermore by definition have $\alpha = \beta \circ q$.
Thus far we have obtained the following commutative diagram:



You may wonder why we haven't defined root space to be $\mathbb{C}_r^n / S_n$ instead of $\mathbb{C}_r^n$. After all, it is counter intuitive to claim that the roots of a polynomial are in any way ordered. There is a reason for this. We'll end up defining loop breaking by studying how loops in $\mathbb{C}_r^n / S_n$ (or $\mathbb{C}_c^n$) correspond with paths in root space. Since S_n has $n!$ elements however, we have a non-uniqueness problem when we try to pick such a path. To solve this, we need to look at covering spaces.

Definition 3.7 (Covering maps)**.** *Let (X, τ_X) and (C, τ_C) be topological spaces. A continuous surjective map $p : C \to X$ is called a covering map if for all $x \in X$ there is a open neighbourhood $U_x \in \tau_X$ such that $p^{-1}(U_x)$ is a union of disjoint opens in C, each of which is mapped homeomorphically to U_x under p.*

Definition 3.8 (covering spaces)**.** *Let (X, τ_X) be a topological space. A covering space $((C, \tau_C), p)$ of X is a topological space (C, τ_C) together with a covering map $p : C \to X$.*

Example 3.9. We define

$$S^1 = \{e^{ix} \in \mathbb{C} \,|\, x \in \mathbb{R}\}$$

The map $p : S^1 \to S^1$, $x \mapsto x^n$ is a covering map. It is clear that the map is continuous and surjective. Take $e^{it} \in S^1$ and $U_t = (e^{i(t-\frac{\pi}{4})}, e^{i(t+\frac{\pi}{4})})$. Then

$$p^{-1}(U_t) = \bigcup_{j=0}^{n-1} (e^{i(t_j - \frac{\pi}{4n})}, e^{i(t_j + \frac{\pi}{4n})})$$

where $t_j = \frac{t+2\pi j}{n} \in p^{-1}(t)$. This forms the sought out union of opens that are homeomorphic with U_t. Therefore (S^1, p) is a covering space.

Lemma 3.10 (path lifting)**.** *Let $p : X \to Y$ be a covering map, $\gamma : [0,1] \to Y$ a path, and $x_0 \in p^{-1}(\gamma(0))$. Then there is a unique path $\tilde{\gamma} : [0,1] \to X$ such that $\tilde{\gamma}(0) = x_0$ and $p \circ \tilde{\gamma} = \gamma$. We call $\tilde{\gamma}$ a lift of γ along p.*

Proof. This proof is found in Proposition 13.7.7 of [8]. □

We are equipped to define what it means for a map to break loops.

3.2 Breaking loops

Definition 3.11 (loop breaking)**.** *Assume that we have a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ \big\downarrow{\scriptstyle p} & \swarrow{\scriptstyle g} & \\ Y & & \end{array}$$

where p is a covering map, f and g are continuous, f is surjective and g is bijective. f^{-1} is said to ***break*** *a loop γ in A if any of the lifts of $g \circ \gamma$ along p is not a loop.*

Remark. Since the diagram commutes, finding a path $\tilde{\gamma}$ in X that is not a loop such that $f(\tilde{\gamma}) = \gamma$ is sufficient to prove that f breaks γ.

Let's return to our previous diagram to study how this applies to our case.

$$
\begin{array}{ccc}
\mathbb{C}_r^n & \xrightarrow{\alpha} & \mathbb{C}_c^n \\
\big\downarrow q & \nearrow_{\beta} & \\
\mathbb{C}_r^n/S_n & &
\end{array}
$$

To start we note that β seemingly maps into the wrong direction. We already know that it is a bijection. As we will later use, it also turns out to be a homeomorphism. Since both the definition of β (Vieta's formulas) and its continuity are significantly more straightforward than those of its inverse however, it is more sensible to draw the arrow this way.

It is somewhat easy to show that for $\mathbb{C}_r^n/S_n$, $\mathbb{C}_r^n$ together with q *almost* fits the definition of covering space. We already know it to be a continuous surjection since it's a quotient map. The issue here is the local homeomorphism requirement, and specifically polynomials with repeated roots.

