Axiomatic Approach to Topological Quantum Field Theory

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Introduction

The idea of topological invariants defined via path integrals was introduced by A S Schwartz (1977) in a special case and by E Witten (1988) in its full power. To formalize this idea, Witten (1988) introduced a notion of a topological quantum field theory (TQFT). Such theories, independent of Riemannian metrics, are rather rare in quantum physics. On the other hand, they admit a simple axiomatic description first suggested by M Atiyah (1989). This description was inspired by G Segal's (1988) axioms for a two-dimensional conformal field theory. The axiomatic formulation of TQFTs makes them suitable for a purely mathematical research combining methods of topology, algebra, and mathematical physics. Several authors explored axiomatic foundations of TQFTs (see Quinn (1995) and Turaev (1994).

Axioms of a TQFT

An $(n+1)$-dimensional TQFT $(\boldsymbol{V},\tau)$ over a scalar field $\boldsymbol{k}$ assigns to every closed oriented n-dimensional manifold X a finite-dimensional vector space $\boldsymbol{V}(X)$ over $\boldsymbol{k}$ and assigns to every cobordism (M,X,Y) a $\boldsymbol{k}$-linear map

$$\tau(M)=\tau(M,X,Y):\boldsymbol{V}(X)\to\boldsymbol{V}(Y)$$

Here a cobordism (M,X,Y) between X and Y is a compact oriented $(n+1)$-dimensional manifold M endowed with a diffeomorphism $\partial M\approx\overline{X}\amalg Y$ (the overline indicates the orientation reversal). All manifolds and cobordisms are supposed to be smooth. A TQFT must satisfy the following axioms.

1. *Naturality* Any orientation-preserving diffeomorphism of closed oriented n-dimensional manifolds $f:X\to X'$ induces an isomorphism $f_\sharp:\boldsymbol{V}(X)\to\boldsymbol{V}(X')$. For a diffeomorphism g between the cobordisms (M,X,Y) and (M',X',Y'), the following diagram is commutative:

$$\begin{array}{ccc}\boldsymbol{V}(X) & \xrightarrow{(g|_X)_\sharp} & \boldsymbol{V}(X')\\ \tau(M)\downarrow & & \downarrow\tau(M')\\ \boldsymbol{V}(Y) & \xrightarrow{(g|_Y)_\sharp} & \boldsymbol{V}(Y')\end{array}$$

2. *Functoriality* If a cobordism (W,X,Z) is obtained by gluing two cobordisms (M,X,Y) and (M',Y',Z) along a diffeomorphism $f:Y\to Y'$, then the following diagram is commutative:

$$\begin{array}{ccc}\boldsymbol{V}(X) & \xrightarrow{\tau(W)} & \boldsymbol{V}(Z)\\ \tau(M)\downarrow & & \uparrow\tau(M')\\ \boldsymbol{V}(Y) & \xrightarrow{f_\sharp} & \boldsymbol{V}(Y')\end{array}$$

3. *Normalization* For any n-dimensional manifold X, the linear map

$$\tau([0,1]\times X):\boldsymbol{V}(X)\to\boldsymbol{V}(X)$$

is identity.

4. *Multiplicativity* There are functorial isomorphisms

$$\boldsymbol{V}(X\amalg Y)\approx\boldsymbol{V}(X)\otimes\boldsymbol{V}(Y)$$
$$\boldsymbol{V}(\emptyset)\approx\boldsymbol{k}$$

such that the following diagrams are commutative:

$$\begin{array}{ccc}\boldsymbol{V}((X\amalg Y)\amalg Z) & \approx & (\boldsymbol{V}(X)\otimes\boldsymbol{V}(Y))\otimes\boldsymbol{V}(Z)\\ \downarrow & & \downarrow\\ \boldsymbol{V}(X\amalg(Y\amalg Z)) & \approx & \boldsymbol{V}(X)\otimes(\boldsymbol{V}(Y)\otimes\boldsymbol{V}(Z))\end{array}$$

$$\begin{array}{ccc}\boldsymbol{V}(X\amalg\emptyset) & \approx & \boldsymbol{V}(X)\otimes\boldsymbol{k}\\ \downarrow & & \downarrow\\ \boldsymbol{V}(X) & = & \boldsymbol{V}(X)\end{array}$$

Here $\otimes=\otimes_{\boldsymbol{k}}$ is the tensor product over $\boldsymbol{k}$. The vertical maps are respectively the ones induced by the obvious diffeomorphisms, and the standard isomorphisms of vector spaces.

