On the Topology of Anyons

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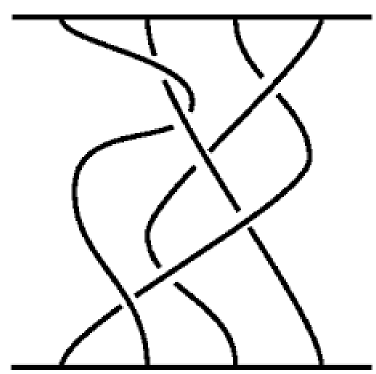

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1 Introduction

In 1977 J.M. Leinaas and J. Myrheim [1] wrote an article about many-body quantum systems with several indistinguishable particles. The notion of symmetry and anti-symmetry of the wave function with respect to the interchanging of two particle coordinates was already consistent with experiment and was physically reasonably well understood. The proper theoretical and mathematical framework however, was not to their satisfaction. The original idea in forming a wave function of a many-particle system is that the physics should not change as two particles are interchanged. This can be expressed as following:

$$|\Psi p(x_1, x_2, \ldots, x_n)|^2 = |\Psi(x_1, x_2, \ldots, x_n)|^2.$$

In this equation p is the permutation-operator that permutes the n variables. This however doesn't have any physical meaning as it is completely unclear what interchanging or permuting two indistinguishable particles means. The only physical consequence that we get out of this notion has been experimentally verified:

$$\Psi p(x_1, x_2, \ldots, x_n) = \pm\Psi(x_1, x_2, \ldots, x_n).$$

However we are nowhere closer to understanding the physics and mathematics behind permuting particles. To get a better understanding of the physics behind the phenomenon we need to take a look at the configuration space $\mathcal{C}_n$ of n particles. We will see that this space is not the same as the Cartesian product of the single particle spaces, certainly when the particles are indistinguishable. In the indistinguishable case we will need to take a look at the action of the symmetric group on $\mathcal{C}_n$.
To get a better understanding of what permuting particles in a many-particle system means we have to consider the fundamental group of the configuration system. The proposition of this thesis is:

$$\pi_1(\mathcal{C}_{\mathbb{R}^d,n}/\mathcal{S}_n) \cong \begin{cases} \mathcal{S}_n & \text{for } d \geq 3 \\ \mathcal{B}_n & \text{for } d = 2. \end{cases}$$

From the equation it is clear that we are considering the Euclidean space in d dimensions. The main focus of this thesis will be the case $d = 2$ as this will give rise to some new physics that do not exist in the 3-dimensional case. This will be a consequence of the properties of the Artin braid group $\mathcal{B}_n$. In the case $d = 2$ the indistinguishable particles will be named Anyons, a term coined by Frank Wilczek in 1982, as they can have any phase when interchanged. This will further be explored in Section 5.

1.1 Structure

In this thesis we will first take a look at what permuting particles means in a topological sense. This will give rise to the question to what the Artin braid group is and how this group can be expressed both algebraically and geometrically. This will be mostly done with help of Christian Kassel and Vladimir Turaev's book: Braid Groups [2].
After this we will still need to prove that the exact definition we find is indeed isomorphic to the fundamental group of the configuration space of n indistinguishable particles in two dimensions.
Finally we will look at some interesting physical consequences that have been a good motivation for me to dive deeper into this subject.

I hope you enjoy reading this as much as I enjoyed studying the subject.

2 Topology of many-particle systems

In this section we will look at the topological properties of a many-particle system. In the quantum case of a identical many-particle system these particles will be indistinguishable. This means we will have to take a look at the full configuration space of all these particles together. We will define what a configuration space is and try to motivate the usage of it as a good model to describe the physics behind permuting particles in the identical many-particle system.

2.1 Configuration Spaces

In this section we will be looking at the configuration space of particles in a d-dimensional Euclidean space. We want our definition to exclude all possible configurations where two particles occupy the same space as we can assume our particles are hard-core.

Definition 2.1. *Let $\mathcal{C}_{\mathbb{R}^d,n}$ be the configuration space of n particles in $\mathbb{R}^d$:*

$$\mathcal{C}_{\mathbb{R}^d,n} := \{(x_1, x_2, \ldots, x_n) \in (\mathbb{R}^d)^n \mid \forall i \neq j, x_i \neq x_j\}.$$

This definition gives us the configuration space of n distinguishable particles For any such n-tuple in $\mathcal{C}_{\mathbb{R}^d,n}$ we can look at the group action of the symmetric group $\mathcal{S}_n$ on this n-tuple. The group action gives an equivalence relation between the permuted n-tuples and the equivalence classes under this group action are exactly the unordered n-tuples in the configuration space or in the physical sense indistinguishable particles. This motivates us to look at the quotient space: $\mathcal{C}_{\mathbb{R}^d,n}/\mathcal{S}_n$.

Definition 2.2. *Let $\widehat{\mathcal{C}}_{\mathbb{R}^d,n}$ be the configuration space of n indistinguishable particles in $\mathbb{R}^d$:*

$$\widehat{\mathcal{C}}_{\mathbb{R}^d,n} := \mathcal{C}_{\mathbb{R}^d,n}/\mathcal{S}_n.$$

Now it becomes apparent that we need clear idea of what it means to permute or interchange particles. To achieve a permutation or swap of two particles we want them to complete paths in their d dimensional space, starting at their original position and ending at the position the particle is permuted with. In the configuration space this means that we want our n-tuple to complete a path $\gamma[0,1]$, ending at the desired permutation of the n-tuple. In our indistinguishable case those paths will even be loops as the symmetric group action gives us an equivalence between the beginning and end of the path. This is why we consider the fundamental group of $\widehat{\mathcal{C}}_{\mathbb{R}^d,n}$. The physical consequences of completing these paths in 2 dimensions will be discussed in Chapter 5.
Chapter 3 and 4 will prove the proposition of the thesis starting at defining the Artin braid group and slowly painting the picture of why braids are the same as loops in the configuration space of 2-dimensional indistinguishable particles.

3 The Artin braid group

In this section we will give an algebraic and a geometric definition of the braid group. We will then try to construct a proof to show that both definitions give isomorphic groups.