Say, we removed elements with repeated coordinates from $\mathbb{C}_r^n$. Let us call the space of repeated root elements E and D_r is the complement of E in $\mathbb{C}_r^n$. Thus

$$E = \{(z_1, \dots, z_n) \in \mathbb{C}_r^n \mid \exists i, j : i \neq j,\ z_i = z_j\}, \quad D_r = \mathbb{C}_r^n \backslash E.$$

Now, this subset E, is what is called *saturated* under q. This means that $q^{-1}(q(E)) = E$, which in our case is simply the observation that distinct roots remain distinct under permutation. This now implies that $q(\mathbb{C}_r^n \backslash E) = q(\mathbb{C}_r^n) \backslash q(E)$.

The map $\tilde{q} : D_r \to D_r/S^n$ is then simply the restriction of q too D_r, which is again surjective and continuous due too being a quotient map. If we now take $i : D_r \to \mathbb{C}_r^n$ and $\tilde{i} : D_r/S_n \to \mathbb{C}_r^n/S_n$ as the natural inclusion maps we have the commutative diagram:

$$
\begin{array}{ccccc}
D_r & \xhookrightarrow{i} & \mathbb{C}_r^n & \xrightarrow{\alpha} & \mathbb{C}_c^n \\
\big\downarrow \tilde{q} & & \big\downarrow q & \nearrow_{\beta} & \\
D_r/S_n & \xhookrightarrow{\tilde{i}} & \mathbb{C}_r^n/S_n & &
\end{array}
$$

Lemma 3.12. *The map $\tilde{q}$ is a covering map.*

Proof. For each $[z] = [(z_1, \dots, z_n)] \in D_r/S_n$ pick $R_{min} = d(i \circ \tilde{q}^{-1}(z), E)$, where $d(\cdot, \cdot)$ is a distance between sets defined with the standard metric on $\mathbb{C}^n$. If we now take U_z as the n-disk of radius $\frac{R_{min}}{2}$ around z (which makes sense on D_r/S_n since the disk would not intersect E even without the restriction to the subspace and is thus identical as how it is defined on $\mathbb{C}^n$), we have an open with the desired properties.
Namely, the pre-image of this U_z is a disjoint union of n-disks of radius $\frac{1}{2}R_{min}$, each of which is centred at one of the $n!$ elements in the pre-image of z.

We can guarantee that these opens are indeed disjoint by the definition of R_{min}. The radius of the open is by construction at most a quarter of the distance to the nearest permutation. Namely, for a permutation σ and a $\tilde{z} \in D_r$

$$||\tilde{z} - \sigma(\tilde{z})|| = ||(\tilde{z}_1 - \sigma(\tilde{z}_1), \dots, \tilde{z}_5 - \sigma(\tilde{z}_5))|| \geq \min_{i \neq j}\{|\tilde{z}_i - \tilde{z}_j|\} \geq 2d(i(\tilde{z}), E).$$

□

Lemma 3.13. *The map β is a homeomorphism.*

Proof. The proof can be found in [1]. □

Let γ be a path in $\mathbb{C}^n_c$. Since β is a homeomorphism, if γ doesn't pass through any polynomial with repeated roots it is homeomorphic with a path $\tilde{\gamma}$ in D_r/S_n. Lemma 3.10 then gives a set of paths (lifts) in D_r.

Definition 3.14. *Define $D_c = \alpha(D_r)$ and let γ be a path in coefficient space $D_c \subset \mathbb{C}^n_c$. We define Γ_γ as the set of lifts of $\tilde{i}^{-1} \circ \beta^{-1} \circ \gamma$.*

We can now rephrase Definition 3.11 to our specific situation.

