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The scalar-plus-compact problem

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Abstract

We present an exposition of the solution to the famous scalar-plus-compact problem as found by Argyros and Haydon in 2011. This is a hereditarily indecomposable $\mathscr{L}_\infty$-space on which every linear operator can be expressed as a compact perturbation of a multiple of the identity, with a dual isomorphic to ℓ_1. In treating its construction, we encounter over 40 years of history in Banach space theory. Indeed, the Argyros-Haydon construction contains ideas from the spaces constructed by Tsirelson, Schlumprecht, Gowers and Maurey and Bourgain and Delbaen. We cover each of these spaces in detail and modify the original notations and proofs in order to provide a uniform exposition.

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Chapter 1

Introduction

This master thesis contains an exposition of the solution of a famous problem in Banach space theory known as the "scalar-plus-compact problem". The goal is to present a "complete" proof of the solution found in 2011 by Argyros and Haydon [3] that can be read and understood start to finish by a student with some advanced knowledge of functional analysis, but without any specific knowledge about isomorphic Banach space theory. In doing so we will encounter over 40 years of history in "weird" Banach space constructions that form a hierarchy of spaces, each solving a specific problem, and culminating in the Argyros-Haydon space X_{AH} that solves the scalar-plus-compact problem.

We will assume that the reader is familiar with the basic concepts of Banach spaces such as (compact) operators, dual spaces, weak and weak* topologies. The rest of this introduction will give a brief overview of the problem and the strategy that will be followed in this thesis.

1.1 Problem description

As in most parts of pure mathematics, one of the principal goals in Banach space theory is to classify when two (subspaces of) Banach spaces are isomorphic (or even isometric), and to decompose given Banach spaces into simpler building blocks. This goes hand in hand with the study of morphisms between Banach spaces, which are the bounded linear operators. In particular, one can study the algebra of bounded linear operators $B(X)$ from a Banach space X to itself. The simplest question one can ask about this algebra is: what operators are in it? This however already turns out to be a difficult question, as is indicated by the fact that the scalar-plus-compact problem was open for over 30 years. The result, which due to the solution is now a theorem, is as follows:

Theorem 1.1.1. *There exists a real Banach space X such that all bounded linear operators $T : X \to X$ can be written as $T = \lambda I + K$, where K is a compact operator.*

Equivalently, one could say that $B(X) = \mathbb{R}I \oplus K(X)$, where $K(X)$ is the ideal of compact operators on X. Or, that the *Calkin algebra* $B(X)/K(X)$ is one-dimensional. If a space X satisfies this property, we will say that it has the *scalar-plus-compact* property. The question of whether such a space exists was first posed by Lindenstrauss in 1976 [22], and it was solved in 2011 by Argyros and Haydon [3]. The question arises naturally in Banach space theory when one considers the Hahn-Banach theorem:

Theorem 1.1.2 (Hahn-Banach theorem)**.** *Let X be a normed vector space, and let Y be a subspace of X. Let $p : X \to \mathbb{R}$ be a sublinear functional, and let $y^* \in Y^*$ be a linear functional such that $y^* \leq p$. Then there exists an extension $x^* \in X$ such that $x^*|_Y = y^*$ and $x^* \leq p$.*

In particular, if X is a general (complex) Banach space, and $x \in X$ is a non-zero element, then the Hahn-Banach theorem implies that there exists a linear functional $x^* \in X^*$ such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$. One can then combine these to construct sums of rank-1 operators on X. Indeed if $x_1, ..., x_n$, $y_1, ..., y_n$ are non-zero elements of X and $x_1^*, ..., x_n^*$ are dual elements as above, then we can define the finite-rank operator

$$x \mapsto \sum_{i=1}^{n} x_i^*(x) y_i.$$

This yields all finite-rank operators on X, the set of which is denoted $F(X)$. Since $B(X)$ is a Banach space, we can also take limits, and so together with the identity operator, the Hahn-Banach theorem implies that at the very least

$$\{\lambda I + F \mid \lambda \in \mathbb{C}, F \in \overline{F(X)}\} \subseteq B(X).$$

Beyond this there is no theoretical guarantee that $B(X)$ contains any other operators. However, most spaces that are studied in practice do admit a much larger algebra of operators, and so the question arises whether one can either prove a stronger result, or find a counterexample. The Argyros-Haydon space, then, is such a counterexample, where we do have a basis and so the set of operators is exactly the set of operators one can construct using the Hahn-Banach theorem.

More generally, a "Hahn-Banach problem" is a problem of the form: given Banach spaces X and $Y \subseteq Z$, when does an operator $T : Y \to X$ have an extension to an operator $\tilde{T} : Z \to X$, i.e. such that $\tilde{T}\big|_Y = T$. Or conversely when does $T : X \to Y$ have a lifting $\tilde{T} : X \to Z$. For the first case, we say that X has the *compact extension property* if such extensions exist whenever T is compact. The Argyros-Haydon space is an example of what is known as an $\mathscr{L}_\infty$-space. We will see that $\mathscr{L}_\infty$-spaces are precisely those spaces that have the compact extension property, and further that Hahn-Banach problems are closely related to the scalar-plus-compact problem.

In posing the scalar-plus-compact problem, Lindenstrauss already noted that such a space would satisfy a number of other properties that are of interest in Banach space theory:

(i) X is not isomorphic to any of its subspaces of finite codimension. (In particular, X solves the *Banach hyperplane problem*.)

(ii) X is *indecomposable*: it cannot be written as the direct sum of two infinite-dimensional subspaces.

(iii) Every operator on $T \in B(X)$ has a non-trivial proper closed invariant subspace. That is a closed subspace $Y \subsetneq X$ such that $Y \neq \{0\}$ and $T(Y) \subseteq Y$.

The questions of whether spaces satisfying any of these properties exist were also open for some time. The first was solved in 1994 by Gowers [16]. The second was solved in 1993 also by Gowers together with Maurey [17] when they constructed a space that is *hereditarily indecomposable* (HI), meaning that all infinite-dimensional subspaces are indecomposable. We will discuss this space in Chapter 4. The third is related to the *invariant subspace problem* on Hilbert spaces. The Argyros-Haydon space X_{AH} is also the first example of a space satisfying this property.

1.2 Strategy

Our starting point for writing this thesis was the overview article by Zsák [34], which gives a nice overview of the history of the scalar-plus-compact problem and the spaces that were constructed in the process. We already have discussed some of the contents of this article in the preceding section. For the contents of this thesis we follow a structure similar to the one of Zsák's paper, but we will go into full detail on the constructions of the spaces and we will also discuss some of the more technical background of the methods used. In particular, we have made an effort to include all the proofs of the main results, and to give a detailed explanation of the key steps in these proofs.

In the process we have made some modifications to the notations of the original papers in order to turn the exposition into a uniform story. Our largest addition in this sense is the proof of hereditary indecomposability of the Gowers-Maurey space X_{GM} in the language of tree analyses as introduced by Argyros [4]. Part of this proof is also the restatement of the results in Schlumprecht's paper [32] using this notation.

We will start in Chapter 2 by giving a brief overview of the theory of bases in Banach spaces and the isomorphic structure of the well-known sequence spaces ℓ_p and c_0. Following up on this, the first construction we will discuss in Chapter 3 is Tsirelson's space T, which provides a counter-example to a question arising from our discussion of the sequence spaces, namely T contains no isomorphic copy of ℓ_p or c_0. This will be the first step in our hierarchy of spaces, and it is a fairly simple construction which nonetheless already contains some of the key ideas that will be used in the later constructions.

Next up is Schlumprecht's space S, which is arbitrarily distortable for any $\lambda > 1$.

This distortability property is a slight tangent, but S will be important for the next space in Chapter 4, which is the Gowers-Maurey space X_{GM}. It is the first example of a hereditarily indecomposable (HI) space. Moreover, all operators on X_{GM} are of the form $\lambda I + S$, where S is strictly singular, so it was already tantilizingly close to having the scalar-plus-compact property.

These three spaces are all constructed by taking the space of finitely supported sequences c_{00}, and then taking a completion with respect to an exotic norm. The three norms all fit neatly into the same framework which makes use finite trees over $\mathbb{N}$ under certain (increasingly complicated) constraints. Further, the arguments used have a combinatorial flavor, with the origins lying in infinite-dimensional Ramsey theory. Most of Argyros' previous work is on this part of the hierarchy.

The other part of the hierarchy are the so-called $\mathscr{L}_p$-spaces, with the case of $p = \infty$ being of particular interest. These are a generalization of the function spaces $L_p(\mu)$ and $C(K)$. This will be the subject of Chapter 5, where we will discuss the Bourgain-Delbaen construction of a certain class of $\mathscr{L}_\infty$-spaces, and the generalization thereof which finally leads to the Argyros-Haydon space X_{AH}. These constructions are not based on the c_{00} space, but on the space of bounded sequences ℓ_∞. This is rather unique within the field of "weird" Banach space constructions. It is in this part of the hierarchy that we see the influence of Haydon's work.

The last chapter will then be dedicated to showing the specific choice of parameters within the generalized Bourgain-Delbaen construction that yields the Argyros-Haydon space. We finish by giving the proof that X_{AH} has the scalar-plus-compact property.

Chapter 2

Preliminaries

In this chapter we cover necessary concepts from Banach space theory that may not be known just from following some courses on functional analysis. The first section acts as a refresher for the completion of a normed space. Then, we will focus on basic sequence techniques and some important results in sequence spaces that arise from these techniques. The last section features some results from infinite Ramsey theory that set the stage for the constructions that follow in Chapter 3.

The definitions, results and proofs in this chapter are drawn mostly from [13] and [26]. Though we also introduce some notational conventions that are more specific to the field of "weird" Banach space constructions.

We will work mostly with sequence spaces. We use the standard notation of writing ℓ_p for the set of real-valued sequences over $\mathbb{N}$ (for convenience we consider $0 \notin \mathbb{N}$) endowed with the $\|\cdot\|_p$ norm, $1 \leq p \leq \infty$, c_0 for the subset of ℓ_∞ of sequences that tend to 0, and c_{00} for the subset of ℓ_1 of sequences that are non-zero in finitely many places. When the index set is not $\mathbb{N}$ but some other set S we will write $\ell_p(S)$, $c_0(S)$ and $c_{00}(S)$.

2.1 Metric completion

In this work we will many times encounter the "completion" of a certain set endowed with a certain norm. This will always refer to the metric completion of the induced metric. It is not hard to see that the resulting complete metric space is in fact a Banach space, thus giving the completion of the norm.

Definition 2.1.1. Let (X, d) be a metric space. Define an equivalence relation on Cauchy sequences in X by $(x_n) \sim (y_n)$ if and only if $\lim_n d(x_n, y_n) = 0$. The *metric completion* is obtained by the set $\hat{X}$ of equivalence classes of Cauchy sequences in X, with the metric $\hat{d}((x_n), (y_n)) = \lim_n d(x_n, y_n)$.
Thus if $(X, \|\cdot\|)$ is any normed space, its completion is found by $(\hat{X}, \|\cdot\|')$ where $\|x\|' = \lim_n \|x_n\|$. One also writes $\overline{X}^{\|\cdot\|}$.

Example 2.1.2. The finitely supported sequences c_{00} are not complete under any of the ℓ_p norms, instead we have the following completions:

(i) $\overline{c_{00}}^{\|\cdot\|_p} \cong \ell_p$ for $1 \leq p < \infty$.

(ii) $\overline{c_{00}}^{\|\cdot\|_\infty} \cong c_0$.

These statements can be checked by showing the density of c_{00} as subset of ℓ_p or c_0.

In general density arguments are very useful when working with the completion, since this allows us to ignore the added complexity of having all elements be equivalence classes of Cauchy sequences. Let us consider one more example of a completion, where we will see that we can ignore the Cauchy sequences and simply write a single infinite sequence:

Example 2.1.3. Consider a norm on c_{00} given by $\|x\| = \sum_{n=1}^{\infty} 2^{-n}|x(n)|$. Let X be the completion of $(c_{00}, \|\cdot\|)$. Then if (x_k) is a single representative from an equivalence class of Cauchy sequences in $(c_{00}, \|\cdot\|)$, we have in particular that $\|x_{k-1} - x_k\| \to 0$ as $k \to \infty$. So we must have that $2^{-n}|x_{k-1}(n) - x_k(n)|$ goes to 0 for each n as $k \to \infty$, which implies that $x(n) = \lim_{k\to\infty} x_k(n)$ is well-defined for each n. And if (y_k) is another representative, we have $\|x_k - y_k\| \to 0$ as $k \to \infty$, so $\lim_{k\to\infty} y_k(n)$ gives the same sequence $x(n)$. Thus we may simply write each element of X as an infinite sequence $(x(0), x(1), ...)$ satisfying $\sum_{n=1}^{\infty} 2^{-n}|x(n)| < \infty$.

This type of argument will be applied implicitly whenever we deal with norm completions of c_{00} in the future. We have essentially shown that the completion above is a Banach space in which the unit vectors form a basis. We will introduce this concept in the next section.

2.2 Basic sequence techniques

In this section we will define the fundamental notion of a basis, the related concept of a basic sequence, and introduce the various types of bases and basic sequences that we will need in later chapters. For the rest of this section, let $(X, \|\cdot\|)$ denote an infinite-dimensional Banach space over the real numbers.

Definition 2.2.1. A sequence $(e_n)_{n\in\mathbb{N}}$ in X is a *basis* if for each $x \in X$ there exists a unique sequence of scalars $(x_n)_{n\in\mathbb{N}}$ such that

$$x = \sum_{n\in\mathbb{N}} x_n e_n.$$

We write e_n^* for the associated *biorthogonal functionals*, which map x to its n-th coordinate x_n. These are elements of the dual space X^*, and they satisfy $e_i^*(e_j) = 1$ if $i = j$ and $e_i^*(e_j) = 0$ otherwise.

It should be noted that not every Banach space has a basis, as was shown by Per Enflo in [12]. However, we will only work with Banach spaces that do have a basis. In particular, since we will work with sequence spaces, the unit vector basis will usually suffice.

Definition 2.2.2. Given a basis $(e_n)_{n\in\mathbb{N}}$ in X, and a finite subset E of $\mathbb{N}$, we define the projection $Ex = \sum_{n\in E} x_n e_n$. In particular, we have the partial sum projections $S_n x = \sum_{i\leq n} x_n e_n$.

By an easy application of the uniform boundedness principle, the partial sum projections satisfy $\sup_n \|S_n\| < \infty$. This gives rise to the following definition:

Definition 2.2.3. Given a basis $(e_n)_{n\in\mathbb{N}}$ in X, define the *basis constant* K_b as the supremum $\sup_n \|S_n\|$. If $K_b = 1$, the basis $(e_n)_{n\in\mathbb{N}}$ is said to be *monotone*.

As we are often interested in the subspace structure of X, the following definition is of great importance:

Definition 2.2.4. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a *basic sequence* if it is a basis for its closed linear span. We denote this span by $[x_n]$, $[x_n]_{n=1}^{\infty}$ or $[x_n]_{n\in\mathbb{N}}$ depending on the index set.

The following test is known as Grunblum's criterion:

Proposition 2.2.5. *A sequence $(x_n)_{n\in\mathbb{N}}$ of non-zero elements of X is a basic sequence if and only if there exists $K > 0$ such that for all integers $m \leq n$ and scalars $a_1, ..., a_n$, we have*

$$\left\|\sum_{k=1}^{m} a_k x_k\right\| \leq K \left\|\sum_{k=1}^{n} a_k x_k\right\|.$$

Proof. ($\Rightarrow$) Let K_b be the basis constant of (x_n) as a basis of $[x_n]$. Then we have

$$\left\|\sum_{k=1}^{m} a_k x_k\right\| = \left\|S_m\left(\sum_{k=1}^{n} a_k x_k\right)\right\| \leq K_b \left\|\sum_{k=1}^{n} a_k x_k\right\|.$$

($\Leftarrow$) From the assumption it follows that distinct elements of the sequence (x_n) are linearly independent, since if $a_1, ..., a_n$ are all non-zero, but $\sum_{k=1}^{n} a_k x_k = 0$, then $\sum_{k=1}^{n-1} a_k x_k = -a_n x_n$ is certainly non-zero, thus $\|a_n x_n\| \leq K \cdot 0$ is a contradiction. This allows us to define the operator s_m from the linear (not closed) span of $(x_n)_{n=1}^{\infty}$ to $[x_k]_{k=1}^{m}$ by

$$s_m\left(\sum_{k=1}^{n} a_k x_k\right) = \sum_{k=1}^{\min(m,n)} a_k x_k.$$

This is well defined since $(x_k)_{k=1}^{m}$ is a basis of the finite-dimensional normed space $[x_k]_{k=1}^{m}$. By density this extends uniquely to $S_m : [x_k] \to [x_k]_{k=1}^{m}$ with $\|S_m\| = \|s_m\| \leq K$. So we have obtained our partial sum projections, and find the biorthogonals since $S_n(y) - S_{n-1}(y) = x_n^*(y)x_n$ for some $x_n^*(y) \in \mathbb{R}$. If we can now show that $S_n(y)$ converges to y for all $y \in [x_n]$, it follows that $y = \sum_{n=1}^{\infty} x_n^*(y)x_n$ is the unique decomposition of y. This certainly holds if y is in the linear span of (x_n), and since the set $\{y \in [x_n] \mid S_n(y) \to y\}$ is closed by uniform boundedness of the S_n, it must be equal to $[x_n]$ by density of the linear span. □

We will now introduce various properties that a basic sequence may have. Each of these will return at some point in the later chapters, so this enumeration serves as a convenient place to look up their definitions.

Definition 2.2.6. Let $(e_n)_{n\in\mathbb{N}}$ be a basis in X, and let $(p_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of integers with $p_0 = 0$. Then a sequence of non-zero vectors $(x_n)_{n\in\mathbb{N}}$ with $x_n \in [e_j]_{j=p_n}^{p_{n+1}-1}$ is called a *block basic sequence* for $(e_n)_{n\in\mathbb{N}}$. If in addition $x_n \in [e_j]_{j=p_n+1}^{p_{n+1}-1}$, it is called a *skipped* block basic sequence.

Definition 2.2.7. Two basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in respective Banach spaces X and Y are *equivalent* if there exists a constant $C > 0$ such that for all finitely supported sequences of scalars $(a_n)_{n\in\mathbb{N}}$ we have

$$\frac{1}{C}\left\|\sum_{n\in\mathbb{N}} a_n y_n\right\| \leq \left\|\sum_{n\in\mathbb{N}} a_n x_n\right\| \leq C\left\|\sum_{n\in\mathbb{N}} a_n y_n\right\|.$$

If $C = 1$ then the basic sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are *isometrically equivalent.*

Definition 2.2.8. A basic sequence $(x_n)_{n\in\mathbb{N}}$ in X is

(i) *normalized* if $\|x_n\| = 1$ for all $n \in \mathbb{N}$.

(ii) *complemented* if $[x_n]$ is a complemented subspace in X.

(iii) *unconditional* if for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$, the sequence $(x_{\sigma(n)})_{n\in\mathbb{N}}$ is also a basis for $[x_n]$.

(iv) *symmetric* if it is unconditional and $(x_n)_{n\in\mathbb{N}}$ is equivalent to $(x_{\sigma(n)})_{n\in\mathbb{N}}$ for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$.

(v) *subsymmetric* if it is unconditional and every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is equivalent to $(x_n)_{n\in\mathbb{N}}$.

(vi) *boundedly complete* if for all scalar sequences $(a_n)_{n\in\mathbb{N}}$ such that the partial sums $\|\sum_{n=1}^{N} a_n x_n\|$ are bounded, we have that $\sum_{n\in\mathbb{N}} a_n x_n$ converges.

(vii) *shrinking* if $\lim_{N\to\infty} \|x^*|_{T_N}\| = 0$ for all $x^* \in [x_n]^*$, where T_N is the tail space $[x_n]_{n>N}$.

We single out the following equivalence because norm estimations play an important role in the main part of this work:

Proposition 2.2.9. *A basic sequence $(x_n)_{n\in\mathbb{N}}$ in X is unconditional if and only if there exists a constant $C > 0$ such that for all sequences of scalars $(a_n)_{n\in\mathbb{N}}$ and weights $(\varepsilon_n)_{n\in\mathbb{N}}$ with $|\varepsilon_n| \leq 1$ for all n, we have*

$$\left\|\sum_{n\in\mathbb{N}} \varepsilon_n a_n x_n\right\| \leq C\left\|\sum_{n\in\mathbb{N}} a_n x_n\right\|.$$

Proof. The proof relies on the fact that the following two are equivalent definitions of unconditional convergence: $\sum_n a_{\sigma(n)}$ converges for all permutations σ and $\sum_n a_{n_i}$ converges for all subsequences n_i. For a proof of this statement, see [26, Prop. 1.c.1.] ($\Rightarrow$) Suppose (x_n) is unconditional. Let $\sum_n a_n x_n$ be any converging sum, call its limit y. Then for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ we by must have

$$\sum_{n\in\mathbb{N}} a_n x_n = \sum_{n\in\mathbb{N}} x_n^*(y)x_n = \sum_{n\in\mathbb{N}} x_{\sigma(n)}^*(y)x_{\sigma(n)} = \sum_{n\in\mathbb{N}} a_{\sigma(n)}x_{\sigma(n)}. \tag{2.1}$$

From this it follows that whenever (ε_n) is a bounded sequence of scalars, the sum $\sum_n \varepsilon_n a_n x_n$ also converges since we may split this in a positive and negative part, and then both parts converge by the above. Now for any $n \in \mathbb{N}$, the map $T_n : [x_n] \to [x_n]$ defined by

$$T_n(x) = x + \sum_{k=1}^{n} (\varepsilon_k - 1)x_k^*(x)x_k$$

is linear and bounded, and so its pointwise limit, which is

$$T(x) = \sum_{n\in\mathbb{N}} \varepsilon_n x_n^*(x)x_n,$$

is also linear and bounded by the Banach-Steinhaus theorem. So taking $C = \|T\|$, we find

$$\left\|\sum_{n\in\mathbb{N}} \varepsilon_n a_n x_n\right\| = \left\|T\left(\sum_{n\in\mathbb{N}} a_n x_n\right)\right\| \leq C\left\|\sum_{n\in\mathbb{N}} a_n x_n\right\|.$$

($\Leftarrow$) Let $\sum_n a_n x_n$ be any converging sum. For (n_i) strictly increasing, define $\varepsilon_{n_i} = 1$ for all i and $\varepsilon_n = 0$ for the remaining values of n. Then for any $m < k$, we have

$$\left\|\sum_{n=m}^{k} \varepsilon_n a_n x_n\right\| \leq C\left\|\sum_{n=m}^{k} a_n x_n\right\|.$$

The right hand side tends to 0 as m, k goes to infinity. Thus $\sum_n \varepsilon_n a_n x_n = \sum_i a_{n_i} x_{n_i}$ is Cauchy, and so it converges. Thus the equalities in (2.1) now show that $(x_{\sigma(n)})$ is also a basis. □

Remark 2.2.10. In light of the above equivalence, one often says an unconditional basis is C-unconditional whenever C is a constant for which the inequality holds. Similarly, one might say bases are C-equivalent, C-symmetric, C-subsymmetric and so on whenever there is such an inequality parametrized by a constant $C > 0$.

Lastly, we will introduce some notational conventions regarding subsets of $\mathbb{N}$ and the supports of elements of X.

Definition 2.2.11. Suppose we have the basis $(e_n)_{n\in\mathbb{N}}$ in X. Given $x \in X$, define the *support* of x by

$$\operatorname{supp}(x) = \{n \in \mathbb{N} \mid e_n^*(x) \neq 0\},$$

and the *range* of x as the smallest interval $\operatorname{ran}(x) \subseteq \mathbb{N}$ such that $\operatorname{supp}(x) \subseteq \operatorname{ran}(x)$.

Definition 2.2.12. For finite subsets A, B of $\mathbb{N}$ we define a partial order by $A < B$ iff $\max A < \min B$ (and analogously for $\leq$). For singletons we will abuse notation and write $n < A$ to mean $\{n\} < A$. With this in mind we say that two finitely supported elements $x, y \in X$ are *successive* if $\text{supp}(x) < \text{supp}(y)$ and in this case we write $x < y$.

Definition 2.2.13. A finite-dimensional decomposition (FDD) of X is a sequence of finite-dimensional subspaces $(M_n)_{n\in\mathbb{N}}$ such that $M_n \cap M_k = \{0\}$ whenever $n \neq k$ and $X = \bigoplus_{n\in\mathbb{N}} M_n$. That is, for each $x \in X$ there is a unique sequence (x_n) such that $x_n \in M_n$ for all n and $x = \sum_{n\in\mathbb{N}} x_n$.

In the context of an FDD, $(M_n)_{n\in\mathbb{N}}$ of X, the support of $x = \sum_n x_n$ will be the set of $n \in \mathbb{N}$ such that $x_n \in M_n \setminus \{0\}$. And the range will again be the smallest interval containing this. A block sequence (w.r.t. the FDD $(M_n)_{n\in\mathbb{N}}$) will be a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\text{ran}(x_{n+1}) > \text{ran}(x)$. Furthermore a *skipped block sequence* is a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\text{ran}(x_{n+1}) > \text{ran}(x) + 1$. Most definitions and results in this section and the next can be extended to the context of an FDD.

2.3 Subspace classification

In this section we will treat the important results about the classical sequence spaces ℓ_p, c_0 and their subspace structure. As mentioned before, we are often interested in the subspace structure of a Banach space X. If we can decompose X into subspaces that we know then we can reduce certain questions to the subspaces.

This is where the concept of a (block) basic sequence becomes very useful, since it allows us to move between subspaces of X and sequences in X. Going from a basic sequence to a subspace is clear. The following proposition shows us what we can expect in the other direction:

Proposition 2.3.1. *Let X be a Banach space with basis $(e_n)_{n\in\mathbb{N}}$. For any closed infinite dimensional subspace Y of X, there is a basic sequence $(x_n)_{n\in\mathbb{N}}$ in Y that is equivalent to a block basic sequence of $(e_n)_{n\in\mathbb{N}}$.*

Proof. For any $p \in \mathbb{N}$, there exists an element $y \in Y$ of the form $y = \sum_{n>p} a_n e_n$. Otherwise Y would be contained in the finite-dimensional space $[e_n]_{n=1}^{p}$ where p is the first integer for which this fails. We construct the block basic sequence as follows: Start with $p_0 = 0$ and $x_1 = \sum_{n>p_0} a_{n,1} e_n \in Y$ such that $\|x_1\| = 1$. Let K_b be the basis constant of $(e_n)_{n\in\mathbb{N}}$. Given x_i, define p_i as the smallest integer such that $\|x_i - S_{p_i}(x_i)\| < \frac{1}{K_b} 4^{-(i+1)}$ and choose $x_{i+1} = \sum_{n>p_i} a_{n,i} e_n \in Y$ with $\|x_{i+1}\| = 1$. Now $y_i = S_{p_i}(x_i) = \sum_{n=p_{i-1}+1}^{p_i} a_{n,i} e_n$ forms a block basis of $(e_n)_{n\in\mathbb{N}}$, and it satisfies $\sum_{n\in\mathbb{N}} \|x_n - y_n\| < \frac{1}{4K_b}$ and $\|y_n\| \geq \frac{1}{2}$ for all n. From this we will show equivalence. Let $y = \sum_n a_n y_n \in Y$. First, note that

$$\max_{n\in\mathbb{N}} |a_n| \leq \max_{n\in\mathbb{N}} 2\|a_n y_n\| = 2 \max_{n\in\mathbb{N}} \|S_n(y) - S_{n-1}(y)\| \leq 4K_b\|y\|,$$

where we have used that a blocking of a basis cannot have a larger basis constant. Define $T : [y_n] \to [y_n]$ by $T(\sum_n a_n y_n) = \sum_n a_n x_n$. Then we have

$$\|y - Ty\| \leq \sum_{n\in\mathbb{N}} |a_n| \|x_n - y_n\| \leq \max_{n\in\mathbb{N}} |a_n| \sum_{n\in\mathbb{N}} \|x_n - y_n\| \leq \|y\|.$$

So $\|I - T\| \leq 1$, i.e. T is an isomorphism on $[y_n]$, which shows in particular that $(x_n)_{n\in\mathbb{N}}$ is an equivalent basis for $[y_n]$. □

And in general we have the following theorem, which is a corollary of the Banach-Mazur theorem:

Theorem 2.3.2. *Every separable infinite-dimensional Banach space contains for every $\varepsilon > 0$ a basic sequence with basis constant less than $1 + \varepsilon$.*

Proof. See [13, Section 1.4]. □

So, the natural question to ask is, can we classify these basic sequences? That is, for each basic sequence, can we say if it is equivalent to a (block) basis in some other classical space which we already know? For the sequence spaces c_0 and ℓ_p we already have a fairly complete answer:

Lemma 2.3.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a normalized block basic sequence in $X = c_0$ or $X = \ell_p$ for some $1 \leq p < \infty$. Then $(x_n)_{n\in\mathbb{N}}$ is isometrically equivalent to the unit vector basis in X. Furthermore, $[x_n]$ is complemented in X.*

Proof. We do the case $X = \ell_p$ for some $1 \leq p < \infty$. Since x_n is normalized, we have

$$\|x_n\|_p = \sum_{i=p_{n-1}+1}^{p_n} |b_i|^p = 1.$$

Thus if we have scalars $(a_n)_{n\in\mathbb{N}}$ we get

$$\left\| \sum_{n\in\mathbb{N}} a_n x_n \right\|_p = \left\| \sum_{n=1}^{\infty} \sum_{i=p_{n-1}+1}^{p_n} a_n b_i e_i \right\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p \sum_{i=p_{n-1}+1}^{p_n} |b_i|^p \right)^{1/p}$$

$$= \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} = \left\| \sum_{n\in\mathbb{N}} a_n e_n \right\|_p,$$

which shows isometric equivalence. Now if we show that the projection $P(y) = \sum_n x_n^*(y) x_n$ is bounded then it follows that $[x_n] = P(\ell_p)$ is complemented. We can pick scalars $(c_n)_{n\in\mathbb{N}}$ such that $x_n^* = \sum_{i=p_{n-1}+1}^{p_n} c_i e_i^*$. Then using the Hölder inequality one can show that

$$|x_n^*(x)| \leq \|x_n^*\|_q \left\| \sum_{i=p_{n-1}+1}^{p_n} x_i e_i \right\|_p$$

holds for all $x \in c_{00}$, where $\|x_n^*\|_q = 1$ and thus

$$\|P(x)\| \leq \left(\sum_{n=1}^{\infty} \sum_{i=p_{n-1}+1}^{p_n} |x_i|^p \right) = \|x\|,$$

which extends to X by density of c_{00}. □

In particular, the above lemma implies that the unit vector basis in ℓ_p and c_0 is symmetric. However, we will apply it to obtain a much stronger result:

Theorem 2.3.4. *Let $X = c_0$ or $X = \ell_p$ for some $1 \leq p < \infty$. Then every complemented infinite-dimensional subspace of X is isomorphic to X itself.*

Proof. Let $Y \subseteq X$ be complemented. By Proposition 2.3.1 there exists a normalized block basic sequence in Y. By Lemma 2.3.3 this sequence spans a complemented subspace Z of X that is isomorphic to X itself. Clearly Z is also complemented in Y. So we may say $Y \cong Z \oplus W$ and $X \cong Y \oplus V$. Thus, since $X \cong X \oplus X$ we have

$$Y \cong Z \oplus W \cong X \oplus W \cong X \oplus X \oplus W \cong X \oplus Y.$$

Now we need the fact that $\ell_p(\ell_p) \cong \ell_p(\mathbb{N} \times \mathbb{N}) \cong \ell_p$ to conclude

$$X \oplus Y \cong \ell_p(X) \oplus Y \cong \ell_p(V) \oplus \ell_p(Y) \oplus Y \cong \ell_p(V) \oplus \ell_p(Y) \cong X,$$

where in the second to last step we have also used $\ell_p(Y) \oplus Y \cong \ell_p(Y)$. The same steps work when $X = c_0$. So we conclude $X \cong X \oplus Y \cong Y$. □

The above property is quite rare in Banach space theory. We say that spaces satisfying the statement in this theorem are *prime*. Not many prime spaces are known. Lastly, we will state a result about the structure of operators between the sequence spaces:

Theorem 2.3.5 (Pitt's theorem)**.** *Let $1 \leq p < r < \infty$, and let X be a closed subspace of ℓ_r. Then any operator $T : X \to \ell_p$ is compact.*

Proof. Since ℓ_r is reflexive, so is X and thus $\operatorname{ball} X$ is weakly compact. Further the weak topology on $\operatorname{ball} X$ is metrizable, so if we show that $x_n \rightharpoonup x$ implies $Tx_n \to Tx$, it follows that $T(\operatorname{ball} X)$ is norm compact. Suppose that $x_n \rightharpoonup 0$ with $\|x_n\| = 1$ for all n but Tx_n does not converge to 0 in norm. Since $(x_n)_{n\in\mathbb{N}}$ is weakly null we have $\lim_{n\to\infty} \|S_k x_n\| = 0$ for all k. So we can find (n_k) and (p_k) such that $\|S_{p_k-1} x_{n_k}\| < \frac{1}{2} 4^{-k}$. and $\|x_{n_k} - S_{p_k} x_{n_k}\| < \frac{1}{2} 4^{-k}$ for all k. (note that the ℓ_p basis is monotone, so we use $K_b = 1$). Then by a similar argument as in Proposition 2.3.1, x_{n_k} is equivalent to the block basic sequence $(y_k = S_{p_k} x_{n_k} - S_{p_{k-1}} x_{n_k})_{k\in\mathbb{N}}$, which by Lemma 2.3.3 is equivalent to the ℓ_r basis. By the same argument we can pass to another subsequence to obtain that Tx_{n_k} is equivalent to the ℓ_p basis. Thus, there are automorphisms U, V on ℓ_p and ℓ_r mapping between these bases, which implies that the inclusion $\iota = UTV^{-1}$ is bounded. Which is a contradiction since $r > p$. □

Corollary 2.3.6. *If X is an infinite-dimensional subspace of any c_0 or ℓ_p, $1 \leq p < \infty$, then X is not isomorphic to a subspace of any other of these spaces.*

Definition 2.3.7. An operator $T : X \to Y$ is called *strictly singular* if there is no infinite-dimensional subspace $E \subseteq X$ such that the restriction $T\big|_E : E \to T(E)$ is an isomorphism.

