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The equivalence of the Lidskii property and the nest approximation property

Thesis BSc Mathematics

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Introduction

In linear algebra, the trace of a matrix A acting on an n-dimensional vector space V can be related to its n eigenvalues $\{\lambda_k\}_{k=1}^n$ by the equation

$$\operatorname{Tr}(A) = \sum_{k=1}^{n} \lambda_k,$$

which we will call the trace equation. For matrices A acting on a complex vector space V, the trace equation is a direct consequence of the existence of the Jordan normal form, combined with the fact that $\operatorname{Tr}(BC) = \operatorname{Tr}(CB)$ for any matrices B and C acting on V. In the case that V is a real vector space, the same equality holds if we allow the eigenvalues to be complex, which can be proven by letting A act on the complexification of V. A different approach towards proving the trace equation is by analyzing the coefficients of the characteristic polynomial of A. Furthermore, the identity $\operatorname{Tr}(BC) = \operatorname{Tr}(CB)$ also implies that the traces are equal whenever we have two different matrix representations A_1 and A_2 of a linear operator A. This shows that the trace of a linear operator is a characteristic of the operator and not of the chosen matrix representation. It would certainly be nice to extend the definition of the trace beyond operators on finite-dimensional vector spaces in such a way that the trace equation is satisfied. This, however, unavoidably leads to different issues we need to resolve.

To make sense of our trace equation for some linear operator A on a vector space V, we clearly must ensure that at least the following three conditions are satisfied:

1. There exists a well-defined (i.e. independent of any chosen representation) notion of a trace.

2. There exists a suitable multiplicity, at least for the nonzero eigenvalues.

3. The nonzero eigenvalues of A, counted according to the multiplicity from 2, are absolutely summable.

Ignoring the question about how to properly define a generalized trace for now, we see that conditions 2 and 3 together already give some problems. For our new trace equation to be consistent with the finite-dimensional case, we want the definition of the multiplicity m_λ of a nonzero eigenvalue λ to be the same for any vector space V. This would lead us to define $m_\lambda = \dim(G_\lambda)$, where G_λ is the generalized eigenspace of λ.

For any infinite-dimensional vector space V, this already implies that the identity operator I_V fails to satisfy the third condition as $\dim(G_1) = \dim(V) = \infty$. Furthermore, there exists an operator A on the sequence space $l^1(\mathbb{N}, \mathbb{R})$ such that $A^2 = 0$ and $\mathrm{Tr}(A) = 1$ when written as an infinite matrix [13, Theorem 2.d.3]. This implies that there exists no such thing as an exotic multiplicity that will make the trace equation valid in all Banach spaces, let alone all vector spaces. This shows that we cannot hope to find a generalization of the trace to infinite dimensional vector spaces, such that the trace equation is satisfied for all linear operators on V. However, not all hope is lost as we may try to restrict ourselves to specific subsets of linear operators such that all three conditions are fullfilled.

For complex Hilbert spaces, this strategy is particularly fruitful and leads to a result known as Lidskii's theorem, in honour of Lidskii who published his proof in 1959 [12] (even though some sources, like Pisier [18], claim that Grothendieck discovered this earlier [7]). For a separable Hilbert space H with orthonormal basis E, we define the trace $\mathrm{Tr}(A)$ of an operator A by

$$\mathrm{Tr}(A) = \sum_{e \in E} \langle Ae, e \rangle.$$

As before, applying to $A = I$ shows that it is too much to ask for this trace to be defined on any bounded linear operator $A \in B(H)$. However, we can define the space of trace-class operators $L^1(H)$ as

$$L^1(H) = \{A \in B(H) : \mathrm{Tr}(|A|) < \infty\},$$

where $|A| = \sqrt{A^*A}$ is defined using the standard square root for positive operators [19, Theorem VI.9]. For any trace-class operator, the trace is convergent and is independent of the choice of E [19, Theorem VI.24]. This implies that Tr is a well-defined functional on $L^1(H)$, hence trace-class operators satisfy our first condition. It can be proven that any trace-class operator is compact [19, Theorem VI.21], which implies that the algebraic multiplicity is finite for all nonzero eigenvalues (see Section 4.1), hence the second condition is also satisfied. Finally, by combining Theorem VI.21 from [19] and Theorem 1.15 from [22], it also follows that the third condition is satisfied. With all three conditions satisfied, at least both sides of the trace equation are well-defined for all operators $A \in L^1(H)$. Proving that equality holds is not trivial, one proof can be found in Simon [22, Section 3] and uses the fact that for operators of the form $I + zA$, with I the identity operator, z a complex scalar and A a trace-class operator, a suitable determinant function det can be defined such that $z \mapsto \det(I + zA)$ is an entire function. The trace equality is then proven by analyzing the coefficients in the analytic expansion of $\det(I + zA)$. This strategy is based on the proof of the trace equation for matrices using the characteristic polynomial.

Lidskii's theorem is a beautiful generalization of the trace equation. However, it only applies to complex Hilbert spaces, which are particularly well-behaved. In general vector spaces, the analysis breaks down at several points, most notably due to the absence of an orthonormal basis, meaning a different trace construction is needed.

In this thesis, we will look at some of the things we *can* say about the generalization of the trace equation in Banach spaces. A very recent paper by Figiel and Johnson [6], only published in 2016, plays a central role in this discussion. This paper proves that two different properties of Banach spaces, namely the Lidskii property and the nest approximation property, are equivalent in complex Banach spaces satisfying the approximation property. A Banach space with the Lidskii property can for now be thought of as a Banach space that allows for a specific generalization of the trace equation. Furthermore, the nest approximation property is a stronger variant of the well-known approximation property. Whereas the approximation property is about approximating the identity operator by finite-rank operators, the nest approximation property also requires these finite-rank operators to leave an arbitrary nest of closed subspaces invariant. Section 3 provides a precise definition of invariant nests and in Section 4 we will properly introduce both properties and give a precise definition of them.

This approach towards generalizing the trace equation is motivated by an article by Erdos [5], in which he gives a proof of Lidskii's theorem different from the one described above. In his proof, Erdos uses results from a paper by Ringrose [20] to decompose compact operators into a normal and a quasi-nilpotent part. He then shows that the trace equation is satisfied for trace-class operators. This is done in two steps. First, Erdos proves that quasi-nilpotent trace-class operators have trace equal to 0. This is done by considering an approximation of the identity operator by finite-rank operators that leave a specific nest of subspaces invariant, which very much resembles the nest approximation property we discuss! Then the proof is concluded by computing the trace of the normal part of the decomposition, which is a trivial computation using the spectral theorem. This approach towards proving Lidskii's theorem is based on Ringrose's construction of invariant nests and the analysis of how compact operators act on them. In Section 3, we will go through this construction in detail and we will see that the result we obtain looks very much like an upper triangular matrix. Hence Erdos' proof is more or less a generalization of the proof for matrices using the Jordan normal form.

The goal of this thesis is to prove the equivalence between the Lidskii property and the nest approximation property, where we follow the proofs of Figiel and Johnson [6]. In the first half of this thesis, we will classify a suitable class of operators on which we can define a trace, the nuclear operators, and prove that this trace is well-defined if and only if the space they are acting on satisfies the approximation property. Therefore, we will first study the approximation property in Section 1, as it will play a prominent role in the rest of this thesis. We will encounter two characterizations of the approximation property and we will prove that these are equivalent. In Section 2, we will construct a trace on the nuclear operators and prove that this trace is well-defined. To do this we will need to introduce the topology of uniform convergence on compact sets (ucc-topology). It will turn out that this topology is intimately connected to one of the characterizations of the approximation property from Section 1. Furthermore, we will characterize all linear functionals that are continuous with respect to the ucc-topology and find that they have striking similarities with the nuclear operators. Exploiting these similarities will allow us to prove that the trace we want to define on the nuclear operators is well-defined whenever the Banach space they are acting on satisfies the approximation property. The second half of this thesis is also divided into two sections. In Section 3 we will introduce nests of subspaces, which we will use to generalize the concept of diagonal coefficients from matrices to compact operators. We will study the main part of the construction in Ringrose's paper [20], which we have already mentioned, to prove a theorem that relates the diagonal coefficients of a compact operator to its eigenvalues. We will see that this creates a sort of analogy to upper triangular matrix representations of linear operators on finite-dimensional spaces. Finally, in Section 4, all the extensive theoretical preparations of the first three sections will be put to use in proving the equivalence of the Lidskii property and the nest approximation property.

In this thesis, we focus on the equivalence between the two properties. It is of course also interesting to discuss examples of Banach spaces, other than Hilbert spaces, that satisfy both of these properties, but this is beyond the scope of this thesis. There exists an article, published by Johnson and Szankowski in 2014 [10], in which they introduce a class of Banach spaces called Γ-spaces and prove that these satisfy the Lidskii property, but this is certainly not trivial.

1 Multiple characterizations of the approximation property

In the first section, we will look at the approximation property for Banach spaces and discuss multiple characterizations of it. In this section, unless stated otherwise, X and Y will be complex Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. However, the subscripts may occasionally be dropped to improve readability. The space of linear operators from Y to X will be denoted by $L(Y,X)$. For a linear operator $T \in L(Y,X)$, we define the operator norm $\|T\| = \sup_{y\in Y,\ \|y\|\leq 1} \|Ty\|$ and the space of bounded operators $B(Y,X) = \{T \in L(Y,X) : \|T\| < \infty\}$. A standard result from functional analysis is that the operator norm really is a norm on $B(Y,X)$ and that $(B(Y,X), \|\cdot\|)$ is complete if $(X, \|\cdot\|_X)$ is. The only topologies considered in this section are the topologies generated by the open balls of the Banach norms and the operator norm. Closures in these topologies will be denoted by $\overline{A}$ for any subset A of these spaces. With these topologies on the Banach spaces, $B(Y,X)$ is precisely the space of continuous linear operators from Y to X. Furthermore, we denote the space of compact operators from Y to X as $K(Y,X)$ and the space of finite-rank operators from Y to X as $F(Y,X)$. For any of these operator spaces, we omit the Y in the notation if $Y = X$. Since any compact operator is bounded, any finite-rank operator is compact and the space of compact operators is closed in the space of bounded operators, we have the following inclusions, ordering the above-mentioned operator spaces: $F(Y,X) \subset \overline{F(Y,X)} \subset K(Y,X) \subset B(Y,X) \subset L(Y,X)$. Where we use the symbol $\subset$ for nonstrict inclusion.

1.1 The approximation property

In existing literature, there are multiple ways in which the approximation property is defined. The one we will use is in line with Megginson [15, p. 330]. However, for example Lindenstrauss and Tzafriri [13, p. 30] and Grothendieck [8, p. 165] use a different definition. The main goal of this section will be to prove Theorem 1.2, which states that these definitions are equivalent. This was first proven by Grothendieck [8] in 1955. In Section 2, we will encounter other characterizations, for which we need some more theory to formulate them.

Definition 1.1. A Banach space X has the *approximation property (AP)*, if for every Banach space Y, the following holds: $\overline{F(Y,X)} = K(Y,X)$.

Theorem 1.2. *Let X be a Banach space. Then the following are equivalent:*

1. *X has the approximation property.*

2. *For every compact $K \subset X$ and every $\epsilon > 0$, there exists some $T_{K,\epsilon} \in F(X)$ such that $\|T_{K,\epsilon}x - x\| < \epsilon$ for all $x \in K$.*

Both Megginson [15, Theorem 3.4.32] and Lindenstrauss [13, Theorem 1.e.4] also give proofs of this, however Lindenstrauss's proof only applies to real Banach spaces. Megginson made some modifications to this proof to also include complex Banach spaces. Therefore, we will follow Megginson's proof in this section. We can prove the implication $2 \implies 1$ immediately. For the converse implication, we need to do more work. It will be proven at the end of Section 1.

Proof of Theorem 1.2, $2 \implies 1$. Suppose 2 holds, so for every compact $K \subset X$ and every $\epsilon > 0$, there exists some $T_{K,\epsilon} \in F(X)$ such that $\|T_{K,\epsilon}x - x\| < \epsilon$ for all $x \in K$. We need to show that for any Banach space Y, we have that $F(Y, X)$ is dense in $K(Y, X)$. Let Y be an arbitrary Banach space and let $A \in K(Y, X)$ be an arbitrary compact operator. Furthermore, let B_Y be the closed unit ball of Y. Compactness of A implies that $K = \overline{AB_Y}$ is compact in X. By assumption, there exists a sequence $\{T_n\}_{n\in\mathbb{N}} \subset F(X)$ such that $\|T_n x - x\| < \frac{1}{n}$ for all $x \in K$ and $n \in \mathbb{N}$. Now consider the sequence $\{A_n\}_{n\in\mathbb{N}}$ defined by $A_n = T_n A$. We see: $\|A_n - A\| = \sup_{y\in Y,\ \|y\|\le 1} \|A_n y - Ay\| = \sup_{y\in Y,\ \|y\|\le 1} \|T_n Ay - Ay\| \le \frac{1}{n}$ as $Ay \in K$ for all $y \in Y$ such that $\|y\| \le 1$. So as $A_n \to A$ as $n \to \infty$ in the operator norm and $A_n \in F(Y, X)$ for all $n \in \mathbb{N}$, it follows that $A \in \overline{F(Y, X)}$, hence $K(Y, X) \subset \overline{F(Y, X)}$. As the converse inclusion always holds, we have equality and thus X has the AP by definition. $\square$

1.2 Convex, balanced and absorbing sets

From now on, we assume $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. If V is a topological vector space over $\mathbb{F}$ and $A \subset V$, $x \in V$ and $\alpha \in \mathbb{F}$ then we define: $x + A = \{x + y : y \in A\}$ and $\alpha A = \{\alpha y : y \in A\}$. Furthermore, if V is a normed space, we denote the closed ball of radius r centered at x by $\overline{B}_r(x)$.

Definition 1.3. Let V be a topological vector space and let $A \subset V$, then:

1. A is *convex* if for all $x, y \in A$ and $t \in [0, 1]$ we have that $tx + (1-t)y \in A$.

2. A is *balanced* if for every $\alpha \in \mathbb{F}$ such that $|\alpha| \le 1$, we have $\alpha A \subset A$.

3. A is *absorbing* if for every $x \in V$, there exists some $s_x \ge 0$ such that for all $t > s_x$ we have $x \in tA$.

4. The *convex hull* of A, denoted by $\mathrm{co}(A)$, is the smallest convex set containing A, so the intersection of all convex sets containing A.

5. The *closed convex hull* of A, denoted by $\overline{\text{co}}(A)$ is the intersection of all closed convex sets containing A.

Some immediate consequences of these definitions are summarized in the next propositions.

Proposition 1.4 ([15, p. 3])**.** *In a topological vector space:*

1. *Arbitrary intersections of convex sets are convex (so the definition of the (closed) convex hull indeed gives a convex set).*

2. *Arbitrary unions and intersections of balanced sets are balanced.*

3. *Scalar multiples of convex sets are convex and scalar multiples of balanced sets are balanced.*

4. *The closed convex hull of any set is closed.*

Proposition 1.5. *Let V be a normed space, then:*

1. *If $C \subset V$ is convex, so is $\overline{C}$.*

2. *If $B \subset V$ is balanced, so is $\overline{B}$.*

Proof. 1. Let $x, y \in \overline{C}$ and $t \in [0, 1]$. As $x, y \in \overline{C}$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in C converging to x and y respectively. Since C is convex, we have $tx_n + (1-t)y_n \in C$ for all $n \in \mathbb{N}$. By letting $n \to \infty$ we see $tx + (1-t)y \in \overline{C}$. So $\overline{C}$ is convex.

2. Let $\alpha \in \mathbb{F}$ such that $|\alpha| \leq 1$ and $x \in \overline{B}$. As $x \in \overline{B}$, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in B converging to x. Since B is balanced, we have $\alpha x_n \in B$ for all $n \in \mathbb{N}$. By letting $n \to \infty$ we see $\alpha x \in \overline{B}$, hence $\alpha\overline{B} \subset \overline{B}$. So $\overline{B}$ is balanced. $\square$

Proposition 1.6. *Let V be a normed space and $A \subset V$, then:*

1. $\overline{\text{co}}(A) = \overline{\text{co}}(\overline{A})$

2. $\overline{\text{co}(A)} = \overline{\text{co}}(A)$

Proof. 1. Since $A \subset \overline{A} \subset \overline{\text{co}}(\overline{A})$ and $\overline{\text{co}}(\overline{A})$ is closed and convex, we have $\overline{\text{co}}(A) \subset \overline{\text{co}}(\overline{A})$ by definition of the closed convex hull. Conversely, since $A \subset \overline{\text{co}}(A)$ and $\overline{\text{co}}(A)$ is closed, we have $\overline{A} \subset \overline{\text{co}}(A)$. Now, since $\overline{\text{co}}(A)$ is closed and convex, by definition, we have $\overline{\text{co}}(\overline{A}) \subset \overline{\text{co}}(A)$. So combining both inclusions gives $\overline{\text{co}}(A) = \overline{\text{co}}(\overline{A})$.

2. We have $A \subset \text{co}(A) \subset \overline{\text{co}(A)}$ and by Proposition 1.5 $\overline{\text{co}(A)}$ is closed and convex. So by definition $\overline{\text{co}}(A) \subset \overline{\text{co}(A)}$. Conversely, since $A \subset \overline{\text{co}}(A)$ and $\overline{\text{co}}(A)$ is convex, by definition we have $\text{co}(A) \subset \overline{\text{co}}(A)$ and as the latter is closed, this yields $\overline{\text{co}(A)} \subset \overline{\text{co}}(A)$. So combining both inclusions gives $\overline{\text{co}(A)} = \overline{\text{co}}(A)$. $\square$

Proposition 1.7. *Let V be a normed space and $A \subset V$ and define*

$$C := \left\{ \sum_{n=1}^{N} t_n x_n : N \in \mathbb{N},\ x_n \in A,\ t_n \geq 0,\ \sum_{n=1}^{N} t_n = 1 \right\}.$$

Then $C = \mathrm{co}(A)$.

Proof. We need to prove that C is the smallest convex set containing A. For all $x \in A$, we have that x is a convex combination as in the definition of C. It follows that $x \in C$, hence $A \subset C$. Furthermore, C is convex; let $x, y \in C$ and $t \in [0,1]$, since $x, y \in C$ we can find $N, M \in \mathbb{N}$, elements $x_n, y_m \in A$ and positive real numbers $r_n, s_m \in [0,1]$ for all $n \in \{1, ..., N\}$ and $m \in \{1, ..., M\}$ such that we can write $x = \sum_{n=1}^{N} r_n x_n$ and $y = \sum_{m=1}^{M} s_m y_m$. Then $tx + (1-t)y = \sum_{n=1}^{N} t r_n x_n + \sum_{m=1}^{M} (t-1) s_m y_m$ is a linear combination as in the definition of C, so $tx + (1-t)y \in C$ implying that C is convex.

It is left to show that C is the smallest convex set that includes A. Now suppose C_1 is a second convex set including A, we are done when we prove that $C \subset C_1$. So let $\sum_{n=1}^{N} t_n x_n$ be an element of C, we want to prove that $\sum_{n=1}^{N} t_n x_n \in C_1$. We proceed by induction on N. Note that for $N = 1$ the claim is trivial, as C_1 contains A. Now let $N > 1$ and suppose that all linear combinations in C of the form $\sum_{n=1}^{N-1} t_n x_n$ are in C_1. Let $\sum_{n=1}^{N} t_n x_n$ be an element of C and without loss of generality, we can assume that $t_n \neq 0$ for all $n \in \{1, ..., N\}$. Define $T = \sum_{n=1}^{N-1} t_n = 1 - t_N \neq 0$, then by the induction hypothesis, we have $T^{-1} \sum_{n=1}^{N-1} t_n x_n \in C_1$ and as C_1 includes A, we also have $x_N \in C_1$. So by convexity of C_1, we have that $\sum_{n=1}^{N} t_n x_n = TT^{-1} \sum_{n=1}^{N-1} t_n x_n + t_N x_N = (1-t_N) T^{-1} \sum_{n=1}^{N-1} t_n x_n + t_N x_N \in C_1$, completing the induction. □

Proposition 1.8. *Let V be a normed space over $\mathbb{F}$ and $A \subset V$ a balanced set, then:*

1. *If $\alpha \in \mathbb{F}$ and $|\alpha| = 1$, then $\alpha A = A$.*
2. $\mathrm{co}(A)$ *is balanced.*

Proof. 1. Let $A \subset V$ be balanced and suppose $|\alpha| = 1$. By definition of a balanced set, we have $\alpha A \subset A$. As $\alpha \in \mathbb{F}$ and $|\alpha| = 1$ we have $\alpha^{-1} \in \mathbb{F}$ and $|\alpha^{-1}| = 1$, so by definition of a balanced set, we also have $\alpha^{-1} A \subset A$, hence $A \subset \alpha A$. Combining both inclusions gives $\alpha A = A$.

2. Let $x \in \mathrm{co}(A)$ and $\alpha \in \mathbb{F}$ with $|\alpha| \leq 1$. Since $x \in \mathrm{co}(A)$, we can find a linear combination as in Proposition 1.7 such that $x = \sum_{n=1}^{N} t_n x_n$. Since A is balanced, we have $\alpha x_n \in A$ for all $n \in \{1, ..., N\}$. So $\alpha x = \sum_{n=1}^{N} t_n \alpha x_n$ is a linear combination as in Proposition 1.7, so $\alpha x \in \mathrm{co}(A)$. Hence $\alpha \mathrm{co}(A) \subset \mathrm{co}(A)$, so $\mathrm{co}(A)$ is balanced. □

Combining Propositions 1.5, 1.6 and 1.8, we can formulate the following corollary.

Corollary 1.9. *Let V be a normed space and $A \subset V$ a balanced subset. Then the following sets are also balanced: $\overline{A}$, $\operatorname{co}(A)$ and $\overline{\operatorname{co}(A)} = \overline{\operatorname{co}}(A) = \overline{\operatorname{co}}(\overline{A})$.*

In the proof of the second implication of Theorem 1.2, compact convex hulls will turn out to be very useful. Therefore, we wish to relate the compactness properties of a set A to the compactness properties of its convex hull, for this we will use Mazur's compactness theorem.

Theorem 1.10 (Mazur's compactness theorem)**.** *Let X be a Banach space and suppose $K \subset X$ is compact, then $\overline{\operatorname{co}}(K)$ is compact.*

A proof of Mazur's compactness theorem can be found in [15]. However, as it relies on lemmas that prove compactness properties of the (closed) convex hull more generally in both the norm and weak topology, it is unnecessarily complicated for our purposes. However, Conway [2, Theorem 4.8] presents a much simpler proof that works for the norm topology. We can reformulate the contents of Mazur's compactness theorem in terms of precompact sets.

Theorem 1.11. *Let X be a Banach space and suppose $A \subset X$ is precompact, then $\operatorname{co}(A)$ is precompact.*

Proof. If A is precompact, then by Mazur's compactness theorem, $\overline{\operatorname{co}}(\overline{A})$ is compact. So by Proposition 1.6, $\overline{\operatorname{co}(A)} = \overline{\operatorname{co}}(\overline{A})$ is compact, hence $\operatorname{co}(A)$ is precompact. □

We saw that given any subset A of normed space V, we can extend this subset to a convex set by looking at the convex hull of A. Moreover, if V is complete and A is precompact, then Mazur's compactness theorem ensures that this convex hull is precompact too. In a similar fashion, we would like to extend a subset to a balanced subset of V while preserving precompactness.

Proposition 1.12. *Let V be a normed space over $\mathbb{F}$ and $A \subset V$, then the set*

$$B := \bigcup\{\alpha A : \alpha \in \mathbb{F}, |\alpha| \leq 1\}$$

is balanced. Furthermore, if A is precompact, so is B.

Proof. To prove B is balanced, we need to show $\beta B \subset B$ for all $\beta \in \mathbb{F}$ such that $|\beta| \leq 1$. Let $\beta \in \mathbb{F}$ be arbitrary such that $|\beta| \leq 1$ and let $y \in B$. Then by definition of B, there exist $\gamma \in \mathbb{F}$ such that $|\gamma| \leq 1$ and $x \in A$ such that $y = \gamma x$. So $\beta y = \beta\gamma x \in B$ as $|\beta\gamma| = |\beta||\gamma| \leq 1$.

So B is balanced. Now suppose A is precompact. We prove that B is precompact by showing all sequences in B have a convergent subsequence. Let $\{y_n\}_{n\in\mathbb{N}} \subset B$. Then for all $n \in \mathbb{N}$ there exist $\alpha_n \in \mathbb{F}$ and $x_n \in A$ such that $|\alpha| \leq 1$ and $y_n = \alpha_n x_n$. Since A is precompact and $\{\alpha \in \mathbb{F} : |\alpha| \leq 1\}$ is compact, there exists a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ such that both $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ and $\{x_{n_k}\}_{k\in\mathbb{N}}$ are convergent, so $\{y_{n_k}\}_{k\in\mathbb{N}}$ is convergent as product of two convergent sequences. It follows that B is precompact. □

Before we move on to finish the proof of Theorem 1.2, we need a few more technical results, relating precompact sets to the closed convex hull of sequences converging to 0. These come straight from Megginson [15]. However, the proofs given here are more detailed.

Proposition 1.13 ([15, Lemma 3.4.29]). *Let X be a Banach space, $\{x_n\}_{n\in\mathbb{N}}$ a sequence converging to 0 and let $H \subset \mathbb{F}$ be either $\{1\}$ or the closed ball with radius ρ, centered at 0. Then*

$$\overline{\mathrm{co}}(\{\alpha x_n : \alpha \in H, n \in \mathbb{N}\}) = \left\{ \sum_{n\in\mathbb{N}} t_n \alpha_n x_n : t_n \geq 0,\ \alpha_n \in H,\ \sum_{n\in\mathbb{N}} t_n \leq 1 \right\},$$

and this closed convex hull is compact.

Proof. Define $C = \left\{\sum_{n\in\mathbb{N}} t_n \alpha_n x_n : t_n \geq 0,\ \alpha_n \in H,\ \sum_{n\in\mathbb{N}} t_n \leq 1\right\}$ and $R = \max\{1, \rho\}$. C is well-defined as $\{x_n\}_{n\in\mathbb{N}}$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ are bounded and $\{t_n\}_{n\in\mathbb{N}}$ is absolutely summable. For all $\alpha \in H$ and $n \in \mathbb{N}$, we have that αx_n is a sum as in the definition of C, therefore $\{\alpha x_n : \alpha \in H, n \in \mathbb{N}\} \subset C$. We first prove that C is closed and convex.

For convexity, let $x, y \in C$ and $t \in [0, 1]$. Then we can write $x = \sum_{n\in\mathbb{N}} r_n \beta_n x_n$ and $y = \sum_{n\in\mathbb{N}} s_n \gamma_n x_n$, where these are sums as in the definition of C. Then $tx + (1-t)y = \sum_{n\in\mathbb{N}}(tr_n\beta_n + (1-t)s_n\gamma_n)x_n$. To prove convexity, we need to prove that this is a sum as in the definition of C. If $t = 0$, $t = 1$ or $H = \{1\}$, this is clear. So suppose $t \in (0, 1)$ and H is a closed ball of radius ρ, centered at 0. Proving this is a sum as in the definition of C means that for all $n \in \mathbb{N}$ we must find $u_n \geq 0$ and $\delta_n \in H$ such that $\sum_{n\in\mathbb{N}} u_n \leq 1$ and $tr_n\beta_n + (1-t)s_n\gamma_n = u_n\delta_n$. If $tr_n + (1-t)s_n = 0$, it follows that $r_n = s_n = 0$ since t and $1-t$ are strictly positive. Put $u_n = 0$ and $\delta_n = 0 \in H$, then $tr_n\beta_n + (1-t)s_n\gamma_n = 0 = u_n\delta_n$. If $tr_n + (1-t)s_n \neq 0$, put $u_n = tr_n + (1-t)s_n$ and $\delta_n = \frac{tr_n\beta_n + (1-t)s_n\gamma_n}{u_n}$. Since

$$\begin{aligned}|\delta_n| = \left|\frac{tr_n\beta_n + (1-t)s_n\gamma_n}{u_n}\right| &\leq \frac{tr_n|\beta_n| + (1-t)s_n|\gamma_n|}{u_n} \\ &\leq \frac{tr_n\rho + (1-t)s_n\rho}{u_n} = \frac{u_n\rho}{u_n} = \rho,\end{aligned}$$

it follows that $\delta_n \in H$. As $\sum_{n\in\mathbb{N}} u_n = \sum_{n\in\mathbb{N}} tr_n + (1-t)s_n \leq 1$, we have that $tx + (1-t)y = \sum_{n\in\mathbb{N}} u_n\delta_n x_n \in C$, so C is convex.