3.1 Algebraic definition

Definition 3.1. *The Artin braid group* $\mathcal{B}_n$ *is the group generated by* $n-1$ *generators* $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ *and the braid relations*

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all $i, j \in \{1, 2, \ldots, n-1\}$ *satisfying* $\mid i-j \mid \geq 2$*, and*

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for all $i \in \{1, 2, \ldots, n-2\}$.

Lemma 3.2. *If* $s_1, \ldots, s_{n-1}$ *are elements of a group* G *satisfying the braid relations, then there is a unique homomorphism* $f : \mathcal{B}_n \to G$ *such that* $s_i = f(\sigma_i)$ *for* $i \in \{1, 2, \ldots, n-1\}$.

Proof. Every element of $\mathcal{B}_n$ can be written as $\sigma_{i_1}^{\pm 1} \sigma_{i_2}^{\pm 1} \ldots \sigma_{i_k}^{\pm 1}$, a product of σ_i's and their inverses. Define f by $f(\sigma_{i_1}^{\pm 1} \sigma_{i_2}^{\pm 1} \ldots \sigma_{i_k}^{\pm 1}) = s_{i_1}^{\pm 1} s_{i_2}^{\pm 1} \ldots s_{i_k}^{\pm 1}$. This is well defined because all s_i satisfy the braid relations. □

If we apply this lemma to the symmetric group $\mathcal{S}_n$, generated by the simple transpositions $s_1, \ldots, s_{n-1} \in \mathcal{S}_n$, where s_i permutes i and $i+1$ and leaves the other elements fixed, we can see there is unique surjective group homomorphism $\pi : \mathcal{B}_n \to \mathcal{S}_n$ such that $s_i = \pi(\sigma_i)$ for $i \in \{1, 2, \ldots, n-1\}$. This holds because all s_i satisfy the braid relations.

From this surjective homomorphism it is clear we can see $\mathcal{B}_n$ as the permutation group with weaker relations. It is for this reason we want to start looking at a space where we permute points over a time interval to get a better geometric idea of the braid group. We will be doing exactly that in the next section.

3.2 Geometric definition

From now on I will denote the closed interval $[0, 1]$ in the set of real numbers $\mathbb{R}$. We also define a *topological interval* as a topological space homeomorphic to $I = [0, 1]$.

Definition 3.3. *A geometric braid on $n \geq 1$ strings is a set $b \subset \mathbb{R}^2 \times I$ formed by n disjoint topological intervals called the strings of b such that the projection $\mathbb{R}^2 \times I \to I$ maps each string homeomorphically onto I and*

$$b \cap (\mathbb{R}^2 \times \{0\}) = \{(1, 0, 0), (2, 0, 0), \ldots, (n, 0, 0)\},$$

$$b \cap (\mathbb{R}^2 \times \{1\}) = \{(1, 0, 1), (2, 0, 1), \ldots, (n, 0, 1)\}.$$

From this definition we can get an idea of what geometric braid will look like. It is clear that every string of braid b connects a point $(i, 0, 0)$ to a point $(s(i), 0, 1)$ for $i, s(i) \in \{1, 2, \ldots, n\}$. Because all begin and end points are unique we can speak of the *underlying permutation* of b which is the permutation of the sequence $(1, 2, \ldots, n)$ to $(s(1), s(2), \ldots, s(n))$. Now we need an idea of an isotopy between geometric braids so we can start constructing a group. We can do this by the standard definition of an isotopy.

Definition 3.4. *Two geometric braids b and b' are isotopic if there is a continuous map $F : b \times I \to \mathbb{R}^2 \times I$ such that for each $s \in I$ the map $F_s : b \to \mathbb{R}^2 \times I$ sending all $x \in b$ to $F(x, s)$ is and embedding whose image is a geometric braid on n strings, $F_0 = \mathrm{id}_b : b \to b$, and $F_1(b) = b'$. Every endpoint of b is mapped to itself by each F_s. Both the map F and the family of geometric braids $\{F_s(b)\}_{s \in I}$ are called an isotopy of $b = F_0(b)$ into $b' = F_1(b)$.*

This definition of isotopy gives rise to an equivalence relation on the class of geometric braids on n strings. The corresponding equivalence classes are called *braids on n strings*. These braids are denoted by β. Next we want to define multiplication on this set of braids on n strings to start the construction of a group.

Definition 3.5. *Let $b_1, b_2 \subset \mathbb{R}^2 \times I$ be two n-string geometric braids. Their product $b_1 b_2$ is defined as the set of points $(x, y, t) \in \mathbb{R}^2 \times I$ such that for $0 \leq t \leq 1/2$, $(x, y, t) \in b_1 b_2 := (x, y, 2t) \in b_1$ and for $1/2 \leq t \leq 1$, $(x, y, t) \in b_1 b_2 := (x, y, 2t - 1) \in b_2$.*

We can see that $b_1 b_2$ is also a geometric braid on n strings. It is also not hard to check that if $b_1 \sim b_1'$ and $b_2 \sim b_2'$ then $b_1 b_2 \sim b_1' b_2'$ with '$\sim$' being equivalence by isotopy. This tells us that the defined product defines a multiplication on the set of braids on n strings.

This multiplication is associative and has a neutral element which is the *trivial braid* $\mathbb{1}_n$ that can be represented by the geometric braid

$$\{1, 2, \ldots, n\} \times \{0\} \times I \subset \mathbb{R}^2 \times I.$$

Now we want to show that the set of braids on n strings with this multiplication is a group and that this group is canonically isomorphic to the Artin braid group $\mathcal{B}_n$.

Let us first try to construct a group out of the set of braids on n strings with the multiplication operation. It remains to prove the existence of inverses. To make this easier we will look at a representation of braids on n strings called the braid diagrams.

3.3 Braid diagrams

Definition 3.6. *A braid diagram on n strands is a set $\mathcal{M} \subset \mathbb{R} \times I$ split as union of n topological intervals called the strands of $\mathcal{M}$ such that the following three conditions are met:*

i The projection $R \times I \to I$ maps each strand homeomorphically onto I.

ii Every point of $\{1, 2, \ldots, n\} \times \{0, 1\}$ is the endpoint of a unique strand.

iii Every point of $\mathbb{R} \times I$ belongs to at most two strands. At each intersection point of two strands, these strands meet transversely, and one of them is distinguished and said to be undergoing, the other strand being overgoing.