Corollary 2.3.8. *If $p \neq r$ every operator $T : \ell_p \to \ell_r$ is strictly singular.*

The argument using partial sum projections we have now applied thrice is often referred to as a *gliding hump* argument. We can also frame this argument in a more general setting and conclude the following: [20]

Theorem 2.3.9. *If X, Y are Banach spaces with respective bases $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $Z \subseteq X$ is a subspace and $T : Z \to Y$ is a non-compact operator, then there exists a normalized block basis $(z_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(Tz_n)_{n\in\mathbb{N}}$ is equivalent to a block basis of $(y_n)_{n\in\mathbb{N}}$.*

We will finish our discussion of basic sequence techniques and the classical sequence spaces with some results about reflexivity and the boundedly complete and shrinking properties mentioned in Definition 2.2.8.

Theorem 2.3.10. *Let X be a Banach space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$. Then the following equivalences hold:*

(i) *$(e_n)_{n\in\mathbb{N}}$ is shrinking if and only if X does not contain a subspace isomorphic to ℓ_1.*

(ii) *$(e_n)_{n\in\mathbb{N}}$ is boundedly complete if and only if X does not contain a subspace isomorphic to c_0.*

Theorem 2.3.11. *Let X be a Banach space with a basis $(e_n)_{n\in\mathbb{N}}$ and biorthogonals $(e_n^*)_{n\in\mathbb{N}}$. Then the following are equivalent:*

(i) *$(e_n)_{n\in\mathbb{N}}$ is shrinking.*

(ii) *$(e_n^*)_{n\in\mathbb{N}}$ is a boundedly complete basis for $[e_n^*]_{n\in\mathbb{N}}$.*

(iii) *$[e_n^*]_{n\in\mathbb{N}} = X^*$.*

Theorem 2.3.12. *Let X be a Banach space with a basis $(e_n)_{n\in\mathbb{N}}$. Then X is reflexive if and only if $(e_n)_{n\in\mathbb{N}}$ is both boundedly complete and shrinking.*

For proofs see [13, Chapter 3].

2.4 Ramsey theory

We now move from basic sequences to the topic of *Ramsey theory*. The purpose of this section is to introduce the central question of Ramsey theory, and to discuss some preliminary results with applications in Banach spaces. We wish to highlight them here to show the dependence of the entire work on this combinatorial foundation. In particular, a result known as Krivine's theorem is crucial to show the existence of certain sequences in the spaces we construct.

We start with some topological definitions on the natural numbers. We write $\mathcal{P}(\mathbb{N})$ for the power set of the natural numbers. This set is endowed with the *Cantor topology* which is inherited from the Cantor space $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ by identifying $A \in \mathcal{P}(\mathbb{N})$ with its characteristic function $\chi_A \in \mathcal{C}$.

Definition 2.4.1. The Cantor topology can be characterized in any of the following ways:

(i) The Cantor topology on $\mathcal{C}$ is generated by the metric $d(f,g) = 2^{-k}$, where k is the smallest number such that $f(k) \neq g(k)$.

(ii) The Cantor topology on $\mathcal{P}(\mathbb{N})$ is generated by the metric $d(A,B) = 2^{-k}$, where $k = \min((A \cup B) \setminus (A \cap B))$.

(iii) The Cantor topology on $\mathcal{C}$ coincides with the topology of pointwise convergence.

(iv) A sequence of sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathbb{N})$ converges to a limit $A \in \mathcal{P}(\mathbb{N})$ with respect to the Cantor topology if and only if the sequence of characteristic functions $(\chi_{A_n})_{n\in\mathbb{N}}$ converges pointwise to χ_A.

Lemma 2.4.2. *The Cantor topology on $\mathcal{C}$ or $\mathcal{P}(\mathbb{N})$ is compact.*

Proof. By Tychonoff's theorem, $\mathcal{C}$ is compact as the product of a countable number of copies of $\{0,1\}$ with the discrete topology. □

Remark 2.4.3. Consider a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathbb{N})$ such that $\min A_n \to \infty$ as $n \to \infty$. Then we have $\lim_{n\to\infty} A_n = \emptyset$. This leads to the convention that $\min \emptyset = \infty$.

For a countably infinite set S we will write $\mathcal{F}(S)$ for the set of all finite subsets of S. Further we have the notation $[S]^r$ for all subsets of size r, including the case $[S]^\omega$ for infinite subsets. We will now introduce a basic version of the dichotomy that is of interest in Ramsey theory:

Theorem 2.4.4 (Ramsey's theorem)**.** *For every $\mathcal{A} \subseteq [\mathbb{N}]^r$ there exists $S \in [\mathbb{N}]^\omega$ such that either $[S]^r \subseteq \mathcal{A}$ or $[S]^r \cap \mathcal{A} = \emptyset$.*

Proof. For $r = 1$ the statement boils down to: for every $\mathcal{A} \subseteq \mathbb{N}$ there exists $S \in [\mathbb{N}]^\omega$ such that either $S \subseteq \mathcal{A}$ or $S \cap \mathcal{A} = \emptyset$. This is easy: if $\mathcal{A}$ is finite, take $S = \mathbb{N} \setminus \mathcal{A}$, and if $\mathcal{A}$ is infinite, take $S = \mathcal{A}$.

Now let $r \geq 2$ and suppose the statement holds for $r-1$. Let $\mathcal{A} \subseteq [\mathbb{N}]^r$. For each $m_1, ..., m_{r-1}$, we can consider the sequence $a_n = \chi_{\mathcal{A}}(\{m_1, ..., m_{r-1}, n\})$ in $\mathbb{R}$, which certainly has a convergent subsequence. By considering an enumeration of $\mathbb{N}^{r-1} = \{x_1, x_2, ...\}$, we obtain for each i a sequence $(n_k^{(i)})$ such that $\chi_{\mathcal{A}}(x_i \cup \{n_k^{(i)}\})$ converges as $k \to \infty$. So by a diagonal argument, we end up with a sequence (n_k) such that $\chi_{\mathcal{A}}(x_i \cup \{n_k\})$ converges for all i. We define $\mathcal{A}' \subseteq [\mathbb{N}]^{r-1}$ by

$$\mathcal{A}' = \left\{ x_i \;\middle|\; \lim_{k\to\infty} \chi_{\mathcal{A}}(x_i \cup \{n_k\}) = 1 \right\}.$$

The induction hypothesis now gives $S' \in [\mathbb{N}]^\omega$ such that either $[S']^{r-1} \subseteq \mathcal{A}'$ or $[S']^{r-1} \cap \mathcal{A}' = \emptyset$. So in other words the limit of $\chi_{\mathcal{A}}(A \cup \{n_k\})$ is the same for all $A \in [S']^{r-1}$. For each such A, define k_A as the smallest number such that $\chi_{\mathcal{A}}(A \cup \{n_k\})$ is equal to its limit for all $k \geq k_A$.

We construct S by picking $m_1, ..., m_{r-1}$ in S', and then given $m_1, ..., m_n \in S$ add $m_{n+1} \in S'$ such that $m_{n+1} \geq k_A$ for all $A \in [\{m_1, ..., m_n\}]^{r-1}$. It follows that $\chi_{\mathcal{A}}(A)$ is the same for all $A \in [S]^r$. Thus $[S]^r \subseteq \mathcal{A}$ or $[S]^r \cap \mathcal{A} = \emptyset$. □

Remark 2.4.5. A slightly more general version of this statement is that if c is any finite colouring of $[\mathbb{N}]^r$, that is $c : [\mathbb{N}]^r \to \{1, ..., k\}$, there is an infinite subset S of $\mathbb{N}$ such that $[S]^r$ is *monochromatic*. The above argument can be applied to this statement as well.

A problem in Ramsey theory is often one of the form "given a certain structure and a colouring of that structure, does there exist a monochromatic substructure?". Further extending the above result would mean considering either finite colorings on $[S]^\omega$ for some infinite set $S \subseteq \mathbb{N}$, or considering colorings that take continuous values, for instance in $[0, 1]$ or some other compact metric space. Neither turn out to be possible in general. For the first case the following counterexample from [18] is illustrative:

Example 2.4.6. We construct a coloring on $[\mathbb{N}]^\omega$ as follows: For $A, B \in [\mathbb{N}]^\omega$ define an equivalence by $A \sim B$ iff the symmetric difference is finite. Let there be a fixed representative for each equivalence class obtained this way. Then we can color A as either 0 or 1 depending on if the symmetric difference of it with the representative of its equivalence class has an odd or even number of elements.
Then, the color of a set changes whenever we add or remove an element. Thus there is no infinite $S \subseteq \mathbb{N}$ for which $[S]^\omega$ is monochromatic.

There are of course ways to remedy the situation and obtain weaker results. In doing so we now enter the domain of infinite-dimensional Ramsey theory. To look at colorings on $[\mathbb{N}]^\omega$ we can simply restrict our interests to colorings that are constructed in a reasonable manner:

Definition 2.4.7. A set $V \subseteq [\mathbb{N}]^\omega$ is a *Ramsey set* if there exists $S \in [\mathbb{N}]^\omega$ such that $[S]^\omega \subseteq V$ or $[S]^\omega \cap V = \emptyset$.

For instance, one sufficient condition for a set to be Ramsey is the following:

Theorem 2.4.8. *If $V \subseteq [\mathbb{N}]^\omega$ is a Borel set with respect to the Cantor topology, then it is a Ramsey set.*

Proof. See [13, Section 11.1]. □

Thus this particularly applies to sets that are open or compact in the Cantor topology, which are the kinds of subsets of $\mathcal{P}(\mathbb{N})$ that we will encounter in later chapters.

For the second case, with continuous colorings, we may consider substructures that are *approximately monochromatic*. [18]

Theorem 2.4.9. *Let K be a compact metric space and $f : [\mathbb{N}]^r \to K$. Then there exists for each $\varepsilon > 0$ an $S \in [\mathbb{N}]^\omega$ such that $d(f(A), f(B)) < \varepsilon$ for all $A, B \in [S]^r$.*

Proof. We can cover K using a finite number of disjoint sets $B_1, ..., B_k$ each of diameter less then ε. Then one gets a finite coloring $c : [\mathbb{N}]^r \to \{1, ..., k\}$ by $c(A) = j$ whenever $f(A) \in B_j$. Ramsey's theorem then gives a set $S \in [\mathbb{N}]^\omega$ such that $[S]^r$ is monochromatic with respect to c. □

Remark 2.4.10. Using the above theorem and a diagonal argument one can obtain a set $S \in [\mathbb{N}]^\omega$ which is monochromatic in the sense that there is an $x \in K$ such that whenever $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(f(A), x) < \varepsilon$ for all $A \in [S]^r$ with $N < A$. See [13, Section 11.1].

The most important application of Ramsey theory for our purposes is Krivine's theorem. It essentially states that every *finite* block sequence in a Banach space with a basis has a subsequence which is equivalent to some ℓ_p^n. However let us first see a more elementary way to apply Ramsey theory to the field of analysis:

Proposition 2.4.11. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that both $(x_{n_k})_{k\in\mathbb{N}}$ and $(x_{n_{k+1}-n_k})_{k\in\mathbb{N}}$ are convergent.*

Proof. The map $f : [\mathbb{N}]^2 \to \mathbb{R}^2$ defined by $f(\{i, j\}) = (x_i, x_{j-i})$ is bounded. Thus by the remark below Theorem 2.4.9 there exists $S \in [\mathbb{N}]^\omega$ and $L \in \mathbb{R}^2$ such that for each $k \in \mathbb{N}$ there exists N_k such that for every $n, m \in S$ greater than N_k we have $\|(x_n, x_{m-n}) - L\| < \frac{1}{k}$. Then picking a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in S such that $n_k > N_k$ for each k gives the conclusion. □

Now we will state Krivine's theorem. First we need another important definition:

Definition 2.4.12. Let X, Y be Banach spaces. A basic sequence $(x_n)_{n\in N} \subseteq X$ is *finitely block represented* in another basic sequence $(y_n)_{n\in N} \subseteq Y$ if for each finite subsequence $(x_{n_i})_{i=1}^k$ and each $\varepsilon > 0$, there are normalized blocks $z_1 < ... < z_k$ of $(y_n)_{n\in N}$ such that $(z_i)_{i=1}^k$ and $(x_{n_i})_{i=1}^k$ are $(1 + \varepsilon)$-equivalent.

Theorem 2.4.13 (Krivine's theorem)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a 1-subsymmetric basic sequence in a Banach space X. Then the unit vector basis of either c_0 or ℓ_p for some $1 \leq p < \infty$ is finitely block represented in $(x_n)_{n\in\mathbb{N}}$.*

Besides using Ramsey's theorem, the proof of Krivine's theorem uses an interesting combinatorial approach. For a treatment see [13, Section 12.3] or [4, Part B, Theorem I.3.8].

Chapter 3

Tsirelson-like constructions

As was briefly mentioned in Section 2.3, the spaces ℓ_p and c_0 are prime, meaning that they are isomorphic to all of their closed subspaces. For quite some time no other prime spaces were known, and it was believed that c_0 and ℓ_p could act as the "building blocks" of all Banach spaces. This theory fell through when Tsirelson published a paper in which he constructs a reflexive Banach space that contains no copies of c_0 or ℓ_p for any p. The dual of this space (which enjoys the same property) is now known as Tsirelson's space T, and it opened the door for many more similar constructions with strange properties.

In this chapter we will first treat the construction of T, and show the above property. Then we will look at a slightly more complicated construction which yields Schlumprecht's space S. Next we will generalize both constructions into what is known as mixed Tsirelson spaces. Lastly we will introduce the concept of a rapidly increasing sequence, which plays a crucial role in the rest of the story.

3.1 Tsirelson's space

This section draws mostly from [14] and [10]. We will start by giving the construction of Tsirelson's space. It is done by inductively defining norms on c_{00}, and having T be the completion of c_{00} with the limit of these norms. First we need to define a certain admissibility rule. This is something we will see repeatedly throughout this chapter.

Definition 3.1.1. A sequence of finite sets $(E_i)_{i=1}^n$ in $\mathbb{N}$ is called *admissible* for Tsirelson's space if $n \leq E_1 < E_2 < ... < E_n$.

Here we have used the notational conventions for subsets of $\mathbb{N}$ as described at the end of Section 2.2.

Definition 3.1.2 (Tsirelson's space)**.** We start with $\|\cdot\|_0 = \|\cdot\|_\infty$. Then, supposing $\|\cdot\|_k$ has been defined for some $k \geq 0$, we define the next norm by

$$\|x\|_{k+1} = \|x\|_k \vee \max\left\{ \frac{1}{2} \sum_{i=1}^{n} \|E_i x\|_k \;\middle|\; (E_i)_{i=1}^n \text{ admissible} \right\}.$$

Here we have used the "join" notation $a \vee b = \max\{a, b\}$ to avoid double brackets. Now the Tsirelson norm on c_{00} is $\|x\|_T = \max_{k\in\mathbb{N}} \|x\|_k$, and Tsirelson's space is the completion of $(c_{00}, \|\cdot\|_T)$.

It is important to note that this maximum is indeed achieved in all cases. This is because for every finitely supported $x \in c_{00}$ there are only finitely many admissible sequences that do not include the case where $E_i x = 0$ for some i. So at some point no further subdivision is possible. This idea of subdividing the support of a vector is something to keep in mind. For now let us see some properties of the unit vectors in this norm:

Proposition 3.1.3. *The unit vectors $(e_i)_{i\in\mathbb{N}}$ form a normalized 1-unconditional basis of T.*

Proof. We will first show that the statement holds for the completion of $(c_{00}, \|\cdot\|_k)$ for all $k \in \mathbb{N}$. Clearly $\|e_i\|_k = 1$ for all i, k. The linear (not closed) span of $(e_i)_{i\in\mathbb{N}}$ is precisely the dense subset c_{00}. Thus $[e_i]_{i\in\mathbb{N}}$ is equal to the completion. Further, it is easy to see that for $x, y \in c_{00}$ with x being an initial part of y, we have $\|x\|_k \leq \|y\|_k$. So Grunblum's criterion (Prop. 2.2.5) gives that $(e_i)_{i\in\mathbb{N}}$ is indeed a basis.

Now we will show by induction that the basis is 1-unconditional. For $k = 0$ it holds because the completion is c_0. For $k \geq 1$, take scalars $(a_i)_{i\in\mathbb{N}}$ and weights $(\varepsilon_i)_{i\in\mathbb{N}}$ in $[-1, 1]$. Let $x_n = \sum_{i=1}^n \varepsilon_i a_i e_i$ and $y_n = \sum_{i=1}^n a_i e_i$. Then by the induction hypothesis $\|x_n\|_{k-1} \leq \|y_n\|_{k-1}$ and $\|Ex_n\|_{k-1} \leq \|Ey_n\|_{k-1}$ hold for all finite sets $E \subseteq \mathbb{N}$. Thus also $\|x_n\|_k \leq \|y_n\|_k$. Taking $n \to \infty$ shows by Prop. 2.2.9 that the basis is 1-unconditional for k.

Lastly, the statements easily extend to T since $\|x\|_T = \max_k \|x\|_k$, and thus all inequalities proven above are preserved. □

Next, we show that the norm on the completion can be described by simply replacing the maxima in the definitions with suprema.

Proposition 3.1.4. *For $x \in T$ and $k \geq 1$ we have the following identity:*

$$\|x\|_k = \|x\|_\infty \vee \sup\left\{ \frac{1}{2} \sum_{i=1}^n \|E_i x\|_{k-1} \;\middle|\; (E_i)_{i=1}^n \text{ admissible} \right\}.$$

Proof. As we noted before, if $x, y \in c_{00}$ with x being an initial part of y, we have $\|x\|_k \leq \|y\|_k$. Thus if $x \in T$, the sequence $(\|S_n x\|_k)_{k\in\mathbb{N}}$ is increasing. So $\|x\|_k = \sup_{n\in\mathbb{N}} \|S_n x\|_k$. Now if $(E_j)_{j=1}^r$ is admissible we may find $n \in N$ such that $E_r \leq n$, and so $\|S_n x\|_k \geq \frac{1}{2}\sum_{j=1}^r \|E_i x\|_{k-1}$. This shows that $\|x\|_k$ is greater than or equal to the supremum. Suppose it is strictly greater. Then there must be an $N \in \mathbb{N}$ such that $n \geq N$ implies $\|S_n x\|_k$ is greater than this supremum. Then in particular we have that $\|S_n x\|_k > \frac{1}{2}\sum_{i=1}^r \|E_i S_n x\|_k$ for all admissible $(E_i)_{i=1}^r$. So by definition $\|S_n x\|_k = \|S_n x\|_{k-1}$ for all $n \geq N$. Thus $\|x\|_k = \|x\|_{k-1}$. Now since $\|x\|_{k-1} \geq \|x\|_{k-2}$

holds for all $x \in T$, we can apply the same argument, just replacing k by $k-1$, to obtain $\|x\|_k = \|x\|_{k-2}$ and so on until $\|x\|_k = \|x\|_0 = \|x\|_\infty$. □

Corollary 3.1.5. *For $x \in T$ we have*

$$\|x\|_T = \|x\|_\infty \vee \sup\left\{ \frac{1}{2} \sum_{i=1}^{n} \|E_i x\|_T \;\middle|\; (E_i)_{i=1}^n \text{ admissible} \right\}.$$

Proof. We get $\|x\|_T = \sup_{k\in\mathbb{N}} \|x\|_k$. Bringing the supremum inside the formula from the proposition gives the result. □

The rest of this section will be dedicated to showing the property that gives Tsirelson's space its significance. Namely that it does not contain any subspace isomorphic to c_0 or ℓ_p, $1 \leq p < \infty$. We first needs to consider the case $p = 1$ separately. Then, we will show that T containing c_0 or ℓ_p for $p > 1$ implies it also contains ℓ_1. For this we first need a theorem about ℓ_1. The statement and proof we give below is adapted from [13].

Theorem 3.1.6 (James's ℓ_1 distortion theorem)**.** *Let $(x_n)_{n\in N}$ be a normalized basic sequence in a Banach space X that is equivalent to the canonical ℓ_1-basis. Then for each $\varepsilon > 0$ there is a normalized block basic sequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that*

$$\left\| \sum_{k=1}^{N} a_k y_k \right\| \geq (1-\varepsilon) \sum_{k=1}^{N} |a_k|$$

for any choice of scalars $(a_k)_{k=1}^N$.

Proof. Let M_n be the least constant such that any $(a_k)_{k=1}^\infty \in c_{00}$ satisfies

$$\sum_{k=n}^{\infty} |a_k| \leq M_n \left\| \sum_{k=n}^{\infty} a_k x_k \right\|.$$

Then M_n is a decreasing sequence with a limit $M \geq 1$. So we can find $N_0 \in \mathbb{N}$ such that $n \geq N_0$ implies $M_n < (1-\varepsilon)^{-\frac{1}{2}} M$. We pick a block sequence $(y_n)_{n\in N}$ by taking $p_0 \geq N_0$ and further $(p_n)_{n\geq 1}$ and $(b_j)_{j\in\mathbb{N}}$ such that

$$y_n = \sum_{j=p_{n-1}-1}^{p_n} b_j x_j$$

satisfies $\|y_n\| = 1$ but $\|y_n\|_1 \geq (1-\varepsilon)^{\frac{1}{2}} M$ for all $n \geq 1$. This is possible by ensuring that $p_n \geq n$. Then such an element must always exist, since otherwise M_n would not be minimal. In particular, we have

$$(1-\varepsilon)^{\frac{1}{2}} M \leq \|y_n\|_1 \leq (1-\varepsilon)^{-\frac{1}{2}} M$$

Now if $(a_k)_{k=1}^N$ are any scalars we find

$$\sum_{k=1}^N |a_k| \leq (1-\varepsilon)^{-\frac{1}{2}} M^{-1} \sum_{k=1}^N |a_k| \cdot \|y_k\|_1$$

$$= (1-\varepsilon)^{-\frac{1}{2}} M^{-1} \sum_{k=1}^N \sum_{j=p_{k-1}+1}^{p_k} |a_k b_j|$$

$$\leq (1-\varepsilon)^{-\frac{1}{2}} M^{-1} M_{p_0+1} \left\| \sum_{k=1}^N \sum_{j=p_{k-1}+1}^{p_k} a_k b_j x_j \right\|$$

$$\leq (1-\varepsilon)^{-1} \left\| \sum_{k=1}^N a_k y_k \right\|,$$

which proves the result. □

The name of the above theorem is linked to the concept of distortion, which we will treat more in depth in the next section. For now it is easier to apply this theorem directly.

Theorem 3.1.7. *T contains no subspace isomorphic to ℓ_1*

Proof. Suppose it did, then by Prop. 2.3.1 and Lemma 2.3.3 there is a block basic sequence $(x_n)_{n\in\mathbb{N}}$ in T which is equivalent to the ℓ_1-basis. Let $\varepsilon < \frac{1}{4}$. Then by Theorem 3.1.6 we get a normalized block basic sequence $(y_n)_{n\in\mathbb{N}}$ such that for all scalars $(a_k)_{k=1}^N$

$$\left\| \sum_{k=1}^N a_k y_k \right\|_T \geq (1-\varepsilon) \sum_{k=1}^N |a_k|$$

Let $a_0 = 1$ and $a_k = \frac{1}{k}$ for $k \geq 1$. Then if $N \geq 1$ this gives

$$\left\| \sum_{k=1}^N a_k y_k \right\|_T \geq (1-\varepsilon) \left(1 + \sum_{k=1}^N \frac{1}{k} \right) \geq 2(1-\varepsilon) > \frac{3}{2}$$

and thus this is also larger than $\left\| \sum_{k=1}^N a_k y_k \right\|_\infty$. So we are in the case of Corollary 3.1.5 where the norm is equal to the supremum, and since the element we are considering is finitely supported, this supremum is attained for a certain admissible $(E_i)_{i=1}^n$. Let $r = \max \operatorname{ran}(y_0)$. If $E_i y_0 = 0$ for all i then

$$\left\| \sum_{k=0}^N a_k y_k \right\|_T = \frac{1}{2} \sum_{i=1}^n \left\| E_i \left(\sum_{k=1}^n a_k y_k \right) \right\|_T \leq \frac{1}{2}(1 + \frac{1}{2} + ...) \leq 1 < 2(1-\varepsilon),$$

which is a contradiction. So $E_i y_0 \neq 0$ for some i and we have $n \leq \min E_1 \leq r$. This leads to the following inequality:

$$\frac{1}{2}\sum_{i=1}^{n}\left\|E_i\left(\sum_{k=0}^{n}a_k y_k\right)\right\|_T \leq 1+\frac{1}{2N}\sum_{i=1}^{n}\left\|E_i\left(\sum_{k=1}^{n}y_k\right)\right\|_T.$$

We proceed the estimation by dividing the indices into the following two sets:

$$A=\{k \mid E_j y_k \neq 0 \text{ for two or more values of } j\},$$

$$B=\{k \mid E_j y_k \neq 0 \text{ for at most one value of } j\}.$$

For $k \in A$ we have $\frac{1}{2N}\sum_{i=1}^{n}\|E_j y_k\|_T \leq \frac{1}{N}$. For $k \in B$ we have $\frac{1}{2N}\sum_{i=1}^{n}\|E_j y_k\|_T \leq \frac{1}{2N}$. Lastly note that the size of A is at most n. So we find:

$$\left\|\sum_{k=0}^{N}a_k y_k\right\|_T = \frac{1}{2}\sum_{i=1}^{n}\left\|E_i\left(\sum_{k=1}^{n}a_k y_k\right)\right\|_T \leq 1+\frac{n}{N}+\frac{N-n}{2N} \leq 1+\frac{N+r}{2N}.$$

The left hand side converges to $\frac{3}{2}$ as $N \to \infty$, which is smaller than $2(1-\varepsilon)$. So we again have a contradiction. □

Now to show that T contains no c_0 or ℓ_p for $p > 1$, we first need the following lemma, which is essentially the statement that T is an *asymptotic* ℓ_1 *space*, meaning that any n vectors "deep enough" act like the basis of ℓ_1^n. See also [2].

Lemma 3.1.8. *Let $n \geq 1$ and let $(y_k)_{k=1}^n$ be normalized elements of T with $n \leq \operatorname{supp}(y_1) < \ldots < \operatorname{supp}(y_n)$. Then for any scalars $(a_k)_{k=1}^n$ we have*

$$\frac{1}{2}\sum_{k=1}^{n}|a_k| \leq \left\|\sum_{k=1}^{n}a_k y_k\right\|_T \leq \sum_{k=1}^{n}|a_k|.$$

Proof. By assumption, the sequence $(E_k = \operatorname{supp}(y_k))_{k=1}^n$ of supports is admissible for T. So by Corollary 3.1.5

$$\left\|\sum_{k=1}^{n}a_k y_k\right\|_T \geq \frac{1}{2}\sum_{k=1}^{n}\left\|E_k\left(\sum_{k=1}^{n}a_k y_k\right)\right\|_T = \frac{1}{2}\sum_{k=1}^{n}\|a_k y_k\|_T = \frac{1}{2}\sum_{k=1}^{n}|a_k|.$$

This shows the first inequality. The second inequality is trivial. □

Theorem 3.1.9. *T contains no subspace isomorphic to c_0 or ℓ_p for $1 < p \leq \infty$.*

Proof. Suppose it did, then again by Prop. 2.3.1 and Lemma 2.3.3 there is a normalized block basic sequence $(x_n)_{n\in\mathbb{N}}$ in T that is equivalent to the basis of ℓ_p or c_0. Then in particular $(x_n)_{n\in\mathbb{N}}$ must be 1-subsymmetric. For any $k \in \mathbb{N}$ we can find

$n_k \in \mathbb{N}$ such that $k \leq \text{supp}(x_{n_k})$. So if we take any scalars $(a_k)_{k=1}^m$ we can use the subsymmetry to find

$$\left\| \sum_{k=1}^m a_k x_k \right\| = \left\| \sum_{k=1}^m a_k x_{n_{k+m-1}} \right\| \geq \frac{1}{2} \sum_{k=1}^m |a_k|,$$

where we have applied the above lemma. This implies that $(x_n)_{n\in\mathbb{N}}$ is equivalent to the ℓ_1-basis, which is a contradiction with Theorem 3.1.7. □

Remark 3.1.10. With a slight change to the above argument it can be shown that T contains no subsymmetric basic sequences at all. So the unit vector basis of T is also not subsymmetric. For the reader interested in logic, we refer to the paper [6] which shows that all subsequences of $(e_n)_{n\in\mathbb{N}}$ that can be bounded by a primitively recursive function are equivalent in T. The paper also gives an example of a subsequence that is not equivalent to the original basis by considering a very rapidly growing function. This shows that the basis of T is almost like a subsymmetric basis if not for its uncomputable subsequences.

Corollary 3.1.11. *T is reflexive.*

Proof. By Theorem 2.3.10 the unit basis of T is both shrinking and boundedly complete. Thus by Theorem 2.3.12, T is reflexive. □

3.2 Schlumprecht's space

The next construction we will consider is that of Schlumprecht's space S. It is known for being the first example of a Banach space that is *arbitrarily distortable*. Before we give the construction, we will discuss the concept of distortability.

Definition 3.2.1. A Banach space $(X, \|\cdot\|)$ is called *λ-distortable* for some $\lambda > 1$ if there exists an equivalent norm $\|\cdot\|'$ on X such that for each infinite-dimensional subspace $Y \subseteq X$ we have

$$\sup \left\{ \frac{\|z_1\|'}{\|z_2\|'} \;\middle|\; z_1, z_2 \in Y, \|z_1\| = \|z_2\| = 1 \right\} \geq \lambda.$$

A norm for which this holds is then called a *λ-distortion* of X. A space which is λ-distortable for all $\lambda > 1$ is called *arbitrarily distortable*.

If we rewrite the statement from Theorem 3.1.6 with this definition we get the following result:

Theorem 3.2.2. *ℓ_1 is not λ-distortable for any $\lambda > 1$.*

Proof. Let $\|\cdot\|'$ be any equivalent norm on ℓ_1. Pick a basic sequence $(x_n)_{n\in\mathbb{N}}$ that is normalized under this norm. Then as in Theorem 3.1.6 we may find $M \geq 1$ and for each $\varepsilon > 0$ a block basic sequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $\|y_n\|' = 1$ but

$$(1-\varepsilon)^{\frac{1}{2}} M \leq \|y_n\|_1 \leq (1-\varepsilon)^{-\frac{1}{2}} M.$$

And furthermore such that

$$(1-\varepsilon)\sum_{k=1}^{\infty}|a_k| \leq \left\|\sum_{k=1}^{\infty}a_k y_k\right\|' \leq \sum_{k=1}^{\infty}|a_k|.$$

In particular we see that if $Y = [y_n]_{n\in\mathbb{N}}$, and $z \in Y$, say $z = \sum_{k=1}^{\infty} a_k y_k$, then

$$\|z\|_1 = \sum_{k=1}^{\infty}|a_k|\cdot\|y_k\|_1 \leq (1-\varepsilon)^{-1}\tfrac{1}{2}M\|z\|',$$

$$\|z\|_1 = \sum_{k=1}^{\infty}|a_k|\cdot\|y_k\|_1 \geq (1-\varepsilon)\tfrac{1}{2}M\|z\|'.$$

Rescaling a bit, we can conclude that there exists $C \geq 1$ such that for each $\varepsilon > 0$ there is a $Y \subseteq \ell_1$ such that for all $z \in Y$ we have

$$(1-\varepsilon)C\|z\|_1 \leq \|z\|' \leq (1+\varepsilon)C\|z\|_1.$$

Thus whenever $z_1, z_2 \in Y = [y_n]_{n\in\mathbb{N}}$ satisfy $\|z_1\|_1 = \|z_2\|_1 = 1$, we have

$$\frac{\|z_1\|'}{\|z_2\|'} \leq \frac{1+\varepsilon}{1-\varepsilon}.$$

Thus this norm cannot be a λ-distortion for any $\lambda > 1$. □

Remark 3.2.3. In the preceding section it would also have been possible to show that Tsirelson's space is distortable for any $\lambda < 2$, and derive Theorem 3.1.7 that way. See also the proof sketch in [34]. It is an open problem whether or not Tsirelson's space is arbitrarily distortable, and more generally whether or not every distortable space is also arbitrarily distortable.

At the time Schlumprecht published his paper [32], the only spaces that were known not to be distortable were c_0 and ℓ_1. (The proof that c_0 is not distortable goes almost analogously.) Further, it was not known whether or not the spaces ℓ_p for $p > 1$ were distortable. This was known as the *distortion problem*. It was solved not much later by Schlumprecht and Odell in [30], who proved that the ℓ_p-spaces for $1 < p < \infty$ are in fact arbitrarily distortable. This somewhat diminishes the result about Schlumprecht's original space S, but it is still important in our story. For one, it is a good example of a mixed Tsirelson space, which we shall cover in generality in the next section. Second, it serves as a starting point for the construction we discuss in Chapter 4.

We now begin with the construction of Schlumprecht's space. The rest of this section will draw from the original paper [32] and from [17].

Definition 3.2.4. We define the class of functions $\mathfrak{F}$ to contain any function $f : [1,\infty) \to [1,\infty)$ that satisfies the following conditions:

(i) $f(1) = 1$ and $f(x) < x$ for $x > 1$.

(ii) $\lim_{x\to\infty} f(x) = \infty$.

(iii) $\lim_{x\to\infty} f(x)/x^q = 0$ for all $q > 0$.

(iv) The function $x/f(x)$ is concave.

(v) $f(xy) \leq f(x)f(y)$ for every $x, y \geq 1$.