For closedness, let $y \in \overline{C}$, so there exists a sequence $\{y^{(m)}\}_{m\in\mathbb{N}}$ such that $y^{(m)} \in C$ for all $m \in \mathbb{N}$ and $y^{(m)} \to y$ as $m \to \infty$. As $y^{(m)} \in C$ we can write $y^{(m)} = \sum_{n\in\mathbb{N}} t_n^{(m)} \alpha_n^{(m)} x_n$ where these are sums as in the definition of C. For fixed $n \in \mathbb{N}$ this yields two sequences $\{t_n^{(m)}\}_{m\in\mathbb{N}} \subset [0,1]$ and $\{\alpha_n^{(m)}\}_{m\in\mathbb{N}} \subset H$. As both $[0,1]$ and H are compact, we can find a subsequence $\{y^{(m_k)}\}_{k\in\mathbb{N}}$ such that $\{t_1^{(m_k)}\}_{k\in\mathbb{N}}$ converges to some $t_1 \in [0,1]$ and $\{\alpha_1^{(m_k)}\}_{k\in\mathbb{N}}$ converges to some $\alpha_1 \in H$. As the same argument now applies to the sequences $\{t_2^{(m_k)}\}_{k\in\mathbb{N}} \subset [0,1]$ and $\{\alpha_2^{(m_k)}\}_{k\in\mathbb{N}} \subset H$, we can find a subsequence $\{y^{(m_{k_l})}\}_{l\in\mathbb{N}}$ such that $\{t_2^{(m_{k_l})}\}_{l\in\mathbb{N}}$ converges to some $t_2 \in [0,1]$ and $\{\alpha_2^{(m_{k_l})}\}_{l\in\mathbb{N}}$ converges to some $\alpha_2 \in H$. By repeating this argument inductively N times, we obtain a subsequence for which the first N pairs of coefficients converge. Now we can make N arbitrarily large. This yields two sequences $\{t_n\}_{n\in\mathbb{N}} \subset [0,1]$ and $\{\alpha_n\}_{n\in\mathbb{N}} \subset H$, where each t_n and α_n are inductively defined in the same way as t_1, t_2, α_1 and α_2. Put $y' = \sum_{n\in\mathbb{N}} t_n\alpha_n x_n$, we will prove that $y = y'$ and $y' \in C$. To prove $y' \in C$ it is only left to prove that $\sum_{n\in\mathbb{N}} t_n \leq 1$. For $N \in \mathbb{N}$, consider $T_N = \sum_{n=1}^N t_n$. By construction, there exists a subsequence $\{y^{(m_k)}\}_{k\in\mathbb{N}}$ such that the first N coefficients $t_n^{m_k}$ converge to t_n. This implies that

$$T_N = \sum_{n=1}^{N} t_n = \lim_{k\to\infty} \sum_{n=1}^{N} t_n^{(m_k)} \leq 1 \quad \forall N \in \mathbb{N}.$$

So by taking $N \to \infty$, it follows that $\sum_{n\in\mathbb{N}} t_n \leq 1$, hence $y' \in C$. We prove that $y = y'$ by showing that $\|y - y'\| = 0$. For this, pick $\epsilon > 0$ and $M \in \mathbb{N}$ such that $\sup_{n>M} \|x_n\| < \frac{\epsilon}{4R}$. By the same argument as before, we can find a subsequence $\{y^{(m_k)}\}_{k\in\mathbb{N}}$ such that the first M pairs of coefficients converge to t_n and α_n for all $n \leq M$. Then we have

$$\begin{aligned}
\|y' - y^{(m_k)}\| &\leq \sum_{n\in\mathbb{N}} \|x_n\| |t_n\alpha_n - t_n^{(m_k)}\alpha_n^{(m_k)}| \\
&= \sum_{n=1}^{M} \|x_n\| |t_n\alpha_n - t_n^{(m_k)}\alpha_n^{(m_k)}| + \sum_{n=M+1}^{\infty} \|x_n\| |t_n\alpha_n - t_n^{(m_k)}\alpha_n^{(m_k)}| \\
&< \sum_{n=1}^{M} \|x_n\| |t_n\alpha_n - t_n^{(m_k)}\alpha_n^{(m_k)}| + \sum_{n=M+1}^{\infty} \frac{\epsilon}{4R} R(t_n + t_n^{(m_k)}) \\
&\leq \sum_{n=1}^{M} \|x_n\| |t_n\alpha_n - t_n^{(m_k)}\alpha_n^{(m_k)}| + \frac{\epsilon}{2}.
\end{aligned}$$

As the first M pairs of coefficients converge, there exists a $K_1 \in \mathbb{N}$ such that for all $k \geq K_1$ the first term of the right-hand side of the last inequality is less than $\frac{\epsilon}{2}$. So for all $k \geq K_1$, it follows that $\|y' - y^{(m_k)}\| < \epsilon$. As $\{y^{(m_k)}\}_{k\in\mathbb{N}}$ is a subsequence of $\{y^{(m)}\}_{m\in\mathbb{N}}$, it also converges to y. This implies there exists a $K_2 \in \mathbb{N}$ such that for all $k \geq K_2$ we have that $\|y - y^{(m_k)}\| < \epsilon$. So for $K = \max(K_1, K_2)$, it follows that $\|y' - y\| \leq \|y' - y^{(m_K)}\| + \|y - y^{(m_K)}\| < 2\epsilon$. As this applies to all $\epsilon > 0$, we have $y = y' \in C$ and therefore we conclude that C is closed.

So C is a closed and convex set containing $\{\alpha x_n : \alpha \in H, n \in \mathbb{N}\}$, to prove $C = \overline{\text{co}}(\{\alpha x_n : \alpha \in H, n \in \mathbb{N}\})$ we need to prove that it is the smallest of such sets. So let C_1 be a closed and convex set such that $\{\alpha x_n : \alpha \in H, n \in \mathbb{N}\} \subset C_1$. We need to prove that $C \subset C_1$. Note that since C_1 is closed, we have $0 \in C_1$. So let $y = \sum_{n\in\mathbb{N}} t_n \alpha_n x_n \in C$ and define $y_N = \sum_{n=1}^{N} t_n \alpha_n x_n$. With $T = \sum_{n=1}^{N} t_n$, we see that $y_N = (1 - T)0 + \sum_{n=1}^{N} t_n \alpha_n x_n \in C_1$ as this is a convex combination of elements in C_1. So since $y_N \to y$ as $N \to \infty$ and C_1 is closed, this implies $y \in C_1$. Hence we have that $C \subset C_1$, proving that C is the desired closed convex hull.

It is left to prove that C is compact. As $\{x_n\}_{n\in\mathbb{N}}$ converges to 0, the set $\{x_n : n \in \mathbb{N}\} \cup \{0\}$ is compact as any cover of open sets contains a finite subcover. Hence $\{x_n : n \in \mathbb{N}\}$ is precompact. By Proposition 1.12, it follows that $\{\alpha x_n : \alpha \in H, n \in \mathbb{N}\} = \bigcup_{\alpha\in H} \alpha\{x_n : n \in \mathbb{N}\}$ is precompact, too. So by combining Mazur's compactness theorem and Proposition 1.6, it follows that $C = \overline{\text{co}}(\{\alpha x_n : \alpha \in H, n \in \mathbb{N}\}) = \overline{\text{co}}\left(\overline{\{\alpha x_n : \alpha \in H, n \in \mathbb{N}\}}\right)$ is compact. □

Proposition 1.14 ([15, Lemma 3.4.30]). *Let X be a Banach space and $A \subset X$ precompact. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to 0 such that $A \subset \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$.*

Proof. If $A = \emptyset$, there is nothing to prove, so assume $A \neq \emptyset$ and A is precompact. Since non-zero scalar multiplication is a homeomorphism, $2A$ is also precompact, hence totally bounded. This implies that we can find $\{x_1, ..., x_{n_1}\} \subset 2A$ such that $2A \subset \bigcup_{j=1}^{n_1} \overline{B}_{\frac{1}{2}}(x_j)$. Now define $A_1 = \bigcup_{j=1}^{n_1} \left((2A \cap \overline{B}_{\frac{1}{2}}(x_j)) - x_j\right)$. Since for all $j \leq n_1$, we have that $2A \cap \overline{B}_{\frac{1}{2}}(x_j) \subset 2A$, it follows that $2A \cap \overline{B}_{\frac{1}{2}}(x_j)$ is precompact too. Since translation is a homeomorphism, we have that $(2A \cap \overline{B}_{\frac{1}{2}}(x_j)) - x_j$ is precompact for all $j \leq n_1$ and thus A_1 is precompact as finite union of precompact sets. Furthermore we see that $A_1 \subset \overline{B}_{\frac{1}{2}}(0)$ and as $A \neq \emptyset$ we also have $A_1 \neq \emptyset$. So we can repeat this procedure with A_1, therefore there are $\{x_{n_1+1}, ..., x_{n_2}\} \subset 2A_1$ such that $2A_1 \subset \bigcup_{j=n_1+1}^{n_2} \overline{B}_{\frac{1}{2^2}}(x_j)$ and we define $A_2 = \bigcup_{j=n_1+1}^{n_2} \left((2A_1 \cap \overline{B}_{\frac{1}{2^2}}(x_j)) - x_j\right)$. Now A_2 is precompact, non-empty and $A_2 \subset \overline{B}_{\frac{1}{2^2}}(0)$. We can continue this construction. Notice that by every iteration, the radius of closed balls decreases by a factor of $\frac{1}{2}$. This yields the sequence $\{x_n\}_{n\in\mathbb{N}}$, which converges to 0 as $A_k \subset \overline{B}_{\frac{1}{2^k}}(0)$ for all $k \in \mathbb{N}$. We prove that $A \subset \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\}) = \left\{\sum_{n\in\mathbb{N}} t_n x_n : t_n \geq 0,\ \sum_{n\in\mathbb{N}} t_n \leq 1\right\}$, where the equality follows from Proposition 1.13. Suppose that $x \in A$, by construction there exists a j_1 with $1 \leq j_1 \leq n_1$ such that $2x - x_{j_1} \in A_1$, so we can find an integer j_2 with $n_1 + 1 \leq j_2 \leq n_2$ such that $4x - 2x_{j_1} - x_{j_2} = 2(2x - x_{j_1}) - x_{j_2} \in A_2$, and so forth. After m iterations and dividing by 2^m, we have that

$$x - \sum_{n=1}^{m} 2^{-n} x_{j_n} \in 2^{-m} A_m \subset \overline{B}_{\frac{1}{4^m}}(0).$$

After taking the limit as $m \to \infty$, it follows that $x = \sum_{n=1}^{\infty} 2^{-n} x_{j_n} \in \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})$, which finishes the proof. □

Proposition 1.15 ([15, Lemma 3.4.31a])**.** *Let X be a Banach space. Then $A \subset X$ is precompact if and only if there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to* 0 *such that $A \subset \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})$.*

Proof. If $A \subset X$ is precompact, then by Proposition 1.14 there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to 0 such that $A \subset \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})$. Conversely, if there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converging to 0 such that $A \subset \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})$. Then by Proposition 1.13, A is a subset of a compact set so A is precompact. □

1.3 Proving the second implication

In the previous subsection, we discussed properties of convex and balanced sets. Furthermore, we looked at (closed) convex hulls, particularly those of sequences converging to 0. In this section, we will use these to prove the forward implication of Theorem 1.2. The strategy for the proof will be to pair every compact $K \subset X$ with a suitable Banach space Y which, as set, is a subset of X. Furthermore, we want to construct Y in such a way that K is contained in the unit ball *in* Y and such that the identity map from Y into X is compact. Then we will see that the approximation property allows us to uniformly approximate the identity map of X on compact sets by finite-rank operators. For the rest of this section, if S is a subset of a vector space V, we denote the linear span of S by $\mathrm{span}(S)$.

Definition 1.16. Let V be a vector space and $A \subset V$ be an absorbing subset. Then the *Minkowski functional* of A, denoted by p_A, is defined as $p_A(x) = \inf\{t : t > 0, x \in tA\}$ for all $x \in V$.

Remark. We require A to be absorbing such that $\{t : t > 0, x \in tA\}$ is non-empty for all $x \in V$. Hence $p_A(x)$ is finite, real-valued, and nonnegative for all $x \in V$.

Definition 1.17. Let V be a vector space, a function $f : V \to \mathbb{R}$ is *positive-homogeneous* if for all $c \in \mathbb{R}$ such that $c > 0$ and all $x \in V$, we have that $f(cx) = cf(x)$. We call f *sublinear* if for all $x, y \in V$ we have that $f(x + y) \leq f(x) + f(y)$.

Proposition 1.18 ([15, Proposition 1.9.14 a])**.** *Suppose that V is a vector space and $A \subset V$ is an absorbing set. Then:*

1. *p_A is positive-homogeneous and $A \subset \{x \in V : p_A(x) \leq 1\}$.*
2. *If A is convex, then p_A is sublinear and $\{x \in V : p_A(x) < 1\} \subset A$.*
3. *If A is both convex and balanced, then p_A is a seminorm on V.*

Using the Minkowski functional, we can construct new Banach spaces.

Proposition 1.19 ([15, Lemma 3.4.38])**.** *Suppose X is a Banach space over $\mathbb{F}$ and $S \subset X$ is nonempty and precompact. Define*

$$K_S = \overline{\text{co}} \left(\bigcup \{\alpha S : \alpha \in \mathbb{F}, |\alpha| \leq 1\} \right),$$

and let $Y = \text{span}(K_S)$. Then:

1. *K_S is compact in X and $S \subset K_S$.*
2. *The vector space Y has a Banach norm $\|\cdot\|_Y$ such that K_S is the closed unit ball in $(Y, \ \|\cdot\|_Y)$.*
3. *The inclusion/identity map from Y into X is compact.*

Proof. 1. Write $B = \bigcup \{\alpha S : \alpha \in \mathbb{F}, |\alpha| \leq 1\}$, then by Proposition 1.12, B is balanced and precompact. So we see that $K_S = \overline{\text{co}}(B) = \overline{\text{co}}(\overline{B})$ by Proposition 1.6 which is compact by Mazur's compactness theorem. Furthermore, we have that $S \subset B \subset \overline{\text{co}}(B) = K_S$.

2. As B is balanced, K_S is also balanced by Corollary 1.9. Moreover, K_S is obviously convex. K_S being balanced and convex also implies that K_S is absorbing in Y. Suppose that $y \in Y$, we will prove there exists $s_y \geq 0$ such that for all $t > s_y$, we have that $y \in tK_S$. Note that since $0 \in K_S$, it follows that for all $t > 0$ we have that $0 \in tK_S$. So if $y = 0$, we can take $s_y = 0$ and we are done. Now suppose $y \neq 0$. Since $Y = \text{span}(K_S)$, we can write $y = \sum_{n=1}^{N} \alpha_n y_n$ for some $N \in \mathbb{N}$, $\alpha_n \in \mathbb{F}$ and $y_n \in K_S$. Since $y \neq 0$, we can assume that $N \neq 0$ and $\alpha_n \neq 0$ for all n such that $1 \leq n \leq N$. We define $M = \max_{1 \leq n \leq N} |\alpha_n|$ and since K_S is balanced, we have that $\frac{\alpha_n}{M} y_n \in K_S$ for all n such that $1 \leq n \leq N$. Now since K_S is convex, it follows that $\frac{y}{NM} = \sum_{n=1}^{N} \frac{\alpha_n}{NM} y_n \in K_S$, hence $y \in NM \cdot K_S$. Now put $s_y = MN$ and suppose that $t > s_y = NM$. By convexity of K_S it follows that $\frac{y}{t} = \frac{MN}{t} \frac{y}{MN} + (1 - \frac{MN}{t})0 \in K_S$, hence $y \in tK_S$. So K_S

is absorbing. Now define $\|y\|_Y = p_{K_S}(y)$ for all $y \in Y$, where p_{K_S} is the Minkowski functional of K_S. By Proposition 1.18 this is a seminorm on Y. However, if $y \in Y$ and $y \neq 0$ we have that $\|y\|_Y > 0$. Suppose not, then by definition of the Minkowski functional, we obtain a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers converging to 0 such that $t_n^{-1}y \in K_S$ for all $n \in \mathbb{N}$. However, this implies that K_S is unbounded with respect to $\|\cdot\|_X$, which contradicts the compactness of $K_S \subset X$. It follows that $\|\cdot\|_Y$ is a norm on Y.

From the first part of Proposition 1.18 it follows that K_S is contained in the closed unit ball of $(Y, \|\cdot\|_Y)$. Conversely, suppose $y \in Y \setminus K_S$. Since K_S is closed, we can find an $\epsilon > 0$ such that the open ball of radius ϵ centered at y with respect to $\|\cdot\|_X$ is disjoint with K_S, where we can assume that $\epsilon < \|y\|_X$. Now take $t > 0$ such that $y \in tK_S$, since $tK_S \subset K_S$ for all $t \leq 1$ it follows that $t > 1$. As $\frac{y}{t} \in K_S$, we have that $\|y - \frac{y}{t}\|_X \geq \epsilon$, hence $1 - \frac{1}{t} \geq \frac{\epsilon}{\|y\|_X}$ so it follows that $t \geq \frac{\|y\|_X}{\|y\|_X - \epsilon} > 1$. By taking the infimum over all such t, we see that $\|y\|_Y > 1$, so y is not contained in the closed unit ball of $(Y, \|\cdot\|_Y)$, completing the proof that K_S is the closed unit ball of $(Y, \|\cdot\|_Y)$.

The only thing left to prove is that $(Y, \|\cdot\|_Y)$ is a Banach space. In the rest of this proof, we will write B_Y instead of K_S if we refer to K_S as subset of Y and just K_S when we mean K_S as subset of X. Since K_S is bounded, there exists a $K > 0$ such that $\|x\|_X \leq K$ for all $x \in K_S$. Now suppose $y \in Y$ and $y \neq 0$, then $\frac{y}{\|y\|_Y} \in B_Y = K_S$, hence $\frac{\|y\|_X}{\|y\|_Y} \leq K$ so $\|y\|_X \leq K\|y\|_Y$ and this identity obviously extends to the case that $y = 0$. Now suppose $(Y, \|\cdot\|_Y)$ is not Banach, so there exists a nonconvergent Cauchy sequence in Y, say $\{v_n\}_{n\in\mathbb{N}}$. By rescaling, we can assume this Cauchy sequence to be in B_Y. By the inequality we have just proven, it follows that $\{v_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in X and lies in K_S. So by compactness of K_S, it follows that $\{v_n\}_{n\in\mathbb{N}}$ has a limit $v \in K_S$. Define $w_n = v_n - v$, then $\{w_n\}_{n\in\mathbb{N}}$ converges to 0 in X and is a nonconvergent Cauchy sequence in Y. So there exists a $\delta > 0$ and a subsequence $\{w_{n_j}\}_{j\in\mathbb{N}}$ such that $\|w_{n_j}\|_Y \geq \delta$ for all $j \in \mathbb{N}$. Consider the sequence $\{z_j\}_{j\in\mathbb{N}}$ defined by $z_j = \|w_{n_j}\|_Y^{-1} w_{n_j}$. As $\|z_j\|_X \leq \delta^{-1}\|w_{n_j}\|_X$ it follows that $\{z_j\}_{j\in\mathbb{N}}$ converges to 0 in X. Furthermore

$$\begin{aligned}
\|z_i - z_j\|_Y &= \left\| \frac{w_{n_i}}{\|w_{n_i}\|_Y} - \frac{w_{n_j}}{\|w_{n_j}\|_Y} \right\|_Y \leq \frac{\|w_{n_i}(\|w_{n_j}\|_Y) - w_{n_j}(\|w_{n_i}\|_Y)\|_Y}{\delta^2} \\
&= \frac{\|w_{n_i}(\|w_{n_j}\|_Y) - w_{n_i}(\|w_{n_i}\|_Y) + w_{n_i}(\|w_{n_i}\|_Y) - w_{n_j}(\|w_{n_i}\|_Y)\|_Y}{\delta^2} \\
&\leq \frac{\left(\|w_{n_i}\|_Y |\|w_{n_j}\|_Y - \|w_{n_i}\|_Y| + \|w_{n_i}\|_Y \|w_{n_i} - w_{n_j}\|_Y\right)}{\delta^2} \\
&\leq \frac{2\|w_{n_i}\|_Y \|w_{n_i} - w_{n_j}\|_Y}{\delta^2},
\end{aligned}$$

where we used the reverse triangle inequality for the last step. As $\{w_{n_i}\}_{i\in\mathbb{N}}$ is bounded, we conclude that $\{z_j\}_{j\in\mathbb{N}}$ is Cauchy in Y. So there exist an integer $j_0 \in \mathbb{N}$ such that for all $i, j \geq j_0$, we have that $\|z_i - z_j\|_Y \leq \frac{1}{2}$. So for all $j \geq j_0$, it follows that $\|2(z_{j_0} - z_j)\|_Y \leq 1$ so $2(z_{j_0} - z_j) \in B_Y = K_S$. So as $z_j \to 0$ in X and K_S is closed in X it follows that $2z_{j_0} \in K_S = B_Y$, so $\|z_{j_0}\|_Y \leq \frac{1}{2}$ which contradicts $\|z_j\|_Y = 1$ for all $j \in \mathbb{N}$. So $(Y, \|\cdot\|_Y)$ is complete.

3. The identity map from Y into X maps the closed unit ball B_Y in Y to the compact set K_S in X, therefore it is a compact map. □

Corollary 1.20. *Let X be a Banach space and let $S \subset X$ be nonempty and precompact. Let Y be the Banach space as constructed in Proposition 1.19. Then the identity map from Y into X is continuous, hence bounded. In other words, there exists a positive constant C such that $\|y\|_X \leq C\|y\|_Y$ for all $y \in Y$, furthermore we can assume that $C \leq \max_{x\in K_S} \|x\|_X$*

Proof. Continuity of the identity map follows directly from the fact that the identity map is compact. The bound on C is found by the same argument as used in the proof of Proposition 1.19. □

Corollary 1.21. *Let X be a Banach space and let $S \subset X$ be nonempty and precompact. Let Y be the Banach space as constructed in Proposition 1.19. If the sequence $\{y_n\}_{n\in\mathbb{N}}$ converges to 0 in Y, then it also does in X. Furthermore, if H is defined as in Proposition 1.13, then the closed convex hull $\overline{\text{co}}(\{\alpha y_n : \alpha \in H, n \in \mathbb{N}\})$ is the same in both spaces.*

Proof. By continuity of the identity map from Y into X, it follows that $\{y_n\}_{n\in\mathbb{N}}$ also converges to 0 in X. So we can apply Proposition 1.13 both to $\{y_n\}_{n\in\mathbb{N}}$ as a sequence in X and as a sequence in Y and see that the closed convex hull is independent of the space. □

Now that we have seen how we can construct Banach spaces from precompact sets, we only need a couple more results to prove the second implication of Theorem 1.2.

Proposition 1.22 ([15, Lemma 3.4.31b])**.** *Suppose X is a Banach space and $A \subset X$ is precompact. Let $\{x_n\}_{n\in\mathbb{N}}$ be as in Proposition 1.15. Then there exists a compact subset $S \subset X$ such that $\overline{\text{co}}(\{x_n : n \in \mathbb{N}\}) \subset S$ and $\{x_n\}_{n\in\mathbb{N}}$ also converges to 0 in Y, where Y is the Banach space constructed from S as in Proposition 1.19.*

Proof. Suppose $A \subset X$ is precompact and $\{x_n\}_{n\in\mathbb{N}}$ is as in Proposition 1.15. From the formula for the closed convex hull found Proposition 1.13 it follows that elements $x_n = 0$ do not affect the shape of the closed convex hull, therefore we may assume $x_n \neq 0$ for all $n \in \mathbb{N}$. Define $y_n = \|x_n\|_X^{-1/2} x_n$ if $\|x_n\|_X < 1$ and $y_n = x_n$ otherwise. Then $\{y_n\}_{n\in\mathbb{N}}$ converges to 0 so

$\{y_n\}_{n\in\mathbb{N}}$ is precompact, therefore $S = \overline{\mathrm{co}}(\{y_n : n \in \mathbb{N}\})$ is compact by Mazur's compactness theorem. If $\|x_n\|_X \geq 1$, we have that $x_n = y_n \in S$ and if $\|x_n\|_X < 1$ then $x_n = \|x_n\|_X^{1/2} y_n \in S$ as S is convex and $0 \in S$. So as $\{x_n : n \in \mathbb{N}\} \subset S$ and since S is closed and convex, it follows that $\overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\}) \subset S$. Now let $(Y, \|\cdot\|_Y)$ be as in Proposition 1.19 and denote the closed unit ball in Y as B_Y. If $\|x_n\|_X < 1$ then $\|x_n\|_X^{-1/2} x_n = y_n \in S \subset B_Y$, hence $\|x_n\|_Y \leq \|x_n\|_X^{1/2}$ and therefore $\{x_n\}_{n\in\mathbb{N}}$ also converges to 0 in Y. □

The last step before we can prove the forward implication of Theorem 1.2 is to show that if Y is a Banach space as constructed in Proposition 1.19, we can approximate the bounded functionals of Y sufficiently well by bounded functionals of X. For this, we will need the so-called separating hyperplane theorem. From now on, if X is a normed space, we denote the dual space of X by X^*.