Definition 3.15 (Polynomial loop breaking)**.** *Let γ be a loop in $D_c \subset \mathbb{C}^n_c$. The inverse map α^{-1} is said to **break** this loop if any of the paths in Γ_γ is not a loop.*

3.3 Transpositions induced by loops

Lemma 3.16. *Given a countable subset $A \subset \mathbb{R}^2$ and two points $a, b \in \mathbb{R}^2 \backslash A$, there exists a circle $C \subset \mathbb{R}^2 \backslash A$ with $a, b \in C$.*

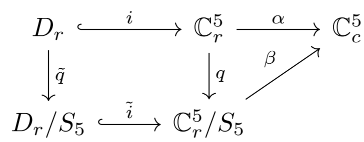
Proof. Take l as the perpendicular bisector of a and b. Take $c_0 \in l$. Since a and b have equal distance to p_0, there is a circle C_0 centred at p_0 that intersects both a and b. Now take $p_1 \in l$ with $p_1 \neq p_0$. Create circle C_1 out of p_1 via the same method. Since C_0 and C_1 have different centres, they are different circles. Say C_0 intersects $x \in A$. If C_1 intersects x as well, C_0 and C_1 would have 3 points in common (x, a, b). Since a circle is uniquely defined by any three points it intersects, this leads to a contradiction as that would imply that $C_0 = C_1$.

This means that any $x \in A$ can only intersect one of such circles. Thus, for any $x \in A$ there is at most one $p \in l$ such that the circle corresponding to p contains x. Since A is countable, it only intersects a countable set of such circles. Since l is not countable, there have to be points on l whose circle doesn't intersect any point in A. Thus, there are circles that intersect a and b but not A. □

We have now arrived at the claim with the longest proof in this document.

Theorem 3.17. *If an algebraic solution to the quintics exists, for any transposition on the set of roots provided by this equation at a polynomial with distinct roots, there is a loop in coefficient space that induces this transposition.*

Proof. Recall that we have the following diagram:



Our theorem states that if we pick an element $p = (r_1, \ldots, r_5) \in D_r$ and a transposition $\sigma \in S_5$, then there is a loop γ in $D_c \subset \mathbb{C}_c^5$ such that Γ_γ contains a path $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}(1) = \sigma(p)$.

Our diagram is commutative. Since we by definition have $\gamma = \beta \circ \tilde{i} \circ \tilde{q} \circ \tilde{\gamma}$, this also means that $\gamma = \alpha \circ i \circ \tilde{\gamma}$. The map i is simply the inclusion map obtained from restricting ourselves to polynomials without repeated roots. Combined this means our problem reduces to finding a path $I\colon [0,1] \to \mathbb{C}_r^n$ with the following properties:

- I does not pass through any polynomial with repeated roots.
- $I(0) = p$ and $I(1) = \sigma(p)$.
- $\alpha \circ I$ is a loop.

If we can construct such a path then $\alpha(I)$ is the γ we seek since the uniqueness granted by Lemma 3.10 guarantees that i^{-1} is contained in $\Gamma_{\alpha(I)}$. Thus, if we find I, we're done.

For $p = (r_1, \ldots, r_5)$, pick a permutation σ that permutes r_i and r_j. By the assumption that σ is a transposition, we know that the other three roots are untouched. For ease of notation we will without loss of generality assume that $r_i = r_1$ and $r_j = r_2$.

Lemma 3.16 guarantees that there is a circle in $\mathbb{C}$ that passes through r_1 and r_2 but not through r_3, r_4 or r_5. Define h_1 as the path from r_1 to r_2 along this circle in the clockwise direction, and h_2 as the path from r_2 to r_1 along the circle in the clockwise direction. Now define $I\colon [0,1] \to \mathbb{C}_r^5$ as

$$I\colon t \mapsto (h_1(t), h_2(t), r_3, r_4, r_5).$$

This by construction starts at p, ends at $\sigma(p)$ and does not pass through polynomials of repeated roots. Furthermore, $\alpha(I)$ is a loop since permutations of roots in root space map to the same polynomial in coefficient space. This finishes the proof.

∎

Remark. An important side-note here is that –since all permutations can be made by the product of transpositions– this proves that there is such a loop for any permutation of the roots provided we can show that loop composition corresponds with taking the product of their induced permutations. This will be covered in the section 4.1

4 Commutator permutations

In this section we seek to study the permutations induced by commutator loops.