5. *Symmetry* The isomorphism

$$\boldsymbol{V}(X\amalg Y)\approx\boldsymbol{V}(Y\amalg X)$$

induced by the obvious diffeomorphism corresponds to the standard isomorphism of vector spaces

$$\boldsymbol{V}(X)\otimes\boldsymbol{V}(Y)\approx\boldsymbol{V}(Y)\otimes\boldsymbol{V}(X)$$

Given a TQFT $(\boldsymbol{V},\tau)$, we obtain an action of the group of diffeomorphisms of a closed oriented n-dimensional manifold X on the vector space $\boldsymbol{V}(X)$. This action can be used to study this group.

An important feature of a TQFT $(\boldsymbol{V},\tau)$ is that it provides numerical invariants of compact oriented $(n+1)$-dimensional manifolds without boundary. Indeed, such a manifold M can be considered as a cobordism between two copies of $\emptyset$ so that $\tau(M)\in Hom_{\boldsymbol{k}}(\boldsymbol{k},\boldsymbol{k})=\boldsymbol{k}$. Any compact oriented $(n+1)$-dimensional manifold M can be considered as a cobordism between $\emptyset$ and ∂M; the TQFT assigns to this cobordism a vector $\tau(M)$ in $\mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{k}, \boldsymbol{V}(\partial M)) = \boldsymbol{V}(\partial M)$ called the vacuum vector.

The manifold $[0,1] \times X$, considered as a cobordism from $\overline{X} \amalg X$ to $\emptyset$ induces a nonsingular pairing

$$\boldsymbol{V}(\overline{X}) \otimes \boldsymbol{V}(X) \to \boldsymbol{k}$$

We obtain a functorial isomorphism $\boldsymbol{V}(\overline{X}) = \boldsymbol{V}(X)^* = \mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{V}(X), \boldsymbol{k})$.

We now outline definitions of several important classes of TQFTs.

If the scalar field $\boldsymbol{k}$ has a conjugation and all the vector spaces $\boldsymbol{V}(X)$ are equipped with natural nondegenerate Hermitian forms, then the TQFT $(\boldsymbol{V}, \tau)$ is Hermitian. If $\boldsymbol{k} = \boldsymbol{C}$ is the field of complex numbers and the Hermitian forms are positive definite, then the TQFT is unitary.

A TQFT $(\boldsymbol{V}, \tau)$ is nondegenerate or cobordism generated if for any closed oriented n-dimensional manifold X, the vector space $\boldsymbol{V}(X)$ is generated by the vacuum vectors derived as above from the manifolds bounded by X.

Fix a Dedekind domain $D \subset \boldsymbol{C}$. A TQFT $(\boldsymbol{V}, \tau)$ over $\boldsymbol{C}$ is almost D-integral if it is nondegenerate and there is $d \in \boldsymbol{C}$ such that $d\tau(M) \in D$ for all M with $\partial M = \emptyset$. Given an almost integral TQFT $(\boldsymbol{V}, \tau)$ and a closed oriented n-dimensional manifold X, we define $S(X)$ to be the D-submodule of $\boldsymbol{V}(X)$ generated by all the vacuum vectors. This module is preserved under the action of self-diffeomorphisms of X and yields a finer "arithmetic" version of $\boldsymbol{V}(X)$.

The notion of an $(n+1)$-dimensional TQFT over $\boldsymbol{k}$ can be reformulated in the categorical language as a symmetric monoidal functor from the category of n-manifolds and $(n+1)$-cobordisms to the category of finite-dimensional vector spaces over $\boldsymbol{k}$. The source category is called the $(n+1)$-dimensional cobordism category. Its objects are closed oriented n-dimensional manifolds. Its morphisms are cobordisms considered up to the following equivalence: cobordisms (M, X, Y) and (M', X, Y) are equivalent if there is a diffeomorphism $M \to M'$ compatible with the diffeomorphisms $\partial M \approx \overline{X} \amalg Y \approx \partial M'$.

TQFTs in Low Dimensions

TQFTs in dimension $0 + 1 = 1$ are in one-to-one correspondence with finite-dimensional vector spaces. The correspondence goes by associating with a one-dimensional TQFT $(\boldsymbol{V}, \tau)$ the vector space $\boldsymbol{V}(pt)$ where pt is a point with positive orientation.

Let $(\boldsymbol{V}, \tau)$ be a two-dimensional TQFT. The linear map τ associated with a pair of pants (a 2-disk with two holes considered as a cobordism between two circles $S^1 \amalg S^1$ and one circle S^1) defines a commutative multiplication on the vector space $\mathcal{A} = \boldsymbol{V}(S^1)$. The 2-disk, considered as a cobordism between S^1 and $\emptyset$, induces a nondegenerate trace on the algebra $\mathcal{A}$. This makes $\mathcal{A}$ into a commutative Frobenius algebra (also called a symmetric algebra). This algebra completely determines the TQFT $(\boldsymbol{V}, \tau)$. Moreover, this construction defines a one-to-one correspondence between equivalence classes of two-dimensional TQFTs and isomorphism classes of finite dimensional commutative Frobenius algebras (Kock 2003).