Theorem 1.23 (separating hyperplane theorem, [19, Theorem V.4])**.** *Let X be a Banach space over $\mathbb{F}$. Let A and B be disjoint convex sets in X. If A is compact and B is closed, there exists a linear functional $\varphi \in X^*$ and a real number b, such that $\Re(\varphi(x)) < b$ for all $x \in A$ and $\Re(\varphi(x)) > b$ for all $x \in B$.*

Lemma 1.24. *Let X be a Banach space and $K \subset X$ compact. Let $\delta > 0$ and let $\{x_n\}_{n\in\mathbb{N}}$ and S be as in Proposition 1.22. Let Y be the Banach space constructed from this S as in Proposition 1.19. Then for every bounded functional $y^* \in Y^*$ there exists a bounded functional $x^* \in X^*$ such that $|y^*x - x^*x| < \delta$ for all $x \in K$.*

Proof. Let $y^* \in Y^*$ be a bounded functional. By Proposition 1.22 it follows that $\{x_n\}_{n\in\mathbb{N}}$ also converges to 0 in Y. As y^* is continuous, we have that $\{y^*x_n\}_{n\in\mathbb{N}}$ converges to 0. This implies there exists an $n_0 \in \mathbb{N}$ such that $|y^*x_n| < \frac{\delta}{2}$ for all $n > n_0$. Let

$$K_{n_0} := 2\delta^{-1}\overline{\mathrm{co}}(\{\alpha x_n : \alpha \in \mathbb{F}, |\alpha| \leq 1, n > n_0\}).$$

By Corollary 1.21, K_{n_0} is well defined in the sense that it makes no difference in which space we take the closed convex hull. Furthermore, by Proposition 1.13 it follows that K_{n_0} is compact. Define

$$C := \{y \in \mathrm{span}(x_1, ..., x_{n_0}) : \Re(y^*y) = 1\}.$$

We claim the following: C is closed in Y and X; C is convex; we can assume C to be nonempty and C and K_{n_0} are disjoint. To prove C is closed in Y, let $y \in \overline{C}$ such that there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in C converging to y. Then it follows that $\Re(y^*y) = \lim_{n\to\infty} \Re(y^*y_n) = 1$, hence $y \in C$. So C is closed in $\text{span}(x_1, ..., x_{n_0})$ as a subspace of Y and as this is a finite-dimensional subspace and hence closed, this implies C is closed in Y. Since finite-dimensional topological vector spaces have a unique Hausdorff topology, we can also view $\text{span}(x_1, ..., x_{n_0})$ as a finite-dimensional subspace of X so by the same argument it follows that C is closed in X. Now pick $x, y \in C$ and $t \in \mathbb{R}$, then it follows that $\Re[y^*(tx+(1-t)y)] = t\Re(y^*x)+(1-t)\Re(y^*y) = 1$ implying that $tx+(1-t)y \in C$ for all $x, y \in C$ and $t \in \mathbb{R}$. In particular, this holds for all $t \in [0, 1]$ implying that C is convex. Now suppose that C is empty, this happens if and only if $\text{span}(x_1, ..., x_{n_0}) \subset \ker(y^*)$. Let $x^* \in X^*$ be the zero functional and let $x \in K \subset \overline{\text{co}}(\{x_n : n \in \mathbb{N}\})$. By Proposition 1.13 it follows we can write $x = \sum_{n\in\mathbb{N}} t_n x_n$ with $t_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} t_n \leq 1$, then we have that $|y^*x - x^*x| = |y^*x| = \sum_{n>n_0} t_n|y^*x_n| \leq \frac{\delta}{2}$. We see that $x^* = 0$ works, therefore we can assume that for at least one n such that $1 \leq n \leq n_0$ we have that $y^*x_n \neq 0$ and hence C is nonempty. To prove C and K_{n_0} are disjoint, we pick an element $y = 2\delta^{-1}x \in K_{n_0}$ such that $x \in \overline{\text{co}}(\{\alpha x_n : \alpha \in \mathbb{F}, |\alpha| \leq 1, n > n_0\})$. By Proposition 1.13 we can write $x = \sum_{n>n_0} t_n\alpha_n x_n$ with $t_n \geq 0$, $|\alpha_n| \leq 1$ and $\sum_{n\in\mathbb{N}} t_n \leq 1$. It follows that $|y^*y| = 2\delta^{-1}|y^*x| \leq 2\delta^{-1}\sum_{n>n_0} t_n|\alpha_n||y^*x_n| < \sum_{n>n_0} t_n|\alpha_n| \leq 1$. So for all $y \in K_{n_0}$ it follows that $\Re(y^*y) \leq |y^*y| < 1$ hence $y \notin C$. The converse argument is the same, so it follows that C and K_{n_0} are disjoint.

This means that we can apply the separating hyperplane theorem to C and K_{n_0} to obtain a real number $b \in \mathbb{R}$ and a bounded functional $x^* \in X^*$ such that $\Re(x^*x) < b$ for all $x \in K_{n_0}$ and $\Re(x^*x) > b$ for all $x \in C$. However, as for all $x, y \in C$ and $t \in \mathbb{R}$ we have that $tx + (1-t)y \in C$, it follows that $\Re x^*$ must be constant on C. Suppose $x, y \in C$ and $\Re(x^*x) \neq \Re(x^*y)$, then the map $t \mapsto t\Re(x^*x) + (1-t)\Re(x^*y)$ is surjective on $\mathbb{R}$, implying that $\Re x^*(C) = \mathbb{R}$, contradicting the existence of b. As $0 \in K_{n_0}$ it follows that $0 \in \Re x^*(K_{n_0})$, so $\Re x^*(C) \neq 0$. By rescaling we can therefore assume that $\Re x^*(C) = 1 = \Re y^*(C)$. We claim that this implies that $x^*x = y^*x$ for all $x \in \text{span}(x_1, ..., x_{n_0})$. First suppose that $x \in \text{span}(x_1, ..., x_{n_0})$ and that $\Re(y^*x) = r \neq 0$, then it follows that

$$\Re(y^*x) = \frac{r}{r}\Re(y^*x) = r\Re\left(y^*\left(\frac{x}{r}\right)\right) = r\Re\left(x^*\left(\frac{x}{r}\right)\right) = \Re(x^*x).$$

Now suppose that $\Re(y^*x) = 0$ and let $z \in C$, then it follows that

$$\begin{aligned}\Re(x^*x) &= \Re(x^*(x + z - z)) = \Re(x^*(x + z)) - \Re(x^*z) \\ &= \Re(y^*(x + z)) - \Re(y^*z) = \Re(y^*(x + z - z)) = 0.\end{aligned}$$

This means that for all $x \in \text{span}(x_1, ..., x_{n_0})$ we have that $\Re(x^*x) = \Re(y^*x)$. Now, this also implies that

$$\Im(x^*x) = \Re(x^*(-ix)) = \Re(y^*(-ix)) = \Im(y^*x),$$

and therefore it follows that $x^*x = y^*x$ for all $x \in \text{span}(x_1, ..., x_{n_0})$. In particular, it follows that $x^*x_n = y^*x_n$ for all n such that $1 \leq n \leq n_0$.

By Corollary 1.9, K_{n_0} is balanced. Let $n > n_0$ and let u_n be defined by $x^*x_n = |x^*x_n|u_n$ where we put $u_n = 1$ if $x^*x_n = 0$. Since K_{n_0} is balanced and $|u_n| = 1$ for all $n > n_0$, it follows from Proposition 1.8 that for all $n > n_0$ we have that $2\delta^{-1}x_nu_n^{-1} \in K_{n_0}$. Hence it follows that $2\delta^{-1}|x^*x_n| = 2\delta^{-1}x^*(x_nu_n^{-1}) < \Re x^*(C) = 1$ and thus that for all $n > n_0$ we have that $|x^*x_n| < \frac{\delta}{2}$. Now, if $x \in K$ there are nonnegative numbers t_n such that $\sum_{n\in\mathbb{N}} t_n \leq 1$ and $x = \sum_{n\in\mathbb{N}} t_nx_n$. It follows that for all $x \in K$

$$|x^*x - y^*x| = \left|\sum_{n\in\mathbb{N}} t_n(x^*x_n - y^*x_n)\right| \leq \sum_{n>n_0} t_n\left(|x^*x_n| + |y^*x_n|\right) < \delta.$$

□

Proof of Theorem 1.2, $1 \implies 2$. Let X be a Banach space and suppose that X has the approximation property. We need to prove that for every compact $K \subset X$ and $\epsilon > 0$ there exists a finite-rank operator $T_{K,\epsilon} \in F(X)$ such that $\|T_{K,\epsilon}x - x\| < \epsilon$ for all $x \in K$. So let $K \subset X$ be an arbitrary compact subset and let $\epsilon > 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ and $S \subset X$ be as in Proposition 1.22. Let Y be the Banach space constructed from S as in Proposition 1.19 and denote the closed unit ball of Y by B_Y. Let I be the identity map from Y into X. Then by Proposition 1.19 we have that $I \in K(Y, X)$. Since X has the approximation property, it follows that there exists a $\Phi_{K,\epsilon} \in F(Y, X)$ such that $\|\Phi_{K,\epsilon} - I\| < \frac{\epsilon}{2}$ where $\|\cdot\|$ denotes the operator norm in $B(Y, X)$. As $K \subset B_Y$ this implies that $\|\Phi_{K,\epsilon}x - x\|_X < \frac{\epsilon}{2}$ for all $x \in K$. Since $\Phi_{K,\epsilon} \in F(Y, X)$ there exist $m \in \mathbb{N}$, $y_1^*, ..., y_m^* \in Y^*$ and $z_1, ..., z_m \in X$ such that $\Phi_{K,\epsilon}x = \sum_{k=1}^m (y_k^*x)z_k$ for all $x \in Y$. Now set $\delta = \epsilon/(2m\max\{\|z_1\|_X, ..., \|z_m\|_X\})$. By Lemma 1.24, there exist functionals $x_k^* \in X^*$ such that $|y_k^*x - x_k^*x| < \delta$ for all $x \in K$. Set $T_{K,\epsilon}x = \sum_{k=1}^m (x_k^*x)z_k$, then for all $x \in K$ it follows that

$$\begin{aligned}
\|T_{K,\epsilon}x - x\|_X &\leq \|T_{K,\epsilon}x - \Phi_{K,\epsilon}x\|_X + \|\Phi_{K,\epsilon}x - x\|_X \\
&\leq \|T_{K,\epsilon}x - \Phi_{K,\epsilon}x\|_X + \frac{\epsilon}{2} \\
&= \left\|\sum_{k=1}^m (x_k^*x - y_k^*x)z_k\right\|_X + \frac{\epsilon}{2} \\
&\leq \sum_{k=1}^m |x_k^*x - y_k^*x|\|z_k\|_X + \frac{\epsilon}{2} \\
&< \delta m \max\{\|z_1\|_X, ..., \|z_m\|_X\} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

2 Well-definedness of the nuclear trace

In this section, we will construct a class of operators on Banach spaces on which we can define a trace. To motivate the construction, we go back to the well-known trace for matrices. Let V be a n-dimensional vector space and let $B = \{e_k\}_{k=1}^n$ be a basis. Let $A : V \to V$ be a linear operator and denote its matrix with respect to the basis B by $\{a_{ij}\}_{i,j=1}^n$. The usual way of defining the trace of A as $\mathrm{Tr}(A) = \sum_{k=1}^n a_{kk}$ is not suitable as we cannot define this in general Banach spaces. A different approach in defining the trace is to decompose the operator A into operators of rank one. If $x = \sum_{k=1}^n x_k e_k \in V$ is an arbitrary vector, then it follows that

$$Ax = \sum_{k=1}^n x_k A e_k = \sum_{k=1}^n \left(\sum_{j=1}^n a_{kj} x_j \right) e_k.$$

So if we define the linear functionals $\{\varphi_k\}_{k=1}^n \subset V^*$ by $\varphi_k(x) = \sum_{j=1}^n a_{kj} x_j$, then we can write $Ax = \sum_{k=1}^n \varphi_k(x) e_k$. Since $\varphi_k(e_k) = \sum_{j=1}^n a_{kj}\delta_{jk} = a_{kk}$, with δ_{jk} the Kronecker delta, it follows that we also define $\mathrm{Tr}(A) = \sum_{k=1}^n \varphi_k(e_k)$. Note that this new definition of the trace can in principle be used in any Banach space, as we eliminated the need for a matrix representation of our operator. However, we now require the operator to have a decomposition in terms of rank one operators.

If X and Y are Banach spaces and we have $x^* \in X^*$ and $y \in Y$, then we define the operator $x^* \otimes y \in B(X, Y)$ by $x^* \otimes y : x \mapsto x^*(x)y$. We see that $x^* \otimes y \in F(X, Y)$ for all $x^* \in X^*$ and $y \in Y$ as its image has dimension 0 or 1. Our discussion above motivates to introduce the following concepts.

Definition 2.1. Let X and Y be Banach spaces and let $A \in B(X, Y)$. Then A is a *nuclear operator* if and only if there are sequences $\{x_n^*\}_{n\in\mathbb{N}}$ in X^* and $\{y_n\}_{n\in\mathbb{N}}$ in Y such that $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$ and $A = \sum_{n=1}^\infty x_n^* \otimes y_n$, which is called the *nuclear representation*. We define $N(X, Y)$ as the space of all nuclear operators $A : X \to Y$.

From Definition 2.1 it is immediately clear that if $A \in N(X, Y)$, then $A \in \overline{F(X, Y)}$. Furthermore, it also follows that A is compact. As is clear from our discussion above, we are interested in these nuclear operators as they seem very suitable for defining a trace.

Definition 2.2. Let X be a Banach space and let $A \in N(X)$ be a nuclear operator, so $A = \sum_{n=1}^\infty x_n^* \otimes x_n$ with $\sum_{n=1}^\infty \|x_n^*\|\|x_n\| < \infty$. Then we define the *nuclear trace* of the representation by

$$\mathrm{Tr}\left(\sum_{n=1}^\infty x_n^* \otimes x_n \right) = \sum_{n=1}^\infty x_n^*(x_n).$$

It should be noted that Definitions 2.1 and 2.2 together do not imply that the nuclear trace is well-defined for a nuclear operator A. The reason for this is that a nuclear representation of a nuclear operator A is not unique and the definition of the nuclear trace depends on the chosen representation. Of course, we would like to prove that the trace is invariant under the chosen representation of a nuclear operator. However, this is not true in general. In this section, we will address the question of the well-definedness of this nuclear trace and we will prove that this is the case if and only if the Banach space X has the approximation property. To do this efficiently, it will prove to be useful to endow the space $B(Y, X)$ with a certain topology τ that captures the behaviour of being uniformly approximable on compact sets. Further references to this topology, in particular closures of sets, will be distinguished from the operator norm topology by means of a superscript τ. So if $A \subset B(Y, X)$, then the closure with respect to the topology τ will be denoted by $\overline{A}^{\tau}$.

2.1 The topology of uniform convergence on compact sets and its continuous linear functionals

To construct the topology mentioned in the introduction of this section, we first need some definitions. The following construction is based on Lindenstrauss and Tzafriri [13, p. 31].

Definition 2.3. Let X and Y be normed spaces over $\mathbb{F}$ and let $K \subset X$ be compact. For all operators $T \in B(X, Y)$, we define $\|T\|_K = \sup_{x \in K} \|Tx\|_Y$.

Lemma 2.4. *Let X and Y be normed spaces over $\mathbb{F}$ and let $K \subset X$ be compact. Then $\|\cdot\|_K$ is a seminorm on $B(X, Y)$.*

Proof. Let X and Y be normed spaces over $\mathbb{F}$ and let $K \subset X$ be compact. Suppose that $S, T \in B(X, Y)$ and $\alpha \in \mathbb{F}$. It follows that

$$\|\alpha T\|_K = \sup_{x \in K} \|\alpha Tx\|_Y = \sup_{x \in K} |\alpha| \|Tx\|_Y = |\alpha| \sup_{x \in K} \|Tx\|_Y = |\alpha| \|T\|_K,$$

and

$$\|S + T\|_K = \sup_{x \in K} \|Sx + Tx\|_Y \leq \sup_{x \in K} (\|Sx\|_Y + \|Tx\|_Y) \leq \|S\|_K + \|T\|_K.$$

Together these two properties imply that $\|\cdot\|_K$ is a seminorm on $B(X, Y)$. $\square$

Remark. 1. The seminorms $\|\cdot\|_K$ need not be norms. Take for example $K = \{x\}$ for some nonzero $x \in X$. If $\dim(X) \geq 2$, we can always find a linear operator $T \in B(X, Y)$ such that $T \neq 0$ but $x \in \ker(T)$. It follows that $T \neq 0$ but $\|T\|_{\{x\}} = 0$.

2. For each nonzero $T \in B(X,Y)$ there exists a compact $K \subset X$ such that $\|T\|_K > 0$. Take for example $K = \{x\}$ with $x \in X$ such that $Tx \neq 0$.

The second remark implies that the set of all seminorms as in Definition 2.3 is separating. So we can use it to construct a locally convex Hausdorff topology on the space $B(X,Y)$.

Definition 2.5. Let X and Y be Banach spaces. Define

$$\mathcal{F} := \{\|\cdot\|_K : K \subset X \text{ compact}\}.$$

We define the *topology of uniform convergence on compact sets* or *ucc topology*, written as τ, as the topology generated by the open balls of all seminorms in $\mathcal{F}$. That is, the open balls of all seminorms in $\mathcal{F}$ are a subbase for τ.

Considering the locally convex space $(B(X,Y),\tau)$ we can wonder what the continuous linear functionals with respect to this topology look like. Most of the remaining part of this subsection will be devoted to answering this question. Furthermore, from now on we write $B_r^{(K)}(T)$ for the open "ball" with radius $r > 0$ centered at $T \in B(X,Y)$ with respect to the seminorm $\|\cdot\|_K$ for some compact $K \subset X$. However, before we start constructing the continuous linear functionals on $(B(X,Y),\tau)$, we might wonder why we care about this topology. One of the reasons is that using this new topology, we can rephrase the property of being uniformly approximable on compact sets very conveniently.

Lemma 2.6. *Let $T \in B(X,Y)$ and suppose we have n positive real numbers $\epsilon_1, ..., \epsilon_n$ and n compact sets $K_1, ..., K_n$. Then there exist an $\epsilon > 0$ and a compact $K \subset X$ such that the following inclusion holds:*

$$B_\epsilon^{(K)}(T) \subset \bigcap_{i=1}^n B_{\epsilon_i}^{(K_i)}(T).$$

Proof. Let $T \in B(X,Y)$ and suppose we have n positive real numbers $\epsilon_1, ..., \epsilon_n$ and n compact sets $K_1, ..., K_n$. Suppose that $K = \bigcup_{i=1}^n K_i$ and that $\epsilon = \min_{1 \leq i \leq n} \epsilon_i$. K is compact as it is a finite union of compact sets. Let $S \in B_\epsilon^{(K)}(T)$ be arbitrary. Then it follows that for all $i \in \mathbb{N}$ such that $i \leq n$ that

$$\|S - T\|_{K_i} \leq \|S - T\|_K < \epsilon \leq \epsilon_i.$$

Therefore it follows that $S \in B_{\epsilon_i}^{(K_i)}(T)$ for all $i \in \mathbb{N}$ such that $i \leq n$, hence it follows that

$$B_\epsilon^{(K)}(T) \subset \bigcap_{i=1}^n B_{\epsilon_i}^{(K_i)}(T).$$

$\square$

Proposition 2.7. *Let $A \in B(X,Y)$ be a linear operator and let $V \subset B(X,Y)$ be a linear subspace. Then the following two assertions are equivalent.*

1. *For every compact $K \subset X$ and every $\epsilon > 0$, there exists some $T_{K,\epsilon} \in V$ such that $\|T_{K,\epsilon}x - Ax\|_Y < \epsilon$ for all $x \in K$.*
2. *$A \in \overline{V}^{\tau}$.*

Proof. We first prove the implication $1 \implies 2$. Assume that for every compact $K \subset X$ and every $\epsilon > 0$, there exists some $T_{K,\epsilon} \in V$ such that $\|T_{K,\epsilon}x - Ax\|_Y < \epsilon$ for all $x \in K$. We will argue by contradiction, so suppose $A \notin \overline{V}^{\tau}$. As $\overline{V}^{\tau}$ is closed, we can use Lemma 2.6 to obtain an $\epsilon > 0$ and compact $K \subset X$ such that $B_{\epsilon}^{(K)}(A) \cap V = \emptyset$. However, this implies that there exists no operator $T \in V$ such that $\|Tx - Ax\|_Y < \frac{\epsilon}{2}$ for all $x \in K$, which contradicts the assumption. So it follows that $A \in \overline{V}^{\tau}$.

To prove $2 \implies 1$, assume that $A \in \overline{V}^{\tau}$. We again argue by contradiction. Suppose there exists a compact $K \subset X$ and an $\epsilon > 0$ such that there is no operator $T \in V$ such that $\|Tx - Ax\|_Y < \epsilon$ for all $x \in K$. This implies that $B_{\epsilon}^{(K)}(A) \cap V = \emptyset$ and therefore it follows that $V \subset B(X,Y) \setminus B_{\epsilon}^{(K)}(A)$. Since the latter is closed, it follows that $\overline{V}^{\tau} \subset B(X,Y) \setminus B_{\epsilon}^{(K)}(A)$. By assumption, this implies that $A \in B(X,Y) \setminus B_{\epsilon}^{(K)}(A)$ which is a contradiction. So for every compact $K \subset X$ and every $\epsilon > 0$, there exists some $T_{K,\epsilon} \in V$ such that $\|T_{K,\epsilon}x - Ax\|_Y < \epsilon$ for all $x \in K$. □

Corollary 2.8. *Let X be a Banach space and let I be the identity operator on X. Then X has the approximation property if and only if $I \in \overline{F(X)}^{\tau}$.*

Proof. This follows from combining Theorem 1.2 with Proposition 2.7 for $A = I$ and $V = F(X)$. □

We see that using the ucc-topology, our second characterization of the approximation property becomes very concise. The ucc-topology also allows us to give to different characterizations of the approximation property.

Theorem 2.9 ([13, Theorem 1.e.4])**.** *Let X be a Banach space. Then the following are equivalent:*

1. *$I \in \overline{F(X)}^{\tau}$.*
2. *$\overline{F(X,Y)}^{\tau} = B(X,Y)$ for all Banach spaces Y.*
3. *$\overline{F(Y,X)}^{\tau} = B(Y,X)$ for all Banach spaces Y.*

Proof. The implications $2 \implies 1$ and $3 \implies 1$ are clear by taking $X = Y$. We prove the remaining implications $1 \implies 2$ and $1 \implies 3$.

$1 \implies 2$: It suffices to prove that $B(X,Y) \subset \overline{F(X,Y)}^{\tau}$ as the converse inclusion is trivial. Let $A \in B(X,Y)$ be a bounded linear operator. Let $\epsilon > 0$ be arbitrary and let $K \subset X$ be compact. As $I \in \overline{F(X)}^{\tau}$, it follows by Proposition 2.7 that there exists a $T \in F(X)$ such that $\|x - Tx\|_X < \epsilon$ for all $x \in K$. This implies that $\|Ax - ATx\|_Y \leq \|A\|\|x - Tx\|_X < \|A\|\epsilon$ for all $x \in K$. As ϵ and K were arbitrary and we have that $AT \in F(X,Y)$, it follows that $A \in \overline{F(X,Y)}^{\tau}$, again by Proposition 2.7. We conclude that $\overline{F(X,Y)}^{\tau} = B(X,Y)$.

$1 \implies 3$: It again suffices to prove that $B(Y,X) \subset \overline{F(Y,X)}^{\tau}$. Let $A \in B(Y,X)$ be a bounded linear operator. Let $\epsilon > 0$ be arbitrary and let $K \subset Y$ be compact. As A is continuous, the image $C = AK \subset X$ is compact. As $I \in \overline{F(X)}^{\tau}$, it follows again that there exists a $T \in F(X)$ such that $\|x - Tx\|_X < \epsilon$ for all $x \in C$. This implies that $\|Ay - TAy\|_X < \epsilon$ for all $y \in K$. As ϵ and K were arbitrary and we have that $TA \in F(Y,X)$, it follows that $A \in \overline{F(Y,X)}^{\tau}$. We conclude that $\overline{F(Y,X)}^{\tau} = B(Y,X)$. $\square$

Returning to the construction of the continuous linear functionals, we need a few preparatory results to prove the final result.

Lemma 2.10. *Let φ be a linear functional on $(B(X,Y),\tau)$. Then φ is continuous if and only if there exist a $C > 0$ and a compact $K \subset X$ such that*

$$|\varphi(T)| \leq C\|T\|_K \quad \forall T \in B(X,Y).$$

Proof. Let φ be a linear functional on $(B(X,Y),\tau)$. To prove the forward implication, let $B_1(0)$ be the open unit ball centred at $0 \in \mathbb{F}$. By continuity of φ, it follows that $\varphi^{-1}(B_1(0))$ is an open neighbourhood of $0 \in B(X,Y)$. So we can find n positive real numbers $\epsilon_1, ..., \epsilon_n$ and n compact sets $K_1, ..., K_n$ such that $\bigcap_{i=1}^n B_{\epsilon_i}^{(K_i)}(0) \subset \varphi^{-1}(B_1(0))$. By applying Lemma 2.6, we can find a compact $K \subset X$ and an $\epsilon > 0$ such that $B_\epsilon^{(K)}(0) \subset \varphi^{-1}(B_1(0))$. So it follows that $\varphi(B_\epsilon^{(K)}(0)) \subset B_1(0)$ and thus by rescaling we find $\varphi(B_1^{(K)}(0)) \subset B_{\epsilon^{-1}}(0)$. If $\|T\|_K = 0$, then for all $m \in \mathbb{N}$ we have that $\varphi(T) \in \varphi(B_{m^{-1}}^{(K)}(0)) \subset B_{(m\epsilon)^{-1}}(0)$ hence $\varphi(T) \in \bigcap_{m\in\mathbb{N}} B_{(m\epsilon)^{-1}}(0) = \{0\}$. Now suppose that $\|T\|_K \neq 0$ then

$$|\varphi(T)| = 2\|T\|_K \left|\varphi\left(\frac{T}{2\|T\|_K}\right)\right| \leq 2\epsilon^{-1}\|T\|_K.$$

By our previous calculation, this inequality obviously extends to the case $\|T\|_K = 0$, proving the forward implication.

For the converse implication, assume there exists a $C > 0$ and a compact $K \subset X$ such that for all $T \in B(X,Y)$ we have that $|\varphi(T)| \leq C\|T\|_K$. Let $U \subset \mathbb{F}$ be open. We prove that $\varphi^{-1}(U)$ is open. Suppose that $T \in \varphi^{-1}(U)$, thus we have that $\varphi(T) \in U$. Since U is open, we can find an $\epsilon > 0$ such that $B_\epsilon(\varphi(T)) \subset U$. By assumption, it follows that $B^{(K)}_{\epsilon C^{-1}}(T) \subset \varphi^{-1}(U)$ hence $\varphi^{-1}(U)$ is open and thus φ is continuous. $\square$

Lemma 2.11. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{n=1}^\infty a_n$ converges. Then there exists a sequence of positive real numbers $\{\eta_n\}_{n\in\mathbb{N}}$ such that $\eta_n \to \infty$ as $n \to \infty$ and $\sum_{n=1}^\infty a_n\eta_n$ converges.*

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{n=1}^\infty a_n$ converges. If only finitely many a_n are nonzero, any divergent sequence will work, therefore we can assume that infinitely many a_n are nonzero. Denote the sum by $S = \sum_{n=1}^\infty a_n$. Define the tails of the series as $t_n = \sum_{m=n}^\infty a_m$. It follows that $t_1 = S$ and as $\sum_{n=1}^\infty a_n$ converges, we also have that $\lim_{n\to\infty} t_n = 0$. As infinitely many a_n are nonzero, it follows that $t_n > 0$ for all $n \in \mathbb{N}$. Now define $\eta_n = 1/\sqrt{t_n}$. It is obvious that the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ diverges. It also follows that

$$\begin{aligned}\sum_{n=1}^\infty a_n\eta_n = \sum_{n=1}^\infty \frac{t_n - t_{n+1}}{\sqrt{t_n}} &= \sum_{n=1}^\infty \frac{(\sqrt{t_n} - \sqrt{t_{n+1}})(\sqrt{t_n} + \sqrt{t_{n+1}})}{\sqrt{t_n}} \\ &\leq 2\sum_{n=1}^\infty (\sqrt{t_n} - \sqrt{t_{n+1}}) = 2(\sqrt{S} - \lim_{n\to\infty}\sqrt{t_n}) = 2\sqrt{S},\end{aligned}$$

where the second-last equality follows from the fact that we have a telescoping series. It follows that the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ is indeed a divergent series such that $\sum_{n=1}^\infty a_n\eta_n$ converges. $\square$

Definition 2.12. Let X be a Banach space. For $p \geq 1$ we define the space $\left(\bigoplus_{n=1}^\infty X\right)_p$ as the space of all sequences $\{x_n\}_{n\in\mathbb{N}}$ in X such that the l_p-norm is finite: $\|\{x_n\}_{n\in\mathbb{N}}\|_p = \left(\sum_{n=1}^\infty \|x_n\|^p\right)^{1/p} < \infty$. We also define $\left(\bigoplus_{n=1}^\infty X\right)_0$ to be the space of sequences $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} x_n = 0$, equipped with the supremum norm $\|\cdot\|_\infty$.

Remark. The spaces defined in Definition 2.12 are also complete, see for example Megginson [15, Appendix C].

Definition 2.13. Let X be Banach space. For all sequences $\{x_n\}_{n\in\mathbb{N}}$ in X, we define the *projection on the i-th coordinate* by the map π_i such that $\pi_i(\{x_n\}_{n\in\mathbb{N}}) = x_i$.