In the rest of this document, we shall apply the tools and setting established thus far to more explicitly study how our hypothetical solution would behave under certain loops and what it would do with the roots of a polynomial. For a given quintic polynomial p, we will now refer to our hypothetical solution as $F(p)$ (previously referred to as α^{-1}), and will refer to a tuple of its roots as $M_p = (r_1, \ldots r_5)$ (previously described as an element in $\mathbb{C}^5_r$, or more specifically an element of $\alpha^{-1}(p)$). Note again that M_p is ordered. Whilst the ordering on M_p is completely arbitrary, we need to have one. We seek to study permutations on the roots, which only makes sense if they have an ordering.

4.1 Permutation products

Definition 4.1 (Paths concatenation)**.** *If γ_1 and γ_2 are paths with $\gamma_1(1) = \gamma_2(0)$, then their* ***concatenation*** *or product $\gamma_1 * \gamma_2$ is*

$$\gamma_1 * \gamma_2(x) = \begin{cases} \gamma_1(2x), & 0 \leq x \leq \frac{1}{2}, \\ \gamma_2(2x-1), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

The ***inverse*** *of a loop γ is defined as $\gamma^{-1}(x) = \gamma(1-x)$.*

Proposition 4.2. *Assume two loops γ_1, γ_2 corresponding with permutations σ_1, σ_2. If both loops are based at the same point then $\gamma_1 * \gamma_2$ corresponds with $\sigma_2 \circ \sigma_1$.*

Proof. Assume we have loops γ_1 and γ_2, which induce permutations σ_1 and σ_2 on a tuple of roots M_p respectively. Thus, if we walk along γ_1, we have that each root $r \in M_p$ is sent to $\sigma_1(r)$. This is once again an element in M_p, meaning that walking γ_2 then sends this to $\sigma_2(\sigma_1(r))$. Thus, walking the composition of both loops corresponds with the being permuted by the product of both permutations. □

Remark. Note that when going from loops to permutations the order of the index reverses. In the product defined on loops, the first element written is the path first traversed. This is inconvenient for when we start working with commutators, since this means that the commutator of loops and that of permutations will have to be defined differently. It is most practical to define loop commutators the natural and invert it with the definition for permutations.

Corollary 4.2.1. *Given a polynomial with 5 distinct roots p, for any permutation on the set $F(p)$ there is a loop in polynomial space that induces this permutation.*

Proof. Using the main result of section 3 we know that for any transposition on $F(p)$ we have a loop that induces this transposition. Group theory tells us that any permutation can be made as a product of transpositions. Applying Proposition 4.2 then extends this property to loops and finishes the proof. □

4.2 Commutators

Definition 4.3 (loop commutators)**.** *The commutator of two loops is defined as*

$$[\gamma_1, \gamma_2] = \gamma_1 * \gamma_2 * \gamma_1^{-1} * \gamma_2^{-1}.$$

Definition 4.4 (permutation commutators)**.** *The commutator of two permutations is defined as*

$$[\sigma_1, \sigma_2] = \sigma_2^{-1} \circ \sigma_1^{-1} \circ \sigma_2 \circ \sigma_1.$$

We will show that certain Commutator loops should still result in permutations after going through our expression $F(p)$.

Lemma 4.5. *There are loops of the form* $[\gamma_1, \gamma_2]$ *that induce non-trivial permutations on the tuple* M_p*.*

Proof. We know that we have for any two roots a loop such that said loop permutes these two roots. In other words, for $M_p = (r_1, ..., r_5)$ we have for all i, j a loop $\gamma_{i,j}$ which induces a $g_{ij} : M_p \to M_p$ such that $g_{ij} : r_i \mapsto r_j$, $r_j \mapsto r_i$, but acts as the identity on all other elements.

Now, since the composition of the loops corresponds with taking the product of their permutations, the commutator loop of $\gamma_{j,k}$ and $\gamma_{i,j}$ $(i \neq j \neq k)$ corresponds to the the permutation $(r_i r_j)(r_j r_k)(r_i r_j)(r_j r_k) = (r_i r_j r_k) \neq (1)$. Thus we have found a commutator loop that induces a nontrivial permutation of a polynomial's roots.