An example is $f(x) = \log_2(1+x)$, but any "log-like" function will do. Next, we have to define a slightly weaker admissibility condition:

Definition 3.2.5. A sequence $(E_i)_{i=1}^n$ of finite subsets of $\mathbb{N}$ is admissible for Schlumprecht's space if they are successive, that is $E_1 < ... < E_n$.

Now we are ready to define the norm on c_{00} with respect to which we will take the completion:

Definition 3.2.6. Let $\|x\|_0 = \|x\|_\infty$ and supposing $\|x\|_k$ is defined, define the next norm on c_{00} by

$$\|x\|_{k+1} = \max\left\{ \frac{1}{f(n)} \sum_{i=1}^{n} \|E_i x\|_k \;\middle|\; (E_i)_{i=1}^n \text{ admissible} \right\}.$$

Now set $\|x\|_S = \max_{k\in\mathbb{N}} \|x\|_k$. Then Schlumprecht's space S is the completion of $(c_{00}, \|\cdot\|_S)$.

The key difference with the Tsirelson norm is the weight $f(n)^{-1}$ before the sum. Just like in the preceding section, the unit vectors form a 1-unconditional basis of S. However, whereas the unit vector basis in T failed to be subsymmetric, this is not the case in S.

Proposition 3.2.7. *The unit vector basis in S is 1-subsymmetric.*

Proof. As before, we proceed with induction on k. For $k = 0$ we have nothing to prove. Assume $(e_n)_{n\in\mathbb{N}}$ is 1-subsymmetric in the completion of $(c_{00}, \|\cdot\|_k)$. Let $(n_j)_{j\in\mathbb{N}}$ be any increasing sequence in $\mathbb{N}$ and let $(a_j)_{j=1}^N$ be a sequence of scalars. Let $x = \sum_{j=1}^N a_j e_{n_j}$ and $y = \sum_{j=1}^N a_j e_j$. Given $E = \{j_1, ..., j_r\} \subseteq \mathbb{N}$, we define the image $E' = \{n_{j_1}, ..., n_{j_r}\}$. Note since $j \mapsto n_j$ is injective, so is $E \mapsto E'$. We also have $E < F$ if and only if $E' < F'$. Further, by the induction hypothesis, we have $\|Ey\|_k = \|E'x\|_k$. Now pick an admissible sequence $(E_i)_{i=1}^n$ that maximizes $\|y\|_{k+1}$. Then $(E_i')_{i=1}^n$ is also admissible and thus

$$\|y\|_{k+1} = \frac{1}{f(n)} \sum_{i=1}^{n} \|E_i y\|_k = \frac{1}{f(n)} \sum_{i=1}^{n} \|E_i' x\|_k \leq \|x\|_{k+1}.$$

Conversely, picking an admissible sequence $(E_i')_{i=1}^n$ that maximizes $\|x\|_{k+1}$ and taking the preimages $(E_i)_{i=1}^n$ gives $\|x\|_{k+1} \leq \|y\|_{k+1}$. Letting $N \to \infty$ shows that $(e_n)_{n\in\mathbb{N}}$ is 1-subsymmetric in the completion of $(c_{00}, \|\cdot\|_{k+1})$. This again easily extends to S since we take the maximum over all $\|\cdot\|_k$. □

Remark 3.2.8. At first glance, it may not be clear why the above argument would not have worked for Tsirelson's space. The problem is the following: it can occur that $n \leq \min E_1'$ but $\min E_1 < n$. So it is not guaranteed that $(E_i)_{i=1}^n$ is admissible for Tsirelson's space whenever $(E_i')_{i=1}^n$ is.

The subsymmetry property is especially useful since it allows us to assume without loss of generality that whenever we deal with an admissible $(E_i)_{i=1}^n$, it is actually a sequence of successive intervals. Any gaps can be eliminated just by reordering the unit vectors, and the norm will stay the same. We also have an implicit formula analogous to Corollary 3.1.5:

Proposition 3.2.9. *For $x \in S$ we have*

$$\|x\|_S = \|x\|_\infty \vee \sup\left\{ \frac{1}{f(n)} \sum_{i=1}^n \|E_i x\|_S \;\middle|\; (E_i)_{i=1}^n \text{ admissible} \right\}.$$

In the next section we will give a proof that these implicit formulas also uniquely define their norms. Now let us see that S is arbitrarily distortable. The trick is to define an alternative norm in which the length of admissible sequences is fixed. First, we need an important result about representability in which we will apply Krivine's theorem (Theorem 2.4.13) from the end of the previous chapter.

Lemma 3.2.10. *ℓ_1 is finitely block represented in every block subspace of S.*

Proof. Let $1 < p < \infty$. For any n blocks $x_1 < ... < x_n$ in S we have

$$\left\| \frac{1}{n^{1/p}} \sum_{i=1}^n x_i \right\|_S \geq \frac{1}{n^{1/p}} \frac{n}{f(n)} = \frac{n^{1-1/p}}{f(n)}.$$

By property (iii) of $\mathfrak{F}$, this lower bound diverges to positive infinity as $n \to \infty$. Whereas if $e_1, ..., e_n$ are the unit basis of ℓ_p^n, we have

$$\left\| \frac{1}{n^{1/p}} \sum_{i=1}^n e_i \right\|_p = 1.$$

Thus for each $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that any n-length block basis in S will not be $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^n. The same argument works for c_0, where we replace $\frac{1}{n^{1/p}}$ by 1. So, applying Krivine's theorem, the only option that remains is that ℓ_1 is finitely block represented in each block subspace of S. □

Theorem 3.2.11. *For each $\ell \in \mathbb{N}$, the norm $\|\cdot\|_\ell'$ on S defined by*

$$\|x\|_\ell' = \sup\left\{ \frac{1}{f(\ell)} \sum_{i=1}^\ell \|E_i x\|_S \;\middle|\; (E_i)_{i=1}^\ell \text{ admissible} \right\}$$

is an $f(\ell)$-distortion on S.

Giving the entire proof now would take too much space. Instead, we will leave the key claim below open, and return to it in Section 3.4 when we have developed some more theory. The direct proof of the claim can be found in the last six pages of Schlumprecht's paper [32].

Proof. Let $Y \subseteq S$ be any subspace. By Lemma 2.3.1, we can pass to a block basic sequence $(y_n)_{n\in\mathbb{N}}$ in Y. Let $\varepsilon > 0$. We will construct $z_1, z_2 \in [y_n]_{n\in\mathbb{N}}$ such that $\|z_1\|_S = \|z_2\|_S = 1$ but $\|z_1\|'_\ell \geq 1-\varepsilon$ and $\|z_2\|'_\ell \leq \frac{1+\varepsilon}{f(\ell)}$.

Let $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be strictly increasing and let $(\varepsilon_n)_{n\in\mathbb{N}}$ decrease to 0 with $\varepsilon_1 < \varepsilon$. By Lemma 3.2.10, we can find a sequence $x_1 < ... < x_{k_1}$ in $[y_n]_{n\in\mathbb{N}}$ that is $(1+\varepsilon_1)$-equivalent to the unit basis of $\ell_1^{k_1}$. Define $x_1 = \frac{1}{k_1}\sum_{i=1}^{k_1} x_i$. Next, pick such a sequence of length k_2 in the block subspace generated by all the elements not used in x_1 that is $(1+\varepsilon_2)$-equivalent, and use this to define x_2, and so on. We obtain a sequence $(x_i)_{i\in\mathbb{N}}$ with $\|x_i\|_S \geq 1-\varepsilon_i > 1-\varepsilon$ for each i and satisfying the following claim:

Claim. There exist $n_1 < ... < n_\ell$ such that $M = \|\sum_{i=1}^{\ell} x_{n_i}\|_S \leq \frac{\ell}{f(\ell)}$.

Then $z_1 = \sum_{i=1}^{\ell} x_{n_i}/M$ is the element we need, since picking $E_i = \mathrm{supp}(x_{n_i})$ gives

$$\|z_1\|_\ell \geq \frac{1}{f(\ell)}\frac{1}{M}\sum_{i=1}^{\ell}\|E_i x_{n_i}\|_S \geq \frac{1}{\ell}\cdot\ell(1-\varepsilon) = 1-\varepsilon.$$

Now to construct z_2 let $n \geq 4\ell\varepsilon^{-1}$ and let Lemma 3.2.10 give a sequence $x_1 < ... < x_n$ that is $(1+\frac{\varepsilon}{2})$-equivalent to the ℓ_1^n-basis. Let $z_2 = \sum_{i=1}^n x_i / \|\sum_{i=1}^n x_i\|_S$. Let $(E_i)_{i=1}^\ell$ be an admissible sequence that maximizes $\|z_2\|_\ell$. By subsymmetry we can assume that each E_i is an interval in $\mathbb{N}$. Thus, since the x_i are also successive, there are for each E_i at most two elements which intersect E_i but are not subsets of E_i. So if we let

$$\tilde{E}_i = \bigcup\{\mathrm{supp}(x_j) \mid \mathrm{supp}(x_j) \subseteq E_i\},$$

it follows that $\|E_i z_2\|_S \leq \|\tilde{E}_i z_2\|_S + \frac{2}{n}$. This is because the (up to two) parts we have thrown away have norm at most $\|\sum_{i=1}^n x_i\|_S^{-1}$, which is at most $\frac{1}{n}$ by the equivalence assumption. Further, this gives that the sequence $(\tilde{E}_i z_2)_{i=1}^\ell$ is a blocking of $(x_i)_{i=1}^n$. So we can estimate $\|z_2\|_\ell$ by:

$$f(\ell)\|z_2\|_\ell = \sum_{i=1}^{\ell}\|E_i z_2\|_S \leq \frac{2\ell}{n} + \sum_{i=1}^{\ell}\left\|\tilde{E}_i z_2\right\|_S$$

$$\leq \frac{\varepsilon}{2} + (1+\frac{\varepsilon}{2})\left\|\sum_{i=1}^{\ell}\tilde{E}_i z_2\right\|_S \leq \frac{\varepsilon}{2} + (1+\frac{\varepsilon}{2})\|z_2\|_S = 1+\varepsilon.$$

□

3.3 Mixed Tsirelson spaces

In the previous two sections we have seen two norm constructions which used an inductive definition. This section will generalize these ideas into a highly extendible method for constructing sequence spaces: Mixed Tsirelson spaces. These were introduced by Argyros and we will follow his treatment from [4].

First, we will give three equivalent definitions of a mixed Tsirelson space. In doing so we will end up moving away from the inductive style of definition we have seen before and instead continue with that of a *norming set* W, which defines a norm by $\|x\|_W = \sup_{f\in W} \langle f, x\rangle$. Then, we introduce an analysis method using trees which makes it easier to reason about the norms of elements in these spaces. Lastly we will give some results about the properties of these spaces under certain conditions.

To start, we will generalize the two admissibility rules we have seen before:

Definition 3.3.1. Let $\mathcal{M}$ be a set of finite subsets of $\mathbb{N}$ that is closed (and thus compact) in the Cantor topology on $\mathcal{P}(\mathbb{N})$ and contains all singletons $\{n\}$. A sequence $(E_i)_{i=1}^n$ of subsets of $\mathbb{N}$ is $\mathcal{M}$-*admissible* if there exists $\{m_i\}_{i=1}^n \in \mathcal{M}$ such that

$$m_1 \leq E_1 < m_2 \leq E_2 < ... < m_n \leq E_n.$$

Example 3.3.2. The *Schreier family* $\mathcal{S} = \{A \subseteq \mathbb{N} \mid \#A \leq \min A\}$ yields the admissibility rule of Tsirelson's space. Here we use the convention that $\min \emptyset = \infty$, so $\emptyset \in \mathcal{S}$, which is required for it to be closed. Obviously the inclusion of the empty set does not matter for the admissibility rule. So above one could also consider sets $\mathcal{M}$ that are relatively closed in $\mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$.

Definition 3.3.3. Let $(\mathcal{M}_n)_{n\in\mathbb{N}}$ be a sequence of sets as in Definition 3.3.1 and let $(\theta_n)_{n\in\mathbb{N}}$ be weights satisfying $0 < \theta_n < 1$ for all $n \in \mathbb{N}$. Define the *norming set* $W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$ to be the smallest set $W \subseteq c_{00}$ satisfying

(i) $\pm e_k \in W$ for all $k \in \mathbb{N}$,

(ii) whenever $f_1, ..., f_n \in W$ are such that $(\mathrm{supp}(f_i))_{i=1}^n$ is $\mathcal{M}_k$-admissible for some $k \in \mathbb{N}$, then also $\theta_k(f_1 + ... + f_n) \in W$. In this case we will also say that $(f_i)_{i=1}^n$ is $\mathcal{M}_k$-admissible.

Then the mixed Tsirelson norm on c_{00} is defined using this norming set W by

$$\|x\|_W = \sup_{f\in W} |\langle f, x\rangle| = \sup_{f\in W} \left|\sum_{i=1}^{\infty} x_i f_i\right|,$$

and the mixed Tsirelson space $T[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$ is the completion of $(c_{00}, \|\cdot\|_W)$.

Remark 3.3.4. If above we have a finite sequence $(\mathcal{M}_n, \theta_n)_{n=1}^r$, there exists an n_0 between 1 and r such that the space given by the length-one sequence $((\mathcal{M}_{n_0}, \theta_{n_0}))$

is isomorphic to $T[(\mathcal{M}_n, \theta_n)_{n=1}^r]$. So we either consider a length-one version, which we denote by $T(\mathcal{M}, \theta)$, or an infinite one. Further, we should note that the topological constraints above are not strictly necessary for our purposes here. They are included in the definition since one can derive certain structural results about $T[(\mathcal{M}_n, \theta_n)_{n=1}^r]$ based only on the topology of $\mathcal{M}$. See also [7].

Example 3.3.5. The spaces we have seen so far are easily obtained from this definition:

(i) Tsirelson's space is $T(\mathcal{S}, \frac{1}{2})$.

(ii) If we write $\mathcal{A}_n = \{A \subseteq \mathbb{N} \mid \#A \leq n\}$ and let $f \in \mathfrak{F}$. Then $T[(\mathcal{A}_n, \frac{1}{f(n)})_{n\in\mathbb{N}}]$ yields Schlumprecht's space.

To show the link with the inductive definitions we have seen before, we will introduce an inductive way of defining the mixed Tsirelson norm and show that it is equivalent.

Definition 3.3.6. Given a sequence $(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}$ as before, define norms on c_{00} by $\|\cdot\|_{*1} = \|\cdot\|_\infty$, and supposing $\|\cdot\|_{*k}$ has been defined, define the next norm by

$$\|x\|_{*k+1} = \|x\|_{*k} \vee \max\left\{\theta_n \sum_{i=1}^r \|E_i x\|_{*k} \;\middle|\; n \in \mathbb{N}, (E_i)_{i=1}^r \;\; \mathcal{M}_n\text{-admissible}\right\}.$$

Then let $\|x\|_* = \max_{k\in\mathbb{N}} \|x\|_{*k}$.

Theorem 3.3.7. *The following ways of defining the mixed Tsirelson norm are equivalent:*

(i) The norming set $\|\cdot\|_W$.

(ii) The inductive limit $\|\cdot\|_*$.

(iii) The implicit norm $\|\cdot\|$ *satisfying*

$$\|x\| = \|x\|_\infty \vee \max\left\{\theta_n \sum_{i=1}^r \|E_i x\| \;\middle|\; n \in \mathbb{N}, (E_i)_{i=1}^r \;\; \mathcal{M}_n\text{-admissible}\right\}$$

for all $x \in c_{00}$.

Proof. (i) $\Leftrightarrow$ (ii): We will show that we can also construct $\|\cdot\|_W$ as an inductive limit of norms $\|\cdot\|_{W_k}$, where at each step $\|\cdot\|_{W_k} = \|\cdot\|_{*k}$. As we have noted before, it holds that $\|x\|_{*k} \leq \|x\|_{*k+1}$. Thus in fact $\|x\|_* = \lim_{k\to\infty} \|x\|_{*k}$. Let $W_1 = \{\pm e_k \mid k \in \mathbb{N}\}$. Then clearly $\|\cdot\|_{*1} = \|\cdot\|_{W_1}$. Once we have defined W_k, define W_k' as the set of vectors of the form $\theta_n(f_1 + ... + f_r)$ where $f_1, ..., f_r \in W_k$ are such that $(\mathrm{supp}(f_i))_{i=1}^r$ is $\mathcal{M}_n$-admissible. Then let $W_{k+1} = W_k \cup W_k'$. We see that

$$\|x\|_{W_{k+1}} = \|x\|_{W_k} \vee \|x\|_{W_k'} = \|x\|_{W_k} \vee \max_{f\in W_k'} |\langle f, x\rangle|.$$

By noting that $f \in W_k$ implies $-f \in W_k$, we can move the absolute values inside of the sum:

$$\|x\|_{W_{k+1}} = \|x\|_{W_k} \vee \max\left\{\theta_n \sum_{i=1}^{r} |\langle f_i, x\rangle| \,\middle|\, f = \theta_n(f_1 + ... + f_r) \in W_k'\right\}$$

$$= \|x\|_{W_k} \vee \max\left\{\theta_n \sum_{i=1}^{r} \|E_i x\|_{W_k} \,\middle|\, n \in \mathbb{N}, (E_i)_{i=1}^r \ \mathcal{M}_n\text{-admissible}\right\} = \|x\|_{*k+1},$$

where for the second to last step one needs to observe that the maximum over W_k inside the sum will yield the same element f_i, otherwise it would not have been part of the maximum in the previous step. The last step is the induction hypothesis. Now the observation that $W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$ is precisely $\bigcup_{k\in\mathbb{N}} W_k$ yields that $\lim_{k\to\infty} \|x\|_{W_k} = \|x\|_W$, so $\|\cdot\|_W = \|\cdot\|_{*k}$.
(ii) $\Rightarrow$ (iii): We have seen this in the case of Tsirelson's space in Section 3.1. The modifications for the general case are left to the reader.
(iii) $\Rightarrow$ (ii): By the above, $\|\cdot\|_*$ satisfies the implicit formula. Let $\|\cdot\|'$ be another norm satisfying the implicit formula. If we show $\|\cdot\|' = \|\cdot\|_*$, we are done. Consider the set $\left\{x \in c_{00} \mid \|x\|' \neq \|x\|_*\right\}$. If this is non-empty, there is an element of minimal support, call it x. Then either $\|x\|' = \|x\|_\infty = \|x\|_*$, or $\|x\|' = \theta_n \sum_{i=1}^r \|E_i x\|'$ for some $\mathcal{M}_n$-admissible $(E_i)_{i=1}^r$ such that $\mathrm{supp}(x)$ is not contained in any E_i. But then $\|E_i x\|' = \|E_i x\|_*$ for all i and thus $\|x\|' = \|x\|_*$ in this case as well, which is a contradiction. □

Out of these three definitions we choose to work with that of the norming set. The main reason is that the norming set definition is easy to extend, which we will see in Chapter 4. Another reason is the set of definitions we cover in the remainder of this section, which concern our first instance of an analysis method using finite trees. The intuitive idea is that every element of the norming set must be constructed in a finite number of steps using one of the operations defined by the pairs $(\mathcal{M}_n, \theta_n)$. We will create a tree structure that records these steps.

Definition 3.3.8. We write $\mathbb{N}^*$ for the set of finite sequences in $\mathbb{N}$. For $\sigma, \tau \in \mathbb{N}^*$ we write $\sigma\tau$ for their concatenation, and if σ is an initial part of τ we write $\sigma \preceq \tau$. Further we write λ for the empty sequence. A *finite tree* is a finite set $\Upsilon \subseteq \mathbb{N}^*$ such that $\lambda \in \Upsilon$ and if $\tau \in \Upsilon$ then for every $\sigma \preceq \tau$ we have $\sigma \in \Upsilon$. We write $S(\sigma)$ for the set of direct successors that are in Υ. Lastly we write $\max \Upsilon$ for the set of leaves, so $\sigma \in \max \Upsilon$ if and only if $S(\sigma) = \emptyset$.

Definition 3.3.9. For $f \in W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$, a *tree analysis* of f is a family $(f_\sigma)_{\sigma\in\Upsilon}$, where $\Upsilon \subseteq \mathbb{N}^*$ is a finite tree, such that:

(i) $f_\lambda = f$

(ii) If $\sigma \in \max \Upsilon$ then $f_\sigma = \pm e_k$ for some $k \in \mathbb{N}$.

(iii) If $\sigma \in \Upsilon \setminus \max \Upsilon$ then the sequence $(f_\tau)_{\tau \in S(\sigma)}$ is $\mathcal{M}_n$-admissible for some $n \in \mathbb{N}$, and $f_\sigma = \theta_n \sum_{\tau \in S(\sigma)} f_\tau$.

Theorem 3.3.10. *Every $f \in W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$ admits a tree analysis.*

Proof. Let W' be the set of all f that admit a tree analysis. The tree $\{\lambda\}$ with $f_\lambda = \pm e_k$ gives $\pm e_k \in W'$. Further if $f_1, ..., f_n \in W'$ are such that $(\mathrm{supp}(f_i))_{i=1}^n$ is $\mathcal{M}_k$-admissible for some k, then given the tree analyses $(f_\sigma^{(i)})_{\sigma\in\Upsilon_i}$ we can join these into the tree $\Upsilon = \{\lambda\} \cup \{i\sigma \mid 1 \leq i \leq n, \sigma \in \Upsilon_i\}$, with $f_\lambda = \theta_k(f_1 + ... + f_n)$ and $f_{i\sigma} = f_\sigma^{(i)}$ to obtain a tree analysis for $\theta_k(f_1 + ... + f_n)$. Thus $\theta_k(f_1 + ... + f_n) \in W'$. So by minimality of the norming set we have $W' = W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$. □

In the next section we will give arguments that use induction on the tree structure of a certain $f \in W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$. For this we will need a few more notational conventions:

Definition 3.3.11. Given $f \in W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$, define the *weight* of f as $w(f) = n$ if $f = \theta_n \sum_{i=1}^n f_i$ for some $\mathcal{M}_n$-admissible sequence $f_1 < ... < f_n$. Further define $w(\pm e_k) = 0$.

Remark 3.3.12. Note that the weight is not necessarily uniquely defined! So this is strictly notation and not a definition of a function from $W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$ to $\mathbb{N}$.

The next definition requires establishing some more intuition. It is obvious that as we move from the root of the tree to its leaves, the ranges of the attached functionals shrink. When we are evaluating the value of some $|\langle f, x\rangle|$, the way that the ranges of the f_σ overlap with the range of x is of interest. Clearly if these ranges are disjoint the value is 0 and the contribution is zero. On the other hand if the range of x is fully contained in that of f_σ, we have not yet reached a point where the tree expansion is giving us any information. The following condition therefore encodes the point (if it exists) at which the range of x no longer fits inside the ranges of the tree elements:

Definition 3.3.13. Given a tree analysis $(f_\sigma)_{\sigma\in\Upsilon}$ and a finitely supported element $x \in T[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$, we say that $\tau \in \Upsilon$ is a *splitting point* for x if it satisfies the following conditions:

(i) $\mathrm{ran}(f_\tau) \cap \mathrm{ran}(x) \neq \emptyset$.

(ii) For all $\sigma \in \Upsilon$ such that $\sigma \preceq \tau$ we have $\mathrm{ran}(f_\sigma) \cap \mathrm{ran}(x) = \mathrm{ran}(f_\tau) \cap \mathrm{ran}(x)$.

(iii) For all $\sigma \in S(\tau)$ we have $\mathrm{ran}(f_\sigma) \cap \mathrm{ran}(x) \subsetneq \mathrm{ran}(f_\tau) \cap \mathrm{ran}(x)$.

If such a splitting point exists it is unique and we denote it by τ_x.

3.4 Rapidly Increasing Sequences

In Section 3.2 we postponed part of the proof that S is arbitrarily distortable. We are now getting close to proving this part using general results about the mixed Tsirelson spaces we defined in the preceding section. In particular, we will see that the sequence we constructed at the start of the proof of Theorem 3.2.11 has a subsequence that satisfies the definition of a *rapidly increasing sequence* (RIS).

Then, the property we claimed about this subsequence will follow from a result about rapidly increasing sequences referred to by Argyros as "the basic inequality". We will prove a specific version of this result that applies to mixed Tsirelson spaces. In both Chapter 4 and Chapter 6 we will see extensions of the concept of a norming set and tree analysis and, as a result, of the rapidly increasing sequences and the basic inequality.

One assumption we need to make before proceeding is that we will work with a mixed Tsirelson space $X = T[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$ in which the sequence of weights $(\theta_n)_{n\in\mathbb{N}}$ is strictly decreasing and converges to 0.

Definition 3.4.1 ((C,ε)-RIS)**.** For $C, \varepsilon > 0$, a block sequence $(x_k)_{k=1}^{\infty}$ in X is a (C,ε)-*rapidly increasing sequence* if $\|x_k\| \leq C$ for all k and there exists a sequence $j_1 < j_2 < \ldots$ such that

(i) $\theta_{j_{k+1}} \cdot \#\mathrm{supp}(x_k) < \varepsilon$, and

(ii) $|\langle f, x_k\rangle| \leq C\theta_i$ whenever $w(f) = i < j_k$

for all k.

The existence of rapidly increasing sequences is not obvious. We will show the existence in the case of Schlumprecht's space. For more generic statements see [4].

Definition 3.4.2. For $C \geq 1$, an element $x \in X$ is a C-ℓ_1^k-*average* if it can be written as $x = \frac{1}{k}\sum_{i=1}^{k} x_i$, where $x_1 < \ldots < x_k$ are normalized blocks that form a basis that is C-equivalent to the standard ℓ_1^k-basis.

It is clear from Lemma 3.2.10 that every block subspace of S contains such ℓ_1^k-averages. We will use these to construct a rapidly increasing sequence of ℓ_1^k-averages:

Proposition 3.4.3. *Every block subspace of Schlumprecht's space S contains for every $\varepsilon > 0$ a $(3,\varepsilon)$-RIS.*

Proof. We use $S = T[(\mathcal{A}_n, \frac{1}{F(n)})_{n\in\mathbb{N}}]$ with $F \in \mathfrak{F}$. We let Lemma 3.2.10 give for each k a $(1+\varepsilon)$-ℓ_1^k-average $x_k = \frac{1}{k}\sum_{i=1}^{k} x_{k,i}$ such that $x_k < x_{k+1}$. We may trivially find $j_1 < j_2 < \ldots$ such that $\varepsilon F(j_{k+1}) > \#\mathrm{supp}(x_{j_k})$ for each k. We also clearly have

$\|x_{j_k}\|_S \leq 1$, so the last step is to show condition (ii). Let $f \in W[(\mathcal{A}_n, \frac{1}{F(n)})_{n\in\mathbb{N}}]$ with $w(f) = r < j_k$. Writing $f = \frac{1}{F(r)} \sum_{m=1}^{r} f_m$, and $E_m = \operatorname{supp}(f_m)$, we estimate

$$|\langle f, x_k\rangle| \leq \frac{1}{F(r)} \sum_{m=1}^{r} |\langle f_m, x_k\rangle| \leq \frac{1}{F(r)} \sum_{m=1}^{r} \|E_m x_k\|_S.$$

For each m define I_m to be the set of all $i \leq j_k$ such that $\operatorname{supp}(x_{k,i}) \subseteq E_m$, and J_m the $i \leq j_k$ where $\operatorname{supp}(x_{k,i}) \cap E_m \neq \emptyset$. Clearly $\sum_{m=1}^{r} \#I_m \leq j_k$. Further we have

$$\|E_m x_k\|_S \leq \frac{1}{j_k} \sum_{i=1}^{j_k} \|E_m x_{k,i}\|_S = \frac{1}{j_k} \sum_{i\in J_m} \|E_m x_{k,i}\|_S \leq \frac{1}{j_k}(\#I_m + 2).$$

Thus

$$\sum_{m=1}^{r} \|E_m x_k\|_S \leq \frac{2r}{j_k} + \frac{1}{j_k} \sum_{m=1}^{r} \#I_m \leq 1 + \frac{2r}{j_k} < 3,$$

which shows $|\langle f, x_k\rangle| \leq \frac{3}{F(r)}$. So $(x_{j_k})_{k=1}^{\infty}$ is a $(3, \varepsilon)$-RIS. □

Theorem 3.4.4 (Basic inequality)**.** *Let $(x_k)_{k=1}^{\infty}$ be a (C, ε)-RIS in X and let $(\lambda_k)_{k=1}^{\infty}$ be scalars. Then for every $f \in W[(\mathcal{M}_n, \theta_n)_{n\in\mathbb{N}}]$ there exists $g_1, g_2 \in c_{00}(\mathbb{N})$ such that $g_1 = h_1$ or $g_1 = e_{k_0} + h_1$ for some h_1 with $k_0 \notin \operatorname{supp}(h_1)$, $\|h_1\|_\infty \leq \theta_{w(f)}$, $\|g_2\|_\infty \leq \varepsilon$, and both g_1, g_2 having non-negative coordinates and satisfying the inequality*

$$\left|\left\langle f, \sum_{k\in\mathbb{N}} \lambda_k x_k \right\rangle\right| \leq C \left\langle g_1 + g_2, \sum_{k\in\mathbb{N}} |\lambda_k| e_k \right\rangle.$$

Proof. The proof uses induction on the tree structure of f. Let $(f_\sigma)_{\sigma\in\Upsilon}$ be a tree analysis for f. For $\tau \in \Upsilon$ we define the sets

$$E_\tau = \{k \in \mathbb{N} \mid \tau = \tau_{x_k}\},$$

$$D_\tau = \bigcup_{\sigma\succeq\tau} E_\sigma.$$

In short, D_τ is the set of indices k for which the splitting point for x_k (if it exists) lies in the subtree with root τ. Whereas E_τ are those indices that are in in D_τ but not in any D_σ for $\sigma \in S(\tau)$. The following two claims follow directly from these definitions:

(a) For every $k \in \mathbb{N}$, if $\operatorname{supp}(f) \cap \operatorname{ran}(x_k) \neq \emptyset$ then $k \in D_\lambda$. Thus $\left\langle f, \sum_{k\in\mathbb{N}} \lambda_k x_k \right\rangle = \left\langle f, \sum_{k\in D_\lambda} \lambda_k x_k \right\rangle$.

(b) For every $\tau \notin \max \Upsilon$ the family

$$\{\{k\} \mid k \in E_\tau\} \cup \{D_\sigma \mid \sigma \in S(\tau)\}$$

of subsets of $\mathbb{N}$ is pairwise disjoint.

We will construct $(g_\tau^1)_{\tau\in\Upsilon}, (g_\tau^2)_{\tau\in\Upsilon}$ such that for all $\tau \in \Upsilon$ we have:

(i) If $\tau \notin \max \Upsilon$, then $g_\tau^1 = h_\tau$ or $g_\tau^1 = e_{k_\tau} + h_\tau$ for some h_τ with $k_\tau \notin \operatorname{supp}(h_\tau)$ and $\|h_\tau\|_\infty \leq \theta_{w(f_\tau)}$.

(ii) $\|g_\tau^2\|_\infty \leq \varepsilon$.

(iii) $\operatorname{supp}(g_\tau^i) \subseteq D_\tau$ and $g_\tau^i(n) \geq 0$ for all $n \in \mathbb{N}$, $i = 1, 2$.

(iv) If $\tau \in \max \Upsilon$ with $D_\tau \neq \emptyset$ then $g_\tau^1 = f_\tau$.

(v) $|\left\langle f_\tau, \sum_{k\in D_\tau} \lambda_k x_k \right\rangle| \leq C \left\langle g_\tau^1 + g_\tau^2, \sum_{k\in D_\tau} |\lambda_k| e_k \right\rangle$.