Lemma 2.14. *Let X be a Banach space and suppose that $i \in \mathbb{N}$. Let π_i be the projection map from Definition 2.13 restricted to $\left(\bigoplus_{n=1}^\infty X\right)_0$. Then $\|\pi_i\| = 1$.*

Proof. Let X be a Banach space and suppose that $i \in \mathbb{N}$. Let π_i be the projection map from Definition 2.13 restricted to $\left(\bigoplus_{n=1}^\infty X\right)_0$. Then for any sequence $\{x_n\}_{n\in\mathbb{N}} \in \left(\bigoplus_{n=1}^\infty X\right)_0$ we have that $\|\pi_i(\{x_n\}_{n\in\mathbb{N}})\| = \|x_i\| \leq$

$\|\{x_n\}_{n\in\mathbb{N}}\|_\infty$. It follows that $\|\pi_i\| \leq 1$. Now let $x \in X$ be a unit vector and let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence defined by $x_n = x\delta_{n,i}$, where $\delta_{n,i}$ denotes the Kronecker delta. Then $\{x_n\}_{n\in\mathbb{N}} \in (\bigoplus_{n=1}^\infty X)_0$ and $\|\{x_n\}_{n\in\mathbb{N}}\|_\infty = 1$. So $\|\pi_i(\{x_n\}_{n\in\mathbb{N}})\| = \|x\| = 1 = \|\{x_n\}_{n\in\mathbb{N}}\|_\infty$, hence $\|\pi_i\| = 1$. $\square$

Proposition 2.15. *Let X be a Banach space and denote its dual space by X^*. Then the spaces $(\bigoplus_{n=1}^\infty X^*)_1$ and $\left((\bigoplus_{n=1}^\infty X)_0\right)^*$ are isometrically isomorphic.*

Proof. Let X be a Banach space and X^* its dual space. Define the map $\Phi : (\bigoplus_{n=1}^\infty X^*)_1 \to \left((\bigoplus_{n=1}^\infty X)_0\right)^*$ as $\{x_n^*\}_{n\in\mathbb{N}} \mapsto \sum_{n=1}^\infty x_n^* \circ \pi_n$. Then by Lemma 2.14 it follows that

$$\left\|\sum_{n=1}^\infty x_n^* \circ \pi_n\right\| \leq \sum_{n=1}^\infty \|x_n^*\|\|\pi_n\| = \sum_{n=1}^\infty \|x_n^*\| < \infty,$$

hence Φ is well defined. Furthermore, linearity of Φ is clear. We are done when we prove that Φ is isometric and surjective.

To prove that Φ is isometric, we need to show that $\|\sum_{n=1}^\infty x_n^* \circ \pi_n\| = \sum_{n=1}^\infty \|x_n^*\|$ for all $\{x_n^*\}_{n\in\mathbb{N}} \in (\bigoplus_{n=1}^\infty X^*)_1$. By the computation above, it remains to show that $\|\sum_{n=1}^\infty x_n^* \circ \pi_n\| \geq \sum_{n=1}^\infty \|x_n^*\|$ for all $\{x_n^*\}_{n\in\mathbb{N}} \in (\bigoplus_{n=1}^\infty X^*)_1$. Let $\{x_n^*\}_{n\in\mathbb{N}} \in (\bigoplus_{n=1}^\infty X^*)_1$ be arbitrary, let $\epsilon > 0$ and let $N \in \mathbb{N}$. Then by definition of the operator norm, there exists $x_i \in X$ such that $\|x_i\| \leq 1$ and $x_i^* x_i > \|x_i^*\| - \frac{\epsilon}{N}$ for all $i \leq N$. Now define $\{y_n\}_{n\in\mathbb{N}}$ by $y_i = x_i$ for all $i \leq N$ and $y_n = 0$ for all $n > N$. It follows that $\{y_n\}_{n\in\mathbb{N}} \in (\bigoplus_{n=1}^\infty X)_0$ and $\|\{y_n\}_{n\in\mathbb{N}}\|_\infty \leq 1$. So we have

$$\begin{aligned}\left\|\sum_{n=1}^\infty x_n^* \circ \pi_n\right\| \geq \left\|\left(\sum_{n=1}^\infty x_n^* \circ \pi_n\right)(\{y_m\}_{m\in\mathbb{N}})\right\| &= \left\|\sum_{n=1}^N x_n^*(x_n)\right\| \\ > \sum_{n=1}^N \|x_n^*\| - \frac{\epsilon}{N} &= -\epsilon + \sum_{n=1}^N \|x_n^*\|.\end{aligned}$$

We conclude that for all $\epsilon > 0$ and $N \in \mathbb{N}$, we have that $\sum_{n=1}^N \|x_n^*\| < \|\sum_{n=1}^\infty x_n^* \circ \pi_n\| + \epsilon$ and thus $\sum_{n=1}^\infty \|x_n^*\| \leq \|\sum_{n=1}^\infty x_n^* \circ \pi_n\|$. This precisely means that Φ is an isometry.

To prove surjectivity let $\varphi \in \left((\bigoplus_{n=1}^\infty X)_0\right)^*$ be an arbitrary functional. If $x \in X$ is a vector and $i \in \mathbb{N}$, define $(x)_i$ as the sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n = x\delta_{n,i}$, where $\delta_{n,i}$ denotes the Kronecker delta. Define the sequence $\{x_n^*\}_{n\in\mathbb{N}}$ by setting $x_n^* x = \varphi((x)_n)$. We need to check that $\{x_n^*\}_{n\in\mathbb{N}} \in (\bigoplus_{n=1}^\infty X^*)_1$ and that $\Phi(\{x_n^*\}_{n\in\mathbb{N}}) = \varphi$. To prove the first assertion, let $\epsilon > 0$ and $N \in \mathbb{N}$. Then for all $i \leq N$ there exists $x_i \in X$ such that $\|x_i\| \leq 1$ and $x_i^* x_i > \|x_i^*\| - \frac{\epsilon}{N}$. Define $x = \sum_{i=1}^N (x_i)_i$. Then $x \in (\bigoplus_{n=1}^\infty X)_0$ and

$\|x\|_\infty \leq 1$. It follows that

$$|\varphi(x)| = \left|\sum_{i=1}^{N} \varphi((x_i)_i)\right| = \left|\sum_{i=1}^{N} x_i^* x_i\right| > -\epsilon + \sum_{i=1}^{N} \|x_i^*\|.$$

We conclude that for all $\epsilon > 0$ and $N \in \mathbb{N}$ we have that $\sum_{i=1}^{N} \|x_i^*\| < \|\varphi\| + \epsilon$ and thus $\sum_{i=1}^{\infty} \|x_i^*\| \leq \|\varphi\|$. It follows that $\{x_n^*\}_{n\in\mathbb{N}} \in \left(\bigoplus_{n=1}^{\infty} X^*\right)_1$. To prove that $\Phi(\{x_n^*\}_{n\in\mathbb{N}}) = \varphi$, let $x = \{x_n\}_{n\in\mathbb{N}} \in \left(\bigoplus_{n=1}^{\infty} X\right)_0$ be arbitrary. Then

$$\begin{aligned}\Phi(\{x_n^*\}_{n\in\mathbb{N}})(x) &= \left(\sum_{n=1}^{\infty} x_n^* \circ \pi_n\right)(x) = \lim_{N\to\infty} \sum_{n=1}^{N} x_n^* \circ \pi_n(x) = \lim_{N\to\infty} \sum_{n=1}^{N} x_n^* x_n \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \varphi((x_n)_n) = \lim_{N\to\infty} \varphi\left(\sum_{n=1}^{N} (x_n)_n\right) = \varphi(x).\end{aligned}$$

The last equality is justified as φ is continuous with respect to the topology induced by the supremum norm and $\lim_{N\to\infty} \left\|x - \sum_{n=1}^{N} (x_n)_n\right\|_\infty = 0$ since $x \in \left(\bigoplus_{n=1}^{\infty} X\right)_0$. $\square$

Having proven these preparatory results, we can find a general form of the continuous linear functionals on the space $(B(X,Y), \tau)$. The theorem and proof given here are from [13].

Theorem 2.16 ([13, Proposition 1.e.3])**.** *Let X and Y be Banach spaces and let τ be the topology of uniform convergence on compact sets in X. Then the continuous linear functionals on $(B(X,Y), \tau)$ are precisely all functionals φ that have a representation in the following form:*

$$\varphi(T) = \sum_{n=1}^{\infty} y_n^*(Tx_n), \quad \{x_n\}_{n\in\mathbb{N}} \subset X, \quad \{y_n^*\}_{n\in\mathbb{N}} \subset Y^*, \quad \sum_{n=1}^{\infty} \|y_n^*\|\|x_n\| < \infty.$$

Proof. First, suppose φ has a representation as in the theorem. We can assume that $x_n \neq 0$ for all $n \in \mathbb{N}$. By Lemma 2.11, there exists a sequence of positive real numbers $\{\eta_n\}_{n\in\mathbb{N}}$ such that $\eta_n \to \infty$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \eta_n \|x_n\|\|y_n^*\| = C < \infty$. Let $K = \{x_n/\|x_n\|\eta_n\}_{n\in\mathbb{N}} \cup \{0\}$. Then K is compact in X as any cover of open sets has a finite subcover. It follows that:

$$|\varphi(T)| \leq \sum_{n=1}^{\infty} \|y_n^*\|\|Tx_n\| = \sum_{n=1}^{\infty} \eta_n \|x_n\|\|y_n^*\|\|T(x_n/\|x_n\|\eta_n)\| \leq C\|T\|_K.$$

By Lemma 2.10 this implies that φ is a continuous linear functional.

Conversely, suppose that φ is a continuous linear functional on $B(X,Y)$. By Lemma 2.10 this implies that there exists a compact $K \subset X$ and a

$C > 0$ such that $|\varphi(T)| \leq C\|T\|_K$ for all $T \in B(X,Y)$. By Proposition 1.15, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, converging to 0, such that $K \subset \overline{\text{co}}(\{x_n\}_{n\in\mathbb{N}})$. Therefore it follows that $|\varphi(T)| \leq C\|T\|_K \leq C\|T\|_{\overline{\text{co}}(\{x_n\}_{n\in\mathbb{N}})}$. So without loss of generality, we can assume that $K = \overline{\text{co}}(\{x_n\}_{n\in\mathbb{N}})$. Define $S : B(X,Y) \to \left(\bigoplus_{n=1}^{\infty} Y\right)_0$ by $T \mapsto \{Tx_n\}_{n\in\mathbb{N}}$. Since $K = \overline{\text{co}}(\{x_n\}_{n\in\mathbb{N}})$, we can use Proposition 1.13 to write $x = \sum_{n\in\mathbb{N}} t_n x_n$ with $t_n \geq 0$ and $\sum_{n\in\mathbb{N}} t_n \leq 1$ for all $x \in K$. So

$$\begin{aligned}
|\varphi(T)| &\leq C\|T\|_K = C \sup_{x\in K} \|Tx\| = C \sup_{\substack{t_n \geq 0 \\ \sum_{n\in\mathbb{N}} t_n \leq 1}} \left\| T\left(\sum_{n\in\mathbb{N}} t_n x_n\right)\right\| \\
&\leq C \sup_{\substack{t_n \geq 0 \\ \sum_{n\in\mathbb{N}} t_n \leq 1}} \sum_{n\in\mathbb{N}} t_n \|Tx_n\| \leq C\|S(T)\|_\infty.
\end{aligned}$$

This implies there exists a well-defined linear functional $\psi : SB(X,Y) \to \mathbb{F}$ such that $\psi(S(T)) = \varphi(T)$ for all $T \in B(X,Y)$. Indeed, suppose that $T, U \in B(X,Y)$ and that $S(U) = S(T)$, then by our calculation above it follows that $|\varphi(T) - \varphi(U)| = |\varphi(T-U)| \leq C\|S(U) - S(T)\|_\infty = 0$. Therefore it follows that $\varphi(U) = \varphi(T)$ and thus ψ is well-defined on $SB(X,Y)$ and can be continuously extended to the closure of $SB(X,Y)$ in $\left(\bigoplus_{n=1}^{\infty} Y\right)_0$. By definition, it follows that $|\psi(S(T))| = |\varphi(T)| \leq C\|S(T)\|_\infty$, hence ψ is bounded. By the Hahn-Banach theorem, we can extend ψ to a bounded functional on $\left(\bigoplus_{n=1}^{\infty} Y\right)_0$ which by Proposition 2.15 corresponds to an element $\{y_n^*\}_{n\in\mathbb{N}} \in \left(\bigoplus_{n=1}^{\infty} Y^*\right)_1$. By using the isometric isomorphism constructed in Proposition 2.15 it follows that

$$\varphi(T) = \psi(S(T)) = \sum_{n=1}^{\infty} y_n^* \circ \pi_n(S(T)) = \sum_{n=1}^{\infty} y_n^*(Tx_n) \quad \forall T \in B(X,Y).$$

As $\{x_n\}_{n\in\mathbb{N}}$ is bounded and $\{y_n^*\}_{n\in\mathbb{N}}$ is absolutely summable, it follows that $\sum_{n=1}^{\infty} \|y_n^*\|\|x_n\| < \infty$ as required. $\square$

Remark. Note that the τ-continuous linear functionals have striking similarities with the nuclear operators. If $A : X \to Y$ is a nuclear operator with a nuclear representation $A = \sum_{n=1}^{\infty} x_n^* \otimes y_n$, then we can define a τ-continuous linear functional φ by $\varphi(T) = \sum_{n=1}^{\infty} x_n^*(Ty_n)$ for all $T \in B(Y,X)$. If $X = Y$, it directly follows that $\operatorname{Tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes y_n\right) = \varphi(I)$. In Section 2.3, we will study this connection in more detail.

2.2 Nuclear operators and the nuclear trace

Having developed quite some machinery concerning the ucc-topology, we can use these results to study nuclear operators. We will spend the rest of this subsection proving the following result, which is also due to Grothendieck

[8] and makes use of the idea that we can view the nuclear trace as the evaluation of some continuous functional at the identity operator.

Theorem 2.17. *Let X be a Banach space. For all nuclear operators $A \in N(X)$, we define the nuclear trace of the operator A as* $\operatorname{Tr}(A) = \operatorname{Tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes x_n\right)$ *where* $\sum_{n=1}^{\infty} x_n^* \otimes x_n$ *is a nuclear representation of A. This trace is well-defined (i.e. representation independent) for all nuclear operators A if and only if X has the approximation property.*

To prove Theorem 2.17, we need some intermediate results. The proof of the following result is from Rudin [21, Theorem 1.13].

Lemma 2.18. *Let X be a topological vector space over $\mathbb{F}$ with topology τ and let $V \subset X$ be a linear subspace. Then $\overline{V}^{\tau}$ is a linear subspace of X.*

Proof. First, note that for any set $S \subset X$ we have the following equivalence: $x \in \overline{S}^{\tau}$ if and only if for all open neighbourhoods U of x, we have $S \cap U \neq \emptyset$ (this is a general feature of topological spaces). This implies that for two subsets A and B of X we have that $\overline{A}^{\tau} + \overline{B}^{\tau} \subset \overline{A+B}^{\tau}$. To prove this, suppose that $a \in \overline{A}^{\tau}$ and $b \in \overline{B}^{\tau}$ and let U be an open neighbourhood of $a+b$. We will prove that $U \cap (A+B) \neq \emptyset$. Since the addition map $+ : X \times X \to X$ is continuous, it follows that the inverse image of U under addition is an open neighbourhood of $(a, b) \in X \times X$. Hence there exist open neighbourhoods U_1 of a and U_2 of b such that $U_1 + U_2 \subset U$. Now as $a \in \overline{A}^{\tau}$ and $b \in \overline{B}^{\tau}$ it follows that we can find elements $x \in A \cap U_1$ and $y \in B \cap U_2$. It follows that $x + y \in (A+B) \cap (U_1 + U_2) \subset (A+B) \cap U$. As $(A+B) \cap U \neq \emptyset$ for all open neighbourhoods of $a+b$, it follows that $a + b \in \overline{A+B}^{\tau}$, we conclude that $\overline{A}^{\tau} + \overline{B}^{\tau} \subset \overline{A+B}^{\tau}$.

Furthermore, in topological vector spaces we also have that for any nonzero scalar α the map $M_\alpha : X \to X$, defined by $x \mapsto \alpha x$, is a homeomorphism. Hence, it follows that for any subset $S \subset X$ we have that $\alpha\overline{S}^{\tau} = \overline{\alpha S}^{\tau}$. For $\alpha = 0$, the same equality holds since $\{0\}$ is closed. We conclude that for all scalars α, we have that $\alpha\overline{S}^{\tau} = \overline{\alpha S}^{\tau}$. Now let α be a scalar and let $x, y \in \overline{V}^{\tau}$ be two vectors. Then by the previous two results, it follows that $\alpha x + y \in \alpha\overline{V}^{\tau} + \overline{V}^{\tau} = \overline{\alpha V}^{\tau} + \overline{V}^{\tau} \subset \overline{\alpha V + V}^{\tau} \subset \overline{V}^{\tau}$. $\square$

Corollary 2.19. *Let X be a Banach space and let $V \subset B(X)$ be a linear subspace. Then $\overline{V}^{\tau}$ is a linear subspace of $B(X)$.*

Remark. Whereas Lemma 2.18 applies to any topological vector space with a topology τ, we now assume τ to be the ucc-topology.

To proceed, we need the following theorem, which is a consequence of the separating hyperplane theorem for locally convex spaces. We will not give proof. This can be found in Rudin [21].

Theorem 2.20 ([21, Theorem 3.5])**.** *Let X be a locally convex space and denote its topology by τ. Let $V \subset X$ be a linear subspace of X. Suppose that $x \notin \overline{V}^{\tau}$. Then there exists a continuous linear functional φ such that $\varphi(x) = 1$ and φ vanishes on V.*

Corollary 2.21. *Let X be a Banach space and let τ be the ucc-topology of $B(X)$. Let $V \subset B(X)$ be a linear subspace and suppose that $A \in B(X)$. Then $A \in \overline{V}^{\tau}$ if and only if each continuous linear functional φ of $(B(X), \tau)$ that vanishes on V also vanishes on A.*

Proof. First, suppose that $A \in \overline{V}^{\tau}$. Let φ be a continuous linear functional of $(B(X), \tau)$ that vanishes on V. It follows that $V \subset \varphi^{-1}(\{0\})$. Continuity of φ implies that $\varphi^{-1}(\{0\})$ is τ-closed. So, it follows that $\overline{V}^{\tau} \subset \varphi^{-1}(\{0\})$ and therefore that $A \in \varphi^{-1}(\{0\})$. Hence φ vanishes on A.

Conversely, suppose that each continuous linear functional φ of $(B(X), \tau)$ that vanishes on V also vanishes on A. Suppose that $A \notin \overline{V}^{\tau}$. By Theorem 2.20, there exists a continuous linear functional that vanishes on V but is non-zero on A, which is a clear contradiction. □

Proposition 2.22. *Let X be a Banach space. Then X has the approximation property if and only if each nuclear representation of the* 0*-operator has nuclear trace* 0*.*

Proof. Let X be a Banach space. Combining Corollary 2.8 and Corollary 2.21 for $A = I$ and $V = F(X)$ gives that X has the approximation property if and only if each τ-continuous functional that vanishes on $F(X)$ also vanishes on I.

Suppose that each τ-continuous functional that vanishes on all finite rank operators also vanishes on I and let $\sum_{n=1}^{\infty} x_n^* \otimes x_n$ be a nuclear representation of the 0-operator. Then it follows that $\sum_{n=1}^{\infty} x_n^*(x)x_n = 0$ for all $x \in X$. Let φ be the continuous functional given by $\varphi(T) = \sum_{n=1}^{\infty} x_n^*(Tx_n)$. Then for all $x^* \in X^*$ and $x \in X$

$$\begin{aligned}\varphi(x^* \otimes x) &= \sum_{n=1}^{\infty} x_n^*(x^* \otimes x(x_n)) = \sum_{n=1}^{\infty} x_n^*(x^*(x_n)x) = \sum_{n=1}^{\infty} x_n^*(x)x^*(x_n) \\ &= x^* \left(\sum_{n=1}^{\infty} x_n^*(x)x_n \right) = x^*(0) = 0.\end{aligned}$$

So φ vanishes on all rank one operators and therefore by linearity it follows that φ vanishes on all finite rank operators. By assumption, it now follows that $\mathrm{Tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes x_n\right) = \varphi(I) = 0$.

Conversely, suppose that each nuclear representation of the 0-operator has nuclear trace 0. Let φ be a τ-continuous linear functional that vanishes on $F(X)$. We are finished when we prove that $\varphi(I) = 0$. By Theorem 2.16 there exist sequences $\{y_n^*\}_{n\in\mathbb{N}}$ in X^* and $\{y_n\}_{n\in\mathbb{N}}$ in X such that

$\sum_{n=1}^{\infty} \|y_n^*\|\|y_n\| < \infty$ and $\varphi(T) = \sum_{n=1}^{\infty} y_n^*(Ty_n)$. As φ vanishes on all finite rank operators, it vanishes in particular on all rank one operators. So it follows that

$$0 = \varphi(x^* \otimes x) = \sum_{n=1}^{\infty} y_n^*(x^* \otimes x(y_n)) = \sum_{n=1}^{\infty} y_n^*(x^*(y_n)x) = \sum_{n=1}^{\infty} y_n^*(x)x^*(y_n)$$
$$= x^* \left(\sum_{n=1}^{\infty} y_n^*(x) y_n \right) \quad \forall x^* \in X^* \; \forall x \in X.$$

As the bounded linear functionals separate all points $y \in X$, it follows that $\sum_{n=1}^{\infty} y_n^*(x)y_n = 0$ for all $x \in X$. Hence $\sum_{n=1}^{\infty} y_n^* \otimes y_n$ is a nuclear representation of the 0-operator, which by assumption has nuclear trace 0. It follows that $\varphi(I) = \mathrm{Tr}\left(\sum_{n=1}^{\infty} y_n^* \otimes y_n\right) = 0$. $\square$

With this result, we can now prove our main result of this section, Theorem 2.17.

Proof of Theorem 2.17. Let X be a Banach space. First, suppose the nuclear trace is well-defined for all nuclear operators $A \in N(X)$. Let $\sum_{n=1}^{\infty} x_n^* \otimes x_n$ be a nuclear representation of the 0-operator. As $\sum_{n=1}^{\infty} x_n^* \otimes 0$ is also a nuclear representation of the 0-operator and the nuclear trace is well-defined by assumption, it follows that

$$\mathrm{Tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes x_n \right) = \mathrm{Tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes 0 \right) = \sum_{n=1}^{\infty} x_n^*(0) = 0.$$

So each representation of the 0-operator has a nuclear trace equal to 0 and therefore by Proposition 2.22 it follows that X has the approximation property.

Conversely, suppose that X has the approximation property. Let $A \in N(X)$ be a nuclear operator. Suppose that $\sum_{n=1}^{\infty} x_n^* \otimes x_n$ and $\sum_{n=1}^{\infty} y_n^* \otimes y_n$ are two nuclear representations of A. Then their difference

$$\sum_{n=1}^{\infty} x_n^* \otimes x_n - \sum_{n=1}^{\infty} y_n^* \otimes y_n = A - A = 0$$

is a nuclear representation of the 0-operator. Since X has the approximation property by assumption, it follows from Proposition 2.22 that $\mathrm{Tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes x_n - \sum_{n=1}^{\infty} y_n^* \otimes y_n\right) = 0$. From Definition 2.2 it follows that the nuclear trace is additive, hence it follows that $\mathrm{Tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes x_n\right) = \mathrm{Tr}\left(\sum_{n=1}^{\infty} y_n^* \otimes y_n\right)$. So all representations of A have the same nuclear trace, hence the nuclear trace of A is well-defined. $\square$

2.3 The Banach space $N(X, Y)$

In the previous subsection, we proved that the nuclear trace is well-defined for all nuclear operators in $N(X)$ if and only if X has the approximation property. Moreover, from the definition of the nuclear trace it is clear that the map $\mathrm{Tr} : N(X) \to \mathbb{F}$ is a linear operator if X has the approximation property. Motivated by this, we can try to equip $N(X, Y)$ with a norm $\|\cdot\|_N$ such that the nuclear trace is a bounded functional on the space $(N(X), \|\cdot\|_N)$. The construction of this is motivated by Diestel, Fourie and Swart [3, p. 10, Proposition 1.14] where the projective tensor product $X^* \widehat{\bigotimes} Y$ is endowed with a similar norm.

Furthermore, we also encountered an intimate connection between nuclear representations of nuclear operators in $N(X, Y)$ and the continuous functionals on $(B(Y, X), \tau)$. This invites us to take a closer look at these spaces, which we will do at the end of this section.

Definition 2.23. Let X and Y be Banach spaces. Then for all nuclear operators $A \in N(X, Y)$ we define the nuclear norm

$$\|A\|_N = \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : A = \sum_{n=1}^{\infty} x_n^* \otimes y_n \right\}.$$

Proposition 2.24. *Let X and Y be Banach spaces. Then for all nuclear operators $A \in N(X, Y)$ we have that $\|A\| \leq \|A\|_N$ and the nuclear norm $\|\cdot\|_N$ is a norm on $N(X, Y)$.*

Proof. Let X and Y be Banach spaces and let $A \in N(X, Y)$ be a nuclear operator. Then for any nuclear representation $A = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ of A we have that $\|A\| \leq \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\|$. When we take the infimum over all nuclear representations of A it follows that $\|A\| \leq \|A\|_N$ for all $A \in N(X, Y)$.

To prove that $\|\cdot\|_N$ is a norm, suppose that $\alpha \in \mathbb{F}$ and $A, B \in N(X, Y)$. It follows that $\|\alpha A\|_N = \inf \{\sum_{n=1}^{\infty} |\alpha| \|x_n^*\| \|y_n\| : \alpha A = \sum_{n=1}^{\infty} x_n^* \otimes \alpha y_n\} = |\alpha| \|A\|_N$. We also have that

$$\begin{aligned} \|A + B\|_N &= \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : A + B = \sum_{n=1}^{\infty} x_n^* \otimes y_n \right\} \\ &\leq \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| + \sum_{n=1}^{\infty} \|u_n^*\| \|v_n\|, \end{aligned}$$

for all representations $A = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ and $B = \sum_{n=1}^{\infty} u_n^* \otimes v_n$. By taking the infima over the representations of A and B we obtain $\|A + B\|_N \leq \|A\|_N + \|B\|_N$. Finally, if $\|A\|_N = 0$ it follows that $\|A\| \leq \|A\|_N = 0$ so $\|A\| = 0$ and hence $A = 0$. So $\|\cdot\|_N$ is a norm on $N(X, Y)$. $\square$

Having constructed the normed space $(N(X,Y), \|\cdot\|_N)$, we can wonder if this space is complete too. This turns out to be the case. We give a proof according to Pietsch [17].

Proposition 2.25 ([17, Lemma 3.1.3])**.** *Let X and Y be Banach spaces. Then the normed space $(N(X,Y), \|\cdot\|_N)$ is Banach.*

Proof. Let X and Y be Banach spaces and let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(N(X,Y), \|\cdot\|_N)$. As $\|T_n - T_m\| \leq \|T_n - T_m\|_N$ for all $n, m \in \mathbb{N}$, it follows that $\{T_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $B(X,Y)$ with the respect to the operator norm. Since Y is complete, so is $B(X,Y)$ and therefore $\{T_n\}_{n\in\mathbb{N}}$ has a $\|\cdot\|$-limit T in $B(X,Y)$.