More broadly, this same argument shows that $[\gamma_1, \gamma_2]$ induces $[\sigma_2, \sigma_1]$, and the set of these elements contains non-trivial permutations. □

5 Commutator radicals

5.1 Loop lifting by radicals

Proposition 5.1. *Loops, inverse loops and loop products are preserved under continuous functions.*

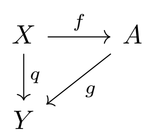
Proof. Let γ be a loop and f a continuous function .
Loops are continuous functions. Since the composite of continuous functions is continuous, we have that $f \circ \gamma$ is continuous. Since f is a function, we have $\gamma(0) = \gamma(1) \implies f \circ \gamma(0) = f \circ \gamma(1)$. Thus $f \circ \gamma$ is a loop.

$\gamma^{-1}(t)$ is defined as $\gamma(1-t)$. Thus $(f \circ \gamma)^{-1}(t)$ is defined as $f \circ \gamma(1-t)$ and is thus an inverse of $f \circ \gamma(t)$.

Furthermore, if $\gamma_1 * \gamma_2$ is a product of loops then since f is a function it has that $f(0) = f(\frac{1}{2}) = f(1)$. Since f is continuous, $f \circ (\gamma_1 * \gamma_2)$ is again a product of loops. □

Proposition 5.2. *Functions can't break loops.*

Proof. Assume we have the a diagram that satisfies the properties given in Definition 3.11:



Assume f^{-1} is a function and breaks the loop γ. Thus, there is lift of $g(\gamma)$ which is not a loop. Call this path $\tilde{\gamma} : [0,1] \to X$. Since the diagram commutes we have $g(\gamma) = q(\tilde{\gamma}) = g(f(\tilde{\gamma}))$. By the injectivity of g we have $f(\tilde{\gamma}) = \gamma$. Since f^{-1} is a function, this gives a contradiction. □

In the upcoming few proofs we will start to explicitly work with nth roots. As previously noted, roots are not functions. Luckily though, they are the inverse of a covering map.

Proposition 5.3. *The map $p : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$, $x \mapsto x^n$ is a covering map.*

Proof. We have the canonical isomorphism $\mathbb{C}\backslash\{0\} \cong S^1 \times (0,\infty)$. Our map p then splits into the map

$$\tilde{p} : S^1 \times (0,\infty) \to S^1 \times (0,\infty),$$

$$(e^{ix}, r) \mapsto (e^{ixn}, r^n).$$

This map already been shown to be a covering map on its first component in example 3.9. The second component is a homeomorphism. Combined this implies that p is a covering map.

□

By Lemma 3.10 this allows us to define what paths are under radicals.

Definition 5.4. *Let γ be a path in $\mathbb{C}\backslash\{0\}$. Then $\sqrt[n]{\gamma}$ is defined as the set of lifts of γ by the covering map from Proposition 5.3. This is called the **radical** of the path γ*

Proposition 5.5. *Let γ_1 and γ_2 be paths with $\gamma_1(1) = \gamma_2(0)$. Then for each lift $\sqrt[n]{\gamma_1}$ there is a lift $\sqrt[n]{\gamma_2}$ such that $\sqrt[n]{\gamma_1}(1) = \sqrt[n]{\gamma_2}(0)$. We define $\sqrt[n]{\gamma_1} * \sqrt[n]{\gamma_2}$ as the concatenation of these corresponding lifts.*

Proof. Since $(\sqrt[n]{\gamma_1}(1))^n = \gamma_2(0)$, Lemma 3.10 guarantees that such a path exists. □

Definition 5.6. *By $\sqrt[n]{\gamma}^{-1}$ we denote the set of lifts of γ traversed in reverse.*

This definition combined with the previous proposition is enough to make sense of the radicals of any arbitrary composition of paths. It should be clear how to make sense of the radical of a commutator loop. When we wish to discuss nested radicals of paths, we look at the set of lifts of each of the lifts that the previous radical provided.