Then by (a) the choice $g_1 = g_\lambda^1$, $g_2 = g_\lambda^2$ satisfies the properties in the statement of the theorem.
We start with the base case. If $\tau \in \max \Upsilon$, with $D_\tau = \emptyset$, set $g_\tau^1 = g_\tau^2 = 0$. Otherwise, $D_\tau = \{k_0\}$ for some $k_0 \in \mathbb{N}$ and $f_\tau = \pm e_p$ for some $p \in \operatorname{ran}(x_{k_0})$. Set $g_\tau^1 = e_{k_0}$, $g_\tau^2 = \varepsilon e_{k_0}$. Then

$$\left|\left\langle f_\tau, \sum_{k\in D_\tau} \lambda_k x_k \right\rangle\right| = |\lambda_{k_0} x_{k_0}(p)| \leq C(1+\varepsilon)|\lambda_{k_0}| = C \left\langle g_\tau^1 + g_\tau^2, \sum_{k\in D_\tau} |\lambda_k| e_k \right\rangle.$$

So (i) through (v) are satisfied.
Now let $\tau \in \Upsilon \setminus \max \Upsilon$ and suppose g_σ^1, g_σ^2 satisfy (i) through (v) for all $\sigma \succ \tau$. We split E_τ into $E_\tau^2 = \{k \in E_\tau \mid j_{k+1} \leq w(f_\tau)\}$ and $E_\tau^1 = E_\tau \setminus E_\tau^2$ to define

$$g_\tau^2 = \sum_{k\in E_\tau^2} \varepsilon e_k + \sum_{\sigma\in S(\tau)} g_\sigma^2.$$

This satisfies (ii) and (iii) by (b).
If $E_\tau^1 = \emptyset$, define $g_\tau^1 = \theta_{w(f_\tau)} \sum_{\sigma\in S(\tau)} g_\sigma^1$. Otherwise, order E_τ^1 as $k_1 < ... < k_l$ and define $g_\tau^1 = e_{k_1} + h_\tau$ where

$$h_\tau = \theta_{w(f_\tau)} \left(\sum_{i=2}^{l} e_{k_i} + \sum_{\sigma\in S(\tau)} g_\sigma^1 \right).$$

Again by (b) this satisfies (i) and (iii). Lastly we need to show that this choice of g_τ^1, g_τ^2 satisfies the inequality in (v). We have

$$\left|\left\langle f_\tau, \sum_{k\in D_\tau} \lambda_k x_k \right\rangle\right| \leq \left|\left\langle f_\tau \sum_{k\in E_\tau} \lambda_k x_k \right\rangle\right| + \theta_{w(f_\tau)} \sum_{\sigma\in S(\tau)} \left|\left\langle f_\sigma, \sum_{k\in D_\sigma} \lambda_k x_k \right\rangle\right|$$

$$\leq \sum_{k\in E_\tau^1} |\langle f_\tau, \lambda_k x_k \rangle| + \sum_{k\in E_\tau^2} |\langle f_\tau \lambda_k x_k \rangle| + \theta_{w(f_\tau)} \sum_{\sigma\in S(\tau)} C \left\langle g_\sigma^1 + g_\sigma^2, \sum_{k\in D_\sigma} |\lambda_k| e_k \right\rangle.$$

By the RIS condition (ii) we have for $k \in E_\tau^1$ that $|\langle f_\tau, \lambda_k x_k\rangle| \leq C\theta_{w(f_\tau)}|\lambda_k|$. Thus

$$\sum_{k\in E_\tau^1} |f_\tau(\lambda_k x_k)| \leq C\theta_{w(f_\tau)} \sum_{k\in E_\tau^1} |\lambda_k| = C\left\langle \theta_{w(f_\tau)} \sum_{i=1}^{l} e_{k_i}, \sum_{k\in D_\tau} |\lambda_k| e_k \right\rangle.$$

Whereas $\|x_k\|_\infty \leq \|x_k\| \leq C$ and RIS condition (i) imply that for $k \in E_\tau^2$ we have

$$|\langle f_\tau, \lambda_k x_k\rangle| \leq |\lambda_k| C\theta_{w(f_\tau)} \cdot \#\mathrm{supp}(x_k) \leq C\varepsilon|\lambda_k|$$

and thus

$$\sum_{k\in E_\tau^2} |\langle f_\tau, \lambda_k x_k\rangle| \leq C\left\langle \sum_{k\in E_\tau^2} \varepsilon e_k, \sum_{k\in D_\tau} |\lambda_k| e_k \right\rangle.$$

Now summing these parts up shows that our choice of g_τ^1, g_τ^2 is correct. □

Remark 3.4.5. If we set the above argument in the context of a mixed Tsirelson space of the form $T[(\mathcal{A}_{n_i}, \theta_i)_{i\in\mathbb{N}}]$ for some sequence $(n_i)_{i\in\mathbb{N}}$, we can conclude a more specific result which says that $h_1 \in W[(\mathcal{A}_{5n_i}, \theta_i)_{i\in\mathbb{N}}]$ with $w(h_1) = w(f)$. The constant 5 here comes from the length of the family of sets used to define h_τ, which is at most five times as long as the family of sets used to define f_τ.

We can now finally prove (up to a constant) the claim in the proof at the end of Section 3.2. This constant obviously does not matter for the original proof. The first part is a direct application of the basic inequality. The second part of the proof is an adaptation of a proof from [17].

Lemma 3.4.6. *For every $n \geq 2$ there exists a rapidly increasing sequence of ℓ_1^k-averages $x_1 < \ldots < x_n$ in Schlumprecht's space S such that $\|\sum_{k=1}^n x_k\|_S \leq \frac{9n}{F(n)}$.*

Proof. Let $x_1 < \ldots < x_n$ be given by Prop. 3.4.3 with $\varepsilon < \frac{1}{F(n)}$. Write $x = \sum_{k=1}^n x_k$. Let $f \in W[(\mathcal{A}_n, \frac{1}{F(n)})_{n\in\mathbb{N}}]$ with $w(f) \geq n$. Then if g_1, g_2 are given by the basic inequality, so $g_1 = e_{k_0} + h_1$ for some k_0 and h_1 with $\|h_1\|_\infty \leq \frac{1}{F(n)}$, we have

$$|\langle f, x\rangle| \leq 3\left\langle g_1 + g_2, \sum_{k=1}^n e_k \right\rangle \leq 3(1 + \varepsilon n + \frac{n}{F(n)}) \leq \frac{9n}{F(n)}.$$

The harder part is the case $w(f) < n$. For such f, define

$$\lambda(f) = \sum_{k=1}^n \frac{\#(\mathrm{supp}(x_k) \cap \mathrm{ran}(f))}{\#\mathrm{supp}(x_k)} \leq n.$$

Noting that $\|x_k\|_\infty \leq j_k^{-1}$ and $\#\mathrm{supp}(x_k) \geq j_k$, it holds that $\#(\mathrm{supp}(x_k) \cap \mathrm{ran}(f)) \leq j_k\lambda(f)$, and so

$$|\langle f, x\rangle| \leq \sum_{k=1}^n \|f\|_\infty \|x_k\|_\infty \#(\mathrm{supp}(x_k) \cap \mathrm{ran}(f)) \leq n\lambda(f).$$

By the above we have for $\lambda(f) \leq 9/F(n)$ that $|\langle f, x\rangle| \leq 9n/F(n)$. For convenience we extend F to $(0,\infty)$ by defining $F(x) = 1$ on $(0,1)$. Next we will show that, when $\lambda(f) \geq F(n)/F(j_1)$, we have

$$|\langle f, x\rangle| \leq \frac{9\lambda(f)}{F(\lambda(f))}.$$

For $w(f) = 0$, i.e. $\#\mathrm{supp}(f) = 1$, we have $\lambda(f) = \frac{1}{\#\mathrm{supp}(x_k)}$ for some k, which is greater than or equal to $\theta_{j_1}/\varepsilon \geq F(n)/F(j_1)$ by RIS condition (i). And further $|\langle f, x_k\rangle| \leq \frac{1}{j_k} \leq \frac{F(n)}{F(j_1)}$. So if the inequality fails, there must be an element of minimal support for which it does. Let f be that element. We can assume $w(f) \geq 2$. Write $m = w(f)$ and $f = \frac{1}{F(m)}\sum_{i=1}^{m} f_i$, $\lambda_i = \lambda(f_i)$ and $\lambda = \lambda(f)$. Then for each i, we have by minimality that either $\lambda_i < F(n)/F(j_1)$ or $|\langle f_i, x\rangle| \leq 9\lambda_i/F(\lambda_i)$. Let A be the set of $1 \leq i \leq m$ for which the first holds, and B the rest. Let $r = \#A$. Observing that $\sum_{i=1}^{m} \lambda_i \leq \lambda$, the concavity of $x/F(x)$ and Jensen's inequality give that

$$\sum_{i\in B} |\langle f_i, x\rangle| \leq (m-r)\frac{\frac{\lambda}{m-r}}{F(\frac{\lambda}{m-r})}.$$

This combined with A gives

$$|\langle f, x\rangle| \leq \frac{r}{F(m)}\frac{F(n)}{F(j_1)} + \frac{m-r}{F(m)}\frac{9\frac{\lambda}{m-r}}{F(\frac{\lambda}{m-r})}.$$

The key thing to note now is that the second term is slightly smaller than $9\lambda/F(\lambda)$, whereas the first term is very small. In fact we can pick j_1 to be as large as needed to ensure the result is always smaller than $9\lambda/F(\lambda)$, since m and r are bounded above by n. This is a contradiction with the minimality of f. So we can conclude that

$$|\langle f, x\rangle| \leq \frac{9\lambda}{F(\lambda)} \leq \frac{9n}{F(n)}.$$

If we further also pick j_1 such that $F(n)/F(j_1)$ is smaller than $9/F(n)$, the two cases we have distinguished above combine to prove the result. □

Remark 3.4.7. It is possible to find a formula for the lower bound on j_1 needed to ensure the inequality above. Gowers and Maurey [17] elect to include this formula as a constraint in their definition of a rapidly increasing sequence. Thus ensuring that every rapidly increasing sequence of length n satisfies the statement above. For our purposes we need the more generic definition of a rapidly increasing sequence.

We have gone through all this effort not only to prove the arbitrary distortability of Schlumprecht's space. The real goal was to showcase the concepts and proof ideas that we will use in future chapters. In the setting of Schlumprecht's space it was still feasible to go into all of the details for these proofs and thus give what we hope is more than enough context for the meaning and use of the constructions defined in this chapter. In the next chapters we will be more liberal in omitting such technical details.

Chapter 4

HI constructions

In this chapter we will cover the next step in the story of solving the scalar-plus-compact problem by discussing the solution of the unconditional basic sequence problem by Gowers and Maurey [17].

It is not obvious that this moves us closer to solving the scalar-plus-compact problem. So the first section will give some motivation by discussing the *hereditary indecomposability* property which is enjoyed both by the Gowers-Maurey space and by the final space of Argyros and Haydon. We will also show that this property implies that every operator is of the form $\lambda I + S$ where S is a strictly singular operator.

Next we will give the construction of the Gowers-Maurey space X_{GM}, which we can do by extending the constructions in the previous chapter. Then we will show that this space is indeed hereditarily indecomposable.

In light of the fact about strictly singular operators it became an open question whether X_{GM} was not already a solution to the scalar-plus-compact problem. The negative answer due to Schlumprecht and Androulakis can be found in [1]. It makes use of some more advanced results based on Ramsey theory about so-called spreading models. The key step is showing that the spreading model for X_{GM} is precisely Schlumprecht's space S. Then constructing a strictly singular operator which is not compact on S also gives one for X_{GM}. We will not cover this proof in this thesis.

4.1 Hereditary Indecomposability

In Banach space theory, a property that a space can have is often called *hereditary* if it also holds for all of the closed infinite-dimensional subspaces of that space. For instance, reflexivity is a hereditary property. The property of interest in this chapter is *hereditary indecomposability*:

Definition 4.1.1. An infinite-dimensional Banach space X is called *indecomposable* if there are no two infinite-dimensional subspaces Y, Z such that $X = Y \oplus Z$. In other

words, whenever $X = Y \oplus Z$, either Y or Z is finite-dimensional. The space is further called *hereditarily indecomposable* if this holds for all closed infinite-dimensional subspaces.

The following was already noted by Lindenstrauss when first stating the scalar-plus-compact problem in [22]:

Theorem 4.1.2. *A Banach space with the scalar-plus-compact property is also indecomposable.*

Proof. Suppose X has the scalar-plus-compact property and suppose Y_1, Y_2 are infinite-dimensional subspaces of X such that $Y_1 \cap Y_2 = \{0\}$ and $X = Y_1 \oplus Y_2$. Then Y_1 and Y_2 must be closed, and the projections $P_1 : X \to Y_1$ and $P_2 : X \to Y_2$ are linear and bounded. We find $P_1 = \lambda I + K$ for some $\lambda \in \mathbb{R}$ and compact operator K. However we must have $P_1 Y_2 = \{0\}$. So for any non-zero $y \in Y_2$, we have $Ky = -\lambda y$, which shows that $Y_2 \subseteq KX$ and thus K can not be compact. □

This is already motivation enough to investigate in the direction of indecomposable spaces. The first known space with this property is that of Gowers and Maurey, who discovered that their solution to the unconditional basic sequence problem was in fact hereditarily indecomposable.

The following equivalence is useful when proving a space has the HI property:

Theorem 4.1.3. *Let X be an infinite-dimensional Banach space. Then X is hereditarily indecomposable if and only if for every pair of infinite-dimensional closed subspaces Y, Z of X and every $\delta > 0$ there exist $y \in Y$ and $z \in Z$ such that $\|y - z\| \leq \delta \|y + z\|$.*

Proof. ($\Rightarrow$) If X is HI then $Y \cap Z \neq \{0\}$. Thus we can pick $y = z \neq 0$ in $Y \cap Z$.
($\Leftarrow$) Suppose Y, Z form a decomposition of a closed subspace $Y \oplus Z$ of X. Then the projection $P : Y \oplus Z \to Y$ must be continuous. However, by the assumption, for any $\delta > 0$ we can find y, z such that

$$\|P(y - z)\| = \|y\| \geq \frac{1}{2}\big|\|y + z\| - \|y - z\|\big| \geq \frac{1 - \delta}{2\delta}\|y - z\|.$$

So $\|P\| \geq \frac{1-\delta}{2\delta}$ for any $\delta > 0$, which shows that P is not bounded. □

For the rest of this section we will follow sections 4 and 5 of [17] covering properties of hereditarily indecomposable spaces (HI spaces), culminating in a proof that all operators on a complex HI space are of the form $\lambda I + S$ for S strictly singular. This result however does not directly extend to real HI spaces, so Gowers and Maurey give a direct proof of this property in the case of X_{GM}. We will however prove a slightly weaker result for real HI spaces, namely that every operator is either strictly singular or Fredholm with index 0.

We will write $\sigma(T)$ for the spectrum of an operator on a complex space. Further we consider the following auxiliary subsets of $\mathbb{C}$:

Definition 4.1.4. Let T be a bounded linear operator on a complex Banach space X. We say $\lambda \in \mathbb{C}$ is *infinitely singular* for T if for every $\varepsilon > 0$ there is an infinite-dimensional subspace Y such that the restriction of $T - \lambda I$ to Y has norm at most ε. We define

$$F_T = \{\lambda \in \mathbb{C} \mid \lambda \text{ not infinitely singular for } T\}.$$

Lemma 4.1.5. *With the above definition we have the following properties:*

(i) F_T *is open in* $\mathbb{C}$.

(ii) If $\lambda \in F_T$ *and if* $(x_n)_{n\in\mathbb{N}}$ *is a bounded sequence in* X *such that* $(T - \lambda I)x_n$ *converges then* $(x_n)_{n\in\mathbb{N}}$ *has a convergent subsequence.*

(iii) If $\lambda \in F_T$ *and* $Y \subseteq X$ *is a closed subspace, then* $(T - \lambda I)Y$ *is also closed.*

(iv) If $\lambda \in \partial\sigma(T) \cap F_T$ *then* λ *is an eigenvalue of* T *with finite multiplicity.*

(v) If $\lambda \in \partial\sigma(T) \cap F_T$ *then* λ *is an isolated point of* $\sigma(T)$.

Proof. The proofs are readily found in [17]. □

Lemma 4.1.6. *If* X *is infinite-dimensional and* $T : X \to X$ *is linear and bounded such that* $\sigma(T) = \{0\}$, *then* 0 *is infinitely singular for* T.

Proof. Suppose $0 \in F_T$. Then $\ker(T)$ must be finite-dimensional, since the norm of T restricted to its kernel is always 0. So T is an isomorphism on a subspace Z of finite codimension. This implies $\inf_{z\in Z} \|Tz\|/\|z\| > 0$, so by considering a multiple we can assume that $\|Tz\| \geq \|z\|$ for all $z \in Z$. But then also $\|T^k z\| \geq \|z\|$ for all $z \in Z$, so $\|T^k\| \geq 1$. This is a contradiction with the spectral radius of T being 0. □

Next we will apply the above lemmas to show that in the infinite-dimensional case there is always at least one infinitely singular value.

Lemma 4.1.7. *If* X *is infinite-dimensional then* $F_T \neq \mathbb{C}$.

Proof. Suppose $F_T = \mathbb{C}$. Then by Lemma 4.1.5(v) all points in $\partial\sigma(T)$ are isolated in $\sigma(T)$ so $\sigma(T)$ is finite. Write $\sigma(T) = \{\lambda_1, ..., \lambda_n\}$ and define the polynomial $P(z) = \prod_{i=1}^n (z - \lambda_i)$. For $\lambda \neq 0$ we have $(P - \lambda)(z) = \prod_{i=1}^n (z - \mu_i)$ where $\mu_i \notin \sigma(T)$ for every i. So $(P - \lambda)(T) = P(T) - \lambda I$ as a product of n invertible operators is invertible, and it follows $\sigma(P(T)) = \{0\}$. Then by Lemma 4.1.6, 0 is infinitely singular for $P(T)$. For $k \geq 1$ let Y_k be an infinite-dimensional subspace of X such that $\|P(T)\|_{Y_k} < \frac{1}{k}$. Pick a normalized basic seqeuence $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in Y_k$ for each k. Then $P(T)x_k \to 0$, but since $P(T) = (T - \lambda_1 I)...(T - \lambda_n I)$, we can apply Lemma 4.1.5(ii) n times to conclude that $(x_k)_{k\in\mathbb{N}}$ has a convergent subsequence, which is a contradiction. □

Theorem 4.1.8. *If* X *is a complex HI space then every bounded linear operator* $T : X \to X$ *can be written as* $T = \lambda I + S$ *where* $\lambda \in \mathbb{C}$ *and* S *is strictly singular.*

Proof. The key claim is that the HI property together with the above lemma implies $\mathbb{C} \setminus F_T = \{\lambda\}$. Indeed, suppose $\lambda_1 \neq \lambda_2$ are infinitely singular for T. Let $0 < \varepsilon < |\lambda_1 - \lambda_2|$ and let Y_1, Y_2 be infinite-dimensional subspaces of X such that $T - \lambda_i I$ restricted to Y_i has norm at most $\frac{1}{2}\varepsilon$. By passing to a further subspace we may assume these are closed. But then $Y_1 \cap Y_2 = \{0\}$ as otherwise we would have for all non-zero $x \in Y_1 \cap Y_2$ that

$$\|(T - \lambda_1 I)x - (T - \lambda_2 I)x\| = |\lambda_1 - \lambda_2| \cdot \|x\| > \varepsilon \|x\|,$$

while by the triangle inequality this should be at most $\varepsilon\|x\|$. Thus $Y_1 \oplus Y_2$ forms a decomposition of a closed subspace of X. So by hereditary indecomposability either Y_1 or Y_2 is finite-dimensional which is a contradiction.

Now let λ be the unique complex number that is infinitely singular for T. We need to show that $T - \lambda I$ is strictly singular, i.e. not an isomorphism on any infinite-dimensional subspace. So suppose $T - \lambda I$ is an isomorphism on some infinite-dimensional subspace Y. By passing to a subspace and a suitable multiplication we may assume Y is closed and $\|T - \lambda I\|_Y = 1$. Similarly let Z be a closed infinite-dimensional subspace such that $\|T - \lambda I\|_Z < 1$. Then it again follows that $Y \cap Z = \{0\}$. Which by hereditary indecomposability is a contradiction. □

Let us now consider the real case. Recall that an operator $T : X \to X$ is called *Fredholm* if there exists an operator S such that both $TS - I$ and $ST - I$ are compact. Equivalently, if the kernel and cokernel of T are finite-dimensional. An operator is *semi-Fredholm* if one of the two holds. One obtains the Fredholm index by $\operatorname{ind}(T) = \dim(\ker(T)) - \dim(\ker(T^*))$. This index is continuous as a map from the set of semi-Fredholm operators with the operator topology to $\mathbb{Z} \cup \{\pm\infty\}$ with the discrete toplogy. See e.g. [11, Chapter IX] for a treatment.

The Fredholm index is also invariant under "small" perturbations. More precisely, an operator U such that $\operatorname{ind}(T) = \operatorname{ind}(T + U)$ for all Fredholm operators T is called an *inessential operator*. It is known [9] that the ideal of strictly singular operators is contained in the ideal of inessential operators. However these are not equal in general. As such the result for real HI spaces we obtain below is slightly weaker than the result we obtained for complex HI spaces. The proof we give is inspired by that of Maurey [27] and uses the complexification, which is defined as follows: [29]

Definition 4.1.9. Let X be a real Banach space. The *complexification* of X is the complex Banach space $X_{\mathbb{C}} = X \oplus iX$ with norm

$$\|x + iy\|_{\mathbb{C}} = \sup \{\|x \cos\theta + y \sin\theta\| \mid \theta \in [0, 2\pi]\}.$$

Whenever $T : X \to X$ is an operator one obtains its complexification $T_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ by $T_{\mathbb{C}}(x + iy) = Tx + iTy$.

This norm in particular has the property that $\|x\|_{\mathbb{C}} = \|x\|$ whenever $x \in X$.

Lemma 4.1.10. *Let X be a real HI space and $T : X \to X$ a bounded operator. If $S = T_{\mathbb{C}}$ then $\mathbb{C} \setminus F_S$ contains at most a pair of complex conjugate elements. That is, $\mathbb{C} \setminus F_S = \{\lambda\}$ for some $\lambda \in \mathbb{R}$ or $\mathbb{C} \setminus F_S = \{\lambda, \overline{\lambda}\}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. We first show that $\mathbb{C} \setminus F_S$ contains at most one real element. Indeed, if $\lambda_1 \neq \lambda_2$ are real elements not in F_S, then we can find closed subspaces Y_1, Y_2 in $X_{\mathbb{C}}$ such that $S - \lambda_i I$ restricted to these subspaces has norm at most $\frac{1}{2}\varepsilon$. However then we can restrict Y_1 and Y_2 to X and obtain subspaces Y_1', Y_2' such that $S - \lambda_i I$ restricted to these subspaces is equal to $T - \lambda_i I$ and has norm at most $\frac{1}{2}\varepsilon$. As we have shown before this is a contradiction with the HI property of X if we choose ε small enough.

Now let $\lambda, \mu \in \mathbb{C} \setminus F_S$. We may assume that λ is not real, and we suppose that $\mu \notin \{\lambda, \overline{\lambda}\}$. Define

$$T_\lambda = T^2 - 2\mathrm{Re}(\lambda)T + |\lambda|^2 I.$$

Then we have a decomposition of $(T_\lambda)_{\mathbb{C}}$ by

$$(S - \lambda I)(S - \overline{\lambda} I)(x + iy) = T_\lambda x + iT_\lambda y.$$

Now if $\varepsilon > 0$ we can find closed subspaces Y_1, Y_2 of X such that both $\|S - \lambda I\|_{Y_1} < \varepsilon$ and $\|S - \overline{\lambda} I\|_{Y_2} < \varepsilon$. Then T_λ restricted to $Y_1 \cap Y_2$ has norm at most $2\varepsilon^2$. Note that $Y_1 \cap Y_2$ is again infinite-dimensional, since otherwise removing the finite-dimensional intersection from one of the two spaces would give a decomposition of a closed subspace. The same holds for T_μ, and taking the intersection again gives an infinite-dimensional subspace Y_ε. Now $T_\lambda - T_\mu = aT + bI$ for some $a, b \in \mathbb{R}$, with a and b not both 0. This operator has norm at most $4\varepsilon^2$ on Y_ε. So if ε is such that $4\varepsilon^2 < b$ this implies that $a \neq 0$.

It follows that $-b/a$ is "close to" a root of the polynomial $(z - \lambda)(z - \overline{\lambda})$. Which would be a contradiction with $\lambda \notin \mathbb{R}$. Let us make this more precise. Write $U = -\frac{b}{a} I$. Then $\|T - U\|_{Y_\varepsilon} < 4\varepsilon^2/a$ and

$$\|T_\lambda - U_\lambda\| \leq \|T - U\|^2 + 2\mathrm{Re}(\lambda)\|T - U\|.$$

So there exists some constant C dependent on λ and a such that

$$\|T_\lambda - U_\lambda\|_{Y_\varepsilon} < C\varepsilon^2$$

And so

$$\|U_\lambda\|_{Y_\varepsilon} < \|T_\lambda\| + C\varepsilon^2 < (C + 2)\varepsilon^2.$$

In particular, we can take ε such that $\|U_\lambda\|_{Y_\varepsilon} < 2|\frac{b}{a} + \frac{b^2}{a^2}|$. Whereas

$$\|U_\lambda\| = \left\| \left(\frac{b}{a}\right)^2 I + 2\frac{b}{a} I + \left(\frac{b}{a}\right)^2 I \right\| = 2\left|\frac{b}{a} + \frac{b^2}{a^2}\right|.$$

This is indeed a clear contradiction. Thus, the assumption that $\mu \notin \{\lambda, \overline{\lambda}\}$ was faulty. We conclude that $\mathbb{C} \setminus F_S = \{\lambda, \overline{\lambda}\}$ for some $\lambda \in \mathbb{C}$. □

Theorem 4.1.11. *If X is a real HI space and $T : X \to X$ is a linear operator, then T is either strictly singular or Fredholm with index 0.*

Proof. Again write $S = T_{\mathbb{C}}$. By the above lemma we need to consider three cases:

(i) $\mathbb{C} \setminus F_S = \{0\}$ in which case T is strictly singular.

(ii) $\mathbb{C} \setminus F_S = \{\lambda\}$ for some non-zero $\lambda \in \mathbb{R}$, in which case $T - \lambda I$ is strictly singular and $T = \lambda I + (T - \lambda I)$ is Fredholm with index 0.

(iii) $\mathbb{C} \setminus F_S = \{\lambda, \overline{\lambda}\}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then $S - \nu I$ is Fredholm with index 0 for all $\nu \notin \{\lambda, \overline{\lambda}\}$. Thus in particular T is Fredholm with index 0.

In case (i) and (ii) the statements about T follow for S as in the proof of Theorem 4.1.8. Then taking the restriction to X gives the result. For (iii), note that $\nu \notin F_S$ implies $S - \nu I$ is at least semi-Fredholm. By Lemma 4.1.5(v) we also have that $\sigma(S)$ outside of $\{\lambda, \overline{\lambda}\}$ consists of isolated points. So for each $\nu \notin \{\lambda, \overline{\lambda}\}$, there is a neighbourhood of U of ν such that $S - \mu I$ is invertible for all $\mu \in U$. In particular $S - \mu I$ is Fredholm with index 0 for all $\mu \in U$. By continuity of the Fredholm index this then extends to $S - \nu I$. For $\nu = 0$ we can then take the restriction to X. □

4.2 The Gowers-Maurey construction

The Gowers-Maurey construction [17] was the first known example of a hereditarily indecomposable Banach space. Originally the goal was to solve the question of whether every Banach space contains an unconditional basic sequence. However Gowers and Maurey obtained a stronger result by constructing an HI space, which consequently does not contain an unconditional basic sequence.

In Section 3.3 we defined a norm on c_{00} using a norming set. In defining this norming set we started with $\{\pm e_k\}$ and then required that the norming set should be closed under certain operations. The Gowers-Maurey construction is a direct extension of this technique in which we add the requirement that the norming set is rationally convex. Also there is another type of operation on so-called "special sequences" which is included in the construction of this norming set.

To stay with the story of tree analyses we will follow the notation that Argyros used in his generalization of the Gowers-Maurey construction [4]. In particular we will give a proof of the "combinatorial lemma" for the Gowers-Maurey space that directly uses induction on the trees. However we will simplify some of Argyros' definitions to stay closer to the original construction.

One such simplification lies in the following: In the applications of mixed Tsirelson spaces that we will see in the remainder of this work, there is always a certain lacunary[1] subset of the natural numbers in play. These are usually sequences where each

[1] The term "lacunary" means "having large holes".

subsequent entry should be at least some exponential larger than the previous entry.

For now we will stick with the lacunary set used by Gowers and Maurey along with a function $F \in \mathfrak{F}$ to define the weights. For the rest of this chapter we fix $F(x) = \log_2(1+x)$, and this lacunary set will be denoted $(n_i)_{i\in\mathbb{N}}$ and is defined to be an increasing sequence of natural numbers such that $F(n_1) > 256$ and $\log\log\log n_{i+1} \geq 4n_i$. The associated weights are $\theta_i = \frac{1}{F(i)}$.

Now let us turn to the aforementioned "special sequences". These arise from a certain coding function σ which was used by Maurey and Rosenthal in their earlier construction of a weakly null sequence with no unconditional subsequence. [28] The precise definition of this function includes a constraint which will be useful later on.

Definition 4.2.1. Let $Q = \{x \in c_{00} \mid \forall n \in \mathbb{N} : x(n) \in \mathbb{Q}, |x(n)| \leq 1\}$. We write Q_s for the set of finite sequences of successive elements of Q. Define the *coding function* to be an injection $\sigma : Q_s \to \{n_{2i} \mid i \in \mathbb{N}\}$ such that whenever $x_1 < ... < x_n$ forms a sequence in Q_s, $k = \sigma(x_1, ..., x_n)$ and $x = \sum_{i=1}^n x_i$, then $\theta_k \cdot \max\operatorname{supp}(x) \leq \|x\|_\infty$.

In the remainder of this section we will fix such a coding function σ and use it to define the special sequences:

Definition 4.2.2. Given a (rationally) convex set $D \subseteq c_{00}$, we define $A_m(D)$ to be the set of vectors of the form $f(m)^{-1}\sum_{i=1}^m x_i$ where $x_1 < ... < x_m$ and $x_i \in D$ for each i. A *special sequence* for D of length k is then a sequence $g_1 < ... < g_k$ such that $g_1 \in A_{n_{2k}}(D)$ and $g_{i+1} \in A_{\sigma(g_1,...,g_i)}(D)$ for each $1 \leq i \leq k-1$. A *special functional* is then a functional of the form $f(k)^{-\frac{1}{2}}\sum_{i=1}^k g_i$ where $g_1 < ... < g_k$ is a special sequence.

We can now define the norming set and thus define the space X_{GM}:

Definition 4.2.3. Let $W_{GM} \subseteq c_{00}$ be the smallest set satisfying:

(i) It contains e_k for all $k \in \mathbb{N}$.

(ii) It is rationally convex.

(iii) It is symmetric (if $f \in W_{GM}$ then $-f \in W_{GM}$).

(iv) If $f \in W_{GM}$ and E is an interval in $\mathbb{N}$ then also $Ef \in W_{GM}$.

(v) Whenever $f_1, ..., f_n \in W_{GM}$ are $\mathcal{A}_n$-admissible for some $n \in \mathbb{N}$, then also $\theta_n(f_1 + ... + f_n) \in W_{GM}$.

(vi) Whenever $g_1, ..., g_{n_{2i+1}}$ is a special sequence in W_{GM} of length n_{2i+1} for some $i \in \mathbb{N}$, then also $\sqrt{\theta_{n_{2i+1}}}(g_1 + ... + g_{n_{2i+1}}) \in W_{GM}$

The Gowers-Maurey space X_{GM} is then the completion of c_{00} under $\|\cdot\|_{W_{GM}}$.

At this point some observations are in order. The first thing to note is the distinction between even and odd elements of the sequence $(n_i)_{i\in\mathbb{N}}$. A special sequence in W_{GM} must be of odd weight, whereas the elements of that sequence must be of even weight. Further, we no longer have a defined weight for all of the elements of W_{GM}. Conditions (ii) through (iv) give a kind of element that does not have a weight. Besides, we have taken the square root of the weight in the case of special sequences.

For this reason we now distinguish four types of elements in W_{GM}. An element $f \in W_{GM}$ has type 0 if $f = \pm e_k$ for some $k \in \mathbb{N}$. It is of type I if it is the result of condition (v), with weight $w(f)$ as before. If it is the result of one of the conditions (ii) through (iv) it is of type II. Lastly it is of type III if it is a special sequence. For the weight we will ignore the square root and write $w(f) = n_{2i+1}$ for a special sequence of length n_{2i+1}.

The special sequences obviously are the most significant change in this construction compared to the previous chapter. We will note here a simple but distinctive "tree-like" property of the special sequences:

Lemma 4.2.4. *Let $(f_k)_{k=1}^m$ and $(g_k)_{k=1}^m$ be two distinct special sequences of the same length. Then for $i \neq j$ we have $w(f_i) \neq w(g_j)$ and there exists t such that $f_k = g_k$ for $k < t$ and $w(f_k) \neq w(g_k)$ for $k > t$.*

Proof. This follows from the injectivity of σ. Indeed, first suppose $i \neq j$, then $\sigma(f_1, ..., f_{i-1})$ is always different from $\sigma(g_1, ..., g_{j-1})$. Next suppose $w(f_t) = w(g_t)$ for some t. Then apparently $\sigma(f_1, ..., f_{t-1}) = \sigma(g_1, ..., g_{t-1})$. Thus by injectivity we must have $f_k = g_k$ for $k < t$. However since the sequences are distinct, there must be a maximal $t < m$ for which this holds. And if such a t does not exist, then we may take $t = 1$ and the statement is still true. □

By very similar arguments as we have seen in Section 3.4, block subspaces of X_{GM} contain ℓ_1^k-averages and rapidly increasing sequences of these averages. Furthermore, the basic inequality (Theorem 3.4.4) still holds for all f of type 0, I or II. One just needs to add a case distinction for functionals of type II. The resulting proof is the one found in [4].

Lastly, the statement in Lemma 3.4.6 can be modified to hold in X_{GM}. The modification is that the length of the rapidly increasing sequence must lie within the interval $[\log(n_{2i}), \exp(n_{2i})]$ for some $i \in \mathbb{N}$, so that it is "far away" from the previous and next special sequence. This is what we will prove in the next few pages. Afterwards we will be able to prove that X_{GM} is an HI space by constructing elements satisfying the equivalent statement in Theorem 4.1.3.

We will need to extend the definition of a tree analysis:

Definition 4.2.5. Let $f \in W_{GM}$. A family $(f_\sigma)_{\sigma\in\Upsilon}$, with Υ a finite tree is called a *tree analysis* of f if the following conditions hold:

(i) $f_\lambda = f$.

(ii) If $\sigma \in \max \Upsilon$ then $f_\sigma = \pm e_k$ for some $k \in \mathbb{N}$.

(iii) If $\tau \preceq \sigma$ then $\mathrm{ran}(f_\sigma) \subseteq \mathrm{ran}(f_\tau)$.

And if $\sigma \notin \max \Upsilon$ then one of the following hold:

(i) f_σ is of type I with $f_\sigma = \theta_{w(f_\sigma)} \sum_{\tau \in S(\sigma)} f_\tau$, where $(f_\tau)_{\tau \in S(\sigma)}$ is $\mathcal{A}_{w(f_\sigma)}$-admissible.

(ii) f_σ is of type II with $f_\sigma = \sum_{\tau \in S(\sigma)} \lambda_\tau f_\tau$, where $(\lambda_\tau)_{\tau \in S(\sigma)}$ is a sequence of rational numbers in $(0,1)$ such that $\sum_{\tau \in S(\sigma)} \lambda_\tau = 1$, and where each f_τ is of type 0, I or III.

(iii) f_σ is of type III with $f_\sigma = \sqrt{\theta_{w(f_\sigma)}} \sum_{\tau \in S(\sigma)} f_\tau$, where $(f_\tau)_{\tau \in S(\sigma)}$ is a special sequence.