In the figures the undergoing strand is represented by a broken line near the crossing; the overgoing strand by a continued line. In my figures the bottom horizontal line will always represent $\mathbb{R} \times \{0\}$, and the top horizontal line will represent $\mathbb{R} \times \{1\}$.

Now we will describe the relationship between braids on n strings and braid diagrams. Each braid diagram $\mathcal{M}$ presents an isotopy class of geometric braids as follows. We can use the identification $\mathbb{R} \times I = \mathbb{R} \times \{0\} \times I \subset \mathbb{R}^2 \times I$ and this is the plane our braid diagram $\mathcal{M}$ lies on. Taking a look at a small neighbourhood around the crossings of M, we can push the undergoing strand into $\mathbb{R} \times (0, +\infty) \times I$ by only increasing the second coordinate and leaving the other two unchanged. This transforms $\mathcal{M}$ into a geometric braid on n strings. The isotopy class of this braid being the braid on n strings presented by $\mathcal{M}$. This braid is denoted by $\beta(\mathcal{M})$. To get from a braid β to a braid diagram pick a braid b that represents β that only has double transversal crossings in the projection of b to $\mathbb{R} \times \{0\} \times I$. At each crossing point of this projection choose the undergoing strand to be the one that was projected by the subarc of b with larger second coordinate. Now this projection of b yields a braid diagram, $\mathcal{M}$, and we can see that $\beta(\mathcal{M}) = \beta$.

3.3.1 Isotopies between braid diagrams

Because we will be working with braid diagrams and the braids β they represent for the final proofs we need to show how the isotopies between braids on n strings and isotopies between braid diagrams correlate. To obtain the right we need a theorem with a very long and precise proof. I will omit this proof but the reader can find it themselves in Braid Groups by Kassel and Turaev. First we define isotopy between braid diagrams.

Definition 3.7. *Two braid diagrams $\mathcal{M}$ and $\mathcal{M}'$ on n strands are said to be isotopic if there is a continuous map $F : \mathcal{M} \times I \to \mathbb{R} \times I$ such that for all $s \in I$ the set $\mathcal{M}_s = F(\mathcal{M} \times s)$ is a braid diagram on n strands, $\mathcal{M}_0 = \mathcal{M}$, and $\mathcal{M}_1 = \mathcal{M}'$. The family of braid diagrams $\{\mathcal{M}_s\}_{s\in I}$ is called an isotopy of $\mathcal{M}_0 = \mathcal{M}$ into $\mathcal{M}_1 = \mathcal{M}'$.*

Even if the reader wasn't familiar with isotopies before this thesis this definition should look similar to the isotopy between geometic braids. To understand their connection we need to introduce the idea of Reidemeister moves and R-equivalence.

We will be looking at the same famous Reidemeister moves from knot theory. As we don't have any loops in our braids because of their projection onto I we will not need the first Reidemeister moves. The second and third Reidemeister moves denoted by Ω_2 and Ω_3 are shown in the figures below. The inverse of them being Ω_2^{-1} and Ω_3^{-1} by reversing the arrow in the figures. Reidemeister moves are known to preserve the corresponding braids up to isotopy. I will not prove this in this thesis. We can now complete the definition of R-equivalence and after state the needed theorem.

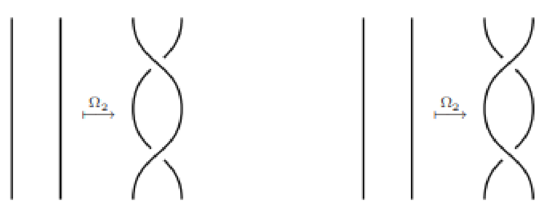

Figure 1: The two different Reidemeister moves Ω_2 from [2]

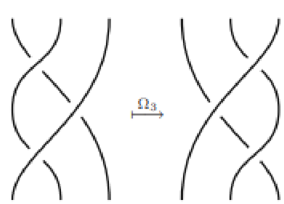

Figure 2: Reidemeister move Ω_3 from [2]

Definition 3.8. *Two braid diagrams $\mathcal{M}$, $\mathcal{M}'$ are R-equivalent if $\mathcal{M}$ can be transformed into $\mathcal{M}'$ by a finite sequence of isotopies and Reidemeister moves* $\Omega_2^{\pm 1}, \Omega_3^{\pm 1}$

Theorem 3.9. *Two braid diagrams present isotopic geometric braids if and only if these diagrams are R-equivalent.*

The proof can be read under Theorem 1.6. in [2].

3.4 The group of braids on n strings

We will now be looking at the set of braids on n strings with multiplication as defined before. With this multiplication the set is a monoid. This monoid will be denoted as $\mathbb{B}_n$. The last thing we need to do is construct a group by finding an inverse for every element.

Lemma 3.10. *Each $\beta \in \mathbb{B}_n$ has a two-sided inverse $\beta^{-1} \in \mathbb{B}_n$.*

Proof. First we need the elementary diagrams σ_i^+ and σ_i^- as shown below.

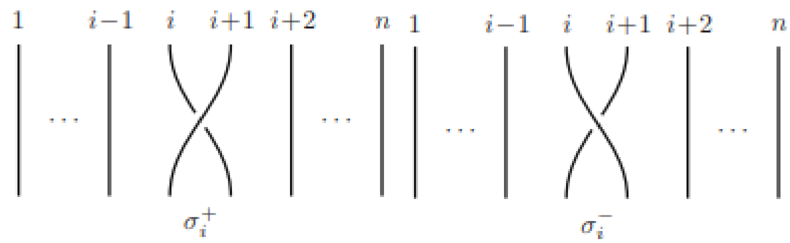

Figure 3: Elementary braids from [2]

They are the diagrams with only one crossing between the i-th and the (i+1)-th string. We claim that $\mathbb{B}_n$ is generated as a monoid by the braids $\sigma_1^+, \ldots, \sigma_{n-1}^+, \sigma_1^-, \ldots, \sigma_{n-1}^-$. Consider a braid β on n strings represented by a braid diagram $\mathcal{M}$. We can find an isotopic braid diagram $\mathcal{M}'$ of this braid diagram $\mathcal{M} \in \mathbb{R} \times I$ that has all its crossings on distinct second coordinates. Then we have a sequence of real numbers