Note that we thus far haven't shown that radicals break a given set of loops. Nor have we provided a method to order the set of lifts that radicals provide. Doing so would be an unnecessary hassle. The only thing that is relevant to us is that unlike the other allowed operations, the framework for loop-breaking is here. This leads to the following:

Proposition 5.7. *If our expression $F(p)$ contains a single radical, that if that radical can't break a loop, the whole function can't.*

Proof. Assume our expression is of the form $f(\sqrt[n]{g(p)})$. Since g is an algebraic expression that does not contain radicals, it is the composition of (continuous) functions. Thus, if p traces a loop, so does $g(p)$ (moreover, composition of loops is maintained as well). The same argument holds for f; if $\sqrt[n]{g(p)}$ traces a loop, so does f, for it is again a composition of (continuous) functions. Thus, if the radical is incapable of breaking loops, so is the whole expression. □

Remark. Note that we abused notation in this proof. $\sqrt[n]{g(p)}$ is technically a set of lifted points rather than an individual element. We of course mean to apply f to the elements in this set and return the set of elements by doing so.

5.2 Commutator lifting

Lemma 5.8. *Let γ be a commutator loop in $\mathbb{C}\backslash\{0\}$. Then all lifts $\sqrt[n]{\gamma}$ are still a loops.*

Proof. Let γ_1 and γ_2 be loops based at the same point. We have to show that $\sqrt[m]{[\gamma_1, \gamma_2]}$ are still loops.

Pick one of the lifts of $\sqrt[m]{[\gamma_1,\gamma_2]}$ and name it $\hat{\gamma}$. We have $\hat{\gamma}:[0,1]\to\mathbb{C}\backslash\{0\}$. Since γ_1 and γ_2 are within $\mathbb{C}\backslash\{0\}$, we can write them in polar coordinates with a radial and angular coordinate. As mentioned prior, the principal root is a continuous function on the positive reals and the radial component is thus guaranteed to preserve loops. The angular component, however, requires a bit more work.
To work it out explicitly, take

$$\begin{aligned}\gamma_1(t) &= r_1(t)\exp[i\theta_1(t)],\\ \gamma_2(t) &= r_2(t)\exp[i\theta_2(t)],\\ \Delta\theta_1(t) &= \theta_1(t)-\theta_1(0),\\ \Delta\theta_2(t) &= \theta_2(t)-\theta_2(0).\end{aligned}$$

(Thus with $\Delta\theta$ we refer to the difference in the angular coordinate after time t on a path compared to the angular coordinate on the path's origin.) Since both γ_1 and γ_2 are loops based at the same point, we have

$$\begin{aligned}r_1(0)=r_1(1)&=r_2(0)=r_2(1),\\ \Delta\theta_1(1)&=2\pi n_1,\ n_1\in\mathbb{Z},\\ \Delta\theta_2(1)&=2\pi n_2,\ n_2\in\mathbb{Z}.\end{aligned}$$

It is now again visible why taking an m-th radical has the potential to break loops. Although it is guaranteed that $\sqrt[m]{r(0)}=\sqrt[m]{r(1)}$ as mentioned previously, $\Delta\theta(1)$ is generally not an integer multiple of $2\pi m$.

For the commutator loop, we have

$$\begin{aligned}[\gamma_1,\gamma_2](t) &= r_{12}(t)\exp[\theta_{12}(t)],\\ \Delta\theta_{12}(t) &= \theta_{12}(t)-\theta_{12}(0),\end{aligned}$$

where as you would expect

$$(r_{12}(t),\theta_{12}(t))=\begin{cases}(r_1(4t),\theta_1(4t)), & 0\le t\le\frac{1}{4},\\ (r_2(4t-1),\theta_2(4t-1)), & \frac{1}{4}\le t\le\frac{1}{2},\\ (r_1(3-4t),\theta_1(3-4t)), & \frac{1}{2}\le t\le\frac{3}{4},\\ (r_2(4-4t),\theta_2(4-4t)), & \frac{3}{4}\le t\le 1.\end{cases}$$