Since $\{T_n\}_{n\in\mathbb{N}}$ is Cauchy in $(N(X,Y), \|\cdot\|_N)$, there exists an increasing sequence of integers N_k such that $\|T_n - T_m\|_N < 1/2^{k+2}$ for all $n, m \geq N_k$. By definition of the nuclear norm, this implies that for all $k \in \mathbb{N}$ we have nuclear representations $T_{N_{k+1}} - T_{N_k} = \sum_{n\in\mathbb{N}} (x_n^{(k)})^* \otimes y_n^{(k)}$ with $\sum_{n\in\mathbb{N}} \|(x_n^{(k)})^*\| \|y_n^{(k)}\| < 1/2^{k+2}$. As this holds for all $k \in \mathbb{N}$, it follows that for all $l \in \mathbb{N}$

$$T_{N_{k+l}} - T_{N_k} = \sum_{m=k}^{k+l-1} (T_{N_{m+1}} - T_{N_m}) = \sum_{m=k}^{k+l-1} \sum_{n\in\mathbb{N}} (x_n^{(m)})^* \otimes y_n^{(m)}.$$

As T_n converges to T, we can take the limit as $l \to \infty$. It follows that

$$T - T_{N_k} = \sum_{m=k}^{\infty} (T_{N_{m+1}} - T_{N_m}) = \sum_{m=k}^{\infty} \sum_{n\in\mathbb{N}} (x_n^{(m)})^* \otimes y_n^{(m)}.$$

As

$$\sum_{m=k}^{\infty} \sum_{n\in\mathbb{N}} \|(x_n^{(m)})^*\| \|y_n^{(m)}\| \leq \sum_{m=k}^{\infty} \frac{1}{2^{m+2}} = \frac{1}{2^{k+1}},$$

it follows that $T - T_{N_k}$ is nuclear and that $\|T - T_{N_k}\|_N \leq 1/2^{k+1}$. So $T = (T - T_{N_k}) + T_{N_k}$ is nuclear. We claim that T is also the limit of $\{T_n\}_{n\in\mathbb{N}}$ with respect to the nuclear norm. Let $\epsilon > 0$ be given. Then there exists a $k \in \mathbb{N}$ such that $1/2^k < \epsilon$. So for all $m \geq N_k$ it follows that $\|T - T_m\|_N \leq \|T - T_{N_k}\|_N + \|T_{N_k} - T_m\|_N \leq 1/2^{k+1} + 1/2^{k+2} < 1/2^k < \epsilon$. It follows that $\{T_n\}_{n\in\mathbb{N}}$ converges to $T \in N(X,Y)$ with respect to the nuclear norm, hence $(N(X,Y), \|\cdot\|_N)$ is complete. □

Remark. Note that we did not use completeness of X in the proof of Proposition 2.25. However, the result is formulated this way as we only consider nuclear operator acting on Banach spaces. If we would allow for nuclear operators on normed spaces, then we would only require completeness of Y.

We will now prove that the nuclear trace is indeed a bounded linear functional on the space $N(X)$ if X has the approximation property and we equip $N(X)$ with the nuclear norm.

Proposition 2.26. *Let X be a Banach space that has the approximation property. Then the nuclear trace $\operatorname{Tr} : N(X) \to \mathbb{F}$ is a bounded linear functional on $(N(X), \|\cdot\|_N)$.*

Proof. Let X be a Banach space that has the approximation property. Then the nuclear trace is well-defined by Theorem 2.17 and is linear by definition. Now suppose that $A \in N(X)$ and that $\epsilon > 0$. By definition of the nuclear norm, there exists a nuclear representation $A = \sum_{n=1}^{\infty} x_n^* \otimes x_n$ such that $\sum_{n=1}^{\infty} \|x_n^*\|\|x_n\| < \|A\|_N + \epsilon$. It follows that

$$|\operatorname{Tr}(A)| = \left|\operatorname{Tr}\left(\sum_{n=1}^{\infty} x_n^* \otimes x_n\right)\right| = \left|\sum_{n=1}^{\infty} x_n^*(x_n)\right| \leq \sum_{n=1}^{\infty} \|x_n^*\|\|x_n\| < \|A\|_N + \epsilon.$$

As $\epsilon > 0$ was arbitrary, it follows that $|\operatorname{Tr}(A)| \leq \|A\|_N$ for all $A \in N(X)$. Therefore it follows that $\|\operatorname{Tr}\| \leq 1$. Hence Tr is a bounded functional of the Banach space $(N(X), \|\cdot\|_N)$. $\square$

As promised, we will have a closer look at the connection between nuclear operators and the linear functionals that are continuous with respect to the ucc-topology. To illustrate this, let X and Y be Banach spaces and $N(X,Y)$ be the corresponding space of nuclear operators. Let $A \in N(X,Y)$ be a nuclear operator. We have already seen that for any representation $A = \sum_{n\in\mathbb{N}} x_n^* \otimes y_n$, where $\{x_n^*\}_{n\in\mathbb{N}}$ is in X^* and $\{y_n\}_{n\in\mathbb{N}}$ is in Y, there exists a functional $\varphi \in (B(Y,X),\tau)^*$ defined by $\varphi(T) = \sum_{n\in\mathbb{N}} x_n^*(Ty_n)$ for all $T \in B(Y,X)$. However, we do not know if different representations of the same nuclear operator correspond to the same functional. In the next theorem, we prove that this is the case if either X or Y has the approximation property.

Theorem 2.27. *Let X and Y be Banach spaces and assume that either X or Y has the approximation property. Consider the map*

$$\Phi : N(X,Y) \to (B(Y,X),\tau)^*$$

defined by $\Phi(A)(T) = \sum_{n\in\mathbb{N}} x_n^*(Ty_n)$ where $\sum_{n\in\mathbb{N}} x_n^* \otimes y_n$ is a nuclear representation of A and $T \in B(Y,X)$. Then Φ is a well-defined (i.e. independent of the choice of nuclear representation) linear isomorphism.

Proof. We prove that Φ is a well-defined map from $N(X,Y)$ to $(B(Y,X),\tau)^*$. By construction, it is then clear that Φ is a linear map. To prove that Φ is a linear isomorphism, we show that Φ is injective. Surjectivity directly follows from Theorem 2.16.

We prove that Φ is well-defined by showing that $\Phi\left(\sum_{n\in\mathbb{N}} x_n^* \otimes y_n\right) = 0$ for any nuclear representation $\sum_{n\in\mathbb{N}} x_n^* \otimes y_n$ of the 0-operator. The well-definedness then follows for arbitrary $A \in N(X,Y)$ as the difference of two different nuclear representations is a nuclear representation of the 0-operator. So let $\sum_{n\in\mathbb{N}} x_n^* \otimes y_n$ be a nuclear representation of the 0-operator and define $\varphi = \Phi\left(\sum_{n\in\mathbb{N}} x_n^* \otimes y_n\right)$. We need to prove that $\varphi = 0$. As either X or Y has the approximation property, it follows from Theorem 2.9 that $\overline{F(Y,X)}^{\tau} = B(Y,X)$. So it suffices to prove that φ vanishes on $F(Y,X)$ as $\ker(\varphi)$ is closed. Now let $y^* \in Y^*$ and $x \in X$ be arbitrary, then

$$\begin{aligned}\varphi(y^* \otimes x) &= \sum_{n\in\mathbb{N}} x_n^*\left(y^* \otimes x(y_n)\right) = \sum_{n\in\mathbb{N}} x_n^*(x) y^*(y_n) \\ &= y^*\left(\sum_{n\in\mathbb{N}} x_n^*(x) y_n\right) = y^*(0) = 0.\end{aligned}$$

So φ vanishes on all rank one operators, hence by linearity φ vanishes on $F(Y,X)$. So $\varphi = 0$, hence Φ is well-defined.

We show that Φ is injective by showing that it has a trivial kernel. Suppose that $A \in N(X,Y)$ is a nuclear operator such that $\Phi(A) = 0$. It follows that $\Phi(A)(T) = 0$ for all $T \in B(Y,X)$. In particular, this holds for all bounded linear operators $T = y^* \otimes x$ where $y^* \in Y^*$ and $x \in X$ are arbitrary. By the same calculations as above, it follows that

$$\Phi(A)(y^* \otimes x) = y^*(Ax) = 0 \quad \forall y^* \in Y^* \; \forall x \in X.$$

As Y^* separates the points in Y, it follows that $Ax = 0$ for all $x \in X$, hence $A = 0$. We conclude that $\ker(\Phi) = \{0\}$ hence Φ is injective. □

The following corollary is an obvious consequence of Theorem 2.27.

Corollary 2.28. *Let X, Y and Φ be defined as in Theorem 2.27 and let $A \in N(X,Y)$ be a nuclear operator. Then the following identities hold:*

1. $\Phi(A)(y^* \otimes x) = y^*(Ax)$ *for any $x \in X$ and $y^* \in Y^*$.*

2. *If $Y = X$, then* $\mathrm{Tr}(A) = \Phi(A)(I)$.

3 Super-diagonal forms for compact operators

In the first two sections, we discussed the approximation property, nuclear operators and the nuclear trace. We now turn to the eigenvalues of compact operators and their invariant subspaces. To study these properly, we will need to introduce so-called nests of invariant closed subspaces, also termed invariant nests. The goal of this section is to show that compact operators on complex Banach spaces can be represented in a way very similar to upper triangular matrices. To prove this, we will follow Ringrose [20], but first we need some preparatory results. In this section, all subspaces are closed, unless stated otherwise. Furthermore, a proper subspace is a subspace that is neither the entire space nor the zero space.

3.1 Lomonosov's theorem

In studying the invariant subspaces for compact operators, a natural first question is whether all operators have such invariant subspaces. Of course, if X is a Banach space and T is a linear operator on X then the zero space $\{0\}$ and the entire space are invariant spaces for the operator T. These are called trivial invariant spaces. The question of whether each linear operator on a Banach space X has a proper invariant subspace was solved in the negative by Enflo, who published his proof in 1987 [4]. However, Aronszajn and Smith proved in 1954 that all compact operators on complex Banach spaces of dimension at least 2 have proper invariant subspaces [1]. Lomonosov generalised this result in 1973 by proving that each compact nonscalar operator on a complex Banach space has a proper hyperinvariant subspace [14]. This result is known as Lomonosov's theorem and we will use this to construct invariant nests.

Definition 3.1. Let X be a Banach space and let T be a bounded operator. A (not necessarily closed) subspace $H \subset X$ is a *hyperinvariant subspace* for T if it is an invariant subspace for all bounded operators commuting with T.

Remark. Since any operator certainly commutes with itself, each hyperinvariant subspace for an operator is an invariant subspace too.

Definition 3.2. Let X be a Banach space. A bounded operator $T \in B(X)$ is a *scalar operator* if it is a scalar multiple of the identity operator.

Theorem 3.3 (Lomonosov's theorem)**.** *Let X be a complex Banach space with* $\dim(X) \geq 2$ *and let T be a nonscalar compact operator on X. Then there exists a hyperinvariant proper subspace for T.*

Remark. Note that the assumptions in our formulation of Lomonosov's theorem differ from those in the formulation given in [16]. The nonzero assumption on the compact operator has been replaced with the assumption of being nonscalar. Furthermore, a restriction on the dimension of the Banach space is added. The restriction on the dimension is due to the obvious reason that spaces with dimension less than or equal to 1 have no proper subspaces. Replacing the nonzero assumption by the nonscalar assumption was done after the author realised that in finite-dimensional Banach spaces, the nonzero assumption is not strong enough. Surely, for infinite-dimensional Banach spaces, replacing nonzero with nonscalar changes nothing as the zero operator is the only compact scalar operator. However, since scalar operators commute with all linear operators and in finite-dimensional spaces all linear operators are compact, it is not difficult to construct counter-examples to Lomonosov's theorem with the weaker assumptions. It also turns out there are other sources, like [11, Section 12], that use the stronger assumptions we found.

The proof we give is due to Hilden [16] and is significantly simpler than Lomonosov's original proof, which used the Schauder fixed-point theorem. We first need a few preparatory results. Peculiarly, Hilden's proof of Lomonosov's theorem is also much simpler than the original proof of the weaker result by Aronszajn and Smith. This is why we use the stronger result.

Proposition 3.4. *Let X be a Banach space and T a bounded operator on X. If $M \subset X$ is an invariant (not necessarily closed) subspace for T, then so is $\overline{M}$.*

Proof. Let X be a Banach space and T a bounded operator on X. Let $M \subset X$ be an invariant subspace for T and let $x \in \overline{M}$. We prove that $Tx \in \overline{M}$. As $x \in \overline{M}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in M such that x_n converges to x as $n \to \infty$. As M is an invariant subspace for T, it follows that $\{Tx_n\}_{n\in\mathbb{N}} \subset M$. By continuity, it follows that $Tx = \lim_{n\to\infty} Tx_n \in \overline{M}$. We conclude that $\overline{M}$ is an invariant subspace for T. □

Other results we will need are the well-known spectral radius formula and the Fredholm alternative. We will not give proofs, but these can be found in various textbooks on functional analysis e.g. Megginson [15, Theorem 3.3.27 + 3.4.24].

Theorem 3.5 (The Spectral Radius Formula)**.** *Let X be a complex Banach space. Then for all bounded operators $T \in B(X)$, the spectral radius $r(T)$ is given by*

$$r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}.$$

Corollary 3.6. *Let X be a complex Banach space. Then for all bounded operators $T \in B(X)$ we have that $r(T) = 0$ if and only if $\lim_{n\to\infty} \|(\alpha T)^n\| = 0$ for all $\alpha \in \mathbb{C}$.*

Proof. Let X be a complex Banach space and let $T \in B(X)$ be a bounded operator. Suppose that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = 0$ and let $\alpha \in \mathbb{C}$ be an arbitrary scalar. If $\alpha = 0$, it is obvious that $\lim_{n\to\infty} \|(\alpha T)^n\| = 0$. So suppose that $\alpha \neq 0$. Choose a positive real ϵ such that $0 < \epsilon < 1$. By definition of the limit, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $\|T^n\|^{1/n} < \epsilon|\alpha|^{-1}$. It then follows that for all $n \geq N$ we have that $\|(\alpha T)^n\| = |\alpha|^n \|T^n\| < \epsilon^n < \epsilon$. Therefore, $\lim_{n\to\infty} \|(\alpha T)^n\| = 0$.

Conversely, suppose that $\lim_{n\to\infty} \|(\alpha T)^n\| = 0$ for all $\alpha \in \mathbb{C}$. Let $\alpha \in \mathbb{C}$ be arbitrary but nonzero. By definition of the limit, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $|\alpha|^n \|T^n\| < 1$. This implies that for all $n \geq N$, we have that $\|T^n\|^{1/n} < |\alpha|^{-1}$. This implies that $\limsup_{n\to\infty} \|T^n\|^{1/n} \leq |\alpha|^{-1}$. As this applies to all nonzero $\alpha \in \mathbb{C}$, we can make α arbitrarily large, hence $\limsup_{n\to\infty} \|T^n\|^{1/n} \leq 0$. It follows that

$$\limsup_{n\to\infty} \|T^n\|^{1/n} \leq 0 \leq \liminf_{n\to\infty} \|T^n\|^{1/n} \leq \limsup_{n\to\infty} \|T^n\|^{1/n}$$

We can conclude that the limit superior and limit inferior are equal and equal 0. Hence, $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0$. □

Theorem 3.7 (The Fredholm Alternative)**.** *Let X be a complex Banach space, let $T \in K(X)$ be a compact operator and let α be a nonzero complex scalar. Then the following are equivalent:*

1. *$\alpha I - T$ is injective.*
2. *$\alpha I - T$ is surjective.*
3. *$\alpha I - T$ is invertible.*

Remark. The Fredholm Alternative as stated here is formulated differently from Megginson [15]. However, Megginson's formulation implies our formulation. Note that parts a and c in Megginson correspond to our parts 1 and 2. Furthermore, the equivalence of injectivity and surjectivity implies equivalence with invertibility by the Bounded Inverse Theorem.

Corollary 3.8. *Let X be a complex Banach space and let $T \in K(X)$ be a compact operator. Then all nonzero elements of the spectrum $\sigma(T)$ are eigenvalues of T.*

With these results established, we can prove Theorem 3.3.

Proof of Theorem 3.3. Let X be a complex Banach space with $\dim(X) \geq 2$ and let T be a nonscalar compact operator on X. We start with a reduction step. Suppose that T has a nonzero eigenvalue λ. We claim that the eigenspace E_λ of this eigenvalue is a proper hyperinvariant subspace for T. To prove this, let A be a bounded operator such that A and T commute and let $x \in E_\lambda$ be an eigenvector of T. It follows that $T(Ax) = A(Tx) = \lambda Ax$, thus $Ax \in E_\lambda$. So E_λ is an invariant subspace for A and therefore it is a hyperinvariant subspace for T. Since T is a nonscalar operator, it follows that $E_\lambda \neq X$. As $E_\lambda = (T - \lambda I)^{-1}(\{0\})$, it follows from continuity of $T - \lambda I$ that E_λ is closed. Hence E_λ is a hyperinvariant proper subspace. Furthermore, if $T/\|T\|$ has a hyperinvariant subspace, then this is also a hyperinvariant subspace for T. So it suffices to consider compact operators of norm 1 without nonzero eigenvalues.

Now assume that T has norm 1 and has no nonzero eigenvalues. By Corollary 3.8, it follows that $\sigma(T) = \{0\}$ and thus that $r(T) = 0$. Choose $x_0 \in X$ such that $\|Tx_0\| > 1$. As $\|T\| = 1$, it follows that $\|x_0\| > 1$. Let B be the closed unit ball centered at x_0. It follows that $0 \notin B$ and since $\|T\| = 1$, it also follows that $0 \notin \overline{TB}$. For all $y \in X$, we define

$$M_y = \{Ay \in X : A \in B(X),\ A \text{ commutes with } T\}.$$

We claim that for all $y \in X$, this is a hyperinvariant subspace for T. To prove this, fix $y \in X$ and choose $v, w \in M_y$. By definition of M_y, this implies there are bounded linear operators C and D, commuting with T, such that $v = Cy$ and $w = Dy$. Since for all $\alpha \in \mathbb{C}$ the operator $\alpha C + D$ commutes with T, it follows that $\alpha v + w = (\alpha C + D)y \in M_y$. Thus M_y is a linear subspace of X. Furthermore, if C commutes with T and $v \in M_y$, we can write $v = Dy$ for some bounded linear operator D commuting with T. As CD also commutes with T, it follows that $Cv = CDy \in M_y$. So M_y is an invariant (but not necessarily closed) subspace for C and therefore a hyperinvariant (but not necessarily closed) subspace for T. By Proposition 3.4, it follows that $\overline{M_y}$ is a hyperinvariant subspace for T for all $y \in X$.

The last step is to prove that there exists a $y \in X$ such that $\overline{M_y}$ is a proper subspace of X. Since the identity operator I of X is bounded and commutes with all operators, it follows that $y \in M_y$ for all $y \in X$. Therefore, if $y \neq 0$ then it follows that $\overline{M_y} \neq \{0\}$. We are left with proving that there exists a $y \in X \setminus \{0\}$ such that M_y is not dense in X. We argue by contradiction. Suppose that for all $y \in X \setminus \{0\}$ we have that M_y is dense in X. Then for all $y \in X \setminus \{0\}$ it follows that $M_y \cap B_1(x_0) \neq \emptyset$. So for all $y \in X$, there exists a bounded linear operator A commuting with T such that $\|Ay - x_0\| < 1$. For all $A \in B(X)$ commuting with T, we define

$$\mathcal{U}(A) = \{y \in X : \|Ay - x_0\| < 1\}.$$

As each nonzero $y \in X$ is contained in at least one of these sets by our previous remark and 0 is certainly contained in none of them as $\|x_0\| > 1$, it follows that the union of all $\mathcal{U}(A)$ is equal to $X \setminus \{0\}$. Furthermore, as the function $f_A : y \mapsto \|Ay - x_0\|$ is continuous for all $A \in B(X)$ and $\mathcal{U}(A) = f_A^{-1}([0,1))$, it follows that $\mathcal{U}(A)$ is open for all A commuting with T. Since

T is compact, $\overline{TB}$ is a compact subset of $X \setminus \{0\}$. So we can find bounded operators $A_1, ..., A_n$ commuting with T such that $\{\mathcal{U}(A_1), ..., \mathcal{U}(A_n)\}$ forms a finite cover of $\overline{TB}$. Since $Tx_0 \in TB$, there exists an $i_1 \leq n$ such that $Tx_0 \in \mathcal{U}(A_{i_1})$. By definition of $\mathcal{U}(A_{i_1})$, it follows that $A_{i_1}Tx_0 \in B$ and thus $TA_{i_1}Tx_0 \in TB$, so there exists an $i_2 \leq n$ such that $TA_{i_1}Tx_0 \in \mathcal{U}(A_{i_2})$. By repeating this argument m times, we can construct a sequence $\{x_m\}_{m\in\mathbb{N}}$ where $x_m = A_{i_m}TA_{i_{m-1}}...A_{i_1}Tx_0 = A_{i_m}A_{i_{m-1}}...A_{i_1}T^m x_0 \in B$. If we set $c = \max_{1\leq i\leq n} \|A_i\|$, it follows that

$$\|x_m\| \leq \|A_{i_m}\|\|A_{i_{m-1}}\|...\|A_{i_1}\|\|T^m\|\|x_0\| \leq c^m\|T^m\|\|x_0\| = \|(cT)^m\|\|x_0\|.$$

Since $r(T) = 0$, Corollary 3.6 implies that

$$\lim_{m\to\infty} \|x_m\| \leq \lim_{m\to\infty} \|(cT)^m\|\|x_0\| = 0.$$

This implies that $0 \in \overline{B} = B$, which contradicts the definition of B. So there exists a nonzero $y \in X$ such that M_y is not dense in X. □

Corollary 3.9. *Let X be a complex Banach space of dimension at least 2 and let T be a compact operator on X. Then T has a proper invariant subspace.*

Proof. We distinguish two cases: T is a scalar operator and T is a non-scalar operator. If T is a scalar operator, any linear subspace of X is an invariant subspace for T. So for any one-dimensional subspace $K \subset X$ it follows that K is a proper invariant subspace for T. If T is a nonscalar operator, Lomonosov's theorem guarantees the existence of a proper hyperinvariant subspace $H \subset X$ for T. As T certainly commutes with itself, H is a proper invariant subspace for T. □

3.2 Nests of subspaces, simple, maximal and invariant nests

In the previous subsection, we saw that compact operators on complex Banach spaces have proper invariant subspaces if the dimension is at least 2. In the coming subsections, we will strengthen this statement vastly. In this subsection, our main goal is to introduce nests and related concepts, which allow us to prove Ringrose's theorems.

Definition 3.10. Let X be a Banach space. A *nest* $\mathcal{N}$ is a set of linear subspaces of X that is totally ordered by inclusion. If T is a bounded operator on X and all subspaces $L \in \mathcal{N}$ are invariant subspaces for T, then $\mathcal{N}$ is an *invariant nest* for T.

Proposition 3.11. *Let X be a Banach space and let $\mathcal{N}$ be a nest of subspaces. Let $\mathcal{N}_0 \subset \mathcal{N}$ be a subnest and define*

$$K = \bigcap_{M \in \mathcal{N}_0} M, \text{ and } L = \overline{\bigcup_{M \in \mathcal{N}_0} M}.$$

Then L and K are subspaces of X and $\mathcal{N} \cup \{K\}$ and $\mathcal{N} \cup \{L\}$ are nests of subspaces. Moreover, if T is a bounded operator and $\mathcal{N}$ is an invariant nest, then so are $\mathcal{N} \cup \{K\}$ and $\mathcal{N} \cup \{L\}$.

Proof. By construction, it follows that L and K are closed subspaces of X. We need to check whether $\mathcal{N} \cup \{K\}$ and $\mathcal{N} \cup \{L\}$ are totally ordered by inclusion. First, consider $\mathcal{N} \cup \{K\}$. As $\mathcal{N}$ is a nest, hence totally ordered by inclusion, we only need to check whether for all $N \in \mathcal{N}$ we either have $N \subset K$ or $K \subset N$. So let $N \in \mathcal{N}$ be a subspace. If there exists an $M \in \mathcal{N}_0$ such that $M \subset N$, then it follows that $K \subset M \subset N$. If such M does not exist, then for all $M \in \mathcal{N}_0$ we have that $N \subset M$. It follows that $N \subset K$. We conclude that $\mathcal{N} \cup \{K\}$ is totally ordered by inclusion and thus a nest. Now consider $\mathcal{N} \cup \{L\}$. By the same argument as above, we need to verify whether for all $N \in \mathcal{N}$, we either have $N \subset L$ or $L \subset N$. So let $N \in \mathcal{N}$ be a subspace. If there exists a $M \in \mathcal{N}_0$ such that $N \subset M$, then it follows that $N \subset M \subset L$. If such M does not exist, then for all $M \in \mathcal{N}_0$ we have that $M \subset N$. It follows that $\bigcup_{M \in \mathcal{N}_0} M \subset N$. As N is closed, this implies that $L \subset N$. We conclude that $\mathcal{N} \cup \{L\}$ is totally ordered by inclusion and thus a nest.

Now let T be a bounded operator and suppose that $\mathcal{N}$ is an invariant nest. To prove our last statement, it is only left to show that K and L are invariant subspaces for T. As $\mathcal{N}_0 \subset \mathcal{N}$ is a subnest of an invariant nest, it follows that $TM \subset M$ for all $M \in \mathcal{N}_0$. Suppose that $x \in K$. Then $x \in M$ for all $M \in \mathcal{N}_0$ and therefore $Tx \in M$ for all $M \in \mathcal{N}_0$. It follows that $Tx \in K$, hence K is an invariant subspace for T. Now suppose that $x \in L$. By definition of L, this implies there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\bigcup_{M \in \mathcal{N}_0} M$ such that x_n converges to x as $n \to \infty$. So for all x_n, there exists an $M \in \mathcal{N}_0$ such that $x_n \in M$. As $\mathcal{N}$ is an invariant nest, it follows that $Tx_n \in M \subset L$. Using the continuity of T, it follows that $Tx = \lim_{n \to \infty} Tx_n \in \overline{L} = L$. We conclude that both K and L are invariant subspaces for T. □

Definition 3.12. Let X be a Banach space and let $\mathcal{N}$ be a nest of subspaces. For all $M \in \mathcal{N}$, we define

$$M_- = \overline{\bigcup \{L \in \mathcal{N} : L \subsetneq M\}}.$$

If the set $\{L \in \mathcal{N} : L \subsetneq M\}$ is empty, we put $M_- = \{0\}$. As M is closed, it follows that $M_- \subset M$. We say that the nest $\mathcal{N}$ is *continuous at* M if $M_- = M$ and that $\mathcal{N}$ is *continuous* if it is continuous at all $M \in \mathcal{N}$.

Proposition 3.13. *Let X be a Banach space and let $\mathcal{N}$ be a nest of subspaces. Let $M \in \mathcal{N}$ be a subspace. If there exists a subspace L of X such that $M_- \subset L \subset M$, then $\mathcal{N} \cup \{L\}$ is a nest.*

Proof. Suppose that there exists a subspace L of X such that $M_- \subset L \subset M$. We need to check whether $\mathcal{N} \cup \{L\}$ is totally ordered. Let $N \in \mathcal{N}$ be a subspace. We distinguish two cases: $M \subset N$ and $N \subsetneq M$. In the first case, it follows that $L \subset M \subset N$, hence it follows that $L \subset N$. In the second case, we have that $N \subset M_-$ by definition of M_-. It follows that $N \subset M_- \subset L$. So we either have $L \subset N$ or $N \subset L$, which implies that $\mathcal{N} \cup \{L\}$ is a nest. □

In Propositions 3.11 and 3.13, we encountered two ways to extend a nest. This raises the question whether or not there are maximal nests; nests that cannot be extended any further.

Definition 3.14. Let X be a Banach space. A nest $\mathcal{N}$ is *maximal* if it is not included in a strictly bigger nest. Furthermore, a nest $\mathcal{N}$ is *simple* if it satisfies the following three properties:

1. $\{0\} \in \mathcal{N}$ and $X \in \mathcal{N}$.
2. For all subnests $\mathcal{N}_0 \subset \mathcal{N}$, we have that $\bigcap_{M \in \mathcal{N}_0} M$ and $\overline{\bigcup_{M \in \mathcal{N}_0} M}$ are in $\mathcal{N}$.
3. $\dim(M/M_-) \leq 1$ for all $M \in \mathcal{N}$.

A nest $\mathcal{N}$ is *complete* if it satisfies the first two properties of a simple nest.