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1,$$

such that the intersection of $\mathcal{M}'$ with $\mathbb{R} \times [t_j, t_{j+1}]$ has exactly one crossing in this intersection. We can conclude this intersection is exactly a diagram σ_i^+ or σ_i^- for an $i \in \{1, 2, \ldots, n-1\}$. This result shows that every braid β can be written as a product of k elementary braids as

$$\beta = \beta(\mathcal{M}) = \beta(\mathcal{M}') = \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \ldots \sigma_{i_k}^{\epsilon_k},$$

where each $\epsilon_j \in \{+, -\}$ and all $i \in \{1, 2, \ldots, n-1\}$. We can see from the diagrams that $\sigma_i^+ \sigma_i^- = \sigma_i^- \sigma_i^+ = 1$. This idea makes it easy to construct a two-sided inverse of β in $\mathbb{B}_n$, namely

$$\beta^{-1} = \sigma_{i_k}^{-\epsilon_k} \ldots \sigma_{i_2}^{-\epsilon_2} \sigma_{i_1}^{-\epsilon_1}.$$

□

With the group $\mathbb{B}_n$ constructed we can start constructing a homomorphism between $\mathbb{B}_n$ and $\mathcal{B}_n$. Because we want to use Lemma 3.2 we still need to show that the elementary elements of $\mathbb{B}_n$ satisfy the braid relations.

Lemma 3.11. *The elements $\sigma_1^+, \dots, \sigma_{n-1}^+ \in \mathbb{B}_n$ satisfy the braid relations: $\sigma_i^+\sigma_j^+ = \sigma_j^+\sigma_i^+$ for all $i, j \in \{1, 2, \dots, n-1\}$ with $|i-j| \geq 2$, and $\sigma_i^+\sigma_{i+1}^+\sigma_i^+ = \sigma_{i+1}^+\sigma_i^+\sigma_{i+1}^+$ for $i \in \{1, 2, \dots, n-2\}$.*

Proof. The first relation gives two isotopic braid diagrams. The second relation gives two braid diagrams differing by the Reidemeister move Ω_3. □

Theorem 3.12. *For $\epsilon \in \{+, -\}$, there is a unique homomorphism $\phi_\epsilon : \mathcal{B}_n \to \mathbb{B}_n$ such that $\phi_\epsilon(\sigma_i) = \sigma_i^\epsilon$ for all $i \in \{1, \dots, n-1\}$. The homomorphism ϕ_ϵ is an isomorphism.*

Proof. Wihout loss of generality we take $\epsilon = +$. From Lemma 3.11 we can use Lemma 3.2 to show the existence and uniqueness of ϕ_+. We have also shown from Lemma 3.10 and its proof that $\sigma_1^+, \dots, \sigma_{n-1}^+$ generate $\mathbb{B}_n$ as a group. This makes ϕ_+ surjective. We can show ϕ_+ is bijective by constructing the map $\psi : \mathbb{B}_n \to \mathcal{B}_n$ such that $\psi \circ \phi_+ = \text{id}$. For the construction we need to represent any $\beta \in \mathbb{B}_n$ with a braid diagram $\mathcal{M}$ which crossings have distinct second coordinates. This will give an expansion of $\mathcal{M}$ in elementary braids $\mathcal{M} = \sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\dots\sigma_{i_k}^{\epsilon_k}$ Now we can construct ψ as following:

$$\psi(\mathcal{M}) = \psi(\sigma_{i_1}^{\epsilon_1}\sigma_{i_2}^{\epsilon_2}\dots\sigma_{i_k}^{\epsilon_k}) = (\sigma_{i_1})^{\epsilon_1}(\sigma_{i_2})^{\epsilon_2}\dots(\sigma_{i_k})^{\epsilon_k},$$

where $(\sigma_i)^+ = \sigma_i$ and $(\sigma_i)^- = \sigma_i^{-1}$.
Now we have to show that the image of $\mathcal{M}$ depends only on the braid β, so it should not change under isotopies of $\mathcal{M}$ as any braid diagram isotopic to $\mathcal{M}$ represents the same braid as we have seen before. Only isotopies that affect the order or amount of elementary braids $\{\sigma_{i_1}, ..., \sigma_{i_k}\}$ might result in a different image so there are three isotopies to check.
First we can change the order of two crossings of $\mathcal{M}$ as long as there are no crossings in between the two and no single strand will be in both crossings. This will replace the term $\sigma_{i_j}^{\epsilon_j}\sigma_{i_{j+1}}^{\epsilon_{j+1}}$ with $\sigma_{i_{j+1}}^{\epsilon_{j+1}}\sigma_{i_j}^{\epsilon_j}$ for $i_j, i_{j+1} \in \{1, 2, \dots, n-1\}$ and $|i_j - i_{j+1}| \geq 2$. Under ψ this new braid diagram and $\mathcal{M}$ will be sent to the same element of $\mathcal{B}_n$ because of the first braid relation of Definition 3.1.
We can also apply the Reidemeister moves Ω_2 and Ω_3 to $\mathcal{M}$ which give isotopic braid diagrams as seen before.
For Ω_2 on $\mathcal{M}$ this will result in adding a term $\sigma_i^+\sigma_i^-$ for an $i \in \{1, \dots, n-1\}$ somewhere in the expansion of $\mathcal{M}$. This will not result in a different image under ψ because this term will cancel itself in the image.
Lastly Ω_3 will replace a $\sigma_i^+\sigma_{i+1}^+\sigma_i^+$ in the expansion of $\mathcal{M}$ with $\sigma_{i+1}^+\sigma_i^+\sigma_{i+1}^+$. This will again result in the same image under ψ because of the second braid relation of Definition 3.1. The same argument holds for Ω_3^{-1}. Now we have shown ψ is a well-defined map and by construction the inverse of ϕ_+. So ϕ_+ is an isomorphism. □

With this result we can start trying to find an isomorphism between the geometric braids and the fundamental group of $\hat{\mathcal{C}}_{\mathbb{R}^d,n}$. This will be done in the next section.

4 The Fundamental group of $\mathcal{C}_{\mathbb{R}^d,n}$ and $\hat{\mathcal{C}}_{\mathbb{R}^d,n}$

In this section we will first take a look at the fundamental group of $\mathcal{C}_{\mathbb{R}^d,n}$ as defined in Definition 2.1 and the connection to the fundamental group of $\hat{\mathcal{C}}_{\mathbb{R}^d,n}$ as defined in Definition 2.2. After this the isomorphism between the fundamental group of $\hat{\mathcal{C}}_{\mathbb{R}^2,n}$ and $\mathbb{B}_n$ can be proven.