Now, since $\Delta\theta_{\gamma*\tilde{\gamma}}=\Delta\theta_\gamma+\Delta\theta_{\tilde{\gamma}}$ and $\Delta\theta_{\gamma^{-1}}=-\Delta\theta_\gamma$ by definition, we have $\Delta\theta_{12}(1)=0$. Then for $j\in\{0,\dots,m-1\}$

$$
\begin{aligned}
\sqrt[m]{[\gamma_1,\gamma_2]}(1) &= \sqrt[m]{r_{12}(1)\exp[i\theta_{12}(1)]}, \\
&= \sqrt[m]{r_{12}(1)}\exp\left[i\frac{\theta_{12}(1)}{m}+\frac{2\pi j}{m}\right], \\
&= \sqrt[m]{r_{12}(1)}\exp\left[i\frac{\theta_{12}(0)}{m}+\frac{2\pi j}{m}+i\frac{\Delta\theta_{12}(1)}{m}\right], \\
&= \sqrt[m]{r_{12}(0)}\exp\left[i\frac{\theta_{12}(0)}{m}+\frac{2\pi j}{m}+0\right], \\
&= \sqrt[m]{[\gamma_1,\gamma_2]}(0).
\end{aligned}
$$

□

Corollary 5.8.1. *An algebraic equation containing only a single radical can't break all loops.*

Proof. This is an immediate consequence of Lemma 4.5, Proposition 5.7, and Lemma 5.8. □

6 Nested commutator radicals

We have shown that under the hypothetical expression F(p), the commutators of loops induce the commutators of permutations, some of which are non-trivial. In other words, commutator loops are not always still loops after passing through $F(p)$.
Moreover, we have shown that one radical alone does not break a commutator loop. A commutator loop under a radical is maybe no longer of the form of a commutator loop, but it is certainly still a loop.
To finish of the proof, we now only need to prove three more claims:

1. The radical of a commutator loop of commutator loops is again a commutator loop.
2. The previous can be continued inductively.
3. Taking all commutators of the set of commutator permutations of S_5 returns the set of commutators of S_5 (which will be shown numerically).

Combining these results shows that we can find commutator loops that induce permutations and are nested an arbitrary N layers of commutator loops deep. Thus, any arbitrary nested amount of radicals will not be able to break all loops and turn them into permutations, despite our hypothetical F(p) having to do so.

Thus, no algebraic expression containing only finite radicals can ever be a solution to the quintic polynomials, which finishes the proof.

We will now cover the former two claims, and then prove the latter claim in section 6.

Lemma 6.1. *Let $\gamma_i, \gamma_j, \gamma_k$, and γ_l be loops in $\mathbb{C}\backslash\{0\}$. The radical of a loop of the form $[[\gamma_i, \gamma_j], [\gamma_k, \gamma_l]]$ is still a commutator loop.*

Proof. let's say γ_1 and γ_2 are commutator loops. Then

$$\begin{aligned}\sqrt[n]{[\gamma_1, \gamma_2]} &= \sqrt[n]{\gamma_1 * \gamma_2 * \gamma_1^{-1} * \gamma_2^{-1}}, \\ &= \sqrt[n]{\gamma_1} * \sqrt[n]{\gamma_2} * \sqrt[n]{\gamma_1}^{-1} * \sqrt[n]{\gamma_2}^{-1}, \\ &= [\sqrt[n]{\gamma_1}, \sqrt[n]{\gamma_2}].\end{aligned}$$

Since γ_1 and γ_2 are commutator loops, their radicals are loops. This means $[\sqrt[n]{\gamma_1}, \sqrt[n]{\gamma_2}]$ is a commutator loop. □

Corollary 6.1.1. *Let $\hat{\gamma}$ be a loop of the form of N layers of nested commutator loops. Then the lifts*

$$\sqrt[n_1]{} \circ \cdots \circ \sqrt[n_N]{} \circ \hat{\gamma}$$

are still a loops.