Remark. If $\mathcal{N}$ is a complete nest, then property 2 implies that $M_- \in \mathcal{N}$ for all $M \in \mathcal{N}$. Moreover, note that any simple nest is complete.

Before we proceed, we could wonder if such maximal nests do exist in the first place. In Proposition 3.16 we will see they actually do. However, the proof of this statement is very non-constructive in the sense that it relies on Zorn's lemma.

Lemma 3.15. *Let X be a Banach space and I a totally ordered index set. Let $\{\mathcal{N}_i\}_{i \in I}$ be an increasing chain of nests of X. Then $\mathcal{N} = \bigcup_{i \in I} \mathcal{N}_i$ is a nest and $\mathcal{N}_i \subset \mathcal{N}$ for all $i \in I$. Moreover, if $\{\mathcal{N}_i\}_{i \in I}$ is an increasing chain of invariant nests for some operator T on X, then $\mathcal{N}$ is also an invariant nest for that operator.*

Proof. We prove that $\mathcal{N}$ is a nest of subspaces of X. The second and third claim then follow trivially. To prove $\mathcal{N}$ is a nest, we need to check whether it is totally ordered with respect to inclusion. Let $M, N \in \mathcal{N}$ be two subspaces. By definition of $\mathcal{N}$, there are $i, j \in I$ such that $M \in \mathcal{N}_i$

and $N \in \mathcal{N}_j$. As $\{\mathcal{N}_i\}_{i\in I}$ is an increasing chain of nests, we either have $\mathcal{N}_i \subset \mathcal{N}_j$ or $\mathcal{N}_j \subset \mathcal{N}_i$. Hence, one of the two nests contains both N and M. As nests are totally ordered, this implies that either $N \subset M$ or $M \subset N$ should hold. We conclude that $\mathcal{N}$ is totally ordered, hence a nest. □

Proposition 3.16. *Let X be a Banach space. Then there exists a maximal nest of subspaces of X. Moreover, if $\mathcal{M}$ is a given nest, then there exists a maximal nest extending $\mathcal{M}$.*

Proof. Let X be a Banach space and let $\mathcal{F}$ be the set of nests of subspaces of X. $\mathcal{F}$ is nonempty as it certainly contains the trivial nest $\{\{0\}, X\}$. $\mathcal{F}$ is a partially ordered set with respect to inclusion. Furthermore, if $\mathcal{F}_0 \subset \mathcal{F}$ is a increasing chain of nests, then by Lemma 3.15, $\mathcal{N}_0 = \bigcup_{\mathcal{N}\in\mathcal{F}_0} \mathcal{N}$ is an upper bound of $\mathcal{F}_0$ in $\mathcal{F}$. This implies that every increasing chain of nests in $\mathcal{F}$ has an upper bound in $\mathcal{F}$ with respect to inclusion. Hence by Zorn's Lemma, there exists a maximal nest $\mathcal{N}_{max}$ in $\mathcal{F}$.

Now suppose that $\mathcal{M}$ is a given nest and let $\mathcal{F}(\mathcal{M})$ be the set of nests containing $\mathcal{M}$. $\mathcal{F}(\mathcal{M})$ is nonempty as $\mathcal{M} \in \mathcal{F}(\mathcal{M})$. The same argument as before now gives that $\mathcal{F}(\mathcal{M})$ has a maximal element $\mathcal{M}_{max}$. We claim that $\mathcal{M}_{max}$ is also maximal in $\mathcal{F}$. Suppose it is not, then there exists a nest $\mathcal{M}'$ in $\mathcal{F}$ such that $\mathcal{M}_{max} \subsetneq \mathcal{M}'$. However, as this implies that $\mathcal{M} \subset \mathcal{M}_{max} \subsetneq \mathcal{M}'$, it follows that $\mathcal{M}'$ extends $\mathcal{M}_{max}$ in $\mathcal{F}(\mathcal{M})$, contradicting the maximality of $\mathcal{M}_{max}$. □

We will study the connection between maximal and simple nests; they will turn out to be the same thing. However, as property 3 of simple nests suggests, we will need some properties of quotient spaces for this. We will quickly revisit some standard results and some preparatory results we need.

If X is a vector space and $M \subset X$ is a linear subspace, then M defines an equivalence relation $\sim_M$ on X via $x \sim_M y$ if and only if $x - y \in M$. With this definition, the quotient space $X/M := X/\sim_M$ has a vector space structure if we define $\alpha(x+M) = \alpha x+M$ and $(x+M)+(y+M) = (x+y)+M$ [15, p. 50]. With these definitions, the quotient map $p : X \to X/M$ defined by $p(x) = x+M$ is a linear map. If X is a normed space and M is a closed subspace, then $\|x + M\| = \inf_{y\in M} \|x - y\|$ defines a norm on X/M [15, Theorem 1.7.4] and the quotient map p is a bounded linear operator with $\|p\| = 1$ if $M \neq X$ [15, Proposition 1.7.12]. We also have that p maps the open unit ball of X onto the open unit ball of X/M [15, Lemma 1.7.11]. Furthermore, if two of the three spaces X, M and X/M are complete, then so is third [15, Theorem 1.7.9].

Proposition 3.17. *Let X be a vector space and $M \subset X$ a subspace. Let $p : X \to X/M$ be the quotient map and let $K \subset X/M$ be a subspace of the quotient space. Then $p^{-1}(K) \subset X$ is a linear subspace.*

Proof. Choose $x, y \in p^{-1}(K)$ and let α be a scalar. Then $x + M \in K$ and $y + M \in K$, hence $\alpha x + y + M \in K$ as K is a linear space. We conclude that $p(\alpha x + y) \in K$, hence $\alpha x + y \in p^{-1}(K)$. So $p^{-1}(K)$ is a linear subspace of X. $\square$

Proposition 3.18. *Let X be a vector space and suppose that $M \subsetneq N \subset X$ are two subspaces. Then $\dim(N/M) > 1$ if and only if there exists a subspace L such that $M \subsetneq L \subsetneq N$. Moreover, if X is a normed space and M and N are closed, L can also be chosen to be closed.*

Proof. Let X be a vector space and suppose that $M \subsetneq N \subset X$ are two subspaces. Suppose that $\dim(N/M) > 1$. Then there exists a one-dimensional subspace $K \subsetneq N/M$. Let $p : N \to N/M$ be the quotient map restricted to N. Set $L = p^{-1}(K) \subset X$. By Proposition 3.17, L is a linear subspace of N, hence of X. As $\dim(K) = 1$, it follows that $\{0\} \subsetneq K$. Combined with surjectivity of p, this yields that $M \subsetneq L$. As we also have that $K \subsetneq N/M$, using surjectivity of p again yields that $L \subsetneq N$. We conclude that $M \subsetneq L \subsetneq N$. If X is a normed space, closedness of L follows from the fact that K is closed as it is finite-dimensional, hence by continuity of p it follows that L is closed in N. As N is closed in X it also follows that L is closed in X.

Conversely, if there exists a subspace L such that $M \subsetneq L \subsetneq N$, then $p(L)$ is a linear subspace of N/M such that $\{0\} \subsetneq p(L) \subsetneq N/M$. Hence $\dim(N/M) > \dim(p(L)) \geq 1$. $\square$

Proposition 3.19. *Let X be a Banach space and let T be a bounded operator on X. Let $M \subset X$ be an invariant subspace for T. Then there exists a well-defined bounded linear operator T_M on X/M such that $T_M \circ p = p \circ T$. Moreover, if T is compact, then so is T_M.*

Proof. Define $T_M : X/M \to X/M$ by $T_M(x + M) = Tx + M$. We first need to verify that T_M is well-defined. Suppose that $x, x' \in X$ with $x + M = x' + M$, hence $x - x' \in M$. As M is an invariant subspace for T, it follows that $Tx - Tx' = T(x - x') \in M$, hence $Tx + M = Tx' + M$. We conclude that $T_M(x + M) = Tx + M = Tx' + M = T_M(x' + M)$, hence T_M is well-defined. By definition of p, it directly follows that $T_M \circ p = p \circ T$. For linearity of T_M, let α be a scalar and let $x + M,\ y + M \in X/M$. Then

$$\begin{aligned}
T_M(\alpha(x + M) + (y + M)) &= T_M(\alpha x + y + M) \\
&= T(\alpha x + y) + M \\
&= \alpha Tx + Ty + M \\
&= \alpha T_M(x + M) + T_M(y + M),
\end{aligned}$$

hence T_M is linear. To prove that T_M is bounded, we calculate the operator norm. For this, we need the fact that the quotient map p maps the open ball $B_r(0)$ in X onto the open ball $B'_r(0)$ in X/M. Let $\epsilon > 0$. It follows that

$$\begin{aligned}
\|T_M\| &= \sup_{\|x+M\|\leq 1} \|T_M(x+M)\| \leq \sup_{\|x+M\|<1+\epsilon} \|T_M(x+M)\| \\
&= \sup_{\|x\|<1+\epsilon} \|T_M(p(x))\| = \sup_{\|x\|<1+\epsilon} \|p(Tx))\| \\
&\leq \|p\|\|T\|(1+\epsilon),
\end{aligned}$$

where $\|p\|$ is equal to 0 or 1 depending on whether $M = X$ or $M \neq X$. As this holds for all $\epsilon > 0$, it follows that $\|T_M\| \leq \|p\|\|T\| = \|T\|$ unless $p = 0$. Hence T_M is a well-defined bounded linear operator on X/M.

Now suppose that T is a compact operator. To prove that T_M is compact too, we show that for each bounded sequence $\{x_n + M\}_{n\in\mathbb{N}}$ in X/M, the sequence $\{T_M(x_n+M)\}_{n\in\mathbb{N}}$ has a convergent subsequence. Let $\{x_n+M\}_{n\in\mathbb{N}}$ be a bounded sequence in X/M. By definition of the quotient norm, we can find a sequence $\{y_n\}_{n\in\mathbb{N}}$ in M such that $\|x_n-y_n\| < \|x_n+M\|+1$. It follows that $\{x_n-y_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X. As T is a compact operator, $p \circ T = T_M \circ p$ is also compact. It follows that $\{T_M \circ p(x_n - y_n)\}_{n\in\mathbb{N}} = \{T_M(x_n+M)\}_{n\in\mathbb{N}}$ has a convergent subsequence. So T_M is compact too. $\square$

These intermediate results allow us to prove that maximal and simple nests are the same. To prove this, we follow Ringrose [20].

Theorem 3.20 ([20, Lemma 1])**.** *Let X be a Banach space. A nest of subspaces of X is maximal if and only if it is simple.*

Proof. Let X be a Banach space and let $\mathcal{N}$ be a nest of subspaces. Suppose that $\mathcal{N}$ is maximal. We check that $\mathcal{N}$ satisfies the three properties of simple nests. It is obvious that we have $\{0\} \in \mathcal{N}$ and $X \in \mathcal{N}$, otherwise $\mathcal{N}$ could have been extended by adding $\{0\}$ or X, contradicting the maximality. By the same argument it follows that $\mathcal{N}$ must also satisfy the second property of simple nests. Suppose it does not. Then there exists a subnest $\mathcal{N}_0 \subset \mathcal{N}$ such that $\bigcap_{M\in\mathcal{N}_0} M$ or $\overline{\bigcup_{M\in\mathcal{N}_0} M}$ is not in $\mathcal{N}$. By Proposition 3.11, it follows that we can extend $\mathcal{N}$, contradicting the maximality. So $\mathcal{N}$ satisfies the first two properties of simple nests. For the third, suppose there exists a $M \in \mathcal{N}$ such that $\dim(M/M_-) > 1$. By Proposition 3.18, it follows there exists a subspace L such that $M_- \subsetneq L \subsetneq M$. Hence by Proposition 3.13, we can extend $\mathcal{N}$ by L, contradicting the maximality of $\mathcal{N}$ again. So $\dim(M/M_-) \leq 1$ for all $M \in \mathcal{N}$ and therefore $\mathcal{N}$ is simple.

Conversely, suppose that $\mathcal{N}$ is simple but not maximal. This implies there exists a subspace L of X such that $L \notin \mathcal{N}$ and $\mathcal{N} \cup \{L\}$ is a nest. Define

$$M = \bigcap\{N \in \mathcal{N} : L \subset N\} \quad \text{and} \quad M' = \overline{\bigcup\{N \in \mathcal{N} : N \subsetneq L\}}.$$

Note that both M and M' are well-defined as both sets $\{N \in \mathcal{N} : L \subset N\}$ and $\{N \in \mathcal{N} : N \subsetneq L\}$ are nonempty since $\{\{0\}, X\} \subset \mathcal{N}$ by the first property of simple nests. By the second property of simple nests, it follows that $M, M' \in \mathcal{N}$ and by construction it follows that $M' \subsetneq L \subsetneq M$. The inclusions must be strict as $L \notin \mathcal{N}$. We claim that $M' = M_-$. The inclusion $M' \subset M_-$ follows from the fact that $N \subsetneq L$ implies that $N \subsetneq M$. Hence

$$M' = \overline{\bigcup\{N \in \mathcal{N} : N \subsetneq L\}} \subset \overline{\bigcup\{N \in \mathcal{N} : N \subsetneq M\}} = M_-.$$

Conversely, suppose that $N \in \mathcal{N}$ and $N \subsetneq M$. As $L \subset N$ implies that $M \subset N$, it follows from contraposition that $N \subsetneq M$ implies $N \subsetneq L$, hence

$$M_- = \overline{\bigcup\{N \in \mathcal{N} : N \subsetneq M\}} \subset \overline{\bigcup\{N \in \mathcal{N} : N \subsetneq L\}} = M'.$$

It follows that $M' = M_-$ and thus we have that $M_- \subsetneq L \subsetneq M$. By Proposition 3.18, it follows that $\dim(M/M_-) > 1$, contradicting the third property of simple nests. Hence $\mathcal{N}$ must be maximal. $\square$

Combining Proposition 3.16 with Theorem 3.20, we can conclude that maximal nests exist in each Banach space X and that we know quite some things about the properties they should have. For our study of the eigenvalues of compact operators, simple nests of invariant subspaces will turn out to be useful. The reason for this might not be obvious at all and is hard to explain with the current results. A detailed explanation will be given in the next subsection. However, before we proceed we should clarify whether such simple nests of invariant subspaces exist. This will be discussed in the next theorem, which is also due to Ringrose [20].

Theorem 3.21 ([20, Theorem 1]). *Let X be a complex Banach space and let T be a compact operator on X. Then there exists a simple nest of invariant subspaces for T.*

Proof. Let X be a complex Banach space and let T be a compact operator on X. Let $\mathcal{F}_T$ be the set of all invariant nests for T. $\mathcal{F}_T$ is nonempty as it contains the trivial nest $\{\{0\}, X\}$ since both $\{0\}$ and X are invariant subspaces for T. $\mathcal{F}_T$ is a partially ordered set with respect to inclusion. Furthermore, if $\mathcal{F}_0 \subset \mathcal{F}_T$ is a increasing chain of nests, then by Lemma 3.15 it follows that $\mathcal{N}_0 = \bigcup_{\mathcal{N} \in \mathcal{F}_0} \mathcal{N}$ is an upper bound of $\mathcal{F}_0$ in $\mathcal{F}_T$. This implies that every increasing chain of nests in $\mathcal{F}_T$ has an upper bound in $\mathcal{F}_T$ with respect to inclusion. By invoking Zorn's Lemma, there exists a maximal nest $\mathcal{N}_{max}$ in $\mathcal{F}_T$.

We claim that $\mathcal{N}_{max}$ is simple. We prove this by verifying the three properties of simple nests separately. As $\{0\}$ and X are certainly invariant subspaces for T, and we can extend any nest by both of them, it directly follows that $\mathcal{N}_{max}$ should contain both $\{0\}$ and X and thus satisfies the first property of simple nests. Let $\mathcal{N}_0 \subset \mathcal{N}_{max}$ be a subnest of invariant subspaces for T. Define

$$K = \bigcap_{M \in \mathcal{N}_0} M, \text{ and } L = \overline{\bigcup_{M \in \mathcal{N}_0} M}.$$

By Proposition 3.11, it follows that $\mathcal{N}_{max} \cup \{K\}$ and $\mathcal{N}_{max} \cup \{L\}$ are invariant nests too. Hence by maximality of $\mathcal{N}_{max}$ it follows that $L, K \in \mathcal{N}_{max}$. So $\mathcal{N}_{max}$ also satisfies the second property of simple nests. It is left to prove that for all $N \in \mathcal{N}_{max}$, we have that $\dim(N/N_-) \leq 1$. We argue by contradiction. Suppose that there exists an $N \in \mathcal{N}_{max}$ such that $\dim(N/N_-) > 1$. As N is a closed subspace of X, it follows that N is a Banach space. Furthermore, as N is an invariant subspace for T, we can restrict the map T to N and view $T : N \to N$ as a compact linear map on the Banach space N. By Proposition 3.11, it follows that $N_- \subset N$ is an invariant subspace for the restriction of T to N. Using Proposition 3.19 with $X = N$ and $M = N_-$, we obtain a compact linear operator T_N on N/N_- such that $T_N \circ p = p \circ T$ where $p : N \to N/N_-$ is the quotient map. Since $\dim(N/N_-) > 1$, Corollary 3.9 guarantees there exists a proper subspace $L_N \subset N/N_-$ that is invariant under T_N. Define $L = p^{-1}(L_N)$. Then by Proposition 3.17 and by continuity of p, it follows that L is a closed subspace of N and hence of X. Furthermore, since $\{0\} \subsetneq L_N \subsetneq N/N_-$ and p is surjective, it follows that $N_- \subsetneq L \subsetneq N$.

To get our contradiction, we show that $\mathcal{N}_{max} \cup \{L\}$ is an invariant nest for T, contradicting the maximality of $\mathcal{N}_{max}$. By Proposition 3.13 and the fact that $N_- \subsetneq L \subsetneq N$, it follows that $\mathcal{N}_{max} \cup \{L\}$ is a nest of subspaces. To prove that $\mathcal{N}_{max} \cup \{L\}$ is an invariant nest, it is left to prove that L is an invariant subspace for T. Let $x \in L$ be arbitrary. By definition of L, it follows that $p(x) \in L_N$ and hence that $T_N \circ p(x) \in L_N$ as L_N is an invariant subspace for T_N. Using Proposition 3.19, it follows that $p \circ T(x) \in L_N$ and therefore that $T(x) \in p^{-1}(L_N) = L$. Hence L is an invariant subspace for T. □

3.3 Diagonal coefficients and eigenvalues of compact operators on complex Banach spaces

In the previous section, we saw that if T is a compact operator on a complex Banach space X, there exists a simple nest of invariant subspaces for that operator T (Theorem 3.21). We will use this to study the eigenvalues of these compact operators. As we will need the existence of simple invariant nests, we will from now on assume that all Banach spaces are complex.

If X is a vector space and $L \subset X$ has codimension 1, it follows from linear algebra that for all $x \in X \setminus L$ the vector spaces X and $\mathbb{C}x \oplus L$ are isomorphic, with $\Phi : \mathbb{C}x \oplus L \to X$ defined by $(\alpha x, y) \mapsto \alpha x + y$ as linear isomorphism. Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a maximal invariant nest for T. It follows that for all $M \in \mathcal{N}$ with $\dim(M/M_-) = 1$ and for all $x \in M \setminus M_-$ the vector spaces M and $\mathbb{C}x \oplus M_-$ are isomorphic. Note that by the Bounded Inverse Theorem, this is also an isomorphism of Banach spaces. Since M is an invariant subspace for T, we have that $Tx \in M$. It follows that for all $x \in M \setminus M_-$ there exists a unique $\alpha_x \in \mathbb{C}$ and $y_x \in M_-$ such that $Tx = \alpha_x x + y_x$.

Proposition 3.22. *Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a maximal invariant nest for T. Let $M \in \mathcal{N}$ be an invariant subspace for T such that $\dim(M/M_-) = 1$. For $x, x' \in M \setminus M_-$, write $Tx = \alpha_x x + y_x$ and $Tx' = \alpha_{x'} x' + y_{x'}$ with $y_x, y_{x'} \in M_-$. Then $\alpha_x = \alpha_{x'}$.*

Proof. Let $x, x' \in M \setminus M_-$ and let p be the quotient map from M to M/M_-. It follows that $p(x), p(x') \neq 0$. As $\dim(M/M_-) = 1$, there exists a nonzero $\gamma \in \mathbb{C}$ such that $p(x) - \gamma p(x') = p(x - \gamma x') = 0$. So $x - \gamma x' \in M_-$ and as M_- is an invariant subspace for T by Proposition 3.11, it follows that $T(x - \gamma x') \in M_-$. So it follows that

$$p \circ T(x - \gamma x') = p(\alpha_x x + y_x - \gamma(\alpha_{x'} x' + y_{x'})) = 0.$$

Rewriting yields that $\alpha_x p(x) - \gamma \alpha_{x'} p(x') = (\alpha_x - \alpha_{x'}) p(x) = 0$. As $p(x) \neq 0$, it follows that $\alpha_x = \alpha_{x'}$. ☐

Corollary 3.23. *Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a maximal invariant nest for T. Then for all $M \in \mathcal{N}$ there exists a complex scalar α_M such that for all $x \in M$ there exists a $y \in M_-$ such that $Tx = \alpha_M x + y$. Moreover, if $\dim(M/M_-) = 1$, this α_M is uniquely defined.*

Proof. We distinguish two cases: $\dim(M/M_-) = 0$ and $\dim(M/M_-) = 1$. If $\dim(M/M_-) = 0$, put $\alpha_M = 0$. As $M = M_-$, it follows that for all $x \in M$ we have that $Tx \in M = M_-$. Hence $Tx = 0x + Tx$, with $Tx \in M_-$. So $\alpha_M = 0$ works.

If $\dim(M/M_-) = 1$, pick $x \in M \setminus M_-$ and define $\alpha_M = \alpha_x$. By Proposition 3.22, it follows that for all $x' \in M \setminus M_-$, there exists a $y \in M_-$ such that $Tx' = \alpha_M x + y$. Now suppose that $x \in M_-$. This implies that $Tx \in M_-$ as M_- is an invariant subspace for T, hence $Tx - \alpha_M x \in M_-$. So there exists a $y \in M_-$ such that $Tx = \alpha_M x + y$. We conclude that for all $x \in M$ there exists a $y \in M_-$ such that $Tx = \alpha_M x + y$. Uniqueness of α_M follows from the fact that α_x is uniquely determined for all $x \in M \setminus M_-$ since M is isomorphic to $\mathbb{C}x \oplus M_-$. ☐

Definition 3.24. Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a maximal invariant nest for T. For all $M \in \mathcal{N}$ the complex scalar α_M as constructed in the proof of Corollary 3.23 is the *diagonal coefficient* of T at M.

Remark. Note that if $\alpha_M \neq 0$, it follows that $\dim(M/M_-) = 1$. Furthermore, note that from Corollary 3.23 it follows that the scalars α_M behave somewhat like the diagonal coefficients of an upper triangular matrix. To illustrate this, consider the vector space $\mathbb{C}^n$ for some natural number $n \in \mathbb{N}$. Let $B = \{e_k : 1 \leq k \leq n\}$ be the standard basis of $\mathbb{C}^n$ and define the subspaces $E_k = \operatorname{span}(e_1, ..., e_k)$. It follows that $\mathcal{N} = \{\{0\}, E_1, ..., E_{n-1}, \mathbb{C}^n\}$ is a maximal nest of subspaces. Let $A \in M_{n\times n}(\mathbb{C})$ be an upper triangular matrix, so $A = \{a_{ij}\}_{i,j=1}^n$ with $a_{ij} = 0$ if $i > j$. Let $k \leq n$ and let $x \in E_k$ be an arbitrary vector. It follows that there are scalars $\{x_j\}_{j=1}^k$ such that we can write $x = \sum_{j=1}^k x_j e_j$. If we let A act on x as matrix, we see that

$$\begin{aligned} Ax = A\left(\sum_{j=1}^k x_j e_j\right) &= \sum_{j=1}^k x_j A e_j = \sum_{j=1}^k x_j \sum_{l=1}^j a_{lj} e_l = \sum_{j=1}^k \sum_{l=1}^j x_j a_{lj} e_l \\ &= \sum_{l=1}^k \sum_{j=l}^k x_j a_{lj} e_l = \sum_{l=1}^k e_l \sum_{j=l}^k x_j a_{lj} = a_{kk} x_k e_k + \sum_{l=1}^{k-1} e_l \sum_{j=l}^k x_j a_{lj} \\ &= a_{kk} x + \sum_{l=1}^{k-1} e_l \left(-a_{kk} x_l + \sum_{j=l}^k x_j a_{lj}\right). \end{aligned}$$

As the last term lies in E_{k-1}, it follows that for all $x \in E_k$, there exists a $y \in E_{k-1}$ such that $Ax = a_{kk}x + y$, where a_{kk} is the k-th diagonal coefficient of the matrix A. Note that this is precisely the sort of equation as in Corollary 3.23 with $\alpha_{E_k} = a_{kk}$ and $M_- = E_{k-1} = E_{k-}$. This explains why the scalars α_M are called diagonal coefficients. It also follows that $\mathcal{N}$ is an invariant nest for A and we know that the diagonal coefficients $\{a_{ii}\}_{i=1}^n$ are precisely the eigenvalues of A. This also explains why we expect invariant nests and diagonal coefficients to be important when studying the eigenvalues of compact operators.

To state the theorem that will be our main goal of this section, we need some more definitions and lemmas. We begin with the following well-known lemma from F. Riesz, the proof of which can be found in Megginson.

Lemma 3.25 (Riesz' Lemma [15, Lemma 3.4.18])**.** *Let V be a normed space and let $W \subsetneq V$ be a proper closed subspace. Let $\theta \in (0,1)$. Then there exists a unit vector $x \in V$ such that $\|x - y\| \geq \theta$ for all $y \in W$.*

Lemma 3.26 ([20, Lemma 2])**.** *Let X be a Banach space, T a compact operator on X, $\mathcal{N}$ a simple invariant nest for T and $\epsilon > 0$. Define $\mathcal{N}_0 \subset \mathcal{N}$ by*

$$\mathcal{N}_0 = \{M \in \mathcal{N} : |\alpha_M| \geq \epsilon\}.$$

Then $\mathcal{N}_0$ contains only finitely many invariant subspaces for T.

Proof. We will argue by contradiction. Suppose that $\mathcal{N}_0$ is infinite. As $\alpha_M \neq 0$ for all $M \in \mathcal{N}_0$, it follows that $\dim(M/M_-) = 1$ for all $M \in \mathcal{N}_0$. Hence M is a normed space and $M_- \subset M$ is a proper closed subspace for all $M \in \mathcal{N}_0$. Let $M \in \mathcal{N}_0$ be an arbitrary subspace. Using Riesz' Lemma, we can find a unit vector $z_M \in M$ such that $\|z_M - y\| \geq \frac{1}{2}$ for all $y \in M_-$. As $\mathcal{N}_0$ is infinite, we can extract a strictly increasing sequence $\{M_n\}_{n\in\mathbb{N}}$ from $\mathcal{N}_0$. As all z_M are unit vectors, the sequence $\{z_{M_n}\}_{n\in\mathbb{N}}$ is bounded. We claim that the sequence $\{Tz_{M_n}\}_{n\in N}$ has no convergent subsequence, contradicting the compactness of T. Suppose that $k, l \in \mathbb{N}$ and that $k \neq l$. As $\mathcal{N}_0$ is a nest, we either have $M_k \subsetneq M_l$ or $M_l \subsetneq M_k$, depending on whether $k \leq l$ or $l \leq k$. Without loss of generality, we can assume that $M_k \subsetneq M_l$. In particular, it follows that $M_k \subset M_{l-}$. As M_k is an invariant subspace for T, it follows that $Tz_{M_k} \in M_k \subset M_{l-}$. By Corollary 3.23, it follows that there exists a $y_{M_l} \in M_{l-}$ such that $Tz_{M_l} = \alpha_{M_l} z_{M_l} + y_{M_l}$. From this, it follows that

$$\begin{aligned}
\|Tz_{M_l} - Tz_{M_k}\| &= \|\alpha_{M_l} z_{M_l} + y_{M_l} - Tz_{M_k}\| \\
&= |\alpha_{M_l}| \|z_{M_l} + \alpha_{M_l}^{-1}(y_{M_l} - Tz_{M_k})\| \\
&\geq \epsilon \|z_{M_l} + \alpha_{M_l}^{-1}(y_{M_l} - Tz_{M_k})\| \geq \frac{1}{2}\epsilon,
\end{aligned}$$

where the last inequality follows from the fact that $\alpha_{M_l}^{-1}(y_{M_l} - Tz_{M_k}) \in M_{l-}$. Hence $\{Tz_{M_n}\}_{n\in\mathbb{N}}$ has no convergent subsequence, contradicting our assumption that T is compact. So $\mathcal{N}_0$ must be finite. □

Definition 3.27. Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a maximal invariant nest for T. For a scalar $\alpha \in \mathbb{C}$, we define the *diagonal multiplicity* d_α to be the number of subspaces $M \in \mathcal{N}$ such that $\alpha_M = \alpha$, where we allow $d_\alpha = \infty$.