4.1 The Pure Braid group

Definition 4.1. *The pure braid group is the kernel of the natural homomorphism* $\pi : \mathbb{B}_n \to \mathcal{S}_n$ *and is denoted by* $\mathbb{P}_n$*. The elements of* $\mathbb{P}_n$ *are the pure braids on* n *strings.*

As we have already proven in chapter 3 that the Artin braid group and the geometric braid group are isomorphic, we can immediately express this definition geometrically without defining the exact generators in the Artin braid group. This will show more useful in proving the main theorem of the thesis.

Lemma 4.2. *A geometric pure braid of* n *strings represents an element of* $\mathbb{P}_n$ *if and only if for all* $i = 1, ..., n$*, the* i*-th string of the braid starting at* $(i, 0, 0) \in \mathbb{R}^2 \times I$ *has endpoint* $(i, 0, 1) \in \mathbb{R}^2 \times I$

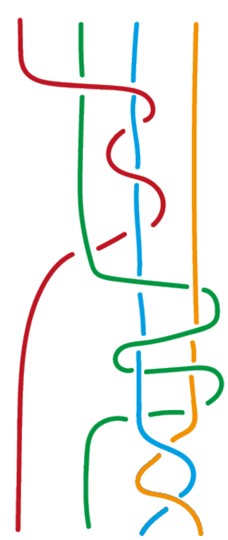

Figure 4: Coloured example of a pure braid

The kernel of the natural projection in Definition 4.1 are exactly the braids where the strings do not permute starting points to different end points.

Proposition 4.3. *The fundamental group of the configuration space of ordered n-tuples in $\mathbb{R}^2$ is isomorphic to the pure braid group.*

$$\pi_1(\mathcal{C}_{\mathbb{R}^2,n}) \cong \mathbb{P}_n$$

Proof. We construct the function $F : \mathbb{P}_n \to \pi_1(\mathcal{C}_{\mathbb{R}^2,n}, ((1,0),(2,0),\ldots,(n,0)))$. F maps the equivalence class under isotopy of any pure geometric braid $b \in \mathbb{R}^2 \times I$ to the equivalence class under homotopy of a path $\gamma : I \to \mathcal{C}_{\mathbb{R}^2,n}$. Here γ maps $t \in I$ to an ordered n-tuple $(u(t)) := ((u_1(t), u_2(t), \ldots, u_n(t))$ where $u_i(t)$ is the natural projection on $\mathbb{R}^2$ of the intersection of b_i and $\mathbb{R}^2 \times \{t\}$, with b_i being the i-th string of b. This path is continuous and $\gamma(0) = \gamma(1) = ((1,0),(2,0),\ldots,(n,0))$ so $[\gamma] \in \pi_1(\mathcal{C}_{\mathbb{R}^2,n}, ((1,0),(2,0),\ldots,(n,0)))$.

To show that this function is well defined we need to show that for any b_1, b_2 if $b_1 \sim b_2$, then $F(b_1) = [\gamma_1] = [\gamma_2] = F(b_2)$, so $\gamma_1 \sim \gamma_2$. If $b_1 \sim b_2$ then there exists a continuous function $f : b_1 \times I \to \mathbb{R}^2 \times I$ with $f(b_1, 0) = b_1$ and $f(b_1, 1) = b_2$. Per construction of γ there also exists a continuous function $\hat{f} : \gamma_1 \times I \to \mathcal{C}_{\mathbb{R}^2,n} \times I$ with $\hat{f}(\gamma_1, t) = \gamma_t$ with γ_t being the path constructed from the braid $f(b_1, t)$ as before. Because $\hat{f}(\gamma_1, 0) = \gamma_1$ and $\hat{f}(\gamma_1, 1) = \gamma_2$, γ_1 and γ_2 are homotopic.

Now it remains to show that F is a group homomorphism and F has an inverse.

To show that F is a group homomorphism we need to show that $F([b_1]\cdot[b_2]) = F([b_1]) \cdot F([b_2]) = [\gamma_1] \cdot [\gamma_2]$ for any $[b_1], [b_2] \in \mathbb{P}_n$. The product of two loops γ_1, γ_2 is defined as

$$(\gamma_1 \cdot \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

For any geometric pure braids b_1, b_2, $[b_1 \cdot b_2] = [b_1] \cdot [b_2]$ and for the loops γ_1, γ_2, $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ holds . This means it is enough to show that for any two geometric pure braids b_1, b_2 with their constructed loops γ_1 belonging to b_1 and γ_2 belonging to b_2 that the loop constructed out of $b_1 \cdot b_2 = \gamma_1 \cdot \gamma_2$.

As in Definition 3.5 we see that the loop constructed from $b_1 \cdot b_2$ will be $\gamma_1(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $\gamma_2(2t-1)$ for $\frac{1}{2} \leq t \leq 1$. This loop is equal to $(\gamma_1 \cdot \gamma_2)(t)$ which proves that F is a group homomorphism. For the inverse we construct an inverse $\hat{F} : \pi_1(\mathcal{C}_{\mathbb{R}^2,n}, ((1,0),(2,0),\ldots,(n,0))) \to \mathbb{P}_n$, which maps the equivalence class under homotopy of a path $\gamma : I \to \mathcal{C}_{\mathbb{R}^2,n}$ to the equivalence class under isotopy of any pure geometric braid $b \in \mathbb{R}^2 \times I$. Again γ maps $t \in I$ to an ordered n-tuple $(u(t)) := ((u_1(t), u_2(t), \ldots, u_n(t))$ and now b_i, the i-th string of the braid b, is equal to $\bigcup_{t \in I}(u_1(t), t)$. Because $\gamma(0) = \gamma(1) = ((1,0),(2,0),\ldots,(n,0))$, for any string b_i the start point is $(i,0,0)$ and endpoint is $(i,0,1)$, which makes $b := \bigcup_{i=1}^{n} b_i$ a pure braid. Showing that $\hat{F}$ is well defined is similar to showing that F is well defined. This will be omitted. Similarly showing that $\hat{F}$ is a group homomorphism again makes use of the fact that multiplying two loops is similar in construction as multiplying two braids. Per construction $\hat{F} \cdot F = \mathrm{id}_{\mathbb{P}n}$ and similarly $F \cdot \hat{F} = \mathrm{id}_{\pi_1(\mathcal{C}_{\mathbb{R}^2,n}, ((1,0),(2,0),\ldots,(n,0)))}$. This gives us the isomorphism $\pi_1(\mathcal{C}_{\mathbb{R}^2,n}, ((1,0),(2,0),\ldots,(n,0))) \cong \mathbb{P}_n$ and because $\mathcal{C}_{\mathbb{R}^2,n}$ is path connected the choice of starting point is irrelevant. ∎

4.2 The Artin Braid Group

With this proof we need a few more ingredients to construct the proof of the final proposition. First we have to prove we are working with a covering map, then we can make use of the lemmas stated after.