Proof. We've already shown this for cases up to $N = 2$, so only the induction step remains.
Let $\hat{\gamma} = [\gamma_1, \gamma_2]$ where γ_1, γ_2 are N-1 levels nested commutator loops. Let $h = \sqrt[n]{} \circ \tilde{h}$, where $\tilde{h}$ is a composition of N-2 radicals. Then

$$h(\hat{\gamma}) = \sqrt[n]{[\tilde{h}(\gamma_1), \tilde{h}(\gamma_2)]}.$$

Via the induction hypothesis, $\tilde{h}(\gamma_1)$ and $\tilde{h}(\gamma_2)$ are (sets of lifts of) loops. Since we know that the radical of a commutator loop is again a loop, $h(\hat{\gamma})$ is a loop. $\square$

7 Nested commutator permutations

I have written some simple code to show that any commutator permutation in S_5 can be written as an arbitrary nested number of commutator permutations. To be more specific, say $\mathcal{A}$ is the set of all commutator permutations of permutations in S_5.

$$\mathcal{A} = \{[\sigma_1, \sigma_2] \mid \sigma_1, \sigma_2 \in S_5\}.$$

Then taking the set of commutators of $\mathcal{A}$ returns $\mathcal{A}$.

$$\mathcal{A} = \{[\sigma_1, \sigma_2] \mid \sigma_1, \sigma_2 \in \mathcal{A}\}.$$

Thus, any commutator permutation can be written as an arbitrary amount of nested commutator permutations.
If you'd wish to check this yourself, running the code and using the command GenPermutations(5) returns the set S_5. The function "Multi_commutate" will return all permutations of elements in a list. The relevant behaviour for our proof is that the list Multi_commutate(Multi_commutate(GenPermutations(5))) contains the same elements as Multi_commutate(GenPermutations(5)). The python code is added in the appendix.

It is alternatively possible to use group theory instead of computation for this final argument. If one were to check the elements in this commutator subgroup, they could observe that it coincides with A_5. Alternating groups are not abelian, and one could further prove that A_5 is simple and thus does not contain any normal subgroups besides the trivial group and itself. Combined, this implies that $[A_5, A_5] = A_5$, which leads to the same conclusion.

Proof of Theorem 2.1. Choose $n > 1$. The previous result shows that it is possible to find a non-trivial permutation σ on the roots of any quintic polynomial with distinct roots, such that σ can be written as n nested layers of commutators of non-trivial permutations. Lemma 4.5 and Corollary 4.2.1 imply that there is a commutator loop made from n nested layers of commutator loops that induce that permutation.

Corollary 6.1.1 tells us that any solution for the quintics would then require $n + 1$ nested radicals to break this loop. Since n is arbitrary, this finishes the proof. □

8 Appendix: Python code

```
import itertools

#Generate S5
def GenPermutations(n):
    set = [i for i in range(1,n+1)]
    permutations = []
    for i in itertools.permutations(set,n):
        permutations.append(list(i))
    return(permutations)


#Multiplication
def Perm_product(a,b):
    return ([a[b[i]-1]
            for i in range(len(a))
            ])

def Perm_multi_product(list):
   N   = len(list)
    N_0 = len(list[0])
    new_list       = [i+1 for i in range(N)]
    new_list[N-1] = [i+1 for i in range(N_0)]
    for i in range(N):
        new_list[i] = Perm_product(list[i],new_list[i-1])
    return new_list[-1]

#Inverse permutation
def Perm_inverse(a):
    b = [0 for i in a]
    for i in range(len(a)):
        b[a[i]-1] = i + 1
    return b


#Remove duplicates
def Remove_dupes(A):
    List = []
    for i in A:
        if i not in List:
            List.append(i)
    return List
```

```
#Commutator
def Commutate(a,b):
    list =  [
            a,                  b,
            Perm_inverse(a),    Perm_inverse(b)
            ]
    return  Perm_multi_product(list)



def Multi_commutate(list):
    b=[]
    for i in list:
        for j in list:
            b.append(Commutate(i,j))
    b = Remove_dupes(b)
    return b
```

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