As before every $f \in W_{GM}$ admits a tree analysis. The only thing to note for the type II functionals is that chaining two rational convex combinations will again give a rational convex combination, thus we can indeed assume that the successors of a type II functional are not of type II. Further we will need the following lemma:

Lemma 4.2.6. *Let $W = W[(\mathcal{M}_n, \theta_n)]$ be a mixed Tsirelson norming set, and let W' be the norming set obtained by adding the condition that W' is rationally convex. Then W and W' induce the same norm on c_{00}.*

Proof. Note that $W' = \mathrm{conv}_{\mathbb{Q}}(W)$. It is clear that $\|\cdot\|_W \leq \|\cdot\|_{W'}$ since $W \subseteq W'$. To show the other inequality, let $f \in W'$ with $f = \sum_{k=1}^n \lambda_k f_k$ for some $f_1, ..., f_n \in W$ and $\lambda_1, ..., \lambda_n \in \mathbb{Q} \cap (0,1)$ such that $\sum_{k=1}^n \lambda_k = 1$. Suppose that for all k we have $|\langle f_k, x\rangle| < |\langle f, x\rangle|$. Then

$$|\langle f, x\rangle| \leq \sum_{k=1}^n \lambda_k |\langle f_k, x\rangle| < |\langle f, x\rangle| \sum_{k=1}^n \lambda_k = |\langle f, x\rangle|,$$

which is a contradiction. Thus there is at least one f_k with $|\langle f_k, x\rangle| \geq |\langle f, x\rangle|$. This shows $\|\cdot\|_W \geq \|\cdot\|_{W'}$. □

Now for the main combinatorial lemma:

Lemma 4.2.7. *Let $n \in [\log(n_{2i}), \exp(n_{2i})]$ for some $i \in \mathbb{N}$ and $\varepsilon < \frac{1}{F(n)}$. Then every block subspace of X_{GM} contains a $(3, \varepsilon)$-RIS of ℓ_1^k-averages $x_1 < ... < x_n$ such that $\|\sum_{k=1}^n x_k\|_{GM} \leq 9n/F(n)$.*

Proof. We start by defining $\Phi : [1, \infty) \to [1, \infty)$ to be the following function:

$$\Phi(x) = \begin{cases} \sqrt{F(x)} & \text{if } x = n_{2i+1} \text{ for some } i \in \mathbb{N}, \\ F(x) & \text{otherwise.} \end{cases}$$

Then define h to be the submultiplicative hull of Φ:

$$h(x) = \inf \{\Phi(x_1) \cdot ... \cdot \Phi(x_k) \mid k \in \mathbb{N}, x_1, ..., x_k \geq x\}$$

Next let $H(x) = x/h(x)$ and define H' to be the concave envelope of H:

$$H'(x) = \sup \{\lambda H(y) + (1-\lambda)H(z) \mid \lambda \in [0,1], \lambda y + (1-\lambda)z = x\}.$$

Then lastly define $G(x) = x/H'(x)$. Doing some calculations shows that G is submultiplicative and that $G \in \mathfrak{F}$ with $\sqrt{F} \leq G \leq \Phi \leq F$. We define the norming set $W = W[(\mathcal{A}_n, \frac{1}{G(n)})_{n\in\mathbb{N}}]$. By the argument in Lemma 3.4.6 it follows that there exists a $(3,\varepsilon)$-RIS of ℓ_1^k-averages $x_1 < ... < x_n$ such that $\|\sum_{k=1}^n x_k\|_W \leq 9n/G(n)$. To finish we will need to show that $\|\cdot\|_{GM} \leq \|\cdot\|_W$ and that $G(n) = F(n)$ on $[\log(n_{2i}), \exp(n_{2i})]$.

By Lemma 4.2.6 we can also consider $W' = \text{conv}_{\mathbb{Q}}(W)$ and show that $\|\cdot\|_{GM} \leq \|\cdot\|_{W'}$. Indeed, if $f \in W_{GM}$ with tree analysis $(f_\sigma)_{\sigma\in\Upsilon}$, then we can construct $g \in W'$ with $|g(n)| \geq |f(n)|$ for all $n \in \mathbb{N}$ as follows: For $\sigma \in \max \Upsilon$ we take $g_\sigma = f_\sigma$. Next for f_σ of type I or III, thus $f_\sigma = \frac{1}{\Phi(w(f_\sigma))}\sum_{\tau\in S(\sigma)} f_\tau$, we take $g_\sigma = \frac{1}{G(w(f_\sigma))}\sum_{\tau\in S(\sigma)} g_\tau$. Lastly for f_σ of type II, thus $f_\sigma = \sum_{\tau\in S(\sigma)} \lambda_\tau f_\tau$ we take $g_\sigma = \sum_{\tau\in S(\sigma)} \lambda_\tau g_\tau$. Then $g = g_\lambda$ is the desired element of W'.

The last claim follows from an extensive calculation using the definition of F, G and the assumption that $\log\log\log n_{i+1} \geq 4n_i$. We refer to [17, Lemma 9] for the details. □

The key to the HI proof will now be to construct a "dependent sequence" in X_{GM}. We will see that this entails a sequence of RIS vectors which itself is a rapidly increasing sequence. For this reason, Zsák calls it a "complexity-2" rapidly increasing sequence [34]. The precise definition we use comes in two parts:

Definition 4.2.8. For $C > 1$, $j \in \mathbb{N}$ and $\theta \in [0,1]$, a pair $(x, f) \in X_{GM} \times W_{GM}$ is called a (C, j, θ)-*exact pair* if

(i) f is of type I with $w(f) = n_j$.

(ii) $1 \leq \|x\| \leq C$ and for $g \in W_{GM}$ we have $|\langle g, x\rangle| \leq 3C\theta_{w(g)}$ if $w(g) < n_j$ and $|\langle g, x\rangle| \leq 3C\theta_{n_j}$ if $w(g) \geq n_{j+1}$.

(iii) $\text{ran}(x) = \text{ran}(f)$ and $\langle f, x\rangle = \theta$.

The second condition is satisfied in particular if $x = c\frac{F(n_j)}{n_j}\sum_{i=1}^{n_j} x_i$, where $(x_i)_{i=1}^{n_j}$ is a (C,ε)-RIS of ℓ_1^k-averages with $\varepsilon < \theta_{n_j}^2$ and $\frac{1}{3} \leq |c| \leq 1$. The first of the inequalities in condition (ii) is seen to be satisfied by noting that in the second part of the proof of Lemma 3.4.6 we could have also concluded that $|\langle f, x\rangle| \leq 9n/F(m)F(n)$, by including this in each step. The second is simpler, it follows from the basic inequality and the observation that $F(n_{j+1}) \geq F(n_j)^2$. We will use this in our construction of an exact pair later on.

Definition 4.2.9. For $C > 1$, $\theta \in [0,1]$, a sequence $(x_k, f_k)_{k=1}^{n_{2i+1}}$ with $x_k \in X_{GM}$, $f_k \in W_{GM}$ is called a (C,θ)-*dependent sequence* if there exists a sequence $(2j_k)_{k=1}^{n_{2i+1}}$ such that

(i) $(f_k)_{k=1}^{n_{2i+1}}$ is a special sequence in W_{GM}.

(ii) Each pair (f_k, x_k) is a $(C, 2j_k, \theta)$-exact pair.

Lemma 4.2.10. *Let $(x_k, f_k)_{k=1}^n$ be a (C,θ)-dependent sequence in X_{GM}. Then*

(i) $(x_k)_{k=1}^n$ forms a $(3C, \theta_n)$-RIS.

(ii) $\| \sum_{k=1}^n x_k \| \geq \theta n F(n)^{-1/2}$.

(iii) $\| \sum_{k=1}^n (-1)^{k-1} x_k \| \leq 3CnF(n)^{-1}$.

Proof. (i) Since $(f_k)_{k=1}^n$ is a special sequence it follows that $n_{2j_{k+1}} = \sigma(f_1, ..., f_k)$ and

$$\theta_{n_{2j_{k+1}}} \#\mathrm{supp}(x_k) \leq \theta_{n_{2j_{k+1}}} \#\mathrm{ran}(f_k) \leq \|f_1\|_\infty \leq \theta_{n_{2j_1}}$$

Again by the definition of a special sequence we have that $j_1 \geq n$ so it follows that this is smaller than θ_n. And if $w(f) = i < n_{2j_k}$ then $|\langle f, x_k \rangle| \leq 3C\theta_i$. So both conditions of an RIS are satisfied for the sequence $j'_k = n_{2j_k}$.
(ii) This is an easy consequence of the fact that $(f_k)_{k=1}^n$ is a special sequence:

$$\left\| \sum_{k=1}^n x_k \right\| \geq F(n)^{-1/2} \sum_{k=1}^n f_k(x_k) = \theta n F(n)^{-1/2}$$

(iii) Write $x = \sum_{k=1}^n (-1)^{k-1} x_k$ and $f = \sum_{k=1}^n f_k$. By $\mathrm{ran}(x_k) = \mathrm{ran}(f_k)$ and $\langle f_k, x_k \rangle = \theta$ we have for all intervals $E \subseteq \mathbb{N}$ that $|\langle f, Ex \rangle| \leq \theta$. Let $g_1, ..., g_n$ be another special sequence of length n, $g = \sum_{k=1}^n g_k$. Let t be maximal such that $f_t = g_t$ or 0 if no such t exists. Then by Lemma 4.2.4 all preceding entries must be equal as well, so we can choose an interval E such that

$$\left| \left\langle \sum_{k=1}^t g_k, x \right\rangle \right| = |\langle f, Ex \rangle| \leq \theta.$$

Further we have $|\langle g_i, x_j \rangle| < n^{-2}$ whenever i or j is greater than $t+1$. Indeed, we have $w(g_i) \neq w(f_j)$ and so by the second condition of an exact pair it holds that $|\langle g_i, x_j \rangle| \leq 3C\theta_{w(g_i) \wedge w(f_j)}$. By the definition of a special sequence we must have $\min\{w(g_i), w(f_j)\} \geq n_{2n}$ so this is certainly less than n^{-2}. Thus in total

$$|\langle g, x \rangle| \leq \theta + |\langle g_{t+1}, x_{t+1} \rangle| + n^2 n^{-2} \leq 2 + C \leq 3Cn/F(n).$$

The other cases (where g is not a special sequence of length n) follow by applying the argument of Lemma 4.2.7 to

$$\Phi'(x) = \begin{cases} \sqrt{F(x)} & \text{if } x = n_{2i+1} \neq n \text{ for some } i \in \mathbb{N}, \\ F(x) & \text{otherwise.} \end{cases}$$

□

Theorem 4.2.11. *X_{GM} is an HI space.*

Proof. Let Y, Z be two infinite-dimensional subspaces of X_{GM} and let $\delta > 0$. We can assume these are block subspaces. If we construct a (C, θ)-dependent sequence $(x_k, f_k)_{k=1}^{n_{2i+1}}$ in X_{GM} such that $x_{2k} \in Y$ and $x_{2k-1} \in Z$ for all k, then if y is the sum of all the evens, z the sum of all the odds, we will have by Lemma 4.2.10 that $\|y - z\| \leq 3C\theta^{-1}F(n_{2i+1})^{-1/2}\|y + z\|$. Thus choosing i large enough such that this is less than $\delta\|y + z\|$ will give the result by Theorem 4.1.3.

Write $k = n_{2i+1}$ for some i large enough. Let $x_{1,1}, ..., x_{1,n_{2k}} \in Y$ be a $(3, \varepsilon_1)$-RIS of ℓ_1^n-averages given by Lemma 4.2.7, where $\varepsilon_1 < \frac{1}{F(n_{2k})}$. We have $\|x_{1,j}\| \geq 1 - \varepsilon_1 > \frac{1}{2}$ for all j. So since $\operatorname{ball} \ell_1(\mathbb{Q}) \cap c_{00} \subseteq W_{GM}$ we can by density pick $f_{1,j}$ such that $\operatorname{ran}(f_{1,j}) = \operatorname{ran}(x_{1,j})$ and

$$\left|\langle f_{1,j}, x_{1,j}\rangle - \frac{1}{2}\right| < n_{2k}^{-1}.$$

From this it follows that if we take $f_1 = \theta_{n_{2k}} \sum_{j=1}^{n_{2k}} f_{1,j}$, we have

$$\left|\left\langle f_1, \frac{F(n_{2k})}{n_{2k}} \sum_{j=1}^{n_{2k}} x_{1,j} \right\rangle - \frac{1}{2}\right| < 1.$$

Thus there exists $c \in \mathbb{R}$ with $\frac{1}{3} \leq |c| \leq 1$ such that $x_1 = c\frac{F(n_{2k})}{n_{2k}} \sum_{j=1}^{n_{2k}} x_{1,j}$ gives $\langle f_1, x_1\rangle = \frac{1}{2}$. Since also $\operatorname{ran}(f_1) = \operatorname{ran}(x_1)$, this means (x_1, f_1) is a $(3, 2k, \frac{1}{2})$-exact pair.

We construct the next pair in the same way, except that now $x_2 \in Z$ and the length is $\sigma(f_1)$ instead of n_{2k}. Next we need $x_3 \in Y$ with length $\sigma(f_1, f_2)$, and so on. Clearly the resulting sequence of functionals $(f_i)_{i=1}^k$ is a special sequence and thus we have a $(3, \frac{1}{2})$-dependent sequence. □

Corollary 4.2.12. *X_{GM} does not contain an unconditional basic sequence.*

Proof. Suppose $(x_n)_{n\in\mathbb{N}}$ is an unconditional basic sequence in X_{GM}. Then the subspaces $[x_{2n-1}]_{n\in\mathbb{N}}$ and $[x_{2n}]_{n\in\mathbb{N}}$ would form a decomposition of X_{GM}. This is a contradiction with the preceding theorem. □

We have now seen the construction and application of exact pairs and dependent sequences within the framework of mixed Tsirelson spaces. In Chapter 6 we will run into these again in the context of the Argyros-Haydon space.

Chapter 5

$\mathscr{L}_\infty$-spaces

So far we have concerned ourselves with Argyros' work on the (extensions of) Tsirelson-like spaces. We now turn to the second major ingredient for the solution of the scalar-plus-compact problem, which is Haydon's work on the construction of $\mathscr{L}_\infty$ spaces. This is an entirely different class of spaces which locally look like ℓ_∞, but globally can have a very different structure.

We recall the notation $L_p(\mu)$ to denote the Banach space of equivalence classes of p-integrable functions on a measure space (Ω, Σ, μ). It is natural to consider any σ-finite measure space. However we can assume without loss of generality that μ is a probability measure. Indeed if μ is σ-finite, we can rescale by a measurable function ϕ such that $\int \phi \, d\mu = 1$ and consider the measure satisfying $d\nu = \phi d\mu$. Then $L_p(\mu)$ is isometric to $L_p(\nu)$ by $U : L_p(\mu) \to L_p(\nu)$ given by $Uf(x) = f(x)(\phi(x))^{-1}$. [13] Further we denote by $C(K)$ the Banach space of continuous real-valued functions on a compact Hausdorff space K.

In the first section of this chapter we give an overview of the $\mathscr{L}_p$-spaces. In particular we will see that the $\mathscr{L}_p$ (resp. $\mathscr{L}_\infty$) spaces are a generalization of the $L_p(\mu)$ (resp. $C(K)$) spaces. Further we will give a nice characterization of the $\mathscr{L}_\infty$-spaces. The next section will give Bourgain and Delbaen's construction of a specific class of $\mathscr{L}_\infty$-spaces. Lastly we will see Argyros and Haydon's generalization of this construction to a larger class of spaces which contains the solution of the scalar-plus-compact problem.

5.1 $\mathscr{L}_p$ spaces

In this section we wish to give an elementary overview of the $\mathscr{L}_p$ spaces. These were introduced by Lindenstrauss and Pełczyński in [23], and a subsequent paper by Lindenstrauss and Rosenthal [24] gave a more detailed treatment. We have found [25, Chapter II.5] to contain the most comprehensive overview of the subject. However, some of the details go back to [21] which uses different notation and terminology.

We will present here a selection of these results, where the focus will lie on the case $p = \infty$ since this is the most important for our purposes.

Definition 5.1.1. Given two Banach spaces X and Y, the *Banach-Mazur distance* between X and Y is defined as

$$d(X,Y) = \inf\left\{\|T\|\|T^{-1}\| \mid T : X \to Y \text{ is an isomorphism}\right\}.$$

If there is no such isomorphism we set $d(X,Y) = \infty$.

This definition is closely related to the concept of basis equivalence which we have used thus far. Indeed if X and Y are Banach spaces with bases which are C-equivalent, then $d(X,Y) \leq C^2$.

Definition 5.1.2. For $1 \leq p \leq \infty$ and $\lambda \geq 1$, a Banach space X is called an $\mathscr{L}_{p,\lambda}$-space if for every finite-dimensional subspace E of X there exists another finite-dimensional subspace F of X, say with $\dim F = n$, such that $E \subseteq F$ and $d(F, l_p^n) \leq \lambda$. X is further called an $\mathscr{L}_p$-space if it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geq 1$.

In the introduction we mentioned that these $\mathscr{L}_p$ spaces are a generalization of the $L_p(\mu)$ and $C(K)$ spaces. This is made precise by the following theorem and the two propositions that follow.

Theorem 5.1.3. *Let X be a Banach space and $1 \leq p < \infty$.*

(i) X is isometric to $L_p(\mu)$ for some σ-finite measure space (Ω, Σ, μ) if and only if X is an $\mathscr{L}_{p,\lambda}$-space for all $\lambda > 1$.

(ii) Every $\mathscr{L}_p$-space is isomorphic to a subspace of $L_p(\mu)$ for some σ-finite measure space (Ω, Σ, μ).

Proof. We will only show $\Rightarrow$ of (i). Let $\lambda > 1$ and let $B \subseteq L_p(\mu)$ be a finite-dimensional subspace with basis $f_1, ..., f_n$. Then there exists an $\varepsilon > 0$ such that for any hyperplane H in B we have $d(f_i, H) \geq \varepsilon$ for at least one i and such that $\varepsilon \leq 1 - \lambda$. Thus if we pick simple functions $s_1, ..., s_n$ such that $\|f_i - s_i\|_p < \varepsilon$ for all i, it follows that $s_1, ..., s_n$ can not lie on the same hyperplane in B and thus are linearly independent. We divide the domain into disjoint measurable sets $A_1, ..., A_N$ of finite measure such that s_i is constant on A_j for all i, j. Then the space $F = \operatorname{span}\{\chi_{A_1}, ..., \chi_{A_N}\}$ is a subspace of $L_p(\mu)$ with $d(F, \ell_p^N) = 1$. Now write $C = \operatorname{span}\{s_1, ..., s_n\}$ and let $P : L_p(\mu) \to C$ be a projection. We can pick $z_1, ..., z_m$ such that $Pz_i = 0$ for all i and the s_i and z_i together form another basis of F. Then we can define an isomorphism $T : F \to TF$ by $T(s_i) = f_i$ and $T(z_i) = z_i$. By construction also $d(F, TF) \leq 1 + \varepsilon$ and thus $B \subseteq TF$ and $d(TF, \ell_p^n) \leq 1 + \varepsilon \leq \lambda$. □

Further, with respect to duality the $\mathscr{L}_p$-spaces behave as one would expect from their namesakes:

Theorem 5.1.4. *Let X be a Banach space and $1 \leq p, q \leq \infty$ be conjugate exponents. Then X is an $\mathscr{L}_p$-space if and only if X^* is an $\mathscr{L}_q$-space.*

We are mostly interested in the case $p = \infty$, so the rest of this section will be dedicated to a more in-depth study of $\mathscr{L}_\infty$-spaces. As such we will not prove the rest of the above results. The results below come mostly from [24], and we have filled some of the gaps in the proofs by consulting [21].

Proposition 5.1.5. *If X is isomorphic to $C(K)$ for some compact Hausdorff space K then it is an $\mathscr{L}_{\infty,\lambda}$-space for all $\lambda > 1$.*

Proof. The proof is similar to the reflexive case. However now we can not use simple functions, since these are not guaranteed to be continuous. Instead we will use a partition of unity to approximate the basis functions. Let $B \subseteq C(K)$ be a finite-dimensional subspace and $f_1, ..., f_n$ be a basis of B. For $\varepsilon > 0$ we can find for each $x \in K$ an open neighborhood U_x such that $|f_i(x_1) - f_i(x_2)| < \varepsilon$ for all i and $x_1, x_2 \in U_x$. Clearly these cover K, so we can find a finite subcovering $U_{x_1}, ..., U_{x_N}$. We assume that this covering is minimal in the sense that no U_{x_i} can be removed without leaving some $x \in K$ uncovered. Take a partition of unity $\phi_1, ..., \phi_N$ subordinate to this covering. That is, $\phi_i \geq 0$, $\operatorname{supp}(\phi_i) \subseteq U_{x_i}$ and $\sum_{i=1}^N \phi_i = 1$. We can then define $t_i = \sum_{j=1}^N f_i(x_j)\phi_j$ to obtain $\|f_i - t_i\|_\infty < \varepsilon$ for all i. Indeed, if $x \in K$ write $I = \{j : x \in U_{x_j}\}$. Then $\phi_j(x) = 0$ for all $j \notin I$ and $|f(x_j) - f(x)| < \varepsilon$ for all $j \in I$. Thus we can estimate as follows:

$$|t(x) - f(x)| = \left|\sum_{j\in I} f(x_j)\phi_j(x) - f(x)\right| \leq \left|\sum_{j\in I} f(x_j)\phi_j(x) - f(x)\sum_{j\in I}\phi_j(x)\right|$$

$$\leq \sum_{j\in I} |f(x_j) - f(x)|\phi_j(x) < \varepsilon \sum_{j\in I}\phi_j(x) = \varepsilon.$$

As before if ε was small enough the $t_1, ..., t_n$ are linearly independent. We now claim that $F = \operatorname{span}\{\phi_1, ..., \phi_N\}$ is a subspace of $C(K)$ with $d(F, \ell_\infty^N) = 1$. From the minimality of the covering it follows that $\|\phi_j\|_\infty = 1$ for all j. Now if $a_1, ..., a_N$ are scalars and $j_0 = \operatorname{argmax}|a_j|$ then we can find $x \in U_{x_{j_0}}$ such that $\phi_{j_0}(x) = 1$, and then we see $\left|\sum_{j=1}^N a_j\phi_j(x)\right| = |a_{j_0}|$. So $\left\|\sum_{j=1}^N a_j\phi_j\right\|_\infty \geq \max|a_j|$. The other inequality is trivial so indeed $d(F, \ell_\infty^N) = 1$. The rest of the proof continues as in the reflexive case. □

Proposition 5.1.6. *If K is compact Hausdorff then every complemented subspace of $C(K)$ is an $\mathscr{L}_\infty$-space.*

Proof. Let X be a complemented subspace of $C(K)$, and let $Q : C(K) \to X$ be a projection. By a result of Pełczyński ([31]) X has a subspace U which is isomorphic to c_0. Say $d(U, c_0) = K_0$. Let $B \subseteq X$ be a finite-dimensional subspace. By the previous proposition, there exists a finite-dimensional subspace W of $C(K)$ such that $B \subseteq W$, $d(W, \ell_\infty^n) \leq 2$ and with a projection $P : C(K) \to W$ with $\|P\| \leq 2$.

We obtain a space $QW \subseteq X$ with dimension $k \leq n$. We can pick a basis of QW

with basis constant at most $2\|Q\|$. We can also find a n-dimensional space $C \subseteq U$ such that $QW \cap C = \{0\}$ and $d(C, \ell_\infty^n) = K_0$. Then concatenating the bases of C and QW we obtain a basic sequence of length $n+k$ and with basis constant at most $K = 2\|Q\| + K_0$. Thus we see that for all $x \in QW$ and $y \in C$ we have

$$\max(\|x\|, \|y\|) \leq K\|x + y\|.$$

It follows that $d(W, C) \leq 2K_0 \leq 2K$. We let $\tau : W \to C$ be an isomorphism with $\|\tau\| \leq 1$ and $\|\tau^{-1}\| \leq 2K$. Define the operator $T = Q\big|_W + \tau(I - PQ\big|_W)$. Then we claim that TW is a subspace of X with $B \subseteq TW$ and $d(TW, \ell_\infty^n)$ less than some constant. Indeed we have $\|T\| \leq 1 + 3\|Q\|$ and for every $x \in W$ we have

$$\|Tx\| \geq \frac{1}{K}\max(\|Qx\|, \|\tau(I - PQ)x\|) \geq \frac{1}{K}\max\left(\|Qx\|, \frac{1}{2K}\|x - PQx\|\right)$$

$$\geq \frac{1}{K}\max\left(\|Qx\|, \frac{1}{2K}(\|x\| - 2\|Qx\|)\right) \geq \frac{\|x\|}{K(K+2)}.$$

Thus $d(W, TW) \leq (1 + 3\|Q\|)(K^2 + 2K)$ and $d(TW, \ell_\infty^n) \leq 2d(W, TW)$ so we are done. □

We will now see that the case $p = \infty$ is equivalent to the existence of certain extensions of operators, and so this paradigm is closely related to the "Hahn-Banach problems" that we have mentioned in the introduction. One such property is injectivity:

Definition 5.1.7. A Banach space X is λ-*injective* if whenever Y is a Banach space such that X is (isometric to) a subspace of Y, there exists a projection $P : Y \to X$ such that $\|P\| \leq \lambda$. The space is *injective* if it is λ-injective for some $\lambda \geq 1$.

There is an alternative definition of injectivity which we will state as a lemma. This is the definition used in e.g. [13] and [5]. The proof below is adapted from [5].

Lemma 5.1.8. *A Banach space X is λ-injective if and only if whenever $Z \subseteq Y$ are Banach spaces and $T : Z \to X$ is a bounded operator, there exists an extension $\tilde{T} : Y \to X$ with $\|\tilde{T}\| \leq \lambda\|T\|$.*

Proof. $\Rightarrow$) We will first show the RHS holds for $\ell_\infty(\Gamma)$ and $\lambda = 1$. Indeed, let $Z \subseteq Y$ be Banach spaces and let $T : Z \to \ell_\infty(\Gamma)$ be a bounded operator. For every $y \in Z$ we have $T(y) = (T_\gamma(y))_{\gamma\in\Gamma}$ for some family of bounded operators $T_\gamma \in Z^*$. Then by the Hahn-Banach theorem there are extensions $\tilde{T}_\gamma \in Y^*$ with $\|\tilde{T}_\gamma\| = \|T_\gamma\|$. We can then define $\tilde{T} : Y \to \ell_\infty(\Gamma)$ by $\tilde{T}(y) = (\tilde{T}_\gamma(y))_{\gamma\in\Gamma}$. Then $\|\tilde{T}\| = \|T\|$.

Now the result follows by noting that we can embed X isometrically into the space $\ell_\infty(\Gamma)$ for Γ large enough. Indeed let $\iota : X \to \ell_\infty(\Gamma)$ denote the embedding. Then if $Z \subseteq Y$ and $T : Z \to X$ is a bounded operator, we can define $U = \iota \circ T$ and obtain an extension $\tilde{U} : Y \to \ell_\infty(\Gamma)$ with $\|\tilde{U}\| = \|U\| = \|T\|$. By injectivity we have a

projection $P : \ell_\infty(\gamma) \to X$ with $\|P\| \leq \lambda$. Then $\tilde{T} = P\tilde{U}$ is the desired extension for T.

$\Leftarrow$) Suppose Y is a Banach space containing X. We can extend the identity operator $I : X \to X$ to an operator $\tilde{I} : Y \to X$ with $\|\tilde{I}\| \leq \lambda$. Then clearly $\tilde{I}$ is a projection. □

Remark 5.1.9. A 1-injective space is also called *isometrically injective*. We see in particular that ℓ_∞ is isometrically injective. Furthermore ℓ_∞^n are the only separable Banach spaces with this property. See also [13, Section 4.3].

We will now state a nice characterization of $\mathscr{L}_\infty$-spaces.

Theorem 5.1.10. *Let X be a Banach space. The following are equivalent:*

(i) X is an $\mathscr{L}_\infty$-space.

(ii) X^{**} is injective.

(iii) X has the compact extension property. That is, if Y is a Banach space with a subspace Z and $T : Z \to X$ is a compact operator, then there exists a compact operator $\tilde{T} : Y \to X$ such that $\tilde{T}\big|_Z = T$.

(iv) Same as (iii) but without the requirement that $\tilde{T}$ is compact.

We will need quite a few ingredients to prove this theorem, so we will first need to treat these. We start with the following theorem which is known as the "principle of local reflexivity". It was introduced by Lindenstrauss and Rosenthal in their treatment of the $\mathscr{L}_p$-spaces [24]. Some more concise proofs have been given since then.

Theorem 5.1.11. *Let X be a Banach space and let $U \subseteq X^{**}$ be a finite-dimensional subspace. Then for every $\varepsilon > 0$ there exists a subspace $E \subseteq X$ and an isomorphism $T : U \to E$ such that $T(x) = x$ for all $x \in U \cap X$ and $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$.*

Proof. See [24] or [13, Section 12.2]. □

We immediately obtain the following corollaries:

Corollary 5.1.12. *If X^{**} is an $\mathscr{L}_\infty$-space, then X is an $\mathscr{L}_\infty$-space.*

Proof. Let $E \subseteq X$ be a finite-dimensional subspace. Then by assumption there exists an $F \subseteq X^{**}$ with $E \subseteq F$ and $d(F, \ell_\infty^n) \leq \lambda$ ($n = \dim F$). Pick any $\varepsilon > 0$. By the principle of local reflexivity, there exists $T : F \to TF \subseteq X$ with $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$. and $T\big|_{F \cap X} = \mathrm{id}$. Thus $E \subseteq TF$ and we find $d(TF, \ell_\infty^n) \leq (1 + \varepsilon)\lambda$. □

Corollary 5.1.13. *Every injective space is an $\mathscr{L}_\infty$-space.*

Proof. If X is any Banach space, we can embed it isometrically into a $C(K)$ space by picking K to be the unit ball of X^* with the weak*-topology. Then by injectivity there is a bounded projection $P : C(K) \to X$. That is, X is a complemented subspace of $C(K)$. So by Proposition 5.1.6 X is an $\mathscr{L}_\infty$-space. □

Finally we need the following result, which is a consequence of the fact that ℓ_∞ is isometrically injective.

Lemma 5.1.14. *Every $\mathscr{L}_\infty$-space has the bounded approximation property.*

Proof. Let X be an $\mathscr{L}_{\infty,\lambda}$-space and let $E \subseteq X$ be a finite-dimensional subspace. Take $F \subseteq X$ with $E \subseteq F$ and $d(F, \ell_\infty^n) \leq \lambda$. Then F is at most λ-injective, so there exists a projection $P : X \to F$ with $\|P\| \leq \lambda$. Thus if $F_1 \subseteq F_2 \subseteq \ldots$ are finite-dimensional subspaces of X with $X = \overline{\bigcup_{n\in\mathbb{N}} F_n}$ and $d(F_n, \ell_\infty^{m_n}) \leq \lambda$ for some increasing sequence $(m_n)_{n\in\mathbb{N}}$, then the projections $P_n : X \to F_n$ obtained this way form a bounded approximation of the identity. □

Proof of Theorem 5.1.10. So far we have seen that (ii) implies (i). It is also clear that (iii) implies (iv).

(iv) ⇒ (ii): We summarize the proof in the following three steps:
Step 1: There exists a $\lambda \geq 1$ such that whenever $Z \subseteq Y$ and $T : Z \to X$ the extension $\tilde{T} : Y \to X$ given by property (iv) satisfies $\|\tilde{T}\| \leq \lambda\|T\|$.
Step 2: Suppose Z is a Banach space containing X^{**}. There exists an extension of $I : X \to X$ to $\tilde{I} : Z \to X^{**}$ with $\|\tilde{I}\| \leq \lambda$.
Step 3: Let $P : X^{****} \to X^{**}$ be the restriction projection, i.e. $P : \phi \mapsto \phi\big|_{X^{**}}$. Then $P\tilde{I}^{**}$ is a projection of Z^{**} onto X^{**} with $\|P\tilde{I}^{**}\| \leq \lambda$.
We conclude that the restriction of $P\tilde{I}^{**}$ to Z is a projection with norm at most λ. Thus X^{**} is λ-injective. The details are found in [21, pp. 12-14].

(i) ⇒ (iii): First suppose $Z \subseteq Y$ and that $T : Z \to X$ is finite rank. We will show that there is a finite rank extension $\tilde{T} : Y \to X$. Indeed, since TZ is finite-dimensional, there is a finite-dimensional subspace $E \subseteq X$ such that $d(E, \ell_\infty^n) \leq \lambda$. Again since ℓ_∞^n is isometrically injective, E is at most λ-injective. Thus by Lemma 5.1.8 there is an extension $\tilde{T} : Y \to E$ with $\|\tilde{T}\| \leq \lambda\|T\|$.

Now if $T : Z \to X$ is compact, we can by Lemma 5.1.14 find a sequence of finite rank operators $T_n : Z \to X$ such that $\|T_n - T\| \to 0$ and $\|T_n\| \leq \lambda\|T\|$. By the previous argument we can find extensions $\tilde{T}_n : Y \to X$. Then the sequence $\tilde{T}_n$ converges in norm to a compact operator $\tilde{T} : Y \to X$ which extends T. □

Our last result in this section is of some significance for the rest of the story. All $\mathscr{L}_\infty$-spaces we are going to construct will be separable and with a dual isomorphic to ℓ_1. This is a consequence of the following theorem:

Theorem 5.1.15. *Let X be a separable $\mathscr{L}_\infty$-space. Then X^* is isomorphic to ℓ_1 if and only if X does not contain a copy of ℓ_1.*

Proof. $\Rightarrow$) If X contains a copy of ℓ_1 then X^* can not be separable, so X^* is not isomorphic to ℓ_1.
$\Leftarrow$) Since X^{**} is injective, we can see X^{**} as a complemented subspace of a $C(K)$ space, like in the proof of Corollary 5.1.13. Thus X^{***} is a complemented subspace of $C(K)^*$, or in other words of $M(K)$, the Banach space of Radon measures on K. Further if we write $J : X \to X^{**}$ for the canonical embedding, then $J^* : X^{***} \to X^*$ is a bounded projection, so X^* is also isomorphic to a complemented subspace of $M(K)$. Thus by a result of Stegall [33, Corollary 1], X^* is isomorphic to either ℓ_1 or $M([0,1])$. However by [33, Corollary 2], X not containing a copy of ℓ_1 implies that X^* can not contain $M([0,1])$, thus the only remaining option is that X^* is isomorphic to ℓ_1. □

Remark 5.1.16. Note that the equivalence in the above theorem is not true for general separable Banach spaces. Indeed, James constructed a counter-example, known as the "James Tree space", that is a separable Banach space containing no copy of ℓ_1 but with a non-separable dual. See also [13, Section 15.4].