Definition 4.4. *Let X be a topological space and G a (discrete) group. An action of G on X is a map $G \times X \to X$, $(g, x) \to gx$ such that $x \to gx$ is continuous for each $g \in G$, and $(gh)x = g(hx)$ and $ex = x$ for all $g, h \in G$, $x \in X$. An action of G on X is called*

- *free if $g \neq e$ implies $gx \neq x$ for all $x \in X$,*
- *properly discontinuous if every $x \in X$ has a neighborhood V such that $gV \cap V = \emptyset \ \forall g \neq e$.*

Lemma 4.5. *Let X be a Hausdorff space and G a finite group. Then every free G-action on X is properly discontinuous.*

Proof. As the action of G on X is free this gives that for all $e \neq g \in G$ and $x \in X$ we have $gx \neq x$. Because X is Hausdorff we can find, for any $g \in G$, two open neighbourhoods U, W with $x \in U$, $gx \in W$ and with $U \cap W = \emptyset$. As each g is a continuous map, $V_g := U \cap g^{-1}(W)$ is also open and we see that $g(V_g) \cap V_g = \emptyset$. Now as G is finite we can construct the open neighbourhood $V := \cap_{g \in G} V_g$ which satisfies $gV \cap V = \emptyset \ \forall g \neq e$. □

Lemma 4.6. *The quotient map $q : \mathcal{C}_{\mathbb{R}^2,n} \to \widehat{\mathcal{C}}_{\mathbb{R}^2,n}$ is a covering map.*

Proof. The action of $\mathcal{S}_n$ on $\mathcal{C}_{\mathbb{R}^2,n}$ is free because any element of an n-tuple that gets permuted with another element of that n-tuple will change its coordinates, since $x_i \neq x_j \ \forall i \neq j$. As $\mathcal{C}_{\mathbb{R}^2,n}$ with its standard topology is a Hausdorff space and $\mathcal{S}_n$ is finite, we can use Lemma 4.5 to show that the action of $\mathcal{S}_n$ on $\mathcal{C}_{\mathbb{R}^2,n}$ is properly discontinuous. Now Theorem 13.7.25 of [3] shows us that q is a covering map. □

Lemma 4.7. *Let $p : \widehat{X} \to X$ be a covering map, $f : I \to X$ a path, and $\widehat{x_0} \in p^{-1}(f(0))$. Then there is a unique continuous lift $\hat{f} : I \to \widehat{X}$ of f such that $\hat{f}(0) = \widehat{x_0}$.*

Lemma 4.8. *Let $p : \widehat{X} \to X$ be a covering map. Assume f, f' are paths in X from x_0 to x_1 and h is a path-homotopy from f to f'. If $\widehat{x_0} \in p^{-1}(x_0)$ and $\hat{f}, \hat{f}'$ of f and f' beginning at $\widehat{x_0}$, then $\hat{f}(1) = \hat{f}'(1)$ and h lifts to a homotopy $\hat{h}$ from $\hat{f}$ to $\hat{f}'$.*

These lemmas and their proofs are given in [3] under Proposition 13.7.7 and Corollary 13.7.10. With these tools we can start to construct the proof of the main proposition of this thesis.

Proposition 4.9. *Let $\widehat{\mathcal{C}}_{\mathbb{R}^2,n}$ be the configuration space of unordered n-tuples as defined in Definition 2.2. Then*