Corollary 3.28. *Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a maximal invariant nest for T. Then every nonzero scalar $\alpha \in \mathbb{C}$ has a finite diagonal multiplicity.*

Proof. For all nonzero $\alpha \in \mathbb{C}$, there exists an $\epsilon > 0$ such that $\epsilon < |\alpha|$. The statement now follows directly from Lemma 3.26. □

The main goal of this section will be to connect the nonzero eigenvalues of a compact operator to its diagonal coefficients. To do this properly, we first need to discuss the multiplicity of eigenvalues. In linear algebra, if we have a linear operator T acting on a finite-dimensional space and an eigenvalue λ, we define the geometric multiplicity of λ to be the dimension of the corresponding eigenspace and we define the algebraic multiplicity to be the dimension of the corresponding generalized eigenspace. However, when we try to generalize these definitions to arbitrary linear operators on possibly infinite dimensional spaces, some caution is required as these quantities may be infinite. We will see, however, that for compact operators these definitions always make sense and are finite. We will cite some results without giving proof, these are due to Zaanen [23].

Definition 3.29. Let T be a compact operator on a Banach space X. Let λ be a nonzero scalar. For all natural numbers n, we define the subspaces $M_n = \ker((T - \lambda I)^n)$ and $L_n = (T - \lambda I)^n X$.

Remark. It is obvious that using the definition above, the sequence $\{M_n\}_{n\in\mathbb{N}}$ is an increasing sequence of closed subspaces of X and that $\{L_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of subspaces of X. Theorem 2 from Zaanen [23, p. 332] guarantees that L_n is also a closed subspace of X for all natural numbers n. Furthermore, using the binomial expansion, $(T - \lambda I)^n$ can be written as $S_n + (-\lambda)^n I$ for some compact operator S_n. It follows that M_n is the eigenspace of the compact operator S_n corresponding to the eigenvalue $-(-\lambda)^n$. Therefore M_n is finite-dimensional for all $n \in \mathbb{N}$.

Definition 3.30. Let X be a Banach space and let T be a compact operator on X. Let λ be a nonzero scalar and let $\{M_n\}_{n\in\mathbb{N}}$ be the sequence of subspaces from Definition 3.29. We define the *geometric multiplicity* of λ as $\dim(M_1)$ and the *algebraic multiplicity* of λ as $\dim\left(\bigcup_{n\in\mathbb{N}} M_n\right)$.

So far, we have not done much. The multiplicities defined in Definition 3.30 are just straightforward generalizations of the standard definitions in linear algebra. By construction, it is clear that the geometric multiplicity is finite for any nonzero scalar. However, we are primarily interested in the algebraic multiplicities of nonzero scalars and it is not at all obvious from the definition that these should be finite. The following two theorems from Zaanen show how the two sequences from Definition 3.29 are connected and that the algebraic multiplicity is indeed finite for any nonzero scalar.

Theorem 3.31 ([23, Theorem 6 + 7, p. 334 - 336])**.** *Let X be a Banach space and let T be a compact operator on X. Let λ be a nonzero scalar and let $\{M_n\}_{n\in\mathbb{N}}$ and $\{L_n\}_{n\in\mathbb{N}}$ be the sequences of subspaces from Definition 3.29. Then there exists a natural number $\nu = \nu(\lambda)$ such that $M_n = M_\nu$ and $L_n = L_\nu$ for all $n \geq \nu$, whereas M_n is proper subspace of M_{n+1} and L_{n+1} is proper subspace of L_n for $n < \nu$.*

Definition 3.32. Let X be a Banach space and let T be a compact operator on X. Let λ be a nonzero scalar and let $\{M_n\}_{n\in\mathbb{N}}$ and $\{L_n\}_{n\in\mathbb{N}}$ be the sequences of subspaces from Definition 3.29. Then the natural number $\nu = \nu(\lambda)$ from Theorem 3.31 is the *index of λ relative to T*.

Corollary 3.33. *Let X be a Banach space and let T be a compact operator on X. Then any nonzero scalar λ has finite algebraic multiplicity.*

Proof. Let $\{M_n\}_{n\in\mathbb{N}}$ be the sequence of subspaces from Definition 3.29 and let ν be the index of λ relative to T. It follows directly from Theorem 3.31 that $\dim\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \dim(M_\nu) < \infty$. Hence the algebraic multiplicity of λ is indeed finite. □

Theorem 3.34 ([23, Theorem 8, p. 336])**.** *Let X be a Banach space and let T be a compact operator on X. Let λ be a nonzero scalar and let $\{M_n\}_{n\in\mathbb{N}}$ and $\{L_n\}_{n\in\mathbb{N}}$ be the sequences of subspaces from Definition 3.29. Let ν be the index of λ relative to T. Then the Banach spaces X and $L_\nu \oplus M_\nu$ are isomorphic.*

Our main goal in this section will be to prove the following Theorem, which is also due to Ringrose. It connects the nonzero diagonal coefficients of a compact operator to its nonzero eigenvalues.

Theorem 3.35 ([20, Theorem 2])**.** *Let X be a complex Banach space, T a compact operator on X and $\mathcal{N}$ a simple invariant nest for T. Then:*

1. *A nonzero scalar $\lambda \in \mathbb{C}$ is an eigenvalue of T if and only if it is a diagonal coefficient of T.*
2. *The diagonal multiplicity of a nonzero scalar $\lambda \in \mathbb{C}$ is equal to its algebraic multiplicity as an eigenvalue of T.*
3. *T is quasi-nilpotent if and only if $\alpha_M = 0$ for all $M \in \mathcal{N}$.*

By combining the third statement of Theorem 3.35 with Corollary 3.23, we get the following result.

Corollary 3.36. *Let X be a complex Banach space, T a compact operator on X and $\mathcal{N}$ a simple invariant nest for T. Then T is quasi-nilpotent if and only if $TM \subset M_-$ for all $M \in \mathcal{N}$.*

The proof of Theorem 3.35 will primarily be divided into a few propositions. We will start by showing that each nonzero diagonal coefficient is an eigenvalue. Then we prove the converse statement, which is significantly more work. Together, these prove the first point of Theorem 3.35. Then we prove the second statement. The third statement then easily follows from the first.

Proposition 3.37 ([20, Lemma 5])**.** *Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a simple invariant nest for T. Let $M \in \mathcal{N}$ be arbitrary. If $\alpha_M \neq 0$, then α_M is an eigenvalue of T.*

Proof. Suppose that $\alpha_M \neq 0$. As $\alpha_M \neq 0$, it follows that $\dim(M/M_-) = 1$. As M is an invariant subspace for T, we can consider the restriction T' of T to M. By Corollary 3.23, it follows that $(T' - \alpha_M I_M)M \subset M_- \subsetneq M$.

Hence $T' - \alpha_M I_M$ is not surjective. As $T' : M \to M$ is a compact operator on the complex Banach space M and α_M is a nonzero scalar, it follows from the Fredholm alternative (Theorem 3.7) that $T' - \alpha_M I_M$ has a nontrivial kernel. Hence α_M is an eigenvalue of T'. As T' is a restriction of T, this implies that α_M is an eigenvalue of T. □

Lemma 3.38. *Let C be a compact topological space, I a totally ordered indexing set and let $\{S_i\}_{i\in I}$ be an increasing or decreasing filtration of nonempty closed subsets of C. Then $\bigcap_{i\in I} S_i$ is nonempty.*

Proof. Suppose $\bigcap_{i\in I} S_i$ is empty, then by using one of De Morgan's laws, we get that

$$C = C \setminus \bigcap_{i\in I} S_i = \bigcup_{i\in I} C \setminus S_i.$$

We see that $\{C \setminus S_i\}_{i\in I}$ is a cover of open sets of C. As C is compact, this implies there exists a finite subset $\{i_1, ..., i_N\} \subset I$, such that

$$C = \bigcup_{k=1}^{N} C \setminus S_{i_k} = C \setminus \bigcap_{k=1}^{N} S_{i_k}.$$

From this we can conclude that $\bigcap_{k=1}^{N} S_{i_k} = \emptyset$. However, as $\{S_i\}_{i\in I}$ is an increasing or decreasing filtration, there exists an integer j such that $1 \le j \le N$ and $S_{i_j} = \bigcap_{k=1}^{N} S_{i_k} = \emptyset$, which is a contradiction. □

Proposition 3.39 ([20, Lemma 3])**.** *Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a simple invariant nest for T. Let $M \in \mathcal{N}$ be an invariant subspace for T and let $\delta > 0$. Then there exists a subspace $L \in \mathcal{N}$ such that $L \subsetneq M$ and such that for all $x \in M_-$ we have that*

$$\|Tx + L\|_L \le \delta \|x\|,$$

where $\|\cdot\|_L$ is the quotient norm on X/L.

Proof. We distinguish two cases: $\dim(M/M_-) = 1$ and $\dim(M/M_-) = 0$. If $\dim(M/M_-) = 1$, then set $L = M_-$. As M_- is an invariant subspace for T, it follows that $\|Tx + M_-\|_{M_-} = \|0 + M_-\|_{M_-} = 0 \le \delta\|x\|$ for all $x \in M_-$.

Now suppose that $\dim(M/M_-) = 0$. We argue by contradiction, so suppose that such L does not exist. Define

$$\mathcal{N}_0 = \{L \in \mathcal{N} : L \subsetneq M\}.$$

By assumption it follows that for all $L \in \mathcal{N}_0$, there exists an $x \in M_-$ such that $\|Tx + L\|_L > \delta\|x\|$. As this clearly does not hold for $x = 0$, we can divide by $\|x\|$ to obtain a unit vector $x' \in M_-$ such that $\|Tx' + L\|_L > \delta$. So the set

$$S_L = \{x \in M_- : \|x\| = 1,\ \|Tx + L\|_L > \delta\}$$

is nonempty for all $L \in \mathcal{N}_0$. We claim that the filtration $\{S_L\}_{L\in\mathcal{N}_0}$ is decreasing. Let $K, N \in \mathcal{N}_0$ and suppose that $K \subset N$. Let $y \in X$ be arbitrary, by definition of the quotient norm, it follows that

$$\|y + K\|_K = \inf_{z\in K} \|y - z\| \geq \inf_{z\in N} \|y - z\| = \|y + N\|_N.$$

So if $x \in S_N$, then $\|x\| = 1$ and $\|Tx + N\|_N > \delta$, hence $\|Tx + K\|_K \geq \|Tx + N\|_N > \delta$. Therefore it follows that $x \in S_K$ and thus we have that $S_N \subset S_K$. It follows that $\{S_L\}_{L\in\mathcal{N}_0}$ is indeed decreasing. Let S be the unit sphere in X, then by definition of S_L, it follows that $S_L \subset S$ for all $L \in \mathcal{N}_0$. As S is bounded, the set $C = \overline{TS}$ is compact by compactness of T. Furthermore, $\{TS_L\}_{L\in\mathcal{N}_0}$ is a decreasing filtration in TS and hence $\{\overline{TS_L}\}_{L\in\mathcal{N}_0}$ is a decreasing filtration of closed sets in C. By Lemma 3.38, it follows that $\bigcap_{L\in\mathcal{N}_0} \overline{TS_L}$ is nonempty. So there exists an $x_0 \in X$ such that $x_0 \in \bigcap_{L\in\mathcal{N}_0} \overline{TS_L} \subset M_-$.

Let $L \in \mathcal{N}_0$ be arbitrary. It follows that $x_0 \in \overline{TS_L}$, hence there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in S_L such that $\{Tx_n\}_{n\in\mathbb{N}}$ converges to x_0 as $n \to \infty$. As $x_n \in S_L$ for all $n \in \mathbb{N}$, it follows that $\|Tx_n + L\|_L > \delta$ for all $n \in \mathbb{N}$. By taking the limit as $n \to \infty$, it follows that $\|x_0 + L\|_L \geq \delta$. So we have that $\|x_0 + L\|_L \geq \delta$ for all $L \in \mathcal{N}_0$. As $x_0 \in M_- = \overline{\bigcup_{L\in\mathcal{N}_0} L}$, there must be a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\bigcup_{L\in\mathcal{N}_0} L$ converging to x_0. In particular, there must exist an $L \in \mathcal{N}_0$ and a $y \in L$ such that $\|x_0 - y\| < \delta$, contradicting that $\|x_0 + L\|_L \geq \delta$ for all $L \in \mathcal{N}_0$. $\square$

Proposition 3.40 ([20, Lemma 4]). *Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a simple invariant nest for T. Let $\lambda \in \mathbb{C}$ be a nonzero eigenvalue of T and $x \in X$ be a corresponding nonzero eigenvector. Let*

$$\mathcal{N}_0 = \{L \in \mathcal{N} : x \in L\}$$

and define $M = \bigcap_{L\in\mathcal{N}_0} L$. Then $M \in \mathcal{N}$, $\dim(M/M_-) = 1$ and $\alpha_M = \lambda$.

Proof. The fact that $M \in \mathcal{N}$ follows directly from the second property of simple nests. We first prove that $\dim(M/M_-) = 1$. We argue by contradiction. Suppose that $\dim(M/M_-) = 0$, hence $M = M_-$. Let $0 < \delta < \frac{1}{2}|\lambda|$. Then by Proposition 3.39, there exists a subspace $L \in \mathcal{N}$ such that $L \subsetneq M$ and $\|Tz + L\|_L \leq \delta\|z\|$ for all $z \in M_-$. As $L \subsetneq M$, it follows that $x \notin L$. This implies that $\mathbb{C}x \cap L = \{0\}$, hence we can view $K = \mathbb{C}x \oplus L$ as a subspace of X. K is a normed space and $L \subset K$ is a closed subspace of codimension 1. Using Riesz' lemma, there exists a unit vector $y' \in K$ such that $\|y' - z\| \geq \frac{2}{3}$ for all $z \in L$. As $\dim(K/L) = 1$ and $x, y' \notin L$, there exists a nonzero $\gamma \in \mathbb{C}$ such that $x - \gamma y' \in L$. Let $y = \gamma y'$. Then it follows that $\|y + L\|_L = |\gamma|\|y' + L\|_L \geq \frac{2}{3}|\gamma| = \frac{2}{3}\|y\|$. Hence $y \in X$ is a vector such that $x - y \in L$ and $\|y\| \leq \frac{3}{2}\|y + L\|_L < 2\|y + L\|_L = 2\|x + L\|_L$. As L is an invariant subspace for T, it follows that $Tx - Ty \in L$. Therefore, we also have

$$\begin{aligned} Ty - \lambda y &= Tx - \lambda x + (Ty - Tx - \lambda y + \lambda x) \\ &= Ty - Tx - \lambda(y - x) \in L, \end{aligned}$$

hence $Ty + L = \lambda(y + L)$. It follows that

$$\|Ty + L\|_L = |\lambda|\|y + L\|_L > \frac{1}{2}|\lambda|\|y\| > \delta\|y\|.$$

However, as $y \in M = M_-$, this contradicts the assumption on L that $\|Tz + L\|_L \leq \delta\|z\|$ for all $z \in M_-$. Therefore it must be the case that $\dim(M/M_-) = 1$.

As $\dim(M/M_-) = 1$ implies that $M_- \subsetneq M$, it follows that $x \notin M_-$. So $M = \mathbb{C}x \oplus M_-$ as vector space. This implies that $\lambda x = Tx = \alpha_M x + y_x$ for some $y_x \in M_-$. As $M = \mathbb{C}x \oplus M_-$, it directly follows that $y_x = 0$ and $\alpha_M = \lambda$, which completes the proof. $\square$

Lemma 3.41. *Let V be a vector space over $\mathbb{F}$ and let $d \in \mathbb{N}$ be such that $d < \dim(V)$. Suppose that we have d linear functionals $\{\varphi_i\}_{i=1}^d$. Then $\bigcap_{i=1}^d \ker(\varphi_i) \neq \{0\}$.*

Proof. We argue by contradiction, so suppose that $\bigcap_{i=1}^d \ker(\varphi_i) = \{0\}$. Let the linear map $\Phi : V \to \mathbb{F}^d$ be defined by $\Phi(x) = (\varphi_1(x), \dots, \varphi_d(x))$. By assumption, it follows that Φ is injective. Define $W = \Phi(V) \subset \mathbb{F}^d$. As Φ is injective, it follows that $W \simeq V$ and therefore that $\dim(V) = \dim(W) \leq \dim(\mathbb{F}^d) = d$, contradicting $\dim(V) > d$. $\square$

Proposition 3.42 ([20, Lemma 6])**.** *Let X be a Banach space, T a compact operator on X and $\mathcal{N}$ a simple invariant nest for T. Let $\lambda \in \mathbb{C}$ be a nonzero scalar. Then the diagonal multiplicity of λ is equal to its algebraic multiplicity as an eigenvalue of T.*

Proof. Let d be the diagonal multiplicity of λ, m the algebraic multiplicity of λ and let ν be the index of λ relative to T. If $\nu = 0$, then λ is not an eigenvalue of T and therefore by Proposition 3.37, it also follows that λ is not a diagonal coefficient. Hence $m = 0 = d$. Now suppose that $\nu \neq 0$. We start with a reduction step. We claim that it suffices to prove the lemma for $\nu = 1$. So suppose the result is true for $\nu = 1$. As T and I commute, we can use the binomial theorem to expand $(T - \lambda I)^\nu = S - \mu I$ where $\mu = -(-\lambda)^\nu$ and S is a compact operator, a polynomial in T. As λ is an eigenvalue of T, it follows that $(T - \lambda I)^\nu = S - \mu I$ has a non trivial kernel. So μ is a nonzero eigenvalue of S. Furthermore, we have that

$$\ker((S - \mu I)^2) = \ker((T - \lambda I)^{2\nu}) = \ker((T - \lambda I)^\nu) = \ker(S - \mu I).$$

Hence the index of μ relative to S equals 1 and the algebraic multiplicity of μ is also equal to m. As S is a polynomial in T, all spaces $M \in \mathcal{N}$ are also invariant subspaces for S. This implies that we can define the diagonal coefficients $\{\sigma_M\}_{M\in\mathcal{N}}$ of S according to Definition 3.24. By definition of the diagonal coefficients $\{\alpha_M\}_{M\in\mathcal{N}}$ of T, it follows that for all $M \in \mathcal{N}$ and for all $x \in M \setminus M_-$ we have that $(T-\lambda I)x = (\alpha_M - \lambda)x + y_1$ with $y_1 \in M_-$. As M_- is also an invariant subspace for $T - \lambda I$, it follows by induction that for all $n \in \mathbb{N}$, there exists a $y_n \in M_-$ such that $(T - \lambda I)^n x = (\alpha_M - \lambda)^n x + y_n$. So

$$Sx = \mu x + (S - \mu I)x = \mu x + (T - \lambda I)^\nu x = (\mu + (\alpha_M - \lambda)^\nu)x + y_\nu.$$

As $y_\nu \in M_-$, it follows by definition of the diagonal coefficients that we have $\sigma_M = \mu + (\alpha_M - \lambda)^\nu$. Hence $\sigma_M = \mu$ if and only if $\alpha_M = \lambda$ and therefore the diagonal multiplicity of μ is also equal to d. As the index of μ relative to S equals 1, it follows by assumption that the diagonal multiplicity of μ is equal to the algebraic multiplicity of μ, hence $m = d$. So it also follows that the diagonal multiplicity of λ is equal to the algebraic multiplicity of λ. Hence it indeed suffices to prove the lemma for the case $\nu = 1$.

So now moreover suppose that $\nu = 1$. We need to prove that $m = d$. Denote the kernel of $T - \lambda I$ with N. By compactness of T, it follows that N is finite-dimensional and by definition of the algebraic multiplicity, it follows that $\dim(N) = m$. For all nonzero $x \in N$, define

$$M(x) = \bigcap \{L \in \mathcal{N} : x \in L\}.$$

From Proposition 3.40, we conclude the following: $M(x) \in \mathcal{N}$, $\alpha_{M(x)} = \lambda$, $\dim(M(x)/M_-(x)) = 1$ and $x \in M(x) \setminus M_-(x)$. Now suppose that $M \in \mathcal{N}$ and that $\alpha_M = \lambda$. We claim that there exists an $x \in N$ such that $M = M(x)$. To prove this, we consider the restriction of T to M, which we will denote by T'. As M is closed in X, it follows that $T' : M \to M$ is a compact map on the Banach space M. Furthermore, denote the restriction of the identity map to M by I_M. As $\alpha_M = \lambda \neq 0$, it follows that $\dim(M/M_-) = 1$. So since $(T' - \lambda I_M)M \subset M_-$, it follows that $T' - \lambda I_M$ is not surjective. It then follows by the Fredholm alternative that $T' - \lambda I_M$ is not injective. So λ is an eigenvalue of T'. As the index of λ relative to T equals 1, it follows that

$$\ker((T' - \lambda I_M)^2) = M \cap \ker((T - \lambda I)^2) = M \cap \ker(T - \lambda I) = \ker(T' - \lambda I_M).$$

We conclude that the index of λ relative to T' is also equal to 1. Let N_M and W_M be the kernel and image of $T' - \lambda I_M$ respectively. Then $W_M = (T' - \lambda I_M)M \subset M_-$ and by Theorem 3.34 we also have that $N_M \oplus W_M$ is isomorphic to M. It now follows that there exists an eigenvector $x \in$

$N_M \cap (M \setminus M_-)$. Suppose $N_M \cap (M \setminus M_-) = \emptyset$. This implies that $N_M \subset M_-$, which would imply that $N_M \oplus W_M \subset M_- \subsetneq M$, which contradicts the fact that $N_M \oplus W_M$ is isomorphic to M. So pick a nonzero eigenvector $x \in N_M \cap (M \setminus M_-)$. As $x \in M$, it follows that $M(x) \subset M$. Moreover, as $x \notin M_-$, it also follows that $M_- \subsetneq M(x)$. Together with $\dim(M/M_-) = 1$, this implies that $M = M(x)$.

We now prove that $m \geq d$. Let $M_1 \subsetneq ... \subsetneq M_d$ be the d subspaces of $\mathcal{N}$ that have λ as diagonal coefficient. By our previous argument, we can find nonzero eigenvectors $x_1, ..., x_d \in N$ such that $M_i = M(x_i)$ for $1 \leq i \leq d$. Now suppose that for some i we have that x_i is a linear combination of $x_1, ..., x_{i-1}$. As $x_1, ..., x_{i-1} \in M_{i-1}$, it follows that $x_i \in M_{i-1}$ and hence that $M_i \subset M_{i-1}$. This is in contradiction with $M_1 \subsetneq ... \subsetneq M_d$ and therefore $x_1, ..., x_d \in N$ must be linearly independent. We conclude that $m = \dim(N) \geq d$.

To prove the other inequality, we again argue by contradiction. Suppose that $m > d$. As $M_i = M(x_i)$ implies that $x_i \in M_i \setminus (M_i)_-$, it follows that M_i and $\mathbb{C}x_i \oplus (M_i)_-$ are isomorphic as Banach spaces. Hence by applying the Hahn-Banach theorem, there exist linear functionals φ_i such that $\varphi_i(x_i) \neq 0$ and $(M_i)_- \subset \ker(\varphi_i)$ for $1 \leq i \leq d$. So if $x \in M_i$ and $\varphi_i(x) = 0$, it follows that $x \in (M_i)_-$. By Lemma 3.41, applied to $V = N$, there exists a nonzero eigenvector $x \in \bigcap_{1 \leq i \leq d} \ker(\varphi_i)$ and by Proposition 3.40, it follows that $\alpha_{M(x)} = \lambda$. So there exists a $1 \leq j \leq d$ such that $M(x) = M(x_j)$. As $\varphi_j(x) = 0$, it follows that $x \in M_-(x_j) = M_-(x)$, which contradicts that $x \in M(x) \setminus M_-(x)$. We conclude that $d \geq m$. Together with the previous inequality, this proves that $m = d$. $\square$

Proof of Theorem 3.35. Let X be a complex Banach space, T a compact operator on X and $\mathcal{N}$ a simple invariant nest for T. By Propositions 3.37 and 3.40, it follows that a nonzero scalar $\lambda \in \mathbb{C}$ is an eigenvalue of T if and only if it is a diagonal coefficient of T. This proves the first part. The second part of the theorem is precisely given by Proposition 3.42.

The only statement left to prove is the third. By Corollary 3.8, it follows that T is quasi-nilpotent if and only if it has no nonzero eigenvalues. By the first statement of this theorem, it follows that T has no nonzero eigenvalues if and only of $\alpha_M = 0$ for all $M \in \mathcal{N}$. Hence T is quasi-nilpotent if and only if $\alpha_M = 0$ for all $M \in \mathcal{N}$. $\square$

4 Equivalence of the Lidskii property and the nest approximation property

In this section, the extensive theoretical preparations we went through in the first three sections will be used to prove our final result. We will prove that for every complex Banach space satisfying the approximation property, the Lidskii property (LP) and the nest approximation property (NAP) are equivalent. This is a very recent result, first published in 2016 by Figiel and Johnson [6]. In this section, we will first discuss the two properties. Then we will prove the equivalence of these properties following the proofs in [6]. In this section, unless stated otherwise, all Banach spaces are assumed to be complex and satisfy the approximation property. This is to ensure that the nuclear trace is well-defined and that Theorem 3.35 applies.

4.1 The Lidskii property and the nest approximation property

4.1.1 The Lidskii property

It is a well-known fact from linear algebra that the trace is equal to the sum of the eigenvalues (counted with algebraic multiplicity) for linear operators on finite-dimensional vector spaces. In 1959, Lidskii proved that a similar statement holds for the so-called trace-class operators on a Hilbert space [12]. This is therefore called Lidskii's theorem. The Lidskii property is the result of trying to generalize this theorem to arbitrary Banach spaces.

To introduce the Lidskii property, suppose that X is a complex Banach space and that T is a compact operator on X. Let $\mathcal{N}$ be a simple invariant nest for T. From Lemma 3.26, it follows that the subnest

$$\mathcal{N}_0 = \{M \in \mathcal{N} : |\alpha_M| > 0\}$$

is countable. Hence the set $\{\alpha_M\}_{M \in \mathcal{N}_0}$ of all nonzero diagonal coefficients, counted according to diagonal multiplicity, is countable. Combined with Theorem 3.35, it follows that $\{\alpha_M\}_{M \in \mathcal{N}_0}$ is the set of all nonzero eigenvalues of T, counted according to algebraic multiplicity. We can summarize this in the following proposition.

Proposition 4.1. *Let X be a complex Banach space and T a compact operator on X. Let $\mathcal{N}$ be a simple invariant nest for T. Then the nest*

$$\mathcal{N}_0 = \{M \in \mathcal{N} : |\alpha_M| > 0\}$$

is countable. This implies the set of nonzero eigenvalues, counted according to their algebraic multiplicity is countable, hence this can be written as $\{\lambda_k\}_{k \in J}$, where either $J = \mathbb{N}$ or there exists an $N \in \mathbb{N}$ such that $J = \{1, \ldots, N\}$.