5.2 The Bourgain-Delbaen construction

In 1980, Bourgain and Delbaen [8] presented a method of constructing $\mathscr{L}_\infty$-spaces with some unexpected properties that answered a number of open conjectures at that time. They construct two examples, with the focus in their paper lying mostly on the first one. This is an $\mathscr{L}_\infty$-space that has the Schur property, which was the first example of such a space. It also therefore does not contain a copy of c_0, so the question posed in [23] and [24] whether every $\mathscr{L}_\infty$-space contains a copy of c_0 was answered in the negative.

Both of their examples are also separable and have the Radon-Nikodym property, which provided a counterexample to a conjecture of Uhl that asks whether every separable Radon-Nikodym space is isomorphic to a subspace of a separable dual space.

The second example has been of most interest in the work following their paper, and it is this construction that we call the "Bourgain-Delbaen construction". This space does not contain a copy of ℓ_1, and is "somewhat reflexive". We will define what this means at the end of this section.

Below we introduce the Bourgain-Delbaen construction as reparametrized by Haydon in [19]. This has become the standard way to denote and work with spaces of this kind. We will see a generalization using this notation in the next section.

Definition 5.2.1 (The set Γ)**.** Let $\Delta_0 = \{0\}$. Then supposing $\Delta_0, ..., \Delta_n$ are defined,

define $\Gamma_k = \bigcup_{i=0}^{k} \Delta_i$ and

$$\Delta_{n+1} = \{n+1\} \times \bigcup_{k=0}^{n-1} (\{k\} \times \Gamma_k) \times \Gamma_n \times \{\pm 1\}.$$

Now let $\Gamma = \bigcup_{n=0}^{\infty} \Delta_n$. So if $\gamma \in \Gamma$ we can write $\gamma = (n, k, \xi, \eta, \varepsilon)$, where $0 \leq k < n-1$, $\xi \in \Gamma_k$, $\eta \in \Gamma_{n-1}$ and $\varepsilon = \pm 1$. We define the following notations for the five coordinates:

$$n = \text{rank}(\gamma), \quad k = \text{cut}(\gamma), \quad \xi = \text{base}(\gamma), \quad \eta = \text{top}(\gamma), \quad \varepsilon = \text{sign}(\gamma).$$

Remark 5.2.2. A potential problem with the above definition is that we have $\Delta_1 = \emptyset$. This is not an issue however as the important part is that Γ_1 is not empty. This is seen in the next definition.

In the next few definitions we define the Bourgain-Delbaen space $X_{a,b}$ as a subspace of $\ell_\infty(\Gamma)$, where a and b are required to satisfy $0 < b < \frac{1}{2} < a < 1$ and $a + b > 1$. Along the way we will also perform the necessary checks to show the construction is well-defined. The strategy is to define extension mappings that take elements of $\ell_\infty(\Gamma_n)$ and spread them out to an element of $\ell_\infty(\Gamma_{n+1})$ in such a way that the maps are uniformly bounded. Then, one can repeat this extension an arbitrary number of times to get an element of $\ell_\infty(\Gamma)$.

Definition 5.2.3 (The maps $i_{m,n}$)**.** We recursively define mappings $u_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Delta_{n+1})$ and $i_n : \ell_\infty(\Gamma_n) \to \ell_\infty(\Gamma_{n+1})$. We start with i_0 being the identity on $\ell_\infty(\{0\})$. Next, given $i_0, ..., i_{n-1}$ and $u_1, ..., u_{n-1}$, we define for each $m < n$ the composition $i_{m,n} = i_{n-1} \circ i_{n-2} \circ ... \circ i_m$, and use this to define

$$u_n(f)(n, k, \xi, \eta, \varepsilon) = af(\xi) + \varepsilon b \left(f(\eta) - (i_{k,n}(f|_{\Gamma_k}))(\eta) \right)$$

and

$$i_n(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \Gamma_n, \\ u_n(f)(\gamma) & \text{if } \gamma \in \Delta_{n+1}. \end{cases}$$

Example 5.2.4. Let us examine the behaviour of these mappings by considering a basic example over the first few values of n. We have $\Delta_0 = \{0\}$, $\Delta_1 = \emptyset$ and $\Delta_2 = \{(2, 0, 0, 0, \pm 1)\}$. Let $f \in \ell_\infty(\{0\})$. We get $i_{0,1} = i_0 = \text{id}_{\ell_\infty(\{0\})}$ and so

$$u_1(f)(2, 0, 0, 0, \varepsilon) = af(0) + \varepsilon b(f(0) - i_{0,1}(f|_{\Gamma_0})(0)) = af(0).$$

Thus, using a lexicographic ordering on $\Gamma_2 = \{0, (2, 0, 0, 0, 1), (2, 0, 0, 0, -1)\}$, we get that $i_1(f) = (f(0), af(0), af(0))$. The next entry has $\#\Delta_3 = 12$, so let us compute one value of $i_2(f)$ for some $f \in \ell_\infty(\Gamma_2)$. Take $\eta = (2, 0, 0, 0, 1)$ and $\gamma = (3, 1, 0, \eta, -1)$. Then

$$\begin{aligned} i_2(f)(\gamma) &= af(0) - b \left(f(\eta) - i_{1,2}(f|_{\Gamma_1})(\eta) \right) \\ &= af(0) - b(f(\eta) - af(\eta)) = af(0) + b(a-1)f(\eta). \end{aligned}$$

□

The well-definedness of the construction hinges on the following proposition:

Proposition 5.2.5. *For $0 \leq m < n$ the norms of the composite mappings $i_{m,n}$ are uniformly bounded by $\lambda = a/(1-2b)$.*

Proof. The constraints on a, b ensure that $\lambda > 1$, and since $\|i_{0,1}\| = 1$, it holds for $n = 1$. Now assume that for some $n \geq 1$ the statement is true for all $m < n$. Let $f \in \ell_\infty(\Gamma_m)$ and $\gamma = (n+1, k, \xi, \eta, \varepsilon) \in \Delta_{n+1}$. Expanding the definition gives

$$|i_{m,n+1}(f)(\gamma)| \leq a|i_{m,n}(f)(\xi)| + b|i_{m,n}(f)(\eta) - (i_{k,n}((i_{m,n}(f))\big|_{\Gamma_k}))(\eta)|.$$

If $k > m$, the second term evaluates to 0, giving $|i_{m,n+1}(f)(\gamma)| \leq a\|i_{m,n}(f)\| \leq \lambda\|f\|$ by the induction hypothesis. If, on the other hand, $k \leq m$, we get $(i_{m,n}(f))\big|_{\Gamma_k} = f\big|_{\Gamma_k}$. Since also $\xi \in \Gamma_k$ we get $i_{m,n}(f)(\xi) = f(\xi)$ and so

$$|i_{m,n+1}(f)(\gamma)| \leq a|i_{m,n}(f)(\xi)| + b|i_{m,n}(f)(\eta) - i_{k,n}(f\big|_{\Gamma_k})(\eta)| \leq a\|f\| + 2b\lambda\|f\|,$$

where we have applied the induction hypothesis to the maps $i_{m,n}$ and $i_{k,n}$. Now since $a = (1-2b)\lambda$, this is at most $\lambda\|f\|$ as required. □

Definition 5.2.6 (The space $X_{a,b}$)**.** We define the map $j_m : \ell_\infty(\Gamma_m) \to \ell_\infty(\Gamma)$ by

$$j_m(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in \Gamma_m, \\ i_{m,n}(f)(\gamma) & \text{if } \gamma \in \Gamma_n \text{ for some } n > m. \end{cases}$$

The uniform boundedness of $i_{m,n}$ ensures the result is always in $\ell_\infty(\Gamma)$. The last step is to take the images $X_m = j_m(\ell_\infty(\Gamma_m))$ and define the space $X_{a,b}$ as the closure of $\bigcup_{m=0}^\infty X_m$ in $\ell_\infty(\Gamma)$.

Theorem 5.2.7. *The space $X_{a,b}$ is a separable $\mathscr{L}_{\infty,\lambda}$-space for $\lambda = a/(1-2b)$.*

Proof. Note that by construction $X_m \subseteq X_{m+1}$. So if E is a finite-dimensional subspace of $X_{a,b}$, there is some minimal m such that $E \subseteq X_m$. Now by Proposition 5.2.5, j_m is an isomorphism between X_m and $\ell_\infty(\Gamma_m)$ with $\|j_m\| = \lambda$ and $\|j_m^{-1}\| = 1$. Thus $d(X_m, \ell_\infty(\Gamma_m)) \leq \lambda$. □

Let us examine the subspace structure of $X_{a,b}$ a little more. The subspace X_m is the image of a projection $S_m : X_{a,b} \to \ell_\infty(\Gamma_m)$ defined by $S_m(f) = j_m(f\big|_{\Gamma_m})$. If we take $P_0 = S_0$ and $P_n = S_n - S_{n-1}$ we get subspaces $M_n = P_n(X_{a,b})$ which form a finite-dimensional decomposition of $X_{a,b}$. (See Definition 2.2.13). More directly, we can write

$$M_n = \left\{ j_n(f) \mid f \in \ell_\infty(\Gamma_n) \text{ and } f\big|_{\Gamma_{n-1}} = 0 \right\}.$$

In particular we see that M_n has a basis given by elements $w_\gamma = j_n(e_\gamma)$. So these also give a basis for the entire space $X_{a,b}$.

Bourgain and Delbaen go on to show some more properties of their construction. We will not go through the proofs here, but we will state the results.

Theorem 5.2.8. *If $b \neq b'$ then $X_{a,b}$ is not isomorphic to any subspace of $X_{a,b'}$.*

This means there is a continuum number of totally incomparable $\mathscr{L}_\infty$-spaces. Further, the following property is important for the next chapter:

Theorem 5.2.9. *$X_{a,b}$ does not contain a copy of ℓ_1.*

Equivalently (Theorem 5.1.15) the dual of $X_{a,b}$ is isomorphic to ℓ_1. Lastly Bourgain and Delbaen show that $X_{a,b}$ is *somewhat reflexive*, meaning that every infinite-dimensional subspace contains a further infinite-dimensional subspace that is reflexive. Later, Haydon [19] showed the following more precise result:

Theorem 5.2.10. *Suppose a, b satisfy $0 < b < \frac{1}{2} < a < 1$ and $a + b > 1$. Let p, q be such that $1/p + 1/q = 1$ and $a^q + b^q = 1$. Then every infinite-dimensional subspace of $X_{a,b}$ contains a subspace isomorphic to ℓ_p.*

5.3 Generalized Bourgain-Delbaen spaces

In this section we will finally break into the contents of the main paper by Argyros and Haydon [3]. They introduce a generalization of the Bourgain-Delbaen construction, based on a mapping from Γ to certain tree-like functionals.

In the preceding section we saw how to construct an $\mathscr{L}_{\infty,\lambda}$-space by using extension mappings on ℓ_∞. Below we will see the dual approach, which starts with a "triangular" basis in ℓ_1 and moves to ℓ_∞ by taking the biorthogonal functionals. We will see that this essentially yields the same construction as in the preceding section. However, we can then reparametrize this with some set Γ similar to the one we used before and generalize the construction to obtain new $\mathscr{L}_\infty$-spaces, among which the space of Argyros and Haydon.

As $\ell_1 \subseteq \ell_\infty^*$, we will denote elements of ℓ_1 with a star to indicate that we consider them as functionals over ℓ_∞.

Definition 5.3.1. A basic sequence $(d_n^*)_{n\in\mathbb{N}}$ in ℓ_1 is called a *triangular basis* if for all $n \geq 1$ we have $\mathrm{supp}(d_n^*) \subseteq \{1, ..., n\}$. The associated projections are denoted by P_n^* which map d_m^* to d_m^* for $m \leq n$ and to 0 for $m > n$.

Such a triangular basis $(d_n^*)_{n\in\mathbb{N}}$ will be a basis for the entire space ℓ_1. Taking the biorthogonal functionals $(d_n)_{n\in\mathbb{N}}$ gives a basic sequence in ℓ_∞. These span a subset of ℓ_∞ which is an $\mathscr{L}_\infty$-space:

Theorem 5.3.2. *If $(d_n^*)_{n\in\mathbb{N}}$ is a triangular basis in ℓ_1 with basis constant M, then the space $X = [d_n]_{n\in\mathbb{N}}$ spanned by the biorthogonal functionals is an $\mathscr{L}_{\infty,M}$-space.*

Proof. Because $e_m^* \in \ell_1^n = [d_1^*, ..., d_n^*]$ we also have $P_n^* e_m^* = e_m^*$ whenever $m \leq n$. This means that we can restrict P_n^* as a projection from ℓ_1 to ℓ_1^n. The resulting operator

$q_n : \ell_1 \to \ell_1^n$ is a quotient map with norm at most M. We then obtain the dual operator $i_n : \ell_\infty^n \to \ell_\infty$ with norm at most M, satisfying for all $m \leq n$ the equality

$$i_n(u)(m) = e_m^*(i_n(u)) = (q_n(e_m^*))(u) = e_m^*(u) = u(m).$$

So i_n is an extension operator, and i_n is an isomorphism onto its image which is exactly $[d_1, ..., d_n] \subseteq X$. Thus $d([d_1, .., d_n], \ell_\infty^n) \leq M$. This shows that X is an $\mathscr{L}_{\infty,M}$-space. □

Definition 5.3.3. Let $(d_k^*)_{k=1}^n$ be a triangular basis for ℓ_1^n. We say an element $c^* \in \ell_1^n$ is a *BD-functional* if there exist $\alpha \in [0,1]$ and $\beta \in [0, \frac{1}{2})$ such that one of the following holds:

(i) $c^* = \alpha e_j^*$ with $1 \leq j \leq n$.

(ii) $c^* = \beta(I - P_k^*)b^*$ with $0 \leq k < n$ and $b^* \in \operatorname{ball} \ell_1^n \cap [e_j^*]_{j=k+1}^n$.

(iii) $c^* = \alpha e_j^* + \beta(I - P_k^*)b^*$ with both of the above.

These look a lot like the extension mappings we used to construct $X_{a,b}$ before. In this form, the BD-functionals encode the structure of a triangular basis. Indeed, the following theorem shows how to use them to construct such a basis.

Theorem 5.3.4. *Let $0 < \theta < \frac{1}{2}$ and let $d_n^* = e_n^* - c_n^*$ in ℓ_1 be such that $c_1^* = 0$ while, for each n, $c_{n+1}^* \in \ell_1^n$ is such that c_{n+1}^* is a BD-functional with $\beta < \theta$ with respect to $(d_m^*)_{m=1}^n$. Then $(d_n^*)_{n=1}^\infty$ is a triangular basis for ℓ_1 with basis constant at most $M = 1/(1-2\theta)$.*

Proof. From the definition of the BD-functionals it is clear that the support of d_n^* is contained in $\{1, ..., n\}$. The rest of the argument is very similar to that in Proposition 5.2.5. Indeed, by induction we can show that $\|P_m^*\| \leq M$ for all m. This shows by Grunblum's criterion (Theorem 2.2.5) that $(d_n^*)_{n=1}^\infty$ is a triangular basis with basis constant at most M. □

We continue with a set Γ such as the one in the preceding section. The generalization is summarized in the next definition:

Definition 5.3.5. Let $\theta < \frac{1}{2}$ and let τ be a mapping defined on $\Gamma \setminus \Delta_0$ such that each $\gamma \in \Delta_{q+1}$ is assigned to a tuple of one of the following forms:

(i) (α, ξ) with $\alpha \in [0,1]$ and $\xi \in \Gamma_q$.

(ii) (p, β, b^*) with $0 \leq p < q$, $\beta \in (0, \theta]$ and $b^* \in \operatorname{ball} \ell_1(\Gamma_n \setminus \Gamma_p)$.

(iii) $(\alpha, \xi, p, \beta, b^*)$ with both of the above but $\alpha > 0$ and $p > 0$.

Then we can define BD-functionals c^*_γ and projections $P^*_{(0,q]}$ by the rules

$$c^*_\gamma = \begin{cases} 0 & \text{if } \gamma \in \Delta_0, \\ \alpha e^*_\xi & \text{if } \tau(\gamma) = (\alpha, \xi), \\ \beta(I - P^*_{(0,p]})b^* & \text{if } \tau(\gamma) = (p, \beta, b^*), \\ \alpha e^*_\xi + \beta(I - P^*_{(0,p]})b^* & \text{if } \tau(\gamma) = (\alpha, \xi, p, \beta, b^*). \end{cases}$$

$$P^*_{(0,q]} d^*_\gamma = \begin{cases} d^*_\gamma & \text{if } \gamma \in \Gamma_q, \\ 0 & \text{if } \gamma \in \Gamma \setminus \Gamma_q, \end{cases}$$

where $d^*_\gamma = e^*_\gamma - c^*_\gamma$.

Lemma 5.3.6. *The above rules uniquely define c^*_γ and $P^*_{(0,q]}$. Furthermore the obtained functionals d^*_γ form a basis of $\ell_1(\Gamma)$ with basis constant at most $M = 1/(1-2\theta)$, the biorthogonals of which span an $\mathscr{L}_{\infty,M}$-space.*

Proof. The result follows from defining a suitable bijection between Γ and $\mathbb{N}$, and then applying Theorem 5.3.4. Indeed, let k_q denote the the size of each Γ_q. Then we wish $\gamma : \mathbb{N} \to \Gamma$ to satisfy the property that $\Delta_q = \{\gamma(n) \mid k_{q-1} < n \leq k_q\}$. In other words we require that γ runs through the Δ_q's in the correct order. This gives a natural isometry $J : \ell_1(\mathbb{N}) \to \ell_1(\Gamma)$ by $J(e^*_n) = e^*_{\gamma(n)}$. We define $c^*_{\gamma(0)} = 0$, and $d^*_{\gamma(0)} = e^*_{\gamma(0)}$. Then if for some n the entries $c^*_{\gamma(m)}$ and $d^*_{\gamma(m)}$ are given for $m \leq n$, we obtain $d^*_n = J^{-1}(d^*_{\gamma(n)}) = e^*_n - J^{-1}(c^*_{\gamma(n)})$. This gives us the projections on $\ell_1(\mathbb{N})$ up to $P^*_{k_q}$ for $q < p$. We obtain the projections on $\ell_1(\Gamma)$ by $P^*_{(0,q]} = JP^*_{k_q}J^{-1}$. This then allows us to define the next $c^*_{\gamma(n+1)}$ by the above rules and $d^*_{\gamma(n+1)} = e^*_{\gamma(n+1)} - c^*_{\gamma(n+1)}$. It is then easy to see that the obtained d^*_n satisfy the conditions of Theorem 5.3.4. □

Definition 5.3.7. The space obtained in the above lemma for a specific choice of Γ and τ is denoted by $X(\Gamma, \tau)$. We further define a finite-dimensional decomposition of $X(\Gamma, \tau)$ by $M_n = [d_\gamma]_{\gamma \in \Delta_n}$. We will thus consider (skipped) block sequences in $X(\Gamma, \tau)$ to be as in Definition 2.2.13.

Example 5.3.8. To reobtain $X_{a,b}$ from this generalized form, we can take Γ as in the preceding section and τ to be the following mapping:

$$\tau(n+1, k, \xi, \eta, \varepsilon) = \begin{cases} (a, \xi, k, b, \varepsilon e^*_\eta) & \text{if } \eta \in \Gamma_n \setminus \Gamma_k, \\ (a, \xi) & \text{if } \eta \in \Gamma_k. \end{cases}$$

As we noted in the preceding section, $X_{a,b}$ has a basis given by $w_\gamma = j_n(e_\gamma)$ for $\gamma \in \Gamma_n$. We will show by induction on the rank of γ that $d_\gamma = w_\gamma$.

For $n = 0$, $d^*_\gamma = e^*_\gamma$, and $w_\gamma = e_\gamma$. Further since $j_n(f)(\delta) = f(\delta)$ whenever $\delta \in \Gamma_n$,

we have that $j_n(e_\gamma)(\gamma') = 0$ when $\gamma' \neq \gamma$ and $\operatorname{rank}\gamma \leq \operatorname{rank}\gamma'$. So $\langle d^*_\gamma, w_{\gamma'}\rangle = 0$ for all other γ' in this case. So indeed w_γ is biorthogonal to d^*_γ, i.e. $w_\gamma = d_\gamma$.

Now let $\gamma = (n+1, k, \xi, \eta, \varepsilon)$. If $\eta \in \Gamma_k$, then $d^*_\gamma = e^*_\gamma - ae^*_\xi - \varepsilon b(I - P^*_{(0,k]})e^*_\eta$. (Where the last term is 0 if $\eta \in \Gamma_k$). Since the ranks of ξ and η are lower than γ, the induction hypothesis gives $\langle e^*_\eta, w_\gamma\rangle = 0$ and $\left\langle e^*_\xi, w_\gamma\right\rangle = 0$. Further if $\eta \in \Gamma_n \setminus \Gamma_k$ then $P^*_{(0,k]}e^*_\eta$ has support in Γ_k and so again $\left\langle P^*_{(0,k]}e^*_\eta, w_\gamma\right\rangle = 0$. Thus $\langle d^*_\gamma, w_\gamma\rangle = 1$.

Now if $\gamma' \neq \gamma$ and $m = \operatorname{rank}\gamma' < \operatorname{rank}\gamma$ we have by the induction hypothesis that $w_{\gamma'} = d_{\gamma'}$. So $\langle d^*_\gamma, w_{\gamma'}\rangle = \langle d^*_\gamma, d_{\gamma'}\rangle = 0$. While the case $\operatorname{rank}\gamma' \geq \operatorname{rank}\gamma$ is already covered by the note above. So indeed $w_\gamma = d_\gamma$.

In total we see that $X_{a,b}$ and $X(\Gamma, \tau)$ have precisely the same basis. So $X_{a,b} = X(\Gamma, \tau)$ as subspaces of $\ell_\infty(\Gamma)$. □

The notation $P^*_{(0,q]}$ is a suggestion that we can define projections on other intervals of $\mathbb{N}$. Indeed we may write $P^*_{(q,\infty)} = I - P^*_{(0,q]}$. With these two we can define P^*_E for any interval $E \subseteq \mathbb{N}$. These have image contained in the span of all d^*_γ for $\gamma \in \Delta_q$, $q \in E$. By the same reasoning as in the proof of Theorem 5.3.2 applied to $P^*_{(0,q]}$ we can then obtain the extension operator $i_q : \ell_\infty(\Gamma_q) \to X(\Gamma, \tau)$ with norm at most M such that $i_q(u)(\gamma) = u(\gamma)$ whenever $\gamma \in \Gamma_q$. Dually, we have projections $P_E : X(\Gamma, \tau) \to \bigoplus_{n\in E} M_n$ defined by $P_E d_\gamma = d_\gamma$ if $\gamma \in \Delta_q$ for some $q \in E$ and $P_E d_\gamma = 0$ otherwise. We will note the following useful facts about these operators:

Lemma 5.3.9. *Let P^*_E, P_E and i_q be as defined above. Then we have the following properties:*

(i) P^*_E *is the dual (or adjoint) operator of* P_E.

(ii) $\|P^*_E\| = \|P_E\|$.

(iii) $P_{(0,q]} = i_q^{-1}$ *on* $[d_\gamma]_{\gamma\in\Gamma_q} = \bigoplus_{n=1}^q M_n$.

(iv) $\|P_{(0,q]}\| = \|i_q\| \leq M$, $\|P_{(q,\infty)}\| \leq M + 1$ *and* $\|P_{(m,n]}\| \leq 2M$.

The easy proofs are left to the reader. Lastly, we circle back to the question of whether the dual of $X(\Gamma, \tau)$ is isomorphic to ℓ_1:

Proposition 5.3.10. *If the FDD $(M_n)_{n=1}^\infty$ of $X(\Gamma, \tau)$ is shrinking, then $X(\Gamma, \tau)^*$ is isomorphic to ℓ_1.*

Proof. The FDD is shrinking iff $\|P^*_{(0,n]} - I\| \to 0$. This clearly implies that the basis $(d_{\gamma(n)})_{n\in\mathbb{N}}$ is shrinking (where we have fixed a bijection $\gamma : \mathbb{N} \to \Gamma$ as in the proof of Lemma 5.3.6). Thus by Theorem 2.3.11 we have $X(\Gamma, \tau)^* = [d^*_{\gamma(n)}]_{n\in\mathbb{N}}$, which as we noted below Definition 5.3.1 is equal to ℓ_1. □

Chapter 6

The Argyros-Haydon space

In this chapter we treat the final solution to the scalar-plus-compact problem due to Argyros and Haydon. As mentioned in the last section of the previous chapter, the Argyros-Haydon space is a specific case of their generalized Bourgain-Delbaen construction. In the first section, we will see the choice of parameters that yields the Argyros-Haydon space X_{AH}. Then, we will see another form of a tree-like structure that we can use to understand the norms of elements of this space, similar to those we used in Chapters 3 and 4. Also in line with those chapters, we will see in Section 3 another form of rapidly increasing and dependent sequences. Finally in Section 4 we move to the operators on X_{AH} and finish with the proof that X_{AH} is indeed a solution to the scalar-plus-compact problem. We will closely follow the original paper, under the assumption that the previous chapters have given the reader the necessary background to understand the construction.

6.1 The Argyros-Haydon construction

We will construct a new instance of a set Γ and a mapping τ as in Definition 5.3.5.

Definition 6.1.1. The sets Δ_{n+1} will be chosen to contain elements of either one of the following two forms:

(i) $\gamma = (n+1, \beta, b^*)$ with $0 < \beta \leq \frac{1}{2}$ and $b^* \in \operatorname{ball} \ell_1(\Gamma_n)$,

(ii) $\gamma = (n+1, k, \xi, \beta, b^*)$ with $0 \leq k < n$, $\xi \in \Gamma_k$, $0 < \beta \leq \frac{1}{2}$ and $b^* \in \operatorname{ball} \ell_1(\Gamma_n \setminus \Gamma_k)$.

We will call these type I and type II elements respectively.

The benefit of this choice of notation for elements in Γ is that the mapping τ follows easily from the definition. One defines $\tau(\gamma) = (0, \beta, b^*)$ in the first case and $\tau(\gamma) = (1, \xi, k, \beta, b^*)$ in the second case. However, there is an infinite number of elements of each of the forms (i) and (ii) above. Thus we need to pick a finite number of them for each n. It is in making this choice that the Tsirelson-like constraints come

into view.

Before we make this choice we need to define some more terminology.

Definition 6.1.2. We recursively define the *age* of an element $\gamma \in \Gamma$ as follows:

(i) If γ is of type I, then $\text{age}(\gamma) = 1$.

(ii) If γ is of type II, then $\text{age}(\gamma) = 1 + \text{age}(\xi)$.

As in Chapter 4 we have need of a certain lacunary subset of the natural numbers. Here we need a sequence of pairs:

Definition 6.1.3. We define the sequence $(m_j, n_j)_{j\in\mathbb{N}}$ to satisfy

(i) $m_1 \geq 4$,

(ii) $m_{j+1} \geq m_j^2$,

(iii) $n_1 \geq m_1^2$,

(iv) $n_{j+1} \geq (16n_j)^{\log_2 m_{j+1}} = m_{j+1}^2 (4n_j)^{\log_2 m_{j+1}}$.

An example of such a sequence is $(m_j, n_j) = (2^{2^j}, 2^{2^{j^2+1}})$. We will see the precise implications of these four requirements in the next section.

Definition 6.1.4. In choosing the elements of Δ_{n+1} we will use the sequence $(m_j, n_j)_{j\in\mathbb{N}}$. In particular we will require that the values of β are of the form m_j^{-1}. Thus for $\gamma \in \Gamma$ we define the *weight* $w(\gamma)$ to be the unique j such that $\beta = m_j^{-1}$.

Besides restricting the weights, we will fix finite subsets of $\ell_1(\Gamma_n \setminus \Gamma_k)$ to choose the b^*'s from. The precise choice is not that important, but we will require that these are 2^{-n}-nets in the unit ball of $\ell_1(\Gamma_n \setminus \Gamma_k)$. Recall that a set A in a metric space X is an ε-net if the $\varepsilon/2$-balls around the elements of A are disjoint while the ε-balls around the elements of A cover X.

Definition 6.1.5. For a strictly increasing sequence $(N_n)_{n\in\mathbb{N}}$ and $0 \leq k < n$ we define the sets $B_{k,n}$ to be the set of all linear combinations

$$\sum_{\eta\in\Gamma_n\setminus\Gamma_k} a_\eta e_\eta^*,$$

where $\sum_{\eta\in\Gamma_n\setminus\Gamma_k} |a_\eta| \leq 1$ and each a_η is a rational with denominator dividing $N_n!$. We further pick $(N_n)_{n\in\mathbb{N}}$ such that for fixed k, the $B_{k,n}$ form a 2^{-n}-net in the unit ball of $\ell_1(\Gamma_n \setminus \Gamma_k)$.

We are now finally ready to give the exact definition of the Argyros-Haydon space:

Definition 6.1.6. We define X_{AH} to be the space $X(\Gamma, \tau)$, where τ will be as at the start of this section and Γ is defined by recursion using $\Delta_0 = \{0\}$ and Δ_{n+1} being the finite set of all γ as in Definition 6.1.1 satisfying the following requirements:

(i) The weight $w(\gamma)$ is between 1 and $n+1$.

(ii) If γ is of type I with even weight then $b^* \in B_{0,n}$.

(iii) If γ is of type II with even weight then $w(\xi) = w(\gamma)$, $\text{age}(\xi) < n_{w(\gamma)}$ and $b^* \in B_{k,n}$.

(iv) If γ is of type I with odd weight then $b^* = e^*_\eta$ for some $\eta \in \Gamma_n$ with $w(\eta) = 4i-2$ and $m^{-1}_{w(\eta)} < n^{-2}_{w(\gamma)}$.

(v) If γ is of type II with odd weight then $w(\xi) = w(\gamma)$, $\text{age}(\xi) < n_{w(\gamma)}$ and $b^* = e^*_\eta$ for some $\eta \in \Gamma_n \setminus \Gamma_k$ with $w(\eta) = 4\sigma(\xi)$.

Here $\sigma : \Gamma \to \mathbb{N}$ is an injection chosen as part of the recursive definition in such a way that $\sigma(\gamma) > \text{rank}(\gamma)$ for all $\gamma \in \Gamma$.

For the rest of this chapter the sets Δ_n and Γ are fixed to be such as in this definition. We obtain, as before, a triangular basis $d^*_\gamma = e^*_\gamma - c^*_\gamma$ with projections P^*_E, P_E and extension mappings i_q as in Lemma 5.3.9. The assumption that $m_1 \geq 4$ implies that the constant M here is at most 2. Thus $\|i_q\| = \|P_{(0,n]}\| \leq 2$, $\|P_{(n,\infty)}\| \leq 3$ and $\|P_{(p,q]}\| \leq 4$ for all $p < q$.

Let us examine the resulting basis $(d_\gamma)_{\gamma\in\Gamma}$ a little more. We see that if we consider the support in terms of the basis instead of the FDD, we have that for $\gamma \in \Delta_{n+1}$ the support of d^*_γ is contained in $\Gamma_n \cup \{\gamma\}$. Therefore by biorthogonality we must have that the support of d_γ is contained in $(\Gamma \setminus \Gamma_n) \cup \{\gamma\}$. However the high availability of even and odd weight elements in each set Δ_n allow us to prove some stronger statements about these supports:

Proposition 6.1.7. $X_{AH} \cap c_0(\Gamma) = \{0\}$.

Proof. We will show that $d_\gamma \notin c_0(\Gamma)$ for all $\gamma \in \Gamma$. Indeed suppose for some $\gamma \in \Delta_{n+1}$ that $d_\gamma \in c_0(\Gamma)$. Then for $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $N > n$ and $|d_\gamma(\eta)| < \varepsilon$ for all $\eta \in \Gamma \setminus \Gamma_N$. So if $\eta \in \Delta_{N+1}$ we must have

$$0 = \langle d^*_\eta, d_\gamma \rangle = \langle e^*_\eta, d_\gamma \rangle - \langle c^*_\eta, d_\gamma \rangle .$$

Thus also $|\langle c^*_\eta, d_\gamma \rangle| < \varepsilon$ for all $\eta \in \Delta_{N+1}$. However we can pick an η of type I with weight 2 along with any $b^* \in B_{0,N}$. Since $B_{0,N}$ is a 2^{-N}-net in the unit ball of $\ell_1(\Gamma_N)$ we can find such an b^* with $\left\|b^* - e^*_\gamma\right\|_1 \leq 2^{-N}$. Thus in particular $b^*(\gamma) > \frac{1}{2}$. Further by flipping signs as needed we can assume that $b^*(\delta)d_\gamma(\delta) \geq 0$ for all $\delta \in \Gamma_N$. Then we have

$$|\langle c^*_\eta, d_\gamma \rangle| = |\langle m_2^{-1} b^*, d_\gamma \rangle| \geq m_2^{-1} b^*(\gamma) d_\gamma(\gamma) > \frac{1}{2} m_2^{-1}.$$

Thus if we pick $\varepsilon < \frac{1}{2} m_2^{-1}$ we have a contradiction. □

The above also implies that the support of each d_γ is an infinite set. In fact we can prove that this set contains elements of all weights:

Proposition 6.1.8. *For every $\gamma \in \Gamma$ and $i \in \mathbb{N}$ there exists an $\eta \in \Gamma$ with weight $w(\eta) = i$ such that $\langle e^*_\eta, d_\gamma \rangle \neq 0$.*

Proof. Let $\gamma \in \Delta_{n+1}$ and let $i \in \mathbb{N}$. If i is even, we can find a type I element η like in the previous proposition for which $\langle c^*_\eta, d_\gamma \rangle \neq 0$ and thus also $\langle e^*_\gamma, d_\gamma \rangle \neq 0$. If i is odd, we can again use the same argument to find a type I element η with $w(\eta) = 4j-2$ for some $j \in \mathbb{N}$ such that $m^{-1}_{4j-2} < n^{-1}_{w(\gamma)}$ for which $\langle e^*_\eta, d_\gamma \rangle \neq 0$. Then we can take η' to be a type I element with weight i and $b^* = e^*_\eta$ to have $\left\langle c^*_{\eta'}, d_\gamma \right\rangle = \langle e^*_\eta, d_\gamma \rangle \neq 0$ and thus $\left\langle e^*_{\eta'}, d_\gamma \right\rangle \neq 0$. □

These two results make it obvious why we should consider the support of an element $x \in X_{AH}$ in terms of the FDD rather than the basis. Otherwise there would not be a concept of finite support or of block sequences.