$$\pi_1(\widehat{\mathcal{C}}_{\mathbb{R}^2,n}) \cong \mathbb{B}_n.$$

Proof. From Lemma 4.6 we see that the quotient map $q : \mathcal{C}_{\mathbb{R}^2,n} \to \widehat{\mathcal{C}}_{\mathbb{R}^2,n}$ is a covering map. We will prove that $\mathbb{B}_n \cong \pi_1(\widehat{\mathcal{C}}_{\mathbb{R}^2,n}, [(1,0),(2,0),\ldots,(n,0)])$ with $q(((1,0),(2,0),\ldots,(n,0))) = [(1,0),(2,0),\ldots,(n,0)]$, the starting point of the loops. Now we have to find a group homomorphism with an inverse in the same way as in the previous proof. First we construct the function p that sends elements from $\mathbb{B}_n$ to the path space of $\mathcal{C}_{\mathbb{R}^2,n}$ in a similar manner as the proof of Proposition 4.3. p maps the equivalence class under isotopy of any geometric braid $b \in \mathbb{R}^2 \times I$ to the equivalence class under homotopy of a path $\gamma : I \to \mathcal{C}_{\mathbb{R}^2,n}$. Here γ maps $t \in I$ to an ordered n-tuple $(u(t)) := ((u_1(t), u_2(t), \ldots, u_n(t))$ where $u_i(t)$ is the natural projection on $\mathbb{R}^2$ of the intersection of b_i and $\mathbb{R}^2 \times \{t\}$, with b_i being the i-th string of b. This path is continuous.
Now the claim is that $f := q \circ p$ is the desired group homomorphism. First we have to check that f maps from $\mathbb{B}_n$ to $\pi_1(\widehat{\mathcal{C}}_{\mathbb{R}^2,n}, [(1,0),(2,0),\ldots,(n,0)])$. Per construction q sends the path γ constructed from b to a path $\hat{\gamma} \in \widehat{\mathcal{C}}_{\mathbb{R}^2,n} \times I$. Now $\hat{\gamma}(0) = \hat{\gamma}(1) = [(1,0),(2,0),\ldots,(n,0)]$ which means that the homotopy class of $\hat{\gamma}$ is indeed an element of $\pi_1(\widehat{\mathcal{C}}_{\mathbb{R}^2,n}, [(1,0),(2,0),\ldots,(n,0)])$. Showing that p is well defined goes in a similar manner as showing that F was well defined in the proof of Proposition 4.3 as isotopy of braid diagrams can be directly translated to homotopy of the paths they map to. Now as q is continuous we have that f is well defined. Here f is a group homomorphism as multiplying or combining braids goes in a similar manner as combining loops, also shown in the proof of Proposition 4.3.
Now we need to construct an inverse of f, let's call it g and prove that this is also well defined. First we take a representation $\hat{\gamma}$ of an equivalence class of loops in $\pi_1(\widehat{\mathcal{C}}_{\mathbb{R}^2,n}, [(1,0),(2,0),\ldots,(n,0)])$. Since q is a covering map we have a unique continuous lift $\gamma : I \to \mathcal{C}_{\mathbb{R}^2,n}$ of $\hat{\gamma}$ such that $q \circ \gamma = \hat{\gamma}$ with $\gamma(0) = ((1,0),(2,0),\ldots,(n,0))$ and $\gamma(1) = \sigma(((1,0),(2,0),\ldots,(n,0)))$ for a unique $\sigma \in \mathcal{S}_n$. From definition γ maps $t \in I$ to an ordered n-tuple $(u(t)) := ((u_1(t), u_2(t), \ldots, u_n(t))$ and now b_i, the i-th string of a braid b, is equal to $\bigcup_{t \in I}(u_1(t), t)$, which makes $b := \bigcup_{i=1}^{n} b_i$ a geometric representation of a braid and its equivalence class a braid.
We construct $g : \pi_1(\widehat{\mathcal{C}}_{\mathbb{R}^2,n}, [(1,0),(2,0),\ldots,(n,0)]) \to \mathbb{B}_n$ as the function that sends the homotopy class of a loop $\hat{\gamma}$ to the isotopy class of a geometric braid b as constructed above.
Now to check g is well defined we have to check that if $\hat{\gamma}_1, \hat{\gamma}_2$ loops in $\widehat{\mathcal{C}}_{\mathbb{R}^2,n}$ and $\hat{\gamma}_1 \sim \hat{\gamma}_2$ then $b_1 \sim b_2$ with b_1 the braid constructed from $\hat{\gamma}_1$ and b_2 constructed from γ_2. From Lemma 4.8 we have that if $\hat{\gamma}_1 \sim \hat{\gamma}_2$ then $\gamma_1 \sim \gamma_2$. Now we can use the argument in the proof of Proposition 4.3 to see that $b_1 \sim b_2$, so g is well defined. As we show g is the inverse of a group homomorphism we also prove that g is a group homomorphism.

All that is left is to show $g \circ f = \mathrm{id}_{\mathbb{B}_n}$ and $f \circ g = \mathrm{id}_{\pi_1(\hat{\mathcal{C}}_{\mathbb{R}^2,n},\dots)}$. Let us start with the first. The map f first sends a braid representation b to a path γ with starting point $((1,0),(2,0),\dots,(n,0))$ and this path will get mapped by q to a loop $\hat{\gamma}$. Now g will first lift this loop $\hat{\gamma}$ to a path with starting point $((1,0),(2,0),\dots,(n,0))$, however this lift is unique so it has to be lifted to the γ we got from f, so $g \circ f = \mathrm{id}_{\mathbb{B}_n}$ follows. For $f \circ g = \mathrm{id}_{\pi_1(\hat{\mathcal{C}}_{\mathbb{R}^2,n},\dots)}$ we can remark that we start with a loop $\hat{\gamma}$ and that this loop gets lifted by g to a path γ and then mapped to a braid representation b. Now f maps this braid representation back to γ per construction and then $\hat{\gamma} = q(\gamma)$ by definition. □

Remark 4.10. *It is still interesting, in my opinion, to take a look at how exactly g is a group homomorphism and preserving the structure of multiplication even though we don't have to explicitly prove it. This will be a bit more informal. For $\hat{\gamma_1}, \hat{\gamma_2}$ loops in $\widehat{\mathcal{C}}_{\mathbb{R}^2,n}$, we first take a look at $g(\hat{\gamma_1}) \cdot g(\hat{\gamma_2})$. First we lift the loops to two paths in $\mathcal{C}_{\mathbb{R}^2,n}$ that start in $((1,0),(2,0),\dots,(n,0))$. These two paths will form one long path however this path is not the same as the braid $g(\hat{\gamma_1}) \cdot g(\hat{\gamma_2})$. We cannot combine or perform multiplication and expect it to preserve a structure in a set of paths that is not a group. We don't even know if this kind of combining is a well defined way of looking at multiplication of paths! This is however all circumvented as we only do the multiplication in the fundamental group or in the braid group and then the problem fixes itself. Now for $g(\hat{\gamma_1} \cdot \hat{\gamma_2})$ the combined loop first gets lifted to a path with starting point $((1,0),(2,0),\dots,(n,0))$ that goes halfway via the endpoint of γ_1 to the endpoint of γ_2 and it's not hard to see that our desired braid gets constructed out of this path.*

5 Anyon statistics and the physical propagation of two dimensional configurations

In this Chapter we first have to sketch why permuting two particles in three dimensions is so different from permuting in two dimensions and what physical consequences this has for wave functions and phase factors. After this we will study an example of why the topological properties of the Artin braid group have consequences for physics in the form of a propagator.

5.1 Anyon statistics

Consider two indistinguishable particles 1 and 2 with a certain wave function $\psi(x_1, x_2)$. If we rotate particle 2 with an arbitrary angle $\Delta\phi$ around particle 1 the wave function will acquire a phase factor: $\psi(x_1, x_2) \to \psi'(x_1, x_2) = e^{i\nu\Delta\phi}\psi(x_1, x_2)$. Here ν is a factor called the *statistics* of ψ. The restrictions on this ν are very different for the cases $d = 2$ and $d \geq 3$, which will be clear from looking at the permutation of particles in both cases. The permuting of two particles x_1 and x_2 in a spatial sense can be done in two ways if $d \geq 2$. The first option is to rotate particle x_2 an angle $\Delta\phi = \pi$ around x_1 and then translate the system's center of mass to the original position.