Remark.

1. There are other ways to prove that compact operators have countably many eigenvalues, without appealing to nests or Theorem 3.35. See for example Megginson [15, Theorem 3.4.23] or Conway [2, Theorem 7.1] for two different approaches.

2. If T is an operator and we write the set of eigenvalues of T as $\{\lambda_k\}_{k\in J}$, then the eigenvalues are always counted according to their algebraic multiplicity. Furthermore, we always assume J to be defined as in Proposition 4.1.

Definition 4.2. Let X be a complex Banach space satisfying the approximation property. X satisfies the *Lidskii property* if for all nuclear operators A with absolutely summable eigenvalues $\{\lambda_k\}_{k\in J}$, the following equality holds:

$$\mathrm{Tr}(A) = \sum_{k\in J} \lambda_k.$$

Remark. In the definition of the Lidskii property, we need to restrict ourselves to the nuclear operators with absolutely summable eigenvalues. This assumption cannot be omitted as any Banach space that is not isomorphic to a Hilbert space has nuclear operators acting on it with non-summable eigenvalues [9, Theorem 3.11].

4.1.2 The nest approximation property

The nest approximation property is a stronger variant of the approximation property we already discussed in the first two sections. For the formulation of the nest approximation property, our starting point will be the second characterization of the approximation property we used. This stated that a Banach X has the approximation property if we can uniformly approximate the identity operator I_X by finite rank operators on each compact subset of X. In the second section, we saw that we can reformulate this statement more concisely as $I_X \in \overline{F(X)}^{\tau}$ when we adopt the ucc-topology τ. To define the nest approximation property, we will only need to impose some more conditions on the operators we use to approximate the identity operator.

Definition 4.3. Let X be a Banach space and $\mathcal{N}$ a nest of closed subspaces of X. We define $B_{\mathcal{N}}(X) \subset B(X)$ as the set of bounded linear operators that leave all subspaces in $\mathcal{N}$ invariant. Furthermore, we define $F_{\mathcal{N}}(X) = F(X) \cap B_{\mathcal{N}}(X)$ and $N_{\mathcal{N}}(X) = N(X) \cap B_{\mathcal{N}}(X)$.

This allows us to define the $\mathcal{N}$-approximation property and subsequently the nest approximation property.

Definition 4.4. Let X be a Banach space and $\mathcal{N}$ a nest of closed subspaces of X. Then X has the *$\mathcal{N}$-approximation property ($\mathcal{N}$-AP)* if $I_X \in \overline{F_{\mathcal{N}}(X)}^{\tau}$.

We say that X has the *nest approximation property (NAP)* if X has the $\mathcal{N}$-approximation property for any nest $\mathcal{N}$.

In the next subsection, we will prove the following result, which is the main result of this thesis.

Theorem 4.5 ([6, Theorem 3.2])**.** *Let X be a complex Banach space that has the approximation property. Then the following are equivalent:*

1. *X has the nest approximation property.*
2. *For every quasi-nilpotent nuclear operator $A \in N(X)$, we have that* $\operatorname{Tr}(A) = 0$.
3. *X has the Lidskii property.*

4.2 Proving the equivalence

In this subsection, we will prove Theorem 4.5. However, we first need some preparatory results.

Definition 4.6. Let X be a Banach space and let $V \subset X$ be a (not necessarily closed) linear subspace. We define the *annihilator of* V, denoted by $V^\perp$, as

$$V^\perp = \{\varphi \in X^* : V \subset \ker(\varphi)\}.$$

Lemma 4.7. *Let X be a Banach space and let $V \subset X$ be a (not necessarily closed) linear subspace. Then $V^\perp = \overline{V}^\perp = \overline{V^\perp}$.*

Proof. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset V^\perp$ be a convergent sequence of functionals and let φ be its limit. It follows that $\|\varphi(x)\| = \|\varphi(x) - \varphi_n(x)\| \leq \|\varphi - \varphi_n\|\|x\|$ for all $x \in V$. As we can make $\|\varphi - \varphi_n\|$ arbitrarily small, it follows that $\varphi(x) = 0$, hence $\varphi \in V^\perp$. So $V^\perp$ is closed, hence we have $V^\perp = \overline{V^\perp}$.

$V^\perp \subset \overline{V}^\perp$ follows from the fact that $V \subset \ker(\varphi)$ implies $\overline{V} \subset \ker(\varphi)$ as the kernel of a bounded functional is closed. As the other inclusion is trivial, it follows that $V^\perp = \overline{V}^\perp$. □

Lemma 4.8. *Let X be a Banach space and let $V \subset X$ be a (not necessarily closed) linear subspace. Then for all $x \in X$, we have that $x \in \overline{V}$ if and only if $x \in \ker(\varphi)$ for all $\varphi \in V^\perp$.*

Proof. The forward implication is a trivial consequence of the fact that $\ker(\varphi)$ is closed for all $\varphi \in V^\perp$.

For the converse implication, we argue by contradiction. Assume that $x \in \ker(\varphi)$ for all $\varphi \in V^\perp$ and suppose that $x \notin \overline{V}$. This implies that we can define $W = \mathbb{C}x \oplus \overline{V}$ and $\varphi : W \to \mathbb{C}$ by $\varphi(\alpha x + v) = \alpha$ for all $\alpha \in \mathbb{C}$

and $v \in \overline{V}$. From this it follows that $\overline{V} \subset \ker(\varphi)$ and that $\varphi(x) = 1$. As $\overline{V}$ is closed, it follows by the Hahn-Banach theorem that we can extend φ to a bounded functional $\hat{\varphi} \in V^{\perp}$ such that $\hat{\varphi}(x) = 1$, contradicting our assumption. □

The following proposition describes the structure of the elements in $F_{\mathcal{N}}(X)$.

Proposition 4.9 ([6, Lemma 1])**.** *Let X be a Banach space and $\mathcal{N}$ a complete nest of subspaces of X. Let $x^* \in X^*$ and $x \in X$ be nonzero. Then:*

1. *$x^* \otimes x \in F_{\mathcal{N}}(X)$ if and only if there exists an $M \in \mathcal{N}$ such that $x \in M$ and $x^* \in (M_-)^{\perp}$.*
2. *If $T \in F_{\mathcal{N}}(X)$ has rank n with $n > 0$, then there exist $x_1, \ldots, x_n \in X$ and $x_1^*, \ldots, x_n^* \in X^*$ such that $T = \sum_{k=1}^{n} x_k^* \otimes x_k$ where the rank one operators $x_k^* \otimes x_k$ are in $F_{\mathcal{N}}(X)$ for all $1 \leq k \leq n$.*

Proof. 1. Suppose that $x^* \otimes x \in F_{\mathcal{N}}(X)$ and define

$$\mathcal{N}_0 = \{L \in \mathcal{N} : x \in L\}.$$

We claim that $M = \bigcap_{L \in \mathcal{N}_0} L$ works. To prove the first part, note that by completeness of $\mathcal{N}$, it follows that $M \in \mathcal{N}$ and by construction it follows that $x \in M$. Now suppose that $N \in \mathcal{N}$ and $N \subsetneq M$, then by construction of M, it follows that $x \notin N$ hence we have that $\mathbb{C}x \cap N = \{0\}$. As $x^* \otimes x \in F_{\mathcal{N}}(X)$, we have that $x^*(y)x \in \mathbb{C}x \cap N$ for all $y \in N$. It follows that $x^*(y) = 0$ for all $y \in N$, hence $N \subset \ker(x^*)$. As this holds for all $N \subsetneq M$ and the kernel of x^* is closed, it follows that $M_- \subset \ker(x^*)$ and therefore we have that $x^* \in (M_-)^{\perp}$.

Conversely, suppose that there exists an $M \in \mathcal{N}$ such that $x \in M$ and $x^* \in (M_-)^{\perp}$. To prove that $x^* \otimes x \in F_{\mathcal{N}}(X)$, we show that each subspace $N \in \mathcal{N}$ is invariant under $x^* \otimes x$. Let $N \in \mathcal{N}$ be a subspace and suppose that $M \subset N$, then obviously for all $y \in N$ we have that $x^*(y)x \in M \subset N$. On the other hand, if $N \subsetneq M$, it follows that $N \subset M_-$ and therefore we have that $x^*(y)x = 0 \in N$ for all $y \in N$. In both cases, it follows that N is an invariant subspace for $x^* \otimes x$. We conclude that $x^* \otimes x \in F_{\mathcal{N}}(X)$.

2. We will proceed by induction on the rank n of T. For $n = 1$, the claim is clear. Suppose we have proven the statement for operators $T \in F_{\mathcal{N}}(X)$ of rank $n - 1 > 0$. Let $T \in F_{\mathcal{N}}(X)$ be an operator of rank n. Denote the unit sphere in the image TX of T by S_T and define

$$\mathcal{N}_1 = \{L \in \mathcal{N} : L \cap S_T \neq \emptyset\}.$$

As $X \in \mathcal{N}$ it follows that $\mathcal{N}_1 \neq \emptyset$, so we can define $M = \bigcap_{L \in \mathcal{N}_1} L$. It is clear that $\{L \cap S_T\}_{L \in \mathcal{N}_1}$ is an increasing filtration of closed nonempty subsets of

S_T. Furthermore, S_T is compact as it is a closed and bounded subset of the finite-dimensional vector space TX. By Lemma 3.38, it follows that $\bigcap_{L\in\mathcal{N}_1}(L \cap S_T) = M \cap S_T$ is nonempty. This implies that there exists a unit vector $x_1 \in M \cap S_T \subset TX$. Extend $\{x_1\}$ to a basis $B = (x_1, \dots, x_n)$ of TX. This implies that there exist linear functionals $x_1^*, \dots, x_n^*$ such that $T = \sum_{k=1}^n x_k^* \otimes x_k$. We claim that $x_1^* \in (M_-)^\perp$. Suppose this claim is true, then by part 1 of this proposition it follows that $x_1^* \otimes x_1 \in F_{\mathcal{N}}(X)$. So $T - x_1^* \otimes x_1 = \sum_{k=2}^n x_k^* \otimes x_k \in F_{\mathcal{N}}(X)$ is an operator of rank $n-1$, to which we can apply our induction hypothesis to complete the proof.

To prove our claim, define

$$\mathcal{N}_2 = \{L \in \mathcal{N} : L \subsetneq M\}.$$

Since we have that $L \cap TX \neq \{0\}$ if and only if $L \cap S_T \neq \emptyset$ for all $L \in \mathcal{N}$, it follows by definition of M that $L \cap TX = \{0\}$ for all $L \in \mathcal{N}_2$. Since $\big(\bigcup_{L\in\mathcal{N}_2} L\big) \cap TX = \bigcup_{L\in\mathcal{N}_2}(L \cap TX) = \{0\}$ and $TL \subset L$ for all $L \in \mathcal{N}_2$, it follows that

$$T\Big(\bigcup_{L\in\mathcal{N}_2} L\Big) \subset \Big(\bigcup_{L\in\mathcal{N}_2} L\Big) \cap TX = \{0\}.$$

This implies that $\bigcup_{L\in\mathcal{N}_2} L \subset \ker(T)$ and as $x_1, \dots x_n$ are linearly independent, it implies that $\bigcup_{L\in\mathcal{N}_2} L \subset \ker(x_i^*)$ for all $1 \leq i \leq n$. So in particularly $\bigcup_{L\in\mathcal{N}_2} L \subset \ker(x_1^*)$. By taking the closure it follows that $M_- = \overline{\bigcup_{L\in\mathcal{N}_2} L} \subset \ker(x_1^*)$, hence $x_1^* \in (M_-)^\perp$. □

Remark. Proposition 4.9 quite explicitly describes the operators in $F_{\mathcal{N}}(X)$ for complete nests $\mathcal{N}$. It also shows that for any complete nest $\mathcal{N}$ the space $F_{\mathcal{N}}(X)$ is nonempty as the Hahn-Banach theorem guarantees that there exist operators of rank 1 in $F_{\mathcal{N}}(X)$.

Using our knowledge about the structure of $F_{\mathcal{N}}(X)$ for complete nests $\mathcal{N}$, we can connect the $\mathcal{N}$-AP to nuclear operators in the following theorem.

Theorem 4.10 ([6, Theorem 2.1])**.** *Let X be a Banach space with the AP and $\mathcal{N}$ a complete nest of subspaces of X. Then X has the $\mathcal{N}$-AP if and only if for all $T \in N(X)$ such that $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$, we have* $\mathrm{Tr}(T) = 0$.

Proof. Assume that X has the $\mathcal{N}$-AP, so $I \in \overline{F_{\mathcal{N}}(X)}^{\tau}$. Suppose that we have a nuclear operator $T \in N(X)$ such that $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$. Let Φ be the map from Theorem 2.27 and define $\varphi = \Phi(T)$. By Corollary 2.28, it follows that $\mathrm{Tr}(T) = \varphi(I)$. So we need to show that $\varphi(I) = 0$. As $I \in \overline{F_{\mathcal{N}}(X)}^{\tau}$, it suffices to prove that φ vanishes on $F_{\mathcal{N}}(X)$. Let $x^* \otimes x \in F_{\mathcal{N}}(X)$ be a rank one operator. By Proposition 4.9 there exists an $M \in \mathcal{N}$ such that $x \in M$ and $x^* \in (M_-)^\perp$. Using the identity in Corollary 2.28, it follows that $\varphi(x^* \otimes x) = x^*(Tx) = 0$ as $TM \subset M_-$. So

φ vanishes on all rank one operators in $F_{\mathcal{N}}(X)$. Hence by combining this with the second part of Proposition 4.9 and the linearity of φ, it follows that φ vanishes on $F_{\mathcal{N}}(X)$. We conclude that $\mathrm{Tr}(T) = \varphi(I) = 0$.

For the converse implication, assume that for all $T \in N(X)$ such that $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$, we have $\mathrm{Tr}(T) = 0$. We argue by contradiction, so suppose that $I \notin \overline{F_{\mathcal{N}}(X)}^{\tau}$. By Theorem 2.20, there exists a continuous linear functional $\varphi \in (B(X), \tau)^*$ such that $\varphi(I) = 1$ and $\varphi \in F_{\mathcal{N}}(X)^{\perp}$. Let Φ again be the map of Theorem 2.27. As Φ is bijective, define $T = \Phi^{-1}(\varphi)$. It follows that $\mathrm{Tr}(T) = \Phi(T)(I) = \varphi(I) = 1$. Now let $x^* \otimes x \in F_{\mathcal{N}}(X)$ be a rank one operator, then $x^*(Tx) = \Phi(T)(x^* \otimes x) = \varphi(x^* \otimes x) = 0$ as $\varphi \in F_{\mathcal{N}}(X)^{\perp}$. Now fix a nonzero $M \in \mathcal{N}$ and $x \in M$, then this implies that $x^*(Tx) = 0$ for all $x^* \in (M_-)^{\perp}$. Hence by Lemma 4.8, it follows that $Tx \in M_-$. As $x \in M$ and $M \in \mathcal{N}$ nonzero were arbitrary, it follows that $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$. By assumption, it follows that $\mathrm{Tr}(T) = 0$, contradicting our choice of T such that $\mathrm{Tr}(T) = 1$. We conclude that $I \in \overline{F_{\mathcal{N}}(X)}^{\tau}$, hence X has the $\mathcal{N}$-AP. $\square$

Before we proceed, we need the following lemma.

Lemma 4.11. *Let X be a Banach space and $\mathcal{N}$ a nest of closed subspaces of X. Suppose that a sequence $\{A_n\}_{n\in\mathbb{N}} \subset B_{\mathcal{N}}(X)$ converges strongly to $A \in B(X)$, then $A \in B_{\mathcal{N}}(X)$.*

Proof. Let $M \in \mathcal{N}$ be a closed subspace and A be defined as above. As $A_n \in B_{\mathcal{N}}(X)$ for all $n \in \mathbb{N}$, it follows that $A_n x \in M$ for all $x \in M$ and $n \in \mathbb{N}$. It now directly follows that $Ax = \lim_{n\to\infty} A_n x \in \overline{M} = M$ for all $x \in M$. We conclude that $AM \subset M$ for all $M \in \mathcal{N}$, hence $A \in B_{\mathcal{N}}(X)$. $\square$

Corollary 4.12. *Let X be a Banach space and $\mathcal{N}$ a nest of closed subspaces of X. Then $B_{\mathcal{N}}(X) \subset B(X)$ is closed (with respect to the operator norm topology).*

Proof. This directly follows from Lemma 4.11 and the fact that norm convergence implies strong convergence in $B(X)$. $\square$

As any maximal nest is complete, we can now combine Theorem 4.10 with Corollary 3.36 to obtain the following result.

Theorem 4.13 ([6, Proposition 2])**.** *Let X be a Banach space with the AP and $\mathcal{N}$ a maximal nest of subspaces of X. Then the following are equivalent:*

1. *X has the $\mathcal{N}$-AP.*
2. *For all quasi-nilpotent $T \in N_{\mathcal{N}}(X)$, we have* $\mathrm{Tr}(T) = 0$.
3. *For all $T \in N_{\mathcal{N}}(X)$ with absolutely summable eigenvalues $\{\lambda_k\}_{k\in J}$, counted according to the algebraic multiplicity, we have* $\mathrm{Tr}(T) = \sum_{k\in J} \lambda_k$.

Proof. We first prove the equivalence 1 $\iff$ 2. By Theorem 4.10, it follows that X has the $\mathcal{N}$-AP if and only if for all $T \in N(X)$ such that $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$, we have $\mathrm{Tr}(T) = 0$.

Assume that for all $T \in N(X)$ such that $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$, we have $\mathrm{Tr}(T) = 0$. Let $T \in N_{\mathcal{N}}(X)$ be quasi-nilpotent. Then by Corollary 3.36 it follows that $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$. Hence by assumption, it follows that $\mathrm{Tr}(T) = 0$.

To prove the converse implication, assume that for all quasi-nilpotent $T \in N_{\mathcal{N}}(X)$, we have $\mathrm{Tr}(T) = 0$. Suppose we have a nuclear operator $T \in N(X)$ such that $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$. We need to prove that $\mathrm{Tr}(T) = 0$. As $TM \subset M_-$ for all nonzero $M \in \mathcal{N}$, it follows that $T \in N_{\mathcal{N}}(X)$ and by Corollary 3.36 it follows that T is quasi-nilpotent. Hence by assumption, it follows that $\mathrm{Tr}(T) = 0$.

We finish the proof by proving the equivalence 2 $\iff$ 3. The implication 3 $\implies$ 2 is trivial by definition of a quasi-nilpotent operator. So it remains to prove the implication 2 $\implies$ 3. So assume that for all quasi-nilpotent $T \in N_{\mathcal{N}}(X)$, we have $\mathrm{Tr}(T) = 0$. Let $A \in N_{\mathcal{N}}(X)$ be a nuclear operator with absolutely summable eigenvalues. Denote these eigenvalues by $\{\lambda_k\}_{k\in J}$, where each eigenvalue is counted according to the algebraic multiplicity and the eigenvalue 0 is excluded. By Theorem 3.35, these eigenvalues precisely correspond to the nonzero diagonal coefficients α_M of A, including their multiplicity. This allows us to define a sequence of *distinct* subspaces $\{M_k\}_{k\in J} \in \mathcal{N}$ such that $\alpha_{M_k} = \lambda_k$. As $\lambda_k \neq 0$, it follows that $\dim(M_k/(M_k)_-) = 1$ for all $k \in J$. This allows us to use Riesz' Lemma to find a unit vector $x_k \in M_k$ such that $\|x_k - y\| \geq \frac{1}{2}$ for all $y \in (M_k)_-$. Since M_k and $\mathbb{C}x_k \oplus (M_k)_-$ are isomorphic as Banach spaces, we can define $x_k^* : M_k \to \mathbb{C}$ by $x_k^*(\alpha x_k + y) = \alpha$ for all $\alpha \in \mathbb{C}$ and $y \in (M_k)_-$. It follows that for all $y \in (M_k)_-$ and nonzero $\alpha \in \mathbb{C}$, we have that

$$|x_k^*(\alpha x_k + y)| = |\alpha| \leq 2|\alpha|\|x_k + \alpha^{-1}y\| = 2\|\alpha x_k + y\|.$$

This inequality also extends to the case that $\alpha = 0$, hence by the Hahn-Banach theorem we can extend x_k^* to a bounded linear functional on X with norm at most 2. Furthermore, it follows that $x_k^*(x_k) = 1$ for all $k \in J$.

Define $B = \sum_{k\in J} \lambda_k x_k^* \otimes x_k$. Then B is a nuclear operator on X since

$$\sum_{k\in J} |\lambda_k|\|x_k^*\|\|x_k\| \leq 2\sum_{k\in J} |\lambda_k| < \infty.$$

By construction it follows that $(M_k)_- \subset \ker(x_k^*)$. So by Proposition 4.9, we have that $\lambda_k x_k^* \otimes x_k \in F_{\mathcal{N}}(X) \subset B_{\mathcal{N}}(X)$ for all $k \in J$. Therefore, it follows by Corollary 4.12 that $B \in B_{\mathcal{N}}(X)$. As both A and B are nuclear operators in $B_{\mathcal{N}}(X)$, it follows that $A - B \in N_{\mathcal{N}}(X)$. We claim that $A - B$ is quasi-nilpotent. If the claim is true, then it follows by assumption that

$0 = \mathrm{Tr}(A - B) = \mathrm{Tr}(A) - \sum_{k\in J} \lambda_k$. So we are finished once we prove the claim.

We prove that $A-B$ is quasi-nilpotent by showing that $(A-B)M \subset M_-$ for all $M \in \mathcal{N}$. Let $M \in \mathcal{N}$ be arbitrary. If $\dim(M/M_-) = 0$, there is nothing to prove as $(A - B)M \subset M = M_-$ since $A - B \in B_{\mathcal{N}}(X)$. So now suppose that $\dim(M/M_-) = 1$. We distinguish two cases: $\alpha_M = 0$ and $\alpha_M \neq 0$.

If $\alpha_M = 0$, it follows that $AM \subset M_-$. Furthermore, it follows that $M \notin \{M_k\}_{k\in J}$. This defines two sets of indices

$$J_1 = \{k : M_k \subset M_-\} \quad \text{and} \quad J_2 = \{k : M \subset (M_k)_-\},$$

that form a partition of J. For arbitrary $x \in M$, it then follows that

$$\begin{aligned} Bx &= \sum_{k\in J} \lambda_k x_k^*(x)x_k = \sum_{k\in J_1} \lambda_k x_k^*(x)x_k + \sum_{k\in J_2} \lambda_k x_k^*(x)x_k \\ &= \sum_{k\in J_1} \lambda_k x_k^*(x)x_k + 0 \in M_-. \end{aligned}$$

So $AM \subset M_-$ and $BM \subset M_-$ implying that $(A - B)M \subset M_-$.

If $\alpha_M \neq 0$, then there exists a $K \in J$ such that $M = M_K$. Again define

$$J_1 = \{k : M_k \subset M_-\} \quad \text{and} \quad J_2 = \{k : M \subset (M_k)_-\},$$

that form a partition of $J \setminus \{K\}$. For arbitrary $x \in M$, it then follows that

$$\begin{aligned} (A - B)x &= Ax - \sum_{k\in J} \lambda_k x_k^*(x)x_k \\ &= Ax - \lambda_K x_K^*(x)x_K - \sum_{k\in J_1} \lambda_k x_k^*(x)x_k - \sum_{k\in J_2} \lambda_k x_k^*(x)x_k \\ &= Ax - \lambda_K x_K^*(x)x_K - \sum_{k\in J_1} \lambda_k x_k^*(x)x_k. \end{aligned}$$

We need to show that this is an element of M_-. As the third term clearly is an element of M_-, it suffices to show that $Ax - \lambda_K x_K^*(x)x_K \in M_-$. As $x_K \in M \setminus M_-$, it follows that $M = \mathbb{C}x_K \oplus M_-$ as vector spaces. So there exists a unique scalar $\alpha \in \mathbb{C}$ and vector $y \in M_-$ such that $x = \alpha x_K + y$. Furthermore, from Corollary 3.23, it follows that there exists a $z \in M_-$ such that $Ax = \lambda_K x + z$. Combining these expressions yields

$$\begin{aligned} Ax - \lambda_K x_K^*(x)x_K &= \lambda_K x + z - \lambda_K x_K^*(x)x_K \\ &= \lambda_K(\alpha x_K + y) + z - \lambda_K x_K^*(\alpha x_K + y)x_K \\ &= \lambda_K(\alpha x_K + y) + z - \lambda_K \alpha x_K \\ &= \lambda_K y + z \in M_-, \end{aligned}$$

hence $A - B$ is indeed quasi-nilpotent. □

Having proven Theorem 4.13, most of the work towards proving Theorem 4.5 is done. We only need a few more results before we can finish the proof.

Lemma 4.14. *Let X be a Banach space and let $\mathcal{N}$ and $\mathcal{M}$ be nests of closed subspaces of X such that $\mathcal{N} \subset \mathcal{M}$. If X has the $\mathcal{M}$-AP, then it also has the $\mathcal{N}$-AP.*

Proof. Suppose that X has the $\mathcal{M}$-AP. As $\mathcal{N} \subset \mathcal{M}$, it is obvious that $F_{\mathcal{M}}(X) \subset F_{\mathcal{N}}(X)$. Since X has the $\mathcal{M}$-AP, it follows immediately that $I \in \overline{F_{\mathcal{M}}(X)}^{\tau} \subset \overline{F_{\mathcal{N}}(X)}^{\tau}$. We conclude that X also has the $\mathcal{N}$-AP. $\square$

Corollary 4.15. *Let X be a Banach space. Then X has the NAP if and only if X has the $\mathcal{N}$-AP for all maximal nests $\mathcal{N}$.*

Proof. By definition of the NAP, the forward implication is trivial. To prove the converse implication, assume that X has the $\mathcal{N}$-AP for all maximal nests $\mathcal{N}$. Suppose that $\mathcal{M}$ is a nest of closed subspaces of X. By Proposition 3.16, there exists a maximal nest $\mathcal{M}_{max}$ such that $\mathcal{M} \subset \mathcal{M}_{max}$. By assumption, X has the $\mathcal{M}_{max}$-AP hence by Lemma 4.14 it follows that X has the $\mathcal{M}$-AP. So X has the $\mathcal{M}$-AP for any nest $\mathcal{M}$, hence X has the NAP. $\square$

Proof of Theorem 4.5. To prove the general statement, we will combine Theorem 3.21 with Theorem 4.13 and Corollary 4.15. We will prove the implications $1 \implies 3$ and $2 \implies 1$. The equivalences then follow as the implication $3 \implies 2$ is obvious.

To prove the implication $1 \implies 3$, assume that X has the NAP. In particular, X has the $\mathcal{N}$-AP for all maximal nests $\mathcal{N}$. Let $A \in N(X)$ be a nuclear operator with summable eigenvalues $\{\lambda_k\}_{k \in J}$. By Theorem 3.21, there exists a maximal nest $\mathcal{N}$ of invariant subspaces of A such that $A \in N_{\mathcal{N}}(X)$. As X has the $\mathcal{N}$-AP, it follows by Theorem 4.13 that $\mathrm{Tr}(A) = \sum_{k \in J} \lambda_k$. As this holds for arbitrary nuclear operators with summable eigenvalues $\{\lambda_k\}_{k \in J}$, it follows that X has the Lidskii property.

To prove the implication $2 \implies 1$, assume that for every quasi-nilpotent nuclear operator $A \in N(X)$, we have that $\mathrm{Tr}(A) = 0$. Let $\mathcal{N}$ be a maximal nest of subspaces of X. It follows that for all quasi-nilpotent nuclear operators $A \in N_{\mathcal{N}}(X)$, we have $\mathrm{Tr}(A) = 0$. By Theorem 4.13, it then follows that X has the $\mathcal{N}$-AP. As $\mathcal{N}$ was an arbitrary maximal nest of subspaces of X, it follows that X has the $\mathcal{N}$-AP for all maximal nests $\mathcal{N}$. By Corollary 4.15, it follows that X has the NAP. $\square$

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