We will now introduce the evaluation analysis, which is our replacement in this setting for the tree analysis methods we have seen before. We will then state the important lemmas from Argyros and Haydon's paper to prepare for the proofs in the next sections.

Proposition 6.1.9. *Let $n \in \mathbb{N}$ and let $\gamma \in \Delta_{n+1}$ have weight j and age $a \leq n_j$. Then there exist natural numbers $0 = p_0 < ... < p_a = n+1$, elements $\xi_0, ..., \xi_a \in \Gamma$ such that $\xi_a = \gamma$, $\xi_r \in \Delta_{p_r}$ and $w(\xi_r) = j$ for all r and functionals $b^*_r \in \operatorname{ball} \ell_1(\Gamma_{p_r-1} \setminus \Gamma_{p_{r-1}})$ such that*

$$e^*_\gamma = \sum_{r=1}^{a} d^*_{\xi_r} + m_j^{-1} \sum_{r=1}^{a} P^*_{(p_{r-1},\infty)} b^*_r = \sum_{r=1}^{a} d^*_{\xi_r} + m_j^{-1} \sum_{r=1}^{a} P^*_{(p_{r-1},p_r)} b^*.$$

If $1 \leq t < a$ then further

$$e^*_\gamma = e^*_{\xi_t} + \sum_{r=t+1}^{a} d^*_{\xi_r} + m_j^{-1} \sum_{r=t+1}^{a} P^*_{(p_{r-1},\infty)} b^*_r.$$

Proof. We proceed by induction on the age a. In the case $a = 1$ we have a type I element, $\gamma = (n+1, m_j^{-1}, b^*)$ and $e^*_\gamma = d^*_\gamma + c^*_\gamma$, where c^*_γ is obtained from the mapping $\tau(\gamma) = (0, m_j^{-1}, b^*)$ and Definition 5.3.5. Thus $c^*_\gamma = m_j^{-1}(I - P^*_{(0,0]})b^* = m_j^{-1} P^*_{(0,\infty)} b^*$. So taking $p_1 = n+1$, $\xi_1 = \gamma$, $b^*_1 = b^*$ and $p_0 = 0$ we have the desired result.
Now if $a > 1$ we have a type II element $\gamma = (n+1, k, \xi, m_j^{-1}, b^*)$ and the BD-functional will be

$$c^*_\gamma = e^*_\xi + m_j^{-1}(I - P^*_{(0,k]})b^* = e^*_\xi + m_j^{-1} P^*_{(k,\infty)} b^*.$$

We can now apply the induction hypothesis to ξ, which has age $a-1$ and by the definition of Δ_n has the same weight as γ, and thus we can append the values $p_a = n+1$, $\xi_a = \gamma$ and $b^*_a = b^*$ to the sequence obtained from the induction hypothesis. □

Definition 6.1.10. For $\gamma \in \Gamma$, we refer to the sequence $(p_r, b_r^*, \xi_r)_{r=1}^a$ obtained in the previous proposition as the *evaluation* of γ. The first identity in the proposition above is called the *evaluation analysis* of γ.

It follows trivially from the definitions so far that if γ has weight j and age a with evaluation $(p_r, b_r^*, \xi_r)_{r=1}^a$, then $\{p_1, ..., p_a\} \in \mathcal{A}_{n_j}$. In fact, if we write $\text{cuts}(\gamma) = \{p_1, ..., p_a\}$, and $\mathcal{C}_j = \{\text{cuts}(\gamma) \mid \gamma \in \Delta_j\}$ then we have

$$\mathcal{C}_j = \{A \subseteq \mathbb{N} \mid \#A \leq n_j \text{ and } \min A \geq j\}.$$

Alternatively we can say that

$$\mathcal{A}_{n_j} = \{\text{cuts}(\gamma) \mid \gamma \in \Gamma \text{ and } w(\gamma) \leq j\}.$$

It is here that we see the connection between the constraints on the elements of Δ_n and the mixed Tsirelson spaces we saw in Chapter 3. Indeed, we see that the second term in the evaluation analysis is a weighted sum of functionals which are supported on intervals $E_r = (p_{r-1}, p_r)$ that form an $\mathcal{A}_{n_j}$-admissible sequence. See also the results in [15, Section 3].

The odd-weight elements in the definition of X_{AH} play a similar role as the special sequences do in the construction of the Gowers-Maurey space. Indeed we have the following analogue of the "tree-like" property stated in Lemma 4.2.4:

Lemma 6.1.11. *Let γ and γ' in Γ be elements of the same weight $2k-1$ and with ages $a \geq a'$. Let $(p_i, e_{\eta_i}^*, \xi_i)_{i=1}^a$ and $(p_i', e_{\eta_i'}^*, \xi_i')_{i=1}^{a'}$ be their respective analyses. Then there exists $1 \leq t \leq a'$ such that $\xi_i = \xi_i'$ for all $i < t$ while $w(\eta_j) \neq w(\eta_i')$ for all j and $t < i \leq a'$.*

Proof. Suppose $w(\eta_j) \neq w(\eta_i')$ for all j and $i \geq 2$. Then the result holds with $t = 1$. Otherwise let $l \geq 2$ be such that $w(\eta_j) = w(\eta_l')$ for some $1 \leq j \leq a$. Then since $w(\eta_j) = w(\eta_l') = 4\sigma(\xi_{l-1}')$ we must also have $j \geq 2$, since $w(\eta_1) = 4k-2$ which is not a multiple of 4. Thus $4\sigma(\xi_{l-1}') = w(\eta_j) = 4\sigma(\xi_{j-1})$ which implies $\xi_{j-1} = \xi_{l-1}'$. Further since $l - 1 = \text{age}(\xi_{l-1}') = \text{age}(\xi_{j-1}) = j - 1$ we must have $j = l$. Then since the evaluation is unique we must have $\xi_i = \xi_i'$ for all $i < l$. The last step is to take t to be the maximal l such that this holds. □

6.2 The auxiliary space

When we first introduced the basic inequality for mixed Tsirelson spaces in Theorem 3.4.4, we remarked that the argument could be adjusted to the context of a mixed Tsirelson space of the form $T[(\mathcal{A}_{n_i}, \theta_i)_{i\in\mathbb{N}}]$. We now need to explore this avenue in order to apply the basic inequality to rapidly increasing sequences in X_{AH} in the next section. We will see that the norming functionals obtained in the basic inequality for X_{AH} can be chosen to lie in the norming set $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$, where $(m_j, n_j)_{j\in\mathbb{N}}$

is as in Definition 6.1.3. The space $T[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ is what Argyros then calls the "auxiliary space". In this section we treat some necessary results regarding the structure of this space, following from the assumptions we made about the sequence $(m_j, n_j)_{j\in\mathbb{N}}$.

Recall that $\mathcal{A}_n = \{A \subseteq \mathbb{N} \mid \#A \leq n\}$. This yields a fairly weak admissibility rule for the tree analysis of a vector in the norming set. Besides the assumptions on the sequence $(m_j, n_j)_{j\in\mathbb{N}}$, the simplicity of these sets $\mathcal{A}_n$ is what allows us to obtain the following two lemmas.

The first of these lemmas is essentially a statement about the distribution of weights within an element of the norming set. If $f \in W$ and $(f_\sigma)_{\sigma\in\Upsilon}$ is a tree analysis, it is intuitively clear that the shape of this tree determines the size of an entry $|f(i)|$. In particular, for one such (non-zero) entry $i \in \mathbb{N}$ there will be a path in the tree from the root f to the leaf e_i. Then the value $|f(i)|$ is completely determined by the weights along this path, $|f(i)| = \theta_1 \cdot \theta_2 \cdot \ldots \cdot \theta_d$. It stands to reason that for a large tree most entries of f will thus become very small. The first of the two lemmas makes this precise for the case of the auxiliary space.

Lemma 6.2.1. *Let $j_0 \in \mathbb{N}$, $f \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ and $g \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$. Then we have the following inequalities:*

(i) $\#\left\{i \in \mathbb{N} \mid |g(i)| > m_{j_0}^{-2}\right\} \leq (3n_{j_0-1})^{\log_2 m_{j_0}-1}$,

(ii) $\#\left\{i \in \mathbb{N} \mid |f(i)| > m_{j_0}^{-1}\right\} \leq (3n_{j_0-1})^{\log_2 m_{j_0}-1}$.

Proof. (i) For any $g \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ write $M_r(g) = \#\{i \in \mathbb{N} \mid |g(i)| > 4^{-r}\}$. We will show by induction on $\#\mathrm{supp}(g)$ that $M_r(g) \leq (3n_{j_0-1})^{r-1}$ for all $r \leq \log_2 m_{j_0}$. Then inequality (i) follows since $4^{-\log_2 m_{j_0}} = m_{j_0}^{-2}$.
Let $g \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$. If $\#\mathrm{supp}(g) = 1$ then $M_r(g) \leq 1$ for all r and there is nothing to prove. Otherwise, $w(g) \geq 2$ and $w(g) \neq j_0$. Writing $h = w(g)$, there thus exists some $n \leq 3n_h$ and a sequence $(g_k)_{k=1}^n$ in $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$ such that

$$g = m_h^{-1} \sum_{k=1}^{n} g_k.$$

If $h > j_0$ then $\|g\|_\infty \leq m_h^{-1}$. By Definition 6.1.3(ii), $m_{j+1} \geq m_j^2$, so this implies $\|g\|_\infty \leq m_{j_0}^{-2}$ and thus $M_r(g) = 0$ for all $r \leq \log_2 m_{j_0}$.
Lastly if $h < j_0$ and $|g(i)| > 4^{-r}$ for some $i \in \mathrm{supp}(g)$, then there is one k for which $m_h^{-1}|g_k(i)| > 4^{-r}$. By Definition 6.1.3(i) we know $m_h \geq 4$, so this implies that $|g_k(i)| > 4^{-(r-1)}$. Thus

$$M_r(g) \leq \sum_{k=1}^{n} M_{r-1}(g_k) \leq n(3n_{j_0-1})^{r-2} \leq 3n_h(3n_{j_0-1})^{r-2} \leq (3n_{j_0-1})^{r-1},$$

where we have applied the induction hypothesis to the g_k which have smaller support than g.
(ii) The second inequality now follows as a special case of (i). If $f \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ we can define $g \in c_{00}$ by

$$g(i) = \begin{cases} f(i) & \text{if } |f(i)| > m_{j_0}^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Noting that the definition of $\mathcal{A}_n$ implies Ef is still in the norming set whenever E is a finite subset of $\mathbb{N}$, it follows that g is in fact contained in $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j<j_0}]$. Thus by (i) we have

$$\begin{aligned} \#\left\{i \in \mathbb{N} \mid |f(i)| > m_{j_0}^{-1}\right\} &= \#\left\{i \in \mathbb{N} \mid |g(i)| > m_{j_0}^{-1}\right\} \\ &\leq \#\left\{i \in \mathbb{N} \mid |g(i)| > m_{j_0}^{-2}\right\} \leq (3n_{j_0-1})^{\log_2 m_{j_0}-1}. \end{aligned}$$

□

The second lemma then applies this to obtain some useful norm estimations. We obtain upper bounds for the norm of a sum of unit vectors. These are just like those we found in Chapter 3 and 4. Indeed where before we had $n/F(n)$, we now get n_{j_0}/m_{j_0}.

Lemma 6.2.2. *Let $j_0 \in \mathbb{N}$ and let $f \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ be an element with weight $w(f) = h$. Then*

$$\left|\left\langle f, n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} e_k\right\rangle\right| \leq \begin{cases} 2m_h^{-1}m_{j_0}^{-1}, & \text{if } h < j_0, \\ m_h^{-1} & \text{if } h \geq j_0. \end{cases}$$

If in addition we assume $f \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$ then

$$\left|\left\langle f, n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} e_k\right\rangle\right| \leq \begin{cases} 2m_h^{-1}m_{j_0}^{-2}, & \text{if } h < j_0, \\ m_h^{-1} & \text{if } h > j_0. \end{cases}$$

Proof. The case $h \geq j_0$ is immediate, since then $\|f\|_\infty \leq m_h^{-1}$. So assume $h < j_0$. As usual, we write $f = m_h^{-1}\sum_{k=1}^n f_k$ with $f_k \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ and $n \leq 3n_h$. Write $E = \left\{i \in \mathbb{N} \mid |f(i)| > m_h^{-1}m_{j_0}^{-1}\right\}$ and $E_k = \left\{i \in \mathbb{N} \mid |f_k(i)| > m_{j_0}^{-1}\right\}$. Then we can estimate

$$\begin{aligned} \left|\left\langle f, n_{j_0}^{-1}\sum_{k=1}^{n_{j_0}} e_k\right\rangle\right| &\leq m_h^{-1}m_{j_0}^{-1} + n_{j_0}^{-1}\sum_{i\in E}|f(i)| \\ &\leq m_h^{-1}m_{j_0}^{-1} + m_h^{-1}n_{j_0}^{-1}\sum_{k=1}^{n}\sum_{i\in E_k}|f_k(i)| \end{aligned}$$

$$\leq m_h^{-1}m_{j_0}^{-1} + m_h^{-1}n_{j_0}^{-1}\sum_{k=1}^{n}\#E_k$$

Then by Lemma 6.2.1 we have $\#E_k \leq (3n_{j_0-1})^{\log_2 m_{j_0}-1}$ for all k. So we may estimate this by

$$\leq m_h^{-1}m_{j_0}^{-1} + m_h^{-1}n_{j_0}^{-1}3n_h(3n_{j_0-1})^{\log_2 m_{j_0}-1}$$
$$\leq m_h^{-1}m_{j_0}^{-1} + m_h^{-1}n_{j_0}^{-1}(3n_{j_0-1})^{\log_2 m_{j_0}}$$

By Definition 6.1.3(iv) we have $n_{j_0}^{-1}(4n_{j_0-1})^{\log_2 m_{j_0}} \leq m_{j_0}^{-2}$ so we get

$$\leq m_h^{-1}m_{j_0}^{-1} + m_h^{-1}m_{j_0}^{-2} \leq 2m_h^{-1}m_{j_0}^{-1}.$$

For the second part, replacing $m_{j_0}^{-1}$ with $m_{j_0}^{-2}$ in the definitions of E and E_k above gives the result. □

6.3 RIS sequences in X_{AH}

We return to the space X_{AH} itself and give the definition of a rapidly increasing sequence in terms of this new space. As before with the proof of the hereditary indecomposability of X_{GM}, the rapidly increasing sequences play a crucial role in the proof of the scalar-plus-compact property.

Definition 6.3.1. Let $I \subseteq \mathbb{N}$ be an interval. A block sequence $(x_k)_{k\in I}$ in X_{AH} is called a C-RIS for some $C > 1$ if $\|x_k\| \leq C$ for all $k \in I$ and there exists an increasing sequence $(j_k)_{k\in I}$ such that

(i) $j_{k+1} > \max \operatorname{ran}(x_k)$

(ii) $|x_k(\gamma)| \leq Cm_i^{-1}$ whenever $w(\gamma) = i < j_k$.

The next steps in Section 3.4 were to show existence of RIS sequences, to prove the basic inequality and obtain the upper bound for the norm of the RIS. Curiously, we do not strictly need to show the existence of RIS sequences in order to prove the scalar-plus-compact property of X_{AH}. Although a similar approach using ℓ_1^k-averages will work, the focus of this section will be on the basic inequality and then on exact pairs and dependent sequences. First, we need two lemmas.

Lemma 6.3.2. *Let $i > 0$ and suppose $x \in X_{AH}$ has the property that $\|x\| \leq C$ and $|x(\gamma)| \leq \delta$ whenever $w(\gamma) = i$. Then for any $q \in \mathbb{N}$ and γ with $w(\gamma) = i$ we have*

$$\left|\left\langle e_\gamma^*, P_{(q,\infty)}x\right\rangle\right| \leq 2\delta + 4Cm_i^{-1}.$$

Proof. Let $(p_r, b_r^*, \xi_r)_{r=1}^a$ be the evaluation of γ. So

$$e_\gamma^* = \sum_{r=1}^{a} d_{\xi_r}^* + m_i^{-1}\sum_{r=1}^{a} P_{(p_{r-1},\infty)}^* b_r^*.$$

If $q \geq p_a$ then $P^*_{(s,\infty)} e^*_\gamma = 0$. If $0 < q < p_1$ then applying $P^*_{(0,q]}$ to e^*_γ gives

$$P^*_{(q,\infty)} e^*_\gamma = (I - P^*_{(0,q]}) e^*_\gamma = e^*_\gamma - m_i^{-1} P^*_{(0,q]} b^*_1.$$

Thus

$$\left|\left\langle e^*_\gamma, P_{(q,\infty)} x \right\rangle\right| \leq |x(\gamma)| + m_i^{-1} \|b^*_1\| \, \|P_{(0,s]}\| \, \|x\| \leq \delta + 2C m_i^{-1}.$$

In the case $p_t \leq q \leq p_{t+1}$ we can use the second part of Proposition 6.1.9 to get

$$e^*_\gamma = e^*_{\xi_t} + \sum_{r=t+1}^{a} d^*_{\xi_r} + m_j^{-1} \sum_{r=t+1}^{a} P^*_{(p_{r-1},\infty)} b^*_r.$$

If we apply $P^*_{(0,q]}$ to this expression we get

$$P^*_{(q,\infty)} e^*_\gamma = e^*_\gamma - e^*_{\xi_t} - m_i^{-1} P^*_{(p_{t+1},q]} b^*_{t+1}.$$

Thus since $w(\xi_t) = w(\gamma) = i$ we have

$$\left|\left\langle e^*_\gamma, P_{(q,\infty)} x \right\rangle\right| \leq 2\delta + 4C m_i^{-1}.$$

□

Lemma 6.3.3. *Let $(x_k)_{k\in I}$ be a C-RIS in X_{AH} and let $i > 0$. Then for any $q \in \mathbb{N}$ and γ with $w(\gamma) = i$ we have*

$$\left|\left\langle e^*_\gamma, P_{(q,\infty)} x_k \right\rangle\right| \leq \begin{cases} 6C m_i^{-1} & \text{if } i < j_k, \\ 3C m_i^{-1} & \text{if } i \geq j_k. \end{cases}$$

Proof. The case $i < j_k$ follows from the previous lemma using $\delta = C m_i^{-1}$. In the case $i \geq j_k$ we can obtain a better estimate as follows. Let $(p_r, b^*_r, \xi_r)_{r=1}^a$ be the evaluation of γ. By RIS condition (2) we have $i \geq \max \operatorname{ran}(x_k)$. Further, since Δ_q only contains elements with weight i if $q \geq i$, it follows that $p_1 \geq i \geq \max \operatorname{ran}(x_k)$. Thus, $P_{[p_r,\infty)} x_k = 0$ for all $r \geq 1$ and so

$$\left\langle d^*_{\xi_r}, P_{(q,\infty)} x_k \right\rangle = \left\langle P^*_{[p_r,\infty)} e^*_{\xi_r}, P_{(q,\infty)} x_k \right\rangle = \left\langle e^*_{\xi_r}, P_{(q,\infty)} P_{[p_r,\infty)} x_k \right\rangle = 0.$$

So of the evaluation analysis we only have the term $P^*_{(0,\infty)} b^*_1$ left. Thus

$$\left|\left\langle e^*_\gamma, P_{(q,\infty)} x_k \right\rangle\right| = m_i^{-1} \left|\left\langle b^*_1, P_{(q,\infty)} x_k \right\rangle\right| \leq 3C m_i^{-1}.$$

□

We are now ready to state the basic inequality for X_{AH}. We also include the special case $W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$ which we have prepared for in the preceding section.

Theorem 6.3.4 (Basic inequality)**.** *Let $(x_k)_{k\in I}$ be a C-RIS in X_{AH}. Let $(\lambda_k)_{k\in I}$ be scalars, $q \in \mathbb{N}$ and $\gamma \in \Gamma$. Then there exist $k_0 \in I$ and a functional $g \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ such that either $g = 0$ or $w(g) = w(\gamma)$ and $k_0 < \operatorname{supp}(g)$, satisfying*

$$\left|\left\langle e_\gamma^*, P_{(q,\infty)} \sum_{k\in I} \lambda_k x_k \right\rangle\right| \leq 4C|\lambda_{k_0}| + 6C \left\langle g, \sum_{k\in I} |\lambda_k| e_k \right\rangle.$$

If moreover there exists a j_0 such that for all subintervals $J \subseteq I$, and all $\xi \in \Gamma$ with $w(\xi) = j_0$ we have

$$\left|\left\langle e_\xi^*, \sum_{k\in J} \lambda_k x_k \right\rangle\right| \leq C \max_{k\in J} |\lambda_k|,$$

then we can choose $g \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$.

Proof. We will use induction on the rank of γ. If $\operatorname{rank}\gamma = 1$ then $P^*_{(q,\infty)} e^*_\gamma = 0$ for all $q \geq 1$. So if $q \geq 1$ we can clearly take $g = 0$ and any $k_0 \in I$. If $q = 0$ then we can at most have that the first entry of the RIS has Δ_1 contained in its range. So letting $k_0 = \min I$, we have

$$\left\langle e_\gamma^*, P_{(q,\infty)} \sum_{k\in I} \lambda_k x_k \right\rangle = \lambda_{k_0} x_{k_0}(\gamma),$$

so this k_0 and $g = 0$ will do.

Now suppose γ has rank greater than 1 with age a and weight h. Let $(j_k)_{k\in I}$ be the increasing sequence associated with the RIS. Suppose that there exists an $i \in I$ such that $j_i \leq h < j_{i+1}$. (The other cases will follow from the proof for this case). We can then estimate the sum over I by splitting it into the parts $k < i$, $k = i$ and $k > i$. For the first part we have by Lemma 6.3.3 that

$$\left|\left\langle e_\gamma^*, P_{(q,\infty)} \lambda_k x_k \right\rangle\right| \leq 3C m_h^{-1} |\lambda_k| \leq 3C m_{j_k}^{-1} |\lambda_k|.$$

Summing over these, using that $m_j \geq 4^j$ and thus $\sum_{j=1}^\infty m_j^{-1} \leq 1/4$, we get

$$\left|\left\langle e_\gamma^*, P_{(q,\infty)} \sum_{k<i} \lambda_k x_k \right\rangle\right| \leq 3C \sum_{k<i} m_{j_k}^{-1} |\lambda_k| \leq C \max_{k<i} |\lambda_k|.$$

For $k = i$ we immediately have

$$\left|\left\langle e_\gamma^*, P_{(q,\infty)} \lambda_i x_i \right\rangle\right| \leq \|P_{(q,\infty)}\| \, |\lambda_i| \, \|x_i\| \leq 3C|\lambda_i|.$$

Thus the first to terms together are bounded by $4C|\lambda_{k_0}|$ whenever $k_0 = \operatorname{argmax}_{k<i} |\lambda_k|$. For the last part we will have to construct our functional g using the induction hypothesis. For convenience let us write $J = \{k \in I \mid k > i\}$ and $y_k = P_{(q,\infty)} x_k$. We have to divide this interval into suitable subintervals $J_1, J_2, \ldots$ such that we can apply

the induction hypothesis. We will do this based on the evaluation of γ. Indeed, let $(p_r, b_r^*, \xi_r)_{r=1}^a$ be the evaluation of γ. Define

$$J_0 = \{k \in J \mid p_r \in \operatorname{ran}(y_k) \text{ for some } 1 \le r \le a\},$$

$$J_r = \{k \in J \mid \operatorname{ran}(y_k) \subseteq (p_{r-1}, p_r)\}.$$

Since y_k is a block sequence, each J_r is a subinterval of J. Further $\#J_0 \le a$, so the family of sets

$$\{\{k\} \mid k \in J_0\} \cup \{J_r \mid 1 \le r \le a\}$$

is $\mathcal{A}_{2a}$-admissible. Clearly if $k \in J$ is not in any of these sets then $y_k(\gamma) = 0$. So using the evaluation analysis we can write

$$\left\langle e_\gamma^*, \sum_{k\in J} \lambda_k y_k \right\rangle = \left\langle e_\gamma^*, \sum_{k\in J_0} \lambda_k y_k \right\rangle + m_h^{-1} \sum_{r=1}^a \left\langle b_r^*, \sum_{k\in J_r} \lambda_k y_k \right\rangle.$$

Focussing on the second term, we note that when $k \in J_r$ we have

$$\left\langle e_\gamma^*, y_k \right\rangle = m_h^{-1} \left\langle b_r^*, P_{(p_{r-1},\infty)} y_k \right\rangle = m_h^{-1} \left\langle b_r^*, P_{(s\vee p_{r-1},\infty)} x_k \right\rangle.$$

This functional b_r^* is a convex combination of functionals $\pm e_\eta^*$ with $p_{r-1} < \operatorname{rank}\eta < p_r$. So there is an $\eta_r \in \Gamma_{p_r-1} \setminus \Gamma_{p_{r-1}}$ such that

$$\left|\left\langle b_r^*, P_{(s\vee p_{r-1},\infty)} x_k \right\rangle\right| \le \left|\left\langle e_{\eta_r}^*, P_{(q\vee p_{r-1},\infty)} x_k \right\rangle\right|.$$

We can apply the induction hypothesis to this η_r, the RIS $(x_k)_{k\in J_r}$ and $q' = q \vee p_{r-1}$. This gives an element $g_r \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ and a $k_r \in J_r$. We define the new functional g by

$$g = m_h^{-1} \left(\sum_{k\in J_0} e_k + \sum_{r=1}^a e_{k_r} + \sum_{r=1}^a g_r \right).$$

Then since $a \le n_h$ this is a sum of at most $3n_h$ elements, so we have that indeed $g \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ with $w(g) = h$. To verify that the final claimed inequality holds, we need the final note that for $k \in J_0$ we have by Lemma 6.3.3 that $|y_k(\gamma)| \le 6Cm_h^{-1}$. Then putting it all together we get

$$\left|\left\langle e_\gamma^*, P_{(q,\infty)} \sum_{k\in J} \lambda_k x_k \right\rangle\right| \le 4C|\lambda_{k_0}| + 6Cm_h^{-1} \sum_{k\in J_0} |\lambda_k| + m_h^{-1} \left| \sum_{r=1}^a \left\langle b_r^*, \sum_{k\in J_r} \lambda_k y_k \right\rangle \right|$$

$$\le 4C|\lambda_{k_0}| + 6Cm_h^{-1} \sum_{k\in J_0} |\lambda_k| + m_h^{-1} \left| \sum_{r=1}^a \left\langle e_{\eta_r}^*, P_{(s,\infty)} \sum_{k\in J_r} \lambda_k y_k \right\rangle \right|$$

$$\le 4C|\lambda_{k_0}| + 6Cm_h^{-1} \left(\sum_{k\in J_0} |\lambda_k| + \sum_{r=1}^a |\lambda_{k_r}| + \sum_{r=1}^a \left\langle g_r, \sum_{k\in J_r} |\lambda_k| e_k \right\rangle \right)$$

$$\leq 4C|\lambda_{k_0}| + 6C\left\langle g, \sum_{k\in J} |\lambda_k| e_k \right\rangle.$$

In the case that the additional assumption holds, but $h \neq j_0$, the same proof works, where in the inductive step we require that the $g_r \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$.
Lastly if $h = j_0$ we can find k_0 and g directly. Indeed there is at most one $l \in I$ for which $q \in \mathrm{ran}(x_l)$ and $P_{(q,\infty)}x_k = 0$ for $k < l$. Thus if we set $J = \{k \in I \mid k > l\}$ we have

$$\left|\left\langle e_\gamma^*, P_{(q,\infty)} \textstyle\sum_{k\in I} \lambda_k x_k \right\rangle\right| \leq |\lambda_l| \, \|P_{(q,\infty)}\| \, \|x_l\| + \left|\left\langle e_\gamma^*, \sum_{k\in J} \lambda_k x_k \right\rangle\right|.$$

Thus if $k_0 = \mathrm{argmax}_{k\in J\cup\{l\}} |\lambda_k|$ we have

$$\left|\left\langle e_\gamma^*, P_{(q,\infty)} \textstyle\sum_{k\in I} \lambda_k x_k \right\rangle\right| \leq 3C|\lambda_{k_0}| + C|\lambda_{k_0}| = 4C|\lambda_{k_0}|.$$

So we see that this choice of k_0 together with $g = 0$ will suffice. □

Corollary 6.3.5. *If $(x_k)_{k\in I}$ be a C-RIS in X_{AH} then for any scalars $(\lambda_k)_{k\in I}$ we have*

$$\left\|\sum_{k\in I} \lambda_k x_k\right\| \leq 10C \left\|\sum_{k\in I} \lambda_k e_k\right\|,$$

where the norm on the right is the norm in $T[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$.

We can now apply what we know from the preceding section about the norm in $T[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\in\mathbb{N}}]$ in order to obtain the upper bounds on the norm of a C-RIS in X_{AH}.

Corollary 6.3.6. *Let $(x_k)_{k=1}^{n_{j_0}}$ be a C-RIS in X_{AH}. Then for every $\gamma \in \Gamma$ with $w(\gamma) = h$ we have*

$$\left|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} x_k(\gamma)\right| \leq \begin{cases} 16Cm_h^{-1}m_{j_0}^{-1} & \text{if } h < j_0, \\ 4Cn_{j_0}^{-1} + 6Cm_h^{-1} & \text{if } h \geq j_0. \end{cases}$$

In particular

$$\left\|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} x_k\right\| \leq 10Cm_{j_0}^{-1}.$$

Proof. This now follows from the basic inequality together with Lemma 6.2.2. Indeed all coefficients λ_k are equal to $n_{j_0}^{-1}$. So there is a $g \in W[(\mathcal{A}_{3n_j}, m_j^{-1})_{j\neq j_0}]$ such that $w(g) = w(\gamma)$ and $k_0 < \mathrm{supp}(g)$ and

$$\left|n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} x_k(\gamma)\right| \leq 4Cn_{j_0}^{-1} + 6C\left\langle g, n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} e_{k_0} \right\rangle.$$

Then Lemma 6.2.2 gives

$$\left| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} x_k(\gamma) \right| \leq \begin{cases} 4Cn_{j_0}^{-1} + 12Cm_{j_0}^{-1}m_h^{-1} & \text{if } h < j_0, \\ 4Cn_{j_0}^{-1} + 6Cm_h^{-1} & \text{if } h \geq j_0. \end{cases}$$

Noting that $n_{j_0} > m_{j_0}^2$ gives the result. □

Corollary 6.3.7. *Let $(x_k)_{k=1}^{n_{j_0}}$ be a C-RIS in X_{AH}. If $(\lambda_k)_{k=1}^{n_{j_0}}$ are scalars with $|\lambda_k| \leq 1$ and having the property that for every interval $J \subseteq \{1, ..., n_{j_0}\}$ and every γ with $w(\gamma) = j_0$ we have*

$$\left| \sum_{k \in J} \lambda_k x_k(\gamma) \right| \leq C \max_{k \in J} |\lambda_k|,$$

then

$$\left\| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} \lambda_k x_k \right\| \leq 10Cm_{j_0}^{-2}.$$

Proof. This is the same as the previous corollary, but with the additional assumption on the coefficients λ_k. So we can apply the second cases of both the basic inequality and Lemma 6.2.2 to get the result. □

As we have noted in Section 6.1, the support of an element in terms of which $x(\gamma)$ are non-zero is not very useful in the context of X_{AH}. For this reason we have thus far used the support and range in terms of which projections onto the subspaces M_n of the FDD are non-zero. There is however a third way to look at the support of an element $x \in X_{AH}$ which Argyros and Haydon call the *local support*. It is defined as follows:

Definition 6.3.8. Let $x \in X_{AH}$ and let $q = \max \operatorname{ran}(x)$, so $x = i_q(u)$ with $u = P_{(0,q]}x$. The *local support* of x is defined as the set $\operatorname{locsupp}(x) = \{\gamma \in \Gamma_q \mid u(\gamma) \neq 0\}$.

We introduce two classes of block sequences based on the local support, both of which will turn out to give rapidly increasing sequences in X_{AH}.

Definition 6.3.9. A block sequence $(x_k)_{k \in N}$ in X_{AH} is said to have *bounded local weight* if there exists an h_0 such that for all $k \in \mathbb{N}$ we have $w(\gamma) \leq h_0$ for all $\gamma \in \operatorname{locsupp}(x_k)$. Further $(x_k)_{k \in N}$ is said to have *rapidly decreasing local weight* if for all $k \in \mathbb{N}$ and all $\gamma \in \operatorname{locsupp}(x_{k+1})$ we have $w(\gamma) > \max \operatorname{ran}(x_k)$.