The formalism of TQFTs was to a great extent motivated by the three-dimensional case, specifically, Witten's Chern–Simons TQFTs. A mathematical definition of these TQFTs was first given by Reshetikhin and Turaev using the theory of quantum groups. The Witten–Reshetikhin–Turaev three-dimensional TQFTs do not satisfy exactly the definition above: the naturality and the functoriality axioms only hold up to invertible scalar factors called framing anomalies. Such TQFTs are said to be projective. In order to get rid of the framing anomalies, one has to add extra structures on the three-dimensional cobordism category. Usually one endows surfaces X with Lagrangians (maximal isotropic subspaces in $H_1(X; \boldsymbol{R})$). For 3-cobordisms, several competing – but essentially equivalent – additional structures are considered in the literature: 2-framings (Atiyah 1989), p_1-structures (Blanchet *et al.* 1995), numerical weights (K Walker, V Turaev).

Large families of three-dimensional TQFTs are obtained from the so-called modular categories. The latter are constructed from quantum groups at roots of unity or from the skein theory of links. *See* Quantum 3-Manifold Invariants.

Additional Structures

The axiomatic definition of a TQFT extends in various directions. In dimension 2 it is interesting to consider the so-called open–closed theories involving 1-manifolds formed by circles and intervals and two-dimensional cobordisms with boundary (G Moore, G Segal). In dimension 3 one often considers cobordisms including framed links and graphs whose components (resp. edges) are labeled with objects of a certain fixed category $\mathcal{C}$. In such a theory, surfaces are endowed with finite sets of points labeled with objects of $\mathcal{C}$ and enriched with tangent directions. In all dimensions one can study manifolds and cobordisms endowed with homotopy classes of mappings to a fixed space (homotopy quantum field theory, in the sense of Turaev). Additional structures on the tangent bundles – spin structures, framings, etc. – may be also considered provided the gluing is well defined.

See also: Braided and Modular Tensor Categories; Hopf Algebras and *q*-Deformation Quantum Groups; Indefinite Metric; Quantum 3-Manifold Invariants; Topological Gravity, Two-Dimensional; Topological Quantum Field Theory: Overview.

Further Reading

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Axiomatic Quantum Field Theory

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Introduction

The term "axiomatic quantum field theory" subsumes a collection of research branches of quantum field theory analyzing the general principles of relativistic quantum physics. The content of the results typically is structural and retrospective rather than quantitative and predictive.

The first axiomatic activities in quantum field theory date back to the 1950s, when several groups started investigating the notion of scattering and *S*-matrix in detail (Lehmann, Symanzik, and Zimmermann 1955 (LSZ-approach), Bogoliubov and Parasiuk 1957, Hepp and Zimmermann (BPHZ-approach), Haag 1957–59 and Ruelle 1962 (Haag–Ruelle theory) (*see* Scattering, Asymptotic Completeness and Bound States and Scattering in Relativistic Quantum Field Theory: Fundamental Concepts and Tools).

Wightman (1956) analyzed the properties of the vacuum expectation values used in these approaches and formulated a system of axioms that the vacuum expectation values ought to satisfy in general. Together with Gårding (1965), he later formulated a system of axioms in order to characterize general quantum fields in terms of operator-valued functionals, and the two systems have been found to be equivalent.

A couple of spectacular theorems such as the PCT theorem and the spin–statistics theorem have been obtained in this setting, but no interacting quantum fields satisfying the axioms have been found so far (in $1+3$ spacetime dimensions). So, the development of alternatives and modifications of the setting got into the focus of the theory, and the axioms themselves became the objects of research. Their role as axioms – understood in the common sense – turned into the role of mere properties of quantum fields. Today, the term "axiomatic quantum field theory" is widely avoided for this reason.

In a long list of publications spread over the 1960s, Araki, Borchers, Haag, Kastler, and others worked out an algebraic approach to quantum field theory in the spirit of Segal's "postulates for general quantum Mechanics" (1947) (*see* Algebraic Approach to Quantum Field Theory).

The Wightman setting was the basis of a framework into which the causal construction of the *S*-matrix developed by Stückelberg (1951) and Bogoliubov and Shirkov (1959) has been fitted by Epstein and Glaser (1973). The causality principle fixes the time-ordered products up to a finite number of parameters at each order, which are to be put in as the renormalization constants.

Already in 1949, Dyson had seen that problems in the formulation of quantum electrodynamics (QED) could be avoided by "just" multiplying the time variable and, correspondingly, the energy variable by the imaginary unit constant ("Wick rotation"). Schwinger then investigated time-ordered Green functions of QED in this Euclidean setting. This approach was formulated in terms of axioms by Osterwalder and Schrader (1973, 1975) (*see* Euclidean Field Theory).

Other extensions of the aforementioned settings are objects of current research (*see* Indefinite Metric,