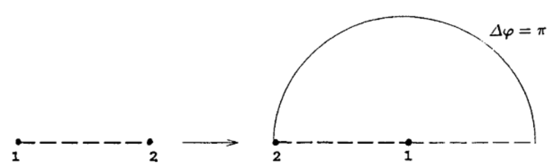

Figure 5: Rotating by $\Delta\phi = \pi$ from [4]

The second option is to rotate x_2 around x_1 with $\Delta\phi = -\pi$.

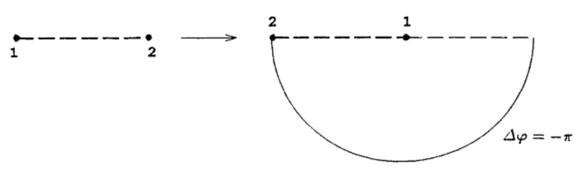

Figure 6: Rotating by $\Delta\phi = -\pi$ from [4]

In three dimensions the rotation paths are homotopic as they can be continuously deformed into each other. This means that in three dimensions we don't expect different physics by whatever option we take and thus we want that: $e^{i\nu\pi}\psi(x_1, x_2) = e^{-i\nu\pi}\psi(x_1, x_2)$ and this gives $\nu = 0, 1 \mod 2$. This results in two different statistics. Either the wave function does not change sign and gets multiplied with 1 for $\nu = 0$ (bosonic statistics) or the wave function gets multiplied with -1 for $\nu = 1$ (fermionic statistics).
This restriction does not hold however for a two dimensional space. Here both paths are not homotopic and so ν can be chosen arbitrarily in principle.
A very interesting physical consequence of this is that both parity(P)- and time(T)-symmetry are violated as the phase acquired by swapping particles changes from $e^{\pm i\nu\pi} \to e^{\mp i\nu\pi}$ under parity or time reversal and for $\nu \neq 0, 1$ there is no symmetry [4].
Now with the newly obtained anyon-statistics we can look at the physical meaning of a system propagating in time under the permuting of particles and why this gives rise to different physics in two dimensions.

5.2 Example: the propagation of an n-particle system in two dimensions

Consider the propagator of a system of configuration q at a time t to a configuration q' to a time t', expressed as a Feynman path integral:

$$K(q, t, q', t') = \int_{q(t)=q;q(t')=q'} \exp\left[\frac{i}{\hbar}\int_t^{t'} \mathcal{L}(\dot{q}, q, \tau)d\tau\right] D[q(t)].$$

Now as we are looking at an example where the beginning and end state are the same we have $q(t) = q(t') = q$ and this describes loops in the two dimensional configuration space $\hat{\mathcal{C}}_{\mathbb{R}^2,n}$ of n particles. As we already know the fundamental group of $\hat{\mathcal{C}}_{\mathbb{R}^2,n}$ is isomorphic to the Artin braid group we can split the amplitudes into subamplitudes of all different possible braids:

$$K(q, t, q, t') = \sum_{b\in\mathbb{B}_n} K_b(q, t, q, t') = \sum_{b\in\mathbb{B}_n} \int_{q;q} \exp\left[\frac{i}{\hbar}\int_t^{t'} \mathcal{L}(\dot{q}_b, q_b, \tau)d\tau\right] D[q_b(t)].$$

It's possible to assign a weight $\chi(b)$, which can be any complex number with modulus 1, to every braid subamplititude as long as this weight preserves the rules of composition of probability. This gives:

$$K(q, t, q, t') = \sum_{b\in\mathbb{B}_n} \chi(b) \int_{q_b(t)=q;q_b(t')=q} \exp\left[\frac{i}{\hbar}\int_t^{t'} \mathcal{L}(\dot{q}_b, q_b, \tau)d\tau\right] D[q_b(t)]. \quad (1)$$

Now for two different braids b_1, b_2 we have that $\chi(b_1) \cdot \chi(b_2) = \chi(b_1 \cdot b_2)$ as $K(q, t, q", t") = \int_{\hat{\mathcal{C}}_{\mathbb{R}^2,n}} dq' K(q, t, q', t')K(q', t', q", t")$. From this we can conclude that $\chi(b)$ must be a one dimensional representation of the braid b.

Now as any braid is constructed from elementary braids we can assign $\chi(\sigma_i) = e^{-i\nu\pi}$ for the elementary braid $\sigma_i = \sigma_i^+$ from Figure 3. Here ν can be any real number as it is not restricted, since $\sigma_i^2 \neq \mathbb{1}_{\mathbb{B}n}$.
Now for the elementary σ_i all winding angles ϕ_{jk} remain constant except for the angle $\phi_{i,i+1}$ which changes by π, so we can rewrite $\chi(\sigma_i)$.

$$\chi(\sigma_i) = \exp\Big[-i\nu\Delta\phi_{i,i+1}\Big] = \exp\Big[-i\nu\sum_{j<k}\Delta\phi_{jk}^{(i)}\Big]$$

with

$$\Delta\phi_{jk}^{(i)} := \phi_{jk}^{(i)}(t') - \phi_{jk}^{(i)}(t) = \pi\delta_{j,i}\delta_{k,i+1}.$$

Here $\phi_{ab}(t)$ is the angle between particle j and particle b at time t. Now this can be generalized to any braid k as it is composed of elementary braids.

$$\chi(b) = \exp\left[-i\nu\sum_{j<k}\int_t^{t'} d\tau\frac{d}{d\tau}\phi_{jk}^{(b)}\right]$$

This equation can be substituted into (1), giving

$$K(q,t,q,t') = \sum_{b\in\mathbb{B}_n}\int_{q;q}\exp\left[\frac{i}{\hbar}\int_t^{t'} d\tau\Big\{\mathcal{L}(\dot{q}_b,q_b,\tau) - \hbar\nu\sum_{j<k}\frac{d}{d\tau}\phi_{jk}^{(b)}\Big\}\right]D[q_b(t)].$$

Now if we define

$$\mathcal{L}' := \mathcal{L}(\dot{q}_b,q_b,\tau) - \hbar\nu\sum_{j<k}\frac{d}{d\tau}\phi_{jk}^{(b)},$$

we see that with the new Lagrangian $\mathcal{L}'$ anyons can be treated as normal three dimensional particles with additional statistics [4]. It is remarkable that these additional statistics are purely topological in nature. As braiding can take place in any propagation of a two dimensional configuration of n indistinguishable particles, this extra term in the Langragian will always appear.

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