We first need the following easy lemma:

Lemma 6.3.10. *Let $\gamma \in \Gamma$ with $w(\gamma) = h$ and assume that $w(\xi) \neq h$ for all $\xi \in \operatorname{locsupp}(x)$. Then $|x(\gamma)| \leq 3m_h^{-1}\|x\|$.*

Proof. Let $q = \max \operatorname{ran}(x)$ and write $u = P_{(0,q]}x$ so $x = i_q(u)$. Then if $\gamma \in \Gamma_q$ we have $x(\gamma) = u(\gamma)$, which implies we must have $x(\gamma) = 0$ since otherwise $\gamma \in \operatorname{locsupp}(x)$ and $w(\gamma) = h$. So in this case we have nothing to prove. Now suppose $\operatorname{rank}(\gamma) > q$. Let the evaluation of γ be $(p_r, b_r^*, \xi_r)_{r=1}^a$. There is a minimal t such that $p_t = \operatorname{rank}(\xi_t) > q$. By the second part of Proposition 6.1.9 we then have

$$e_\gamma^* = e_{\xi_t}^* + \sum_{r=t+1}^{a} d_{\xi_r}^* + m_h^{-1} \sum_{r=t+1}^{a} P_{(p_{r-1},\infty)}^* b_r^*.$$

By similar arguments as before $P_{(p_{r-1},\infty)}x = 0$ and $\left\langle d_{\xi_r}^*, x \right\rangle = 0$ for all $r \geq t+1$. So $e_\gamma^* = e_{\xi_t}^*$. If $t = 1$ then $e_{\xi_t}^* = m_h^{-1} b_1^*$. Whereas if $t > 1$ we may write $e_{\xi_t}^* = e_{\xi_{t-1}}^* + m_h^{-1} b_t^*$. We note that $\xi_{t-1} \in \Gamma_q$ by minimality of t, but with $w(\xi_{t-1}) = h$, so $\left\langle e_{\xi_{t-1}}^*, x \right\rangle = x(\xi_{t-1}) = 0$. Thus in both cases we have $x(\gamma) = m_h^{-1} \left\langle b_t^*, P_{(p_{t-1},\infty)}x \right\rangle$. Since $\|b_t^*\| = 1$ and $\|P_{(p_{t-1},\infty)}\| \leq 3$ this gives the desired estimate. □

Proposition 6.3.11. *Let $(x_k)_{k\in\mathbb{N}}$ be a bounded block sequence in X_{AH}. If $(x_k)_{k\in\mathbb{N}}$ has either bounded local weight or rapidly decreasing local weight, then $(x_k)_{k\in\mathbb{N}}$ is a C-RIS for some $C > 1$.*

Proof. Let the norms of $(x_k)_{k\in\mathbb{N}}$ be a bounded above by some $M > 0$. First suppose that $(x_k)_{k\in\mathbb{N}}$ has rapidly decreasing local weight. Let j_k be the minimum weight of an element $\gamma \in \operatorname{locsupp}(x_k)$. Since $j_{k+1} > \max \operatorname{ran}(x_k)$, RIS condition (i) is already satisfied. Also if $h < j_k$ and $w(\gamma) = h$ then $|x_k(\gamma)| \leq 3m_h^{-1}\|x_k\| \leq 3Mm_h^{-1}$ by the previous lemma. So if we take $C = 3M$ we have that $(x_k)_{k\in\mathbb{N}}$ is a C-RIS.

Now suppose that the block sequence has bounded local weight. Let j_1 be the upper bound for the weights. Then define $j_{k+1} = 1 + \max \operatorname{ran}(x_{k-1})$ so that we have RIS condition (i). Now if $w(\gamma) = h$ and $h < j_k$ then either $h > j_1$ in which case we have $|x_k(\gamma)| \leq 3Mm_h^{-1}$, or $h \leq j_1$ in which case $|x_k(\gamma)| \leq \|x_k\| \leq m_h^{-1} m_{j_1} \|x_k\|$. So $C = m_{j_1} M$ is the suitable constant for which $(x_k)_{k\in\mathbb{N}}$ is a C-RIS. □

As a result we can prove the following proposition, which is a crucial step in showing the scalar-plus-compact property of X_{AH}.

Proposition 6.3.12. *Let Y be a Banach space and let $T : X_{AH} \to Y$ be a bounded linear operator. If $\|Tx_k\| \to 0$ for every RIS $(x_k)_{k\in\mathbb{N}}$ in X_{AH}, then $\|Tx_k\| \to 0$ for every bounded block sequence $(x_k)_{k\in\mathbb{N}}$ in X_{AH}.*

Proof. It is enough to show that every bounded block sequence $(x_k)_{k\in\mathbb{N}}$ has a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ for which $\|Tx_{k_j}\| \to 0$. Indeed then if $(x_k)_{k\in\mathbb{N}}$ is a block sequence such that $\|Tx_k\| \not\to 0$, then there must be a subsequence such that $\|Tx_{k_j}\| \geq \varepsilon$ for some $\varepsilon > 0$. But then this is again a block sequence, so there is a further subsequence such that this norm does converge to 0, which is a contradiction.

Now to construct this subsequence, write $u_k = P_{(0,q_k]}x_k$ where $q_k = \max \operatorname{ran}(x_k)$. For $N \in \mathbb{N}$ we can split each entry into $u_k = v_k^N + w_k^N$ where

$$v_k^N(\gamma) = \begin{cases} u_k(\gamma) & \text{if } w(\gamma) \leq N, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad w_k^N(\gamma) = \begin{cases} u_k(\gamma) & \text{if } w(\gamma) > N, \\ 0 & \text{otherwise.} \end{cases}$$

We can then write $x_k = y_k^N + w_k^N$ where $y_k^N = i_{q_k}(v_k^N)$ and $z_k^N = i_{q_k}(w_k^N)$. Then $\|y_k^N\| \leq 2\|x_k\|$ and $\|w_k^N\| \leq 2\|x_k\|$. It is clear that each $(y_k^N)_{k\in\mathbb{N}}$ has bounded local weight for any N. So $\|Ty_k^N\| \to 0$ for each N. By a diagonal argument we get $(k_N)_{N\in\mathbb{N}}$ such that $\|Ty_{k_N}^N\| \to 0$ as $N \to \infty$. Next if we take $N_1 = 1$ and $N_{j+1} = q_{k_{N_j}}$ we get that the sequence $(z_{k_{N_j}}^{N_j})_{j\in\mathbb{N}}$ has rapidly decreasing local weight. So the subsequence $x_{k_{N_j}} = y_{k_{N_j}}^{N_j} + w_{k_{N_j}}^{N_j}$ satisfies $\|Tx_{k_{N_j}}\| \to 0$. □

Corollary 6.3.13. *The dual of X_{AH} is isomorphic to ℓ_1.*

Proof. By Proposition 5.3.10 it is enough to show that the FDD in X_{AH} is shrinking. This is equivalent to showing that every bounded block sequence (with respect to the FDD) is weakly null. (See [13, Section 3.2]). So let $\phi \in X_{AH}^*$ and let $(x_k)_{k\in\mathbb{N}}$ be a C-RIS, and suppose that $|\phi(x_k)| \not\to 0$. By passing to a subsequence and potentially flipping the sign of ϕ we may assume $\phi(x_k) \geq \varepsilon$ for some $\varepsilon > 0$ and for all k. Then we are still dealing with a C-RIS, so for $j_0 \in \mathbb{N}$ we have

$$n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} \phi(x_k) \geq \varepsilon.$$

However by Corollary 6.3.6 we should have

$$\left| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} \phi(x_k) \right| \leq \|\phi\| \left\| n_{j_0}^{-1} \sum_{k=1}^{n_{j_0}} x_k \right\| \leq 10C \|\phi\| m_{j_0}^{-1},$$

which converges to 0 as $j_0 \to \infty$. This is a contradiction, so we can now apply the previous proposition to conclude that $\phi(x_k) \to 0$ for every bounded block sequence $(x_k)_{k\in\mathbb{N}}$ and every $\phi \in X_{AH}^*$. □

The next step is to redefine the exact pairs and dependent sequences in terms of X_{AH}. At this point the definitions should no longer be surprising.

Definition 6.3.14. Let $C > 0$ and let $\varepsilon \in \{0, 1\}$. A pair $(x, \eta) \in X_{AH} \times \Gamma$ is called a (C, j, ε)-*exact pair* if

(i) $\left|\left\langle d_\xi^*, x \right\rangle\right| \leq C m_j^{-1}$ for all $\xi \in \Gamma$,

(ii) $w(\eta) = j$,

(iii) $\|x\| \leq C$ and $x(\eta) = \varepsilon$.

(iv) For every $\xi \in \Gamma$ with $w(\xi) = i \neq j$ we have $|x(\xi)| \leq Cm_{i\wedge j}^{-1}$.

For the scalar-plus-compact property we will only need to consider the case $\varepsilon = 0$. We can construct dependent pairs for this case using an RIS sequence. Note again the similarity between this construction and the one in Chapter 4.

Lemma 6.3.15. *Let $(x_k)_{k=1}^{n_{2j}}$ be a skipped block C-RIS in X_{AH}. Let $q_0 < q_1 < ... < q_{n_{2j}}$ be such that $\mathrm{ran}(x_k) \subseteq (q_{k-1}, q_k)$. Write*

$$z = \frac{m_{2j}}{n_{2j}} \sum_{k=1}^{n_{2j}} x_k.$$

Further let $b_k^* \in B_{q_{k-1},q_k-1}$ such that $\langle b_k^*, x_k\rangle = 0$ for each $1 \leq k \leq n_{2j}$. Then there exist $\xi_1, ..., \xi_{n_{2j}}$ such that the element $\eta = \xi_{n_{2j}}$ has analysis $(q_k, b_k^*, \xi_k)_{k=1}^{n_{2j}}$ and (z, η) is a $(16C, 2j, 0)$-exact pair.

Proof. For every $\gamma \in \Gamma$ there is at most one k for which $\langle d_\gamma^*, x_k\rangle \neq 0$. So

$$\left|\langle d_\gamma^*, z\rangle\right| \leq \frac{m_{2j}}{n_{2j}} \left|\langle d_\gamma^*, x_k\rangle\right| \leq \frac{m_{2j}}{n_{2j}} \|d_\gamma^*\|\|x_k\| \leq Cm_{2j}^{-1},$$

which is the first condition. Next, we may simply pick $\xi_1 = (q_1, m_{2j}^{-1}, b_1^*)$ and for $2 \leq k \leq n_{2j}$ we may pick $\xi_k = (q_k, q_{k-1}, \xi_{k-1} m_{2j}^{-1}, b_k^*)$. Then $\eta = \xi_{n_{2j}}$ has the desired analysis with $w(\eta) = 2j$. Since $\mathrm{ran}(x_k) \subset (q_{k-1}, q_k)$ we then have $\left\langle d_{\xi_i}^*, x_j \right\rangle = 0$ for all i, j, and $P_{(q_{i-1},q_i)}^* x_j = 0$ whenever $i \neq j$ and $P_{(q_{i-1},q_i)}^* x_j = x_j$ when $i = j$. So

$$z(\eta) = \langle e_\eta^*, z\rangle = n_{2j}^{-1} \sum_{k=1}^{n_{2j}} \langle b_k^*, x_k\rangle = 0.$$

Further by Corollary 6.3.7 we have $\|z\| \leq 10C$. So condition (iii) is satisfied. Finally, to show condition (iv) we need to use that $(x_k)_{k=1}^{n_{2j}}$ is a C-RIS. Indeed, if $w(\gamma) = h \neq 2j$ then we can apply Corollary 6.3.7 with $j_0 = 2j$ to get

$$|x(\gamma)| = \frac{m_{2j}}{n_{2j}} \left|\sum_{k=1}^{n_{2j}} x_k(\gamma)\right| \leq \begin{cases} 16Cm_h^{-1} & \text{if } h < 2j, \\ 4Cm_{2j}n_{2j}^{-1} + 6Cm_{2j}m_h^{-1} & \text{if } h > 2j. \end{cases}$$

The second of these cases we can further estimate by $10Cm_{2j}^{-1}$. So in both cases we have $|x(\gamma)| \leq 16Cm_{h\vee 2j}^{-1}$. □

Definition 6.3.16. Let $C > 0$ and let $\varepsilon \in \{0, 1\}$. A sequence $(x_k)_{k=1}^{n_{2j_0-1}}$ in X_{AH} is called a $(C, 2j_0 - 1, \varepsilon)$-*dependent sequence* if there exist $0 = p_0 < p_1 < ... < p_{n_{2j_0-1}}$ together with $\eta_k \in \Gamma_{p_k-1} \setminus \Gamma_{p_{k-1}}$ and $\xi_k \in \Delta_{p_k}$ for $1 \leq k \leq n_{2j_0-1}$ such that

(i) for each k, $\mathrm{ran}(x_k) \subseteq (p_{k-1}, p_k)$,

(ii) the element $\xi = \xi_{n_{2j_0-1}}$ has analysis $(p_k, e^*_{\eta_k}, \xi_k)_{k=1}^{n_{2j_0-1}}$,

(iii) (x_1, η_1) is a $(C, 4j_1 - 2, \varepsilon)$-exact pair,

(iv) for each $2 \leq k \leq n_{2j_0-1}$ the element (x_k, η_k) is a $(C, 4j_k, \varepsilon)$-exact pair.

Lemma 6.3.17. *A $(C, 2j_0 - 1, \varepsilon)$-dependent sequence in X_{AH} is a C-RIS.*

Proof. Take $j'_1 = 4j_1 - 2$ and $j'_k = 4j_k$ for $2 \leq k \leq n_{2j_0-1}$. We have $j'_{k+1} = 4\sigma(\xi_i) > \operatorname{rank} x_i = p_i$, from which it follows that $j'_{k+1} > \max \operatorname{ran}(x_k)$. So condition (i) is satisfied. Condition (ii) is satisfied by the definition of an exact pair. □

We now finally arrive at the payoff for considering the second case $j \neq j_0$ in the previous results about the norms in X_{AH} and the auxiliary space. It allows us to prove here that the 0-dependent sequences satisfy a much stronger upper bound than a general rapidly increasing sequence.

Proposition 6.3.18. *Let $(x_k)_{k=1}^{n_{2j_0-1}}$ be a $(C, 2j_0 - 1, 0)$-dependent sequence in X_{AH}. Then*

$$\left\| n_{2j_0-1}^{-1} \sum_{k=1}^{n_{2j_0-1}} x_k \right\| \leq 30C m_{2j_0-1}^{-2}.$$

Proof. We wish to apply Corollary 6.3.7 with $\lambda_i = 1$ and $2j_0 - 1$. So we need to show that the coefficients satisfy the condition of the lemma. Let $J \subseteq \{1, ..., n_{2j_0-1}\}$ and let $\gamma' \in \Gamma$ with $w(\gamma') = 2j_0 - 1$. We will show that

$$\left| \sum_{i \in J} x_i(\gamma') \right| \leq 3C.$$

Let ξ_i, η_i, p_i and j_i be such as in the definition of a dependent sequence. Let $\gamma = \xi_{n_{2j_0-1}}$ and let $(p'_i, e^*_{\eta'_i}, \xi'_i)_{i=1}^{a'}$ be the analysis of γ'. Note that $a' \leq n_{2j_0-1}$. Thus we can apply Lemma 6.1.11 to get a $1 \leq t \leq a'$ such that $(p'_i, \eta'_i, \xi'_i) = (p_i, \eta_i, \xi_i)$ for all $i < t$ while $w(\eta_k) \neq w(\eta'_i)$ for all $i \leq a'$ when $k > t$. So, for $k < t$ we have $\operatorname{ran}(x_k) \subseteq (p_{k-1}, p_k) = (p'_{k-1}, p'_k)$ and so

$$x_k(\gamma') = x_k(\gamma) = m_{2j_0-1}^{-1} \left\langle P^*_{(p_{k-1},\infty)} e^*_{\eta_k}, x_k \right\rangle = m_{2j_0-1}^{-1} x_k(\eta_k) = 0.$$

On the other hand for $k > t$ we can write out the evaluation analysis

$$|x_k(\gamma')| \leq \sum_{i \leq a'} \left| \left\langle d^*_{\xi'_i}, x_k \right\rangle + m_{2j_0-1}^{-1} \left\langle e^*_{\eta'_i}, P_{(p'_{i-1},\infty)} x_k \right\rangle \right|,$$

which we can approximate by applying Def. 6.3.14(i) to get

$$\left| \left\langle d^*_{\xi'_i}, x_k \right\rangle \right| \leq C m_{w(\eta_k)}^{-1},$$

and Def. 6.3.14(iv), noting $w(\eta_i') \neq w(\eta_k)$ to get

$$\left| m_{2j_0-1}^{-1} \left\langle e_{\eta_i'}^*, P_{(p_{i-1}', \infty)} x_k \right\rangle \right| \leq 6C m_{2j_0-1}^{-1} m_{w(\eta_i') \wedge w(\eta_k)}^{-1}.$$

Together these are at most $2C \min\{w(\eta_1), w(\eta_1')\} \leq 2C n_{2j_0-1}^{-2}$. So in total we have

$$\left| \sum_{k \in J} x_k(\gamma') \right| \leq \|x_t\| + \sum_{\substack{k \in J \\ k > t}} \sum_{i \leq a'} 2C n_{2j_0-1}^{-2} \leq C + 2C = 3C.$$

□

Remark 6.3.19. The above argument, together with a 1-dependent sequence, will give the HI property of X_{AH}. Indeed if $(x_k)_{k=1}^{n_{2j_0-1}}$ is a $(C, 2j_0 - 1, 1)$-dependent sequence, then it is easy to see

$$\| n_{2j_0-1}^{-1} \sum_{k=1}^{n_{2j_0-1}} x_k \| \geq m_{2j_0-1}^{-1},$$

while replacing $3C$ by $4C$ above one can show

$$\| n_{2j_0-1}^{-1} \sum_{k=1}^{n_{2j_0-1}} (-1)^k x_k \| \leq 40C m_{2j_0-1}^{-2}.$$

By the same argument as in Chapter 4, this leads to the fact that X_{AH} is hereditarily indecomposable. The details are found in [3, Section 8].

6.4 The operators on X_{AH}

We turn to the final proof that X_{AH} has the scalar-plus-compact property. The strategy will be to show, using the results about rapidly increasing sequences, that for an operator $T : X_{AH} \to X_{AH}$ and an RIS (x_i) in X_{AH}, there exist scalars λ_i such that $\|Tx_i - \lambda_i x_i\| \to 0$. Then showing that the λ_i converge to some limit λ will give the scalar such that $T - \lambda I$ is compact. The existence of these scalars is equivalent to the statement that the distance between Tx_i and $\mathbb{R}x_i$ converges to zero, where the distance is defined as follows:

Definition 6.4.1. For a Banach space X and two points $x, y \in X$ we define

$$\operatorname{dist}(y, \mathbb{R}x) = \inf_{\lambda \in \mathbb{R}} \|y - \lambda x\|.$$

For the next two lemmas it will be easier to make the switch to work with elements of X_{AH} with rational coordinates, that is we work in $X_{AH} \cap \mathbb{Q}^\Gamma$. By construction, each d_γ is contained in $X_{AH} \cap \mathbb{Q}^\Gamma$, so that $X_{AH} \cap \mathbb{Q}^\Gamma$ is dense in X_{AH}. The result of the second lemma will allow us to construct a 0-dependent sequence if we assume $\operatorname{dist}(Tx_i, \mathbb{R}x_i) \not\to 0$. In the proposition thereafter we will use this to derive a contradiction.

Lemma 6.4.2. *Let $x \in X_{AH} \cap \mathbb{Q}^\Gamma$ and $y \in X_{AH}$ be such that* $\mathrm{ran}(x)$ *and* $\mathrm{ran}(y)$ *are both contained in some interval $(m,n]$. If* $\mathrm{dist}(y, \mathbb{R}x) > \delta$, *then there exists $b^* \in \mathrm{ball}\, \ell_1(\Gamma_n \setminus \Gamma_m)$ with rational coordinates such that $b^*(x) = 0$ and $b^*(y) > \frac{1}{2}\delta$.*

Proof. Let $u, v \in \ell_\infty(\Gamma_n \setminus \Gamma_m)$ be such that $x = i_n(u)$ and $y = i_n(v)$. Then for any scalar $\lambda \in \mathbb{R}$ we have $\|y - \lambda x\| \leq \|i_n\| \, \|v - \lambda u\|$. So $\mathrm{dist}(v, \mathbb{R}u) > \frac{1}{2}\delta$. Applying the Hahn-Banach theorem to the finite-dimensional space $\ell_\infty(\Gamma_n \setminus \Gamma_m)$ we find $a^* \in \mathrm{ball}\, \ell_\infty(\Gamma_n \setminus \Gamma_m)$ with $a^*(u) = 0$ and $a^*(v) > \frac{1}{2}\delta$. Then by density we can approximate a^* by a b^* with rational coordinates having the same properties. □

Lemma 6.4.3. *Let $T : X_{AH} \to X_{AH}$ be a bounded operator and $(x_i)_{i\in\mathbb{N}}$ be a C-RIS in $X_{AH} \cap \mathbb{Q}^\Gamma$. Suppose that* $\mathrm{dist}(Tx_i, \mathbb{R}x_i) > \delta > 0$ *for all $i \in \mathbb{N}$. Then for all $j, p \in \mathbb{N}$ there exist $z \in [x_i]_{i\in\mathbb{N}}$, $q > p$ and $\eta \in \Delta_q$ such that*

(i) (z, η) *is a* $(16C, 2j, 0)$*-exact pair,*

(ii) $\mathrm{ran}(z) \subseteq (p, q]$,

(iii) $(Tz)(\eta) > \frac{7}{16}\delta$,

(iv) $\|(I - P_{(p,q)})Tz\| < m_{2j}^{-1}\delta$,

(v) $\left\langle P^*_{(p,q]} e^*_\eta, Tz \right\rangle > \frac{3}{8}\delta$.

Proof. Since $(x_i)_{i\in\mathbb{N}}$ is a block sequence, Tx_i is weakly null. This implies that for $\varepsilon > 0$ there exist $p < q_0 < q_1 < \ldots$ and a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ such that $\mathrm{ran}(x_{n_i}) \subseteq (q_{i-1}, q_i)$ and

$$\left\|(I - P_{(q_{i-1},q_i)})Tx_{n_i}\right\| < \varepsilon.$$

Indeed note that

$$\left\|(I - P_{(a,b)})Tx_i\right\| \leq \|P_{(0,a]}Tx_i\| + \|(I - P_{(0,b]})Tx_i\|,$$

where the first term goes to 0 as $i \to \infty$ while the second term goes to 0 as $b \to \infty$. Thus given q_{i-1} we can first find n_i such that $\mathrm{ran}(x_{n_i}) \subseteq (q_{i-1}, \infty)$ and $\|P_{(0,q_{i-1}]}Tx_{n_i}\| < \frac{1}{2}\varepsilon$. Then we can find $q_i > q_{i-1}$ such that $\mathrm{ran}(x_{n_i}) \subseteq (q_{i-1}, q_i)$ and $\|(I - P_{(0,q_i]})Tx_{n_i}\| < \frac{1}{2}\varepsilon$.

We take $\varepsilon = \frac{1}{5}m_{2j}^{-2}\delta$ and suppose by passing to a subsequence that $(x_i)_{i\in\mathbb{N}}$ already satisfies the above property. This choice of upper bound will lead to inequalities (iii) through (v) after the construction of z and η. First note that $m_{2j} \geq 16$, so $\varepsilon \leq \frac{1}{5}(\frac{1}{16})^2\delta = \frac{1}{1280}\delta$. Thus $\mathrm{dist}(P_{(q_{i-1},q_i)}Tx_i, \mathbb{R}x_i) > \frac{1279}{1280}\delta$. Applying the previous lemma we obtain for each i a functional $b_i^* \in \mathrm{ball}\, \ell_1(\Gamma_{q_i-1} \setminus \Gamma_{q_{i-1}})$ with rational coordinates such that $\langle b_i^*, x_i\rangle = 0$ and $\left\langle b_i^*, P_{(q_{i-1},q_i)}Tx_i\right\rangle > \frac{1279}{2560}\delta$.

Now we may take a further subsequence $(x_{n_i})_{i\in\mathbb{N}}$ and new values q_i' such that $b^*_{n_i} \in B_{q_{i-1}', q_i'-1}$. We do this by $n_1 = 0$ and $q_0' = q_0$, and then given q_{i-1}' and n_i we may

find $q_i' > \max\{q_{i-1}', q_i\}$ large enough so that the coordinates of $b_{n_i}^*$ have denominators dividing $N_{q_i'-1}!$. Then we can simply take n_{i+1} such that $\mathrm{ran}(x_{n_{i+1}}) \subseteq (q_i', \infty)$. Again we assume by passing to this subsequence that x_i, q_i and b_i^* already satisfy these properties.

Lastly we assume $q_1 \geq 2j$. Then we are now in a position to apply Lemma 6.3.15, giving us a $(16C, 2j, 0)$-exact pair (z, η), where z is the weighted average

$$z = \frac{m_{2j}}{n_{2j}} \sum_{i=1}^{n_{2j}} x_i.$$

We have $\varepsilon \leq \frac{1}{80} m_{2j}^{-1} \delta$. We use this to estimate $(Tx_i)(\eta)$ by

$$\begin{aligned}
(Tx_i)(\eta) &= \langle e_\eta^*, P_{(q_{i-1},q_i)} Tx_i \rangle + \langle e_\eta^*, (I - P_{(q_{i-1},q_i)}) Tx_i \rangle \\
&\geq \langle e_\eta^*, P_{(q_{i-1},q_i)} Tx_i \rangle - \left\| (I - P_{(q_{i-1},q_i)}) Tx_i \right\| \\
&= m_{2j}^{-1} \langle b_i^*, P_{(q_{i-1},q_i)} Tx_i \rangle - \left\| (I - P_{(q_{i-1},q_i)}) Tx_i \right\| \\
&> (\frac{1279}{2560} - \frac{1}{80}) m_{2j}^{-1} \delta > \frac{7}{16} m_{2j}^{-1} \delta.
\end{aligned}$$

Thus we see that $(Tz)(\eta) > \frac{7}{16}\delta$. Next, if we take $q = q_{n_{2j}}$, then $\mathrm{ran}(z) \subseteq (p, q]$ and we obtain inequality (iv) since

$$\begin{aligned}
\left\| (I - P_{(p,q]}) Tx_i \right\| &= \left\| (P_{(0,p]} + P_{(q,\infty)})(I - P_{(q_{i-1},q_i)}) Tx_i \right\| \\
&\leq 5 \left\| (I - P_{(q_{i-1},q_i)}) Tx_i \right\| < m_{2j}^{-2} \delta.
\end{aligned}$$

So that $\|(I - P_{(p,q)}) Tz\| < m_{2j}^{-1} \delta$. Lastly we have (v) since

$$\left\langle P_{(p,q]}^* e_\eta^*, Tz \right\rangle \geq (Tz)(\eta) - \left\| (I - P_{(p,q]}) Tz \right\| > \frac{7}{16}\delta - m_{2j}^{-1}\delta \geq \frac{3}{8}\delta.$$

This concludes the lemma. □

Proposition 6.4.4. *Let $T : X_{AH} \to X_{AH}$ be a bounded operator and let $(x_i)_{i \in \mathbb{N}}$ be a C-RIS in X_{AH}. Then $\mathrm{dist}(Tx_i, \mathbb{R}x_i) \to 0$ as $i \to \infty$.*

Proof. By density, we may assume that $x_i \in X_{AH} \cap \mathbb{Q}^\Gamma$ for all $i \in \mathbb{N}$. Suppose that $\mathrm{dist}(Tx_i, \mathbb{R}x_i) > \delta > 0$ for all $i \in \mathbb{N}$. We will carefully pick indices to satisfy the conditions of a dependent sequence. First off, we pick j_0 such that

$$m_{2j_0-1} > (4 \cdot 16 \cdot 30) C \|T\| \delta^{-1}$$

and then j_1 such that $m_{4j_1-2} > n_{2j_0-1}^2$. Let Lemma 6.4.3 with $j = 2j_1 - 1$ and $p = p_0 = 0$ give q_1, z_1 and η_1. Let $p_1 = q_1 + 1$ and define $\xi_1 = (p_1, m_{2j_0-1}^{-1}, e_{\eta_1}^*) \in \Delta_{p_1}$. Now we can continue inductively by defining $j_{i+1} = \sigma(\xi_i)$, and applying Lemma 6.4.3 again with $j = 2j_{i+1} - 1$ and $p = p_i$ to get q_{i+1}, z_{i+1} and η_{i+1}. Naturally we take

$p_{i+1} = q_{i+1} + 1$, and then we define $\xi_{i+1} = (p_{i+1}, p_i, \xi_i, m_{2j_0-1}^{-1}, e^*_{\eta_{i+1}}) \in \Delta_{p_{i+1}}$.

We continue this until we have the $(16C, 2j_0 - 1, 0)$-dependent sequence $(z_i)_{i=1}^{n_{2j_0-1}}$. Indeed, it is clear from (i) and (ii) of Lemma 6.4.3 that properties (i), (iii) and (iv) of Definition 6.3.16 are satisfied. Lastly property (ii) follows from an inductive argument much like the one in the proof of Proposition 6.1.9.

Now if we write

$$z = n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} z_i,$$

then by Proposition 6.3.18 we have $\|z\| \leq (30 \cdot 16) C m_{2j_0-1}^{-2}$. And if we write $\gamma = \xi_{n_{2j_0-1}}$. Then by property (ii) we have

$$e^*_\gamma = \sum_{i=1}^{n_{2j_0-1}} d^*_{\xi_i} + m_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} P^*_{(p_{i-1},p_i)} e^*_{\eta_i}.$$

Since $m_{4j_i} > m_{4j_1-2} > n_{2j_0-1}^2$ we can estimate $(Tz)(\gamma)$ by

$$(Tz)(\gamma) = n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} (Tx_i)(\gamma)$$

$$\geq n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} \left\langle P^*_{(p_{i-1},p_i)} e^*_\gamma, Tx_i \right\rangle - \left\| (I - P_{(p_{i-1},p_i)})(Tx_i) \right\|$$

$$\geq n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} m_{2j_0-1}^{-1} \left\langle P^*_{(p_{i-1},p_i)} e^*_{\eta_i}, Tx_i \right\rangle - m_{4j_1-2}^{-1} \delta$$

$$\geq \delta n_{2j_0-1}^{-1} \sum_{i=1}^{n_{2j_0-1}} (\frac{3}{8} m_{2j_0-1}^{-1} - n_{2j_0-1}^{-2})$$

$$\geq \frac{1}{4} m_{2j_0-1}^{-1} \delta.$$

This is a contradiction since now

$$\|Tz\| \geq \frac{1}{4} m_{2j_0-1}^{-1} \delta > (4 \cdot 30 \cdot 16C)^{-1} \delta m_{2j_0-1} \|z\|,$$

while by assumption $\|T\| < (4 \cdot 30 \cdot 16C)^{-1} \delta m_{2j_0-1}$. □

Theorem 6.4.5. *Let $T : X_{AH} \to X_{AH}$ be a bounded operator. Then there exists a scalar λ such that $T - \lambda I$ is compact.*

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a normalized RIS in X_{AH}. By the previous proposition we have that $\operatorname{dist}(Tx_i, \mathbb{R}x_i) \to 0$ as $i \to \infty$. This implies that there exist scalars λ_i such that $\|Tx_i - \lambda_i x_i\| \to 0$. Suppose that the λ_i do not converge to some limit λ. Then by passing to a subsequence we may assume that $|\lambda_i - \lambda_{i+1}| > \epsilon$ for all $i \in \mathbb{N}$. Now the sequence $(y_i = x_{2i-1} + x_{2i})_{i\in\mathbb{N}}$ is again an RIS. So there in turn exist μ_i such that $\|Ty_i - \mu_i y_i\| \to 0$. Thus

$$\|(\lambda_{2i} - \mu_i)x_{2i} + (\lambda_{2i-1} - \mu_i)x_{2i-1}\|$$

$$\leq \|Tx_{2i} - \lambda_{2i}x_{2i}\| + \|Tx_{2i-1} - \lambda_{2i-1}x_{2i-1}\| + \|Ty_i - \mu_i y_i\| \to 0.$$

Since $(x_i)_{i\in\mathbb{N}}$ is a block sequence we can find for each i a split point l_i such that $P_{(0,l_i]}y_i = x_{2i-1}$ and $P_{(l_i,\infty)}y_i = x_{2i}$. Then since the RIS is normalized we have

$$|\lambda_{2i-1} - \mu_i| = \|(\lambda_{2i-1} - \mu_i)x_{2i-1}\|$$

$$\leq \|P_{(0,l_i]}\| \, \|(\lambda_{2i} - \mu_i)x_{2i} + (\lambda_{2i-1} - \mu_i)x_{2i-1}\|,$$

which tends to 0, and the same holds for $|\lambda_{2i} - \mu_i|$. This implies that $|\lambda_{2i} - \lambda_{2i-1}| \to 0$, which is a contradiction.

Now we need to show that this scalar $\lambda = \lim_{i\to\infty} \lambda_i$ is the same for all rapidly increasing sequences. Indeed suppose that there exists another RIS $(x'_i)_{i\in\mathbb{N}}$ and a scalar λ' such that $\|Tx'_i - \lambda' x'_i\| \to 0$. Then we can find $(i_k)_{k\in\mathbb{N}}$ such that the sequence $(y_k)_{k\in\mathbb{N}}$ defined by

$$y_k = \begin{cases} x_{i_k} & \text{if } k \text{ is odd} \\ x'_{i_k} & \text{if } k \text{ is even} \end{cases}$$

is again an RIS. Then there is a third scalar μ such that $\|Ty_k - \mu y_k\| \to 0$. But this implies that $\lambda = \mu = \lambda'$. Thus we see that $\|(T - \lambda I)x_i\| \to 0$ for all rapidly increasing sequences $(x_i)_{i\in\mathbb{N}}$. By Proposition 6.3.12 we obtain $\|(T - \lambda I)x_i\| \to 0$ for all bounded block sequences $(x_i)_{i\in\mathbb{N}}$. Thus by the same arguments that yield Theorem 2.3.9, noting that the gliding hump argument easily extends to the case of an FDD, this implies that $T - \lambda I$ is compact. □

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